AN ANALYTICAL STUDY OF ELECTROMAGNETIC VECTOR FIELD PROPAGATION IN A NONLINEAR ELECTRON PLASMA

Thesis by

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ABSTRACT

From the equations of hydrodynamics and electrodynamics, a system of a coupled nonlinear equations governing the propagation of plane electromagnetic waves in a collisionless electron plasma is obtained. It is shown that solitary wave solutions exist for both the longitudinal and transverse components of the electromagnetic field. It is found that the velocity of the electromagnetic vector solitary wave depends on the amplitudes of all components of the field linearly. The relations among the longitudinal and transverse components that support the solitary waves are determined for different values of plasma temperature. It is shown that while transverse solitary waves cannot exist, except when they are supported by longitudinal waves, the latter can exist by themselves. The dynamics of the plasma electrons during the passage of a longitudinal wave is analyzed and the interaction of such waves with each other is studied. An upper bound on the amplitudes of these waves is obtained. The uniqueness and stability of the longitudinal waves are demonstrated. A Lagrangian density function and two conservation laws for the longitudinal wave equation are found. Frequency spectra of the solitary waves are calculated and their low frequency content is emphasized.

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1. INTRODUCTION

The study of electromagnetic wave propagation in a plasma, that is, in an ionized gas, has been the subject of a large number of papers and books. We shall here only mention some monographs which contain extensive bibliographies, namely books by Ginzburg [1970], Tsytovich [1970], Lifschitz and Pitaevskii [1981] and Spitzer [1962]. Besides the fact that studies of electromagnetic wave propagation in plasma enhance our understanding about dispersive media, waves in plasma play an exceptionally important role in astrophysics and in the physics of the ionosphere [Ginzburg, 1979]. Although the development of plasma wave theory began with the consideration of linear effects only, soon it was shown that a linear theory does not answer the fundamental question of how significant plasma instabilities are and of how they effect plasma parameters [Tsytovich, 1970, Ch.1]. Moreover, a linear theory does not explain the conversion of longitudinal waves into transverse waves, which is a purely nonlinear effect that plays an important role in the interpretation of many astrophysical phenomena, such as chromospheric flares and the emission from supernovae, whose investigation can provide us with considerable information about the origin of cosmic rays [Tsytovich, 1970, Ch. 1]. Since the concept of solitary wave (or soliton), which is a product of nonlinear wave theory, was introduced in physics an extensive search for electromagnetic solitary waves has been undertaken in plasma physics, because of their very interesting and promising features, namely their consistent shape, their constant velocity, and their ability to interact cleanly. Based on Ambartsumian's hypothesis that stellar flares and other stellar instabilities are due to the energy transport from the core of the star (prestellar matter) to its outer layer, Papas [1976] introduced the idea that this energy is transported by solitons. His idea is based on the fact that since the content of the star is plasma, which is a highly dispersive media, it is unlikely that ordinary electromagnetic waves could travel enormously large distances through the stellar matter to the outer layer of the star, because they would disperse in the process of propagation and leave no trace on the surface of the star. When solitons release their energy in the chromosphere, the charged particles obtain enormous amounts of energy and therefore they start to radiate inordinately. The mechanism of such radiation most likely resembles the mechanism that produces a gigantic electromagnetic pulse due to high altitude nuclear explosion in the earth athmosphere, which has become the subject of considerable interest [Marable et al., 1972].

Another important application of solitary waves in plasma is electromagnetic wave propagation in the ionosphere. This phenomenon can play an exceptional role in communication between spacecraft and earth, since there are limitations on the frequency of ordinary electromagnetic waves travelling through plasma [Papas, 1965, Ch. 6]. The transmission of information through the ionosphere by means of solitary waves would be free of undesirable distortions due to nonlinear and dispersive effects because solitary waves are remarkably stable entities. The study of solitary waves in laboratory plasma has recently attracted much attention from the theoretical and the practical points of view.

Derivations of equations governing electromagnetic wave propagation in plasma are based, generally speaking, on a combination of the Maxwell equations that describe electromagnetic wave propagation and the hydrodynamic equations that characterize the dynamics of a plasma as a fluid.

The present work represents the first, to the best of our knowledge, treatment of electromagnetic vector solitary waves in an unbiased plasma, where the study of the existence and properties of such vector fields is based on the Maxwell equations and the equations of hydrodynamics, including, as it will be shown later, pressure and nonlinear Lorentz force terms. By vector solitary wave we mean some electromagnetic field quantity, say electric field **E**, all components of which are solitary waves. The main difficulty here, as pointed out by Whitham [1973, pp. 213-214], is the coupling between different components of the same vector field, which gives rise to the interaction between these components.

In order to obtain a single (scalar) equation for some electromagnetic field quantity in a plasma different approaches are used in the literature to approximate the hydrodynamic equations for various states of a plasma. The complexity and variety of approaches and assumptions are based, primarily, on the complex nature of a plasma itself, since it is characterized by electrical, mechanical and also thermal properties. Davis et al. [1958] in their study of hydromagnetic shock waves in a cold, collisionless plasma consider infinite plane compressional waves travelling perpendicular to a uniform biased magnetic field, where the transverse component of the electric field is taken to be a constant, and obtain a scalar solitary wave solution for a magnetic field in the direction of propagation. A large number of papers investigate solitons in a cold collisionless plasma by assuming that these solitons are governed by the Korteweg de-Vries (KdV) equation:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0\,, \tag{1.1}$$

where u denotes the magnitude of some perturbation. Gardner and Morikawa [1960] arrive at (1-1) by considering only one component of the electric field in the first-order approximation of a collisionless cold plasma in a biased magnetic field, and by neglecting the pressure term in the equation of motion. Lamb [1980, Ch. 6.3], Lifschitz and Pitaevskii [1981, Ch. 38, 39], Ikezi et al. [1970] obtain equation (1.1) for ion acoustic waves, that is for fluctuations in the ion density in a two-component plasma, using perturbation expansions for scalar electric field, ion velocity, and density, and ignoring the nonlinear Lorentz force. Again, using (1-1) as the equation for the finite perturbations in plasmas, Berezin and Karpman [1967] found the conditions for the decomposition of disturbances into solitons, and gave an heuristic method for predicting their number and speed. Zabusky and Kruskal [1965], assuming periodic initial conditions, observed nonlinear interaction among solitary wave pulses in their numerical investigation of KdV equation. Studying a turbulent plasma, Gibbons et al. [1977] make extensive use of the conservation laws to study interaction processes between ion acoustic waves and Langmuir solitons, which result from the equations describing the nonlinear state of plasma instability [Zakharov, 1972]. As in many other investigations they neglect the nonlinear Lorentz force in the hydrodynamic approximation. Solitary hydromagnetic waves, propagating parallel and at an angle to a uniform magnetic field in a cold collision-free plasma were derived by Saffman [1961a] and Saffman [1961b], where it was concluded that in the latter case the solitary waves are unstable. The only experimental work, known to us, with regard to plasma solitons is work by Ikezi [1978], where using electric discharge in argon gas he obtains small amplitude ion-acoustic density solitons.

It should be emphasized that in what follows we shall have in mind a nonrelativistic gaseous plasma for the description of which it is sufficient to use the classical approximations. It is just this kind of plasma which one usually encounters in astrophysics, and in the physics of the ionosphere.

2. FORMULATION OF THE PROBLEM

2.1 Derivation of the Vector Field Equation

To consider nonlinear effects in plasma, we assume that only the plasma electrons contribute to the plasma current, since the relation between plasma current and electric field in the linear approximation arises solely from the displacement of the plasma electrons.

The equation of motion of the electrons in a collisionless plasma under the influence of external electric and magnetic fields is

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w} \cdot \nabla)\mathbf{w} = \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{w} \times \mathbf{H}\right) - \frac{1}{mn_o}\nabla p \,. \tag{2.1}$$

where $\mathbf{w}$ is the electron velocity; n is the electron density; n_o is the electron density of a homogeneous plasma; $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields respectively; c is the velocity of light; p is the electron pressure, and m and e are the electron mass and charge respectively. The equation of continuity of the electrons is

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{w}) = 0 \,. \tag{2.2}$$

The plasma current is assumed to include the contribution of every single electron and therefore can be written as

$$\mathbf{j} = en\mathbf{w} \tag{2.3}$$

Furthermore, it is assumed also that plasma is of infinite extent. The electromagnetic field in plasma is expressed by Maxwell equations:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial H}{\partial t}, \tag{2.4}$$

and

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.5}$$

Eliminating the magnetic field **H** from above system we obtain

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t}. \tag{2.6}$$

If we write the expression for **j**, generated in the plasma, in the form

$$\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2 + \mathbf{j}_3 + \cdots \tag{2.7}$$

where subscript 1 indicates that this part of the current contains linear terms with respect to the total field, subscript 2 indicates that this part of the current contains quadratic terms with respect to the total field, etc., the equations (2.6) and (2.7) will constitute nonlinear equation in **E**, describing the electrodynamics of nonlinear wave propagation in plasma.

Equations (2.1) and (2.2) together with (2.3) and (2.7) can be used to find the nonlinear current to any order in the field **E**. This will enable us to substitute expression for the current into equation (2.6) and, therefore, to obtain a single equation for electric field **E**. At this point it is convenient to transform equations (2.1), (2.2) and (2.3) from $(\mathbf{r}, t)$ space into $(\mathbf{k}, \omega)$ space expanding the variables in Fourier series :

$$\mathbf{w}(\mathbf{r},t) = \int \mathbf{w}(\vec{k}) e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, d\vec{k}, \tag{2.8}$$

where $\vec{k}$ is a four-vector $\vec{k} = (\mathbf{k},\omega)$, $d\vec{k} = d\mathbf{k}d\omega$ and $\mathbf{w}(\vec{k})$ is a counterpart of $\mathbf{w}(\mathbf{r},t)$ in a $\vec{k}$ space:

$$\mathbf{w}(\vec{k}) = \frac{1}{(2\pi)^4}\int \mathbf{w}(\mathbf{r},t) e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, d\mathbf{r}dt. \tag{2.9}$$

Equations (2.1), (2.2) and (2.3) in a $\vec{k}$ space become correspondingly,

$$-i\omega\mathbf{w}(\vec{k}) + i\int (\mathbf{w}(\vec{k}_1)\cdot\mathbf{k}_2)\,\mathbf{w}(\vec{k}_2)\, d\lambda - \frac{e}{m}\mathbf{E}(\vec{k})$$

$$-\frac{e}{mc}\int \mathbf{w}(\vec{k}_1)\times \mathbf{H}(\vec{k}_2)\,d\lambda - \frac{i}{mn}\mathbf{k}p(\vec{k}) = 0\,, \tag{2.10}$$

$$\omega n(\vec{k}) = \mathbf{k}\cdot\left(\int n(\vec{k}_1)\,\mathbf{w}(\vec{k}_2)\,d\lambda\right), \tag{2.11}$$

$$\mathbf{j}(\vec{k}) = e\int n(\vec{k}_1)\mathbf{w}(\vec{k}_2)\,d\lambda\,. \tag{2.12}$$

where $d\lambda = d\vec{k}_1\, d\vec{k}_2\, \delta(\vec{k}-\vec{k}_1-\vec{k}_2)$ and $\delta(\vec{k})$ is Dirac's delta function.

The second integral in (2.10) represents the Lorentz' force in a $\vec{k}$ space. To rewrite this integral in terms of the electric field we use Maxwell equation (2.4) in a $\vec{k}$ space

$$\frac{1}{c}\mathbf{H}(\vec{k}) = \frac{1}{\omega}\mathbf{k}\times\mathbf{E}(\vec{k})\,. \tag{2.13}$$

Accordingly the second integrant in (2.10) becomes

$$\frac{\mathbf{w}(\vec{k}_1)}{c}\times\mathbf{H}(\vec{k}_2) = \frac{1}{\omega_2}\mathbf{w}(\vec{k}_1)\times(\mathbf{k}_2\times\mathbf{E}(\vec{k}_2))$$

$$= \frac{\mathbf{k}_2}{\omega_2}(\mathbf{w}(\vec{k}_1)\cdot\mathbf{E}(\vec{k}_2)) - \frac{1}{\omega_2}\mathbf{E}(\vec{k}_2)(\mathbf{k}_2\cdot\mathbf{w}(\vec{k}_1))\,. \tag{2.14}$$

Substituting (2.14) into (2.10) we obtain

$$\mathbf{w}(\vec{k}) = \frac{ie}{\omega m}\mathbf{E}(\vec{k}) + \frac{ie}{\omega m}\int\frac{\mathbf{k}_2}{\omega_2}(\mathbf{w}(\vec{k}_1)\cdot\mathbf{E}(\vec{k}_2))\,d\lambda$$

$$+\frac{1}{\omega}\int(\mathbf{k}_2\cdot\mathbf{w}(\vec{k}_1))\left(\mathbf{w}(\vec{k}_2)-\frac{ie}{m\omega_2}\mathbf{E}(\vec{k}_2)\right)d\lambda + \frac{1}{mn_o\omega}\mathbf{k}p(\vec{k})\,. \tag{2.15}$$

Clearly, the terms containing the integrals describe nonlinear effects. We can neglect them in the first approximation to obtain

$$\mathbf{w}(\vec{k}) = \mathbf{w}_o(\vec{k}) + \mathbf{w}_1(\vec{k})\,, \tag{2.16}$$

$$\mathbf{w}_o(\vec{k}) = 0\,;\quad \mathbf{w}_1(k) = \frac{ie}{\omega m}\mathbf{E}(\vec{k}) + \frac{1}{mn_o\omega}\mathbf{k}p(\vec{k})\,, \tag{2.17}$$

where subscripts 0 and 1 play the same role as in (2.7). Substituting the first approximation (2.17) into nonlinear terms in (2.15) we find the second-order approximation with respect to the external field, namely,

$$\mathbf{w}_2(\vec{k}) = \frac{-e^2}{m^2\omega}\int \frac{1}{\omega_1\omega_2}\mathbf{k}_2\left(\mathbf{E}(\vec{k}_1)\cdot\mathbf{E}(\vec{k}_2)\right)d\lambda$$

$$+\frac{c_o^2}{\omega mn}\int\frac{1}{\omega_1\omega_2}\mathbf{k}_2\left(\mathbf{k}_1\left(\mathbf{k}_1\cdot\mathbf{E}(\vec{k}_1)\right)\cdot\mathbf{E}(\vec{k}_1)\right)d\lambda \tag{2.18}$$

$$+\frac{1}{\omega}\int\mathbf{k}_2\cdot\left[\frac{ie}{\omega_1 m}\mathbf{E}(\vec{k}_1)+\frac{ic_o^2}{\omega ne4\pi}\mathbf{k}(\mathbf{k}\cdot\mathbf{E}(\vec{k}))\right]\frac{ic_o^2}{\omega_2 n_o e}\mathbf{k}_1(\mathbf{k}_1\cdot\mathbf{E}(\vec{k}_1))d\lambda,$$

where we assumed that passage of the wave in plasma causes one-dimensional adiabatic changes, so that

$$p = 3\kappa Tn = c_o^2 mn\ , \quad c_o = \sqrt{3\frac{\kappa T}{m}}, \tag{2.19}$$

where κ is the Boltzmann's constant. In the derivation of (2.18) we used the Maxwell equation

$$\nabla\cdot\mathbf{E} = -4\pi ne\ . \tag{2.20}$$

The assumption that leads to (2.19) reflects the adiabatic law $p \sim n^\gamma$ and since $\gamma = (N+2)/N$, where N is the number of degrees of freedom, we have $\gamma = 3$. The consequence of this assumption, as we will see later, is that the dispersion relation for an infinite homogeneous plasma derived by the present method agrees with the dispersion relation obtained from the Boltzmann equation[1] for the case of long wavelengths. Neglecting the term of order $0(\frac{1}{n_o^2})$ and cancelling appropriate terms in (2.18) we obtain

$$\mathbf{w}_2(\vec{k}) = \frac{-e^2}{2m^2\omega}\int\frac{1}{\omega_1\omega_2}\mathbf{k}\left(\mathbf{E}(\vec{k}_1)\right)\cdot\mathbf{E}(\vec{k}_2)\,)\,d\lambda\ , \tag{2.21}$$

where symmetrization $\mathbf{k}_2 \rightarrow \frac{1}{2}(\mathbf{k}_1+\mathbf{k}_2) = \frac{1}{2}\mathbf{k}$ was used. If we expand the density n similarly to the expansions for $\mathbf{J}$ and $\mathbf{w}$, namely,

1. For the description of the Boltzmann's equation see Lifshitz and Pitaevskii [1981], or Ginzburg [1960].

$$n = n_o(\vec{k}) + n_1(\vec{k}) + n_2(\vec{k}) + \cdots \quad , \tag{2.22}$$

the zeroth approximation corresponding to a homogeneous plasma $n_o(r,t) = n_o = \text{const}$, it follows that

$$n_o(\vec{k}) = \left(\frac{1}{2\pi}\right)^4 \int n_o(r,t)\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, d\mathbf{r}dt = n_o \delta(\vec{k}) \tag{2.23}$$

The first approximation follows from the continuity equation

$$\omega n_1(\vec{k}) = \mathbf{k} \cdot \int (n_o(\vec{k}_1)\, \mathbf{w}_1(\vec{k}_2) + n_1(\vec{k}_1)\, \mathbf{w}_o(\vec{k}_2))\, d\lambda \ . \tag{2.24}$$

With the use of (2.17), equation (2.23) becomes

$$n_1(\vec{k}) = \frac{in_o e}{m\omega^2} \left[(\mathbf{k} \cdot \mathbf{E}(\vec{k})) + \frac{c_o^2}{4\pi e\omega} \mathbf{k} \cdot \left[\mathbf{k}\, (\mathbf{k} \cdot \mathbf{E}(\vec{k})) \right] \right] . \tag{2.25}$$

Hence the first approximation for the nonlinear plasma current is

$$\begin{aligned} \mathbf{j}_1(\vec{k}) &= e \int n_o(\vec{k}_1)\, \mathbf{w}_1(\vec{k}_2)\, d\lambda \\ &= e n_o \int \mathbf{w}_1(\vec{k}_2)\, \delta(\vec{k} - \vec{k}_1 - \vec{k}_2)\, \delta(\vec{k}_1)\, d\vec{k}_1 d\vec{k}_2 \\ &= e n_o \mathbf{w}_1(\vec{k}) \\ &= e n_o \left[\frac{ie}{\omega m} \mathbf{E}(\vec{k}) + \frac{ic_o^2}{\omega n e 4\pi} \mathbf{k}\, (\mathbf{k} \cdot \mathbf{E}(\vec{k})) \right] . \end{aligned} \tag{2.26}$$

Similarly, with the aid of (2.12), the second order current term can be approximated by

$$\begin{aligned} \mathbf{j}_2(\vec{k}) &= e \int \left[n_1(\vec{k}_1)\, \mathbf{w}_1(\vec{k}_2) + n_o(\vec{k}_1)\, \mathbf{w}_2(\vec{k}_2) \right] d\lambda \\ &= \frac{-\omega_p^2 e}{8\pi m} \int \frac{1}{\omega_1 \omega_2} \left[\frac{1}{\omega} \mathbf{k}\ (\mathbf{E}(\vec{k}_1) \cdot \mathbf{E}(\vec{k}_2)) + \mathbf{E}(\vec{k}_1)\, \frac{\mathbf{k}_2 \cdot \mathbf{E}(\vec{k}_2)}{\omega_2} \right. \\ &\quad \left. + \mathbf{E}(\vec{k}_2)\, \frac{\mathbf{k}_1 \cdot \mathbf{E}(\vec{k}_1)}{\omega_1} \right] d\lambda , \end{aligned} \tag{2.27}$$

where $\omega_p^2 = \dfrac{4\pi n_o e^2}{m}$ denotes plasma frequency. Transforming $\mathbf{j}_1(\vec{k})$ and $\mathbf{j}_2(\vec{k})$

back into (**r**,t) space, using (2.9), and substituting the transformed expressions in (2.6) we obtain

$$c^2\nabla \times \nabla \times \mathbf{E} + 3a_e^2\omega_p^2\nabla (\nabla \cdot \mathbf{E}) + \frac{\partial^2\mathbf{E}}{\partial t^2} + \omega_p^2\mathbf{E} = \frac{\omega_p^2 e}{2m}\frac{\partial}{\partial t}(\mathbf{Z} + 2\psi\nabla \cdot \varphi), \quad (2.28)$$

where $a_e^2 = c_o^2 / 3\omega_p^2 = \kappa T / 4\pi n_o e^2$ is electron Debye radius, κ is the Boltzmann's constant, T is the temperature, and

$$\frac{\partial \mathbf{Z}}{\partial t} = \nabla (\psi \cdot \psi) \quad (2.29)$$

$$\frac{\partial \psi}{\partial t} = \mathbf{E} \quad (2.30)$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \mathbf{E} \quad (2.31)$$

Equation (2.28) describes electromagnetic wave propagation in an isotropic, collisionless electron plasma in the presence of Lorentz force and pressure force, both appearing in (2.1). The term $\mathbf{w} \times \mathbf{H}$ and the second term in (2.1) constitute the nonlinearity that appears on the right-hand side of equation (2.28).

This report is concerned primarily with an investigation of the electromagnetic waves, governed by equation (2.28). In particular, the main effort is focused on the study of the stationary wave solutions.

