A DIRECT METHOD FOR DESIGNING OPTIMAL CONTROL SYSTEMS THAT ARE INSENSITIVE TO ARBITRARILY LARGE CHANGES IN PHYSICAL PARAMETERS

Thesis by

Uy-Loi Ly

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ABSTRACT

A new concept has been developed for designing optimal feedback controllers that will be insensitive to given, arbitrarily large variations in physical parameters. The method uses as a single figure of merit the expected value of a quadratic performance index, the minimization of which determines directly (without trial and error) the desired set of feedback gains. These values of the feedback gains (where such exist) guarantee at the outset closed-loop stability for all possible values of physical parameters in the prescribed domain of uncertainty.

The new method extends the well known method for the optimal regulator design where physical parameters have single, precisely known values, to the case where they may have a range of values. In addition, it encompasses (as a special case) the Minimax design developed also for handling systems whose physical parameters may have a range of values (which the Minimax explores by trial and error while the new method accounts automatically for the entire range).

An essential feature of the new procedure is that it includes exactly in its cost criterion whatever effects accompany large departures in the plant parameters from their nominal values. This is why the new method is able to guarantee stability over the whole range of parameter values, where perturbation techniques are not.

The feasibility and usefulness of the new design technique are illustrated by numerical examples in which control systems are designed for second-order plants each of whose parameters may have a given range of values. Comparisons which results using standard optimal design and the Minimax technique are given.

ABSTRACT (Continued)

Application to high-order systems will need to be accompanied by further development of appropriate computational procedures.

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LIST OF SYMBOLS

Symbol	Definition
a_{ij}	ij element of the output weighting matrix
A	output weighting matrix
b	element of the control weighting matrix
b_{ij}	ij element of the control weighting matrix
B	control weighting matrix
C_i	components of the feedback gain matrix for single input systems
C_{ij}	ij element of the feedback gain matrix
C	feedback gain matrix
C^{ℓ}	feedback gain vector at the ℓ^{th} iteration
$\mathcal{C}$	controllability matrix
D	domain of uncertainty
D^{ℓ}	normalized search direction vector at the ℓ^{th} iteration
E( )	expected value operator
f_i	components of the open loop plant matrix
F(p)	open loop plant matrix
g	component of the control distribution matrix
G(p)	control distribution matrix
h_i	integration step size of the i^{th} variable parameter
h^{ℓ}	search step size at the ℓ^{th} iteration
H(p)	output distribution matrix
i,j	integral indices
j	purely imaginary number ($=\sqrt{-1}$)
$J(x_0,p,C)$	integral quadratic index

LIST OF SYMBOLS (Continued)

Symbol	Definition
$\hat{J}(x_o,C)$	expected value of the integral quadratic cost index
k	constant of proportionality
k	number of iterations (in Appendix B)
K_i	scalar or 2x2 block matrix
K	nonsingular coordinate transformation matrix
m	number of control inputs
M	number of points in the parameter domain
n	number of state variables
N_i	number of integration steps in the i^{th} variable parameter
Δp_i	interval of uncertainty in the i^{th} variable parameter
p	parameter vector
p_o	nominal parameter value
P	upper partition of the eigenvector matrix
Q	weighting matrix for the trajectory sensitivity vector
Q	lower partition of the eigenvector matrix (in Appendix A)
r	number of uncertain parameters
R	parameter covariance matrix
S_{ij}	ij element of the Lyapunov matrix
$S(p,C)$	Lyapunov matrix
t	time
t_o	initial time
U_j	normalized vector along the direction of the j^{th} coordinate axis
$u(t)$	control input vector

LIST OF SYMBOLS (Continued)

Symbol	Definition
v_o	initial vector in the power method
v_k	vector obtained at the k^{th} iteration
$w(p)$	probability density function of the parameter vector
$w_o(p)$	distribution function defined inside the domain of uncertainty
x_o	initial state
$x(t)$	state vector
$y(t)$	output vector

Greek Symbols

α_i	weighted coefficient of the i^{th} parameter
$\delta(\ )$	Dirac delta function
δ	prescribed level of accuracy
$\lambda(S)$	eigenvalue of S
$\lambda_n(k)$	approximation of the largest eigenvalue λ_n achieved at the k^{th} iteration
Δ	incremental volume in the parameter domain
ν	normalization factor
$\sigma(t)$	trajectory sensitivity vector
$\Vert\ \Vert$	norm operator

Subscripts

o	nominal or initial condition
-	vector notation

LIST OF SYMBOLS (Continued)

Superscripts

Symbol	Definition
-	complex conjugate of
T	Transpose of

LIST OF FIGURES

I. INTRODUCTION

Development of a usable method for designing feedback controllers that will be insensitive to variations in the parameters of the physical system being controlled has been a major concern for designers in the control field. In practice, the problem of sensitivity arises from the fact that there are always uncertainties in the knowledge of physical constants characteristic of the dynamical system to be controlled. In addition, certain system parameters may be changed or may vary during normal system operation. In the design of aircraft flight control systems in particular, there are uncertainties about the actuator dynamics, about aerodynamics, and about structural dynamics. Moreover, some of these will be very different for different speeds and altitudes of flight. It is apparent that any careful design should incorporate in it an explicit criterion representing the effect of system variations or uncertainties. Extensive work has been done in this area but, in general, no consensus has been reached as to which methods should be used in approaching the problem of minimum sensitivity control system design (1-10).*

A direct attack on the problem is to begin with the so-called modern control approach, a central feature of which is that a single figure of merit, called a "cost function," is established at the outset, to replace and represent the many separate performance criteria typical of classical design. The design objective then reduces to finding those control parameters that minimize the cost function. This is called the

*Numbers in parentheses designate references at the end of the paper.

optimal design, and numerous detailed methods have been developed for achieving it efficiently when physical parameters are known exactly.

In applying modern control theory to systems whose parameters are uncertain, two means of incorporating sensitivity are commonly used, trajectory sensitivity and integrated mean square response. The first approach consists of augmenting the state vector to include the trajectory sensitivity vector $\underline{\sigma}(t)$ defined as the derivative of the state vector with respect to the variable parameters $\underline{p}$, evaluated at the nominal condition $\underline{p}_0$, i.e.,

$$\underline{\sigma}^T(t) = \left[\frac{\delta \underline{x}^T}{\delta p_1}, \frac{\delta \underline{x}^T}{\delta p_2}, \dots, \frac{\delta \underline{x}^T}{\delta p_r}\right]\Bigg|_{\underline{p}=\underline{p}_0} \tag{1.1}$$

and of augmenting the usual cost functional,

$$J(\underline{p}_0,\underline{x}_0,\underline{u}) = \tfrac{1}{2}\int_0^\infty [\underline{x}^T(t)A\underline{x}(t)+\underline{u}^T(t)B\underline{u}(t)]dt \tag{1.2}$$

to incorporate a quadratic sensitivity term:

$$J'(\underline{p}_0,\underline{x}_0,\underline{u}) = \tfrac{1}{2}\int_0^\infty [\underline{x}^T(t)A\underline{x}(t)+\underline{u}^T(t)B\underline{u}(t)+\underline{\sigma}^T(t)Q\underline{\sigma}(t)]dt \tag{1.3}$$

Different formulations of the linear feedback control law in the state and sensitivity vectors are considered (3-8) for the minimization of the augmented cost functional $J'(\underline{p}_0,\underline{x}_0,\underline{u})$. It is shown in (2) that for a linear feedback control of the form $\underline{u}(t)=[C\underline{x}(t)+D\underline{\sigma}(t)]$, no finite state-independent feedback matrices, C and D, can be found which will satisfy the necessary conditions of optimality.

The implicit assumption usually made in these approaches is that all higher order derivatives of the sensitivity vector with respect to the variable parameters, $\underline{p}$, are negligible, consequently restricting the range of applicability of the design to a small neighborhood about the nominal parameter value. Moreover, the development of the controller design requires additional hardware to implement signals that approximate the trajectory sensitivity state vector (4).

The second common approach to design for insensitivity is based upon the optimization of an integrated mean square response with an appropriate choice of the control gains. In this case the effect of small parameter variations is modeled as a random external disturbance to the open loop plant. The variable parameters are often considered as components of a gaussian distributed random vector with mean at the nominal value and known covariance (9, 10). Again, small perturbations of the state vector about the nominal trajectory are assumed to result from the random disturbances created by the uncertainties in the plant parameters. The usefulness of this design for relatively large parameter variations remains an open question.

This thesis describes a new approach to controller design which reduces performance sensitivity not just to small, but to large parameter variations. The main objective of this approach is the desensitization of the closed loop system over a finite but arbitrarily large range of plant variations while retaining the attractive structure of a linear feedback controller. The method uses the expected value of a normalized quadratic performance index as a cost function, the minimization of which determines a set of feedback gains which render the closed loop plant asymptotically stable over the entire domain of uncertainties. Furthermore, this design provides a better compromise between the performance cost and its sensitivity to parameter variations than the Minimax design suggested in a recent Honeywell report (1), which has been considered the best approach to date for handling large parameter variations.

A second order control system is used to illustrate the effectiveness of design by the new method in reducing the closed loop system performance sensitivity to variations in plant parameters. Using two numerical examples, a comparison of the reduction in performance sensitivity is made between the conventional optimal quadratic design based upon the nominal parameter values, the present design, and the Minimax design mentioned in (1).

While the examples presented in this thesis demonstrate the conceptual value and rigor of the new method introduced here, numerical techniques for applying it efficiently, especially for higher order systems, need to be developed further to make the method an effective design tool.

The new method is described in Section II and the general procedures for applying it are outlined in Section III. Added insight is provided by certain simplifications that obtain for second order systems. These are developed in Section IV.

In Section V are given the numerical examples and comparisons with other methods. Section VII recommends several directions for further work to extend the method and improve its usefulness.

II. DESCRIPTION OF THE DESIGN METHOD

The design concept will first be described in some generality and later applied to specific cases of practical interest. The procedure is developed within the formalism of state variable modeling techniques where the controlled plant dynamics are described by the following deterministic equation (Fig. 1),

$$\dot{\underline{x}}(t) = F(\underline{p})\underline{x}(t) + G(\underline{p})\underline{u}(t) \qquad (2.1)$$

with initial condition $\underline{x}(t_0)=\underline{x}_0$, where

$\underline{x}(t)$ is a state vector of dimension n

$F(\underline{p})$ is an nxn open-loop plant matrix

$G(\underline{p})$ is an nxm control distribution matrix

$\underline{u}(t)$ is a control vector of dimension m

$\underline{p}$ is a parameter vector of dimension r.

Note that the mathematical development assumes $\underline{x}_{ref} = 0$; that is, the development is directed toward the closed loop stability of the system, not its signal following properties. In the following analysis, the variable parameter $\underline{p}$ is allowed to take on any constant, state-independent values lying in a prescribed region D.

The deterministic plant output responses are given by

$$\underline{y}(t) = H(\underline{p})\underline{x}(t) \qquad (2.2)$$

where $\underline{y}(t)$ is an output vector of dimension q

$H(\underline{p})$ is a qxn output distribution matrix.

We assume for simplicity that all states $\underline{x}(t)$ are accessible for feedback purposes, and we consider a linear feedback control law of the form

$$\underline{u}(t) = -C\underline{x}(t) \qquad (2.3)$$

where C is an mxn control gain matrix.

The closed loop plant becomes $\dot{\underline{x}}(t) = [F(\underline{p})-G(\underline{p})C]\underline{x}(t)$ (2.4)

It can be shown that if the system is completely controllable (and $\underline{p}$ is known), the system closed loop poles may be assigned arbitrarily by a proper choice of the linear feedback gain matrix, C. This design technique is known as the method of pole assignment. A condition for controllability may be stated as follows (11-18):

The linear time invariant system described in (2.1) is said to be completely controllable if the controllability matrix

$C(\underline{p}) = [G(\underline{p})\ F(\underline{p})G(\underline{p})\ F^2(\underline{p})G(\underline{p})\ \ldots\ F^{n-1}(\underline{p})G(\underline{p})]$ has maximum rank n.

To achieve closed loop stability the controller gains, C, are chosen such that all the eigenvalues of $[F(\underline{p})-G(\underline{p})C]$ lie in the open left half of the complex plane. This choice of C also guarantees the existence and boundedness of the following commonly used quadratic performance index

$$J(\underline{p},\underline{x}_0,\underline{u}) = \tfrac{1}{2}\int_0^\infty [\underline{y}^T(t)A\underline{y}(t)+\underline{u}^T(t)B\underline{u}(t)]dt \qquad (2.5)$$

where A is a qxq real symmetric positive semi-definite matrix, and B is an mxm real symmetric positive definite matrix.

In regulator design, the matrices A and B are often chosen initially according to the following rule of thumb suggested by Bryson and Ho (11):

$$A_{ij} = 0 \quad i\neq j \quad , A_{ii} = \frac{1}{y_{i_{max}}^2} \qquad i=1,2,\ldots,q \qquad (2.6)$$

$$B_{ij} = 0 \quad i\neq j \quad , B_{jj} = \frac{1}{u_{j_{max}}^2} \qquad j=1,2,\ldots,m \qquad (2.7)$$

where $y_{i_{max}}$ and $u_{j_{max}}$ are the maximum permissible magnitudes of the i^{th} output and j^{th} control, respectively.

Substituting (2.2) and (2.3) into (2.5) yields

$$J(\underline{p},\underline{x}_0,C) = \tfrac{1}{2}\int_0^\infty \underline{x}^T(t)[H^T(\underline{p})AH(\underline{p})+C^TBC]\underline{x}(t)dt \qquad (2.8)$$

It can be easily shown that the performance index, $J(\underline{p},\underline{x}_0,C)$, can be written in terms of the initial condition $\underline{x}_0$, as (11-18)

$$J(\underline{p},\underline{x}_0,C) = \tfrac{1}{2}\underline{x}_0{}^T S(\underline{p},C)\underline{x}_0 \qquad (2.9)$$

where the real nxn symmetric positive semi-definite matrix, $S(\underline{p},C)$, satisfies the following Lyapunov matrix equation:

$$S(\underline{p},C)[F(\underline{p})-G(\underline{p})C]+[F(\underline{p})-G(\underline{p})C]^T S(\underline{p},C)=-[H^T(\underline{p})AH(\underline{p})+C^TBC] \qquad (2.10)$$

which may be solved numerically by the eigenvector decomposition method suggested by Bryson and Hall (19). Appendix A describes the eigenvector decomposition method used in the solution of (2.10). The accuracy of the computed matrix $S(\underline{p},C)$ depends heavily on the quality of the eigenvalue-eigenvector subroutine used. A drawback of the eigenvector decomposition method is that one has to solve for the eigenvalues and eigenvectors of an augmented system matrix of size 2n. This algorithm may turn out to be very costly in terms of storage requirements and computational time for cases where n is large. Alternatively, the solution $S(\underline{p},C)$ to (2.10) may be written explicitly in terms of the stable closed loop transition matrix (11-18), namely

$$S(\underline{p},C)= \int_0^\infty \exp[F(\underline{p})-G(\underline{p})C]^T t\cdot[H^T(\underline{p})AH(\underline{p})+C^TBC]\cdot \exp[F(\underline{p})-G(\underline{p})C]t\cdot dt \qquad (2.11)$$

The expression given in (2.11) clearly indicates the dependence of the matrix $S(\underline{p},C)$ upon the parameter vector $\underline{p}$ and on the choice of the suboptimal control gain matrix C. However, this representation of $S(\underline{p},C)$ is not computationally attractive. For other methods of solving the Lyapunov matrix equation (2.10), the reader should consult the excellent text written by Barnett and Storey (21).

So far we have discussed the evaluation of the quadratic performance index, $J(\underline{p},\underline{x}_0,C)$, in terms of the matrices $F(\underline{p})$, $G(\underline{p})$, $H(\underline{p})$, A, B and C. We shall now indicate how one may use this cost function to determine a unique set of state variable feedback gains, C. In the conventional optimal quadratic regulator design, the set of feedback gains is obtained from the minimization of the cost functional $J(\underline{p},\underline{x}_0,u)$, given in (2.5), assuming that the plant parameters have their nominally chosen values, that is $\underline{p}=\underline{p}_0$. This design presumes, then, an exact knowledge of the plant parameters. The resulting optimal feedback gain matrix, C, is given by (11-18)

$$C = B^{-1}G^T(\underline{p}_0)S(\underline{p}_0) \tag{2.12}$$

where $S(\underline{p}_0)$ satisfies the well known steady state Riccati equation obtained upon substitution of (2.12) into (2.10), namely

$$S(\underline{p}_0)F(\underline{p}_0)+F^T(\underline{p}_0)S(\underline{p}_0)-S(\underline{p}_0)G(\underline{p}_0)B^{-1}G^T(\underline{p}_0)S(\underline{p}_0) = -H^T(\underline{p}_0)AH(\underline{p}_0) \tag{2.13}$$

It turns out that the choice of the optimal gains is independent of the initial condition $\underline{x}_0$ (22). In other words, the set of feedback gains, C, given by (2.12) and (2.13) minimizes the cost function, $J(\underline{p}_0,\underline{x}_0,C)$, for any set of initial conditions $\underline{x}_0$. The impetus behind the use of a quadratic cost index is that its minimization leads to a linear feedback controller of the form given in (2.3). Furthermore, for a single input system (i.e., $\underline{u}(t)$ a scalar) the optimal quadratic controller design possesses the following properties in the frequency domain: an infinite gain margin and a phase margin of at least 60° (10, 13).

