ON THE STATISTICAL THEORY OF TURBULENCE

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TABLE OF CONTENTS

ABSTRACT

The present work starts with a study of isotropic turbulence which was introduced by G. I. Taylor in 1935. The different notions of averages are critically examined. The notion of stochastic average is then introduced and the general transport equation is developed. After a detailed study of kinematics of turbulence, the concept of correlation and spectrum, the correspondence between the Karman-Howarth equation and the spectrum equation is made. The turbulence decay is studied. A theory for turbulence decay at large Reynolds number is proposed. In the study of turbulence spectrum, different assumptions on the transfer function are critically discussed and the solution using Heisenberg's assumption is obtained explicitly. The spectrum is further studied by trying to fit the turbulence phenomenon into a general scheme of stochastic processes. In the second part of the work, an entirely different approach to the statistical theory is made. Linearized vorticity transport theory is developed and finally the non-linear effects in turbulence are studied.

INTRODUCTION

Osborne Reynolds in his classical papers introduced the notion of turbulent* flow (Ref. 1). Since then the subject has been one of the outstanding fields in hydrodynamics in the past few decades. Admittedly our knowledge about the general turbulence phenomenon is still quite meagre today. Taylor defined turbulence as an irregular motion which in general makes its appearance in fluids, gases or liquids, when they flow past solid surfaces or even where neighboring streams of the same fluid flow past or over one another. Taylor stated that the actual motion is usually so irregular that very little is known of its details. Then why are we interested in such an irregular phenomenon. A best answer to this is to quote von Karman's Wright Memorial Lecture (Ref. 3).

"Turbulence is far from being the only irregular motion or irregular phenomenon which physics tries to analyze. As a matter of fact, regular motion is a rather exceptional case in nature. Even the laminar or streamline motion appears as regular motion only to the human observer who looks at the molecular world so far away and with such rough instruments that he is able only to see the average motion of matter and to measure the average values of physical quantities and simple laws for the average motion. Instead of "in spite of the irregular character," we rather have to say, "because." --- The next question is, what is the practical effect of the presence

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*The word "turbulent" is due to Lord Kelvin (Ref. 2).

of irregular motion, why are we interested in the irregular motion which is superposed on the main motion. The answer is that the presence of irregular motion radically changes the order of magnitude of the frictional resistance, the heat transfer and the diffusion of fluids. Hence the turbulent problem reaches practically into all fields of engineering in which fluid motion plays an important role, whether the fluid is the medium in which the motion of a body takes place, or is being transported for some purpose, or is an agent in a process involving heat transfer, diffusion, mixing, dissolving evaporation combustion etc."

The study of turbulence has been approximately divided into two branches. One is the study of development of turbulence and the other is the study of fully developed turbulent flow. Of the former elaborate theories on laminar stability have been developed. The problem on the stability of two-dimensional parallel flows for incompressible fluid has been clarified by Lin. Details should be referred to Lin's paper (Ref. 4). The work has been extended to the compressible flow by Lees and Lin. To study the transition from laminar to turbulent in a boundary layer, effects of surface roughness, curvature, free stream turbulence etc. have to be investigated. The reader is referred to Goldstein Vols. I and II (Ref. 5) and Dryden's report (Ref. 6).

The study of fully developed turbulent flow is further divided into two branches. One is phenomenological, the Prandtl's mixing length theory and Taylor's vorticity transfer theory are typical in this category. Phenomenological theories owe their origin to kinetic theory of gases e.g. the concept of mean free path. The other is statistical. The aim of the statistical theory is to find the method representing the turbulence field by considering different mean values and probability distribution of quantities connected with the motion. A promising beginning on statistical theory was made by Taylor who first introduced the concept of isotropic turbulence (Ref. 7). Recent advances in statistical theory were largely due to Kolmogoroff (Ref. 8).

There remains one outstanding method for studying fully developed turbulent flow. This method does not fit into either of the two categories mentioned in the last paragraph. However this method is of great generality and can be applied to both phenomenological and statistical. This method is known as Karman's similarity principle (Ref. 9). The principle says the motion at all instances is similar with an appropriate change in length scale. In particular it was assumed that the fluctuations were similar to each other throughout the field of flow so that the conditions of the flow in the neighborhood of two points differ only by a multiplicative factor of the magnitude of the fluctuations and by a length characteristic. Accepting the Karman similarity hypothesis we then have the logarithmic distribution of mean velocity in a two-dimensional channel which has been checked beautifully with experiments. In the later course of the present work we will again use the principle to some extent.

The present work begins with a study of isotropic turbulence. The notion of stochastic average and the general transport equation is then developed. After a full discussion of correlation and spectrum, the correspondence between the Karman-Howarth equation and the spectrum equation is made. Possible extensions of the present knowledge on turbulence decay and spectrum are made. In the last part of the work, an entirely different approach to the statistical theory is used. Linearized theory for vorticity transport is developed and finally the non-linear effects are studied.

PART I FUNDAMENTALS OF TURBULENCE

We shall begin with the following definitions:

1.1 Homogeneous Turbulence: Turbulence is defined to be homogeneous when the average value of any function of observable quantities in relation to a particular set of axes is unaltered by a translation of the given set of axes.

1.2 Isotropic Turbulence:* Isotropic turbulence may be defined by the condition that average value of any function of the velocity components and their derivatives at a particular point, defined in relation to a particular set of axes is unaltered if the axes of reference are rotated in any manner and if the coordinate system is reflected in any plane through the origin.

It is seen that the definition for homogeneity is of global character while the definition for isotropic turbulence is entirely local. These definitions obviously depend on what we mean by average. First we must emphasize the notion of average is necessary. It is not feasible for us to follow the motion in detail. Furthermore even if the details were known, they will be of very little practical value to us, if for example we are only interested in the total amount of mass transfer in a turbulent flow etc.

1. Notion of Averages

Intuitively we want the notion of averages to satisfy the

*This definition was given by Karman and Howarth (Ref. 10).

following postulates. Let us denote the average of a quantity A by $\overline{A}$ then

(a) The average operator is linear, namely the average of the sum of two quantities is the sum of the two averages. Symbolically

$$\overline{\alpha A + \beta B} = \alpha \overline{A} + \beta \overline{B} \tag{1}$$

where α and β are two constants.

(b) The average of an averaged quantity is the same as the averaged quantity before second average, i.e.

$$\overline{\overline{A}} = \overline{A} \tag{2}$$

This is an iterative property.

(c) Associativity

$$\overline{\overline{A}\, B} = \overline{A}\, \overline{B} \tag{3}$$

i.e., the average of a product $[\,B$ and another averaged quantity $\overline{A}\,]$, is the product of these two respective averages.

(d) Average operators commute with the differential and integral operators.

Reynolds has introduced three kinds of averages. The first is known as a time average and is defined as follows.

For any function of time $u(t)$ the time average of $u(t)$ is defined by the following integral

$$\overline{u} = \frac{1}{\tau} \int_0^{\tau} u(t+t')\, dt' \tag{4}$$

The second is space average. For example if we want to speak of space average of velocities at a given point in a three-dimensional turbulent flow. Then we first assign to ourselves a neighborhood of that specified point and the average is defined by the quotient of the volume integral of velocity and the corresponding volume of the neighborhood $v(N)$

$$\bar{u} = \frac{1}{v(N)} \iiint_N u(x, y, z) \, dx\, dy\, dz \tag{5}$$

The third one is a combination of these two notions of average, i.e.

$$\bar{u} = \frac{1}{v(N)\tau} \int_0^\tau \iiint_N u(x, y, z; \, t+t') \, dx\, dy\, dz\, dt' \tag{6}$$

The time average is very useful especially when we want to correlate between theory and experiment, because in the usual experiments only time averages are actually measured. The definition nevertheless suffers a theoretical drawback, namely the definition does not suggest in general a proper choice of time interval τ . Similarly in defining the space average the proper choice of the size of the neighborhood is left open. In the experiment we have to determine the time resolution for each particular setup. A detailed discussion of the above three notions are given in Appendix I.

In statistical mechanics we have yet another kind of average called ensemble average. The concept of ensemble was invented by Gibbs for the purpose of calculating thermodynamic properties of a system consisting of very large numbers of molecules. For stationary stochastic processes an appropriate ensemble average (e.g., each element of the ensemble may have the same a priori probability) can be identified with observable time averages. This idea has been substantiated and actually used in some of the recent turbulence measurements by Liepmann. The notion of averages just mentioned is related to the concept of stochastic or probability average. The details of probability theory, which is nothing but measure theory will not be reproduced here. The reader is suggested to refer to some standard treatise on the subject such as Refs. 11 and 12. We now propose to investigate further on the notion of stochastic average. Some preliminary definitions will first be given.

(a) Random Variable. u is a random variable if it is associated with a unique probability distribution

$$P(\xi) = P(-\infty < u \leq \xi) \tag{7}$$

Eq. (7) reads: $P(\xi)$ is the probability that u has a value less then or equal to ξ.

(b) Stochastic Process. $u(t)$ defines a stochastic process (t is the parameter) if for each fixed t_0, $u(t_0)$ is a random variable.

(c) Stochastic Average. The stochastic average of a function of $u(t)$ (we denote this function by $F(u)$) is defined by the following Stieljes integral

$$\overline{F(u)} = \int_{-\infty}^{\infty} F(\xi)\, dP(\xi; t) \tag{8}$$

The definitions here are given for one-dimensional stochastic processes. The extension to the multi-dimensional case is quite obvious.

If P is differentiable with respect to ξ, when we say in this case we have a probability density $\rho(\xi, t)$

$$\rho(\xi, t) = \frac{d P(\xi; t)}{d \xi} \tag{9}$$

The definition for the stochastic averages then becomes

$$\overline{F(u)} = \int_{-\infty}^{\infty} F(\xi) \, \rho(\xi; t) \, d\xi \tag{10}$$

The postulates (a), (b) and (c) are easily verified provided F and ρ as functions of u and ξ are smooth enough. The fourth postulate will lead us to the general transport equation which will be discussed later. It is also clear, that the definition depends on our knowledge of the density function. For practical purposes we may assume the existence of the density function, then proceed to get our results and finally identify the stochastic averages of dynamical variables with those experimentally observed by appealing to the so-called ergodic theory. The virtue of this formulation is its simplicity from the mathematical point of view without worrying about what should be the appropriate time interval or space neighborhood to be used in time or space averages respectively.

2. General Transport Equations

In the following discussion we shall assume the probability densities always exist. Since we are primarily interested in hydrodynamics, the notations are used in accordance with the usual convention namely, x stands for position coordinates, u the velocity, and a the acceleration.

Let us suppose the flow field consists of infinitely many turbulent particles. The following probability density functions are defined

$$\rho_1 (x ; t)$$

$$\rho_2 (x , u ; t)$$

$$\rho_3 (x, u, a ; t)$$

$\rho_1 \, dx$ is the probability that a turbulent particle is at a position between x and $x + dx$ at time t.

$\rho_2 \, dx \, du$ is the probability that a turbulent particle is at a position between x and $x + dx$ with a velocity between u and $u + du$.

$\rho_3 \, dx \, du \, da$ is the probability that a turbulent particle is at a position between x and $x + dx$, with a velocity between u and $u + du$ and with an acceleration between a and $a + da$.

More briefly $\rho_1(t)$, $\rho_2(t)$ and $\rho_3(t)$ define the probability measure in the respective underlying spaces, (x), (x, u) and (x, u, a).*

Let $f(x, u, t)$ be any function of the arguments x, u, t where x and u are the position and velocity of a turbulent particle, f may be called a flow function.

Then

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*The following relations may be worth noting, i.e.

$$\int_{-\infty}^{\infty} \rho_2 (x, u ; t) \, du = \rho_1 (x , t)$$

$$\int_{-\infty}^{\infty} \rho_3 (x, u, a ; t) \, da = \rho_2 (x, u ; t)$$

$$\frac{d}{dt} f(x,u,t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} u + \frac{\partial f}{\partial u} a \tag{11}$$

u and a are stochastic derivatives of x and u respectively. All x, u and a for a given turbulent particle are random variables.

From postulate (d) we have

$$\frac{d}{dt} \overline{f(x,u,t)} = \overline{\frac{\partial f}{\partial t}} + \overline{\frac{\partial f}{\partial x} u} + \overline{\frac{\partial f}{\partial u} a} \tag{12}$$ *

By definition of stochastic average, the left hand side of Eq. (12) becomes

$$\frac{d}{dt} \overline{f(x,u,t)} = \frac{d}{dt} \iint_{-\infty}^{\infty} f(x,u,t)\, \rho_2(x,u;t)\, dx\, du$$

$$= \iint_{-\infty}^{\infty} \frac{\partial f(x,u,t)}{\partial t} \rho_2(x,u;t)\, dx\, du + \iint_{-\infty}^{\infty} f(x,u;t) \frac{\partial \rho_2}{\partial t}\, dx\, du$$

The right hand side becomes

$$\iint_{-\infty}^{\infty} \frac{\partial f}{\partial t} \rho_2(x,u,t)\, dx\, du + \iint_{-\infty}^{\infty} \frac{\partial f}{\partial x} u\, \rho_2(x,u,t)\, dx\, du + \iiint_{-\infty}^{\infty} \frac{\partial f}{\partial u} a\, \rho_3(x,u,a;t)\, dx\, du\, da$$

Denote the conditional mean of acceleration by A **

$$A = \frac{\int_{-\infty}^{\infty} a\, \rho_3(x,u,a;t)\, da}{\rho_2(x,u;t)} \tag{13}$$

*If we use Dedebant's definitions for stochastic derivatives (cf. Ref. 13) then the postulate (d) is superfluous.

**For the definition of conditional mean see Kolmogoroff (Ref. 11).

Eq. (12) becomes

$$\iint f(x,u;t)\,\frac{\partial \rho_2}{\partial t}\,dx\,du - \iint \frac{\partial f}{\partial x}\,u\,\rho_2\,dx\,du - \iint \frac{\partial f}{\partial u}\,A\,\rho_2\,dx\,du = 0$$

The second and third terms may be further transformed by partial integration and assuming $u\rho_2 f$ and $A\rho_2 f$ vanish when $x = \pm\infty$ and $u = \pm\infty$ respectively, then

$$\iint f(x,u;t)\left\{\frac{\partial \rho_2}{\partial t} + u\,\frac{\partial \rho_2}{\partial x} + \frac{\partial}{\partial u}(A\,\rho_2)\right\} dx\,du \equiv 0 \tag{14}$$

Since f may be any integrable function a necessary and sufficient condition that the above identity is true is

$$\frac{\partial \rho_2}{\partial t} + u\,\frac{\partial \rho_2}{\partial x} + \frac{\partial}{\partial u}(A\,\rho_2) = 0 \tag{15}$$

This equation is a particular type of Fokker-Planck equation which described how the probability density $\rho_2(x,u)$ changes in the course of time in (x,u) space. It is obvious from the equation that a complete determination of ρ_2 as an initial or boundary value problem will require a definite knowledge of the conditional mean of acceleration, i.e. a first moment of the probability density $\rho_3(x,u,a)$. This feature is typical in all statistical considerations of hydrodynamic problems.

We may proceed from Eq. (14) in a slightly different direction. Let us define the following conditional averages (denoted by $\overline{}^{c}$ over the averaged quantity)

$$
\begin{aligned}
\rho_1 \overline{f}^{c} &= \int f \rho_2 \, du \\
\rho_1 \overline{fu}^{c} &= \int f u \rho_2 \, du \\
\rho_1 \overline{\frac{\partial f}{\partial t}}^{c} &= \int \frac{\partial f}{\partial t} \rho_2 \, du \\
\rho_1 \overline{u \frac{\partial f}{\partial x}}^{c} &= \int u \frac{\partial f}{\partial x} \rho_2 \, du \\
\rho_1 \overline{A \frac{\partial f}{\partial u}}^{c} &= \int A \frac{\partial f}{\partial u} \rho_2 \, du
\end{aligned}
\tag{16}
$$

We then have for example

$$
\begin{aligned}
\iint f(x,u,t) \frac{\partial \rho_2}{\partial t} \, dx \, du &= \frac{\partial}{\partial t} \iint f \rho_2 \, dx \, du - \iint \frac{\partial f}{\partial t} \rho_2 \, dx \, du \\
&= \frac{\partial}{\partial t} \int \rho_1 \overline{f}^{c} \, dx - \int \rho_1 \overline{\frac{\partial f}{\partial t}}^{c} \, dx
\end{aligned}
$$

if the differentiation under the integral sign is justified etc.