2.2 Scalarization

To reduce (2.28) to the scalar form, let's assume that the electric field vector **E** consists of two components: longitudinal, in the assumed direction of propagation of the wave, and transverse, in the direction normal to the direction of propagation. Also let x denote the direction of propagation and z the direction of the transverse component of the electric field (see Fig.1). Clearly the electric field can be expressed now as $\mathbf{E}(x,t) = \left[E^L(x,t) \;\; 0 \;\; E^T(x,t) \right]$. The equalities (2.31), (2.30) and (2.29) along with the above assumptions yield the following

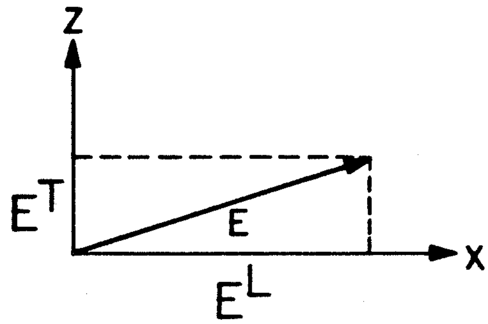

Figure II.1 The longitudinal and transverse components of the electric field.

scalarized version of (2.26) in terms of the "potential" function φ:

$$L_1 \varphi^L = \frac{\omega_p^2 e}{m} \left(2 \varphi_{xt}^L \varphi_t^L + \varphi_{tt}^L \varphi_x^L + \varphi_{tx}^T \varphi_t^T \right)$$

$$L_2 \varphi^T = \frac{\omega_p^2 e}{m} \left(\varphi_{tt}^T \varphi_x^L + \varphi_t^T \varphi_{xt}^L \right) , \qquad (2.32)$$

where φ^L and φ^T are the longitudinal and transverse components of the potential function φ respectively, the subscripts x and t denote partial differentiation with respect to x and t, and

$$L_1 = \frac{\partial^4}{\partial t^4} - 3\,\omega_p^2 a_e^2 \frac{\partial^4}{\partial x^2 \partial t^2} + \omega_p^2 \frac{\partial^2}{\partial t^2} ,$$

and

$$L_2 = \frac{\partial^4}{\partial t^4} - c^2 \frac{\partial^4}{\partial x^2 \partial t^2} + \omega_p^2 \frac{\partial^2}{\partial t^2} ,$$

are linear operators. System (2.32) can be simplified by noting that for any wave profile moving with a speed U, the t and x derivatives are related by

$$\frac{\partial}{\partial t} = - U \frac{\partial}{\partial x} . \qquad (2.33)$$

With help of (2.33), system (2.32) becomes

$$L_{1m}\psi^L = \frac{\omega_p^2 e}{m}\left(3\psi_x^L \psi^L + \psi_x^T \psi^T\right)$$

$$(2.34)$$

$$L_{2m}\psi^T = \frac{\omega_p^2 e}{m}\left(\psi^T \psi^L\right)_x$$

where ψ^L and ψ^T are components of the potential function ψ, defined in (2.28), and

$$L_{1m} = \frac{\partial^3}{\partial t^3} - 3\,\omega_p^2 a_e^2 \frac{\partial^3}{\partial x^2 \partial t} + \omega_p^2 \frac{\partial}{\partial t},$$

and

$$L_{2m} = \frac{\partial^3}{\partial t^3} - c^2 \frac{\partial^3}{\partial x^2 \partial t} + \omega_p^2 \frac{\partial}{\partial t}.$$

Thus we shall study system (2.34), that clearly exhibits the coupling between longitudinal and transverse components of the field. This coupling constitutes one of the main difficulties of the problem.

Consider the linear parts of system (2.34), namely,

$$L_{1m}\psi^L = 0\,,$$

and

$$L_{2m}\psi^T = 0\,.$$

The corresponding dispersion relations are

$$\omega = \omega_p\,(1 + 3a_e^2 k^2)^{1/2}\,, \qquad (2.35)$$

and

$$\omega = (\omega_p^2 + c^2 k^2)^{1/2}\,, \qquad (2.36)$$

where k stands for a wavenumber.

With $a_e^2 k^2 << 1$, (2.35) can be approximated by

$$\omega \approx \omega_p \left(1 + \frac{3}{2} a_e^2 k^2\right) . \qquad (2.37)$$

Dispersion relations (2.36) and (2.37), that correspond to longitudinal and transverse waves respectively, are obtained here directly from the system (2.35). Lifshitz and Pitaevskii [1981] derive relations (2.36) and (2.37) from expressions for the dielectric constants in the transverse and longitudinal directions; ε^T and ε^L respectively, considering collisionless and isotropic plasmas. In a case when plasma is cold we have $a_e = 0$ and (2.38) reduces to

$$\omega = \omega_p . \qquad (2.38)$$

Equation (2.38) shows that frequency ω of the longitudinal wave is independent of the wave number k. Therefore, there is no dispersion in (2.38), whereas (2.36) and (2.37) exhibit the properties of dispersive waves, namely ω (k) is real and $\frac{d^2\omega(k)}{dk^2} \neq 0$. Longitudinal waves with dispersion relation (2.38) are known as plasma oscillations, [Tsytovich 1970, ch. 1.]. Sometimes they are referred to as Langmuir type waves [Lifshitz and Pitaevskii, 1981, § 32]. It is clear that the term $\frac{3}{2} a_e^2 k^2$ in (2.37) is the result of inclusion of the pressure term in (2.1), which in turn produces the term $3 a_e^2 \omega_p^2 \frac{\partial^3}{\partial x^2 \partial t}$ in the linear operator L_{1m}, since $\nabla p \sim \nabla \rho$, $\rho \sim \nabla \cdot \mathbf{E}$ and therefore $\nabla p \sim \nabla (\nabla \cdot \mathbf{E}) = E^L_{xx}$ (ρ denotes charge density).

Before we start investigating systems (2.32) or (2.34), an important observation should be made: $\psi^L = 0$ implies $\psi^T = 0$, and $\psi^T = 0$ does not imply $\psi^L = 0$. This means that in a nonlinear plasma transverse waves cannot exist by themselves, whereas longitudinal waves can.

3. LONGITUDINAL WAVES

3.1 Solitary Waves

Using the following transformations of dependant variables

$$\Psi^L = u + u_o, \quad \Psi^L = \frac{3|e|}{m}\psi^L, \tag{3.1}$$

and

$$\Psi^T = v + v_o, \quad \Psi^T = \frac{|e|}{3m}\psi^T, \tag{3.2}$$

where u_o and v_o are constants (independent of x and t), system (2.33) becomes

$$u_t + u_o u_x + uu_x + 3v_o v_x + 3vv_x + \frac{1}{\omega_p^2}u_{ttt} - 3a_e^2 u_{xxt} = 0, \tag{3.3}$$

$$v_t + \frac{1}{3}v_x u_o + \frac{1}{3}v_x u + \frac{1}{3}u_x v + \frac{1}{3}u_x v_o + \frac{1}{\omega_p^2}v_{ttt} - \frac{c^2}{\omega_p^2}v_{xxt} = 0. \tag{3.4}$$

New potentials u and v correspond to the longitudinal and transverse components of the field **E** respectively and they have dimension of velocity. Constants u_o and v_o should be regarded as characteristic velocities for the respective components of the field. To the physical interpretation of these constants we shall return later.

By assuming that

$$|\Psi_x^L \Psi^L| >> |\Psi_x^T \Psi^T|, \tag{3.5}$$

and using (3.1.) and (3.2), equation (3.3) reduces to

$$u_t + u_o u_x + uu_x + \frac{1}{\omega_p^2}u_{ttt} - 3a_e^2 u_{xxt} = 0. \tag{3.6}$$

Condition (3.5) means that here the attention is limited to fields that are predominantly longitudinal. Solutions of constant shape and moving with constant velocity are found by assuming that

$$u(x,t) = \bar{u}(x - Ut) = \bar{u}(\zeta)\,. \tag{3.7}$$

With an assumption (3.7), (3.6) becomes

$$- U\bar{u}' + u_o\bar{u}' + \bar{u}\,\bar{u}' + \frac{U}{\omega_p^2}(3a_e^2\,\omega_p^2 - U^2)\,\bar{u}''' = 0\,, \tag{3.8}$$

where primes denote differentiation w. r. t. the variable ζ. Integrating (3.8), then multiplying the result by u' and again integrating give

$$h\bar{u}'^2 = -\frac{1}{3}\bar{u}^3 + (U - u_o)\,\bar{u}^2 - 2A\bar{u} - 2B$$

$$= C(\bar{u}) = 0\,, \tag{3.9}$$

where

$$h = \frac{U}{\omega_p^2}(3a_e^2\,\omega_p^2 - U^2) = \frac{U}{\omega_p^2}(c_o^2 - U^2)\,. \tag{3.10}$$

where c_o is defined in (2.19), and A and B are constants of integration. When A and B are different from zero, the solution of (3.9) can be expressed in terms of Jacobian elliptic functions sn (ζ, m) or cn (ζ, m), where m is the modulus of the elliptic function - parameter that depends on the distribution of the roots of the cubic polynomial on the right-hand side of (3.9). Solutions, bounded between two of the mentioned roots, exist when all three roots of $C(\bar{u})$ are real. Since we want $\bar{u} \to 0$ as $\zeta \pm \infty$, constants A and B should be zero (actually, we can allow u $\to$ const. as $\zeta \pm \infty$, since the required electric field is a derivative of u, however by proper choice of constants u_o and v_o we can require $u \to 0$ as $\zeta \to \infty$).

Without loss of generality we assume the roots of $C(\bar{u})$ to be α, 0, 0, as shown on Fig. III.1 .With such a choice equation (3.9) can be integrated using the transformation

$$\bar{u} = \alpha\,\text{sech}^2\vartheta\,. \tag{3.11}$$

With help of (3.11), (3.9) integrates to

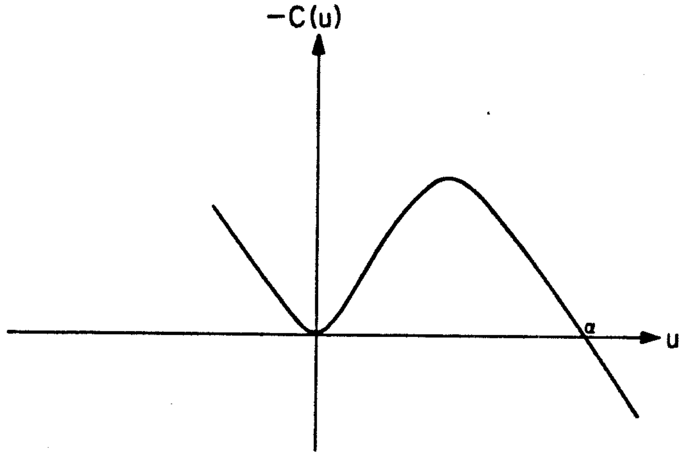

Figure III.1 Distribution of the roots of cubic polynomial C(u).

$$\bar{u} = \alpha \operatorname{sech}^2 \left\{ \left[\frac{\alpha}{12U(c_o^2 - U^2)} \right]^{1/2} \omega_p \,(x - Ut) \right\} . \tag{3.12}$$

Equation (3.9) also implies that

$$U = u_o + \frac{1}{3}\alpha , \tag{3.13}$$

showing that the velocity of the wave is proportional to its amplitude α. Equation (3.13) is the remnant of the dispersion relation in this nonperiodic case. Analyzing (3.9) and (3.13) we arrive at :

$$c_o^2 > U^2 \rightarrow h > 0 \rightarrow \alpha > 0 \rightarrow u_o > 0 \rightarrow U < c_o \rightarrow 0 < \alpha < 3(c_o - u_o) \rightarrow u_o < c_o \tag{3.14}$$

$$c_o^2 < U^2 \rightarrow h < 0 \rightarrow \alpha < 0 \rightarrow u_o > 0 \rightarrow U > c_o \rightarrow 3(c_o - u_o) < \alpha < 0 \rightarrow u_o > c_o \tag{3.15}$$

In the case when $h > 0$, increase in α leads to increase in U. When $h < 0$, increase in α leads to decrease in U. Therefore, whether 3-dimensional thermal velocity of electrons is greater or smaller than the velocity of the wave, makes a qualitative difference. For definiteness we will first consider case when $h > 0$ and later we shall return to the case $h < 0$.

With the aid of (2.28) and (3.1) the longitudinal component of the electric field is

$$E^L = \frac{2\alpha^{3/2}\omega_p Um}{3r^{1/2}|e|}\,\mathrm{sech}^2\left[(\frac{\alpha}{r})^{1/2}\omega_p(x-Ut)\right]\tanh\left[(\frac{\alpha}{r})^{1/2}\omega_p(x-Ut)\right], \tag{3.16}$$

where

$$r = 12U\,(c_o^2 - U^2)\,. \tag{3.17}$$

From the Maxwell equation

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{3.18}$$

it follows that only the longitudinal component of vector **E** contributes to charge density function ρ. From equations (3.16) and (3.18) we obtain charge density

$$\rho = \frac{\alpha^2\omega_p^2 Um}{6\pi r|e|}\left\{3\,\mathrm{sech}^4\left[(\frac{\alpha}{r})^{1/2}\omega_p(x-Ut)\right] - 2\mathrm{sech}^2\left[(\frac{\alpha}{r})^{1/2}\omega_p(x-Ut)\right]\right\}. \tag{3.19}$$

If we denote by ζ_m and ζ_n the abscissas of the peak values of E^L and ρ respectively, then

$$\zeta_m = \frac{1}{\omega_p}(\frac{r}{\alpha})^{1/2}\,\mathrm{arsinh}\,\frac{1}{\sqrt{2}}. \tag{3.20}$$

Clearly ζ_n is proportional to ζ_m. The expression (3.20) shows that as α decreases, solitary waves, given by (3.12), 3.16) and (3.19), become broader and move more slowly, whereas an increase in α narrows the wave profiles and makes them move faster. This dependence of shape and velocity on amplitude is a typical property of nonlinear waves.

The additive velocity u_o, which we call the characteristic velocity, might be interpeted here as the mean thermal velocity of the electrons, namely

$$u_o = \sqrt{\frac{\kappa T}{m}}, \tag{3.21}$$

which determines the velocity of the electrons in the absence of external fields. Since equation (3.6) is invariant under the transformation

$x \to -x$, $u \to -u$, and $u_o \to -u_o$, it governs also the propagation of similar waves having opposite sign and moving in the opposite direction, and thus the general expression for the velocity of wave is

$$U = \pm \left[\sqrt{\frac{\kappa T}{m}} + \frac{1}{3}\alpha\right], \tag{3.22}$$

where "+" sign corresponds to the waves propagating in positive x direction whereas " - " sign corresponds to the waves propagating in the negative x direction.

The "degree of nonlinearity" can be represented by the dimensionless parameter ν defined by

$$\nu = \left(\frac{\alpha}{r}\right)^{1/2} \omega_p d\ , \tag{3.23}$$

where d is a characteristic length. For a solitary wave with $d \approx \left(\frac{r}{\alpha\omega_p^2}\right)^{1/2}$ this parameter is of order 1. Large ν indicates the dominance of nonlinear processes over dispersive processes and small ν indicates dominance of dispersive effects For example, by increasing α we can achieve dominance of nonlinearity over the dispersion.

With an aid of (3.14) and (3.22) we obtain the following bound on the amplitude:

$$|\alpha| < 3\,(\sqrt{3}-1)\sqrt{\frac{\kappa T}{m}}\ . \tag{3.24}$$

From the approximations used in deriving (2.26) we have

$$|\rho| \le n_o\,|e|\ , \tag{3.25}$$

where n_o is the electron density of a homogeneous plasma. Combining (3.25) and (3.19) we obtain a slightly more restrictive bond on α, namely

$$|\alpha| \leq 2\sqrt{\frac{\kappa T}{m}}, \tag{3.26}$$

and the corresponding upper bound on E^{L2}

$$|E| \leq 2\times 10^{8}\,(2\pi\kappa T n_o)^{1/2}\,. \tag{3.27}$$

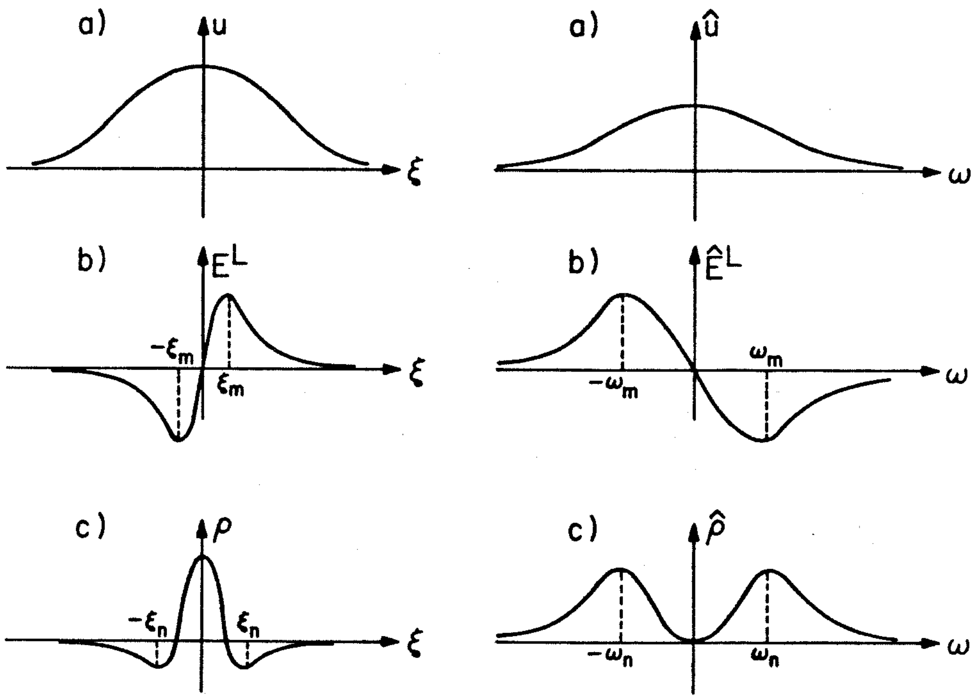

Figure III.2 The potential function (a), the longitudinal component of the electric field (b) and the charge density function (c) in a time and frequency domain.

It is of interest to study the frequency spectra of ρ and the fields u and E^L. Fig. III.2 represents the graphs of these quantities and their Fourier tranforms, where the presence of low frequencies is evident. Moreover, the peak values of $E^L(\omega)$ occur at $\omega_m < \omega_p$ for any

$$\alpha < \frac{3}{4+\pi^2}\left[2\left(\sqrt{\pi^4+4\pi^2}-1\right)-\pi^2\right]\sqrt{\frac{\kappa T}{m}} \approx 2.512\sqrt{\frac{\kappa T}{m}}, \tag{3.28}$$

which includes the total permissible range for α. Hence the maxima of $E^L(\omega)$

2. Equation (3.27) is given in v/m, κ is given in MKS units (J/degree).

always occur at $\omega_m < \omega_p$, in contrast with the linear case where frequencies below ω_p do not propagate without attenuation. It should be mentioned that the case (3.15) leads to inconsistency with the assumption (3.21) that leads to (3.22), otherwise the analysis for this case is identical to the analysis for the case (3.14).

3.2 Dynamics of the Electron Distribution

In this paragraph we will study the local distribution of the electrons in some strip Δ x (see Fig. III-3.)

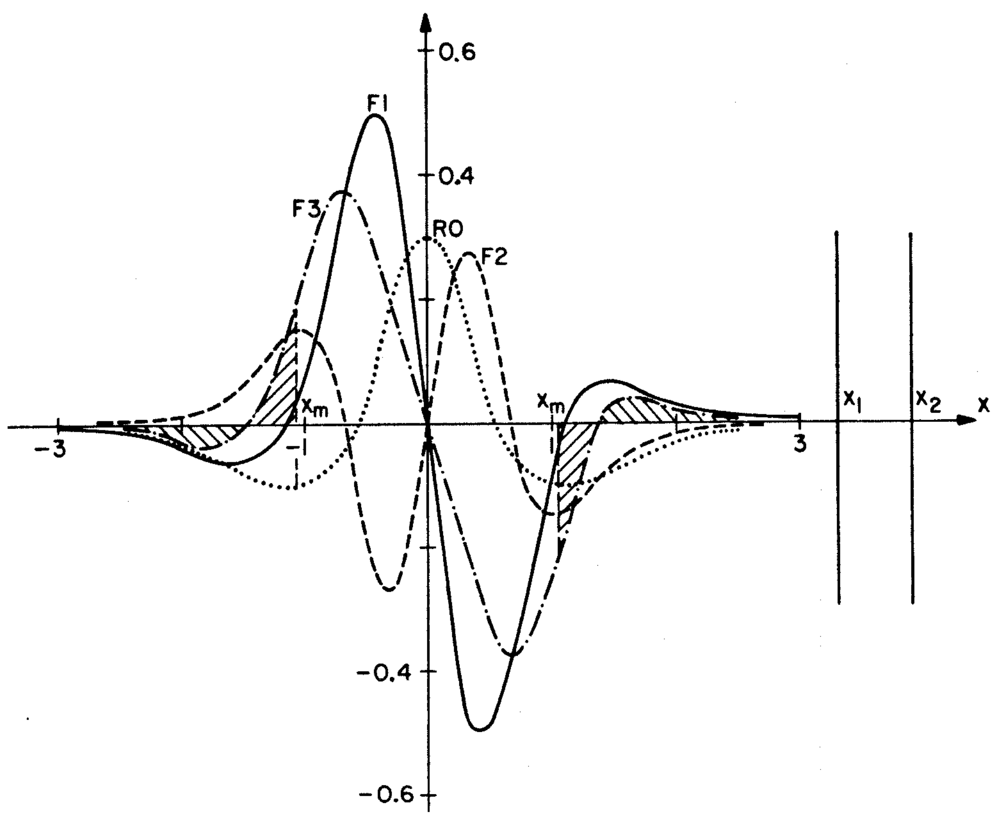

Figure III.3 The pressure (F1) and electrostatic (F2) force densities, their sum (F = F1 + F2) and charge density distribution.