If the matrices $F(\underline{p}),G(\underline{p}),H(\underline{p})$ deviate from their nominal values (i.e., $\underline{p}\neq\underline{p}_0$), the above design is no longer optimal, and the performance index evaluated at an off-nominal condition may deteriorate significantly. In other words, the regulator which is optimal for the nominal parameter values may be highly sensitive to parameter variations. This suggests that one should consider a new criterion in which a trade-off between the performance index and its sensitivity could be achieved. This concept is schematically shown in Fig. 2.

To include the effect of parameter variations in the performance index, we consider the following cost function

$$\hat{J}(\underline{x}_0,C)=E[J(\underline{p},\underline{x}_0,C)]=E[\tfrac{1}{2}\underline{x}_0^{\ T}S(\underline{p},C)\underline{x}_0]=\tfrac{1}{2}\underline{x}_0^{\ T}E[S(\underline{p},C)]\underline{x}_0 \qquad (2.14)$$

where E[ ] is the expected value operator over the parameter space R^r, in which $\underline{p}$ has the probability density function $w(\underline{p})$ defined as follows

$$w(\underline{p}) = \begin{cases} w_0(\underline{p}) & , \quad \underline{p}\in D \\ 0 & , \quad \underline{p}\notin D \end{cases} \qquad (2.15)$$

i.e., the probability that the controlled plant takes on values outside the specified bounded domain, D, is null, where $w_0(\underline{p})$ is an arbitrary integrable function in R^r such that

$$\begin{cases} w_0(p) \geq 0 \\ \int_D w_0(\underline{p})d\underline{p} = 1 \end{cases} \tag{2.16}$$

Figure 3 illustrates a case where r=2.

Cost functional (2.14) was mentioned in (2) but not discussed in any detail. The present study proposes the use of a linear state variable feedback controller of the form given in (2.3) to minimize the cost functional $\hat{J}(\underline{x}_0,C)$. The optimization of $\hat{J}(\underline{x}_0,C)$ will lead to a set of feedback controller gains, C, which are dependent upon the initial state $\underline{x}_0$. Given a particular initial condition $\underline{x}_0$, the optimal feedback gains can be determined. It is clear that a feedback system optimized for one initial state may not be satisfactory for another. To ensure the validity of the feedback system for all initial disturbances we introduce a supremum norm defined as follows

$$\|Q\| = \sup_{\|\underline{x}_0\|=1} (\underline{x}_0{}^T Q \underline{x}_0) = \lambda_{max}(Q) \tag{2.17}$$

where Q is an arbitrary symmetric positive semi-definite matrix.

It can be seen that $E[S(\underline{p},C)]$ in (2.14) is a symmetric positive semi-definite matrix since $S(\underline{p},C)$ is always positive semi-definite if $[F(\underline{p})-G(\underline{p})C]$ is a stability matrix (20), and since $w_0(\underline{p})$ is a non-negative function of the parameter vector $\underline{p}$. Therefore $\|E[S(\underline{p},C)]\| = \lambda_{max}(E[S(\underline{p},C)]) \geq 0$.

The set of feedback controller gains, C, obtained from the minimization of $\sup_{\|\underline{x}_0\|=1} \hat{J}(\underline{x}_0,C)$, i.e.,

$$\underset{C}{\text{Min}}\ \underset{\|\underline{x}_0\|=1}{\text{Sup}}\ \hat{J}(\underline{x}_0,C)=\tfrac{1}{2}\ \underset{C}{\text{Min}}\|E[S(\underline{p},C)]\|=\tfrac{1}{2}\underset{C}{\text{Min}}\lambda_{max}(E[S(\underline{p},C)]) \tag{2.18}$$

guarantees the stability of the feedback control system for all $\underline{p}$ lying in the domain D. In general, conditions for the existence of such a set of controller gains, C, need further study. The examples in Section V illustrate cases where such gains clearly exist.

As described previously, the insensitive design requires a priori knowledge of the domain of uncertainties, D, and of the distribution of uncertainties $w_0(\underline{p})$. The latter requirement should reflect the level of confidence the designer has in each of the parameters $\underline{p}$, derived primarily from his experience and his understanding of the actual physical system. Some typical examples of the domain D and of the probability density function $w_0(\underline{p})$ one may consider are:

(a) If some statistical properties of the parameter uncertainties are known, specifically their mean $E(\underline{p})=\underline{p}_0$ and their covariance matrix $E[(\underline{p}-\underline{p}_0)(\underline{p}-\underline{p}_0)^T]=R$, where R is a symmetric positive definite matrix, then one might choose (Fig. 4a)

$$w_0(\underline{p})=\frac{1}{\nu}\exp[-(\underline{p}-\underline{p}_0)^T R^{-1}(\underline{p}-\underline{p}_0)] \text{ for all } \underline{p} \text{ in D} \tag{2.19}$$

where

$$D = \{\underline{p}/(\underline{p}-\underline{p}_0)^T R^{-1}(\underline{p}-\underline{p}_0)\leq d, d>0, \text{ constant}\}$$

$$\nu = \int_D \exp[-(\underline{p}-\underline{p}_0)^T R^{-1}(\underline{p}-\underline{p}_0)]d\underline{p}$$

In general the domain D is a region bounded by a hyperquadratic surface δD of dimension r-1, where δD is a contour of constant $w_0(\underline{p})$ in this case.

(b) If the system parameters $\underline{p}$ are known to be equally likely to lie anywhere within a specified range of uncertainty about their nominal value, say $\underline{p}=\underline{p}_0\pm\Delta\underline{p}$, then (Fig. 4b)

$$w_0(\underline{p})=\frac{1}{\nu} \text{ for all } \underline{p} \text{ in D (uniform distribution)} \tag{2.20}$$

where

$$D = \{p/|\underline{p}_i-\underline{p}_{o_i}|\leq\Delta\underline{p}_i \quad , \quad i=1,2,\ldots,r\}$$

$$\nu = 2^r\cdot\Delta p_1\Delta p_2\ldots\Delta p_r$$

In general the domain D is bounded by a hyper "cube" δD of dimension r-1.

(c) If a discrete set of parameter values is of interest and the relative weights α_i on each of them are known, then the following probability density function is appropriate (Fig. 4c):

$$w_0(\underline{p}) = \sum_{i=1}^{M} \alpha_i\cdot\delta(\underline{p}-\underline{p}_i) \quad \text{for all } \underline{p} \text{ in D} \tag{2.21}$$

where D is an arbitrary domain containing all the parameters $\underline{p}_i$(i=1,2,... M) and $\delta($ $)$ is the Dirac delta function. Note that the coefficients α_i are chosen such that

$$\begin{cases} 0\leq\alpha_i\leq1 \quad , \quad i=1,2,\ldots,M \\ \sum_{i=1}^{M} \alpha_i = 1 \end{cases}$$

(d) If one wants to design a feedback controller to handle the worst case that may happen to the closed loop system, then (Fig. 4d)

$$w_0(\underline{p}) = \delta(\underline{p}-\underline{p}_w) = \delta(p_1-p_{w_1})\ \delta(p_2-p_{w_2})\ldots\ \delta(p_r-p_{w_r}) \tag{2.22}$$

for all $\underline{p}$ in D, where D is any closed bounded domain containing $\underline{p}_w$, the plant parameter corresponding to the worst case.

The worst case parameter is defined as the set of parameter values

$\underline{p}_w$ such that (1, 23)

$$J(\underline{p}_w,\underline{x}_o,C^*) \geq J(\underline{p},\underline{x}_o,C^*) \quad \text{for all } \underline{p} \text{ in } D \tag{2.23}$$

and

$$J(\underline{p}_w,\underline{x}_o,C) \geq J(\underline{p}_w,\underline{x}_o,C^*) \quad \text{for all } C \tag{2.24}$$

The values of $\underline{p}_w$ and C*, if they exist, can be determined iteratively using the algorithm suggested in (23).

This approach requires a priori knowledge of the worst case value of the plant parameters. Determination of these parameter values $\underline{p}_w$ is difficult when the dimensions n and r are large. This design is identical to the Minimax design mentioned in (1).

(e) In the special case where one knows precisely the plant parameters ($\underline{p}=\underline{p}_o$), then (Fig. 4e)

$$w_o(\underline{p}) = \delta(\underline{p}-\underline{p}_o) = \delta(p_1-p_{o_1})\,\delta(p_2-p_{o_2})\ldots\delta(p_r-p_{o_r}) \tag{2.25}$$

for all $\underline{p}$ in D where D is any closed bounded domain containing $\underline{p}_o$. This design is, of course, the conventional quadratic regulator design.

The probability density functions $w_o(\underline{p})$ discussed in cases (a) and (b) are nonzero over a finite domain of uncertainty. Thus the feedback control system developed according to (2.18) will be adequately designed to cope with a changing or unknown environment.

For systems where very large changes in the physical parameters are expected (e.g., autopilots), the usual procedure is to design programmed changes in control gains versus flight conditions (for example) as measured by simple sensors of dynamic pressure, Mach number and the like. The present method enhances such a procedure by providing reduced sensitivity of system performance to variations in the values of physical parameters from the nominal ones associated with each flight condition.

III. DESIGN PROCEDURES

This section will briefly summarize the above design technique and describe its implementation. The design method involves two basic computational processes: evaluation of $E[S(\underline{p},C)]$, and determination of a set of feedback gains, C, which minimizes the norm of $E[S(\underline{p},C)]$.

III.1 Evaluation of $E[S(\underline{p},C)]$: Since $E[S(\underline{p},C)]$ is a real symmetric positive semi-definite matrix, only $\frac{1}{2}n(n+1)$ of its elements need to be determined in order to completely specify $E[S(\underline{p},C)]$; that is, one has to evaluate the following $\frac{1}{2}n(n+1)$ integrals

$$E[S_{ij}(\underline{p},C)] = \int_D w_o(\underline{p})S_{ij}(\underline{p},C)d\underline{p} \quad \text{for } 1\leq i\leq j\leq n \tag{3.1.1}$$

where the $\frac{1}{2}n(n+1)$ elements of the matrix $S(\underline{p},C)$ are solutions to the Lyapunov matrix equation given in (2.10). To evaluate (3.1.1) exactly, the Lyapunov matrix equation must be solved for all values of $\underline{p}$ in D. This constitutes the major impediment to a practical implementation of this technique.

A simple zeroth-order numerical integration scheme applied to (3.1.1) illustrates the difficulty encountered in the implementation of this technique:

$$E[S_{ij}(\underline{p},C)]= \int_D w_o(\underline{p})S_{ij}(\underline{p},C)d\underline{p} \quad \text{for } 1\leq i\leq j\leq n$$

$$\approx \sum_{i_r=0}^{N_r-1} \sum_{i_{r-1}=0}^{N_{r-1}-1} \cdots \sum_{i_1=0}^{N_1-1} w_o(p_1'+(i_1+\tfrac{1}{2})h_1,\ldots, \ldots,p_r'+(i_r+\tfrac{1}{2})h_r)$$

$$\cdot S_{ij}(p_1'+(i_1+\tfrac{1}{2})h_1,p_2'+(i_2+\tfrac{1}{2})h_2,\ldots,p_r'+(i_r+\tfrac{1}{2})h_r,C)\cdot\Delta \tag{3.1.2}$$

where

$$\Delta = h_1h_2\ldots h_r$$

h_i = Integration step size of the i^{th} parameter and

$S_{ij}(p_1'+(i_1+\frac{1}{2})h_1, p_2'+(i_2+\frac{1}{2})h_2, \ldots, p_r'+(i_r+\frac{1}{2})h_r, C)$ satisfies the Lyapunov matrix equation (2.10) for

$\underline{p}^T = [p_1'+(i_1+\frac{1}{2})h_1, p_2'+(i_2+\frac{1}{2})h_2, \ldots, p_r'+(i_r+\frac{1}{2})h_r]$ with

$i_1=0,1,\ldots,N_1-1;\ i_2=0,1,\ldots,N_2-1;\ \ldots\ ;\ i_r=0,1,\ldots,N_r-1.$

As a result, to completely evaluate one integral for a given set of C, we need to solve (2.10) a total of $N_1N_2\ldots N_r$ times. In order to make truncation errors in the integration acceptably small, step size h_i $(i=1,2,\ldots,r)$ must be sufficiently small, i.e., N_i sufficiently large. Assuming that the number of operations required to solve (2.10) is proportional to n^3, then the entire integration cost amounts to approximately $kN_1N_2\ldots N_rn^3$ operations where k is a constant of proportionality.

Higher order numerical integration schemes may be used to evaluate $E[S(p,C)]$. The real advantage of using higher order quadrature formulas is that one may be able to reduce the number of integration steps N_i $(i=1,2,\ldots,r)$ while keeping truncation errors within reasonable bounds, consequently lowering the overall cost of the design synthesis. Further investigation into this problem of numerical integration is recommended for cases where r is large.

The norm of $E[S(\underline{p},C)]$ is found by solving for its largest eigenvalue, which may be efficiently determined using the numerical power method (28) as described in Appendix B.

III.2 Determination of the controller gain matrix C: The design technique outlined above requires the minimization of a cost functional $\hat{J}(\underline{x}_0,C)$ with respect to a set of parameters $C_{ij}(i=1,2,\ldots,m;$

$j=1,2,\ldots,n$). If the domain of uncertainty D is a small neighborhood about the nominal value $\underline{p}_o$, then the existence of a set of constant control gains C which stabilizes the closed loop system for all $\underline{p}$ in D may be asserted. But for arbitrarily large variations in the parameters, it is not clear whether such a set of gains C can always be found. Further research into sufficient conditions for the existence of C is needed in order to broaden the applicability of the sensitivity reduction technique.

Here we shall simply assume that a set of control gains C which minimizes the cost function $\hat{J}(\underline{x}_o,C)$ can be found. Various minimization techniques for nonlinear functions are available (24). Two approaches to finding a set of feedback gains, C, which minimizes $\sup\limits_{\|x_o\|=1} \hat{J}(\underline{x}_o,C)$ or equivalently the norm of $E[S(\underline{p},C)]$, are given. They are the hill descent method and the Fletcher-Powell method.

III.2.a <u>The hill descent, univariate or relaxation method</u> (24): The minimization procedure consists of the evaluation of the function to be minimized at a set of points selected according to a sequential procedure. These test points then provide information about the function and its minimum. The first test point is usually picked in an arbitrary manner. To minimize computing time, a good initial guess is highly desirable. The remaining test points are determined iteratively from the equation

$$\underline{C}^{\ell+1} = \underline{C}^{\ell} + h^{\ell}\underline{D}^{\ell} \qquad (3.2.1)$$

where $\underline{C}^{\ell}$ is an mn-dimensional vector whose components are the independent, unconstrained controller gains $C_{11},C_{12},\ldots,C_{1n},C_{21},C_{22},\ldots,C_{2n},\ldots,C_{m1},C_{m2},\ldots,C_{mn}$ at the ℓ^{th} iteration, h^{ℓ} is a positive constant representing the magnitude of the ℓ^{th} iteration step size and $\underline{D}^{\ell}$ is an mn-dimensional normalized direction vector. For convenience the initial step size is chosen to be unity.

In each iteration we employ a fixed sequence of normalized search direction vectors, in which we change only one coordinate at a time, that is

$$\underline{D}^{\ell} = \pm\underline{U}_j^{\ell} \qquad (3.2.2)$$

where

$$\underline{U}_j^{\ell^T} = [0\ 0\ 0\ \ldots 0\ 1\ 0\ \ldots 0] \qquad ,j=1,2,\ldots,mn$$

$$\qquad\qquad j^{th} \uparrow \text{column}$$

If in an iteration the function evaluated at all test points specified in (3.2.1) exceeds the value of the function found in the previous iteration, then the step size in the next iteration is reduced by half and the search procedure repeated (Fig. 5). The procedure is terminated when the step size h^{ℓ} gets below a certain prescribed margin. This method was used in the numerical solution of the second order examples discussed in Section V.

III.2.b <u>The Fletcher-Powell method</u> (25-27): This procedure allows the user to find the minimum of a function of several variables without calculating the derivatives. An approximate conjugate direction is developed at each iteration to ensure a fast convergence rate from a bad initial approximate to the minimum. For further details, the reader is referred to the cited references. This method shows great promise in providing an efficient computational algorithm for the solution of the feedback gains C.