Eq. (14) then implies

$$
\frac{\partial}{\partial t}(\rho_1 \overline{f}^{c}) + \frac{\partial}{\partial x}(\rho_1 \overline{fu}^{c}) = \rho_1 \left[\overline{\frac{\partial f}{\partial t}}^{c} + \overline{u \frac{\partial f}{\partial x}}^{c} + \overline{A \frac{\partial f}{\partial u}}^{c} \right] \tag{17}
$$

If the turbulent particle is not restricted in one dimension, then in general we have (for simplicity ρ_1 will be denoted by ρ hereafter)

$$
\frac{\partial}{\partial t}(\rho \overline{f}^{c}) + \frac{\partial}{\partial x_\kappa}(\rho \overline{f u_\kappa}^{c}) = \rho \left[\overline{\frac{\partial f}{\partial t}}^{c} + \overline{u_\kappa \frac{\partial f}{\partial x_\kappa}}^{c} + \overline{A_\kappa \frac{\partial f}{\partial u_\kappa}}^{c} \right] \tag{18}
$$

This is a generalized transport equation for any flow function f . This equation is analogous to the Boltzmann equation in the kinetic theory of gases. Two important special cases are worth mentioning.

(i) $f = 1$ Eq. (18) reduces to the ordinary form of equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\kappa} \rho \overline{u_\kappa}^c = 0 \tag{19}$$

if ρ is identified as mass density

(ii) $f = u_i$, then we have

$$\frac{\partial}{\partial t}(\rho \overline{u_i}^c) + \sum_\kappa \frac{\partial}{\partial x_\kappa}(\rho \overline{u_i u_\kappa}^c) = \rho R_i \tag{20}$$

where R_i is the forcing function

$$R_i = \left[\overline{\frac{\partial u_i}{\partial t}}^c + \overline{u_\kappa \frac{\partial u_i}{\partial x_\kappa}}^c + \overline{A_\kappa \frac{\partial u_i}{\partial u_\kappa}}^c \right] \tag{21}$$

Following Reynolds we assume u_i can be separated into two parts, the mean velocity $\overline{u_i}^c = U_i = \int u f_2(x, u)\, du$ and the fluctuating part u_i' . Eq. (20) can be written in a more conventional form

$$\frac{\partial U_i}{\partial t} + \sum_\kappa U_\kappa \frac{\partial U_i}{\partial x_\kappa} = R_i - \frac{1}{\rho} \sum_\kappa \frac{\partial}{\partial x_\kappa}(\rho \overline{u_i' u_\kappa'}^c) \tag{22}$$

This equation coincides with the Reynolds equation for the mean mean motion if we identify ρ as the ordinary density, the term $\rho \overline{u_i' u_\kappa'}^c$ on the right hand side is just the Reynolds stress.

Similarly we may find transport equations for many other interesting quantities such as the turbulent energy, temperature, entropy etc. However in introducing the concept of entropy care must be taken (cf. Ref. 14). First the turbulence represents a dissipative system and cannot be interpreted as in thermodynamic equilibrium state, and second we can hardly attach any meaning of the fluctuating entropy as in the case of fluctuating velocity.

In concluding the discussion on the notion of averages, it is worth pointing out that to adopt either the stochastic average or the ordinary time average is rather a matter of taste. Mathematicians will tend to use stochastic average, because this notion is mathematically precise. However, in order that the derived results may be compared with experiments - which is the ultimate goal of every physical theory - one has to rely on the so-called ergodic theory, and this branch of mathematics has not been satisfactorily explored yet. The ordinary time average may be understood as a smoothing process. The averaged quantity can be again considered as a variable which varies slowly with time. With this understanding the time interval τ used in the averaging process should be small compared to any characteristic time interval during which an appreciable change in the large occurs.

3. Concepts of Correlation and Spectrum

Before we shall finally embark on the dynamics of turbulent flow, still some further preliminary concepts such as correlation and spectrum have to be introduced. The first question is how do we define a correlation between various quantities and secondly if we know these correlations how much information have we obtained for the quantities we are interested in.

Mathematically a correlation $\psi(x_1, \cdots\cdots, x_n)$ for n random variables $(x_1, \cdots\cdots, x_n)$ may be defined as follows

$$\psi(x_1, \cdots\cdots, x_n) = \underbrace{\int\cdots\int}_{n} P_n(x_1, \cdots\cdots, x_n) \prod_{i=1}^{n} (x_i - \overline{x_i})\, dx_i \tag{23}$$

where $\overline{x_i}$ is the expectation value of the random variable x_i , $i = 1, 2, \cdots\cdots, n$

$$\bar{X}_i = \int x_i \, P_i(x_i) \, dx_i \tag{24}$$

where $P_n(x_1, \cdots, x_n)$ is the probability density in the n-dimensional space and $P_i(x_i)$ is the probability density for x_i .

If all these random variables are statistically independent then

$$P_n = \prod_{i=1}^{n} P_i(x_i) \tag{25}$$

and consequently the correlation is zero. However, the statement that "if the random variables have zero correlation then they are statistically independent" is not true. It is very easy to construct an example such that the correlation of two random variables are zero but nevertheless they are statistically dependent.

If x 's are functions of time, the correlation is more conveniently defined in terms of time average for practical purposes, i.e.

$$\psi(x_1, x_2, \cdots, x_n) = \overline{\prod_{i=1}^{n} (x_i - \bar{x}_i)} \tag{26}$$

where bar denotes the time average.

The use of correlation is best illustrated by the following simple examples.

(1) Let us first investigate the mass diffusion inside a one-dimensional homogeneous turbulent field. To simplify the problem assume the mass is concentrated at a certain point (taken as origin) at $t = 0$. We then inquire how does the mass diffuse when $t > 0$? Assume the mass consists of very many mass particles, then the question is how these particles distribute among themselves.*

*The first investigation of this sort was made by G. I. Taylor (Ref. 15).

Denote by $u(t)$ the velocity of a mass particle, $x(t)$ the displacement of the particle.

Assume for mathematical convenience

$$\overline{u(t)} = 0 \qquad \overline{u^2(t)} = \psi_0$$

Again bar denotes the time average. Since

$$x(t) = \int_0^t u(\tau)\, d\tau \tag{27}$$

then

$$\bar{x}(t) = \int_0^t \overline{u(\tau)}\, d\tau = 0$$

which is of course true, but this is not an interesting result. For all practical cases, we definitely know mass does diffuse and we want to get some measure of the rate of diffusion. The measure that we ordinarily take is the dispersion. By definition dispersion is equal to $\overline{x^2(t)}$ here. We have

$$\overline{x^2(t)} = \overline{\left(\int_0^t u(\tau_1)\, d\tau_1\right)\left(\int_0^t u(\tau_2)\, d\tau_2\right)}$$

$$= \int_0^t \int_0^t \overline{u(\tau_1)\, u(\tau_2)}\, d\tau_1\, d\tau_2 \tag{28}$$

$\overline{u(\tau_1)\, u(\tau_2)}$ is the correlation function in the present case and is known as the Lagrangian correlation function. Denote $\psi(\tau_1, \tau_2) = \overline{u(\tau_1)\, u(\tau_2)}$.

For a stationary* process

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*Physically a stationary process means that the underlying mechanism which causes the fluctuations does not change in the course of time. Mathematically this corresponds to say that the probability distributions and hence correlations are invariant with respect to a translation of time axis.

$$\psi = \psi(|\tau_1 - \tau_2|)$$

and

$$\psi(0) = \psi_0$$

Eq. (28) then becomes

$$\overline{X^2(t)} = \int_0^t \int_0^t \psi(|\tau_1 - \tau_2|)\, d\tau_1\, d\tau_2$$

$$= 2\int_0^t (t-\tau)\, \psi(\tau)\, d\tau \tag{29}$$

We see if $\psi(\tau)$ is given, then we may get by (29)

$$\overline{X^2(t)} = F(t)$$

and the dispersion at any time is known. Even in the case we have very incomplete information about $\psi(\tau)$, we can still discuss some of the qualitative behaviors of Eq. (29).

We see for small value of t

$$\overline{X^2(t)} \doteq \psi_0\, t^2$$

and for large value of t

$$\overline{X^2(t)} \sim 2t\int_0^\infty \psi(\tau)\, d\tau - 2\int_0^\infty \tau\, \psi(\tau)\, d\tau$$

Denote $\psi(\tau) = \psi_0\, R(\tau)$ where $R(\tau)$ is the normalized correlation coefficient, then for large t

$$\overline{X^2(t)} \sim 2\psi_0 \left[t\int_0^\infty R(\tau)\, d\tau - \int_0^\infty \tau\, R(\tau)\, d\tau \right]$$

$$= 2\psi_0 \left[t\,T - T_1^2 \right]$$

where T is the correlation time and T_1^2 is the first moment of the correlation function.

The above example shows that to introduce a double correlation gives some knowledge of kinematical description. In turbulence we are in general interested in two kinds of correlations. One is double correlation and the other is triple correlation. The triple correlations are necessary for a complete understanding of the dynamics of turbulent flow. This will become obvious later on.

(2) The example is this. Suppose we have a definite harmonic function $u(t) = \sqrt{2} \sin \omega_0 t$. We then inquire what is the correlation function $\psi(t,\tau) = \overline{u(t)\ u(t+\tau)}$. Formally we have

$$\psi(t, \tau) = \overline{u(t)\ u(t+\tau)} = \cos(\omega_0 \tau)$$

This again represents a stationary process. Now suppose we know the correlation, what do we know about the structure of the original process. In order to answer this question we shall first establish the following lemma.

Lemma: For a stationary continuous stochastic process $f(t)$, the correlation function ψ is positive definite.*

Proof: By definition

*We call a function $\psi(x)$ positive definite if

(i) Continuous in any finite interval and bounded $-\infty < x < \infty$.
(ii) Hermitean symmetry, i.e. $\psi^*(-x) = \psi(x)$ where * denotes complex conjugate and
(iii) For any arbitrary points $x_1, x_2, \dots, x_m$ ($m = 1, 2, 3, \dots$) and any arbitrary complex numbers $\rho_1, \dots \rho_m$, the condition

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \psi(x_\mu - x_\nu)\, \rho_\mu \rho_\nu^* \geq 0$$

$$\psi(\tau) = \overline{f(t)\ f^*(t+\tau)} \tag{a}$$

The condition (1) is obvious.

Take the complex conjugate of (a)

$$\psi^*(\tau) = \overline{f^*(t)\ f(t+\tau)}$$

hence

$$\psi^*(-\tau) = \overline{f(t-\tau)\ f^*(t)} = \psi(\tau)$$

since the process is stationary.

To verify the third condition, we have

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \psi(\tau_\mu - \tau_\nu)\, \rho_\mu\, \rho_\nu^* = \overline{\left| \sum_{\mu=1}^{m} f(t-\tau_\mu)\, \rho_\mu \right|^2} \geq 0$$

Any positive definite functions $\psi(\tau)$ can be represented by a Fourier Stieljes integral as follows (Ref. 16)

$$\psi(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau}\, dF(\omega) \tag{30}$$

$F(\omega)$ is bounded and in general can be decomposed into two parts $F(\omega) = D(\omega) + S(\omega)$, where $D(\omega)$ is a step function representing the discontinuous part and $S(\omega)$ is continuous. We say that $F(\omega)$ gives the spectral resolution of the given stationary process. In the second example we quoted above we have

$$F(\omega) = \begin{cases} 0 & ; \quad \omega < -\omega_0 \\ \frac{1}{2} & ; \quad |\omega| < \omega_0 \\ 1 & ; \quad \omega > \omega_0 \end{cases}$$

We see immediately that there is one characteristic frequency ω_0 in the given process, however this is all the information we can get from this representation. Because there may be two or more processes which give rise to the same correlation coefficient, i.e. the knowledge of the correlation coefficient cannot determine the process uniquely.*

Furthermore if $F(\omega)$ has a derivative $F'(\omega)$ and $F'(\omega)$ is an even function of ω then

$$\psi(\tau) = \int_0^\infty 2F'(\omega) \cos\omega\tau \, d\omega$$

$2F'(\omega)$ is usually defined as the spectrum density of the process and is denoted by $E(\omega)$, i.e.

$$\psi(\tau) = \int_0^\infty E(\omega) \cos(\omega\tau) \, d\omega \tag{31}$$

In this way we have clearly demonstrated the relation between the correlation coefficient and the corresponding spectrum. If $\psi(\tau)$ represents the velocity correlation $\psi(\tau) = \overline{u(t)\,u(t+\tau)}$ then

$$\psi(0) = \overline{u^2(t)} = \int_0^\infty E(\omega)\, d\omega \tag{32}$$

$E(\omega)$ may be interpreted as actual energy density i.e. $E(\omega)\,d\omega$ is the amount of contribution to $\overline{u^2(t)}$ between the frequencies ω and $\omega + d\omega$. G. I. Taylor in 1938 first introduced the concept of spectrum into the theory of turbulence (Ref. 17), and thereafter it has

*For example the process $u(t) = \sqrt{2}\cos(\omega_0 t + \phi)$ where ϕ is a random variable, and ϕ_s are uniformly distributed $0 \leq \phi < 2\pi$, then this process has the same correlation coefficient $\psi(\tau) = \cos\omega_0\tau$.

remained one of the interesting objectives for the workers in the field. Actually to speak of correlation or of spectrum is rather a matter of preference, as long as they provide the same intuitive physical picture.

PART II CRITICAL REVIEW OF THE THEORY OF ISOTROPIC TURBULENCE

1. Kinematics of Turbulence

Kinematics in general deals with mathematical or graphical descriptions of motion. In fluid mechanics we are particularly interested in the velocity or vorticity field of flow at every instant. The fact that the turbulent velocities are highly irregular hints to us to seek an alternative form of description. It turns out that the concept of correlation that we just described is very useful in this respect. Namely, we want to get some information of how the velocities at different points in space are correlated at the same time or how the velocity at a given instant is correlated to the velocity at the same point but at a later time in order to calculate the rate of heat transfer, mass diffusion etc. These are indeed the concepts of space correlation and time correlation. The use of correlation as an unknown variable in hydrodynamics was first proposed by the late A. Friedman (Ref. 18), however he could not carry through his idea to any practical results. von Karman in 1937 (Ref. 19) was the first one who used correlation with success and also with reasonable simplicity. An excellent reformulation of Karman's original paper was given by H. P. Robertson in 1940 (Ref. 20).

Karman introduced the now well-known Karman correlation tensors as follows, (in the case of the homogeneous and isotropic turbulence field)

(1) Double velocity correlation tensor

$$R_{ij} = \frac{\overline{u_i u_j'}}{\overline{u^2}} \tag{33}$$

(2) Triple velocity correlation tensor

$$T_{ijk} = \frac{\overline{u_i u_j \; u_k'}}{(\overline{u^2})^{3/2}} \tag{34}$$

The same notations as in Ref. 19 are used.

The assumption that the turbulent field is homogeneous implies that R_{ij} and T_{ijk} are functions of ξ_i ($\xi_i = x_i' - x_i$, $i = 1, 2, 3$). Furthermore, on account of isotropy R_{ij} and T_{ijk} must be of the following form (isotropic tensor)

$$\begin{aligned} R_{ij} &= R_1 \xi_i \xi_j + R_2 \delta_{ij} \\ T_{ijk} &= T_1 \xi_i \xi_j \xi_k + T_2 \delta_{ij} \xi_k + T_3 \delta_{ik} \xi_j + T_4 \delta_{jk} \xi_i \end{aligned} \tag{35}$$

where R_1 , R_2 , T_1 , T_2 , T_3 and T_4 are scalar functions of the distance r between the two points P and P' . The isotropic tensors of first or higher ranks can be easily expressed in similar forms by using Robertson's invariant theory. It should be pointed out that not only isotropic tensors are useful in studying isotropic turbulence but also the skew isotropic tensors. The skew isotropic tensors transform as isotropic tensors in proper rotation, but they take opposite sign to isotropic tensors on reflexion in the origin. A typical example is the velocity and vorticity correlation $\overline{u_i \omega_j'}$ where u_i is the i-th component of the velocity vector at point P and ω_j' is the j-th component of the vorticity vector at point P' .

Karman has obtained the following results for homogeneous isotropic turbulence

$$R_1 = \frac{f-g}{r^2} \quad , \quad R_2 = g \quad ;$$

$$T_1 = \frac{k-h-2q}{r^3} \quad , \quad T_2 = \frac{h}{r} \quad ; \quad T_3 = T_4 = \frac{q}{r} \tag{36}$$

Notations are explained in Ref. 19.

If we assume the fluid is incompressible, additional simplification can be made. The equation of continuity then states that the velocity is a solenoidal vector. Consequently the velocity correlations are solenoidal tensors, i.e. $R_{ij,i} = R_{ij,j} = 0$. Using this condition the double or triple correlations can be expressed solely in terms of a single scalar function. We have

$$\begin{aligned} g &= f + \frac{r}{2}\frac{\partial f}{\partial r} \\ k &= -2h \\ q &= -h - \frac{r}{2}\frac{\partial h}{\partial r} \end{aligned} \tag{37}$$

Hence

$$R_{ij} = -\frac{1}{2r}\frac{\partial f}{\partial r}\xi_i\xi_j + \left(f + \frac{r}{2}\frac{\partial f}{\partial r}\right)\delta_{ij}$$

and a corresponding expression for T_{ijk} .