Let the strip be bounded by two planes $x = x_1$ and $x = x_2$. The total force density F consists of pressure force density F1 ~ F_{xx} and electrostatic force density F2 ~ ρE. When wave is far away from the strip (far to the left from the strip), F is positive and very small, such that the plasma inside the strip is almost neutral. When wave gets closer to the strip (approaches the strip from the left) F increases and it tends to push the electrons to the right, so that the number of electrons entering the strip exceeds the number of electrons exiting the strip, thus creating the excess of the electrons in the strip. When F achieves its maximum the difference between the flux of electrons in and out of the strip is maximum. To the left of its maximum F decreases, although it retains its sign which implies that more electrons are entering the strip, making it more negative (note, that the flux of electrons out of the strip also continues but with slower rate). When F = 0, due to their inertia, electrons continue to penetrate into the strip. This process of inreasing the number of electrons in the strip continues until $x = x_m$, at which $\rho=\rho_{min}$. At this point we have

$$\int_{x_o}^{\infty} F dx = - \int_{x_m}^{x_o} F dx \, ,$$

where x_o is the abscissa to the right from x_m ; where F vanishes. Showing that the energies of the electron in the region $[x_o, \infty)$ and $[x_m, x_o]$ are the same. At $x = x_m$ we reach the balance between exiting and entering electrons. From x_o all the way to the origin of the coordinate system F is negative, so it pulls the electrons to the left. In the region $[x_m, 0]$ electrons start moving to the left and the flux of the electrons out of the strip exceeds the flux of the electrons entering the strip from the right, thus making the strip less negative.

At $x = 0$, $F = 0$ and $\rho=\rho_{max}$, almost all electrons freed the strip, and amount of the electrons entering the strip is equal to the amount of the electrons leaving the strip.

From 0 to x_5 F changes its sign to positive, which results on applying force on the electrons to the right, thus decreasing the flux of the electrons out of the strip to the right. This implies that the strip becomes more negative. At the point where F reaches maximum there is already an excess of the electrons in the strip. This process continues up to the point $x = -x_m$, where from the conservation point of view we conclude that the electrons have zero velocity, and from $-x_m$ they start moving in the opposite direction, e.g. to the right of the strip. Similar to the region where x is positive, we have

$$\int_{-x_o}^{-x_m} F dx = -\int_{-\infty}^{x_o} F dx \, ,$$

where $-x_o$ stands for the abscissa to the left from $-x_m$, where F vanishes.

3.3 Periodic Wavetrains

The study of periodic wavetrains that are basic solutions in dispersive problems, can lead to the determination of the structure of the equation (3.6) - whether it is hyperbolic or elliptic, thus providing us with important information about stability of the waves. It also can yield the dispersion relation, which is not a trivial task, since equation (3.6) is nonlinear. Among other things periodic wavetrains are used in developing the modulation theory.

It is of interest to see whether periodic wavetrains, governed by (3.6) exist or not. To do so, instead of considering equation (3.6), we will work with following equation

$$u_t + uu_x + \frac{1}{\omega_p^2} u_{ttt} - 3a_e^2 u_{xtt} = 0 \qquad (3.29)$$

which can be obtained directly from (3.6) by simple transformation $u \rightarrow u - u_o$,[3] and use the expansion

$$\frac{u}{c_o/\sqrt{3}} = \xi = \varepsilon \xi_1(\vartheta) + \varepsilon^2 \xi_2(\vartheta) + \varepsilon^3 \xi_3(\vartheta) + \cdots \qquad \vartheta = kx - \omega t \qquad (3.30)$$

in equation (3.29). $\frac{c_o}{\sqrt{3}} = \sqrt{\frac{\kappa T}{m}}$ is the electron thermal velocity. As a result of

3. Here we retain letter u for a new variable to avoid the extensive use of new letters.

expansion (2.30) we obtain the following hierarchy

$$\omega \xi_1' + \left[\frac{\omega^3}{\omega_p^2} - 3a_e^2 k^2 \omega \right] \xi_1''' = 0$$

$$\omega \xi_2' + \left[\frac{\omega^3}{\omega_p^2} - 3a_e^2 k^2 \omega \right] \xi_2''' = k \frac{c_o}{\sqrt{3}} \xi_1' \xi_1 \tag{3.31}$$

$$\omega \xi_3' + \left[\frac{\omega^3}{\omega_p^2} - 3a_e^2 k^2 \omega \right] \xi_3''' = k \frac{c_o}{\sqrt{3}} (\xi_1 \xi_2)'$$

and so on. The primes in (3.31) denote differentiation w.r.t. ϑ.

With $\omega = \omega_o(k) = \omega_p \sqrt{1 + 3a_e^2 k^2}$, which represents the dispersion relation for the linearized equation (see (2.36)), the solution to the first equation in (3.31) is $\xi_1 = \cos\vartheta$. Similarly we obtain the next term in (3.30)

$$\xi_2 = \frac{-kc_o/\sqrt{3}}{12\omega_p \sqrt{1 + 3a_e^2 k^2}} \cos 2\vartheta . \tag{3.32}$$

Right hand side of the third equation in (3.31) has a resonant term (proportional to sin ϑ), which gives rise to a secular term in the solution. To avoid this we use Stokes' expansion for ω (see Stokes G.G. (1847))

$$\omega = \omega_o(k) + \varepsilon^2 \omega_2(k) + \cdots \tag{3.33}$$

Using (2.33) system (2.31) becomes

$$\omega_o \xi_1' + \left[\frac{\omega_o^2}{\omega_p^2} - 3a_e^2 k^2 \right] \omega_o \xi_1''' = 0$$

$$\omega_o \xi_2' + \left[\frac{\omega_o^2}{\omega_p^2} - 3a_e^2 k^2 \right] \omega_o \xi_2''' = k \frac{c_o}{\sqrt{3}} \xi_1' \xi_1 \tag{3.34}$$

$$\omega_o \xi_3' + \left[\frac{\omega_o^2}{\omega_p^2} - 3a_e^2 k^2 \right] \omega_o \xi_3''' = k \frac{c_o}{\sqrt{3}} (\xi_1 \xi_2)' - \omega_2 \xi_1' - 3\omega_2 \left[\frac{\omega_o^2}{\omega_p^2} - a_e^2 k^2 \right] \xi_1'''$$

and so on. Again, with the same $\omega_o(k)$ we have $\xi_1 = \cos\vartheta$, ξ_2 is given by (3.32) and the third equation in (2.34) becomes

$$\xi_3' + \xi_3''' = \frac{1}{\omega_o}\left[\left(\frac{k^2c_o^2/3}{24\omega_p\sqrt{1+3a_e^2k^2}} + \omega_2 - \frac{3\omega_o^2\omega_2}{\omega_p^2} + 3a_e^2k^2\omega_2\right)\sin\vartheta\right.$$

$$\left. + \frac{k^2c_o^2/3}{8\omega_p\sqrt{1+3a_e^2k^2}}\sin 3\vartheta\right]. \tag{3.35}$$

Clearly, to avoid the secular terms we should take

$$\omega_2 = \frac{k^2c_o^2/3}{48\omega_p(1+3a_e^2k^2)^{3/2}}. \tag{3.36}$$

With the choice of (3.36) ξ_3 is

$$\xi_3 = \frac{k^2c_o^2/3}{192\omega_p^2(1+3a_e^2k^2)}\cos 3\vartheta. \tag{3.37}$$

Therefore (3.30) becomes

$$\frac{u}{c_o/\sqrt{3}} = \varepsilon\cos\vartheta - \varepsilon^2\frac{kc_o/\sqrt{3}}{12\omega_p\sqrt{1+3a_e^2k^2}}\cos 2\vartheta + \varepsilon^3\frac{k^2c_o^2/3}{192\omega_p^2(1+3a_e^2k^2)}\cos 3\vartheta + \ldots, \tag{3.38}$$

and

$$\omega = \omega_p\sqrt{1+3a_e^2k^2} + \varepsilon^2\frac{k^2c_o^2/3}{48\omega_p(1+3a_e^2k^2)^{3/2}} + \cdots \tag{3.39}$$

It can be seen from (3.38) and (3.39) that the expansion parameter is really $\frac{\varepsilon k c_o}{\omega_p\sqrt{1+3a_e^2k^2}}$, which corresponds to the modulus of the elliptic funtions, discussed earlier [Whitham, 1973, pp. 457-463], which in turn is proportional to the amplitude of the wave. Thus the nonlinear dispersion relation (3.39) shows the crucial dependence on the amplitude.

According to Whitham [1967, ch. 14.2], the sign of the term $\omega_o''\omega_2$ determines the stability of periodic wavetrain. When $\omega_o''\omega_2 > 0$, equation exhibits hyperbolic behavior, which leads to well-posed problems in the wave propagation context. When $\omega_o''\omega_2 < o$, equation becames elliptic, which leads to ill-posed problems in the wave propagation context because of imaginary characteristics. It means that small perturbations will grow in time and in this sense the periodic wavetrain is unstable. It can be easily shown that in our case $\omega_o''\omega_2 > 0$ implies that

$$\frac{3a_e^2k^2}{1 + 3a_e^2k^2} < 1 \tag{3.40}$$

which is always true. Thus (3.29) exhibits hyperbolic behavior for any value a_e and k, and the less $a_e^2k^2$ is, the more hyperbolic it is, the more stable periodic wavetrain is.

3.4 Interacting Solitary Waves

We will investigate the interaction of the solitary waves, governed by (3.6) or ultimately by (3.29), using the approach that was utilized by G.B. Whitham [1973] in the study of Korteweg de-Vries equation, and also by R. Hirota [1972] in his study of Boussinesq equation. For convenience we rescale independent variables x and t to normalize the coefficients of equation (3.29) to obtain

$$u_t + uu_x + u_{ttt} - u_{xxt} = 0 \tag{3.41}$$

Using the transformation

$$u = \eta_x \tag{3.42}$$

and then integrating the resulting equation w.r.t. x gives

$$\eta_t + \eta_x^2 + \eta_{ttt} - \eta_{xxt} = 0 \tag{3.43}$$

The nonlinear transformation

$$\eta = -12(\log F)_t \tag{3.44}$$

yields the following equation for a new variable F

$$(F_t+F_{ttt}-F_{xxt})_t F^3-(F_t+F_{ttt}-F_{xxt})F_tF^2+(12F_{tt}-2F_{xx})F_t^2F-(3F_{tt}-F_{xx})F_{tt}F^2$$

$$-2(F_{tt}F_x-F_{ttx}F)F_xF+4F_{xt}F_xF_tF-4F_{tx}^2F^2-(3F_{ttt}-F_{xxt})F_tF^2-6F_t^4=0 \qquad (3.45)$$

It can be shown that if

$$F=1+\exp\left[-\sqrt{1+\alpha^2}(x-x_m)+\alpha t\right]$$

$$=1+\exp\left[-(\vartheta-\vartheta_o)\right], \qquad x_m=\frac{\vartheta_o}{\sqrt{1+\alpha^2}}, \qquad (3.46)$$

where ϑ_o and α are parameters, then the solution for a single solitary wave can be written as

$$u=3\alpha\sqrt{1+\alpha^2}\,\mathrm{sech}^2\left(\frac{\vartheta-\vartheta_o}{2}\right), \qquad (3.47)$$

where $\vartheta=\sqrt{1+\alpha^2}x-\alpha t$ is a phase function. Furthermore F, given by (2.46), satisfies the following equation

$$F_t+F_{ttt}-F_{xxt}=0\,. \qquad (3.48)$$

Moreover, when $\alpha << 1$, the same F satisfies also the whole equation (3.45). To study interaction we take

$$F=1+F^{(1)}+F^{(2)}+\cdots\,. \qquad (3.49)$$

Substituting (2.49) into (2.45) we obtain the hierarchy

$$(F_t^{(1)}+F_{ttt}^{(1)}-F_{xxt}^{(1)})_t=0$$

$$(F_t^{(2)}+F_{ttt}^{(2)}-F_{xxt}^{(2)})=(3F_{tt}^{(1)}-F_{xx}^{(1)})F_{tt}^{(1)}-2F_x^{(1)}F_{ttx}^{(1)}+4F_{tx}^{(1)2}$$

$$+(3F_{ttt}^{(1)}-F_{xxt}^{(1)})F_t^{(1)}, \qquad (3.50)$$

and so on.

If we take $F^{(1)}=f_1+f_2$, where $f_j=\exp\left[-\sqrt{1+\alpha_j^2}\,(x-x_{mj})+\alpha_j t\right]$ $j=1,2,\ldots,$ the second equation in (2.50) becomes

$$F_t^{(2)} + F_{ttt}^{(2)} - F_{xxt}^{(2)} = -3(\alpha_1-\alpha_2)^2 f_1 f_2 . \qquad (3.51)$$

The solution to (2.51) can be easily obtained, if we will seek it in a form $F = Af_1f_2$, where A is a constant. The result is

$$F^{(2)} = \frac{3\dfrac{(\alpha_1-\alpha_2)^2}{(\alpha_1+\alpha_2)^2}}{1+2\sqrt{1+\alpha_1^2}\,\sqrt{1+\alpha_2^2} - 2\alpha_1\alpha_2} f_1 f_2 . \qquad (3.52)$$

The next equation in the hierarchy has zero on the right hand side, therefore F can be approximated by

$$F = 1 + f_1 + f_2 + \frac{3\dfrac{(\alpha_1-\alpha_2)^2}{(\alpha_1+\alpha_2)^2}}{1+2\sqrt{1+\alpha_1^2}\,\sqrt{1+\alpha_1^2} - 2\alpha_1\alpha_2} f_1 f_2 . \qquad (3.53)$$

Hence (2.53) represents two solitary waves and is a solution of (2.45). The corresponding expression for u is

$$\frac{u}{12} = \frac{\alpha_1\sqrt{1+\alpha_1^2}\, f_1 + \alpha_2\sqrt{1+\alpha_2^2}\, f_2}{[1 + f_1 + f_2 + \gamma(\alpha) f_1 f_2]^2}$$

$$+ \frac{\left[\gamma(\alpha)\left(\sqrt{1+\alpha_1^2}+\sqrt{1+\alpha_2^2}\right)(\alpha_1+\alpha_2) + \sqrt{1+\alpha_1^2}\,(\alpha_1-\alpha_2) + \sqrt{1+\alpha_2^2}\,(\alpha_2-\alpha_1)\right] f_1 f_2}{[1+f_1+f_2+\gamma(\alpha) f_1 f_2]^2}$$

$$+ \frac{\gamma(\alpha)\left[\alpha_2\sqrt{1+\alpha_2^2}\, f_1^2 f_2 + \alpha_1\sqrt{1+\alpha_1^2}\, f_1 f_2^2\right]}{[1 + f_1 + f_2 + \gamma(\alpha) f_1 f_2]^2}, \qquad (3.54)$$

where

$$\gamma(\alpha) = \frac{3(\alpha_1-\alpha_2)^2/(\alpha_1+\alpha_2)^2}{1+2\sqrt{1+\alpha_1^2}\,\sqrt{1+\alpha_2^2} - 2\alpha_1\alpha_2} .$$

Equation (2.54) describes the interaction of two solitary waves governed by (3.6). For a single solitary wave we have

$$\frac{u}{12} = \frac{\alpha\sqrt{1+\alpha^2}}{(1+f)^2} f , \qquad (3.55)$$

with a maximum of f = 1, and

maximum amplitude of u being $3\alpha\sqrt{1+\alpha^2}$,

position of a maximum at $x = x_m + \frac{\alpha}{\sqrt{1+\alpha^2}}t$,

velocity of the wave being $\frac{\alpha}{\sqrt{1+\alpha^2}}$.

Equation (3.54) approximates a solitary wave with α_1 for regions where $f_1 \approx 1$ and f_2 is either large or small, and for regions where $f_2 \approx 1$ and f_1 is either large or small it describes a solitary wave with α_2.

When $f_1 \approx 1$ and $f_2 << 1$ equation (2.54) reduces to

$$\frac{u}{12} = \frac{\alpha_1\sqrt{1+\alpha_1^2}\, f_1}{(1+f_1)^2}, \qquad (3.56)$$

and when $f_1 \approx 1$ and $f_2 >> 1$ equation (2.54) becomes

$$\frac{u}{12} = \frac{\alpha_1\sqrt{1+\alpha_1^2}}{(1+\tilde{f}_1)^2}\tilde{f}_1, \qquad (3.57)$$

where $\tilde{f}_1 = \gamma(\alpha)f_1$. The latter is the solitary wave α_1 where x_{1m} is shifted by

$$\tilde{x}_{1m} = x_{1m} - (1+\alpha_1^2)^{-1/2}\ln\gamma^{-1}(\alpha).$$

Similar considerations, in regions where $f_2 \approx 1$ and f_1 is either large ($f_1 >> 1$) or small ($f_1 << 1$), imply the shift in x_{2m}:

$$\tilde{x}_{2m} = x_{2m} - (1+\alpha_2^2)^{-1/2}\ln^{-1}\gamma(\alpha).$$

The interaction regions are the regions where $f_1 \approx 1$ and $f_2 \approx 1$. Regions where both f_1 and f_2 are small ($f_1 << 1$, $f_2 << 1$) or large ($f_1 >> 1$, $f_2 >> 1$), $u \approx 0$.

If we assume that wave α_2 is faster than α_1, i.e. $\alpha_2 > \alpha_1 > 0$ then as $t \to -\infty$, $f_1 > f_2$ and (3.54) yields a

solitary wave α_1 at $x = x_{1m} + \frac{\alpha_1}{\sqrt{1+\alpha_1^2}}t$ for regions where $f_1 \approx 1$, $f_2 << 1$, solitary wave α_2 at $x = x_{2m} - \frac{1}{\sqrt{1+\alpha_1^2}} \ln\gamma^{-1}(\alpha) + \frac{\alpha_2}{\sqrt{1+\alpha_2^2}} t$ where $f_1 >> 1, f_2 \approx 1$.

At the same limit there is no interaction region ($f_1 \approx 1$, $f_2 \approx 1$), and $u \approx 0$ when both f_1 and f_2 are small ($f_1 << 1$, $f_2 << 1$), or large ($f_1 >> 1$, $f_2 >> 1$).

As $t \to \infty$, $f_1 < f_2$, and (3.54) yields a

solitary wave α_1 at $x = x_{1m} - \frac{1}{\sqrt{1+\alpha_1^2}} \ln\gamma^{-1}(\alpha) + \frac{\alpha_1}{\sqrt{1+\alpha_1^2}} t$ where $f_1 \approx 1$, $f_2 >> 1$ and,

solitary wave α_2 at $x = x_{2m} + \frac{\alpha_2}{\sqrt{1+\alpha_2^2}} t$, where $f_1 << 1$, $f_2 \approx 1$.

Finally, it is easy to see that the coordinates of the interaction region are

$$t = \frac{x_{2m} - x_{1m}}{\frac{\alpha_1}{\sqrt{1+\alpha_1^2}} - \frac{\alpha_2}{\sqrt{1+\alpha_2^2}}},$$

and

$$x = \frac{x_{2m}\frac{\alpha_1}{\sqrt{1+\alpha_1^2}} - x_{1m}\frac{\alpha_2}{\sqrt{1+\alpha_2^2}}}{\frac{\alpha_2}{\sqrt{1+\alpha_2^2}} - \frac{\alpha_1}{\sqrt{1+\alpha_1^2}}}.$$

At these coordinates two solitary waves merge and then reemerge. ($f_1 \approx 1$, $f_2 \approx 1$). Interactions of more than two solitary waves can be studied by starting with $F^{(1)} = 1 + f_1 + f_2 + \cdots + f_N$, where N is the number of solitary waves under investigation, and then using exactly the same arguments as in the case of two solitary waves. In general, as time increases, the arbitrary sequence of solitons will reorder itself in such a way that solitons with larger amplitudes (fast solitons) will get ahead, followed by solitons with smaller height (slow solitons), and the initial and final number of solitons remains the same.

Extensive numerical approaches were made in connection with phenomena of solitary wave interactions. B. Fornberg and B.G. Whitham (1977) developed a numerical method for the KdV equation with periodic initial value problem that enables investigation of solitary wave interaction. Zabusky and Kruskal [1965] and Zabusky [1967], using numerical analysis in their investigation of KdV equation, observe nonlinear interactions among "solitary-wave pulses" propagating in nonlinear dispersive media for variety of initial conditions. Although we have obtained the qualitative features of the soliton interaction, governed by (3.41), further numerical verification of the interaction is needed.

3.5 Stability

If we assume that

$$|u_{ttt}| << |3a_e^2 \omega_p^2 u_{xxt}| , \tag{3.58}$$

and properly scale dependent and independent variables in (3.6) we obtain

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{3.59}$$

Assumption (3.58) is a more strict version of condition (3.14). We adapt (3.58) because it is consistent with the previous assumption (3.14) and also because it significantly simplifies the stability analysis. This equation has been studied by Benjamin et al. [1971]. It was shown that solutions to the initial value problem for this equation have better smoothness properties then those of the KdV equation. In foregoing analysis we will adopt Benjamin's approach to stability analysis for the KdV equation to show that solitons that are governed by (3.59) is also stable. The context of this paragraph in its entirety will be based on the analysis, conducted by Benjamin [1972] regarding the KdV equation.