IV. SIMPLIFICATION OF THE METHOD FOR A SECOND ORDER CONTROL SYSTEM

Consider a second order control system represented by the following set of linear differential equations with constant coefficients

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f_1 & f_2 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix} u(t) \tag{4.1}$$

where u(t) is a scalar control input. The variable parameter vector is $\underline{p}^T=(f_1,f_2)$. The domain of excursions is given by (Fig. 6)

$$D: \begin{cases} f_{1_{min}} \leq f_1 \leq f_{1_{max}} \\ f_{2_{min}} \leq f_2 \leq f_{2_{max}} \end{cases} \tag{4.2}$$

The output response consists of the entire state vector, i.e., $\underline{y}(t)=\underline{x}(t)$ or H=I where I is the identity matrix. We assume a linear controller of the form

$$u(t) = -[C_1x_1(t)+C_2x_2(t)] \tag{4.3}$$

where C_1 and C_2 are to be determined using the design method described in Section II. Let the weighting matrices A and B be of the following form

$$A= \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad \text{and } B=(b) \tag{4.4}$$

where the elements a_{11}, a_{12}, a_{22} and b may be chosen according to (2.6) and (2.7). In expanded form, the Lyapunov matrix equation (2.10) becomes a set of linear system equations whose unknowns are the elements of the symmetric matrix $S(\underline{p},C)$.

$$\begin{cases} 2(f_1-gC_1)\, S_{12} & =-(a_{11}+bC_1^{\,2}) \quad (4.5a) \\ S_{11} + (f_2-gC_2)\, S_{12} + (f_1-gC_1)\, S_{22} & =-(a_{12}+bC_1C_2) \quad (4.5b) \\ 2\, S_{12} + 2(f_2-gC_2)\, S_{22} & =-(a_{22}+bC_2^{\,2}) \quad (4.5c) \end{cases}$$

The simultaneous solution of (4.5) yields

$$\begin{cases} S_{11}=-(a_{12}+bC_1C_2)+\frac{f_1-gC_1}{2(f_2-gC_2)}(a_{22}+bC_2^{\,2}) \\ \qquad -\frac{a_{11}+bC_1^{\,2}}{2(f_2-gC_2)}+\frac{f_2-gC_2}{2(f_1-gC_1)}(a_{11}+bC_1^{\,2}) & (4.6a) \\ S_{12}=-\frac{a_{11}+bC_1^{\,2}}{2(f_1-gC_1)} & (4.6b) \\ S_{22}=-\frac{a_{22}+bC_2^{\,2}}{2(f_2-gC_2)}+\frac{a_{11}+bC_1^{\,2}}{2(f_1-gC_1)(f_2-gC_2)} & (4.6c) \end{cases}$$

For simplicity we assume a uniform probability density function $w_o(p)$ inside D. This is shown in Fig. 6. The magnitude of $w_o(\underline{p})$ is given by $\frac{1}{d_1d_2}$ where d_1 and d_2 are defined below. Then the integration in (3.1.1) can be evaluated explicitly. We obtain, upon integration of $S(\underline{p},C)$,

$$E(S_{11})=-(a_{12}+bC_1C_2)+\frac{a_{22}+bC_2^{\,2}}{4d_1d_2}(f_1-gC_1)^2\Big|_{f_{1_{min}}}^{f_{1_{max}}}\log(f_2-gC_2)\Big|_{f_{2_{min}}}^{f_{2_{max}}}$$

$$-\frac{a_{11}+bC_1^{\,2}}{2d_2}\log(f_2-gC_2)\Big|_{f_{2_{min}}}^{f_{2_{max}}} \qquad (4.7a)$$

$$+\frac{a_{11}+bC_1^{\,2}}{4d_1d_2}(f_2-gC_2)^2\Big|_{f_{2_{min}}}^{f_{2_{max}}}\log(f_1-gC_1)\Big|_{f_{1_{min}}}^{f_{1_{max}}}$$

$$E(S_{12}) = -\frac{a_{11}+bC_1^{\,2}}{2d_1}\log(f_1-gC_1)\Bigg|_{f_{1_{min}}}^{f_{1_{max}}} \tag{4.7b}$$

$$E(S_{22}) = -\frac{a_{22}+bC_2^{\,2}}{2d_2}\log(f_2-gC_2)\Bigg|_{f_{2_{min}}}^{f_{2_{max}}} + \frac{a_{11}+bC_1^{\,2}}{2d_1d_2}\log(f_1-gC_1)\Bigg|_{f_{1_{min}}}^{f_{1_{max}}}\log(f_2-gC_2)\Bigg|_{f_{2_{min}}}^{f_{2_{max}}} \tag{4.7c}$$

where $d_i=f_{i_{max}}-f_{i_{min}}$ (i=1,2).

The norm of $E[S(\underline{p},C)]$ is easily found by solving for the largest eigenvalue. It is given by the following expression

$$\|E[S(\underline{p},C)]\| = \lambda_{max}(E[S(\underline{p},C)]) = \frac{E(S_{11})+E(S_{22})+\sqrt{\Delta}}{2} \tag{4.8}$$

where $\Delta=[E(S_{11})-E(S_{22})]^2+4[E(S_{12})]^2$

Note that in this second order case it was possible to obtain analytical expressions for $S_{ij}(1\leq i\leq j\leq 2)$ as functions of the parameter vector $\underline{p}$ and of the control gains C_i. An analytical expression for $E[S(p,C)]$ was then obtained assuming a uniform distribution for $\underline{p}$.

For a given controller design (with $\underline{p}$ known) the performance cost given in (2.8) is bounded by the following two limits

$$\inf_{\|\underline{x}_0\|=1} J(\underline{p},\underline{x}_0,C)\leq J(\underline{p},\underline{x}_0,C)\leq \sup_{\|\underline{x}_0\|=1} J(\underline{p},\underline{x}_0,C) \tag{4.9}$$

or

$$\inf_{\|\underline{x}_0\|=1} \tfrac{1}{2}\underline{x}_0^{\,T}S(\underline{p},C)\underline{x}_0\leq J(\underline{p},\underline{x}_0,C)\leq \sup_{\|\underline{x}_0\|=1} \tfrac{1}{2}\underline{x}_0^{\,T}S(\underline{p},C)\underline{x}_0 \tag{4.10}$$

or $\frac{1}{2}\lambda_{min}(S(\underline{p},C)) \leq J(\underline{p},\underline{x}_o,C) \leq \frac{1}{2}\lambda_{max}(S(\underline{p},C))$ (4.11)

with normalized initial conditions $\underline{x}_o$, i.e., $\|\underline{x}_o\|=1$, where

$$\lambda_{min}(S(\underline{p},C)) = \frac{S_{11}+S_{22}-\sqrt{\Delta}}{2} \quad (4.12)$$

and

$$\lambda_{max}(S(\underline{p},C)) = \frac{S_{11}+S_{22}+\sqrt{\Delta}}{2} \quad (4.13)$$

with $\Delta = (S_{11}-S_{22})^2+4S_{12}{}^2$.

These bounds on the performance cost will be used in the subsequent comparative study of the optimal regulator design based on nominal parameter values, the present design, and the Minimax design, for second order examples.

V. NUMERICAL EXAMPLES

Consider the following two numerical examples of the second order control system described in the previous section.

V.1 <u>Case A:</u> In this case the open-loop plant is stable. The nominal parameter values are f_1=-2 and f_2=-1. The domain D of possible values of the parameters defined to be

$$D: \{(f_1,f_2)/-4 \le f_1 \le 0; -2 \le f_2 \le 0\} \qquad \text{and } g = 1.$$

The parameter values corresponding to the worst case, defined in (2.23), are found by evaluating the optimal cost function over a finite set of parameter values in D. The set of parameter values that satisfies approximately the Minimax condition stated in (2.23) is chosen to be the worst case for the closed loop system. We get roughly f_{1_w}=0 and f_{2_w}=0. For convenience the following weighting matrices A and B are used

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = (10), \text{ i.e., } a_{11}=a_{22}=1, a_{12}=0 \text{ and } b=10.$$

The open loop eigenvalues are located at $\lambda_{1,2}$=-0.5±j1.213. The linear controller gains C corresponding to the optimal regulator design based upon nominal parameter values, the present design and the Minimax design, and their associated closed loop eigenvalues are tabulated in Table 1. An increase in the gain constants is seen to accompany the reduction in performance sensitivity. Figure 7 shows a root locus mapping of the region D. Both the optimal regulator design based upon nominal parameter values and the present design lead to a stable closed loop system for all parameter variations in D. The stability boundaries for each controller design are shown in Fig. 8. The time responses of the states and of the control effort to the initial condition $\underline{x}_0^T$=(1,0) are shown in Fig. 9 for the nominal plant parameter values and in Fig. 10 for an off-nominal plant parameter value, for the above three controller designs. A large expenditure in the feedback control effort u(t) is observed in the Minimax design; thus this design tends to be more sensitive to measurement noise than the present sensitivity reduction design. Table 2 illustrates the balance achieved between the performance cost and its sensitivity in the three controller designs.

V.2 Case B: In this case the open loop plant is unstable. The nominal parameter values are $f_1=-2$ and $f_2=1$. The domain D of possible values of the parameters is defined to be

$$D: \{(f_1,f_2)/\ -3 \le f_1 \le -1;\ 0. \le f_2 \le 2.5\} \qquad \text{and } g = 1.$$

Estimates of the worst case parameter values are found to be $f_{1_w}=-3$ and $f_{2_w}=2.5$. For convenience, the weighting matrices given in case A are used. The open loop eigenvalues of the system are located at $\lambda_{1,2}=+0.5\pm j1.323$. The linear controller gains corresponding to the above three designs and their associated closed loop eigenvalues are tabulated in Table 3. Figure 11 shows a root locus mapping of the region D. Clearly the optimal regulator design based upon nominal parameter values is incapable of handling large parameter variations: The resulting closed loop system becomes unstable for some parameter values in D. On the other hand, the present controller design stabilizes the closed loop system for all parameter values in D. The stability boundaries for each controller design are shown in Fig. 12. The time responses of the states and the control effort to the initial condition $\underline{x}_0^T=(1,0)$ using the above three controller designs are shown in Fig. 13 for the nominal plant parameter values and in Fig. 14 for some typical off-nominal plant parameter values (not the worst case values). Table 4 illustrates the balance achieved between the performance cost and its sensitivity in the three designs. An infinite cost value for the performance index represents instability in the feedback-controlled system.

Table 1.

Controller Gains Chosen for Case A by the Three Design Methods

Linear Controller Gains	Optimal Design for $F=F_o$	New Design	Minimax Design
C_1	0.025	0.121	0.316
C_2	0.072	0.210	0.886
Closed loop eigen-values for $F=F_o$	-0.54±j1.32	-0.60±j1.32	-0.93±j1.21

Table 2.

The Cost J Associated with Control Designed for Case A by the Three Design Methods

	Range of J for the Nominal Parameter Values		Range of J for the Worst Case Parameter Values	
Design	$\min_{\Vert \underline{x}_o \Vert =1} J(\underline{p}_o,\underline{x}_o,C)$	$\max_{\Vert \underline{x}_o \Vert =1} J(\underline{p}_o,\underline{x}_o,C)$	$\min_{\Vert \underline{x}_o \Vert =1} J(\underline{p}_w,\underline{x}_o,C)$	$\max_{\Vert \underline{x}_o \Vert =1} J(\underline{p}_w,\underline{x}_o,C)$
Optimal Design for $F=F_o$	0.332	0.885	3.564	144.5
New Design	0.375	0.940	1.455	13.48
Minimax Design	1.175	1.978	0.662	4.971

Table 3.

Controller Gains Chosen for Case B by the Three Design Methods

Linear Controller Gains	Optimal Design for $F=F_o$	New Design	Minimax Design
C_1	0.025	0.592	0.017
C_2	2.072	3.937	5.026
Closed loop eigenvalues for $F=F_o$	$-0.54 \pm j1.32$	$-1.47 \pm j0.66$	$-2.01 \pm j1.23$

Table 4.

The Cost J Associated with Control Designed for Case B by the Three Design Methods

	Range of J for the Nominal Parameter Values		Range of J for the Worst Case Parameter Values	
Design	$\min_{\|\underline{x}_o\|=1} J(\underline{p}_o,\underline{x}_o,C)$	$\max_{\|\underline{x}_o\|=1} J(\underline{p}_o,\underline{x}_o,C)$	$\min_{\|\underline{x}_o\|=1} J(\underline{p}_w,\underline{x}_o,C)$	$\max_{\|\underline{x}_o\|=1} J(\underline{p}_w,\underline{x}_o,C)$
Optimal Design for $F=F_o$	10.36	20.86	∞	∞
New Design	13.41	24.44	27.36	87.05
Minimax Design	15.78	31.90	25.13	75.60

VI. CONCLUSION

A minimum sensitivity control system design has been developed. It provides the control designer a powerful but simple tool in the synthesis of control systems that will be insensitive to large parameter variations. In contrast to existing sensitivity reduction techniques which are valid only to a first order approximation, the present controller design ensures stability of the controlled system over the entire design range of possible parameter values, provided only that such a control exists.

The new method has been shown to produce control systems having better performance over the possible range of plant parameter values than would an optimal controller based on the nominal plant parameters. In particular, cases have been demonstrated where the latter system becomes unstable for some possible values of the plant parameters, while the new design provides good stable performance for all values.

There has been shown also a better trade-off in the system performance and its sensitivity with the new design than with the Minimax design. In the latter, significant loss in the overall system performance occurs due to unnecessarily large control feedback gains. However, further development of the computational schemes used is still needed to make this new sensitivity reduction technique into a practical and useful design tool for higher order systems.

VII. RECOMMENDATIONS FOR FURTHER STUDY

The following are some recommendations for possible extensions and applications of the new design concept:

1. Develop an efficient computer program for the synthesis of the above minimum sensitivity design for higher order systems with several uncertain parameters. The program should include efficient numerical algorithms for the following operations:
   - Solution of the Lyapunov matrix equation (2.10)
   - Numerical integration of $E[S(\underline{p},C)]$ in (3.1.1)
   - Determination of the maximum eigenvalue of $E[S(\underline{p},C)]$
   - Determination of a set of feedback gains, C, which minimize $\lambda_{max}(E[S(\underline{p},C)])$.
2. Develop a similar insensitive design in which state estimates are used instead of the actual states in the implementation of the controller. Investigate the dynamic coupling thus developed between the controller and the estimator.
3. Based on the above concept of reduced sensitivity design, develop a similar technique for a discrete-time feedback control system. In this case the state equation (2.1) is a difference equation and the feedback control law is now a linear function of the full state evaluated at the sampling times. Extend the results to include a discrete-time state estimator. Investigate the dynamic coupling thus developed between the digital controller and the discrete-time state estimator.
4. Investigate the possible use of this design concept in the selection of sampling rate for a digital control system.

RECOMMENDATIONS FOR FURTHER STUDY (Continued)

5. Investigate the effect of external disturbances or inputs upon this minimum sensitivity design.
6. Extend this design concept to reference-following systems where the locations of both the poles and the zeros of the closed-loop system are of interest.

APPENDIX A.

Solution of the Lyapunov Matrix Equation

The well known Lyapunov matrix equation plays an important role in the stability theory of linear dynamical systems. For continuous, time invariant linear feedback control systems,

$$\dot{\underline{x}}(t) = (F-GC)\underline{x}(t) \qquad (A.1)$$

The Lyapunov matrix equation derived in section II has the form

$$-(F-GC)^T S - S(F-GC) = H^T AH + C^T BC \qquad (A.2)$$

where (F-GC) is an nxn stability matrix, i.e., its eigenvalues lie in the open left half of the complex plane, S and $(H^T AH+C^T BC)$ are nxn symmetric positive semi-definite matrices. The existence and uniqueness of S are shown in (20). The solution of (A.2) can be written as

$$S = QP^{-1} \qquad (A.3)$$

where P and Q are nxn matrices such that the columns of

$$\begin{bmatrix} P \\ Q \end{bmatrix},$$

a 2nxn matrix, form the n eigenvectors of the 2nx2n associated matrix

$$R = \begin{bmatrix} -(F-GC) & 0 \\ H^T AH+C^T BC & (F-GC)^T \end{bmatrix} \qquad (A.4)$$

corresponding to those eigenvalues of R with positive real parts (19,20). In general, the eigenvalues and eigenvectors of R are complex. If $\underline{v}$ is a complex eigenvector corresponding to the complex eigenvalue λ, then $\overline{\underline{v}}$, the complex conjugate of $\underline{v}$, is the eigenvector corresponding to $\overline{\lambda}$, the complex conjugate of λ, since the matrix R is real. To avoid the use of complex arithmetic, we note that the solution S given in (A.3) remains invariant under any nonsingular linear transformation of

$$\begin{bmatrix} P \\ Q \end{bmatrix} \text{; that is, if } \begin{bmatrix} P^* \\ Q^* \end{bmatrix} = \begin{bmatrix} P \\ Q \end{bmatrix} K \tag{A.5}$$

where K is an nxn nonsingular matrix, then S*=S, since

$$S^* = Q^* \, P^{*-1} = (QK)(PK)^{-1} = QKK^{-1}P^{-1} = QP^{-1} = S \tag{A.6}$$

Let us choose the matrix K to be of the following form

$$K = \begin{bmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & \cdot \end{bmatrix} \tag{A.7}$$

where K_i is a scalar or a 2x2 matrix block depending on whether the eigenvectors in the corresponding columns of $\begin{bmatrix} P \\ Q \end{bmatrix}$ are real or complex conjugate pairs. More precisely,

$$\begin{cases} K_i = (1) & \text{if } \underline{v}_i \text{ is real} \\ K_i = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}j \\ \frac{1}{2} & -\frac{1}{2}j \end{bmatrix} & \text{if } \underline{v}_i \text{ and } \underline{v}_{i+1} \ (=\overline{\underline{v}}_i) \text{ are a pair of complex conjugate eigenvectors} \end{cases}$$

where $j = \sqrt{-1}$.

For complex eigenvectors, $\underline{v}_i = \underline{\xi}_i + j\underline{\eta}_i$, $\underline{v}_{i+1} = \overline{\underline{v}}_i$, and

$$\begin{bmatrix} P^* \\ Q^* \end{bmatrix} = [\underline{\xi}_1 \underline{\eta}_1 \ \ldots \underline{\xi}_i \underline{\eta}_i \ \ldots \ \ldots] \tag{A.8}$$

Therefore it is valid to use the real and imaginary parts of the complex eigenvectors as columns of the matrix $\begin{bmatrix} P \\ Q \end{bmatrix}$.

The symmetric matrix S is then solved as the solution of the following linear equation

$$SP = Q \tag{A.9}$$

or $$P^T S = Q^T \quad . \tag{A.10}$$

APPENDIX B

The Power Method

The power method is an efficient numerical procedure to determine the largest eigenvalue of an nxn symmetric positive semi-definite matrix A (28). Let $\lambda_i (i=1,2,\ldots,n)$ be the eigenvalues of A such that

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \tag{B.1}$$

and let $\underline{x}_i$ be the corresponding eigenvectors, i.e., $A\underline{x}_i = \lambda_i \underline{x}_i$ for $i=1,2,\ldots,n$. We note that the set of eigenvectors $\{\underline{x}_i, i=1,n\}$ spans the entire n-dimensional vector space. Let $\underline{v}_o$ be an arbitrary vector which can be written as a linear combination of the eigenvectors $\underline{x}_i$, such that

$$\underline{v}_o = \sum_{i=1}^{n} \alpha_i \underline{x}_i \tag{B.2}$$

where α_i are scalar constants and α_n is nonzero.