So far we have not mentioned the properties of the isotropic vector, i.e. the isotropic tensor of first rank. This class includes, for example, the correlation between a scalar fluctuating quantity at a point and the velocity at a different point. von Karman and Howarth show that in incompressible homogeneous and isotropic turbulence the correlation between pressure fluctuating and velocity is identically zero on account of the equation of continuity. Actually, the argument there is quite general and under these assumptions, i.e. incompressible homogeneous and isotropic, the correlation between any scalar quantity and velocity is identically zero.*

It is seen that R_{ij} or T_{ijk} can be expressed in terms of only one scalar function, hence any operator operating on these tensors can be replaced by an appropriate operator operating on the corresponding scalar functions. For example, the vorticity correlation can be expressed in terms of velocity correlation as follows: by definition

$$\Omega_{ij} = \overline{\omega_i \omega_j'} \tag{38}$$

where ω_i is the i th vorticity component at point $P(x_1, x_2, x_3)$ and ω_j' is the j th vorticity component at point $P'(x_1', x_2', x_3')$.

We have

$$\omega_i = \epsilon_{ilk} \frac{\partial u_k}{\partial x_l}$$

$$\omega_j' = \epsilon_{jmn} \frac{\partial u_n'}{\partial x_m}$$

hence

$$\begin{aligned}\Omega_{ij} &= - \epsilon_{ilk} \epsilon_{jmn} \frac{\partial^2}{\partial \xi_l \partial \xi_m} \overline{u_k u_n'} \\ &= \overline{u^2} \left\{ \frac{1}{2r} \frac{\partial^3 f}{\partial r^3} + \frac{2}{r} \frac{\partial^2 f}{\partial r^2} - \frac{2}{r^3} \frac{\partial f}{\partial r} \right\} \xi_i \xi_j + \overline{u^2} \left\{ - \frac{r}{2} \frac{\partial^3 f}{\partial r^3} - 3 \frac{\partial^2 f}{\partial r^2} - \frac{2}{r} \frac{\partial f}{\partial r} \right\} \delta_{ij} \\ &= - \nabla_\xi^2 \, \overline{u^2} R_{ij}\end{aligned}$$

*It is easy to show that the skew isotropic vector is identically zero for homogeneous turbulence, even though the fluid is compressible.

It is not difficult to verify that the Laplacian of R_{ij} an isotropic tensor of second rank is an isotropic solenoidal tensor of second rank $\frac{-\Omega_{ij}}{\overline{u^2}}$ with a different characteristic scalar function. If we write

$$\frac{-\Omega_{ij}}{\overline{u^2}} = -\frac{1}{2r}\frac{\partial \tilde{f}}{\partial r}\xi_i\xi_j + \left(\tilde{f} + \frac{r}{2}\frac{\partial \tilde{f}}{\partial r}\right)\delta_{ij} \tag{39}$$

then

$$\tilde{f} = Df$$

D is a scalar operator and in the present case $D = \frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}$. In certain cases we find it is more intuitive and convenient to use the vorticity correlations rather than the velocity correlation.

2. Dynamics of Isotropic Turbulence and the Dynamic Invariant

Using Friedman's idea one now seeks to find an equation satisfied by the double and triple correlations. The equation was first given by von Karman and Howarth in 1938 (Ref. 10) using the Navier-Stokes equation as the basis for the development. It should be mentioned that some people (for example Bass Ref. 12) raise the question that the turbulent velocity is not a differentiable function of time and hence the Navier-Stokes equation cannot serve as a starting point. However, it was pointed out by Kirkwood that for all practical purposes the observed fluctuating velocity is already a small time average (small compared to some characteristic time in a given turbulent field), consequently any irregularity including the non-differentiability property is smoothed out by this process. It is only essential that two different time intervals are distinguished (cf. the end of section 2 Part I). Furthermore the success of the Karman-Howarth equation in explaining the turbulent energy decay and the fact that "the measured values of λ and G (Ref. 21) (notations will be explained later) both from correlation and spectrum measurement check with each other respectively for λ and G well within the experimental error" indicates very little doubt about the differentiability of fluctuating velocity.

The Karman-Howarth equation which relates the double and triple correlations is the following:

$$\frac{\partial(\overline{u^2}f)}{\partial t} + 2(\overline{u^2})^{3/2}\left(\frac{\partial k}{\partial r} + \frac{4k}{r}\right) = 2\nu\left(\frac{\partial^2(\overline{u^2}f)}{\partial r^2} + \frac{4}{r}\frac{\partial(\overline{u^2}f)}{\partial r}\right) \qquad (40)^*$$

Eq. (40) can be rewritten in terms of the trace $Tr.$ of the correlation tensor $\overline{u^2}\,R_{ij}$. The trace is defined as follows:

$$Tr. = \overline{u^2}\sum_{i=1}^{3} R_{ii} = \overline{u^2}\,(f+2g) \qquad (41)$$

The trace is an intrinsic property of two points in a homogeneous turbulence and invariant under unitary transformation. Its use will facilitate to establish the correspondence between the energy spectrum equation and the Karman-Howarth equation (cf. section 5). We have

$$\frac{\partial Tr.}{\partial t} - 2(\overline{u^2})^{3/2}\left(\frac{\partial Q}{\partial r} + \frac{2Q}{r}\right) = 2\nu\left(\frac{\partial^2 Tr.}{\partial r^2} + \frac{2}{r}\frac{\partial Tr.}{\partial r}\right) \qquad (42)$$

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*For derivations see Ref. (10). On account of the non-linear terms in the Navier-Stokes equation, the equation for time rate of change of double correlation necessarily involves the triple correlations. Similarly the equation for time rate of change of triple correlations involves the quadruple correlations and so on. Hence the Karman-Howarth equatioh, is in this sense, indeterminate as long as we do not have an additional independent relation between the double and triple correlation.

where $Q = k + 2g$, k and g are triple correlations defined by Karman.

Eq. (40) is the basic dynamic equation for studying homogeneous isotropic turbulent flow. The next question is does there exist any dynamic invariant? In classical mechanics, the Hamiltonian or the total energy (potential energy plus kinetic energy) is an invariant of motion in a non-dissipative system. However, in a turbulent flow the mean energy of fluctuation is continuously dissipated by the action of viscosity and becomes a part of molecular heat energy. Hence we cannot expect the total mean fluctuation of energy to be invariant, instead we have to seek some alternatives. Experimentally the highly diffusive character due to vortex stretching etc. is a well-recognized fact in any turbulent flow. If we may define some sort of length characteristic for turbulence, then we expect this length to increase as time increases. It is very probable that a certain combination of length and turbulent intensity (according to Dryden's definition the turbulence intensity is equal to $\frac{\sqrt{\overline{u^2}}}{U}$ where U is a reference velocity) is invariant in the entire decay process. Such an invariant was first discovered by Loitsiansky (Ref. 22).

Suppose that we multiply the Karman-Howarth equation by r^4 and integrate the whole equation with respect to r from zero to infinity and if the order of time differentiation and space integration can be interchanged and the following conditions are satisfied

(i) As $r \to \infty$; $r^4 h \to 0$ and $r^4 \frac{\partial f}{\partial r} \to 0$

(ii) As $r = 0$; $r^4 h = 0$ and $r^4 \frac{\partial f}{\partial r} = 0$

then

$$\frac{d}{dt}\int_0^\infty r^4\ \overline{u^2}\ f(r)\ dr = 0$$

Therefore

$$\int_0^\infty r^4\ \overline{u^2}\ f(r)\ dr = \Lambda = \text{constant} \tag{43}$$

Similarly we can find

$$\int_0^\infty r^6\ \frac{\overline{\omega^2}\,\lambda^2}{5}\ \tilde{f}(r)\ dr = 10\,\Lambda = \text{const} \tag{43a}$$

where $\overline{\omega^2} = \overline{\omega_1^2} = \overline{\omega_2^2} = \overline{\omega_3^2}$ in isotropic turbulence. $\tilde{f}(r)$ is the longitudinal vorticity correlation $\left(\tilde{f}(r) = \frac{\overline{\omega_1(x_1,x_2,x_3)\ \omega_1'(x_1+r,x_2,x_3)}}{\overline{u^2}}\right)$.

We see immediately that $\int_0^\infty r^4 f(r)\ dr$ has a dimension of (L^5). An appropriate length scale L^* can be introduced such that

$$L^{*5} = \int_0^\infty r^4\ f(r)\ dr$$

hence

$$\Lambda = \overline{u^2}\,L^{*5}$$

It seems at the present time that all possible invariants (except those trivial invariants arising from the equation of continuity) are not independent from Loitsiansky's invariant. We shall see later how the Loitsiansky invariant plays an important role in the behavior of turbulence decay and spectrum of turbulence expecially in connection with large eddies in a turbulent flow.

The ordinary measure for turbulence scale is the correlation length L defined by

$$L = \int_0^\infty f(r)\, dr$$

Assuming L is proportional to L^* , we then have*

$$\Lambda \sim \overline{u^2} L^5$$

Λ in this form has been experimentally measured (see Ref. 21). The results show that Λ is practically a constant

3. Kolmogoroff's Theory of Local Isotropy

The recent advances in the statistical theory of turbulence were largely due to Kolmogoroff's theory of local isotropy. In the following we shall give a qualitative description of Kolmogoroff's theory and in sections 4 and 5 we will show the use of the theory in determining the shape of turbulence spectrum and the rate of turbulent energy decay, especially for the cases of very large Reynolds number.

Turbulent motion is an irregular fluctuating motion. One may suppose superposing on the mean motion, there are "large eddies" which contain the bulk of turbulent energy. The "large eddies" are unstable and superposing on large eddies, there are smaller eddies and so on. The "smaller eddies" derive their energy from the large eddies by the inertia action. The viscous dissipation becomes more important when the eddy sizes become smaller, and finally all the turbulent energy will pass on to the molecular energy of heat by the action of viscosity. The above vaguely described process - better known as the

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*If the spectrum is a true optical spectrum, then L is actually proportional to L^* .

Cascade process - is one of the characteristic features of all turbulent flows. A concrete example is given in Appendix II.

One may suspect due to the random character of breaking down of large eddies that the smaller eddies are subjected to approximately space isotropic conditions* despite that the flow as a whole is not isotropic, e.g. turbulent shear flow. This is indeed the supposition of Kolmogoroff (Ref. 8) and experiments by Townsend at Cambridge (Refs. 23 and 24), Laufer at GALCIT (Ref. 25) and Corrsin at Johns Hopkins University (Ref. 26) seem to verify Kolmogoroff's idea. Furthermore within small time intervals it might be possible to consider this particular regime of turbulence as approximately stationary despite that the flow as a whole is markedly non-stationary. This is indeed also the concept of equilibrium turbulent spectrum which was developed by Heisenberg (Ref. 27).

It is clear that all above statements do not make sense, unless we have specifically defined what we mean by "smaller."

It is probably more proper to speak of high wave number components of turbulence instead of speaking of small eddies. The low wave number components will correspond intuitively to the large eddies. In this paper these names will be used synonymously. By high frequency components we mean those components whose length scales L are much smaller than the overall characteristic length L_0 of turbulence which we may define by dimensional reasoning as

*For a mathematical definition of local isotropy see Kolmogoroff's original paper (Ref. 8).

$$L_0 = \frac{u'^3}{\varepsilon} = \frac{\lambda}{15} R_\lambda \qquad R_\lambda = \frac{u'\lambda}{\nu} \tag{44}$$ *

where R_λ is called the Reynolds number of turbulence.

Kolmogoroff suggested that the characteristic length L_1 and velocity V_1 scales at high frequencies are determined solely by the kinematic viscosity ν and the energy dissipation ε, again by dimensional reasoning

$$L_1 = \nu^{\frac{3}{4}} \varepsilon^{-\frac{1}{4}} = \frac{\lambda}{(15)^{\frac{1}{4}}} R_\lambda^{-\frac{1}{2}} \qquad V_1 = \nu^{\frac{1}{4}} \varepsilon^{\frac{1}{4}} = (15)^{\frac{1}{4}} u' R_\lambda^{-\frac{1}{2}} \tag{45}$$ *

Then the existence of high wave number components in the Kolmogoroff sense requires

$$L_1 << L_0 \tag{46}$$

We see that the relations are consistent if $R_\lambda >> 1$.

Based on the physical picture discussed at the beginning of the section, Kolmogoroff proposed the first hypothesis of similarity. This hypothesis states that all the statistical properties for small eddies are uniquely determined by the quantities ν and ε. He further suggested that a "low"** wave number end of the small eddies

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* $L_0 u'$ is the type of turbulent diffusion coefficient suggested by Dr. von Karman in 1937 (Ref. 3).

$u' = \sqrt{\overline{u^2}}$

λ is the microscale of turbulence defined by Taylor (Ref. 7).

**In this range the characteristic length L is such that $L >> L_1$ but still $L << L_0$.

the effect of viscosity is irrelevant, because the turbulent energy transfer mechanism is independent of viscosity, and also in this range the viscous dissipation of energy is small compared to the energy transfer due to inertia effect. This is his second hypothesis of similarity. The consequences of both hypothesis and their applications will be discussed later.

4. Turbulence Decay

Problems on decay of turbulence were first investigated by G. I. Taylor in 1935. He predicted a linear law of decay of turbulence behind the grids* in a wind tunnel (Ref. 7). This law has been compared with experimental results with satisfactory accuracy. However, all these experiments show a similar behavior that at large distances from the grid, the turbulent energy decays faster than those predicted by the linear law. Of course there is no a priori reason to believe that linear law should apply to the entire range of decay process. Nevertheless the experimental facts do indicate the basic mechanisms for decay are not the same for both initial period of decay and the final period of decay. As already pointed out, the Karman-Howarth equation is the basic equation in the study of isotropic turbulence, let us now investigate what are the possible deductions from the Karman-Howarth equation on the decay of turbulence.

Using the power series expansions around $r=0$ for correlation functions f and h

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*It is understood that this law does not apply to regions too close to the grid where the shadow effect of the grid is important.

$$f(r) = 1 + \frac{r^2}{2!} f_0'' + \frac{r^4}{4!} f_0^{IV} + \ldots\ldots$$

$$h(r) = \frac{r^3}{3!} h_0''' + \frac{r^5}{5!} h_0^{V} \tag{47}$$

The Karman-Howarth equation (Eq. 40) becomes

$$\begin{aligned} \frac{d\overline{u^2}}{dt} &= 10\,\nu\,\overline{u^2}\,f_0'' \\ \frac{d\,\overline{u^2} f_0''}{dt} + \frac{14}{3}\left(\overline{u^2}\right)^{3/2} h_0''' &= \frac{14}{3}\,\nu\,\overline{u^2}\,f_0^{IV} \end{aligned} \tag{48}$$

$$\cdot\ \cdot\ \cdot\ \cdot$$

The first equation is then the equation for turbulent energy decay. The second equation represents the decay of vorticity.* Eq. (48) can be rewritten as follows:

$$\frac{d\overline{u^2}}{dt} = -10\,\nu\,\frac{\overline{u^2}}{\lambda^2} \tag{49a}$$

$$\frac{d\overline{\omega^2}}{dt} - 70\,h_0'''\left(\overline{u^2}\right)^{3/2} = -\frac{14}{3}\,\nu\,\overline{\omega^2}\,\lambda^2 f_0^{IV} \tag{49b}$$

Some remarks may be made in connection with the above equations. At first glance the turbulent energy decay depends only on the viscous terms, where the net effect of inertia terms in energy decay seem to be equal to zero. However, it is generally believed that the non-linear terms are responsible for the transfer of energy from low frequencies to high frequencies where the viscosity is mainly operative. This will be clear after we derive the equation for time variation of λ presently. Eq. (49b) can be interpreted as follows: The first term represents the time rate of change of vorticity, the second term represents the increase of vorticity due to the vortex stretching and

*The mean square of the fluctuating vorticity components $\overline{\omega^2} = \sum_{i=1}^{3} \overline{\omega_i^2}$ can be expressed in isotropic turbulence in terms of u' and λ by $\overline{\omega^2} = -15\,\overline{u^2}\,f_0'' = 15\,\overline{u^2}/\lambda^2$.

the term on the right hand side represents the viscous dissipation of vorticity.

The equation for time variation of the micro-scale can be easily obtained by combining Eqs. (49a) and (49b) and eliminating $\overline{u^2}$, the result is

$$\frac{d\lambda^2}{dt} = \nu \left(\frac{14}{3} G - \frac{7}{3} S R_\lambda - 10 \right) \tag{50}$$

where

$$G = \lambda^4 f_0^{iv}$$

$$R_\lambda = \frac{u' \lambda}{\nu} \quad *$$

$$S = 2 h_0''' \lambda^3$$

S can be expressed in a slightly different form by noting that in isotropic turbulence

$$\overline{\left(\frac{\partial u}{\partial x}\right)^3} = -2 \left(\overline{u'^2}\right)^{3/2} h_0'''$$

hence

$$S = - \frac{\overline{\left(\frac{\partial u}{\partial x}\right)^3}}{\left[\overline{\left(\frac{\partial u}{\partial x}\right)^2}\right]^{3/2}} \tag{51}$$

One obvious inference is this, if one thinks of above average as stochastic average, then $-S$ is the ordinary skewness factor for the probability distribution of $\frac{\partial u}{\partial x}$. It is immediately seen that is closely connected with the triple correlation or the non-linear terms in the Navier-Stokes equation. Physically S represents exactly

*Notations are due to Batchelor and Townsend (Ref. 28).

the deformation of vortex tubes. From all present day experiments (Refs. 29, 30, 31) we generally conclude that the probability distribution of fluctuating velocity is Gaussian (which implies the skewness is equal to zero). Unlike common stochastic processes, the distribution of the derivative is not Gaussian (Refs. 29, 32). However, one should not be disturbed by the fact, as the author points out presently that S is a real characteristic of turbulent flow representing the interaction between different Fourier components of the fluctuating velocity is a dissipative field. It is really amazing that the distribution of velocity is Gaussian.