The solitary wave solution $\bar{u}$ will be compared with a class of solutions u(x,t) that evolve from some initial conditions that are close, in some sense, to those for $\bar{u}$. Intuitively, the stability of $\bar{u}$ can be seen as if at t = 0, $\bar{u}$ and u are close to each other, then $\bar{u}$ and u will remain to be close to each other at later times. Mathematically (Liapunov definition) the stability of $\bar{u}$ means that given an arbitrary small $\varepsilon > 0$, there exists such a $\delta(\varepsilon) > 0$, that if

$$\left[d_1(u,\bar{u}) \right]_{t=0} \leq \delta(\varepsilon) ,$$

then

$$\left[d_2(u,\bar{u}) \right]_{t>0} \leq \varepsilon , \tag{3.60}$$

where $d_1(u,\bar{u})$ and $d_2(u,\bar{u})$ represent some kind of a measure of a distance between u and $\bar{u}$. The stability of $\bar{u}$ can be assumed if there exist some constants $\gamma_1 > 0$, $\gamma_2 > 0$, such that, when $[d_1(u,\bar{u})]_{t=0} \leq D$, then

$$\gamma_1 [d_1(u,\bar{u})]^2 \geq I(u,\bar{u}) \quad \text{at} \ t = 0$$

(3.61)

$$\gamma_2 [d_2(u,\bar{u})]^2 \leq I(u,\bar{u}) \ \text{ for } \ t \geq 0 ,$$

where $I(u,\bar{u})$ is a certain functional, which is invariant with time when u is a solution of (3.59), and $\bar{u}$ is a solitary-wave solution of (3.59). First inequality in (3.61) shows that the initial value $d_1(u,\bar{u})$ fixes an upper bound on the invariant $I(u,\bar{u})$, whereas in the second inequality of (3.61) we see an upper bound on $d_2(u,\bar{u})$. Some additional information on metrics d_1 and d_2 is given in the Appendix A. The result in the form (3.60) means not only the uniform boundedness in time of variance of the difference $\Delta = u - \bar{u}$, and its x derivative, but in a view of (A-2) (see Appendix A), (3.60) means that the magnitude of Δ is everywhere uniformly bounded. In order to determine stability of the solitary wave solution of (3.59) we need to find two nonlinear functionals that are invariant with time for solutions of (3.59). It should be mentioned, that the idea of using two invariants for stability purposes was first introduced by Boussinesq [1877], and later adopted by Keulegan and Patterson [1940]. Multiplying (3.59) by u and then integrating the result w.r.t. x between $x = -\infty$ and $x = \infty$ we obtain

$$\int_{-\infty}^{\infty} (uu_t + u^2u_x - uu_{xxt})\, dx = 0 \tag{3.62}$$

Integrating by parts and assuming that u, u_x, u_{xt} all vanish as $x \to \pm\infty$, we reduce (3.62) to

$$\int_{-\infty}^{\infty} (uu_t + u_xu_{xt})\, dx = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} (u^2 + u_x^2)\, dx = 0\,,$$

and hence

$$M(u) = \int_{-\infty}^{\infty} (u^2 + u_x^2)\, dx = \text{const.} \tag{3.63}$$

Next, we define

$$y(x,t) = \int_{-\infty}^{x} u_t(s,t)\, ds\,,$$

so that (3.59) can be rewritten as

$$y_x - y_{xxx} + (u + \frac{1}{2}u^2)_x = 0 \tag{3.64}$$

Multiplying (3.64) by y, then integrating between $x = -\infty$ and $x = \infty$ using integration by parts, and assuming y_{xx} and y vanish as $x \to \pm\infty$, since u and u_{xt} are assumed to vanish at $x = \pm\infty$, we can show that

$$\int_{-\infty}^{\infty} (u + \frac{1}{2}u^2)\, y_x dx = \int_{-\infty}^{\infty} (u + \frac{1}{2}u^2)\, u_t dx = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} (u^2 + \frac{1}{3}u^3)\, dx = 0\,,$$

and therefore

$$N(u) = \int_{-\infty}^{\infty} (u^2 + \frac{1}{3}u^3)\, dx = \text{const.} \tag{3.65}$$

The functionals N(u) and M(u) will play a key role in determination of a stability of the solitary wave solution of (3.59). Putting $u = \bar{u} + \Delta$ and restricting u so that $M(u) = M(\bar{u})$ we can determine the first variations of M and N as follows

$$\delta M = M(u) - M(\bar{u}) = M(\bar{u} + \Delta) - M(\bar{u})$$

$$= \int_{-\infty}^{\infty} (2\bar{u}\Delta + \Delta^2 + 2\bar{u}_x\Delta_x + \Delta_x^2)\, dx = 0 \tag{3.66}$$

$$-\delta N = N(\bar{u}) - N(u) = N(\bar{u}) - N(\bar{u} + \Delta)$$

$$= \int_{-\infty}^{\infty} \left\{\bar{u}^2 + \frac{1}{3}\bar{u}^3 - \left[(\bar{u} + \Delta)^2 + \frac{1}{3}(u + \Delta)^3\right]\right\} dx$$

$$= \int_{-\infty}^{\infty} (-2\bar{u}\Delta - \Delta^2 - \bar{u}^2\Delta - \Delta^2\bar{u} - \frac{1}{3}\Delta^3)\, dx \tag{3.67}$$

Adding (U + 1) times (3.66) to (3.67) and integrating by parts the third term in (3.66) we can isolate the terms that are linear in Δ, thus obtaining

$$-\delta N = \int_{-\infty}^{\infty} \left[2\bar{u}U - 2\bar{u}''(1+U) - \bar{u}^2\right] \Delta dx + \int_{-\infty}^{\infty} \left[\Delta^2 U + \Delta_x^2 (U+1) - \Delta^2\bar{u}\right] dx$$

$$-\frac{1}{3}\int_{-\infty}^{\infty} \Delta^3 dx \tag{3.68}$$

The first integrand in the brackets is equal to zero identically, since $\bar{u}$ is the solution to (3.59), for if in (3.59) we assume $u = \bar{u}[x-(1+U)t]$ and integrate the result once with respect to the new variable $x-(1+U)t$, we can see that the first integrand is zero. Note that by assumption $\delta M = 0$, therefore $-\delta N + (1+U)\,\delta M = -\delta N$ in (3.68). Finally we can rewrite (3.68) in a shorter form

$$-\delta N = \int_{-\infty}^{\infty} \left[\Delta_x^2 (1+U) - \Delta^2(\bar{u}-U)\right] dx - \frac{1}{3}\int_{-\infty}^{\infty} \Delta^3 dx = \delta^2 N + \delta^3 N \tag{3.69}$$

With the aid of (B-1), (B-2) and (B-3) we can obtain an upper bound on δN, remembering that $\bar{u}$ is a non-negative function. It is

$$-\delta N \le \int_{-\infty}^{\infty} \left[\Delta_x^2 (1+U) + \Delta^2 U\right] dx + \frac{1}{3}\sup_x|\Delta| \int_{-\infty}^{\infty} \Delta^2 dx$$

$$\le (1+U)\|\Delta\|^2 + \frac{1}{3\sqrt{2}}\|\Delta\|^3, \tag{3.70}$$

and if $\|\Delta\| \le D$, then we have

$$-\delta N \le \gamma \,||\Delta||^2 \;\text{ with }\; \gamma = (1 + U + \frac{1}{3\sqrt{2}} D)\,. \tag{3.71}$$

Therefore, taking $d_1(u,\bar{u}) = ||u - \bar{u}|| = ||\Delta||$ we have the form of estimate that is compatible with the first inequality in (3.61).

To get a lower bound on δN, we first observe that since $\bar{u}$ is an even function (3.69) can be rewritten as

$$\begin{aligned} -\delta N(\Delta) &= +\delta^2 N(\Delta) - \frac{1}{3}\int_{-\infty}^{\infty} \Delta^3 dx \\ &\ge \delta^2 N(\Delta) - \frac{1}{3}\sup_x |\Delta| \int_{-\infty}^{\infty} \Delta^2 dx \\ &\ge \delta^2 N(r) + \delta^2 N(s) - \frac{1}{3}\sup_x\, |\Delta| \int_{-\infty}^{\infty} \Delta^2 dx\,, \end{aligned} \tag{3.72}$$

where r and s are even and odd part of Δ respectively. With the use of (B-3), inequality can further be reduced to

$$-\delta N \ge \delta^2 N(r) + \delta^2 N(s) - \frac{||\Delta||^3}{3\sqrt{2}}\,. \tag{3.73}$$

The lower bound estimates for $\delta^2 N(r)$ and $\delta^2 N(s)$ are determined in Appendix C, and they are

$$\delta^2 N(r) \ge -l_1 ||\Delta||^4 + \frac{4}{5}\int_{-\infty}^{\infty} \left[(1 + U) r_x^2 + U r^2 \right] dx\,, \tag{3.74}$$

where l_1 is given below (C.23), and

$$\delta^2 N(s) \ge U ||s||^2\,. \tag{3.75}$$

Combining the inequalities (3.74) and (3.75) we obtain

$$\delta^2 N(\Delta) \ge -l_1 ||\Delta||^4 + \frac{4}{5}\int_{-\infty}^{\infty} \left[(1 + U) r_x^2 + U r^2 \right] dx + U ||s||^2\,. \tag{3.76}$$

Therefore (3.73) becomes

$$-\delta N(\Delta) \ge \delta^2 N(r) + \delta^2 N(s) - \frac{||\Delta||^3}{3\sqrt{2}}$$

$$= \delta^2 N(\Delta) - \frac{\|\Delta\|^3}{3\sqrt{2}}$$

$$\geq -l_1\|\Delta\|^4 + \frac{4}{5}\int_{-\infty}^{\infty}\left[(1+U)r_x^2 + Ur^2\right]dx + U\|s\|^2 - \frac{\|\Delta\|^3}{3\sqrt{2}} \quad (3.77)$$

$$\geq -l_1\|\Delta\|^4 + \frac{4}{5}m\|r\|^2 + U\|s\|^2 - \frac{\|\Delta\|^3}{3\sqrt{2}}$$

$$\geq -l_1\|\Delta\|^4 - l_2\|\Delta\|^3 + l_3\|\Delta\|^2 \quad (3.78)$$

$$= f(\|\Delta\|)\ ,$$

where $m = 1 + U$, $l_3 = \min\left(\frac{4}{5}\ m,\ U\right)$, $l_2 = \frac{1}{3\sqrt{2}}$, and we used an equality $\|\Delta\|^2 = \|r\|^2 + \|s\|^2$.

The right-hand side of (3.78) attains maximum value at

$$\|\Delta\| = \|\Delta\|_o = \frac{3l_2}{8l_1}\left(\sqrt{1 + \frac{32l_1l_3}{9l_2^2}} - 1\right)$$

and maximum is $f(\|\Delta\|_o) = f_o$. Let us assume that the arbitrary value $\frac{3l_2}{8l_1}$ (smaller than $\|\Delta\|_o$) will be considered as an upper bound for $\|\Delta\|$. For this value of $\|\Delta\|_1$ we have

$$f(\|\Delta\|_1) = -\frac{1}{8^4 l_1^3}(81l_2^4 + 216l_2^3) + \frac{9l_3l_2^2}{64l_1^2}\ .$$

If $\|\Delta\| \leq \left[\frac{3l_2}{8l_1}\right]$, then $l_1 \leq \frac{3l}{8\|\Delta\|}$, and (3.78) becomes

$$-\delta N(\Delta) \geq -\frac{11l_2}{8}\|\Delta\|^3 + l_3\|\Delta\|^2 = g(\|\Delta\|)\ . \quad (3.79)$$

The right-hand side of (3.79) achieves maximum at $\|\Delta\| \approx \frac{l_3}{2l_2}$, where $g(\|\Delta\| = \frac{l_3}{2l_2}) \approx \frac{5}{64}\frac{l_3^3}{l_2^2}$. Assuming that the arbitrary value $\|\Delta\|_1 = \frac{l_3}{4l_2}$ (smaller that $||\Delta||$ at which $g(||\Delta||$ is max.) will be considered as an upper bound for $||\Delta||$.

Then $g(||\Delta||_1) \approx \frac{1}{24}\frac{l_3^3}{l_2^2}$, and since

$$||\Delta|| \leq \frac{1}{4}\frac{l_3}{l_2}, \tag{3.80}$$

it follows that

$$l_2 \leq \frac{l_3}{4||\Delta||},$$

and therefore (3.79) becomes

$$-\delta N \geq \frac{21}{32} l_3 ||\Delta||^2 . \tag{3.81}$$

Using (C.1) and (3.81) in one hand and (5.2) and (3.71) on the other hand we obtain

$$I(\Delta) \geq \frac{21}{32} l_3 \, [d_2(u,\bar{u})]^2 , \tag{3.82}$$

and

$$I(\Delta) \leq \gamma_1 \, [d_1(u,\bar{u})]^2 , \tag{3.83}$$

where $I(\Delta) = -\delta N(\Delta)$. This is the required result in accord with system (3.61). Using the right-hand side of (3.79) and (3.70) we can specify D to be the positive root of the following equation

$$(1 + U)D^2 + \frac{1}{3\sqrt{2}} D^3 = \frac{1}{24}\frac{l_3^3}{l_2^2}, \tag{3.84}$$

and hence the initial condition

$$\Big[d_1(u,\bar{u}) \Big]_{t=0} \leq D \tag{3.85}$$

ensures that $I(\Delta)$ satisfies (3.81). Comparing (3.84) with the right-hand side of (3.79), it is seen that $D < \frac{l_3}{4l_2}$, which implies that $||\Delta||$ has an initial value less than $\frac{l_3}{4l_2}$. Moreover, $\|\Delta\|$ can be assumed to vary continuously with time, and thus the conditions providing the inequality (3.81) are satisfied for all t. This is the justification for (3.82) and (3.83), which hold if (3.85) is satisfied. The system (3.82)-(3.83) shows, therefore, that as long as $M(u) = M(\bar{u})$ throughout the evolution of the wave, the solitary wave solution $\bar{u}$ is stable.

Putting $u = \bar{u} + \varepsilon\eta$ in KdV equation and linearizing in ε, Jeffrey and Kakutani [1970] examine the linearized equation for η and show that there are no runaway solutions in the form $\tilde{\eta}(x)e^{\sigma t}$ with $Re(\sigma) > 0$. Although this result does not contradict with u being stable, it does not prove stability either.

3.6 Uniqueness

We want to show that C^∞ solutions of (3.6), which together with their x derivatives tend to zero as $x \to \pm\infty$, are uniquely determined by their initial values. Since it is more convenient to work with (3.59) instead of (3.6) we will show the uniqueness for (3.59), which whould imply the uniqueness for (3.6). Let y be another solution of (3.59) :

$$y_t + yy_x + y_{ttt} - y_{xxt} = 0 \tag{3.86}$$

Subtracting (3.86) from (3.59) we obtain

$$w_t + uw_x + wy_x + w_{ttt} - w_{xxt} = 0\,, \tag{3.87}$$

where $w = u - y$. Integrating (3.87) with respect to x from $-\infty$ to $+\infty$ gives

$$\frac{d}{dt}\int_{-\infty}^{\infty} w dx + \frac{d^3}{dt^3}\int_{-\infty}^{\infty} w dx - \frac{d}{dt}\int_{-\infty}^{\infty} w_{xx} dx + \int_{-\infty}^{\infty} wy_x dx + \int_{-\infty}^{\infty} uw_x dx = 0\,. \tag{3.88}$$

Using integration by parts we can show that the last three integrals are zero, assuming that both w and w_x tend to zero as $x \to \pm\infty$, since

$$\int_{-\infty}^{\infty} wy_x dx + \int_{-\infty}^{\infty} uw_x dx = \int_{-\infty}^{\infty} wy_x dx + \int_{-\infty}^{\infty} u dw$$

$$= \int_{-\infty}^{\infty} wy_x dx + uw \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} wdu$$

$$= \int_{-\infty}^{\infty} wy_x dx - \int_{-\infty}^{\infty} wd(w+y)$$

$$= \int_{-\infty}^{\infty} wdy - \frac{1}{2} w^2 \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} wdy = 0 .$$

and

$$\frac{d}{dt} \int_{-\infty}^{\infty} w_{xx} dx = \frac{d}{dt} \left\{ w_x \Big|_{-\infty}^{\infty} \right\} = 0.$$

Therefore (3.88) reduces to

$$\frac{d}{dt} \left[F(t) + \frac{d^2}{dt^2} F(t) \right] = 0 , \tag{3.89}$$

where

$$F(t) = \int_{-\infty}^{\infty} wdx. \tag{3.90}$$

Equation (3.89) implies that

$$\frac{d^2}{dt^2} F(t) + F(t) = \text{const.} ,$$

and therefore

$$F(t) = Ae^{it} + Be^{-it} + \text{const.}$$

If initial conditions are zero, i.e. $F(0) = 0$, this means that

$$A + B + \text{const.} = 0$$

and since we have assumed that $w \to 0$ as $x \to \pm \infty$, $A = B = 0$, and $F(t) = \text{const} = 0$. Therefore $w = 0$ for all t. In other words if u and y correspond to the same initial waveform, say $g(x)$, so that $w = u - y$ is identically zero at $t = 0$, then $F(0) = 0$, and since (3.90) should hold for all t and $w \to 0$ as $x \to \pm \infty$, $F(t) = 0$, and therefore $u \equiv y$ for all finite t.

3.7 Conservation Laws and Lagrangian

Equation (3.6) represents a particular case of a more general class of equations

$$u_t = G(u) \,, \tag{3.91}$$

where G(u) is a nonlinear operator. In most of the equation that belong to class (3.91) and have a soliton solution, operator G contains explicitly the function u along with its various spatial derivatives. Examples of such equation are KdV, Boussinesq equation, nonlinear Schrodinger equation etc. For thorough review of these equations reader can be referred to Miura [1976], Miura [1967], Miura et al. [1967], Hirota [1972], Whitham [1973], Lax [1968], Zakharov and Shabat [1972]. In contrast with the above mentioned equations, (3.6) differs from them since if written in the form (1), operator G would contain also time derivatives of u along with its spatial derivatives. The only other equation known to us, that is similar to (3.6) in this respect and has a solitary wave solution is Born-Infield equation

$$u_{xx}(1 - u_t^2) + 2u_x u_t u_{xt} - (1+u_x^2)u_{tt} = 0 \,, \tag{3.92}$$

which was derived in the spatial dimensions by Born and Infield [1934] as a non-linear modification of the Maxwell equations. Most of the above equations are known to have common properties such as possession of solitary wave solutions, clean soliton interactions, an infinite number of conservation laws and also they all can be derived from a suitable Lagrangian density function.

For the sake of completeness, we will briefly discuss the concept of conservation law and Lagrangian density in connection to equation (3.6). Consider a pair of (nonlinear) operators $D(u)$ and F(u). If

$$\frac{\partial}{\partial t} D + \frac{\partial}{\partial x} F = 0 \tag{3.93}$$

for any u satisfying (1), then (3) is said to be a conservation law, where $D(u)$ is the conserved density and F is the conserved flux. The functional

$$I(u) = \int_{-\infty}^{\infty} D\left[u(x,t)\right] dx \tag{3.94}$$

is said to be a constant of motion provided the integral exists and integrand satisfies appropriate boundary conditions at $x = \pm\infty$.

Conservation laws and constants of motion provide simple and efficient methods to study both quantitative and qualitative properties of solitons [Miura, 1967], namely stability, evolution of solitons and decomposition of solitons [Benjamin, 1972]. There also seems to be a close relation between conservation laws and interaction of solitons [Miura, 1976]. Zabusky and Kruskel [1963] first guessed and then numerically verified clean soliton interaction just by studying the conservation laws for the KdV equation. The two conservation laws for the normalized version of (3.6), namely (3.41) are

$$(u + u_{tt})_t + (\frac{1}{2}u^2 - u_{xt})_x = 0\ , \tag{3.95}$$

and

$$\left[(u^2)_{tt} - 3u_t^2 + u^2 + u_x^2\right]_t + \left[\frac{2}{3}u^3 + (u^2)_{xt} + 2u_t u_x\right]_x \tag{3.96}$$

Conservation law (3.95) follows directly from (3.59) simply by grouping the appropriate terms, and identity (3.96) can be obtained by multiplying (3.59) by 2u and rearranging the appropriate terms. Assuming u, and u_x both tend to zero as $x \to \pm\infty$ we have

$$I(u) = \int_{-\infty}^{\infty}\left[(u^2)_{tt} - 3u_t^2 + u^2 + u_x^2\right] dx = \text{const.} \tag{3.97}$$

which is a constant of motion. Perhaps there are more conservation laws associated with (3.59), than there are for the other evolution equations, mentioned earlier.

It is known that one way of characterizing nonlinear wave systems in which dispersion (or energy storage effects) dominate and dissipative effects are neglected, is to demonstrate the existence of Lagrangian density L, or so-called "energy function" from which the equation defining the system can be derived [Scott et al., 1973]. A Lagrangian density L is an explicit function of u and its time and space derivatives, namely $u_t, u_x, u_{xx}, u_{tx}, u_{tt}, \ldots$. Scott et al. [1973] show that if the equation under consideration can be derived from Lagrangian density, that does not explicitly depend on time, then this equation may be considered lossless, or conservative in the conventional sense. This fact is well known in classical mechanics [Landau, 1967] and it states that $\frac{dE}{dt} = \frac{\partial L}{\partial t}$, where E stands for total energy of a system. This concept of Lagrangian density is widely used in a study of nonlinear dispersive wave propagation. Whitham [1967] presents the resonant near-linear interaction theory in terms of the Lagrangian density function. In the other paper Whitham [1974] uses various manipulations of the average Lagrangian density in the study of slowly varying wave trains.