It can be shown by induction that upon repeated multiplication of the vector $\underline{v}_o$ by the matrix A, we get

$$\begin{aligned}
\underline{v}_1 = A\underline{v}_o &= \sum_{i=1}^{n} \alpha_i \lambda_i \underline{x}_i \\
\underline{v}_2 = A\underline{v}_1 &= \sum_{i=1}^{n} \alpha_i \lambda_i^2 \underline{x}_i \\
&\ \ \vdots \\
\underline{v}_k = A\underline{v}_{k-1} &= \sum_{i=1}^{n} \alpha_i \lambda_i^k \underline{x}_i \sim \alpha_n \lambda_n^k \underline{x}_n \\
\underline{v}_{k+1} = A\underline{v}_k &= \sum_{i=1}^{n} \alpha_i \lambda_i^{k+1} \underline{x}_i \sim \alpha_n \lambda_n^{k+1} \underline{x}_n
\end{aligned} \tag{B.3}$$

since $\lambda_n \geq \lambda_i (i=1,2,...,n)$, and k is sufficiently large. At the $(k+1)^{st}$ iteration, the approximate value of λ_n is given by

$$\lambda_n \sim \lambda_n(k+1) = \frac{i^{th} \text{ component of } \underline{v}_{k+1}}{i^{th} \text{ component of } \underline{v}_k} \quad \text{for some } i.$$

The iterative scheme is terminated when the absolute difference between successive approximations of λ_n is less than some prescribed value δ, i.e., $|\lambda_n(k+1)-\lambda_n(k)|<\delta$. To avoid exceedingly large numbers in the components of $\underline{v}_k$, the vector $\underline{v}_k$ should be renormalized after a few iterations. The procedure is repeated with the normalized vector $\underline{v}_k$.

In our case we apply the power method to the solution of the largest eigenvalue of $E[S(\underline{p},C)]$ given in (3.1.1).

REFERENCES

1. Harvey, C.S. and R.E. Pope,"Study of Synthesis Techniques for Insensitive Aircraft Control Systems," NASA Report CR-2803, April 1977.

2. Newmann, M.M., "On Attempts to Reduce the Sensitivity of the Optimal Linear Regulator to a Parameter Change," Int. J. of Control, vol. 11, no. 6, 1970, pp. 1079-1084.

3. Kreindler, E., "Closed Loop Sensitivity Reduction of Linear Optimal Control Systems," IEEE Trans. on Automatic Control, vol. AC-13, no. 3, June 1968.

4. Kreindler, E., "On Minimization of Trajectory Sensitivity," Int. J. of Control, vol. 8, July 1968, pp. 89-96.

5. Sarman, V.V.S. and E.L. Deekshatulu, "Sensitivity Design of Optimal Linear Systems," Int. J. of Control, 1968, pp. 653-658.

6. Kreindler, E., "On the Definition and Application of the Sensitivity Function," J. of Franklin Institute, vol. 285, no. 1, January 1968.

7. Luh, J.Y.S. and E.R. Cross, "Optimal Controller Design for Minimum Trajectory Sensitivity," Int. J. of Control, 1968, pp. 557-558.

8. Byrne, P.C. and M. Burke, "Optimization with Trajectory Sensitivity Considerations," IEEE Trans. on Automatic Control, vol. AC-21, April 1976, pp. 282-283.

9. Hadass, Z., "Design Method for Minimizing Sensitivity in Plant Parameter Variations," AIAA Journal, vol. 13, no. 10, 1975.

10. Hadass, Z., "Design of State Feedback Controllers Including Sensitivity Reduction with Applications to Precision Pointing," SUDAAR no. 482, Stanford University, August 1974.

11. Bryson, A.E. and Y.C. Ho, "Applied Optimal Control: Optimization, Estimation and Control," Blaisdell Publishing Company, Waltham, Massachusetts, 1969.

12. Hsu, J.C., "Modern Control Principles and Applications," McGraw Hill, Inc., 1970.

13. Anderson, B.D.O. and J.B. Moore, "Linear Optimal Control," Prentice Hall, Englewood Cliffs, New Jersey, 1971.

14. Kwakernaak, H. and R. Sivan, "Linear Optimal Control Systems," Wiley Interscience, New York, 1972.

15. Woerkom, P.Th.L.M. Van, "Survey of Modern Control and Observation Theory for Disturbed, Target-Following Aerospace Systems. Part One: Basic Theory," NLR TR 76041 U, March 1976.

16. Greensite, A., "Elements of Modern Control Theory," Spartan Books, 1970.

17. Wiberg, D.M., "State Space and Linear Systems," Schaums Outline Series, McGraw Hill Book Company, 1971.

18. Chen, C.T., "Introduction to Linear System Theory," Holt, Rinehart, and Winston, Inc., 1970.

19. Bryson, A.E. and W.E. Hall, "Optimal Control and Filter Synthesis by Eigenvector Decomposition," SUDAAR no. 436, Stanford University, November 1971.

20. Bar-Ness, Y. and G. Langholz, "The Solution of the Matrix Equation XC-BX=D as an Eigenvalue Problem," Int. J. Systems Science, vol. 8, no. 4, 1977, pp. 385-392.

21. Barnett, S. and C. Storey, "Matrix Methods in Stability Theory," Thomas Nelson and Sons Ltd., 1970.

22. Jameson, A., "Optimization of Linear Systems of Constrained Configuration," Int. J. of Control, vol. 11, no. 3, 1970, pp. 409-421.

23. Salmon, D.M., "Minimax Controller Design," IEEE Trans. on Automatic Control," vol. AC-13, no. 4, August 1968.

24. Spang, H.A.III, "A Review of Minimization Techniques for Nonlinear Functions," SIAM Review, vol. 4, no. 4, October 1962.

25. Fletcher, R. and C.M. Reeves, "Function Minimization by Conjugate Gradients," Computer Journal, vol. 7, 1964, pp. 149-154.

26. Powell, M.J.D., "An Efficient Method for Finding the Minimum of a Function of Several Variables without Calculating Derivatives," Computer Journal, vol. 7, 1964, pp. 155-162.

27. Fletcher, R. and M.J.D. Powell, "A Rapidly Convergent Descent Method for Minimization," Computer Journal, vol. 6, 1963, p. 163.

28. Householder, A.S., "The Theory of Matrices in Numerical Analysis," Dover Publications, Inc., New York, 1964.

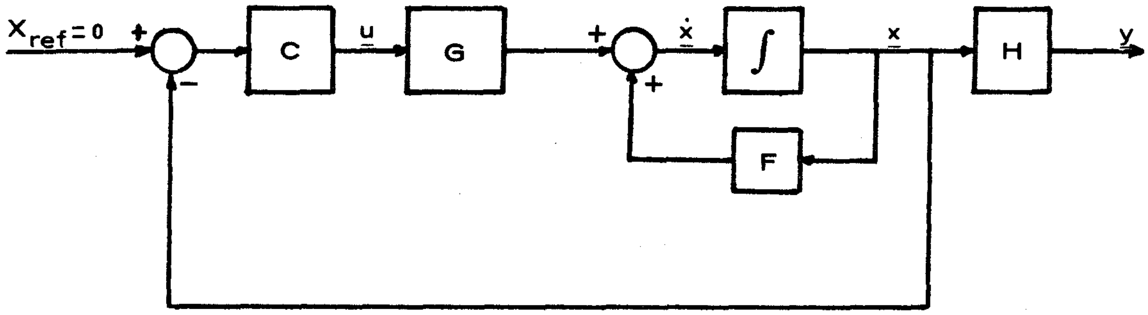


Fig. 1 Schematic diagram of the feedback control system.

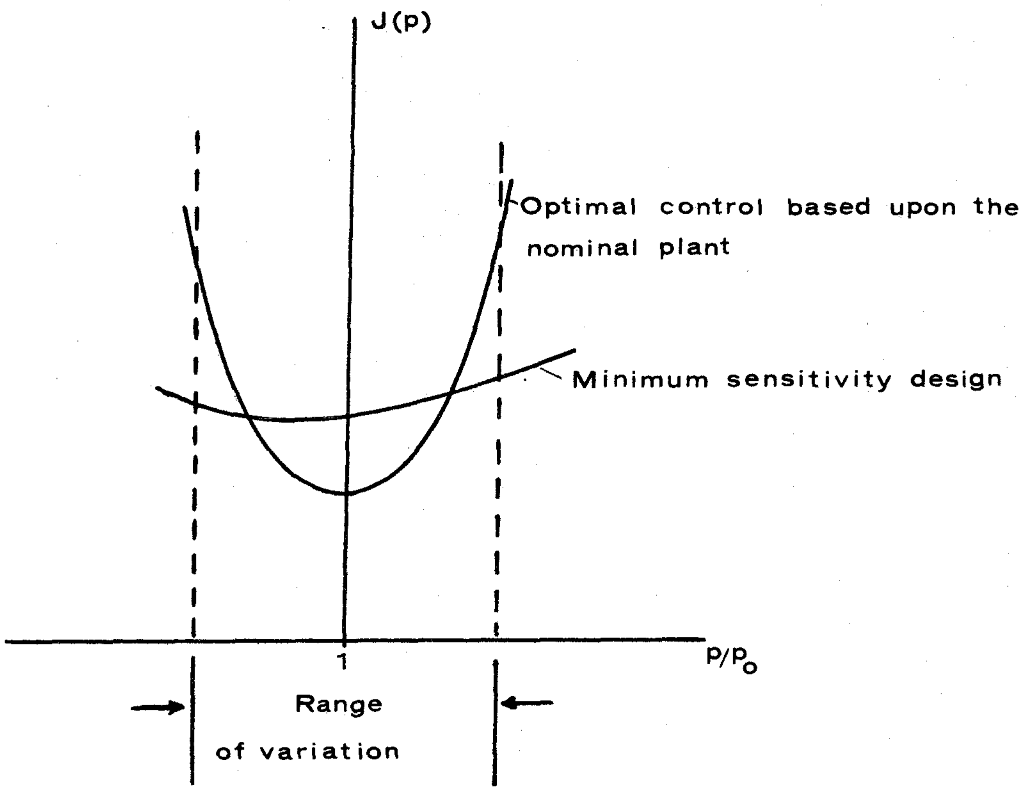


Fig. 2 Trade-off between measure of performance and sensitivity.

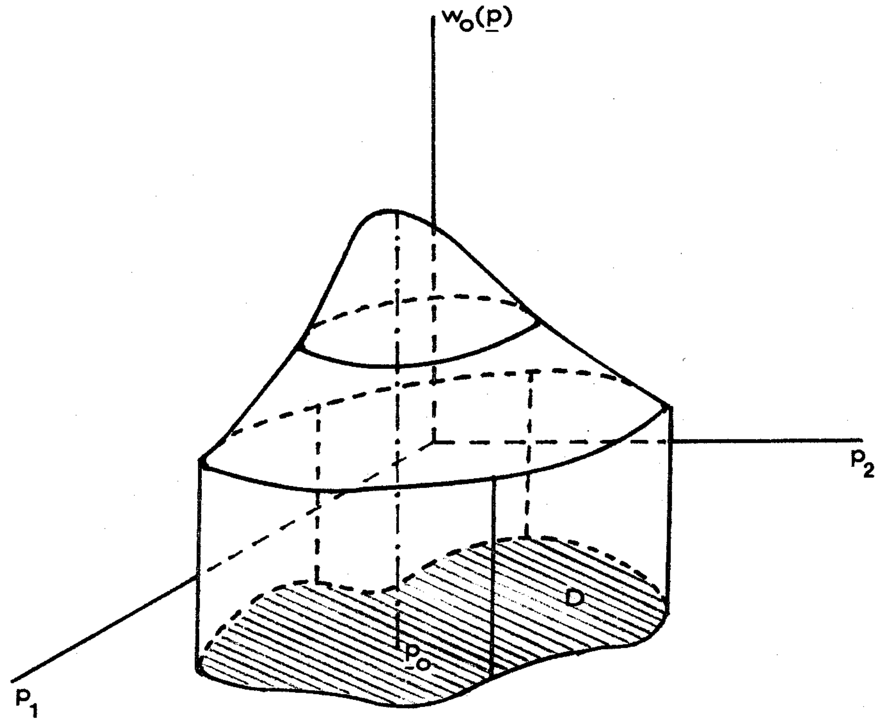


Fig. 3 Probability density function $w_o(p)$ for r=2.

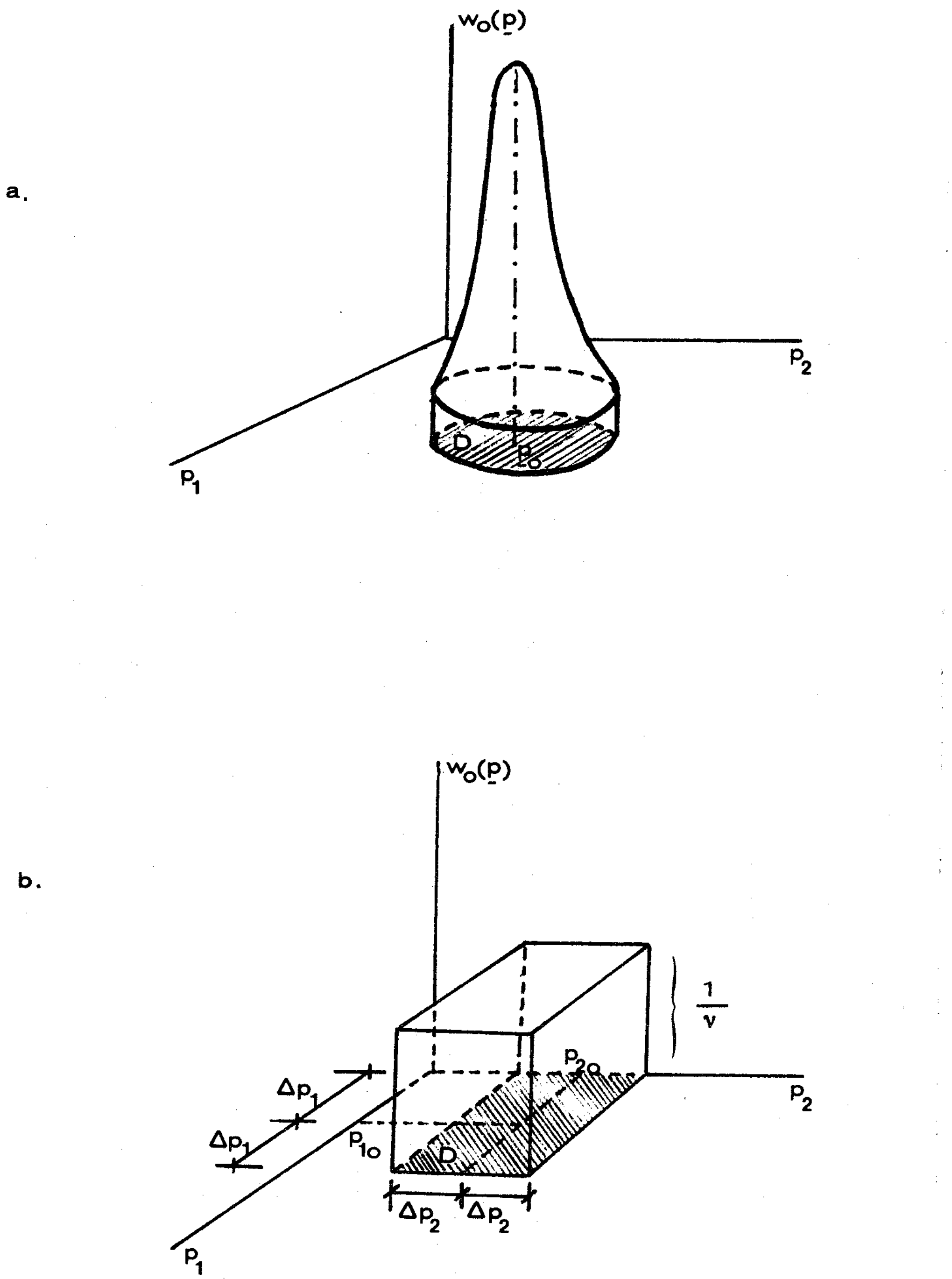
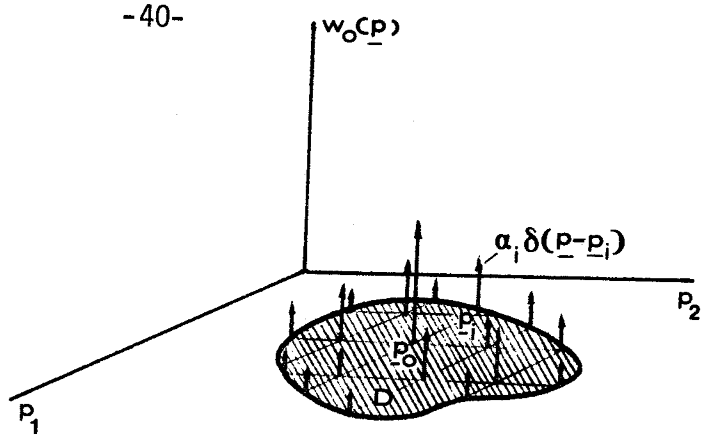
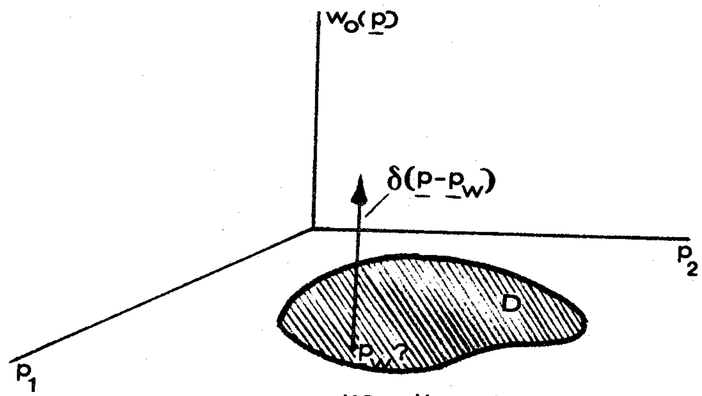


Fig. 4 Different types of domain D and distribution function $w_0(p)$.

c.



d.



e.