Any further discussion on the decay will necessitate to dip more into the nature of the correlation function. First we express the Karman-Howarth equation in a non-dimensional form.

Let V and L be the reference velocity and length and introduce the non-dimensional units $\overline{u_0^2}$, r_0 and τ as follows

$$\overline{u^2} = V^2 \; u_0^2$$

$$r = L \; r_0$$

$$t = \frac{L}{V} \; \tau$$

After simplification the equation becomes

$$\frac{\partial (f \, \overline{u_0^2})}{\partial \tau} + 2(\overline{u_0^2})^{3/2}\left(\frac{\partial h}{\partial r_0} + \frac{4h}{r_0}\right) = \frac{2}{R}\left(\frac{\partial^2 (f \, u_0^2)}{\partial r_0^2} + \frac{4}{r_0}\frac{\partial (f \, u_0^2)}{\partial r_0}\right) \tag{52}$$

where $R = \frac{VL}{\nu}$ is a reference Reynolds number of turbulence. We assume in the following that R is of the same order of magnitude as R_λ .

4a. Cases of Large Reynolds Number

Define the wave number of a given turbulent component κ by $2\pi/L$ where L is the length scale of the given turbulent component. Then the criterion for the high wave number reads (refer to section 3)

$$\kappa \gg \frac{30\pi}{\lambda}\frac{1}{R_\lambda}$$

Hence as R_λ becomes larger and larger most of the turbulent components can be considered such that they obey Kolmogoroff's first similarity hypothesis. By dimensional reasoning we can express the correlation functions f and h as follows

$$\begin{aligned} \overline{u^2}\,(1 - f(r)) &= \nu^{\frac{1}{2}}\,\varepsilon^{\frac{1}{3}}\; F\left(\frac{r}{\nu^{\frac{3}{4}}\,\varepsilon^{-\frac{1}{4}}}\right) \\ (\overline{u^2})^{3/2}\, h(r) &= (\nu\,\varepsilon)^{\frac{3}{4}}\; H\left(\frac{r}{\nu^{\frac{3}{4}}\,\varepsilon^{-\frac{1}{4}}}\right) \end{aligned} \tag{53}$$

Furthermore the characteristic wave number of the high frequency components is

$$\kappa_1 = \frac{2\pi}{L_1} = \frac{2\pi\,(15)^{\frac{1}{4}}}{\lambda}\,(R_\lambda)^{\frac{1}{2}}$$

Hence as R_λ becomes sufficiently large then most turbulence components can be treated as low wave number components of the small eddies. Consequently Kolmogoroff's second similarity hypothesis applies, i.e. terms on the right hand sides of Eq. (53) should be independent of the viscosity. The only possible combinations are

$$\begin{aligned} \overline{u^2}\,(1 - f(r)) &= C_1\,(\varepsilon r)^{2/3} \\ (\overline{u^2})^{3/2}\, h(r) &= C_2\,(\varepsilon r) \end{aligned} \tag{54}$$

C_1 and C_2 are two constants the same for all fields of turbulence with large Reynolds numbers. The first of the above equations has been experimentally verified by Liepmann (Ref. 29), and Batchelor and Townsend (Ref. 33).

When the Reynolds number is large, the terms on the right hand side of Eq. (52) can be neglected.* The Karman-Howarth equation then becomes

$$\frac{\partial(\overline{u^2} f)}{\partial t} + 2(\overline{u^2})^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4h}{r}\right) = 0 \tag{55}$$

Substituting Eq. (54) into Eq. (55)

$$\frac{d\overline{u^2}}{dt} + 10\, C_2\, \varepsilon = 0$$

or

$$\frac{d\overline{u^2}}{dt} = -\frac{150\,\nu\,\overline{u^2}}{\lambda^2}\, C_2 \tag{56}$$

Comparing with Eq. (49a), we obtain a definite value for C_2 , i.e.

$$C_2 = \frac{1}{15} \tag{57}$$

A slightly different approach is the use of the idea of self-preservation of correlation coefficients. This idea was first introduced and used by von Karman in 1938. He arrived at a law of decay with an arbitrary exponent in the time variable. A closer investigation will show that this exponent is definite at least in the large Reynolds number case.

Assume in the turbulent flow with large Reynolds numbers that we

*To be sure this is a singular perturbation problem. However, it is different from the classical boundary layer theory, the problem here does not involve any definite boundary conditions.

have only one characteristic length which is proportional to λ the micro-scale. For simplicity, this characteristic length may be taken as λ itself. Since both extremely low wave number and extremely high wave number components do not play important roles in this picture, the above assumption does not seem unreasonable.

Assume

$$f = f(\xi)$$

$$h = h(\xi)$$

and

$$\xi = \frac{r}{\lambda}$$

Eq. (55) becomes

$$\lambda \frac{d\overline{u^2}}{dt} f - \overline{u^2} \frac{d\lambda}{dt} \xi f' + 2 (\overline{u^2})^{3/2} (h' + 4\xi^{-1} h) = 0 \tag{58}$$

where prime denotes the differentiation with respect to the argument ξ. The idea of self-preservation is compatible only if the above equation is a differential equation with one independent variable ξ. Hence the following equations must be true

$$\frac{\lambda}{(\overline{u^2})^{3/2}} \frac{d\overline{u^2}}{dt} = \alpha \tag{a}$$

$$\frac{1}{(\overline{u^2})^{1/2}} \frac{d\lambda}{dt} = \beta \tag{b}$$

α and β are two time independent constants. However, there does not seem any good reason to believe that α and β are entirely independent. In the following we shall suppose that they are not independent and hence Eqs. (a) and (b) are really one equation. The other appropriate equation in this connection will be Eq. (49a)

$$\frac{d\overline{u^2}}{dt} = -10\nu \frac{\overline{u^2}}{\lambda^2} \tag{c}$$

Eqs. (a) and (c) imply that

$$\lambda = \frac{-10\nu}{\alpha(\overline{u^2})^{1/2}} \tag{d}$$

Substituting (d) into (b) we have

$$\frac{10\nu}{\alpha(\overline{u^2})^{3/2}} \frac{d(\overline{u^2})^{1/2}}{dt} = \beta \tag{e}$$

Solving for $\overline{u^2}$ in (e)

$$\frac{1}{\overline{u^2}} = \frac{1}{\nu}\left(\frac{-2\beta\alpha}{10} t + c\right) \tag{f}$$

The integration constant C appearing in Eq. (f) can be chosen as zero by a proper shift of time axis. Since $\overline{u^2}$ is necessarily a positive quantity, α and β must be of opposite sign

Finally, we have the decay law

$$\frac{1}{\overline{u^2}} = -\frac{\beta\alpha}{5\nu} t \tag{59a}$$

Using Eqs. (c) and (d)

$$\lambda^2 = 10\nu t \tag{59b}$$

and

$$\alpha = -2\beta \tag{59c}$$

Eq. (58) can be rewritten as

$$2\beta f + \beta\xi f' - 2(h' + 4\xi^{-1}h) = 0 \tag{60}$$

Eqs. (59a) and (59b) have been experimentally verified by Batchelor and Townsend (Ref. 33).

From Eq. (d) we have

$$\beta = \frac{5}{R_\lambda} \tag{61}$$

Hence in the present theory, we require R_λ as a constant during the decay process.

Coming back to Eq. (50) and using Eq. (59b), we have

$$\frac{14}{3} G - \frac{7}{3} S R_\lambda = 20 \tag{62}$$

According to Kolmogoroff's theory, all dimensionless parameters specified solely by smaller eddies should be absolute constants. In the case of very large Reynolds number R_λ (which is the case that we discuss here), most of the eddies can be considered as "smaller" (cf. section 3). Hence G and S should be approximately constant. Again this requires that R_λ is a constant during a decay process. The constancy of the quantities G and S in the initial period of decay has been approximately verified by Batchelor and Townsend (Ref. 34) and by Liepmann, Laufer and Liepmann (Ref. 29). The present theory is consistent with Kolmogoroff's theory.

It is noted that the above theory gives the asymptotic law of decay when the Reynolds number of turbulence is very large. In all present day experiments R_λ shows a tendency to decrease after a certain time interval and eventually R_λ becomes so low that to neglect the viscous terms in the Karman-Howarth equation is no longer justified.

There remains one arbitrariness to be settled, that is the arbitrary constant which appears in Eq. (f). A closer investigation shows that c can only depend on the mean velocity $\mathcal{U}$ and the mesh size M in a turbulent flow behind a grid in a wind tunnel, i.e.

$$C = \varkappa_1 \frac{M}{U}$$

$\varkappa_1$ is a proportional constant. Similarly Eq. (59b) becomes

$$\lambda^2 = 10\nu\left(t - \varkappa\frac{M}{U}\right) \tag{63}$$

where $\varkappa$ is an absolute constant for very large Reynolds numbers. If we may replace t by $\frac{x}{U}$ then

$$\lambda^2 = 10\frac{\nu M}{U}\left(\frac{x}{M} - \varkappa\right) \tag{64}$$

Hence we have, at a fixed mean velocity

$$\frac{\lambda_1^2}{\lambda_2^2} = \frac{M_1}{M_2} \tag{64a}$$

where λ_1 is the micro-scale when the grid of mesh size M_1 is used and λ_2 is the micro-scale when the mesh size M_2 is used. The λ's are measured at the same $\frac{x}{M}$. For a fixed mesh size

$$\frac{\lambda_1^2}{\lambda_2^2} = \frac{U_2}{U_1} \tag{64b}$$

again λ's are measured at the same $\frac{x}{M}$ in respective cases. λ_1 and λ_2 are the micro-scales corresponding to mean velocities U_1 and U_2. Similar equations for $\overline{u^2}$ can be obtained in a similar fashion.

4b. Cases of Very Small Reynolds Number

This part of turbulence is easily understood and is also least interesting. Strictly speaking the name turbulence is not very appropriate in the sense that the turbulence is essentially a non-linear phenomenon. In this range the non-linear effect is believed unimportant, i.e. we can neglect the triple correlation terms in the Karman-Howarth equation, hence

$$\frac{\partial(\overline{u^2} f)}{\partial t} = 2\nu\left(\frac{\partial^2(\overline{u^2} f)}{\partial r^2} + \frac{4}{r}\frac{\partial(\overline{u^2} f)}{\partial r}\right) \tag{65}$$

This is exactly the equation for diffusion of heat in five-dimensional spherical symmetry cases. The fundamental solution is

$$\frac{1}{(8\pi\nu t)^{5/2}} e^{-\frac{r^2}{8\nu t}}$$

Denote $\overline{u^2} f$ at time $t_0 = 0$ by $\overline{u_0^2}\, f(r, t_0)$ and suppose this is known then we can write down immediately $u^2 f$ at any later time.

$$\overline{u^2} f(r,t) = \frac{1}{(8\pi\nu t)^{5/2}} \int \overline{u_0^2}\, f(\vec{r}' - \vec{r}, t_0)\, e^{-\frac{r'^2}{8\nu t}}\, d\tau(\vec{r}') \tag{66}$$

$d\tau(\vec{r}')$ is the element of volume in the five-dimensional space and the integration is carried out over the full space.

If we are only interested in the decay of energy, we may simply put $r = 0$ in the above equation, we have

$$\overline{u^2} = \frac{1}{48\sqrt{2\pi}\,(\nu t)^{5/2}} \int_0^\infty \overline{u_0^2}\, f(r', t_0)\, e^{-\frac{r'^2}{8\nu t}}\, r'^4\, dr' \tag{67}$$

Assume $\int_0^\infty r^6 f(r)\, dr$ exists, then

$$\overline{u^2} = \frac{\Lambda}{48\sqrt{2\pi}\,(\nu t)^{5/2}}\left(1 + O\left(\frac{1}{t}\right)\right) \tag{67a}$$

where O is the mathematical order symbol and Λ is the Loitsiansky invariant. Using Eq. (49a) we have

$$\lambda^2 \cong 4\nu t \tag{68}$$

Also

$$L^* \cong 2.6\,(\nu t)^{\frac{1}{2}} \tag{69}$$

The solution for this problem has been obtained by Karman and Howarth (Ref. 10) in a different form by using the concept of self-preservation. In this range there is likely only one scale of length determined by the viscosity and hence the correlation function is completely similar. The result is

$$f(\chi) = 2^{15/4} \chi^{-5/2} e^{-\frac{\chi^2}{16}} M_{(10\alpha - \frac{5}{4}), \frac{3}{4}} \left(\frac{\chi^2}{8} \right) \tag{70}$$

and

$$\chi = \frac{r}{\sqrt{\nu t}}$$

A comparison of the derived decay law shows that α must be equal to $\frac{1}{4}$. The solution (70) is then much simplified. We have

$$f(r) = e^{-\frac{r^2}{8\nu t}} \tag{71}$$

This is a Gaussian correlation with the dispersion $4\nu t$. From this result, we can again derive the formulas for λ and G . By Taylor series expansion

$$f(r) = 1 - \frac{r^2}{8\nu t} + \frac{1}{2!} \frac{r^4}{64\nu^2 t^2} + \cdots$$

then we have

$$\begin{aligned} \lambda^2 &= 4\nu t \\ G &= 3 \end{aligned} \tag{72}$$

As a check, recall that we have an equation for the diffusion of micro-scale

$$\frac{d\lambda^2}{dt} = \nu \left(\frac{14}{3} G - \frac{7}{3} S R_\lambda - 10 \right) \tag{73}$$

if $R_\lambda << 1$, then

$$4 \cong \left(\frac{14}{3} G - 10 \right) \qquad \text{or} \qquad G \cong 3$$

It should be noted that in all actual cases $G \geq 3$.

4c. Case of Medium Reynolds Number

In the previous discussions on very large and very small Reynolds number cases, we have seen that the decay laws in the two are quite different. In the one case $\overline{u^2}$ is proportional to t^{-1} and in the other case $\overline{u^2}$ is proportional to $t^{-5/2}$. If we imagine the turbulence starts to decay at very large Reynolds numbers, then the present case just represents the transition period between the previous two. Consequently, both inertia forces and viscous forces are effective and of the same order of magnitude and hence neither terms in the Karman-Howarth equation can be neglected. We know the behaviors of large eddies (non-viscous) and small eddies (viscous) are not the same. The concept of self-preservation does not lead to any definite result in the present case. It is also difficult to estimate the time duration of the transition period. There does seem that at least two characteristic lengths exist which eventually will coincide with each other in the last stage of the decay process.

A rough approximation can be made by using a single* characteristic length which is related to the Loitsiansky invariant and dimensional analysis. The following decay laws were proposed by Kolmogoroff

*This is indeed an over simplification which can hardly represent the real physical situation, see also comments by Batchelor and Townsend (Ref. 38).

(Ref. 35) and Frenkiel (Refs. 36, 37)

$$\overline{u^2} = 1.67\,\Lambda^{\frac{2}{7}}\,(\gamma t)^{-\frac{10}{7}}$$
$$L^* = .902\,\Lambda^{\frac{1}{7}}\,(\gamma t)^{\frac{2}{7}} \qquad (74)$$

where γ is a constant.

As a final remark the Loitsiansky invariant may be correct in an infinite turbulent field, however, in actual experimental set-ups the invariant need not be true due to the limited size of the apparatus.

5. Turbulence Spectrum

G. I. Taylor was the first one who introduced the concept of spectrum into turbulent flow study. His spectrum E (now better known as Taylor spectrum) can be expressed as the one-dimensional Fourier transform of f -correlation (comparing section 3, Part I), i.e.

$$E(k_1) = \frac{2}{\pi}\,\overline{u^2}\int_0^\infty f(r)\cos(k_1 r)\,dr \qquad (75)$$

By Fourier inversion theorem

$$\overline{u^2}\,f(r) = \int_0^\infty E(k_1)\cos(k_1 r)\,dk_1 \qquad (76)$$

$E(k_1)\,dk_1$ may be interpreted as the contribution to $\overline{u^2}$ from those turbulence components whose wave numbers in the u -direction lie between k_1 and $k_1 + dk_1$. Taylor's spectrum can be experimentally measured by using ordinary hot-wire techniques and passing the output through a harmonic analyzer. Since turbulence is essentially a three-dimensional phenomenon, it is obvious that Taylor's spectrum will not be suitable for a theoretical investigation.

Heisenberg in 1948 first studied the three-dimensional turbulence spectrum and gave very instructive formulation (Ref. 27). We have seen in section 3 Part I that there exists a definite relation between the correlation function and the spectrum function. The formulation can then be made straight forward if we start from the Karman-Howarth equation.