Introducing a new potential

$$u = \eta_x$$

normalized (3.59) becomes

$$\eta_{xt} + 6\eta_x\eta_{xx} + \eta_{xttt} - \eta_{xxxt} = 0 \qquad (3.98)$$

To seek the Lagrangian density function, from which (3.98) can be derived, we consider the variational principle

$$\delta J = \delta \int\int_R L\,(\eta, \eta_t, \eta_x, \eta_{tt}, \eta_{xx}, \eta_{tx})\,dxdt \qquad (3.99)$$

for a function $\eta(x,t)$. The corresponding Euler-Lagrange equation is [Whitham, 1973, p. 391].

$$L_\eta - \frac{\partial}{\partial t} L_{\eta_t} - \frac{\partial}{\partial x} L_{\eta_x} + \frac{\partial^2}{\partial t^2} L_{\eta_{tt}} + \frac{\partial^2}{\partial x^2} L_{\eta_{xx}} + \frac{\partial^2}{\partial x \partial t} L_{\eta_{xt}} = 0 . \qquad (3.100)$$

A suitable Lagrangian density function corresponding to (3.98) is

$$L = -\frac{1}{2}\eta_t\eta_x - \eta_x^3 - \frac{1}{2}\eta_{tx}(\eta_{tt} - \eta_{xx}) , \qquad (3.101)$$

which can be verified directly by substituting (3.101) into (3.100). Note, that while the majority of the evolution equation, known to us, including the KdV and Boussinesq equations, are invariant under the Galilean transformation

$$u = u' + u_o , \quad x = x' + Vt , \quad t = t' ,$$

equation (3.6) and (3.41) are not. The only other, evolution equations cited in literature that are not Galilean invariant are Modified KdV and Born-Infeld equations [Barbashov and Charnikov, 1966], as far as we know.

4. TRANSVERSE WAVES

4.1 Case When Longitudinal Waves are Dominant

As it was shown earlier, in a nonlinear plasma, transverse waves cannot exist all by themselves, although they can exist together with longitudinal waves. In the case when $|uu_x| >> |vv_x|$, which is ultimately the condition (3.5), we prefer to start with the second equation in (2.32). If we rewrite the right-hand side of it as $(\varphi^T \varphi^L_x)_t$ and then integrate both sides with respect to the variable t we obtain the following equation for φ^T :

$$-c^2 \varphi^T_{xxt} + \varphi^T_{ttt} + \omega_p^2 \varphi^T_t = \frac{\omega_p^2 e}{m} \varphi^T_t \varphi^L_x + G \,, \tag{4.1}$$

where G is a constant of integration, which at this point will be assumed to be zero. Using the transformations (3.1) and (3.2) together with the potential representation $\varphi_t = \psi$ and also assuming $v_o = 0$, it follows from (4.1) that

$$-c^2 v_{xx} + v_{tt} + \omega_p^2 v = \frac{\omega_p^2}{3U} (u + u_0)\, v \,, \tag{4.2}$$

where u, as it is shown earlier, is a solution of the longitudinal wave equation. Assuming a travelling wave solution

$$v(x,t) = \bar{v}\,(x - Ut) = \bar{v}\,(\zeta) \tag{4.3}$$

and also substituting the solitary wave solution for u, as given by (3.12), we find that equation (4.2) reduces to

$$\frac{d^2}{d\zeta^2}\bar{v} + \frac{\omega_p^2}{c^2 - U^2}\left\{-1 + \frac{1}{3U}\left[\,u_o + \alpha\,\mathrm{sech}^2\left(\frac{\alpha}{r}\right)^{1/2} \omega_p \zeta\right\}\bar{v} = 0 \right. \tag{4.4}$$

where r is given by (3.17). At this point it is convenient to use the change of variables

$$X = \left(\frac{\alpha}{r}\right)^{1/2} \omega_p \zeta .$$

Accordingly, (4.4) becomes

$$\frac{d^2\bar{v}}{dX^2} = \left[\lambda_1^2 - \lambda_2(1+\lambda_2)\,\mathrm{sech}^2 X\right]\bar{v} , \tag{4.5}$$

where

$$\lambda_1^2 = \left(1 - \frac{u_o}{3U}\right)\frac{r}{\alpha(c^2 - U^2)} , \tag{4.6}$$

and

$$\lambda_2(1+\lambda_2) = \frac{r}{3U(c^2 - U^2)} . \tag{4.7}$$

To ensure that condition (3.5) is satisfied we require the following boundary conditions on $\bar{v}$:

$$\frac{d\bar{v}}{dX} = 0 \qquad \text{at } X = 0 \tag{4.8}$$

along with the requirement that v and its all derivatives vanish as $X \to \pm\infty$.

We easily recognize that (4.5) represents a scattering problem with $\mathrm{sech}^2 X$ as its potential function. Formula (4.6) shows the dependence of the eigenvalue λ_1 on the amplitude α of the longitudinal wave u, which again indicates the coupling between the two components E^L and E^T. Formally, when $\lambda_1 = \lambda_2$ equation (4.5) has the bound-state solution

$$\bar{v} = (\text{const}) \cdot \mathrm{sech}^{\lambda_1} X . \tag{4.9}$$

In general the potential $\mathrm{sech}^2 X$ with an appropriate sign (as in our case) is an attractive potential in a sense that it gives a possibility of having eigenfunctions that are bounded. Solution (4.9) also shows coupling between the longitudinal and transverse waves. For example, in the above mentioned case when $\lambda_1 = \lambda_2$, the width of the transverse wave depends on the amplitude of the longitudinal wave. Generally speaking there are many classes of bounded solutions of (4.5) for different values of λ_1 and λ_2. Lamb [1980] surveys the whole class of solitary wave solutions to the problem (4.5).

The constant in (4.9) can be determined from initial conditions, so that the energy of some initial pulse would be divided into energies carried by longitudinal and transverse waves. Next we consider the following problem

$$\frac{d^2\Psi^T}{dX^2} + \left[-\lambda_1^2 + \tilde{\lambda}^2 \operatorname{sech}^2 X \right] \Psi^T = G \,, \tag{4.10}$$

where $\tilde{\lambda}_2^2 = \lambda_2 (1 + \lambda_2)$. This equation can be obtained similarly to the one obtained in (4.5), except here we use different dependent variable Ψ^T, given earlier by (3.2), and we retain a constant of integration G. We will seek solution of (4.10) in the form

$$\Psi^T = v_o + \operatorname{sech}^{\eta} X \cdot W(X) \,. \tag{4.11}$$

The choice of a such Ψ^T stems from a fact that by the analogy with Ψ^L, we can also require Ψ^T to approach a constant (v_o) as $X \to \pm \infty$, since the actual field, which can be obtained by differentiating (4.11) according to (2.28), will go to zero as $X \to \pm \infty$ providing W(X) does not force the second term in (4.11) go to infinity. If we assume that v_o can be interpreted as the mean thermal velocity of the electrons, i.e. $v_o = \sqrt{\frac{\kappa T}{m}}$, similar to u_o in (3.21), then it follows from (4.10) and (4.11) that

$$G = -\lambda_1^2 \sqrt{\frac{\kappa T}{m}} = -\lambda_1^2 v_o \,. \tag{4.12}$$

Substituting (4.11) into (4.10), and taking $\lambda_1 = \mp \eta$ we obtain

$$\operatorname{sech}^{\eta} X \left[W'' - 2\eta \tanh W' + (\lambda_2^2 - \eta - \eta^2) W \cdot \operatorname{sech}^2 X \right]$$

(4.13)

$$= G + v_o\,(\lambda_1^2 - \tilde{\lambda}_2^2\,\mathrm{sech}^2 X)\,, \tag{4.13}$$

Using the tranformation

$$z = \frac{1}{2}(1 - \tanh X) \tag{4.14}$$

equation (4.13) becomes

$$4z^2(1-z)^2 W'' + (1-2z)(1-z)4z + 4\eta z(1-z)\,(1-2z)W' + (\tilde{\lambda}_2^2 - \eta - \eta^2)4z(1-z)W$$

$$= \left[\, G + v_o\lambda_1^2 - v_o\tilde{\lambda}_2^2\, 4z(1-z) \,\right] 4z(1-z)^{-\frac{\lambda_1}{2}}\,, \tag{4.15}$$

where the primes now indicate differentiation with respect to the new variable z. If we divide all terms of (4.15) by 4z(1-z) we obtain

$$z(1-z)W'' + \left[\, 1 + \eta - z(2+2\eta) \,\right] W' - (\lambda_1^2 + \lambda_1 - \tilde{\lambda}_2^2)\, W =$$

$$= \left[\, G + v_o\lambda_1^2 - v_o\tilde{\lambda}_2^2 4z(1-z) \,\right] \left[\, 4z(1-z) \,\right]^{-\frac{\lambda_1}{2} - 1} \tag{4.16}$$

Noting that $\lambda_1 = \eta$ (see above (4.13)) and using (4.12) (4.16)

Noting that $\lambda_1 = \eta$ (see above (4.13)) and using (4.12) we can further reduce (4.16) thus obtaining

$$z(1-z)W'' + \left[\, d - z(a+b+1) \,\right] W' - abW = -v_o\tilde{\lambda}_2^2 \left[\, 4z(1-z) \,\right]^{-\frac{\lambda_1}{2}}, \tag{4.17}$$

where

$$d = 1 + \lambda_1\,, \tag{4.18}$$

$$a + b + 1 = 2(1 + \lambda_1) = 2d\,, \tag{4.19}$$

$$ab = \lambda_1^2 + \lambda_1 - \tilde{\lambda}_2^2\,. \tag{4.20}$$

Introducing the new variable

$$y = -\frac{W}{v_o\tilde{\lambda}_2^2}\cdot 2^{\lambda_1} \tag{4.21}$$

we can rewrite (4.17) in the following form

$$z(1-z)y'' + \left[d - z(a+b+1)\right] y' - aby = z^{s-1}(1-z)^{\tau-1}, \tag{4.22}$$

where

$$s = \tau = 1 - \frac{\lambda_1}{2}. \tag{4.23}$$

Equation (4.22) represents nonhomogeneous hypergeometric equation. For the detailed study of such equation we refer reader to Bailey [1935]. Expanding $(1-z)^{\tau-1}$ in a power series in z we find a particular integral

$$Q_{1,s}^{(\tau)}(a,b;d;z) = z^s \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} z^{n+m}, \tag{4.24}$$

where[1]

$$A_{mn} = \frac{\Gamma(1+n-\tau)\Gamma(a+s+m+n)\Gamma(b+s+m+n)\Gamma(s+n)\Gamma(s+d+n-1)}{\Gamma(1-\tau)\Gamma(a+s+n)\Gamma(b+s+n)\Gamma(s+1+m+n)\Gamma(s+d+n+m)n!} \tag{4.25}$$

The solution (4.24) can be further reduced to

$$Q_1 = Q_{1,s}^{(\tau)} = z^s \sum_{n=1}^{\infty} B_n z^n, \tag{4.26}$$

where

$$B_n = \frac{\Gamma(a+s+n)\Gamma(b+s+n)\Gamma(s)\Gamma(d+s-1)}{\Gamma(a+s)\Gamma(b+s)\Gamma(s+n+1)\Gamma(s+d+n)}\cdot$$

$$\times\, {}_3F_{2,(n+1)}(1-\tau, s, s+d-1; s+a, s+b; 1). \tag{4.27}$$

In (4.27).${}_3F_{2,(n+1)}$ stands for the first (n+1) terms of the generalized hypergeometric series with the given parameters. Therefore the general solution of (4.10) is

1. From here on an extensive use of results on hypergeometric functions is utilized. e.g. see Bailey [1935], or Babister [1967].

$$\Psi^T = v_o + \text{sech}^{\lambda_1} X \left[C_1 F_1 + C_2 \left\{ \frac{1}{2}(1-\tanh X) \right\}^{1-d} F_2 - 2^{-\lambda_1} v_o \tilde{\lambda}_2^2 Q_1 \right] \qquad (4.28)$$

where C_1 and C_2 are constants, and

$$F_1 = F\left[a,b,d; \frac{1}{2}(1-\tanh x)\right], \qquad (4.29)$$

and

$$F_2 = F\left[a-d+1, b-d+1; 2-d; \frac{1}{2}(1-\tanh X)\right], \qquad (4.30)$$

are shorthand for hypergeometric functions - two linearly independent solutions of the homogeneous hypergeometric equation. Since X lies in the range $(-\infty, \infty)$ the variable z, given in (4.14) lies in a circle

$$|z| \leq 1 \qquad (4.31)$$

We should require $C_2 = 0$, since

$$\lim_{X\to\infty} \left\{ \text{sech}^{\lambda_1} X \left[\frac{1}{2}(1-\tanh X) \right]^{1-d} F\left[a-d+1, b-d+1; 2-d; \frac{1}{2}(1-\tanh X)\right] \right\},$$

$$= \lim_{X\to\infty} \frac{\text{sech}^{\lambda_1} X}{\left[\frac{1}{2}(1-\tanh X)\right]^{\lambda_1}} \cdot \lim_{X\to\infty} F_2$$

$$= \lim_{X\to\infty} \frac{\left[\dfrac{\text{sech} X}{\frac{1}{2}(1-\tanh X)}\right]}{\left[\dfrac{\text{sech} X}{\frac{1}{2}(1-\tanh X)}\right]^{1-\lambda_1}} \cdot \lim_{z\to o} \left[1 + \frac{ab}{d} z - \frac{a(a+1)(b+1)b}{d(d+1)} z^2 + \cdots \right]$$

$$= \lim_{X\to\infty} \left[\frac{\text{sech} X}{\frac{1}{2}(1-\tanh X)}\right]^{\lambda_1}$$

$$= \lim_{X\to\infty} \frac{2\tanh X}{\operatorname{sech} X} = \infty \;,$$

where the fact that $\lim_{z\to o} F_2 = 0$, and the L'Hospital rule were used. With $C_2 = 0$, (4.28) becomes

$$\Psi^T = v_o + \operatorname{sech}^{\lambda_1} X\,[\; C_1F_1 - 2^{-\lambda_1} v_o \tilde{\lambda}_2^2 Q_1 \;] \tag{4.32}$$

and therefore

$$E^T = \lambda_1 \mu \frac{m}{|e|} \operatorname{sech}^{\lambda_1} X \tanh X [\, C_1F_1 - AQ_1 \,]$$

$$- \frac{1}{2}\mu \frac{m}{|e|} \operatorname{sech}^{\lambda_1+2} X \left\{ \frac{ab}{d} C_1F_3 - A[\; (s-1)Q_2 + (1-\tau)Q_3 \;] \right\}, \tag{4.33}$$

where

$$F_3 = F\left[a+1, b+1, d+1; \frac{1}{2}(1-\tanh X)\right],$$

$$Q_2 = Q_{f,s-1}^{(\tau)}\left[a+1, b+1, d+1; \frac{1}{2}(1-\tanh X)\right],$$

$$Q_3 = Q_{f,s}^{(\tau-1)}\left[a+1, b+1, d+1; \frac{1}{2}(1-\tanh X)\right],$$

and

$$A = 2^{-\lambda_1} v_o \tilde{\lambda}_2^2 , \tag{4.34}$$

$$\mu = -U\left(\frac{\alpha}{r}\right)^{1/2} \omega_p \,. \tag{4.35}$$

4.2 Numerical Results

Note that for the computational purposes a and b can be obtained directly from (4.19) and (4.20). They are

$$a = \frac{1+2\lambda_1}{2} + \frac{1}{2}\sqrt{1+4\lambda_2^2} \tag{4.36}$$

and

$$b = \frac{1+2\lambda_1}{2} \pm \frac{1}{2}\sqrt{1+4\lambda_2^2} \tag{4.37}$$

It is convenient, for the computational purposes, to express all hypergeometric functions in (4.32) and (4.33) in the form that can be easily executed on computer. With an aid of (4.18), (4.36) and (4.37) we have

$$F_1 = F(a,b,d;z) = 1 + \frac{ab}{1\cdot d}z + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot d(d+1)}z^2$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot d(d+1)(d+2)}z^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(d)_n} z^n = \frac{\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)z^n}{\Gamma(d+n)n!}, \tag{4.38}$$

where

$$(a)_n = (a+1)(a+2)(a+3)\cdots(a+n-1)\ ,(a)_o = 1$$
$$(b)_n = (b+1)(b+2)(b+3)\cdots(b+n-1)\ ,(b)_o = 1$$
$$(d)_n = (d+1)(d+2)(d+3)\cdots(d+n-1)\ ,(d)_o = 1$$

Series (4.38) converges for $|z| < 1$, and it converges conditionally when $z = 1$. On the circle of convergence ($z = 1$), series (4.38) converges absolutely when Re $(d - a - b) > 0$, and converges conditionally when $-1 <$ Re $(d - a - b) \leq 0$. Since z lies inside a unit circle, the convergence is guaranteed for $\lambda_1 < 0$ and series (4.38) conditionally converges for $\lambda_1 > 0$, since Re $(d - a - b)$ = Re $(-\lambda_1) = -\lambda_1$ (λ is real). From (4.6) it is clear that λ_1^2 is positive, therefore we can select $\lambda_1 < 0$ to get an absolute convergence. Using the same arguments as in derivation of (4.38) we obtain

$$F_3 = F(a+1,b+1,d+1;z) = \frac{d\Gamma(d)}{ab\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a+n)(b+n)\Gamma(a+n)\Gamma(b+n)}{(d+n)\Gamma(d+n)n!} z^n, \tag{4.39}$$

$$Q_1 = z^{1-\frac{\lambda_1}{2}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{nm} \, , \tag{4.40}$$

$$Q_2 = z^{-\frac{\lambda_1}{2}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{nm} \frac{(1-\frac{\lambda_1}{2}+n+m)}{(n-\frac{\lambda_1}{2})} , \tag{4.41}$$

and

$$Q_3 = z^{1-\frac{\lambda_1}{2}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{nm} \frac{(n+\frac{\lambda_1}{2})(1-\frac{\lambda_1}{2}+a+n+m)}{\frac{\lambda_1}{2}(1-\frac{\lambda_1}{2}+a+n)(1-\frac{\lambda_1}{2}+b+n)(2+\frac{\lambda_1}{2}+n+m)}$$

$$\times \frac{(1-\frac{\lambda_1}{2}+b+n+m)(1+\frac{\lambda_1}{2}+n)}{\frac{\lambda_1}{2}(1-\frac{\lambda_1}{2}+a+n)(1-\frac{\lambda_1}{2}+b+n)(2+\frac{\lambda_1}{2}+n+m)} , \tag{4.42}$$

where

$$P_{nm} = \frac{\Gamma(\frac{\lambda_1}{2}+n)\Gamma(1+a-\frac{\lambda_1}{2}+m+n)\Gamma(1+b-\frac{\lambda_1}{2}+m+n)}{\Gamma(\frac{\lambda_1}{2})\Gamma(1+a-\frac{\lambda_1}{2}+n)\Gamma(1+b-\frac{\lambda_1}{2}+n)\left[\Gamma(2-\frac{\lambda_1}{2}+n+m\right]^2 n!}$$

$$\times \frac{\Gamma(1-\frac{\lambda_1}{2}+n)\Gamma(1+\frac{\lambda_1}{2}+n)}{\Gamma(\frac{\lambda_1}{2})\Gamma(1+a-\frac{\lambda_1}{2}+n)\Gamma(1+b-\frac{\lambda_1}{2}+n)\left[\Gamma(2-\frac{\lambda_1}{2}+n+m\right]^2 n!} z^{n+m} . \tag{4.43}$$

In order to satisfy condition (3.5) we select C_1 such that E^T $(X = 0) = 0$. Hence from (4.33) and (4.18) - (4.20) it follows that

$$C_1 = \frac{A\lambda_1\left[Q_3(X{=}0) - Q_2(X{=}0) \right]}{2(1-\frac{\tilde{\lambda}_2^2}{1+\lambda_1}) F_3(X{=}0)} , \tag{4.44}$$

where

$$F_3(X=0) = F(a+1,b+1;d+1;\frac{1}{2})$$

$$= F(a+1,b+1;\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2};\frac{1}{2})$$

$$= F(a+1,b+1;\frac{1}{2}(a+1+b+1+1);\frac{1}{2})$$

$$= \pi^{1/2}\,\frac{\Gamma\left[\frac{1}{2}+\frac{1}{2}(a+1)+\frac{1}{2}(b+1)\right]}{\Gamma\left[\frac{1}{2}+\frac{1}{2}(a+1)\right]\Gamma\left[\frac{1}{2}+\frac{1}{2}(b+1)\right]}$$

$$= \pi^{1/2}\,\frac{\Gamma\left[\frac{3}{2}+\frac{1}{2}a+\frac{1}{2}b\right]}{\Gamma\left[1+\frac{1}{2}a\right]\Gamma\left[1+\frac{1}{2}b\right]},$$

and $Q_2(X=0)$ and $Q_3(X=0)$ can be obtained directly by summing up series (4.41) and (4.42) for $z = \frac{1}{2}$.