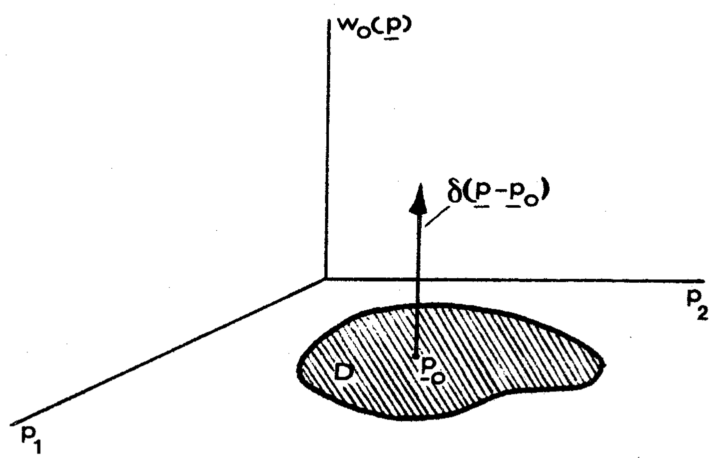


Fig. 4 (Continued)

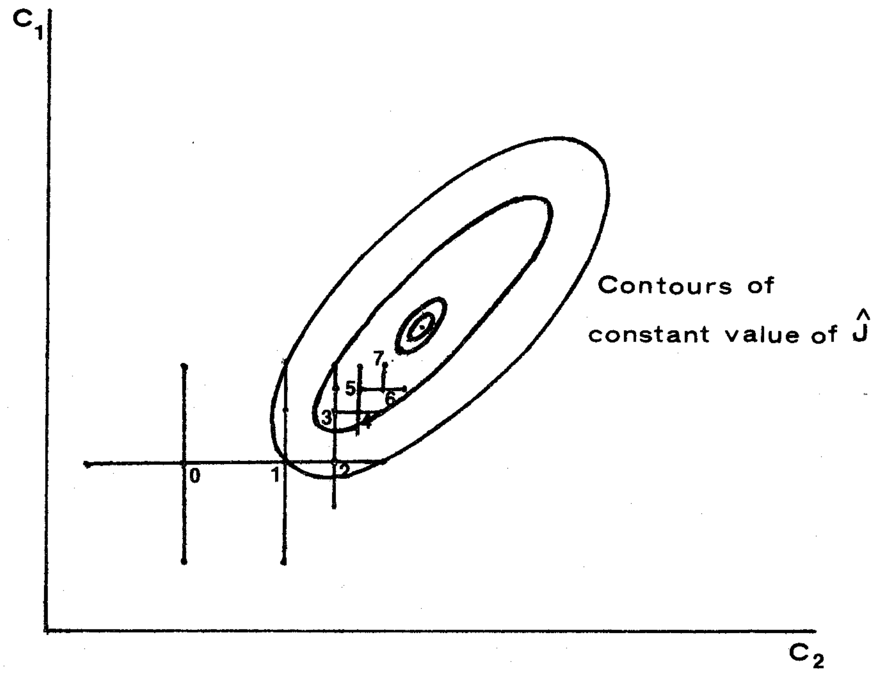


Fig. 5 A hill descent or univariate search.

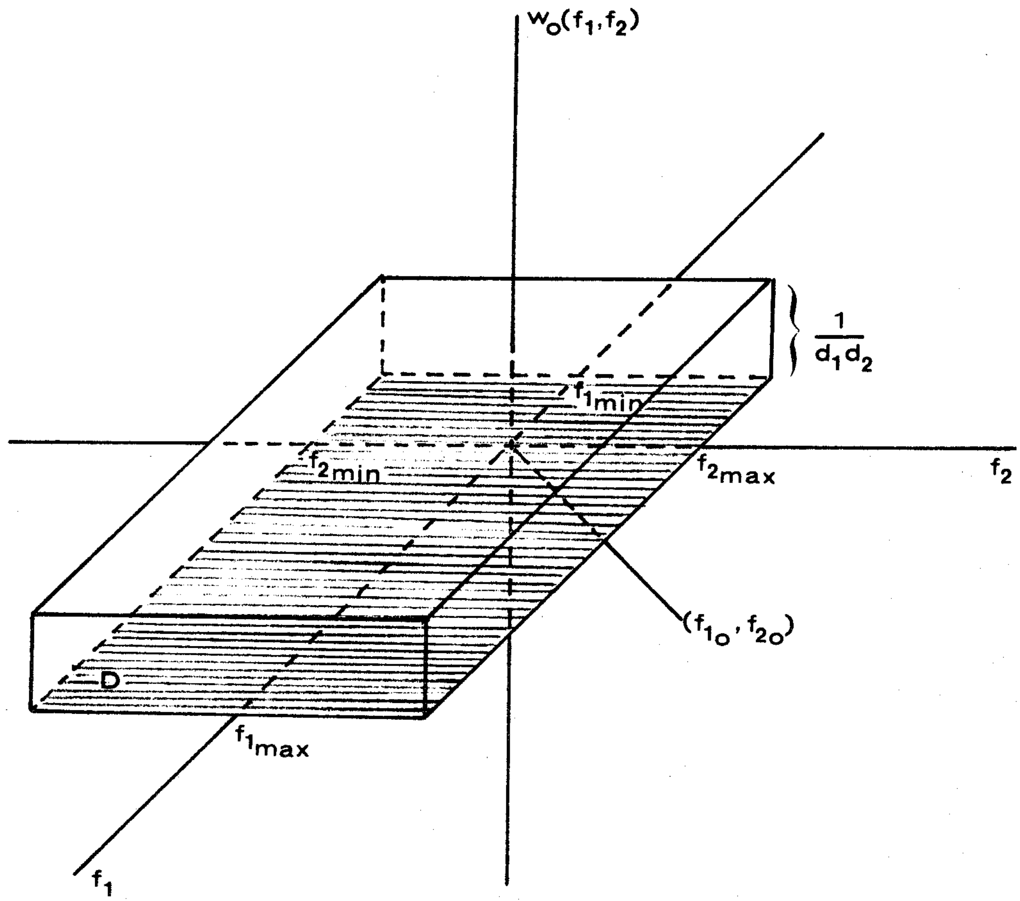


Fig. 6 Weighting distribution function $w_o(p)$ for the second order examples.

——— Optimal design based upon the nominal parameter values

- - - - - New design

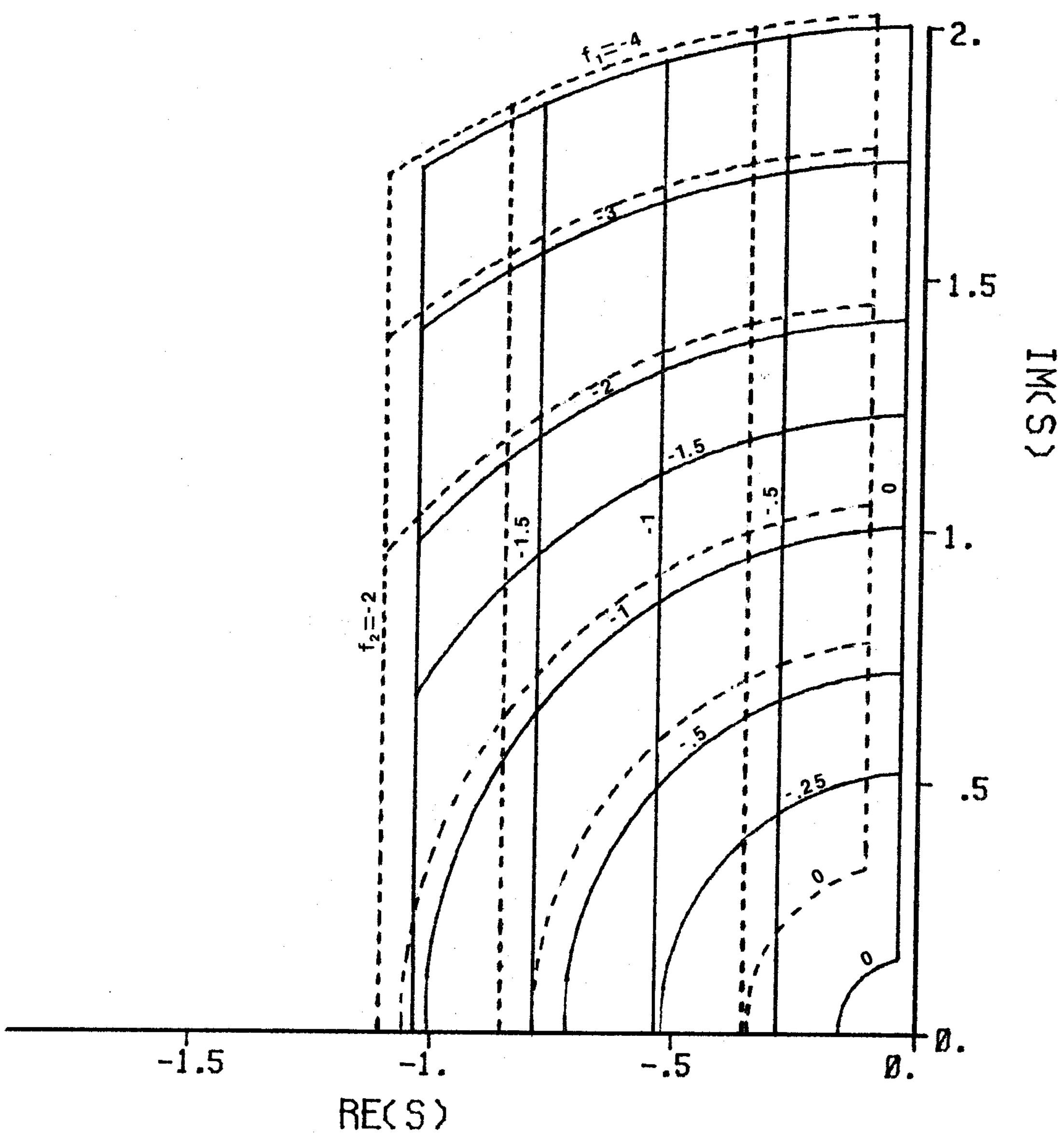


Fig. 7 Root locus mapping of the region D for Case A.

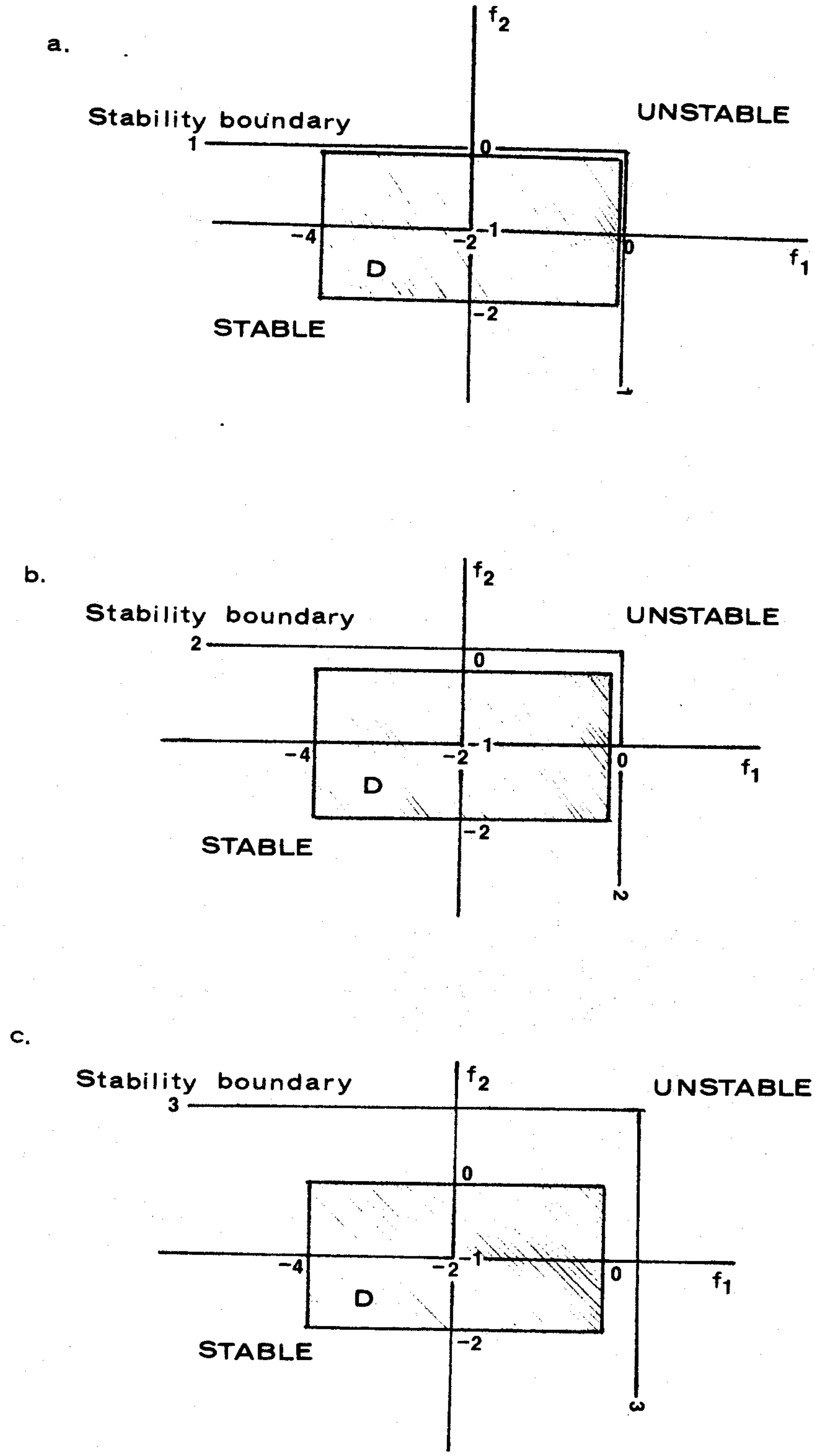


Fig. 8 Stability boundaries of the three closed loop designs in Case A. (a) Optimal design based upon the nominal parameter values, (b) New design, (c) Minimax design.