First, we shall establish the relationship between correlation and spectrum in the three-dimensional case. From analogy in the one-dimensional case, we take formally the three-dimensional Fourier transform of Karman's correlation tensor.

Definition of Φ_{ij}

$$\Phi_{ij}(\vec{\kappa}) = \frac{1}{8\pi^3}\iiint R_{ij}^{*}(\vec{r})\, e^{-i(\kappa r)}\, d\tau(\vec{r}) \qquad (77)^{**}$$

where

$R_{ij}^{*} = \overline{u_i u_j'}$

u_i the i-th component of fluctuating velocity at $P(\vec{R})$

u_j the j-th component of fluctuating velocity at $P'(\vec{R}')$

$\vec{r} = \vec{R}' - \vec{R}$

$\vec{\kappa}$ denotes the wave number vector

(κr) the inner product of two vectors $\vec{\kappa}$ and $\vec{r}$,

$$(\kappa r) = k_1 x_1 + k_2 x_2 + k_3 x_3$$

$d\tau(\vec{r})$ element volume in the physical space

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*The i appearing in the exponent should not be confused with the subscript i.

**This was first introduced by Kampe de Feriet (Ref. 39) for homogeneous turbulence and also by Batchelor (Ref. 40).

Φ_{ij} exists if $R_{ij}(\vec{r})$ is absolutely integrable, i.e.

$$\iiint_{-\infty}^{\infty} | R_{ij}(\vec{r}) | \, d\tau(\vec{r}) < \infty$$

Using the inversion formula

$$R_{ij}(\vec{r}) = \iiint_{-\infty}^{\infty} \Phi_{ij}(\vec{\kappa}) \, e^{i(\kappa r)} \, d\tau(\vec{\kappa}) \tag{78}$$

Putting $\vec{r} = 0$ in Eq. (78) we have

$$\overline{u_i u_j} = \iiint_{-\infty}^{\infty} \Phi_{ij}(\vec{\kappa}) \, d\tau(\vec{\kappa}) \tag{79}$$

$\Phi_{ij}(\vec{\kappa}) \, d\tau(\vec{\kappa})$ can then be interpreted as the contribution to the Reynolds stress* $\overline{u_i u_j}$ from those Fourier components having wave numbers lying between κ_i and $\kappa_i + d\kappa_i$ $(i = 1, 2, 3)$. In particular, $\frac{1}{2} \sum_{i=1}^{3} \Phi_{ii}$ is the spectrum density of the total energy of turbulence.

It can easily be verified that the equation of continuity $\nabla \cdot \vec{u} = 0$ can be expressed in terms of Φ_{ij} in the wave number space as follows

$$\begin{aligned} \kappa_i \, \Phi_{ij} &= 0 \\ \kappa_j \, \Phi_{ij} &= 0 \end{aligned} \tag{80}$$

Eq. (80) expresses the orthogonality relation between spectrum components and the wave number vector.

So far in the present discussions we have not restricted ourselves to the consideration of isotropic turbulence. In isotropic

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*For simplicity the density ρ in the present discussion is assumed to be equal to 1 .

turbulence Φ_{ij} is not a very interesting quantity, instead we ask ourselves what is the amount of turbulent energy contributed from a spherical shell in wave space, i.e. those Fourier components having their norm of the wave vector lying between κ and $\kappa + d\kappa$. This question can be answered immediately.

Rewriting Eq. (79)

$$\overline{u_i u_j} = \int_0^\infty d\kappa \left[\iint \Phi_{ij}(\vec{\kappa})\, d\Sigma(\vec{\kappa}) \right]$$

$d\Sigma(\kappa)$ is an element surface on a sphere of radius κ in wave space or

$$\frac{1}{2} \sum_i \overline{u_i^2} = \int_0^\infty d\kappa \left[\iint \frac{1}{2} \sum_i \Phi_{ii}(\vec{\kappa})\, d\Sigma(\vec{\kappa}) \right] \tag{81}$$

The quantity we are searching now is the spectrum $\mathcal{F}$ formulated by Heisenberg and we shall frequently refer to it as Heisenberg's spectrum to distinguish from Taylor's spectrum. From Eq. (81), it is obvious that

$$\mathcal{F} = \frac{1}{2} \sum_{i=1}^{3} \iint \Phi_{ii}(\vec{\kappa})\, d\Sigma(\vec{\kappa}) \tag{82}$$

Substituting Φ_{ii} in Eq. (82) by using Eq. (77)

$$\mathcal{F} = \frac{1}{\pi} \int_0^\infty Tr.\, kr \sin kr\, dr \tag{83}$$

When expressed in terms of f correlation alone

$$\mathcal{F} = \frac{\overline{u^2}}{\pi} \int_0^\infty f \left(kr \sin kr - k^2 r^2 \cos kr \right) dr \tag{84}$$

Comparing Eqs. (75) and (84) we obtain a relation between Heisenberg's and Taylor's spectrum

$$\mathcal{F} = \frac{1}{2}\left[k^2 \frac{\partial^2 E(k)}{\partial k^2} - k \frac{\partial E(k)}{\partial k} \right] \qquad (85)^*$$

Heisenberg has obtained an integral relation in a different way (Ref. 27). One essential difference should be noted, that is that Heisenberg's spectrum is equal to zero when k is zero, while Taylor's spectrum does not equal zero when k_1 is zero. The reason is fairly obvious. In the Heisenberg case this simply means the Fourier component with infinite wave length does not exist, while in Taylor's case the total energy contribution from those Fourier components, which have in one direction wave numbers equal to zero, is by no means zero.** In fact

$$E(0) = \frac{2\overline{u^2}}{\pi} L$$

where $L = \int_0^\infty f(r)\,dr$, L is the correlation length.

Having thus clarified the definitions of different spectrum functions, we are now in a position to investigate deeper into the nature of the spectrum. An equation for the time variation of spectrum can be formally obtained by taking some appropriate transform of the

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*Heisenberg's integral relation is of the following form

$$E(k_1) = \int_{k_1}^{\infty} \frac{dk}{k^3} (k^2 - k_1^2)\, \mathcal{F}(k)\, dk \qquad (A)$$

It was pointed out by von Karman (Ref. 41) that this relation represents the geometrical fact that all oblique waves with wave length $2\pi/k < 2\pi/k_1$ necessarily contribute, in the one-dimensional analysis, to waves with length $2\pi/k_1$. Also, from Eq. (A) we know $E(k_1)$ has a maximum at $k_1 = 0$.

**See Appendix III.

Karman-Howarth equation. Multiplying Eq. (42) by $\frac{1}{\pi} kr \sin kr$ and integrating with respect to r from zero to infinity, after some integration by parts and using Eq. (83) we have (compare also with Ref. 42)

$$\frac{\partial \mathcal{F}}{\partial t} + W_\kappa = -2\nu\kappa^2 \mathcal{F} \tag{86a}$$

$$W_\kappa = \frac{2(\overline{u^2})^{3/2}}{\pi}\int_0^\infty Q(r)(kr\cos kr - \sin kr)\, k\, dr \tag{86b}$$

If we integrate the above equation from 0 to k with respect to κ we have

$$\left(\frac{\partial}{\partial t}\int_0^\kappa \mathcal{F}(\kappa')\, d\kappa'\right) + \int_0^\kappa W_\kappa(\kappa')\, d\kappa' = -\int_0^\kappa 2\nu\kappa'^2 \mathcal{F}(\kappa')\, d\kappa' \tag{87}$$

and

$$\int_0^\kappa W_\kappa\, d\kappa = \frac{2(\overline{u^2})^{3/2}}{\pi}\int_0^\infty \left[3kr\cos kr + k^2r^2\sin kr - 3\sin kr\right]\frac{Q(r)}{r^2}\, dr$$

Eq. (87) can be interpreted as follows: $\int_0^\kappa \mathcal{F}(\kappa')\, d\kappa'$ represents the energy content for all Fourier components with wave numbers* less than κ. Hence Eq. (87) states that the time rate of change of the energy content between 0 and κ is equal to the rate of viscous dissipation in that region plus the rate of amount of energy transferred to those Fourier components having wave numbers higher than κ. The transfer term is represented by $\int_0^\kappa W_\kappa\, d\kappa$. We have seen the difficulty in the Karman-Howarth equation is the triple correlation. Here again the difficulty in the present spectrum approach is the transfer term W_κ and indeed the triple correlations and transfer

*More appropriately the norm of the wave number vector.

terms are related through Eq. (86b). However, we have made some progress so long as the spectrum does provide a more intuitive physical picture which will allow us to make more reasonable assumptions.

Some information about the spectrum can be obtained immediately without the detailed knowledge of the transfer function. If we expand formally Eqs. (83) and (86) in the Taylor series in κ we have

$$\mathcal{F} = \frac{k^2}{\pi}\int_0^\infty T_{r,}\, r^2\,dr - \frac{k^4}{3!\,\pi}\int_0^\infty T_{r,}\, r^4\,dr + \cdots$$

$$W_\kappa = \frac{2(\overline{u^2})^{3/2}}{\pi}\left[-\frac{k^4}{6}\int_0^\infty Q\, r^3\,dr + \frac{k^6}{30}\int_0^\infty Q\, r^5\,dr + \cdots\right] \tag{88}$$

The Taylor series expansion can be continued as long as the respective integrals appearing as the coefficients converge. On account of the equation of continuity the following results can be easily verified.

$$\int_0^\infty T_{r,}\, r^2\,dr = 0$$

$$\int_0^\infty Q\, r^3\,dr = 0$$

$$\int_0^\infty T_{r,}\, r^4\,dr = -2\Lambda$$

Then for small values of κ

$$\mathcal{F} = \frac{k^4}{3\pi}\Lambda + O(k^6)$$

$$W_\kappa = \frac{k^6}{15\pi}(\overline{u^2})^{3/2}\,\lambda^6 + O(k^8) \tag{89}$$

where

$$\lambda^6 = \int_0^\infty Q\, r^5\,dr$$

Substituting in Eq. (86a) we have

$$\frac{d\Lambda}{dt} = 0 \tag{90}$$

This gives a different derivation of the Loitsiansky invariant. The result that we just obtained states that the spectrum at very low wave numbers is proportional to the 4th power of the wave number k, if the convergence conditions for Loitsiansky's invariant are satisfied, and furthermore the proportionality remains constant throughout the decay process. However, apart from atmospheric or interstellar turbulence the significance of this result is doubtful due to the limited extension of the geometrical configuration in the particular cases involved.

The next simple case is the case of very large Reynolds numbers to which the Kolmogoroff theory applies. Then for high wave numbers the spectrum should be uniquely determined by ν the viscosity and ε the total energy dissipation. By dimensional reasoning the characteristic length L and the characteristic velocity V are

$$\begin{aligned} V &= \nu^{\frac{1}{4}} \varepsilon^{\frac{1}{4}} \\ L &= \nu^{\frac{3}{4}} \varepsilon^{-\frac{1}{4}} \end{aligned} \tag{91}$$

Again from dimensional reasoning the spectrum $\mathcal{F}$ is of the form

$$\begin{aligned} \mathcal{F} &= V^2 L \quad F(\kappa L) \\ &= \nu^{\frac{5}{4}} \varepsilon^{\frac{1}{4}} \quad F(\kappa \nu^{\frac{3}{4}} \varepsilon^{-\frac{1}{4}}) \end{aligned} \tag{92}$$

By Kolmogoroff's second similarity hypothesis, the spectrum at the low wave number region of the high wave number region should be independent of viscosity. Then we arrive at the famous -5/3 power law, i.e.

$$\mathcal{F} = C\,\varepsilon^{\frac{2}{3}}\,\kappa^{-\frac{5}{3}} \tag{93}$$

The region of the applicability of the above power law can be extended to very low and very high wave numbers provided the Reynolds number of turbulence is large enough (see section 3).

In order that the result (Eq. (93)) can be compared with the experimentally measured power spectrum W, some modifications will have to be introduced. First we must have a characteristic frequency ω_0 and this is easily done if we introduce the mean velocity U of the turbulent flow behind a grid. For instance we may define $\omega_0 = \frac{U}{L}$ where L is defined in Eq. (91). Secondly we note that Eq. (85) is homogeneous in k. We naturally expect that any power laws derived for $\mathcal{F}$ will in general be preserved for E. So the argument we just went through for $\mathcal{F}$ can be directly applied to W. Hence we have

$$W = \nu^{\frac{5}{4}}\,\varepsilon^{\frac{1}{4}}\,U^{-1}\,\mathcal{W}\left(\omega\nu^{\frac{3}{4}}\varepsilon^{-\frac{1}{4}}U^{-1}\right) \tag{94}$$

For the part of the spectrum where the viscosity is irrelevant, we have

$$W = C_1\,\varepsilon^{\frac{2}{3}}\,U\,\omega^{-5/3} \tag{95}$$

C_1 is a constant for all turbulent flow behind a grid.

Before going into further detail some specific assumptions on the transfer term W_K will have to be made.

First we postulate the transition function $\Theta(\kappa \to \kappa')$ exists, i.e. the function which represents the amount of energy transferred per unit time from the Fourier components having their wave numbers between κ and $\kappa + d\kappa$ to those Fourier components having their wave numbers between κ' and $\kappa' + d\kappa'$. In general Θ will be a function of time. From definition

$$\Theta(\kappa \rightarrow \kappa') = -\Theta(\kappa' \rightarrow \kappa) \qquad (96)$$

and W_κ can be expressed as

$$W_\kappa = \int_0^\kappa \Theta(\kappa' \rightarrow \kappa)\, d\kappa' - \int_\kappa^\infty \Theta(\kappa \rightarrow \kappa')\, d\kappa' \qquad (97)$$

The first term on the right hand side represents the energy transferred into κ and $\kappa + d\kappa$ from the components having their wave numbers lower than κ, and the second term represents the energy transferred out to higher wave numbers. It follows from Eq. (97)

$$\int_0^\infty W_\kappa \, d\kappa = 0 \qquad (98)$$

Eq. (98) is of course a true identity, it expresses the fact that the net transfer effect over complete wave number range is zero.

Second we assume the transition function shall be a function of four variables $\mathcal{F}(\kappa)$, $\mathcal{F}(\kappa')$, κ and κ'. This assumption probably cannot be exact. We shall see in the third part of the present paper the transfer term can be expressed as a convolution integral of velocity spectrum. This means that the values $\mathcal{F}(|\kappa \pm \kappa'|)$ will likely appear in the argument of the Θ function. However, we shall accept this as a first approximation.

Third we assume Θ can be expressed as

$$\Theta = -c\, \mathcal{F}^{\alpha}(\kappa)\, \mathcal{F}^{\alpha'}(\kappa')\, \kappa^{\beta}\, \kappa'^{\beta'} \qquad (99)\ ^*$$

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*This assumption was first introduced by von Karman (Ref. 43).

From dimensional argument $\alpha + \alpha' = \frac{3}{2}$, $\beta + \beta' = \frac{1}{2}$

5a. Heisenberg's Approach - The Concept of Equilibrium Spectrum

Heisenberg in 1946 introduced a very interesting assumption on the transfer function $\int_0^\kappa W_\kappa \, d\kappa$. The idea was based on the concept of eddy viscosity. He assumed that the action of small eddies on large eddies is equivalent to an additional variable viscosity coefficient. By dimensional analysis we have

$$\int_0^\kappa W_\kappa \, d\kappa = \varkappa \int_\kappa^\infty \sqrt{\frac{\mathcal{F}(\kappa'')}{\kappa''^3}} \, d\kappa'' \int_0^\kappa \mathcal{F}(\kappa') \, 2\kappa'^2 \, d\kappa'$$

Hence the spectrum equation becomes

$$S_k = \frac{\partial}{\partial t} \int_0^k \mathcal{F}(k', t) \, dk' = -\left(\nu + \varkappa \int_\kappa^\infty \sqrt{\frac{\mathcal{F}(\kappa'')}{\kappa''^3}} \, d\kappa'' \right) \int_0^\kappa 2\kappa'^2 \mathcal{F}(\kappa') \, d\kappa' \qquad (100)$$

where $\varkappa$ is a constant of order 1.*

It is not difficult to verify that this corresponds to our case $\alpha = \frac{1}{2}$, $\beta = -\frac{3}{2}$. Heisenberg then proceeded using Weizsacker's result (Ref. 45) that for large values of κ , S_k must be independent of κ because most part of the energy is contained in long wave length regions, i.e. small κ region. This idea leads to the concept of the equilibrium spectrum. A more careful examination will show that the Heisenberg equilibrium range corresponds essentially to the range we defined in section 3 where we may apply Kolmogoroff's similarity hypothesis. Rewritting Eq. (100) and for large κ , S_κ approaches

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*Lee has recently made an estimate of $\varkappa$ (Ref. 44).

to a constant. Let us denote the constant by $(-\varepsilon)$ which is of course the total energy dissipation

$$\varepsilon = \left(\nu + \varkappa \int_{\kappa}^{\infty} \sqrt{\frac{\mathcal{F}(\kappa'')}{\kappa''^3}}\, d\kappa''\right) \int_0^{\kappa} 2\kappa'^2 \mathcal{F}(\kappa')\, d\kappa' \tag{101}$$

Indeed in the low wave number part of the equilibrium range in which the viscous effects can be neglected, the solution is obtained immediately

$$\mathcal{F} = \left(\frac{8\varepsilon}{9\varkappa}\right)^{2/3} \kappa^{-5/3} \tag{102}$$

which is the Kolmogoroff result (Eq. (93)). For very high wave numbers Heisenberg gave an approximate solution. However, the exact solution can be obtained easily (see Appendix IV). The solution is

$$\mathcal{F}(\kappa) = \frac{\left(\frac{8\varepsilon}{9\varkappa}\right)^{2/3} \kappa^{-5/3}}{\left[1 + \left(\frac{\kappa}{\kappa_c}\right)^4\right]^{4/3}} \tag{103}$$ *

$$\kappa_c = \left(\frac{3\varkappa^2 \varepsilon}{8\nu^3}\right)^{1/4}$$

We see for $\frac{\kappa}{\kappa_c} \gg 1$, then

$$\mathcal{F}(\kappa) \approx \left(\frac{\varkappa \varepsilon}{2\nu^2}\right)^2 \kappa^{-7} \tag{104}$$

Since the Heisenberg equilibrium range is essentially Kolmogoroff's similarity range Eqs. (103) and (104) should check with the experimental measured spectrum even in a shear flow, if the Heisenberg assumption is correct. The measurements by Laufer (Ref. 25) in a

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*This was independently worked out and at the same time published by Chandrasekhar (Refs. 46, 47).

two-dimensional channel seem to verify this result satisfactorily.