Because of the complex nature of the hypergeometric functions it is difficult to see the graphical representation of the solution (4.33). However with the help of the relationships (4.40)-(4.45) we can evaluate E^T, for different values of λ_1 and $\tilde{\lambda}_2$, numerically. Fig. IV.1 shows the crucial dependence of E^T and E^L on amplitude of the longitudinal potential function for a given temperature T and electron density n. The transverse field wavefront in Fig. IV.1 (the positive part of E^T) is big compared to its tail (the negative part of E^T), in contrast with the longitudinal wave E^L, for which we have the absolute symmetry with respect to the origin. It can be explained, perhaps, due to the fast convergence of the hypergeometric series in the region $0 \leq X < \infty$ $(0 < z < \frac{1}{2})$, and relatively slow convergence rate of the same series in the region $-\infty < X < 0$. $(\frac{1}{2} < z < 1)$. Graphs for E^T in Fig. IV.1 were obtained by summing only 10 terms in the hyper-

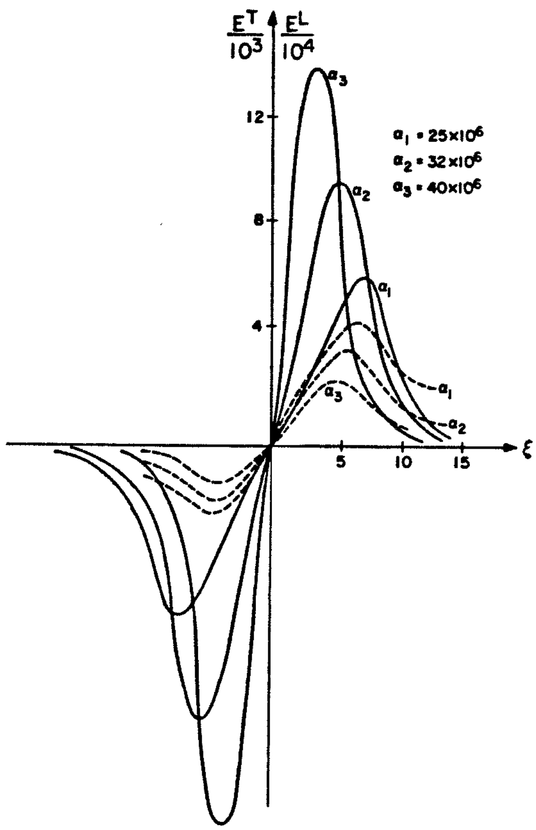

Figure IV.1 The longitudinal (solid lines) and transverse (dashed lines) electric fields for different values of α. The graphs for E^T are plottes for T = 10^6 K, n = 8 x 10^{18}, and by summing up only 10 terms in the expressions (4.43) - (4.45).

geometric series, since adding more terms creates the overflow in the computer. It is noted, that increase of number of terms in the series enlarges the tail (negative part) of E^T, yet retaining the wavefront of E^T almost unchanged. This leads us to believe that adding up sufficient number of terms in hypergeometric series would lead to the symmetrization of E^T with respect to the origin, and, therefore, it would make the shapes of E^T and E^L look alike.

Fig. IV.2 shows dependence of the amplitudes of E^L and E^T on temperature, for a given electron density n and amplitude α of the potential $\bar{u}$. While $|E^L_{max}|$ is decreasing, as temperature goes up, $|E^T_{max}|$ increases. To retain consistency with (3.14) the values of T were limited from below by T = 10°K. Although it is not shown here, the numerical results indicate that increase in α produces a shift in both of the curves $|E^T_{max}|$ and $|E^L_{max}|$ upwards, whereas decrease in α

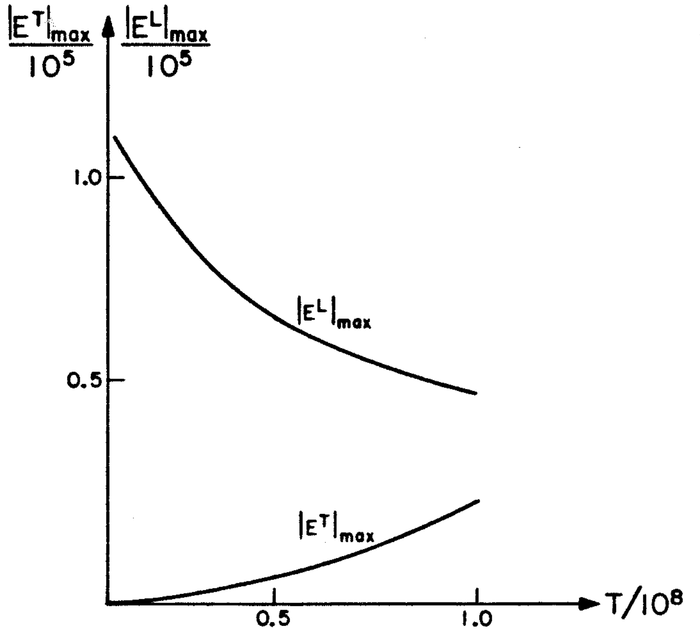

Figure IV.2 The dependence of the maximum values of E^T and E^L on temperature. The plots are drawn for $\alpha = 3x10^7$ and $n = 8x10^{18}$.

shifts these curves downwards. As we showed earlier, transverse waves cannot exist in a nonlinear plasma all by themselves. This is confirmed simply by investigating (4.5) or (4.10), and the solution (4.33) under the condition $\alpha \to 0$ (remember that X is a function of α also!), namely $E^T \to 0$ as $\alpha \to 0$.

4.3 General Case

In this paragraph the objective is to solve (3.3) and (3.4) without using condition (3.5). We will be seeking for the solitary wave solutions for the longitudinal and transverse waves, u and v respectively. If we use (2.32), with U replaced by V, and integrate it once, equation (3.3) can be put in a form

$$-Vu + u(u_o + \frac{1}{2}u) + 3v(v_o + \frac{1}{2}v) - \frac{V}{\omega_p^2}(u_{tt} - c_o^2 u_{xx}) = 0. \qquad (4.46)$$

We seek solution of (4.49) in a form

$$u = \alpha\, \mathrm{sech}^2 A\zeta + v(\zeta)\,, \tag{4.47}$$

where[2]

$$A = \left(\frac{\alpha}{r}\right)^{1/2}\omega_p = \left[\frac{\alpha}{12V(c_o^2 - V_2)}\right]^{1/2}\omega_p \tag{4.48}$$

and, $\zeta = x - Vt$. Note that we have assumed the new velocity of the wave V instead of U, defined in (3.7), since it is reasonable to assume that velocity of a two-component wave (E^L and E^T) should in general differ from velocity of the longitudinal wave, however the former one should reduce to latter one in the limit when $E^T \to 0$, in other words $V \to U$ as $E^T \to 0$. We did not think it was necessary to do these adjustments in the case when waves were predominantly longitudinal.

The choice of (4.47) is justified because when $v \to 0$, u approaches the already obtained solution (3.12), which was found under the assumption that the transverse wave is much smaller (or is zero) than the longitudinal wave. Substituting (4.47) in (4.46) we obtain

$$\begin{aligned} &v(3v_o + u_o - V) + 2v^2 + (c_o^2 - V^2)\frac{V}{\omega_p^2}v'' + \alpha v\, \mathrm{sech}^2 A\zeta \\ &\quad - V\alpha\, \mathrm{sech}^2 A\zeta + \frac{1}{2}\alpha^2\, \mathrm{sech}^4 A\zeta + u_o\,\alpha\, \mathrm{sech}^2 A\zeta \\ &\quad + \frac{2V\alpha A^2}{\omega_p^2}(c_o^2 - V^2)(2\,\mathrm{sech}^2 A\zeta - 3\,\mathrm{sech}^4 A\zeta) = G\,, \end{aligned} \tag{4.49}$$

where G is a constant of integration and primes denote differentiation with respect to ζ. If we multiply both sides of (4.49) by v' and then integrate the whole equation with respect to ζ we get

$$\frac{V(c_o - V^2)}{2\omega_p^2}v'^2 = -a_1v^3 + a_2v^2 + a_3v + a_4 - \left\{ \int \alpha v v'\, \mathrm{sech}^2 A\zeta d\zeta \right.$$

2. See (3.12) and (3.17).

$$+\int v'\left[\frac{1}{2}\alpha^2 \operatorname{sech}^4 A\zeta + (u_o\alpha - V\alpha)\operatorname{sech}^2 A\zeta\right] d\zeta$$

$$+\int \frac{\alpha^3}{6} v'(2\operatorname{sech}^2 A\zeta - 3\operatorname{sech}^4 A\zeta)d\zeta \Bigg\}, \tag{4.50}$$

where

$$a_1 = \frac{2}{3}\ ,\ a_2 = \frac{1}{2}(V - 3v_o - u_o)\ ,\ a_3 = G\ ,\ a_4 = K\ , \tag{4.51}$$

K being a constant of integration. To solve (4.53) we assume

$$\int \alpha v v' \operatorname{sech}^2 A\zeta d\zeta + \int v'\left[\frac{1}{2}\alpha^2 \operatorname{sech}^4 A\zeta + (u_o\alpha - V\alpha)\operatorname{sech}^2 A\zeta\right] d\zeta$$

$$+\int \frac{\alpha^2}{6} v'(2\operatorname{sech}^2 A\zeta - 3\operatorname{sech}^4 A\zeta)d\zeta$$

$$= b_1 v^3 + b_2 v^2 + b_3 v + b_4 = B(v) \tag{4.52}$$

The assumption (4.52) based on the fact that if we can find such coefficients b_i, where i = 1,2,3,4, of the cubic B(v) so that (4.55)is satisfied providing us with some function $v(\zeta)$, then the right hand side of (4.50) will be represented by some other cubic

$$C(u) = B(u) + \tilde{A}(u)\ , \tag{4.53}$$

where $\tilde{A}(u) = -a_1 v^3 + a_2 v^2 + a_3 v + a_4$ as given in (4.50). This will allow us to solve (4.50) and then match its solution with the solution, obtained by solving (4.52) if the latest can be done! Differentiating (4.52) with respect to ζ and then dividing the result by v' we obtain

$$3b_1 v^2 + (2b_2 - \alpha \operatorname{sech}^2 A\zeta) + (V - u_o - \frac{\alpha}{3})\alpha \operatorname{sech}^2 A\zeta - b_3 = 0 \tag{4.54}$$

Equation (4.54) is quadratic for v, and therefore can be easily solved. If we take $b_3 = 0$ and

$$b_2 = \frac{3}{2} b_1 (V - u_o - \frac{\alpha}{3}) \tag{4.55}$$

then it follows from (4.54) that

$$v = \frac{1}{3b_1} \alpha \operatorname{sech}^2 A\zeta . \tag{4.56}$$

By examining (4.55) we immediately recognize that when the transverse component of the wave is zero, then the expression in parentheses is zero, requiring that $b_1 \to \infty$ so that v would vanish.

With the assumption (4.52) and with use of (4.53), equation (4.50) becomes

$$h_1 v'^2 = -v^3 + \frac{a_2 - b_2}{a_1 + b_1} v^2 , \tag{4.57}$$

where we put $a_3 = a_4 = b_3 = b_4 = 0$, and

$$h_1 = \frac{V(c_o^2 - V^2)}{2\omega_p^2 (a_1 + b_1)} . \tag{4.58}$$

Denoting

$$\beta = \frac{a_2 - b_2}{a_1 + b_1} , \tag{4.59}$$

and using exactly the same procedures as in (3.9)-(3.12), we obtain

$$v = \beta \operatorname{sech}^2 \left[\frac{1}{2} \left(\frac{\beta}{h_1} \right)^{1/2} (x - Vt) \right] , \tag{4.60}$$

and therefore

$$E^T = \Psi_t^T = \frac{3m}{|e|} (\Psi^T)_t = \frac{3m}{|e|} [(v + v_o)]_t = \frac{3m}{|e|} v_t$$

$$= \frac{3m}{|e|} \frac{\beta^{3/2} V}{h_1^{1/2}} \operatorname{sech}^2 \left[\frac{1}{2} \left(\frac{\beta}{h_1} \right)^{1/2} \zeta \right] \tanh \left[\frac{1}{2} \left(\frac{\beta}{h_1} \right) \zeta \right] . \tag{4.61}$$

Comparing (4.56) with (4.60), and making use of (4.55) yields

$$a_2 - b_2 = \frac{1}{3}\alpha, \tag{4.62}$$

and therefore

$$\beta = \frac{\alpha}{3b_1} = \frac{\alpha(V - u_o - \frac{\alpha}{3})}{V - 3v_o - u_o - \frac{2\alpha}{3}}, \tag{4.63}$$

and (4.60) becomes

$$v = \beta \operatorname{sech}^2 A\zeta \tag{4.64}$$

Since a_2 is given by (4.51), condition (4.62) fixes the value for b_2. Thus, if condition (4.63) is satisfied, then (4.60) is a solution of (4.46), and therefore it follows from (4.47) and (4.60) that

$$u = (\alpha + \beta) \operatorname{sech}^2 A\zeta . \tag{4.65}$$

To conclude that u and v, given by (4.64) and (4.60) respectively, are solutions of (3.3) and (3.4), we need to show that these u and v satisfy (3.4) as well. To do so we first simplify (3.4) by using (2.32), with U replaced by V, and then integrating result once with respect to x to obtain

$$-Vv + \frac{1}{3}(u_o v + v_o u + vu) + \frac{V}{\omega_p^2}(c^2 - U^2) v_{xx} = H_1 , \tag{4.66}$$

where H_1 is a constant of integration. Substituting (4.65) and (4.60) in (4.66) we obtain

$$\left[-V\beta + \frac{1}{3}u_o\beta + \frac{1}{3}v_o(\alpha + \beta) + \frac{4\beta A^2 V(c^2 - V^2)}{\omega_p^2} \right] \operatorname{sech}^2 A\zeta$$

$$+ \left[\frac{1}{3}\beta(\beta + \alpha) - \frac{1}{2}\alpha\beta \frac{c^2 - V^2}{c_o^2 - V^2} \right] \operatorname{sech}^4 A\zeta = 0 , \tag{4.67}$$

where A, as before, is given by (4.48). Clearly to satisfy (4.67) we require the terms in square brackets to vanish. This leaves us with a system of three nonlinear algebraic equations (third being (4.63)) :

$$\beta(\beta+\alpha) = \frac{3}{2}\beta\alpha\gamma\,, \tag{4.68}$$

$$\beta(V-\frac{1}{3}u_o-\frac{1}{3}v_o) = \frac{1}{3}\beta\alpha\gamma-\frac{1}{3}v_o\alpha\,, \tag{4.69}$$

and

$$\beta = \alpha\frac{V-u_o-\frac{\alpha}{3}}{V-3v_o-u_o-\frac{2\alpha}{3}}\,, \tag{4.70}$$

where

$$\gamma = \frac{c^2-V^2}{c_o^2-V^2} \tag{4.71}$$

4.4 Generalization of the Wave Velocity

Investigation of the system (4.68)-(4.70) leads us to the following interesting observations:

1. (4.68) shows that if

a) $\beta = 0$ then $0 = 0$,

b) $\alpha + \beta = 0$ then $\beta = 0$(also $\alpha = 0$)

2. (4.69) shows that if

a. $\beta = 0$, then $v_o = 0$. We can think of v_o as of a constant (independent of x and t), and is zero iff $\beta = 0$.

b. $(\alpha + \beta) = 0$, then $\beta(V - \frac{1}{3}v_o - \frac{1}{3}u_o) = \beta\frac{\alpha\gamma}{3} + \beta\frac{v_o}{3}$. Thus $\beta = 0$.

3. (4.70) shows that if

a. $\beta = 0$, then $V = u_o + \frac{\alpha}{3}$. ($V \to U$). Only longitudinal wave exists.

b. $(\alpha+\beta)=0$, then $\beta=-\beta\dfrac{V-u_o-\frac{\alpha}{3}}{V-3v_o-u_o-\frac{2\alpha}{3}}$, so that $\beta=0$, and thus $\alpha=0$.

This leads us to the following important conclusions

i. longitudinal wave can exist by itself,

ii. transverse wave does not exist unless longitudinal wave does exist.

To further simplify system (4.68)-(4.70) let us equate terms $\beta\alpha\gamma$ in (4.68) and (4.69). Accordingly we have

$$\frac{2}{3}\beta(\beta+\alpha)=3\beta\left(V-\frac{1}{3}u_o-\frac{1}{3}v_o\right)+v_o\alpha\,,$$

which can be put in a form

$$V=u_o+\frac{1}{3}\alpha+\frac{2}{9}\beta+\frac{1}{3}v_o-\frac{1}{9}\alpha-\frac{2}{3}u_o-v_o\frac{\alpha}{3\beta}$$

$$=U+\sigma(\alpha,\beta,v_o,u_o)\,, \tag{4.72}$$

where U, as before, stands for the velocity of a solitary longitudinal wave, and $\sigma(\alpha,\beta,v_o,u_o)$ is an excess velocity due to the presence of the transverse wave.

$$\sigma(\alpha,\beta,v_o,u_o)=\frac{2}{9}\beta+\frac{1}{3}v_o-\frac{1}{9}\alpha-\frac{2}{3}u_o-v_o\frac{\alpha}{3\beta}\,. \tag{4.73}$$

We want $\lim_{\beta\to 0}\sigma=0$. This will bring a new condition for v_o. Using (4.72) and (4.73) we deduce from (4.70) that

$$\beta=\alpha\frac{\sigma}{\sigma-\frac{\alpha}{3}-3v_o}\,, \tag{4.74}$$

and therefore system of equations (4.68) and (4.74) should be solved subject to condition (4.73). Note that if we assume some form for σ, for example $\sigma=K\alpha\beta$, where K is some constant, then (4.73) yields

$$v_o = \beta \left[3K\alpha\beta - \frac{2}{3}\beta + \frac{1}{3}\alpha + 2u_o \right] / (\beta - \alpha) \tag{4.75}$$

With this choice of σ we have

$$\lim_{\beta \to 0} \sigma = \lim_{\beta \to 0} v_o = 0 \, . \tag{4.76}$$

Substituting $\sigma = K\alpha\beta$ into (4.74), and (4.72) into (4.68), we obtain

$$\beta = \frac{K\alpha^2\beta}{K\alpha\beta - \dfrac{3\beta(3K\alpha\beta + 2u_o - \alpha)}{\beta - \alpha} - \dfrac{\alpha}{3}} \tag{4.77}$$

$$\beta(\beta + \alpha) = \frac{3}{2}\beta\alpha \frac{c^2 - (u_o + \dfrac{1}{3\alpha} + K\alpha\beta)^2}{c_o^2 - (u_o + \dfrac{1}{3\alpha} + K\alpha\beta)^2} \, . \tag{4.78}$$

From (4.77) it follows that

$$K = \frac{3\alpha\beta^2 - 6\beta^2 u_o - \dfrac{\alpha\beta}{3}(\beta - \alpha)}{9\beta^3\alpha - \alpha\beta(\alpha - \beta)} \, . \tag{4.79}$$

If we put (4.79) into (4.78) we obtain polynomial

$$\sum_{m=0}^{7} l_m \alpha^m = 0 \, , \tag{4.80}$$

where

$$\begin{aligned}
l_o &= 72u_o^2\beta^5 + 72u_o^3\beta^4 + 72u_o^4\beta^3 - 1152\beta^5c_o^2 \, , \\
l_1 &= 384\beta^5u_o + 420\beta^4u_o^2 + 36\beta^3u_o^3 + 36\beta^2u_o^4 - 1728\beta^4c_o^2 + 576\beta^4c^2 \, , \\
l_2 &= 512\beta^2 + 264\beta^4u_o + 192\beta^3u_o^2 - 72\beta^2u_o^3 - 360\beta^3c_o^2 + 288\beta^3c^2 \, , \\
l_3 &= 448\beta^4 - 180\beta^3u_o - 24\beta^2u_o^2 - 36\beta u_o^3 + 288\beta^2c_2^2 - 108\beta^2c^2 \, , \\
l_4 &= 50\beta^3 + 6\beta u_o^2 - 144\beta^2u_o + 54c_o^2\beta - 36\beta c^2 \, , \\
l_5 &= 9u_o^2 - 35\beta^2 - 6\beta u_o - 9c^2 - 18c_o^2 \, ,
\end{aligned} \tag{4.81}$$

$l_6 = 6u_o - 4\beta$,

$l_7 = 1$.

If we assume $u_o = \sqrt{\frac{\kappa T}{m}}$, as before, then specifying β and $\mathring{T}$ we can determine α, being a root of (4.80). Numerous computer runs were made in order to determine the values of α that would support a given β, for different temperatures, numerically. Fig. IV.3 shows this dependence for respective values of T_i, i = 1,2,3, being 1000°,2000° and 4000° respectively.[3] The plots indicate that the relationship between α and β is linear for almost entire ranges of tried values of $T \in [1., 10^9]$, and $\beta \in [0, 10^8]$, except for $\beta < 20$, and $\beta > 5 \times 10^7$, where the slopes of the curves slightly increase. Fig. IV.4 shows the values of excess velocity σ as a function of β for a different temperatures T.

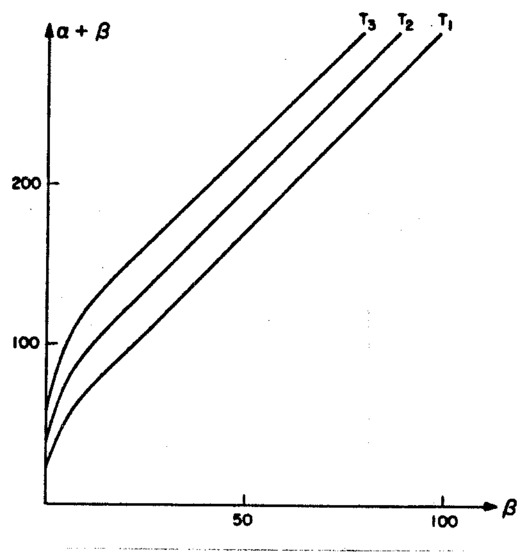

Figure IV.3 Dependence of $\alpha+\beta$ on β for different temperatures $T_n = 10^n$.