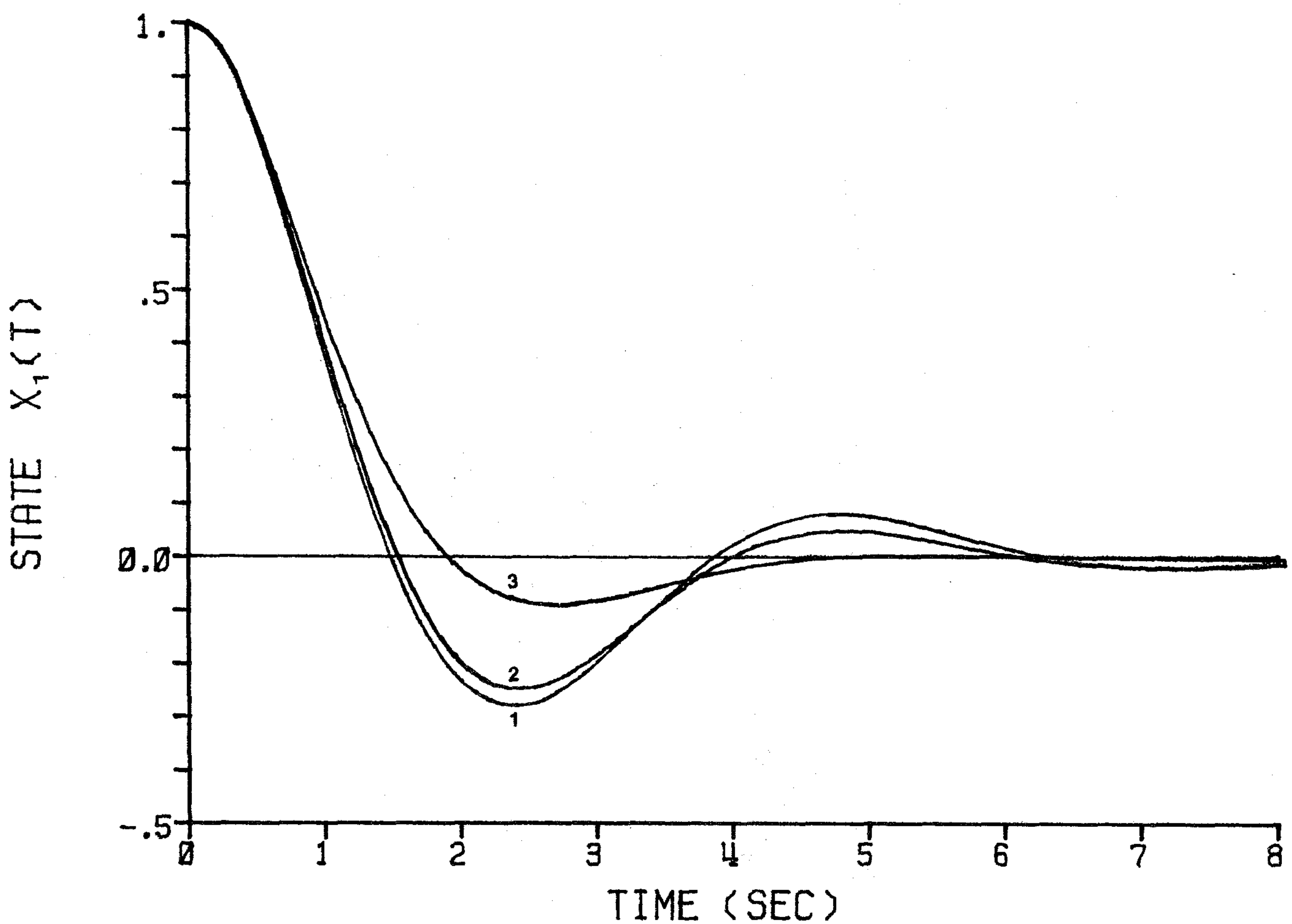


Fig. 9a Time response of the state variable $x_1(t)$ to the initial condition $x_0^T=(1,0)$ for the nominal plant in Case A. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

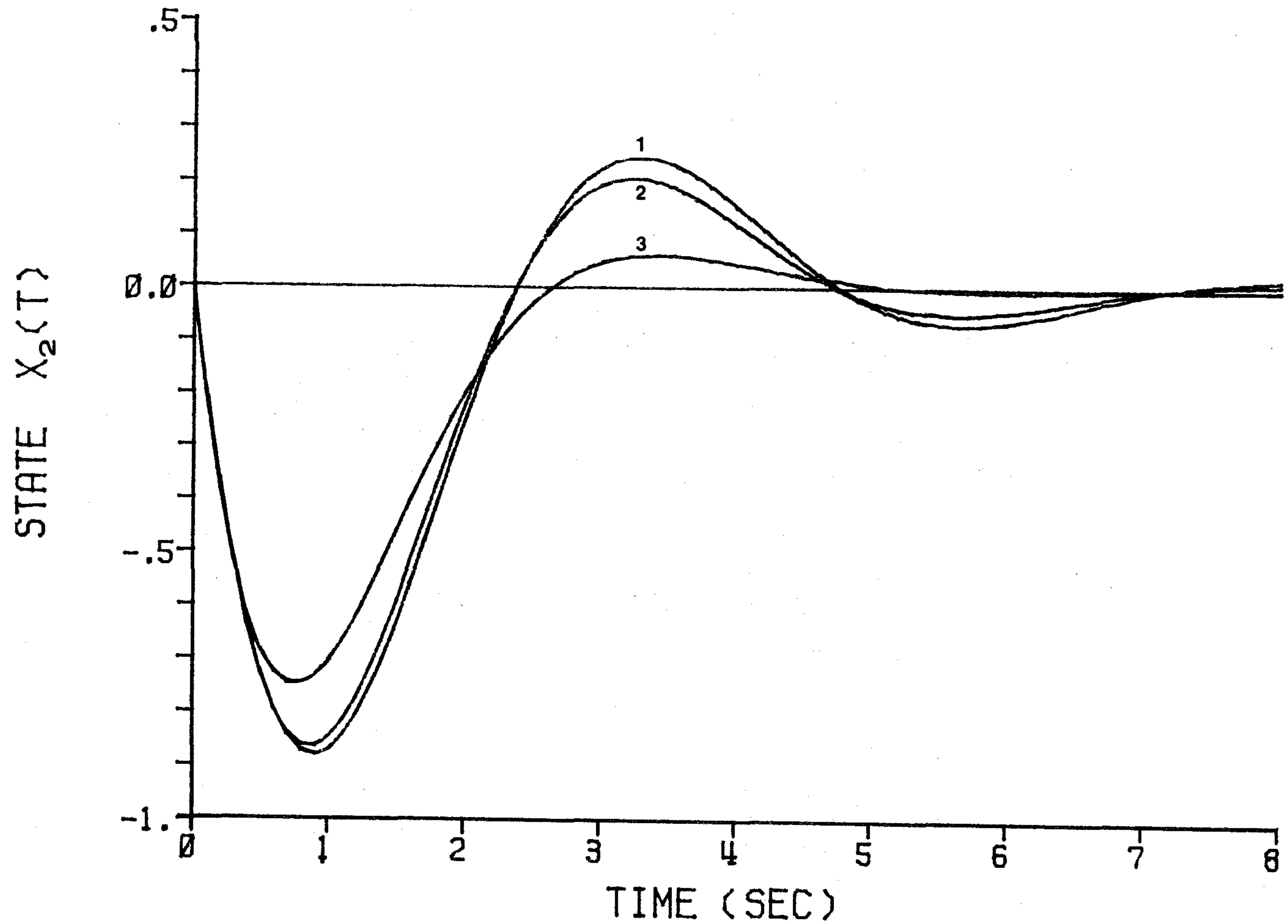


Fig. 9b Time response of the state variable $x_2(t)$ to the initial condition $x_o^T=(1,0)$ for the nominal plant in Case A. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

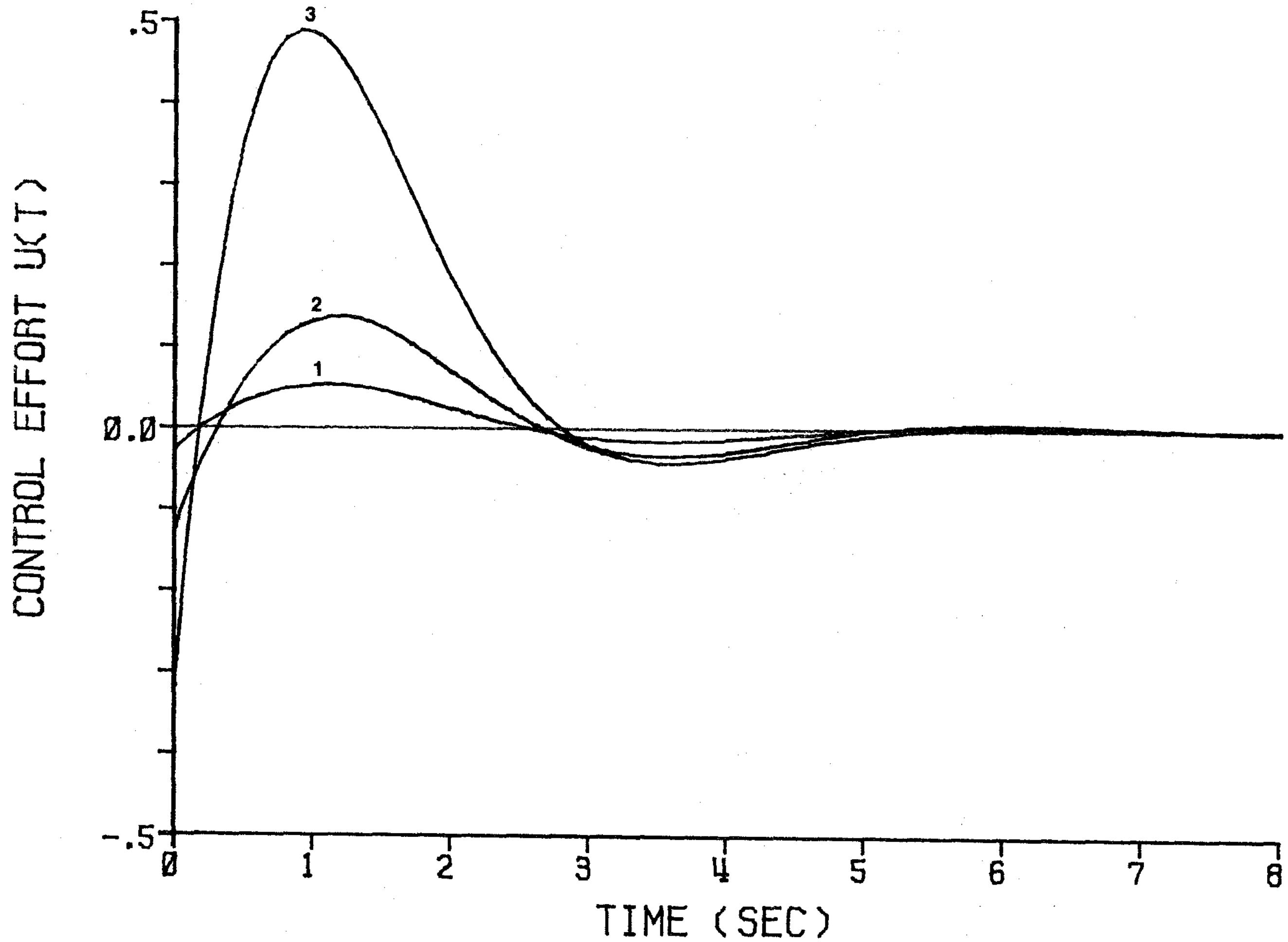


Fig. 9c Time response of the control effort u(t) to the initial condition $x_0^T=(1,0)$ for the nominal plant in Case A. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

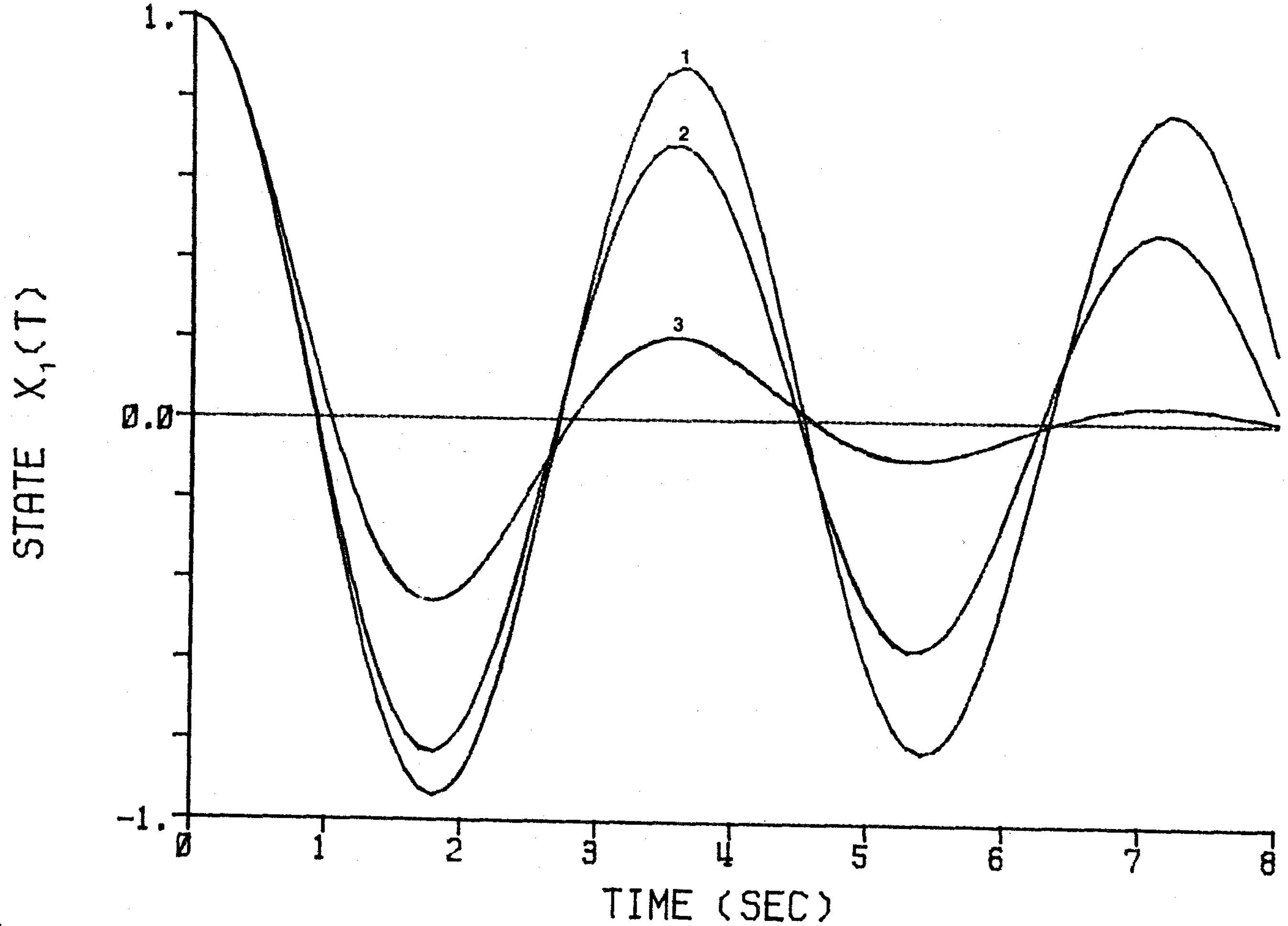


Fig. 10a Time response of the state variable $x_1(t)$ to the initial condition $x_o^T=(1,0)$ for an off-nominal plant $(f_1,f_2)=(-3,0)$ in Case A. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New Design, (3) Minimax Design.

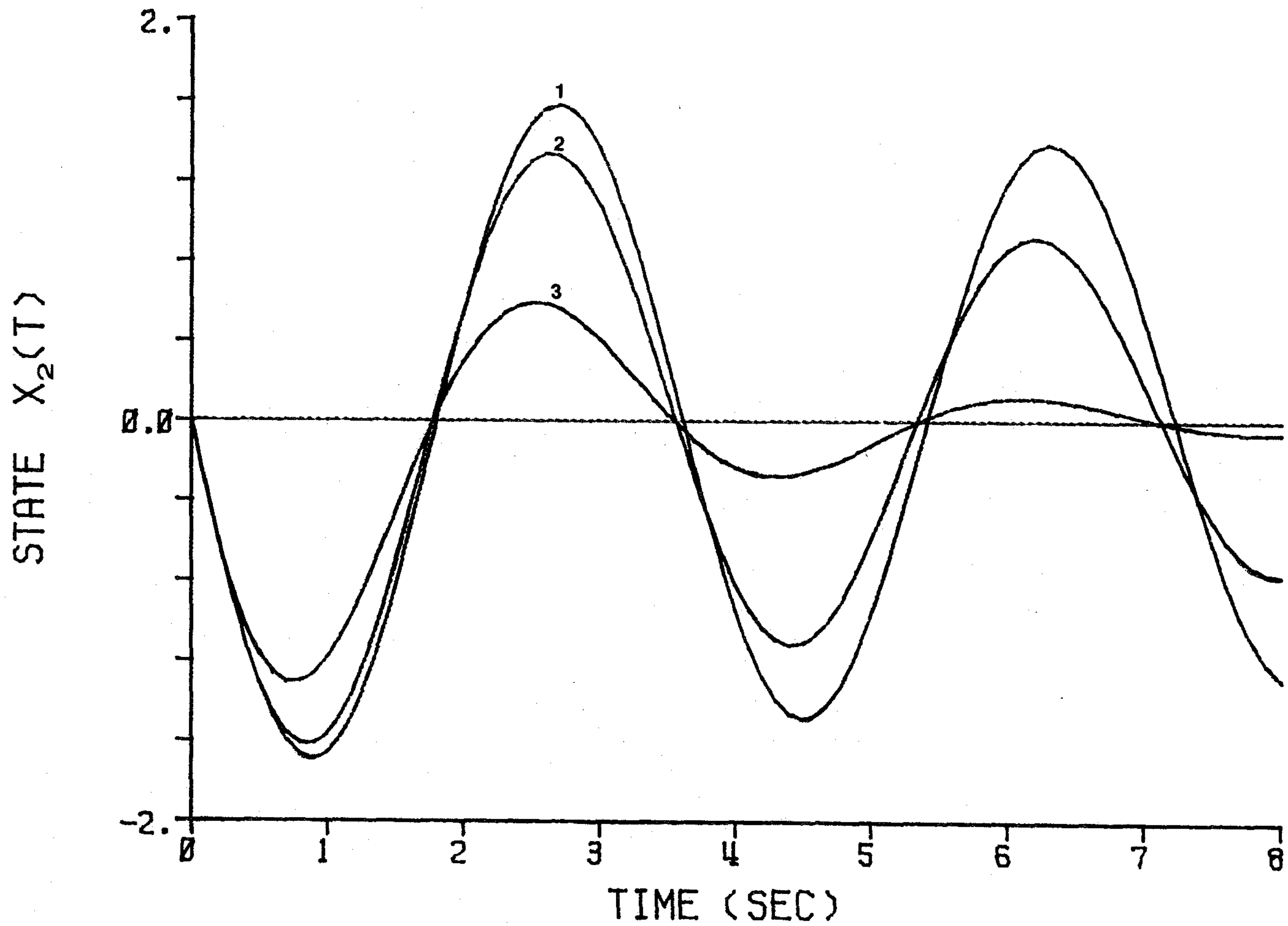


Fig. 10b Time response of the state variable $x_2(t)$ to the initial condition $x_0^T=(1,0)$ for an off-nominal plant $(f_1,f_2)=(-3,0)$ in Case A. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