Obukhov (Ref. 48) in 1941 made a different assumption, which in the present notation

$$\int_0^\kappa W_\kappa \, d\kappa = 2\kappa_1 \left[\int_0^\kappa \kappa^2 \mathcal{F}(\kappa) \, d\kappa \right]^{1/2} \int_\kappa^\infty \mathcal{F}(\kappa) \, d\kappa \qquad (105)$$

We may interpret this assumption as follows: In a two-dimensional shear motion the turbulent energy transferred from the mean flow can be expressed as

$$\iint \rho \, \overline{u'v'} \, \frac{d\mathcal{U}}{dy} \, dx \, dy$$

$\rho \overline{u'v'}$ is the Reynolds shear stress and $\frac{d\mathcal{U}}{dy}$ is the mean velocity gradient. An obvious analogy can be drawn at once. $\left[\int_0^\kappa \kappa^2 \mathcal{F}(\kappa) \, d\kappa \right]^{1/2}$ represents the mean velocity gradient of the macro-components* of turbulence (which corresponds to $\frac{d\mathcal{U}}{dy}$) and $\int_\kappa^\infty \mathcal{F}(\kappa) \, d\kappa$ represents the shear stress acted on the macro-components of turbulence by the micro-components* of turbulence. However, the analogy is not complete, besides it is contradictory to the theory of local isotropy. Because in this analogy we have tacitly assumed that the shear spectrum and energy spectrum are similar even at very high wave numbers. This fact obviously precluded the possibility of local isotropy. Direct measurements of shear spectrums have been made by Corrsin (Ref. 26) and Laufer (Ref. 25). Their results show that the shear spectrums fall off to zero much more rapidly than the corresponding

*Macro and micro here simply denote the wave number is less or greater than κ .

energy spectrums. Consequently, Eq. (105) may not be a good assumption. Indeed if we try to use Heisenberg's concept of equilibrium spectrum and Obukhoff's assumptions, the results are indefinite at very high wave numbers. Although Obukhoff's assumption does lead to the -5/3 law for the non-viscous range of the spectrum, this only means the assumption is dimensionally correct (see Appendix IV).

At present Heisenberg's assumption is alone able to predict the spectrum distribution at extremely high frequencies. It seems instructive to consider what the spectrum distribution is at low wave numbers by using the same assumption on the transfer function. It is fairly obvious that the low wave number regions cannot be included in the equilibrium range especially when the turbulence is decaying because the low wave number range of the spectrum contains most of the turbulent energy. Consequently the term $\frac{\partial \mathcal{F}}{\partial t}$ cannot be neglected. The complete spectrum equation becomes

$$\frac{\partial}{\partial t}\int_0^{\kappa} \mathcal{F}(\kappa', t)\, d\kappa' = -\left(\nu + \varkappa \int_{\kappa}^{\infty} \sqrt{\frac{\mathcal{F}(\kappa'')}{\kappa''^3}}\, d\kappa''\right) \int_0^{\kappa} \mathcal{F}(\kappa')\, 2\kappa'^2\, d\kappa' \tag{106}$$

Assumption on the transfer function alone is not enough for the present investigation. We now introduce a second important assumption that the turbulence at all stages is similar. The concept of similarity was first introduced by von Karman. This concept has been extensively used in all fields of hydrodynamics. It means the following: "The physical mechanism underlying the process is the same for all time, but the scales are changing. The scales may be length scale, time scale etc." With this additional assumption we now assume (as in Ref. 49)

$$\mathcal{F} = \frac{(VL)^2}{\sqrt{\nu t}} \widetilde{\mathcal{F}}(k\sqrt{\nu t}) \tag{107}$$ *

V and L are characteristic velocity and length respectively. Let

$$k\sqrt{\nu t} = x$$

and

$$R = \frac{4\varkappa V L}{\nu}$$

R is the Reynolds number of turbulence in this case. Eq. (106) now becomes

$$2\int_0^x \widetilde{\mathcal{F}}\, dx - x\widetilde{\mathcal{F}}(x) = \left\{4 + R\int_x^\infty \sqrt{\frac{\widetilde{\mathcal{F}}}{x^3}}\, dx\right\} \int_0^x \widetilde{\mathcal{F}} x^2\, dx \tag{108}$$

Now, introducing the viscous dissipation $z = \int_0^x \widetilde{\mathcal{F}} x^2\, dx$ as the independent variable and $W = \widetilde{\mathcal{F}} x^3$ as the dependent variable, the above integral equation can be transformed into the following non-linear ordinary differential equation

$$W^{\frac{3}{2}} \frac{d^2 W}{dz^2} + \left[\frac{Rz}{2} - 2W^{\frac{1}{2}}\right]\frac{dW}{dz} + \left[2zR + 8W^{\frac{1}{2}} - 2RW\right] = 0 \tag{109}$$

Discussions on Eq. (109)

The range of independent variable z is finite $0 \le z \le z_0$ $z_0 = \int_0^\infty \widetilde{\mathcal{F}} x^2\, dx$ except when $R = \infty$, then $z_0 = \infty$ because there is infinite dissipation.

(a) For infinite Reynolds number we have

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*This assumption implies a linear law of decay of turbulent energy, i.e. $\overline{u^2} \propto 1/t$, and consequently cannot be valid when the Reynolds number is very low.

$$\frac{z}{2}\frac{dw}{dz} + 2z - 2w = 0 \tag{109a}$$

This equation can be easily solved, and the result is $\tilde{\mathcal{F}} \propto x^{-5/3}$ as expected.

(b) For very small Reynolds number; in the limit we put $R = 0$ then we have

$$w\frac{d^2w}{dz^2} - 2\frac{dw}{dz} + 8 = 0 \tag{109b}$$

This equation can also be solved and the result is $\tilde{\mathcal{F}} \propto x e^{-2x^2}$. Compare the result with Eq. (89). We should expect that when x is small $\tilde{\mathcal{F}} \propto x^4$, due to the equation of continuity. However, this* is not the case here. This fallacy hints that the extremely low wave number portion of the spectrum cannot be included in the similarity range in the sense of Eq. (107).

(c) For any finite Reynolds number; w is positive for $0 < z < z_0$, $w = 0$ when $z = 0$ also $w \to 0$ when $z \to z_0$. For very small w, and assuming that in the actual solution $\frac{d^2w}{dz^2}$ and $\frac{dw}{dz}$ are bounded, then Eq. (109) can be simplified to the following form

$$\frac{Rz}{2}\frac{dw}{dz} + 2zR = 0 \tag{109c}$$

If $R \neq {0 \atop \infty}$, then Eq. (109c) implies either

$$z = 0$$

or

$$\frac{dw}{dz} = 4 \tag{110}$$

*The linear variation here can be proved quite generally for all finite Reynolds numbers.

The results (Eq. (110)) are independent of the Reynolds number. $z = 0$ is a trivial result. The other, $\frac{dW}{dz} = -4$, leads to $\mathcal{F} \propto X^{-1}$ as $X \to \infty$. So this shows that the concept of equilibrium spectrum is consistent with the similarity assumptions.

Before concluding the present discussion, we should keep in mind that the equilibrium spectrum cannot persist for an indefinite length of time in a decaying field of turbulence. This is because of the limited amount of energy supply. However, the general tendency in the transition period is the different characteristic lengths in turbulence which may exist at the initial period will tend to the same order of magnitude and very probably the spectrum will be completely similar in the final period of decay.

Assume that we can neglect the transfer terms W_k in the final period of decay, then the spectrum equation is

$$\frac{\partial \mathcal{F}}{\partial t} = -2\nu k^2 \mathcal{F} \tag{111}$$

The general solution of the above equation is

$$\mathcal{F} = \mathcal{F}^*(\kappa)\, e^{-2\nu\kappa^2 t} \tag{112}$$

$\mathcal{F}^*$ is a function of κ only. Since $\mathcal{F} \propto \kappa^4$ at low wave numbers, we have

$$\mathcal{F} = C\,\kappa^4 e^{-2\nu\kappa^2 t} \tag{113}$$

This result can also be directly obtained by applying the Fourier transform to the correlation function (see Ref. 50).

5b. Spectrum Functions and Their Relations to Stochastic Processes

Dryden (Ref. 51) and Liepmann (Ref. 21) found experimentally that the Taylor spectrum $E(k_1)$ can often be best approximated by the following empirical formula

$$E(k_1) = \frac{2\,\overline{u^2}}{\pi} \frac{L}{1 + L^2 \kappa_1^2} \tag{114}$$

for a wide range of low wave numbers κ_1 . L is the correlation length defined as $L = \int_0^\infty f(r)\,dr$. Using Eq. (85) we then have the Heisenberg spectrum expressed by the following empirical formula

$$\mathcal{F}(k) = \frac{8\,\overline{u^2}\, L^5 \kappa^4}{\pi\,(1 + L^2 \kappa^2)^3} \tag{115}$$

This means that apart from the equilibrium range of spectrum which does not contain an appreciable amount of energy, the energy containing part of the spectrum can be best approximated by Eq. (115). It is obvious that Eq. (115) does satisfy the continuity requirement, i.e. $\mathcal{F} \propto C \kappa^4$ when $L\kappa \ll 1$. Furthermore since L and L^* are proportional, then the coefficient C is proportional to Loitsiansky's invariant which is a low wave number characteristic (compare Eq. (89)).

At any rate Eq. (114) represents a reasonable spectrum in addition to the simplicity. We have tried very hard in the previous discussions on turbulence spectrum to make more or less restrictive assumptions on the transfer function W_κ . The question is "can we get some information on the turbulence spectrum from general theory of stochastic processes and then extrapolate backward to find out more exact knowledge about the transfer function?"

One immediately recognizes Eq. (114) as the Fourier transform of the exponential correlation coefficient. We have seen in section 3 Part I a complete information about correlation coefficient does not insure a unique interpretation of the stochastic process. For the time being there are two well-known types of stochastic processes which lead to exponential correlation. One occurs in the theory of communication, the random telegraph signal with Poisson distribution (Ref. 52). This process has a close analogy with the action of grid which is responsible for turbulence producing in actual experiments. The other occurs in the theory of one-dimensional Brownian motion. It can be proved that if the given one-dimensional stochastic process $u(t)$ is stationary and Markoffian,* then the correlation is of the exponential type $e^{-\beta|t-s|}$ provided the joint probability of $u(t)$ and $u(s)$ is double Gaussian (for proof see Appendix V).

*For the definition of the Markhoff process see Kolmogoroff (Ref. 11).

PART III ON THE NAVIER-STOKES EQUATION WITH RANDOM BOUNDARY CONDITIONS

1. General Introduction

The present part of the paper is concerned with the investigations of the effect of the random disturbances produced in a certain part of the domain of fluid upon the solutions on the Navier-Stokes equation as a whole. By random boundary conditions we mean that the boundary conditions are not known exactly, but instead we have certain statistical informations such as the probability distributions and correlation coefficients. The problem is then to determine the probability distribution and the correlations away from the boundary. The present investigation was first suggested by the fact that all the present day experiments for isotropic turbulence are carried out in wind tunnels behind grids, hence the turbulence thus produced is connected with the action of grids if it is not all distorted by the inertia effects (in the latter part of the work simple non-linear examples are studied). It has been pointed out (cf. the end of Part II) that if the irregular shedding of vortices by the grid is something like a random telegraph signal with a Poisson distribution then the simple optical spectrum is indeed a very plausible turbulence spectrum.

2. Method of Analysis

The kernel of the recent turbulence research on isotropic turbulence is to understand how the modulation and phase interaction between different Fourier components take place. Consequently the most appropriate direct approach is to use Fourier analysis.

Assume the compressibility effect can be neglected, the fundamental equations to be used are

(1) Equation of Continuity

$$\nabla \cdot \vec{V} = 0 \tag{1}$$

(2) Equation of Motion

$$\frac{D\vec{V}}{Dt} = -\nabla \frac{P}{\rho} + \nu \nabla^2 \vec{V} \tag{2}$$

We choose the following coordinate system. Let the z axis coincide with the direction of mean motion or the wind tunnel axis and the xY plane correspond to the grid plane.* The grid is supposed to have a periodic spacing of their rods both in X and Y direction. Naturally this suggests that it is reasonable to assume that the boundary conditions at $z = 0$ are periodic in x and Y, for simplicity we assume the period is 1 in both directions. The analysis given in the following can be easily extended to the cases where the boundary conditions can be expressed in terms of the Fourier integral.

Assume that we have a constant mean flow in the z direction with velocity u_0. Hence we have

$$\begin{aligned} \vec{V} &= \vec{u} + u_0 \vec{e}_z \\ P &= p + \overline{P} \end{aligned} \tag{3}$$

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* The system of coordinates is chosen with an aim to approximate the actual experimental case.

where

$\vec{e}_z$ is the unit vector in the z direction

$\vec{u}$ is the fluctuation of V about its mean $u_0 \vec{e}_z$

$\overline{P}$ mean pressure, p is the fluctuating part

Eqs. (1) and (2) then become

$$\nabla \cdot \vec{u} = 0 \tag{4}$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + u_0 \frac{\partial \vec{u}}{\partial z} = - \nabla \frac{p}{\rho} + \nu \nabla^2 \vec{u} \tag{5}$$

We are interested in finding out the turbulence field for the region

$$\begin{aligned} 0 &< z < \infty \\ -\infty &< y < \infty \\ -\infty &< x < \infty \end{aligned}$$

The boundary conditions are as follows:

(I) $\vec{u} = 0$, $z < 0$, i.e. the grid does not introduce any upstream influence. To be sure this is only a mathematical idealization, in the actual case $\vec{u} \neq 0$, but the flow can be considered as laminar.

(II) $\vec{u} \to 0$ as $z \to \infty$; this condition expresses the fact that the turbulence decays downstream.

(III) At $z = 0$, $\vec{u} = \vec{u}_B(x, y, 0, t)$ periodic in x and y with periods l , where $\vec{u}_B$ for each fixed time is a random variable, i.e. only the probability of $\vec{u}_B$ is given.

Remark:

If $\vec{u}_B$ in (III) is a perfect definite vector function, and if the inertia terms in the Navier-Stokes equation can be neglected, then from the theory of linear partial differential equations, the conditions (II) and (III) alone guarantee the existence of a unique solution. In the non-linear case there is no guarantee for uniqueness.

3. Pressure and Velocity Spectrums and Their Relationship

We define the velocity spectrum $\vec{\Gamma}(\vec{k},t)$ and the pressure spectrum $\Pi(\vec{k},t)$ as follows:

$$\vec{\Gamma}(\vec{k},t) = \int_{-\infty}^{\infty}\int_0^1\int_0^1 \vec{u}(\vec{r})\, e^{2\pi i(\vec{k}\cdot\vec{r})}\, dx\, dy\, dz \qquad (6a)$$

$$\Pi(\vec{k},t) = \int_{-\infty}^{\infty}\int_0^1\int_0^1 \frac{p}{\rho}(\vec{r})\, e^{2\pi i(\vec{k}\cdot\vec{r})}\, dx\, dy\, dz \qquad (6b)\ *$$

where $\vec{r}$ is the position vector in the physical plane $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ $\vec{k} = n\vec{e}_x + m\vec{e}_y + k_1 e_z$; n , m integers, k_1 is a continuous real variable.