In most of the regions of interest σ depends on β inearly, and for every temperature $T_n = 10^n$, n is real, there is a region, that can be approximated by the following inequalities

3. Complex roots, along with negative roots were excluded from consideration. Also excluded were the roots that were too big (greater than the speed of light).

$$3 \cdot 10^{\frac{n}{2}+3} < \beta_n < 5 \cdot 10^{n-3}, \tag{4.82}$$

where $\sigma < 0$, and V changes its sign. We call this region the instability band. At the vicinities of the respective bands (4.82), the graphs exhibit obvious non-linearities. Another interesting observation is that

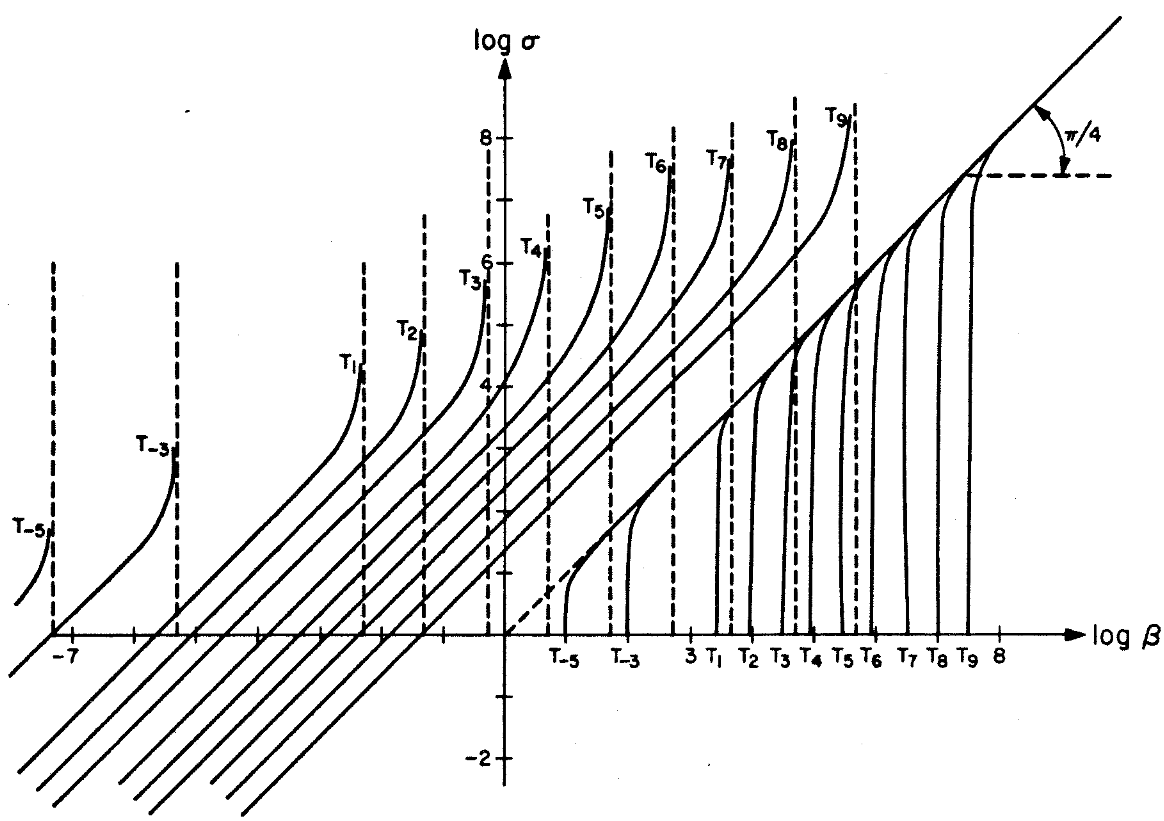

Figure IV.4 The excess velocity σ as a function of β.

all the curves, right from the region of instability, converge to the line $\sigma = \beta$, whereas all the curves, that lie to the left from the region of instability, as they approach this region from the left, tend to converge to the line $\beta = \beta_{n-}$, where β_{n-} is the lower bound of the respective region (4.82). The fact that all the curves, that are to the left of the region (4.82) and lie above the line $\sigma = \beta$, indicates that up to $\beta = \beta_{n-}$ there is some coefficient of proportionality, say μ, between σ and β, such that $\sigma = \mu\beta$, $(\mu \geq 1)$ where μ would depend on temperature: as larger the temperature the smaller is μ. In the limit when T → ∞, μ tends to 1. Thus velocity V of the two-component wave also depends linearly on amplitude of the transverse component.

The presence of transverse wave guarantees the existence of the magnetic

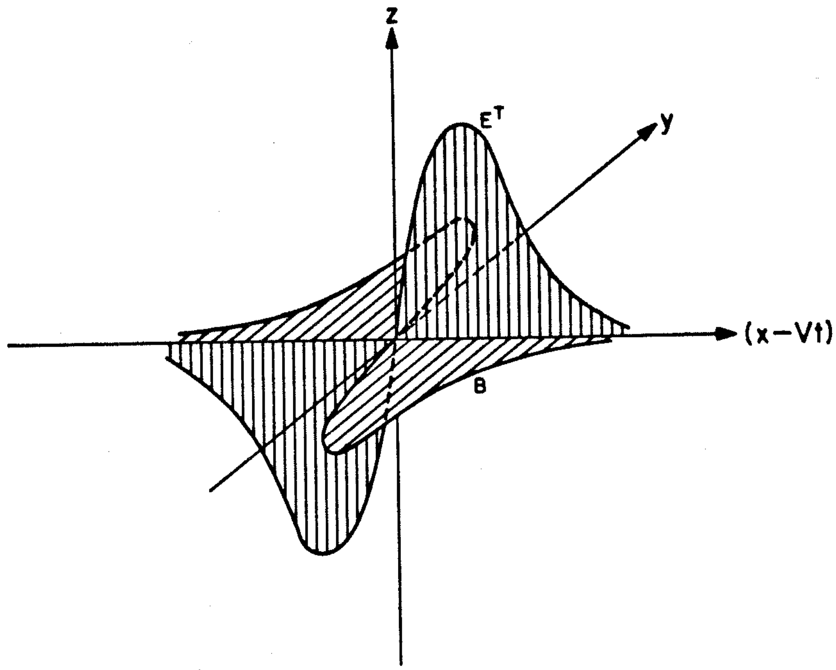

Figure IV.5 E^T and B in x - V t coordinate system

field, since according to the Maxwell's equation

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial H}{\partial t}, \tag{4.83}$$

while the longitudinal component of the **E**-field does not contribute to the magnetic field, since **E** = **E**(x) only. It follows from (4.83) and (4.61) that

$$\mathbf{H} = -\mathbf{e}_y \frac{c}{V} E^T = -\mathbf{e}_y \frac{3m}{|e|} \frac{\beta^{3/2}}{h_1^{1/2}} \mathrm{sech}^2 \left[\frac{1}{2} \left(\frac{\beta}{h_1} \right)^{1/2} \zeta \right] \tanh \left[\frac{1}{2} \left(\frac{\beta}{h_1} \right)^{1/2} \zeta \right], \tag{4.84}$$

where $\mathbf{e}_y$ stands for a unit vector in the y direction. Fig. IV.5 shows E^T and B in the (x - Vt) coordinate frame.

5. SOME PHYSICAL REMARKS

5.1 Motion of the Electron in a Solitary Field

The differential equation of motion of a charged particle is

$$m\,\frac{d\mathbf{v}}{dt} = e\mathbf{E} + \frac{e}{c}\mathbf{v}\times\mathbf{H}\,, \tag{5.1}$$

where $\mathbf{v} = (v_1\ \ v_2\ \ v_3)^T$ is a velocity of a particle. The scalarized version of (5.1) is

$$m\,\frac{dv_1}{dt} = e\,E^L - \frac{1}{c}v_3H\,,$$

$$m\,\frac{dv_3}{dt} = e\,E^T + \frac{1}{c}v_1H\,, \tag{5.2}$$

since there is no y and z dependence. To determine the law of motion of a particle of charge e and mass m in the field of a plane electromagnetic wave, it is convenient to use the following transformation

$$t' = t - \frac{\mathbf{n}\cdot\mathbf{r}}{V}\,,\quad \frac{dt'}{dt} = 1 - \frac{v_1}{V}\,, \tag{5.3}$$

where $\mathbf{n}$ is a unit vector in the direction of propagation. Accordingly system (5.2) becomes

$$m\frac{dv_1(t')}{dt'}\left[1 - \frac{v_1}{V}\right] = e\left[E^L(t') - \frac{v_3(t')}{c}H(t')\right],$$

$$m\frac{dv_3}{dt'}\left[1 - \frac{v_1(t'))}{V}\right] = e\left[E^T(t') + \frac{v_1(t')}{c}H(t')\right], \tag{5.4}$$

Using the relation

$$H = -\frac{c}{V}E^T\,, \tag{5.5}$$

which follows directly from the Maxwell equation (2.4), term $\left[1 - \frac{v_1}{V}\right]$ can be factored out on the right-hand side of the second equation in (5.4) thus reducing the second equation to

$$m\frac{dv_3}{dt'} = eE^T(t') . \tag{5.6}$$

Therefore the transverse velocity and the transverse displacement of electron are

$$v_3 = \frac{e}{m}\int^{t'} E^T(\tau)\,d\tau , \tag{5.7}$$

$$z(t') = \frac{1}{m}\int^{t'} p_3\,d\tau , \tag{5.8}$$

where

$$p_3 = e\int^{t'} E^T(\tau)\,d\tau$$

is the component of the momentum of the particle in the **E**,**H** plane. Using (5.7) the first equation in (5.4) becomes

$$\frac{dv_1}{dt'}\left[1 - \frac{v_1}{V}\right] = Q(t') , \tag{5.9}$$

where

$$Q(t') = \frac{e}{m}\left[E^L(t') + \frac{e}{mV}E^T(t')\int^{t'} E^T(t')\,d\tau\right]$$

$$= \frac{e}{m}\left[E^L(t') + \frac{1}{V}E^T(t')v_3\right] . \tag{5.10}$$

Equation (5.9) can be easily integrated to yield

$$v_1 = V\left\{1 \pm \left[1 - \frac{2}{V}\int^{t'} Q(\tau)\,d\tau\right]^{1/2}\right\} , \tag{5.11}$$

where it was assumed that at t= 0, the particle was at rest at the origin. Since

$$E^L(t') = \alpha_0 \operatorname{sech}^2 qt' \tanh qt' , \tag{5.12}$$

and

$$E^T(t') = \beta_o \operatorname{sech}^2 q t' \tanh q t' , \tag{5.13}$$

where

$$\alpha_o = -\frac{2(\alpha+\beta) m \omega_p V}{|e|} \sqrt{\frac{\alpha}{r_1}} , \tag{5.14}$$

$$\beta_o = -\frac{2\beta m \omega_p V}{|e|} \sqrt{\frac{\beta}{h_1}} \tag{5.15}$$

and

$$q = -AV, \quad (\text{A is given by } (4.48)) ,$$

it follows from (5.9) and (5.10) that

$$v_3 = \rho_3 \tanh^2 q t' + C_1 , \tag{5.16}$$

where

$$\rho_3 = \frac{e \beta_o}{mq} , \tag{5.17}$$

and C_1 is a constant of integration. Clearly $C_1 = -\rho_3$ because v_3 vanishes as $t' \to \pm\infty$. Thus, we have

$$v_3 = -\rho_3 \operatorname{sech}^2 q t' . \tag{5.18}$$

Accordingly (5.10) becomes

$$\begin{aligned} Q(t') &= \frac{e}{m}\left[E^L(t') - \frac{\rho_3}{V} \operatorname{sech}^2 q t' E^T(t') \right] \\ &= \frac{e}{m}\left[E^L(t') - \frac{\rho_3 \beta_o}{V} \operatorname{sech}^4 q t' \tanh q t' \right] , \end{aligned} \tag{5.19}$$

and therefore (5.11) reduces to

$$v_1 = V\left\{ 1 - \left[1 - \frac{2e}{Vm}\left(\frac{\alpha_o}{q} \tanh^2 qt' - \frac{\rho_3\beta_o}{Vq}\left(-\frac{1}{4}\operatorname{sech}^4 qt'\right) + C_2 \right) \right]^{1/2} \right\}$$

$$= V\left\{1 - \left[1 - \frac{2e}{Vm}\left[\rho_1 \tanh^2 qt' + \rho_2 \operatorname{sech}^4 qt' + C_2\right]\right]^{1/2}\right\}, \tag{5.20}$$

where

$$\rho_1 = \frac{\alpha_o}{q}, \tag{5.21}$$

$$\rho_2 = \frac{\rho_3 \beta_o}{4Vq}, \tag{5.22}$$

and C_2 is a constant of integration. We choose $C_2 = \frac{9V^2m^2}{4e^2} - \rho_1$ and an appropriate sign in (5.11) since $v_1 = -V$ as $t' \to \pm\infty$. Hence (5.20) reduces to

$$v_1 = V\left\{1 - \left[1 + \frac{2|e|}{Vm}\left[-\rho_1 \operatorname{sech}^2 qt' + \rho_2 \operatorname{sech}^4 qt' + \frac{9V^2m^2}{4e^2}\right]\right]^{1/2}\right\}. \tag{5.23}$$

Integrating (5.18) and (5.23) we obtain displacements of the electron in the z and x directions respectively

$$z(t') = -\frac{\rho_3}{q}(1 + \tanh qt') \tag{5.24}$$

and

$$x(t') = V\int^{t'} \left\{1 - \left[1 + \frac{2|e|}{m}\left[\rho_2 \operatorname{sech}^4 qt' - \rho_1 \operatorname{sech}^2 qt' + \frac{9V^2m^2}{4e^2}\right]\right]^{1/2}\right\} dt', \tag{5.25}$$

where constant of integration in (5.24) is chosen so that $z(t')$ vanishes as $t \to -\infty$. Formula (5.24) implies that as the solitary wave propagates, electron undergoes some finite dispacement $z = \frac{-2\rho_3}{q}$ in the transverse directon. Formula (5.23), however, indicates that $v_1 = -V$ as $t' \to \pm\infty$, which implies that in these two limits electron is stationary since v_1 represents the longitudinal component of the electron velocity in the coordinate frame that moves with velocity V. Hence in the longitudinal direction electron also undergoes finite dispacement that is given by (5.25).

5.2 Electromagnetic Energy Density

The quantity $\nabla \cdot \mathbf{S}$ represents the rate of change of the electromagnetic energy density W, that is

$$\nabla \cdot \mathbf{S} = \frac{\partial}{\partial t} W \,, \tag{5.26}$$

where $\mathbf{S}$ is the Poynting's vector that is defined by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \,,$$

and it is

$$\mathbf{S} = \mathbf{e}_1 \beta_o \gamma_o \operatorname{sech}^4 A\zeta \tanh^2 A\zeta + \mathbf{e}_3 \alpha_o \gamma_o \operatorname{sech}^4 A\zeta \tanh^2 A\zeta \,,$$

where $\mathbf{e}_1$ and $\mathbf{e}_3$ are unit vectors in the x and z directions respectively, and γ_o is a shorthand for the amplitude of the magnetic field, namely

$$\gamma_o = -\frac{3m}{|e|} \frac{\beta^{3/2}}{h_1^{1/2}} \,.$$

Since $\mathbf{S}$ depends on x and t only, and their derivatives relate through the wave velocity V, (5.26) can be reduced accordingly

$$\frac{\partial}{\partial x} S_1 = V \frac{\partial}{\partial x} W \,,$$

and therefore

$$W = \frac{1}{V} S_1 = \frac{1}{V} \beta_o \gamma_o \operatorname{sech}^4 A\zeta \tanh^2 A\zeta \,, \tag{5.27}$$

where S_1 stands for the component of the Poynting vector in the direction of wave propagation, namely in x direction. Although the amplitude of the longitudinal wave does not appear in (5.27) it does contribute to the energy density since β_o has to be supported by longitudinal component α_o. In the case when waves are predominantly (or completely) longitudinal we have $W = 0$, which implies that there is no energy propagation, and we rather have energy transportation.

6. CONCLUSIONS

Starting from the Maxwell equations and the equations of hydrodynamics we have obtained a nonlinear vector differential equation, governing electromagnetic wave propagation in a collisionless electron plasma. Based on the analysis of this equation we draw the following conclusions:

In a nonlinear collisionless plasma transverse electromagnetic waves cannot exist, unless they are accompanied by longitudinal waves; however, longitudinal waves can exist by themselves. The same is true for solitary electromagnetic waves. The velocity of a vector solitary electromagnetic wave depends linearly on the amplitude of each component of the field. Moreover, the relationship among the amplitudes of the longitudinal and transverse waves is likewise linear for physically reasonable ranges of amplitude and plasma temperature. While the increase in plasma temperature has a tendency to decrease the amplitude and to broaden the longitudinal wave profile, it leads to greater amplitudes of the transverse waves in the case where the waves are predominantly longitudinal. The longitudinal solitary waves are stable in the Lyapunov sense, and the analysis of a periodic wavetrain solution indicates that the equation for these waves is hyperbolic in nature. The longitudinal wave equation can be derived from a Lagrangian density function; accordingly, the wave equation is "lossless", or conservative in the conventional sense. The longitudinal solitons, whose magnitudes are limited by the electron density and plasma temperature, are uniquely determined by their initial values. Frequency spectra of the solitary waves show the dominance of low frequencies. Although the qualitative features of the soliton interaction are obtained, further numerical verification of the interaction is needed.

APPENDIX A

SELECTION OF THE MATRICS

The metric that we use to measure the distance between u and $\bar{u}$ is a functional, that should satisfy the following axioms:

$$d(u,\bar{u}) \geq 0$$

$$d(u,\bar{u}) = d(\bar{u},u) ,$$

and

$$d(u,\bar{u}) \leq d(u,u_o) + d(u_o,\bar{u}) , \quad \text{(triangle inequality)} ,$$

where u, $\bar{u}$ and u_o are defined on the whole real axis . It is clear that $d(u,\bar{u})$ is a function of time, since u and $\bar{u}$ are functions of time.

We select the metrics $d_1(u,\bar{u})$ and $d_2(u,\bar{u})$ to be

$$d_1(u,\bar{u}) = ||u - \bar{u}|| , \tag{A.1}$$

and

$$d_2(u,\bar{u}) = \inf \ ||\tau u - \bar{u}|| , \tag{A.2}$$

where the symbol ||.|| means the norm in the Banach space $W_2^1(R)$, (see Appendix B), and τ stands for a group of translations along the x-axis, such that

$$\tau f(x) = f(x - \xi) , \tag{A.3}$$

where ξ is any real number.

Choice of a metric (5.2) allows to discriminate the stability problem respecting only the shape of solitary waves. Although the choice of metric may be to a large extent arbitrary, it is often crucial to stability analysis. There are dynamical systems, in which the motion is stable with respect to one metric but not with respect to another [Nemat-Nasser and Hermann, 1966]. The set of functions, obtained by such a translation constitutes some quotient space P, subspace of $W_2^1(R)$, each element of which is an equivalent class of functions that are translations of each other. For example

$$\bar{u} = 3U\,\mathrm{sech}^2\left\{\frac{1}{2}\sqrt{\frac{U}{1+U}}\,[\,x-(1+U)\,t\,]\right\}, \tag{A.4}$$

which is solitary wave solution of (3.59), is the same element of P for all t, and can be represented by $\bar{u} = 3U\,\mathrm{sech}^2\left\{\frac{1}{2}\sqrt{\frac{U}{1+U}}\,x\right\}$. We want the solitary wave solution to be stable with respect to the metric d_2, which is actually defined in a subspace P, where the difference $\Delta = u-\bar{u}$ represents the difference between the shapes of u and $\bar{u}$.

APPENDIX B

SOME RESULTS FROM FUNCTIONAL ANALYSIS

We assume that for all t, the solution u is, as a function of x, an element of the Sobler function space $W_2^1(R)$. Therefore $W_2^1(R)$ is the Banach space of functions f(x), that are square-integrable $[f \in L_2(R)]$. The norm in Banach space is defined by

$$||f|| = \left[\int_{-\infty}^{\infty} (f^2 + f'^2)\, dx \right]^{1/2} \tag{B.1}$$

Since $[f \in L_2(R)]$ and $[f' \in L_2(R)]$, their Fourier transforms $\hat{f}(k)$ and $\hat{f}'(k) = -ik\hat{f}$ respectively are also the L_2 functions. Using the Parsival's theorem, the norm in $W_2^1(R)$ is

$$||f|| = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}|^2 (1 + k^2)\, dk \right]^{1/2}. \tag{B.2}$$

Therefore $\hat{f} \equiv (1 + k^2)\hat{f}(1 + k^2)^{-1/2} dk$ is an L_1 function, since it is the product of two L_2 functions. Hence f(x), which is inverse Fourier transform of $\hat{f}(k)$, is a continuous function of x, which vanishes as $x \rightarrow \pm \infty$.[4]

By expanding f as the inverse Fourier transform of $\hat{f}$, arranging f as the product of two L_2 functions and then using the Schwarz inequality we obtain

$$\sup_{x \in R} |f(x)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}|\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1+k^2)^{1/2} |\hat{f}| (1+k^2)^{-1/2} dk$$

$$\leq \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} (1+k^2) |\hat{f}|^2 dk \right]^{1/2} \left[\int_{-\infty}^{\infty} (1+k^2)^{-1/2} dk \right]^{1/2} = \frac{||f||}{\sqrt{2}} \tag{B.3}$$

Inequality (B.3) represents an explicit upper bound for $|f|$.