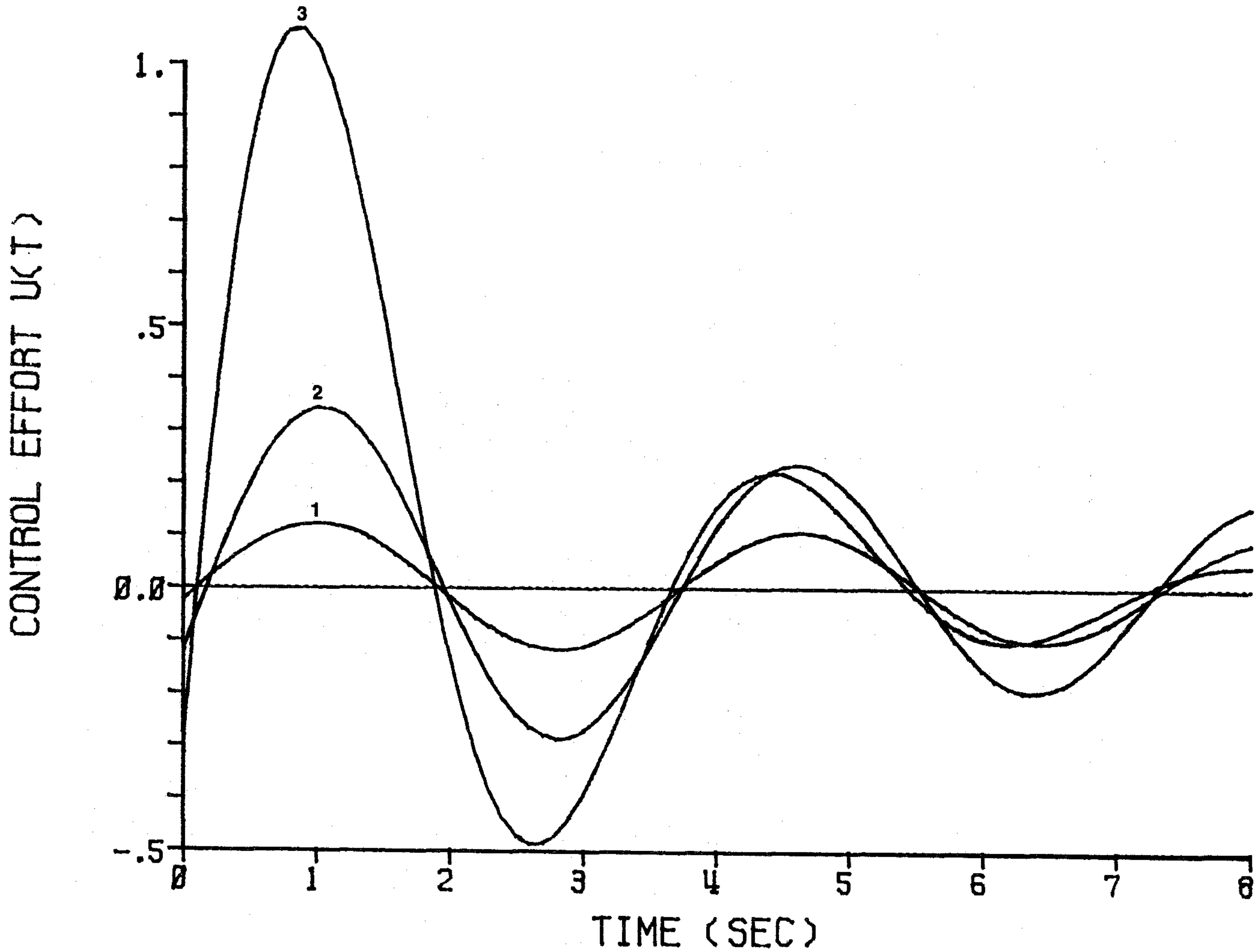


Fig. 10c Time response of the control effort u(t) to the initial condition $x_0^T=(1,0)$ for an off-nominal plant $(f_1,f_2)=(-3,0)$ in Case A. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

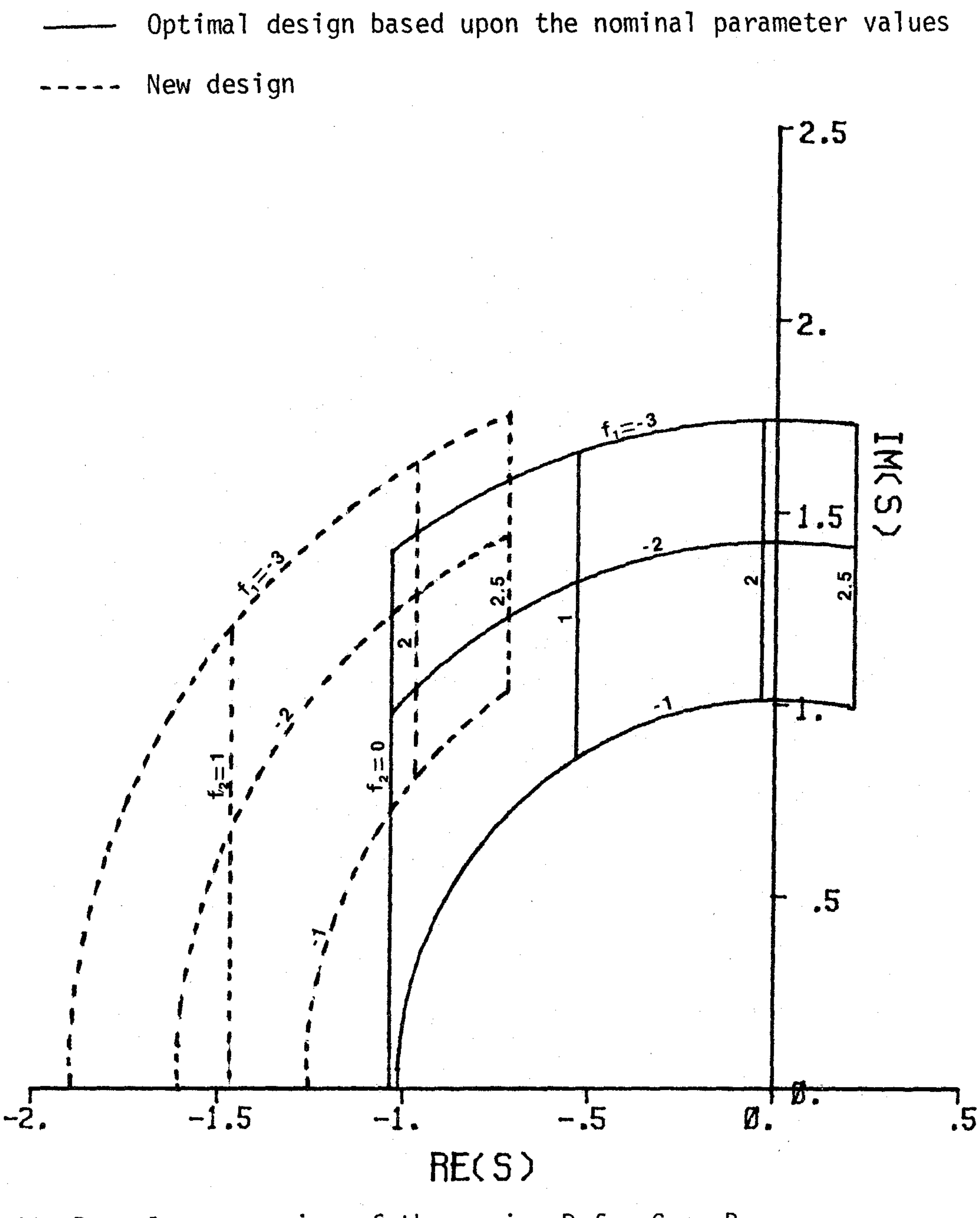


Fig. 11 Root locus mapping of the region D for Case B.

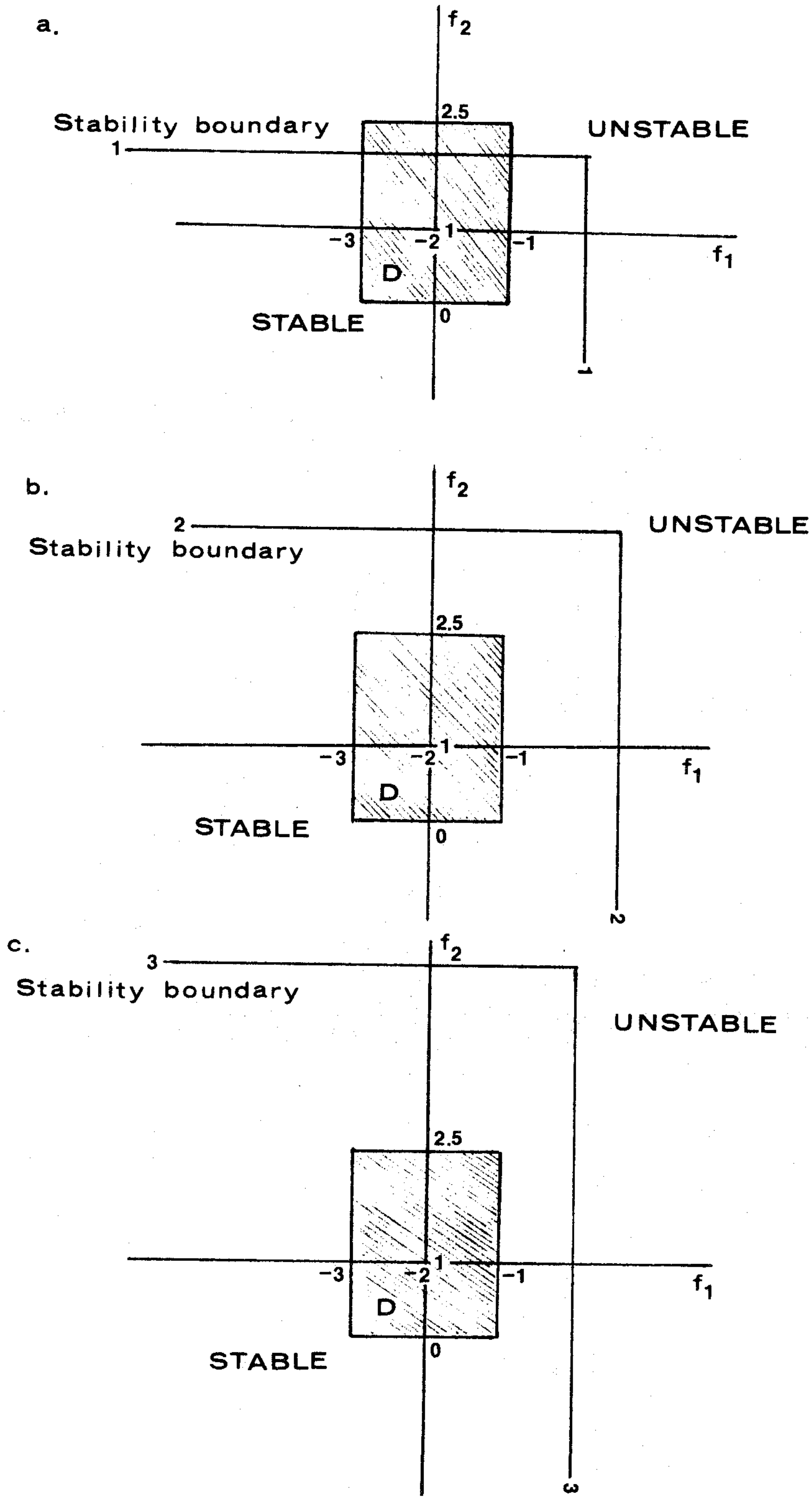


Fig.12 Stability boundaries of the three closed loop designs in Case B. (a) Optimal design based upon the nominal parameter values, (b) New design, (c) Minimax design.

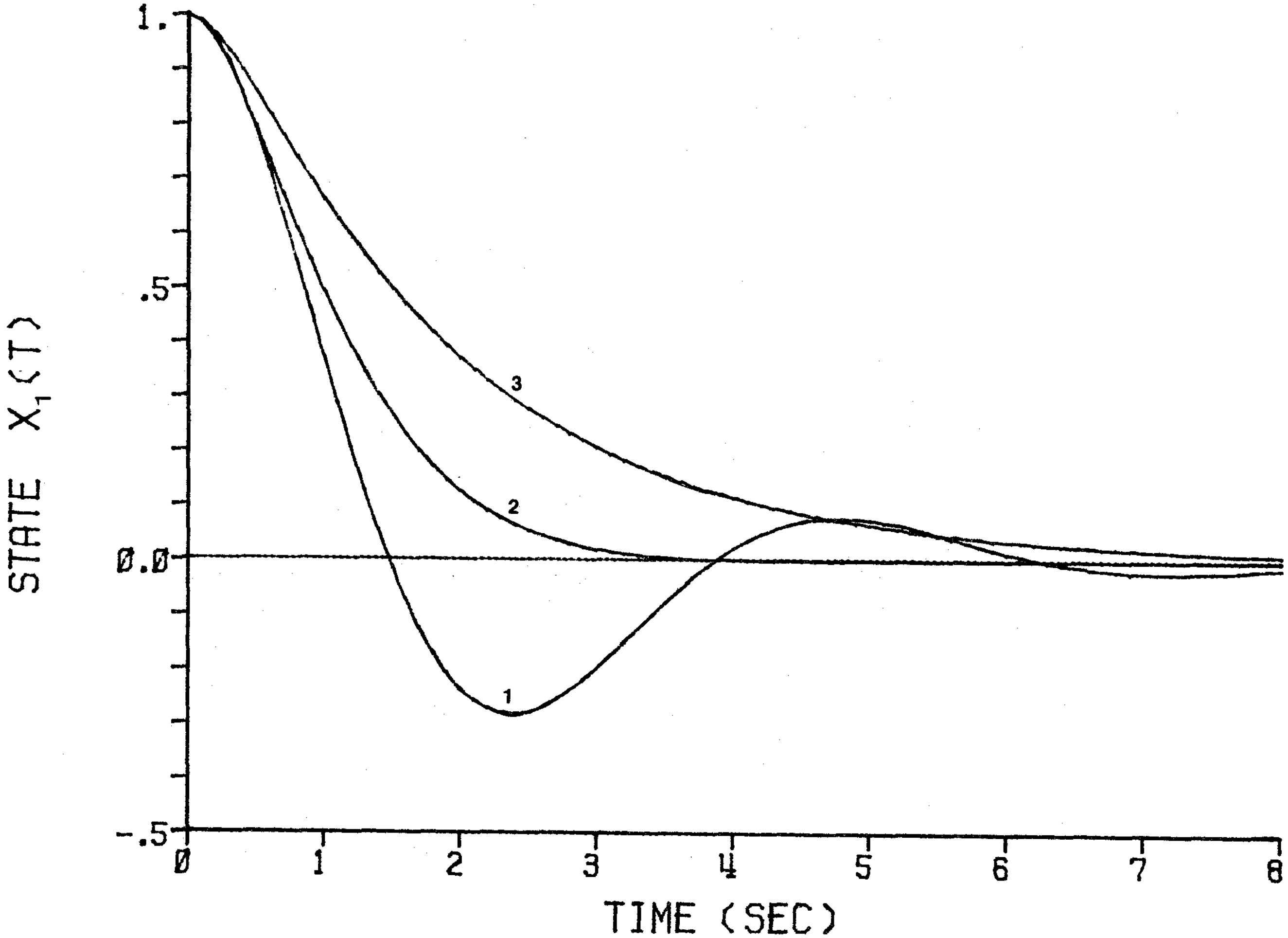


Fig. 13a Time response of the state variable $x_1(t)$ to the initial condition $x_0^T=(1,0)$ for the nominal plant in Case B. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

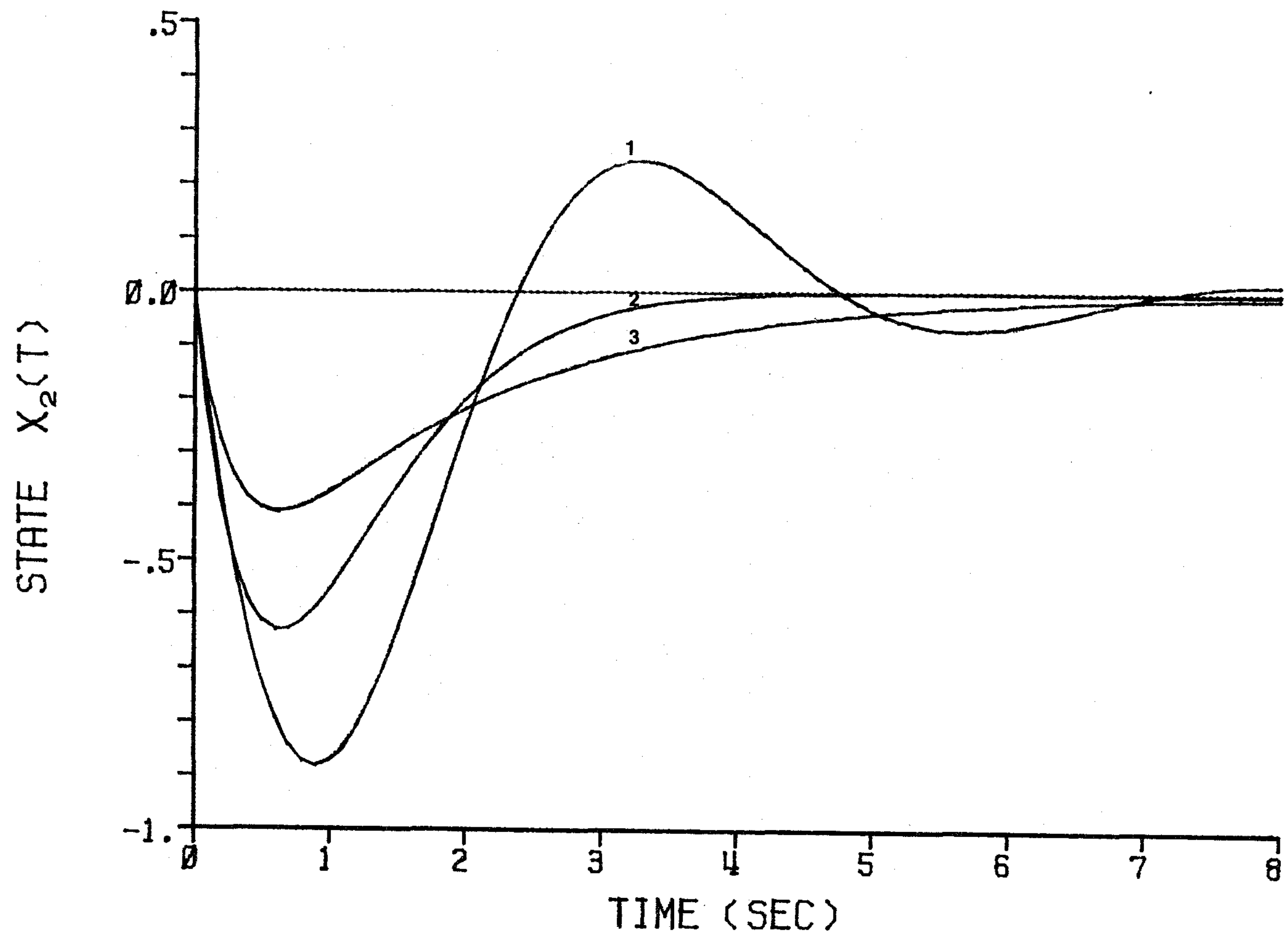


Fig. 13b Time response of the state variable $x_2(t)$ to the initial condition $x_o^T=(1,0)$ for the nominal plant in Case B. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