A relation between these two spectrum is immediately obtained by combining the equation of continuity and the equations of motion and taking appropriate transform of the resulting equation. We have

$$\nabla\cdot\vec{u} = 0 \qquad (4)$$

$$\nabla^2\frac{p}{\rho} + \nabla\cdot(\vec{u}\cdot\nabla\vec{u}) = 0 \qquad (7)$$

Taking the transform of Eq. (4) as follows

*In this definition we choose $p=0$ for $z<0$. Since only pressure gradient is important, it is clear that we can consider any appropriate pressure as zero pressure.

$$\int_0^\infty \int_0^1 \int_0^1 (\nabla \cdot \vec{u})\, e^{2\pi i (\vec{k}\cdot\vec{r})}\, d\vec{r} = 0$$

The result is

$$\vec{k} \cdot \vec{\Gamma}(\vec{k}, t) = \frac{i}{2\pi}\, \gamma_3(k_2) \tag{8}$$

where

$$\gamma_3(\vec{k}_2) = \int_0^1 \int_0^1 u_z(x, y, 0, t)\, e^{2\pi i (\vec{k}_2\cdot\vec{r})}\, dx\, dy$$

$$\vec{k}_2 = n\vec{e}_x + m\vec{e}_y$$

u_z is the fluctuating velocity in the z direction.

Taking the similar transform of Eq. (7) we have

$$\Pi = -\frac{1}{k^2}\, \vec{k} \cdot \sum_{n', m'} \int_{-\infty}^{\infty} \left[\vec{k} \cdot \vec{\Gamma}(\vec{k} - \vec{k}')\right] \vec{\Gamma}(\vec{k}')\, dk_1 + A(\vec{k}, t) \tag{9}$$

where

$$A(\vec{k}, t) = \frac{1}{4\pi k^2} \Big\{ \sigma_z + 2\pi i k_1 \sigma + 2\pi i\, \vec{k}_2 \cdot \sum_{n'} \sum_{m'} \vec{\gamma}(\vec{k}_2 - \vec{k}_2')\, \gamma_3(\vec{k}_2')$$

$$- \sum_{n'} \sum_{m'} 2\gamma_3(\vec{k}_2 - \vec{k}_2')\, \gamma_{3z}(\vec{k}_2')$$

$$+ 2\pi i\, \vec{k} \cdot \sum_{n'} \sum_{m'} \gamma_3(k_2 - k_2')\, \vec{\gamma}(\vec{k}_2') \Big\}$$

and

$$\sigma = \int_0^1 \int_0^1 \left[\frac{p}{\rho}\, e^{2\pi i (\vec{k}_2\cdot\vec{r})}\right]_{z=0} dx\, dy$$

$$\sigma_z = \int_0^1 \int_0^1 \frac{1}{\rho} \frac{\partial p}{\partial z}(x, y, 0, t)\, e^{2\pi i (\vec{k}_2\cdot\vec{r})}\, dx\, dy$$

$$\vec{\gamma}\,(\vec{k}_2) = \int_0^1\!\!\int_0^1 \vec{u}\,(x, y, 0, t)\; e^{2\pi i\,(\vec{k}_2\cdot\vec{r})}\, dx\, dy$$

$$\gamma_{3z}\,(\vec{k}_2) = \int_0^1\!\!\int_0^1 \frac{\partial u_z}{\partial z}\,(x, y, 0, t)\; e^{2\pi i\,(\vec{k}_2\cdot\vec{r})}\, dx\, dy$$

$A(\vec{k}, t)$ may be interpreted as an action function which constitutes a part of the pressure spectrum depending only on the character of the boundary conditions.

4. Fundamental Equations in the Spectrum Analysis of Turbulent Flow

Having obtained the relation between the pressure spectrum and the velocity spectrum, we then take the transform of the Navier-Stokes equation and arrive at the following integro-differential equation for the velocity spectrum. The required equation is

$$\frac{\partial \vec{\Gamma}(\vec{k},t)}{\partial t} + \beta(\vec{k})\,\vec{\Gamma}(\vec{k},t) - 2\pi i \sum_{n'}\sum_{m'} \int_{-\infty}^{\infty} \vec{k}\cdot\vec{\Gamma}(\vec{k}-\vec{k}')\left\{\frac{1}{m} - \frac{\vec{k}\vec{k}}{k^2}\right\}\cdot\vec{\Gamma}(\vec{k}')\, dk_1'$$

$$= \vec{B}\,(\vec{k}, t) + u_0\,\vec{\gamma} = \vec{B}'\,(\vec{k}, t) \tag{10}$$

$$\vec{B}(\vec{k},t) = \sigma\vec{e}_z + 2\pi i\,\vec{k}\,A(k,t) + \sum_{n'}\sum_{m'} \gamma_3\,(\vec{k}_2 - \vec{k}_2')\,\vec{\gamma}\,(\vec{k}_2') - \nu\,[\,\vec{\gamma}_2 - 2\pi i\, k_1\vec{\gamma}\,] \tag{11}$$

where

$$\beta(\vec{k}) = 4\pi^2 \nu k^2 - 2\pi i\, k_1 u_0$$

$$\vec{\gamma}_2\,(\vec{k}_2) = \int_0^1\!\!\int_0^1 \frac{\partial \vec{u}}{\partial z}\,(x, y, 0, t)\; e^{2\pi i\,(\vec{k}_2\cdot\vec{r})}\, dx\, dy$$

$$\gamma_3\,(k_2) = \int_0^1\!\!\int_0^1 u_z\,(x, y, 0, t)\; e^{2\pi i\,(\vec{k}_2\cdot\vec{r})}\, dx\, dy$$

$u_0\vec{\gamma} + \vec{B}(\vec{k},t)$ may be called a grid function. It is believed that the statistical behvaior of $\vec{B}'(\kappa,t)$ does determine the complete statistical behavior of the entire turbulent flow field downstream. Referring to Eq. (11), we see that the first two terms on the right hand side represent the effect of pressure fluctuations, the third term represents the incipient inertia effect and the last term represents the local dissipation at the grid plane.

Eq. (10) can be further simplified by using the equation of continuity. In this simplification the convolution terms in $\vec{B}(\vec{k},t)$ disappear. We have

$$\frac{\partial\vec{\Gamma}}{\partial t} + \beta(\vec{k})\,\vec{\Gamma} - 2\pi i\sum_{n'}\sum_{m'}\int_{-\infty}^{\infty}\vec{k}\cdot\vec{\Gamma}(\vec{k}-\vec{k}')\left\{\underset{\sim}{1} - \frac{\vec{k}\vec{k}}{k^2}\right\}\cdot\vec{\Gamma}(k')\,dk_1'$$
$$= \left\{\underset{\sim}{1} - \frac{\vec{k}\vec{k}}{k^2}\right\}\chi(\vec{k}) + \frac{\vec{k}}{k^2}\frac{i}{2\pi}\left[\frac{\partial\gamma_3}{\partial t} + \beta(\vec{k})\,\gamma_3\right] \tag{12}$$

where

$$\chi(\vec{k}) = u_0\vec{\gamma} - \nu\vec{\gamma}_z + 2\pi i\,k_1\nu\vec{\gamma} + \sigma\vec{e}_z$$

The advantage of the present formulation is that we now have a clear idea of where we should introduce the statistical element into the ordinary hydrodynamic theory. With a slight modification, the equation can be applies to an infinite turbulent field (in the sense of Part II of the present paper). If we assume that we nowhere have a feed in of turbulent energy, then $\vec{B}'(\vec{k},t) = 0$. Furthermore we assume that there is no mean motion, i.e. $u_0 = 0$, the non-homogeneous Eq. (10) then becomes

$$\frac{\partial \vec{\Gamma}(\vec{k},t)}{\partial t} + 4\pi^2 \nu k^2 \vec{\Gamma} - 2\pi i \iiint_{-\infty}^{\infty} \vec{k}\cdot \vec{\Gamma}(\vec{k}-\vec{k}')\left\{\frac{1}{\sim} - \frac{\vec{k}\vec{k}}{k^2}\right\}\cdot \vec{\Gamma}(\vec{k}')\, d\vec{k}' = 0 \quad \textbf{(13)}\ ^*$$

and the equation of continuity

$$\vec{k}\cdot \vec{\Gamma}(\vec{k},t) = 0 \qquad (14)$$

Eq. (14) states that the Fourier component at every instant is perpendicular to the wave vector (compare with Eq. (80), Part II). In certain cases this fact does help us to visualize the geometrical picture in taking averages. The term

$$2\pi i \iiint_{-\infty}^{\infty} \vec{k}\cdot \vec{\Gamma}(\vec{k}-\vec{k}')\ \frac{\vec{k}\,\vec{k}}{k^2}\cdot \vec{\Gamma}(\vec{k}')\, d\vec{k}'$$

represents the effect of pressure in the way the equation is derived.

In all practical cases we are not primarily interested in the instantaneous values of the velocity spectrum, but rather in the mean energy distribution for various wave numbers. The total energy contained in the turbulent flow E is equal to (assume density $\rho = 1$)

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \langle (\vec{u}\cdot\vec{u}) \rangle_{Av}\, d\vec{r}$$

$$= \int_{-\infty}^{\infty} \mathcal{F}(k)\, dk \qquad (15)$$

Using Parseval's theorem

$$\frac{1}{2}\int_{-\infty}^{\infty} \langle \vec{u}\cdot\vec{u} \rangle_{Av}\, d\vec{r} = \frac{1}{2}\int_{-\infty}^{\infty} \langle \vec{\Gamma}(\vec{k})\cdot \vec{\Gamma}(-\vec{k}) \rangle_{Av}\, d\vec{k}$$

*The doubly infinite sum n', m' in Eq. (10) has been replaced by an integral in Eq. (13). It is obvious that we can make such a change if the Fourier transform exists.

Since $\vec{u}$ is real

$$\vec{\Gamma}(\vec{k}) = \vec{\Gamma}^*(-\vec{k}) \tag{16}$$

then

$$E = \frac{1}{2}\int \left\langle \vec{\Gamma}(\vec{k})\cdot\vec{\Gamma}^*(\vec{k})\right\rangle_{Av} d\vec{k}$$

Hence the energy spectrum may be identified as

$$\mathcal{F}(\vec{k},t) = \frac{1}{2}\left\langle \vec{\Gamma}(\vec{k})\cdot\vec{\Gamma}^*(\vec{k})\right\rangle_{Av} \tag{17}$$

and

$$\mathcal{F}(\vec{\infty},t) = 0$$

Multiplying Eq. (13) by $\vec{\Gamma}^*\cdot$ and adding the result to the product of $\vec{\Gamma}\cdot$ and the conjugate of Eq. (13), we then have the spectrum equation

$$\frac{\partial\mathcal{F}}{\partial t} + W_k = -8\pi^2\nu k^2\mathcal{F} \tag{18}$$

W_k represents the transfer term and is given in terms of small time average in the following expression

$$W_k = \left\langle 2\pi\, \mathcal{I}_m \left\{ \vec{\Gamma}^*(\vec{k})\cdot\int_{-\infty}^{\infty}\left(\vec{k}\cdot\vec{\Gamma}(\vec{k}-\vec{k}')\right)\vec{\Gamma}(\vec{k}')\,d\vec{k}'\right\rangle_{Av} \tag{19}$$

W_k given in this form should be compared with the different assumptions we made in section 5 Part II. It is now clearly seen that the transfer term is related to the triple correlation of the velocity spectrum. We note that in the energy equation the pressure term does not appear on account of the equation of continuity. Therefore the pressure fluctuations have no direct effect whatsoever on the transfer mechanism which is responsible for continuous modulation among different

*The average operator may be either a small time average or an ensemble average.

Fourier components in a decaying turbulent field, if there is no external supply of turbulent energy from the mean motion. This criteria applies to isotropic turbulence as well as to non-isotropic turbulence since we have not made any specific assumption about the isotropy. It is probable then that the fluctuation of pressure is connected with the problem of interaction between the field of sound and turbulence. Batchelor (Ref. 40) has proved that the pressure effect does not change the total energy contribution by any small region of wave number space for an incompressible homogeneous turbulence field. The only limitation to the present theory is that the underlying turbulence field can be subjected to a Fourier analysis.

4a. Stationary Spectrum

Using Eqs. (8), (10) and (11) the equation for the variation of instantaneous spectrum $\mathcal{F}'$ (not the averaged quantity) in the non-homogeneous case can be written as follows

$$\frac{\partial \mathcal{F}'}{\partial t} + 8\pi^2 \nu k^2 \mathcal{F}' + W_k' = D'(k,t) + 2\, R_e \left[\vec{\Gamma}^* \cdot (\vec{B} + u_0 \vec{\gamma}) \right] \tag{20}$$

where $D'(k,t)$ represents the effect of pressure

$$\mathcal{F}' = \frac{1}{2} \vec{\Gamma}(\vec{k}) \cdot \vec{\Gamma}^*(\vec{k})$$

$$D'(k,t) = - R_e \left\{ \sum_{n'} \sum_{m'} \int_{-\infty}^{\infty} \left[\vec{k} \cdot \vec{\Gamma}(\vec{k} - \vec{k}') \right] \gamma_3^*(\vec{k}_2) \frac{\vec{k}}{k^2} \cdot \vec{\Gamma}(\vec{k}') \, dk_1 \right.$$

and

$$W_k' = 2\pi\, \mathcal{I}_m \left\{ \vec{\Gamma}^*(\vec{k}) \cdot \sum_{n'} \sum_{m'} \int_{-\infty}^{\infty} \left(\vec{k} \cdot \vec{\Gamma}(\vec{k} - \vec{k}') \right) \vec{\Gamma}(\vec{k}') \, dk_1 \right.$$

Eq. (18) is still non-linear and in fact it is just another phase of the Navier-Stokes equation. Similarly Eq. (86a) (Part II) is another phase of the Karman-Howarth equation. At the present time it does not seem possible to find the complete solution of the equation. We may assume, as in the actual experiment, that the flow in a wind tunnel reaches stationary state after a certain transient period. By stationary state, here, we mean the mean values at any given point inside the wind tunnel are independent of time. It is not difficult to see that in the homogeneous case (i.e. an infinite turbulent field without the external forcing function such as the grid in the present case) the stationary state in the sense we just described is in general not possible. We should keep this difference in mind in order that a decent comparison between the theory in Part II and the experiments can be made.

Accepting the stationary assumption and taking the averages of Eq. (20) we have

$$8\pi^2 \nu \kappa^2 \mathcal{F} + W_\kappa = \mathcal{D}(\vec{k}, t) + R_e \left\langle \vec{\Gamma}^* \cdot \left(\vec{B} + u_0 \vec{\gamma} \right) \right\rangle_{Av} \tag{21}$$

The first term on the left hand side represents the viscous dissipation of a given Fourier component, W_k is the energy transfer due to inertia effect, $\mathcal{D}(\vec{k}, t)$ represents the effect of pressure (this effect is out if we do not have the grid forcing function) and the last term represents the correlation term with the boundary forcing functions.

W_k, $\mathcal{D}(\vec{\kappa}, t)$ are believed to be important at high Reynolds numbers. However for very small Reynolds numbers we may neglect these terms and Eq. (21) becomes "approximately" linear. The only non-linear character of this equation is retained through the boundary forcing functions. This assumption is justified when we are interested in the turbulence field far downstream from the grid and also when the turbulence level is low. Hence Eq. (21) becomes

$$8\pi^2 \nu k^2 \mathcal{F} = \left\langle Re \quad \vec{\Gamma}^* \cdot (\vec{B} + u_0 \vec{\gamma}) \right\rangle_{Av}$$

or

$$\mathcal{F} = \frac{1}{8\pi^2 \nu k^2} \left\langle Re \quad \vec{\Gamma}^* \cdot (\vec{B} + u_0 \vec{\gamma}) \right\rangle_{Av} \tag{22}$$

Eq. (22) clearly shows the very dependence of the spectrum function on the random forcing function in the case of small Reynolds numbers. True, this is a consequence of linearization. Nevertheless this does indicate that "there is no a priori reason to expect the turbulence spectrum will approach Gaussian at far distances downstream from the grid. The effect of the grid seems to be an important factor in low Reynolds number turbulent flow in a wind tunnel." Indeed the measurements by Liepmann, Laufer and Liepmann showed that the turbulence spectrum at $\frac{z}{M} = 1{,}000$ with mesh size 0.141 cm and mean velocity u_0 = 630 cm per sec. agrees poorly with the Gaussian spectrum (Ref. 21). The agreement is especially poor at low frequencies. This fact strengthens the belief that the large eddies produced by the grid are of the first order and unaffected by the non-linear and viscous terms. The small eddies damp much faster than the large eddies. Hence any irregularities of large eddies will markedly show up at far distances downstream. This argument incidentally explains why in Townsend's measurements (Ref. 50) the agreement between the Gaussian correlation curve and the experimentally observed correlation curve become worse instead of better at large $\frac{r}{\lambda}$, if the measurement is carried further downstream.

5. Attempts to Relate Turbulence with the Theory of Brownian Motion*

It is noted that if we linearize the fundamental equation (10), the resulting equation which we will get is just the equation of motion of a free Brownian particle, known as the Einstein-Langevin equation (Ref. 53)

$$\frac{\partial \vec{\Gamma}(\vec{k},t)}{\partial t} + \beta(\vec{k})\,\vec{\Gamma}(\vec{k},t) = \vec{B}_L(\vec{k},t) \tag{23}$$

$$\beta(\vec{k}) = \nu k^2 - i u_0 k_1$$

$\vec{B}_L(\vec{k},t)$ is the linearized boundary forcing function.