4. See Riemann-Lebesque theorem for integrals, e.g. Rudin [1966, § 9.6].

APPENDIX C

LOWER BOUND ESTIMATION FOR THE FUNCTIONALS

Before we estimate $\delta^2 N(r)$ and $\delta^2 N(s)$ it is important to observe that if expanded, (3.66) can provide us with an important constraint on the even function r(x) and its derivative, namely,

$$\int_{-\infty}^{\infty} (\Delta^2 + \Delta_x^2)\, dx = -2 \int_{-\infty}^{\infty} (\bar{u}\Delta + u_x \Delta_x)\, dx$$
$$= -2 \int_{-\infty}^{\infty} (\bar{u}r + u_x r_x)\, dx + \int_{-\infty}^{\infty} (\bar{u}s + \bar{u}_x s_x)\, dx$$
$$= -2 \int_{-\infty}^{\infty} (\bar{u}r + \bar{u}_x r_x)\, dx\,, \qquad (C.1)$$

since $\Delta(x) = r(x) + s(x)$, $\Delta(-x) = r(x) - s(x)$, and $\int_{-\infty}^{\infty} (\bar{u}s + \bar{u}_x s_x)\, dx = 0$ for the integrand is an odd function. Condition (C.1) will play an important role in estimating the lower bound for $\delta^2 N(r)$ used as the constraint on r(x). The estimation of the lower bounds for $\delta^2 N(r)$ as well as $\delta^2 N(s)$ will be based on the spectral theory of singular boundary value problems. Consider the integral

$$J = \frac{1}{2}\left(\frac{1+U}{U}\right)^{1/2} \int_{-\infty}^{\infty} \left[4r'^2 + \left(\mu \frac{U}{1+U} - \frac{Q}{3(1+U)}\bar{u} \right) r^2 \right] dx\,, \qquad (C.2)$$

where μ and q are constants. We want to obtain the lower bound for J, instead of computing the lower bound for $\delta^2(r)$ which is given by

$$\delta^2 N(r) = \int_{-\infty}^{\infty} [\, (1+U) r_x^2 + (U - \bar{u}) r^2 \,]\, dx\,, \qquad (C.3)$$

as seen in (3.69). To have $\delta^2 N(r) \geq J$ we want the negative part of the integrand in (C.2) be larger than the negative part of the integrand in (C.3) compared with the terms in r_x^2 respectively in (C.2) and (C.3). The proper choice of constant Q should do it. Introducing the new variable $y = \frac{1}{2}\sqrt{\frac{U}{1+U}}\, x$ we can rewrite (C.2) in the following form

$$J = \int_{-\infty}^{\infty} \left[r_y^2 + (\mu - Q\,\mathrm{sech}^2 y)\, r^2 \right] dy\,. \tag{C.4}$$

We want to expand $\mathrm{sech}^2 y$ term as a linear combination of the eigenfunctions of the associated eigenvalue problem

$$\frac{d^2\eta}{dy^2} + (\lambda + Q\,\mathrm{sech}^2 y)\,\eta = 0\,, \qquad \frac{d\eta(o)}{dy} = 0\,, \tag{C.5}$$

where $\eta(y;\lambda)$ is defined on $[0, \infty)$ and is required to remain bounded as $y \to \infty$, and λ is real. The problem (C.5) is a well-known problem and solutions to it, for a given values of Q are available [see Morse and Feshbach, 1953, p. 768]. In our case eigenvalue spectrum consists of a finite number of discrete negative values, together with a continuous spectrum on the interval $[0, \infty)$. Since $r \in L_2\,[0, \infty)$ there exists a transform pair[5]

$$R(\lambda) = \int_0^{\infty} \eta(y;\lambda)\, r(y)\, dy\,, \tag{C.6}$$

and

$$r(y) = \int_{-\infty}^{\infty} \eta(y;\lambda)\, R(\lambda)\, d\rho(\lambda)\,, \tag{C.7}$$

where the spectral function $\rho(\lambda)$ is non-decreasing on the whole interval $-\infty < \lambda < \infty$, has a jump at each negative eigenvalue, non-decreasing at any $\lambda \le 0$ not an eigenvalue, and is continuous for $\lambda \ge 0$. It is shown (see Titchmarsch 1962, § 2.21) ; that since r and r' both vanish as $y \to \pm\,\infty$, and $r \in W_2^1\,[0, \infty)$ we have

$$\int_0^{\infty} (r'^2 - Qr^2\,\mathrm{sech}^2 y)\, dy = \int_{-\infty}^{\infty} R^2 \lambda\, d\rho(\lambda) \tag{C.8}$$

The only value of Q in (C.5) that enables us to express the $\mathrm{sech}^2 y$ as a linear combination of the eigenvalues of the problem (C.5) is 20. This coefficient also ensures the negative part of the integrand in (C.2) be larger than the corresponding term in (c.3), compared with r'^2 terms. The two negative

5. See, for example, Coddington and Levinson [1955, ch. 9], or Titchmarsh [1962, part 1, ch. 2,3].

eigenvalues and their respective eigenfunctions in (C.5) with Q = 20 are

$$\lambda_1 = -16\,, \quad \eta_1 = \frac{\sqrt{35}}{4}\operatorname{sech}^4 y$$

$$\lambda_2 = -\,4\,, \quad \eta_2 = \frac{\sqrt{5}}{2\sqrt{2}}(6\operatorname{sech}^2 y - 7\operatorname{sech}^4 y)\,, \tag{C.9}$$

when the coefficients are chosen to normalize the eigenfunction, so that

$$\int_0^\infty \eta_i^2(y)\,dy = 1\,. \quad i = 1,2\,.$$

Hence expansion (C.7) becomes

$$r(y) = R_1\eta_1(y) + R_2\eta_2(y) + \int_0^\infty \eta(y;\lambda)\,R(\lambda)\,d\rho(\lambda)\,, \tag{C.10}$$

where

$$R_i = R(\lambda_i) = \int_0^\infty \eta_i(y;\lambda)\,r(y)\,dy\,, \tag{C.11}$$

and the Parseval's equality becomes [6]

$$\int_0^\infty r^2(y)\,dy = R_1^2 + R_2^2 + \int_0^\infty R^2(\lambda)\,d\rho(\lambda) \tag{C.12}$$

Therefore (C.8) reduces to

$$\int_0^\infty (r' - 20\,r^2\operatorname{sech}^2 y)\,dy = -16\,R_1^2 - 4\,R_2^2 + \int_0^\infty R^2(\lambda)\,\lambda\,d\rho(\lambda)\,, \tag{C.13}$$

and hence using (C.4), (C.8), (C.12) and (C.13) we obtain

$$J = \int_0^\infty (\mu + \lambda)\,R^2(\lambda)\,d\rho(\lambda) + (\mu - 16)\,R_1^2 + (\mu - 4)\,R_2^2\,. \tag{C.14}$$

To further reduce (C.14) we first note that $\operatorname{sech}^2 y$ can be expressed as a linear combination of the eigenfunctions η_1 and η_2, namely

$$\operatorname{sech}^2 y = \frac{14}{3\sqrt{35}}\eta_1 + \frac{1}{3}\sqrt{\frac{2}{5}}\,\eta_2\,. \tag{C.15}$$

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6. For any pair of functions r and $\tilde{r}$ the Parseval's equality is

$$\int_{-\infty}^\infty r\tilde{r}\,dy = \int_{-\infty}^\infty R\tilde{R}\,d\rho(\lambda)\,.$$

If we multiply both sides of (C.15) by r(y) and integrate both sides from o to ∞ and also use (C.11) we obtain

$$R_2 = -\sqrt{14}R_1 + 3\sqrt{\frac{5}{2}}\int_o^\infty r\,\mathrm{sech}^2 y\,dy\,. \tag{C.16}$$

Applying the same change of variables $y = \frac{1}{2}\sqrt{\frac{U}{1+U}}x$ in the constraint (C.1) we obtain

$$\int_o^\infty r\bar{u}dy = -\frac{1}{2}\int_o^\infty (\Delta^2 + \frac{1}{4}\frac{U}{1+U}\Delta_y^2)\,dy - \frac{1}{4}\int_o^\infty \bar{u}_y r_y\,dy\,. \tag{C.17}$$

Noting that $\bar{u} = 3U\,\mathrm{sech}^2 y$ and integrating the last integral by parts we get

$$\int_o^\infty r\,\mathrm{sech}^2 y dy = -\frac{1}{6U}\int_o^\infty (\Delta^2 + \frac{1}{4}\frac{U}{1+U}\Delta_y^2)\,dy + \frac{1}{4}\int_o^\infty u_{yy}r dy\,. \tag{C.18}$$

But since

$$\int_o^\infty \bar{u}_{yy} r dy = \int_o^\infty (4\,\mathrm{sech}^2 y - 6\,\mathrm{sech}^4 y)\,r dy$$

$$R_2 = \frac{1}{2}\sqrt{\frac{5}{2}}\int_o^\infty (6\,\mathrm{sech}^2 y - 7\,\mathrm{sech}^4 y)\,r dy$$

and

$$R_1 = \frac{\sqrt{35}}{4}\int_o^\infty r\,\mathrm{sech}^4 y dy\,,$$

equation (C.18) can be written in a following form

$$\int_o^\infty r\,\mathrm{sech}^2 y dy = -\frac{1}{6U}\int_o^\infty (\Delta^2 + \frac{U}{4(1+U)}\Delta_y^2)\,dy + \frac{\sqrt{2}}{3\sqrt{5}}R_2 - \frac{4}{3\sqrt{35}}R_1\,,$$

and therefore (C.16) becomes

$$R_1 = -\frac{a}{U}\int_o^\infty (\Delta^2 + \frac{U}{4(1+U)}\Delta_y^2)\,dy\,,$$

where

$$a = \frac{\sqrt{5}}{2\sqrt{2}\,(\sqrt{14} + \frac{2\sqrt{2}}{\sqrt{7}})} .$$

Thus (C.14) reduces to

$$J = \int_0^\infty (\mu+\lambda)R^2(\lambda)d\rho(\lambda) + (\mu-16)\frac{a^2}{U^2}\left[\int_0^\infty (\Delta^2 + \frac{U}{(1+U)4}\Delta_y^2)dy\right]^2 + (\mu-4)R_2^2 . \qquad (C.19)$$

Selecting $\mu = 4$ makes the last term vanish and implies also that the first integral is non-negative, and therefore (C.19) becomes

$$J \geq -12\frac{a^2}{U^2}\left[\int_0^\infty (\Delta^2 + \frac{U}{4(1+U)}\Delta_y^2)\,dy\right]^2 . \qquad (C.20)$$

Since $\frac{U}{4(1+U)} < 1$, we have

$$J \geq -12\frac{a^2}{U^2}\,||\Delta||^4 . \qquad (C.21)$$

Comparing (C.2) and (C.3) with $\mu = 4$ and $Q = 20$ it can be easily shown that (C.3) can be written in terms of J as follows

$$\delta^2 N(r) = \frac{3}{5}\sqrt{U(1+U)}\,J + \frac{4}{5}\int_0^\infty (1+U)r_x^2dx + \frac{4}{5}U\int_0^\infty r^2dx . \qquad (C.22)$$

With the aid of (C.21), equation (C.22) becomes

$$\delta^2 N(r) \geq -l_1||\Delta||^4 + \frac{4}{5}\int_0^\infty \left[(1+U)r_x^2 + Ur^2\right]dx , \qquad (C.23)$$

where

$$l_1 = \frac{36}{5}\,\frac{\sqrt{1+U}a^2}{U^{3/2}} .$$

To obtain the contribution to $\delta^2 N(\Delta)$ from odd functions, namely $\delta^2 N(s)$, we note that u as an element of P can represent any translation, so we can choose such a translation that expedites the needed comparison between u and $\bar{u}$. We specify such a constant ν so that

$$\int_{-\infty}^{\infty} \{ u(x-\nu) - \bar{u}(x) \}^2 dx = \inf_{\xi \in R} \int_{-\infty}^{\infty} \{u(x-\xi) - \bar{u}(x) \}^2 dx . \qquad (C.24)$$

The infinum is achieved for some finite ν, since u and $\bar{u}$ are elements of $W_2^1(R)$ and therefore converge to zero as $x \to \pm \infty$. Moreover, it also implies that at some $\zeta = \nu$ the right-hand side of (C.24) is stationary w.r.t. ξ, therefore

$$\frac{d}{d\xi} \int_{-\infty}^{\infty} \{ u(x-\zeta) - \bar{u}(x) \}^2 dx = 0 \quad \text{at } \xi = a ,$$

thus giving

$$\int_{-\infty}^{\infty} u'(x-\nu) \{ u(x-\nu) - \bar{u}(x) \} dx = -\int_{-\infty}^{\infty} u'(x-\nu)\, \bar{u}(x)\, dx \qquad (C.25)$$

$$= \int_{-\infty}^{\infty} u(x-\nu)\, \bar{u}'(x) dx = 0 ,$$

and since $u(x-\nu) = \bar{u}(x) + \Delta(x)$, where $\Delta(x) = r(x) + s(x)$, we obtain

$$\int_{-\infty}^{\infty} s\bar{u}'(x) dx = 0 . \qquad (C.26)$$

The constraint (C.26) will be used in a same manner as (C.1) was used in estimating the contribution to $\delta^2 N(\Delta)$ from even functions. It is clear now that (C.24) together with (C.2) implies that

$$d_2(u,\bar{u}) \le \|\Delta\| . \qquad (C.27)$$

To obtain an estimate on $\delta^2 N(s)$ we use (3.70) and the fact that $\|\Delta\|^2 = \|r\|^2 + \|s\|^2$, and therefore

$$\delta^2 N(s) = 2 \int_0^{\infty} \Big[(1+U)s_x^2 + (U-\bar{u})s^2 \Big] dx$$

$$= 2 \int_0^{\infty} (s_x^2 - \bar{u}s^2)\, dx + 2U \int_0^{\infty} (s_x^2 + s^2) dx$$

$$= 2 \left\{ \int_0^{\infty} (s_x^2 - \bar{u}s^2) dx + \frac{1}{2} U \|s\|^2 \right\} . \qquad (C.28)$$

Using the same spectral theory (see (C.10)-(C.13)) we obtain

$$= \int_0^\infty (s_x^2 - \bar{u}s^2)dx = \int_{-\infty}^{\infty} S^2 \lambda \, d\rho(\lambda)$$

$$= \sum_{\lambda_i < 0} S_i^2 \lambda_i + \int_0^\infty S^2 \lambda \, d\rho(\lambda) \,, \qquad \text{(C.29)}$$

where $\rho(\lambda)$ is a spectral function, defined earlier, and

$$S_i = \int_{-\infty}^{\infty} s \, w_i \, dx \,, \qquad \text{(C.30)}$$

where w_i are the eigenfunctions corresponding to the negative eigenvalues of the following eigenvalue problem, defined on $[0, \infty)$:

$$w'' + (\bar{u} + \lambda U) w = 0 \,, \quad w(0) = 0 \,. \qquad \text{(C.31)}$$

The only negative eigenvalue in (C.31) is $\lambda_1 = -1$ and corresponding eigenfunction is

$$w_1 = (\text{const}) \left[3U \sqrt{\frac{U}{1+U}} \operatorname{sech}^2 \left[\sqrt{\frac{U}{1+U}} x \right] \tanh \left[\sqrt{\frac{U}{1+U}} x \right] \right] = (\text{const}) \cdot \bar{u}' \,.$$

Thus (C.30) and (C.26) give

$$S_1 = \text{const} \cdot \int_{-\infty}^{\infty} s \bar{u}' \, dx = 0 \,. \qquad \text{(C.32)}$$

Therefore the summation term in (C.29) vanishes and (C.28) becomes

$$\delta^2 N(s) = \int_0^\infty S^2 \lambda \, d\rho(\lambda) + U \|s\|^2$$

$$\geq U \|s\|^2 \,, \qquad \text{(C.33)}$$

since $\int_0^\infty S^2 \lambda d\rho(\lambda) \geq 0$.

REFERENCES

1. Babister, A.W. 1967. *Transcendental Functions*, Macmillan Co. N.Y.

2. Bailey, W.N. 1935. *Generalized Hypergeometric Series*, Cambridge, [Engl.], The University Press.

3. Barbashov, B.M. and N.A. Chernikov, 1966. Solution and quantization of a nonlinear two-dimensional model for a Born-Infield type field. Soviet Phys. J.E.T.P. 23, 861-869.

4. Benjamin, T.B., J.L. Bona and J.J. Bahoney, 1971. Model equations for long waves in nonlinear dispersive systems. Phil. Trans. R. Soc. Lond. A272, 47-78.

5. Benjamin, T.B. 1972. The stability of solitary waves. Proc R. Soc. Lond. A. 328, 153-183.

6. Berezin, Yu.A. and V.I. Karpman. 1967. Nonlinear evolution of disturbances in plasmas and other dispersive media. Soviet Phys. J.E.T.P. 24, No. 25, 1049-1056.

7. Born, M. and L. Infeld. 1934. Foundations of a new field theory. Proc. R. Soc. London, A147, 522-546.

8. Boussinesq, J. 1877. Essai sur la theorie des eaux courentes. mem. pres. div. Sav. Acad. Sci. Inst. Fr. 23, 1.

9. Coddington, E.A. and N. Levinson. 1955. *Theory of Ordinary Differential Equations*. McGraw-Hill, New York.

10. Davis, L., R. Lüst, and A. Schlüter. 1958. The structure of hydromagnetic shock waves. Z. Naturforschg. 13A, 916-936.

11. Fornberg, B. and G.B. Whitham. 1977. A Numerical and Theoretical Study of Certain Nonlinear Wave Phenomena. Phil. Trans. of Roy. Soc. of London, Vol. 289A, 373-404.

12. Gardner, C.S. and G.K. Morikawa. 1960. Similarity in the asymptotic behaviour of collision-free hydromagnetic waves and water waves. Courant Inst. of Math. Sc., Report NYO 9082.

13. Gibbons, J., S.G. Thornhill, J.J. Wardrop, and D. Ter Haar. 1977. On the theory of Langmuir solitons. J. Plasma Phys. Vol. 17, part 2, 171-184.

14. Ginzburg, V.L. 1970. *Propagation of Electromagnetic Waves in Plasmas*. Pergamon Press, Oxford.

15. Ginzburg, V.L. 1979. *Theoretical Physics and Astrophysics.* Pergamon Press, Oxford.

16. Hirota, R. 1972. Exact N-soliton solutions of the wave equation of long waves in shallow water and nonlinear lattices. J. Math. Phys. 14, No. 7, 810-814.

17. Hoffman, A.L. 1967. A single fluid model for shock formation in MHD shock tubes. J. Plasma Phys. 1, 193-207.

18. Ikezi, H., R.J. Taylor, and D.R. Baker. 1970. Formation and interaction of ion-acoustic solitons. Phys. Rev. Lett. 25, 11-14.

19. Ikezi, H. 1978. Experiments on Solitons in Plasmas, in *Solitons in Action*, ed. Loungren K. and A.C. Scott. Academic Press.

20. Jeffrey, A. and T. Kakutani. 1970. Stability of the Burgers shock wave and the Korteweg-de Vries soliton. Ind. Univ. Math. J. 20, 463-467.

21. Keulegan, G.H. and G.W. Patterson. 1940. Mathematical theory of irrotational translation waves. J. Res. Natn. Bur. Stand. 24, 27-42.

22. Landau, L.D. and E.M. Lifschitz. 1976. *Mechanics.* Pergamon Press.

23. Lamb, Jr. G.L. 1980. *Elements of Soliton Theory.* John Wiley & Sons, Inc.

24. Lax, P.D. 1968. Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure and Appl. Math. 21, 467-490.

25. Lifschitz, E.M. and L.P. Pitaevskii. 1981. *Physical Kinetics.* Pergamon Press.

26. Marable, J.N., J.K. Baird, and D.B. Nelson, 1972. Effects of electromagnetic pulse (EMP) on a power system. Oak Ridge Nat. Lab. December.

27. Miura, R.M., C.S. Gardner, and M.D. Kruskal. 1967. Korteweg-de Vries Equation and generalizations. II. Existence of conservation laws and constants of motion. J. Math. Phys. 9, No. 8, 1204-1209.

28. Miura, R.M. 1967. Korteweg-de Vries equation and generalization. I. A remarkable explicit nonlinear transformation. J. Math. Phys. 9, No. 8, 1202-1204.

29. Miura, R.M. 1976. The Korteweg-de Vries equation: a survey of results. SIAM Rev. 18, No.3, 412-459.

30. Morse, P.M. and H. Feshbach. 1953. *Methods of Theoretical Physics.* McGraw-Hill, New York.

31. Nemat-Nasser, S. and G. Hermann. 1966. On the stability of equilibrium of continuous systems. Ing.-Arch. 35, 17-23.

32. Papas, C.H. 1965. *Theory of Electromagnetic Wave Propagation.* McGraw-Hill, New York.

33. Papas, C.H. 1976. On the Electrodynamical Implications of Flare Stars, in *Flare Stars*, ed. L.B. Mirzoian, 175-180.

34. Saffman, P.G. 1961a. Propagation of a solitary wave along a magnetic field in a cold collision-free plasma. J. Fluid Mech., 11, 16-20.

35. Saffman, P.G. 1961b. On hydrodynamic waves of finite amplitude in a cold plasma. J. Fluid Mech. 11, 552-566.

36. Scott, A.E., F.Y.F. Chu and D.W. McLaughlin. 1973. The soliton: a new concept in applied science. Proc. IEEE, 61, No. 10, 1443-1483.

37. Spitzer, L. Jr. 1962. *Physics of Fully Ionized Gases.* Interscience Publishers.

38. Stokes, G.G. 1847. On the theory of oscillatory waves. Camb. Trans. 8, 441-473.

39. Titchmarsh, E.C. 1962. *Eigenfunction Expansions* (2nd ed.). Oxford University Press.

40. Tsytovich, V.N. 1970. *Nonlinear Effects in Plasma.* Plenum Press.

41. Whitham, G.B. 1965. A general approach to linear and nonlinear dispersive waves using a lagrangian. J. Fluid Mech. 22, 273-282.

42. Whitham G.B. 1967. Variational methods and applications to water waves. Proc. R. Soc. A299, 6-25.

43. Whitham G.B. 1970. Two-timing, variational principles and waves. J. Fluid Mech. 44, 373-395.

44. Whitham, G.B. 1973. *Linear and Nonlinear Waves.* John Wiley & Sons, Inc.

45. Zabusky, N.J. and M.D. Kruskal. 1965. Interaction of solitons in a collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. 15, 240-243.

46. Zabusky, N.J. 1967. A synergetic approach to problems of nonlinear dispersive wave propagation and interaction. Proc. Symp. on nonlinear partial differential equations. W.F. Ames, ed., Academic Press, New York, 223-258.

47. Zakharov, V.E. and A.B. Shabat. 1972. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Soviet Phys. J.E.T.P. 34, 62-69.

48. Zakharov, V.E. 1972. Kinetic equation for solitons. Soviet Phys. J.E.T.P., 33, pp. 538-541.