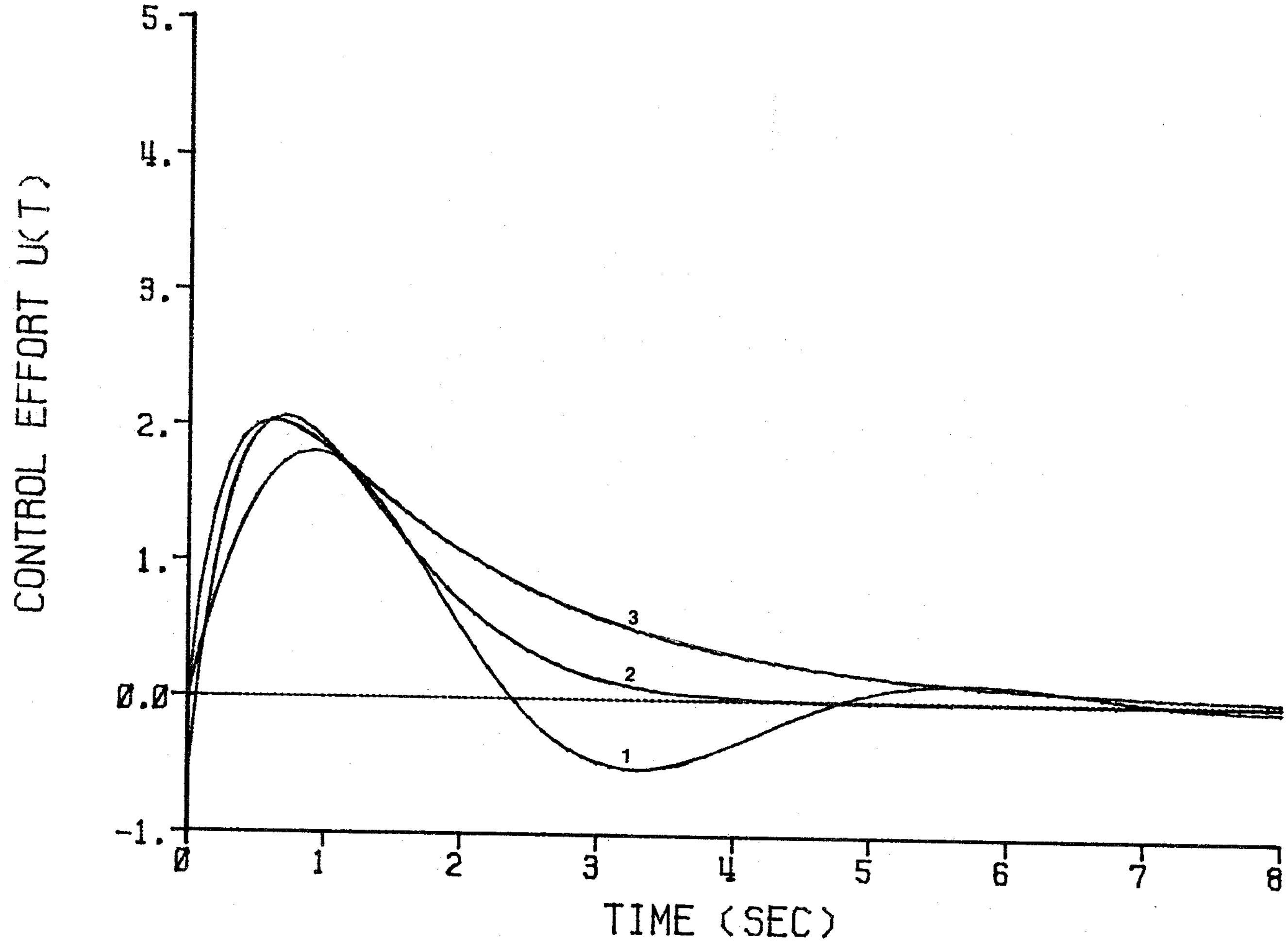


Fig. 13c Time response of the control effort u(t) to the initial condition $x_o^T=(1,0)$ for the nominal plant in Case B. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

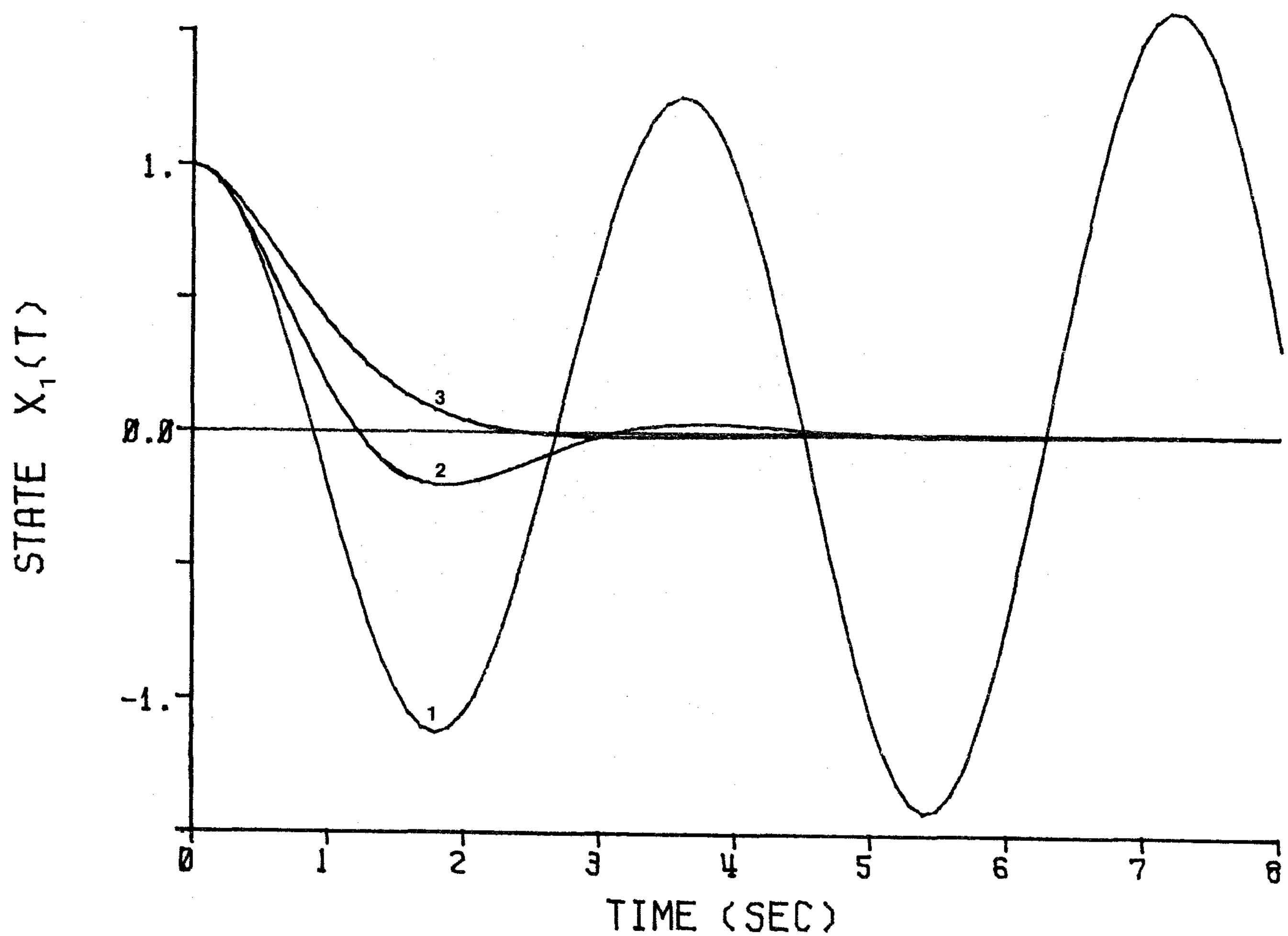


Fig. 14a Time response of the state variable $x_1(t)$ to the initial condition $x_o^T=(1,0)$ for an off-nominal plant $(f_1,f_2)=(-3,2.2)$ in Case B. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

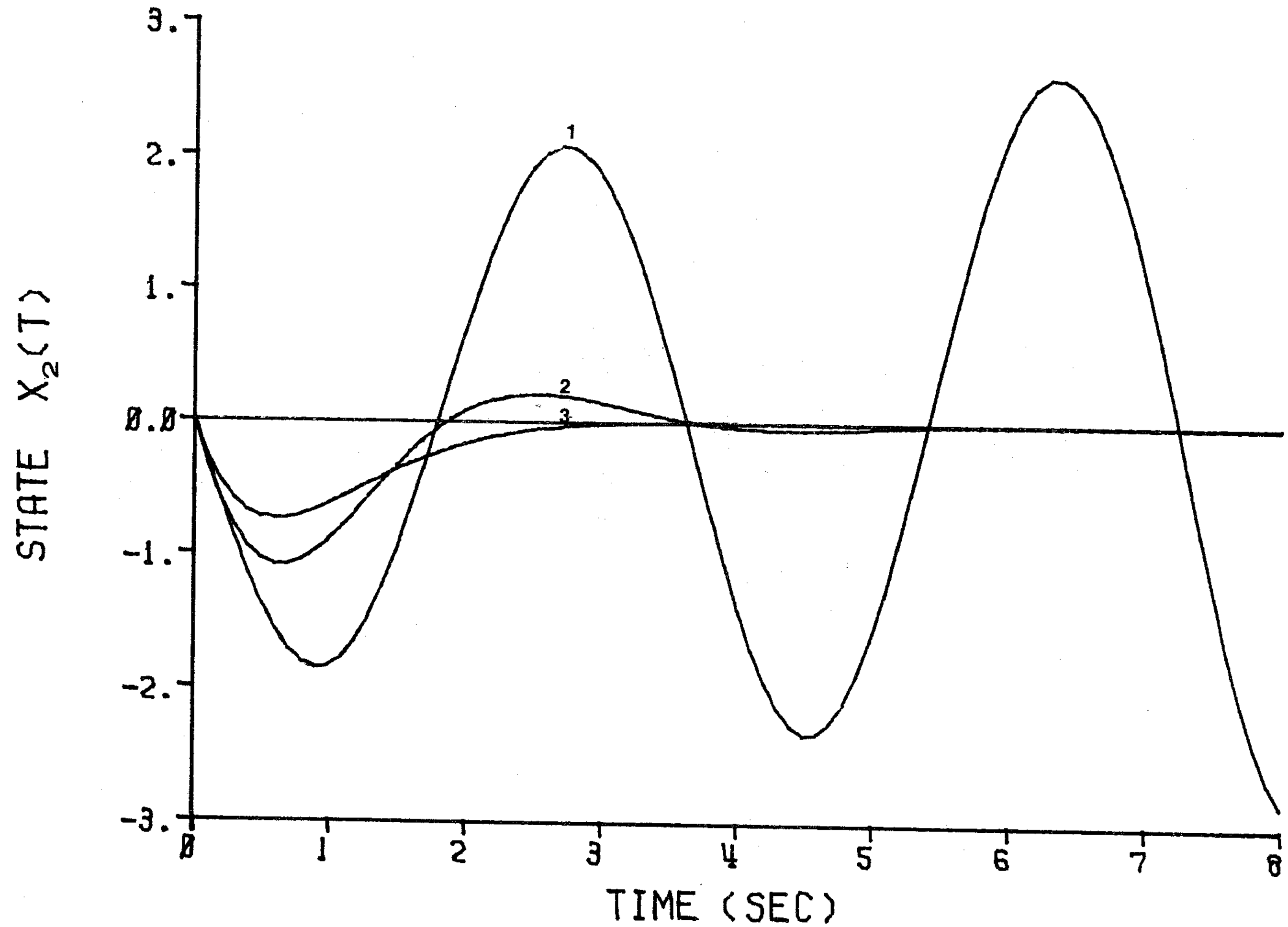


Fig. 14b Time response of the state variable $x_2(t)$ to the initial condition $x_0^T=(1,0)$ for an off-nominal plant $(f_1,f_2)=(-3,2.2)$ in Case B. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.

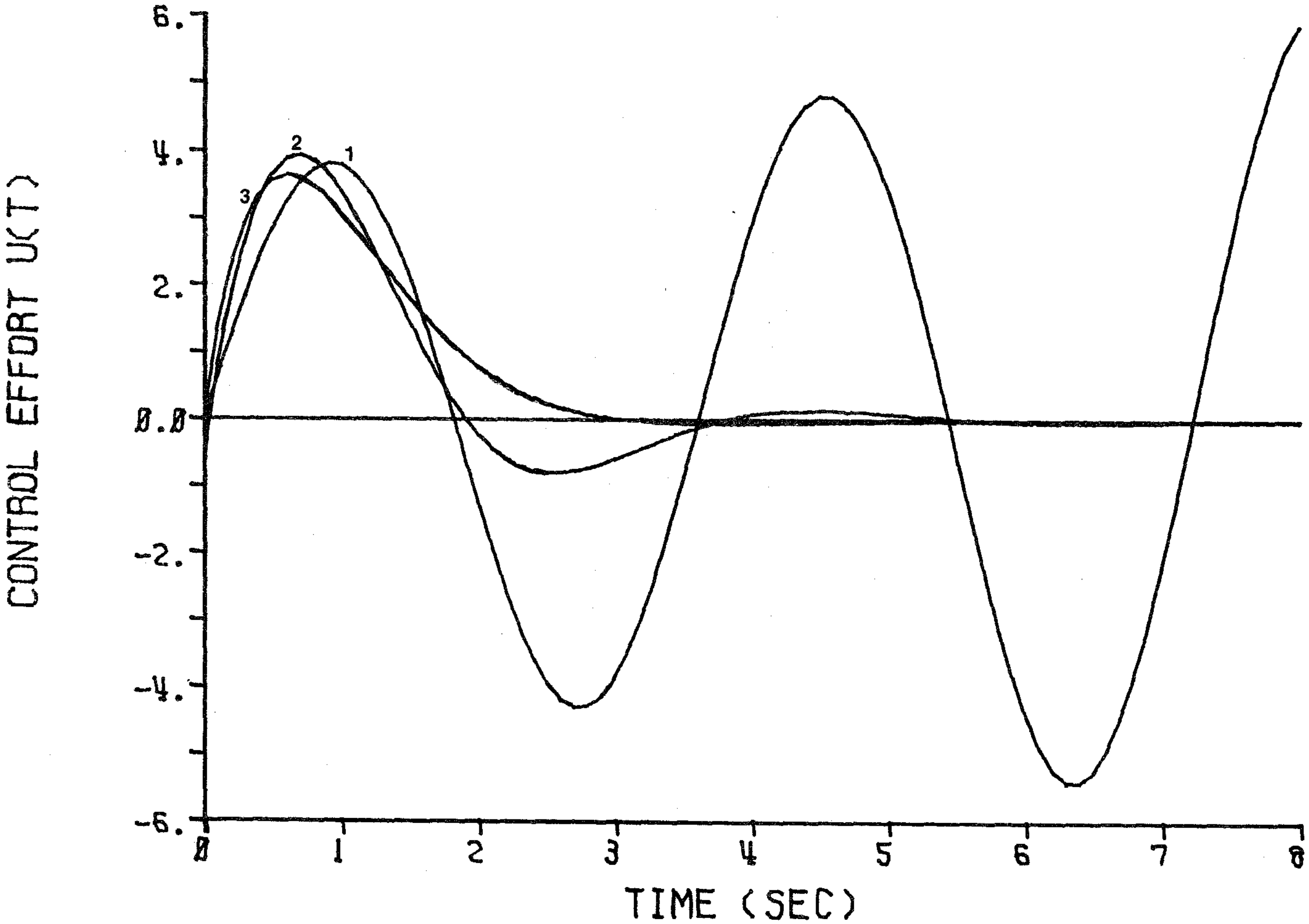


Fig. 14c Time response of the control effort to the initial condition $x_0^T=(1,0)$ for an off-nominal plant $(f_1,f_2)=(-3,2.2)$ in Case B. The plots are for: (1) Optimal design based upon the nominal parameter values, (2) New design, (3) Minimax design.