The effect of linearization is to decouple the interaction between different Fourier components. This fact enables us to consider each Fourier component separately. This treatment will also serve as a first approximation to the case where there is a weak interaction between these components. Nevertheless, the investigation of Eq. (23) will give us some information regarding the random character of turbulence. Eq. (23) is not a differential equation in the ordinary sense, simply because we do not know exactly the time history of $\vec{B}_L(\vec{k},t)$. Some statistical behaviors of $\vec{B}_L(\vec{k},t)$ may be assumed and this leads to a type of stochastic differential equation. The appropriate problem for this equation is "given $\vec{\Gamma}(\vec{k},t) = \vec{\Gamma}(\vec{k},0)$ at $t=0$, we ask what is the probability distribution of $\vec{\Gamma}(\vec{k},t)$ at any later time $t>0$." To be sure we have to talk about distribution

*In this section $2\pi\vec{k}$ has been replaced by $\vec{k}$ in order to save writings.

functions of the real and imaginary parts of $\vec{\Gamma}(\vec{k},t)$ separately. For all practical purposes we are especially interested in the limiting distribution which will in general correspond to the experimentally observed distribution.

We may mention in this connection another type of stochastic problem for differential equations which will be illustrated in the next section. In this problem, the differential equation itself does not involve explicitely random functions, however the boundary or initial conditions that we deal with are not sharply defined. Hence the solutions obtained herewith can only be interpreted statistically. Both types are useful in treating the statistical theory of turbulence.

The following arguments are tentative. They are plausible rather than unique. We assume

(1) $\vec{B}_L(\vec{k},t)$ is independent of $\vec{\Gamma}(\vec{k},t)$. Physically this assumption means that the flow downstream from the grid is in no way to effect the forcing mechanism of the grid.

(2) There exists a time interval $\Delta\tau$, during which $\vec{B}_L(\vec{k},t)$ has undergone very many fluctuations while $\vec{\Gamma}(\vec{k},t)$ changes slowly. Specifically we assume the correlation function $\underset{\sim}{M} = \overline{\vec{B}_L(\vec{k},t)\vec{B}_L(\vec{k},t+\tau)}$ $= \underset{\sim}{M}_0\,\delta(\tau)$ * where $\underset{\sim}{M}_0$ is independent of t (i.e. the process is stationary). $\delta(\tau) = 1$ when $\tau = 0$, and zero when $\tau \geq \Delta\tau$. $\Delta\tau$ is small compared to the characteristic time during which $\vec{\Gamma}(\vec{k},t)$ has an appreciable change. Physically this second assumption means that flow conditions downstream do not have an immediate response to

*This means that the forcing function has a white frequency spectrum.

the rapid change of the boundary forcing functions. This seems reasonable especially when the approximate steady condition in a wind tunnel is established.

Let

$$\vec{\Gamma} = \vec{\Gamma}_1 + i\,\vec{\Gamma}_2$$

$$\vec{B}_L = \vec{B}_1 + i\,\vec{B}_2$$

where $\vec{\Gamma}_1$, $\vec{\Gamma}_2$, $\vec{B}_1$ and $\vec{B}_2$ are now real quantities. The solution of Eq. (22) consists of two parts. The first part consists of a solution to the homogeneous equation and staisfies certain initial conditions. For $|\kappa| > 0$ this part will be damped out exponentially when $t > 0$. <u>Hence the limiting distribution function will not depend on the initial condition ($|\kappa| > 0$).</u> The second part consists of the solution to the non-homogeneous equation and satisfies the homogeneous initial condition. This part is not damped out as time t approaches infinity and is of interest in studying the limiting distribution function. Hence as $t \to \infty$, $\kappa > 0$

$$\vec{\Gamma}_1(\vec{k},t) = \int_0^t e^{-\nu\kappa^2\xi}\left[\vec{B}_1(t-\xi)\cos(u_0 k_1 \xi) - \vec{B}_2(t-\xi)\sin(u_0 k_1 \xi)\right] d\xi$$

$$\vec{\Gamma}_2(\vec{k},t) = \int_0^t e^{-\nu\kappa^2\xi}\left[\vec{B}_2(t-\xi)\cos(u_0 k_1 \xi) + \vec{B}_1(t-\xi)\sin(u_0 k_1 \xi)\right] d\xi \qquad (24)$$

In order to obtain the limiting distribution, we apply the following lemma, which was proved by Chandrasekhar (Ref. 53) which states that if $\vec{\Gamma}$ and $\vec{B}_L$ are real, and the second assumption on $\vec{B}_L$ is valid, and if

$$\vec{\Gamma} = \int_0^t \psi(\xi)\,\vec{B}(\xi)\,d\xi$$

Then the probability distribution P of $\vec{\Gamma}(\vec{k},t)$ is given by $t \to \infty$

$$P(\vec{\Gamma}) = \frac{1}{\left[4\pi g \int_0^t \psi^2(\xi)\, d\xi\right]^{3/2}} \exp\left(-\frac{|\vec{\Gamma}|^2}{4g\int_0^t \psi^2(\xi)\, d\xi}\right)$$

where g is proportional to temperature in the case of Brownian motion. In the present case g is related to the correlations of boundary forcing functions.

The limiting distributions that we obtain are for the real and imaginary parts of $\vec{\Gamma}$ (for $\kappa > 0$);

$$P(\vec{\Gamma}_1) = \frac{1}{\left[4\pi(g_1\alpha_1^2 + g_2\alpha_2^2)\right]^{3/2}} \exp\left(-\frac{|\Gamma_1|^2}{4g_1\alpha_1^2 + 4g_2\alpha_2^2}\right) \tag{25a}$$

$$P(\vec{\Gamma}_2) = \frac{1}{\left[4\pi(g_2\alpha_1^2 + g_1\alpha_2^2)\right]^{3/2}} \exp\left(-\frac{|\Gamma_2|^2}{4g_2\alpha_1^2 + 4g_1\alpha_2^2}\right) \tag{25b}$$

where g_1 and g_2 can be shown to be equal to

$$\begin{aligned} g_1 &= \frac{1}{\tau}\int_0^\tau\int_0^\tau \left\langle \vec{B}_1(\xi)\, \vec{B}_1(\xi')\right\rangle_{Av} d\xi\, d\xi' \\ g_2 &= \frac{1}{\tau}\int_0^\tau\int_0^\tau \left\langle \vec{B}_2(\xi)\, \vec{B}_2(\xi')\right\rangle_{Av} d\xi\, d\xi' \end{aligned} \tag{26}$$

τ is an appropriate time interval. The average operator here refers to ensemble average

$$\begin{aligned} \alpha_1^2 &= \frac{1}{4}\left[\frac{1}{\nu\kappa^2} + \frac{\nu\kappa^2}{\nu^2\kappa^4 + u_0^2 k_1^2}\right] \\ \alpha_2^2 &= \frac{1}{4}\left[\frac{1}{\nu\kappa^2} - \frac{\nu\kappa^2}{\nu^2\kappa^4 + u_0^2 k_1^2}\right] \end{aligned} \tag{27}$$

In deriving Eqs. (25a) and (25b) we have also assumed that there is no correlation between the amplitude of the boundary forcing function and its phase. This assumption is probably valid.

Hence the distributionsof $\vec{\Gamma}_1(\vec{k})$ and $\vec{\Gamma}_2(\vec{k})$ will approach Gaussian, however the approach is not uniform for all κ . The approach is especially slow when κ is small. When $\kappa = 0$, the initial condition is not damped out, consequently the limiting Gaussian distribution can hardly be realized. For the moment let us suppose that we have a zero initial condition and that the turbulence consists of denumerable numbers of Fourier components. Furthermore, if the sum of the dispersions of components is finite, then the fluctuating velocity has a Gaussian distribution.

The second deduction that we can make from assumptions (1) and (2) is the frequency spectrum and the time correlation coefficient. We do this by using a time Fourier analysis of Eq. (23). Let

$$\begin{aligned} \vec{B}_L(\vec{\kappa},t) &= \sum_n \vec{B}_n(\kappa)\, e^{i\omega_n t} \\ \vec{\Gamma}(\vec{\kappa},t) &= \sum_n \vec{\Gamma}_n(\kappa)\, e^{i\omega_n t}. \end{aligned} \tag{28}$$

Substituting Eq. (28) into Eq. (23) we have

$$\vec{\Gamma}_n(\vec{\kappa}) = \frac{\vec{B}_n(\vec{\kappa})}{\beta(\kappa) + i\omega_n} \tag{29}$$

Hence the time spectrum of $\vec{\Gamma}(k,t)$ is

$$\vec{\Gamma}_n(\kappa)\,\vec{\Gamma}_n^*(\kappa) = \frac{\vec{B}_n(\kappa)\,\vec{B}_n^*(\kappa)}{\nu^2\kappa^4 + (u_1 k_1 - \omega_n)^2} \tag{30}$$

Let us denote $\vec{\Gamma}_n(\kappa)\,\vec{\Gamma}_n^*(\kappa)$ by $\underset{\sim}{\mathcal{F}}_n(\kappa)$ (spectral tensor) and $\underset{\sim}{\mathcal{F}}^0 = \vec{B}_n(\kappa)\,\vec{B}_n^*(\kappa)$ and furthermore restrict outselves to talk about positive frequencies, then

$$\underset{\sim}{\mathcal{F}_n} = \underset{\sim}{\mathcal{F}_n}^{\circ} \left[\frac{1}{\nu^2 K^4 + (u_0 k_1 - \omega_n)^2} + \frac{1}{\nu^2 K^4 + (u_0 k_1 + \omega_n)^2} \right] \tag{31}$$

It is seen that if ν is a small dimensional coefficient there exists a sharp resonance at $\omega_n = u_0 k_1$. This probably explains the reason for introducing the mean velocity in defining a characteristic frequency (see section 5, Part II). It is also worthwhile to notice that $\underset{\sim}{\mathcal{F}_n}$ is inversely proportional to the square of frequency when the time frequency is very large in comparison with the rest of the terms in the denominator.

From the general theory of Brownian motion (Ref. 54) the correlation tensor is of the following form

$$\underset{\sim}{R_K}(\tau) = \overline{\vec{\Gamma}_K(t)\, \vec{\Gamma}_K^{*}(t+\tau)} = \underset{\sim}{R_K}(0)\, e^{-\beta(K)\,|\tau|} \tag{32}$$

if the process is Markhoffian. If we separate the real and imaginary parts

$$\underset{\sim}{R_K}(\tau) = \underset{\sim}{R_{K1}}(\tau) - i\, \underset{\sim}{R_{K2}}(\tau)$$

hence

$$\begin{aligned} \underset{\sim}{R_{K1}}(\tau) &= \overline{\vec{\Gamma}_{K1}(t)\, \vec{\Gamma}_{K1}(t+\tau)} + \overline{\vec{\Gamma}_{K2}(t)\, \vec{\Gamma}_{K2}(t+\tau)} \\ \underset{\sim}{R_{K2}}(\tau) &= \overline{\vec{\Gamma}_{K1}(t)\, \vec{\Gamma}_{K2}(t+\tau)} - \overline{\vec{\Gamma}_{K1}(t+\tau)\, \vec{\Gamma}_{K2}(t)} \end{aligned} \tag{33}$$

and

$$\underset{\sim}{R_{K1}}(0) = \underset{\sim}{R_K}(0)$$

$$\underset{\sim}{R_{K2}}(0) = \underset{\sim}{0}$$

Comparing Eqs. (32) and (33) we have

$$
\begin{aligned}
\underset{\sim}{R}_{K1}(\tau) &= \underset{\sim}{R}_{K}(0)\, e^{-\nu K^2|\tau|} \cos\,(u_0 k_1 \tau) \\
\underset{\sim}{R}_{K2}(\tau) &= \underset{\sim}{R}_{K}(0)\, e^{-\nu K^2|\tau|} \sin\,(u_0 k_1 \tau)
\end{aligned}
\tag{34}
$$

It is not difficult to verify that the spectral tensor $\mathcal{J}$ in Eq. (31) corresponds to the Fourier transform of the real part of the correlation tensor $\underset{\sim}{R}_K(\tau)$. Similarly the energy spectrum is also completely determined by the real part of the correlation function. The imaginary part of the correlation is believed to be the time phase correlation of each individual space Fourier component. This assertion is not in contradiction with the assumptions we made on the boundary forcing functions since the time scales are different in the two cases. In actual cases the phase interaction among different Fourier components is of interest, nevertheless $\underset{\sim}{R}_{K2}(\tau)$ can be served as a rough indication of the more complicated problems.

It should be pointed out that a more exact treatment than that given here is possible. The ordinary space correlations downstream can be expressed in terms of the generalized correlation functions* at the boundary, i.e. the grid plane.**

6. Linearized Vorticity Transport in a Turbulent Flow

Sometimes it is found more convenient to consider the vorticity

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*The generalized correlation function is a mixed space and time correlation function.

**This was pointed out by Professor J. G. Kirkwood.

or the curl of velocity rather than the velocity itself. Let us denote $\vec{u}$ the velocity and $\vec{\omega}$ the vorticity, then

$$\vec{\omega} = \nabla \times \vec{u} \tag{35}$$

A vorticity spectrum $\vec{\Omega}$ may be defined as follows (cf. section 3)

$$\vec{\Omega} = \int_{-\infty}^{\infty}\int_{0}^{1}\int_{0}^{1} \vec{\omega}\, e^{2\pi i(\vec{k}\cdot\vec{r})}\, dx\, dy\, dz$$

$$= \int_{0}^{\infty}\int_{0}^{1}\int_{0}^{1} \vec{\omega}\, e^{2\pi i(\vec{k}\cdot\vec{r})}\, dx\, dy\, dz \tag{36}$$

The relation between vorticity spectrum and velocity spectrum can be easily obtained. We have

$$\vec{\Omega} = -\vec{k} \times \vec{\Gamma}(\vec{k},t) - \vec{k} \times \left(2\pi\,\vec{\gamma} - \delta\,\vec{e}_z\right) \tag{37}$$

where

$$\delta = \int_{0}^{1}\int_{0}^{1} (u_z)_{z=0}\, e^{2\pi i(\vec{k}_1\cdot\vec{r})}\, dx\, dy$$

u_z is the velocity in the z direction.

The equation for vorticity spectrum can be directly obtained from the velocity spectrum equation. However, the author will not proceed further on the spectrum discussion at present. The following analysis will be carried out in the physical plane.

We shall frequently refer to x_1, x_2 and x_3 axes as previous x, y, z axis, and ω_i and u_i respectively vorticity and velocity in the x_i direction, $i = 1, 2$ or 3. The Helmholtz vorticity equation is

$$\frac{\partial \omega_i}{\partial t} + u_k \frac{\partial \omega_i}{\partial x_k} - \omega_k \frac{\partial u_i}{\partial x_k} = \nu\, \Delta\, \omega_i \tag{38}$$

We choose our coordinate system such that the $x_1 x_2$ plane corresponds to the grid plane and x_3 in the direction of mean motion. When the grid is absent, we have a parallel uniform flow in the direction with velocity $\mathcal{U}$. The effect of the grid is to produce the cross components of velocity and a fluctuation in the longitudinal components. In other words, the effect of the grid is to produce a random vorticity or velocity field. Denoting the fluctuating components with primes we have

$$u_1 = u_1'$$

$$u_2 = u_2'$$

$$u_3 = \mathcal{U} + u_3'$$

Letting M denote the mesh size at a sufficient distance (e.g. $\frac{z}{M} \sim 100$) downstream from the grid we assume

$$\frac{u_i'}{\mathcal{U}} \ll 1 \qquad\qquad \frac{\partial u_i'}{\partial x_k} \ll \frac{\mathcal{U}}{M}$$

and $\frac{\partial \omega_i}{\partial x_k}$, $\frac{\partial \omega_i}{\partial t}$ and $\frac{\partial^2 \omega_i}{\partial x_j^2}$ are of the same order as $\frac{\partial u_i'}{\partial x_k}$.
With these assumptions, Eq. (38) becomes linearized if we keep only first order terms. The resulting equations are

$$\frac{\partial \omega_i}{\partial t} + \mathcal{U}\frac{\partial \omega_i}{\partial x_3} = \nu\left(\frac{\partial^2 \omega_i}{\partial x_1^2} + \frac{\partial^2 \omega_i}{\partial x_2^2} + \frac{\partial^2 \omega_i}{\partial x_3^2}\right) \tag{39}$$

i = 1, 2 or 3. We see that all vorticity components satisfy the same linear equation. For simplicity we shall drop the subscript i , and it is understood that ω may refer to any of the three components

$$\frac{\partial \omega}{\partial t} + \mathcal{U}\frac{\partial \omega}{\partial z} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2}\right) \tag{40}$$

6a. Non-Viscous Solution

If the viscosity is very small, such that the terms on the right hand side can be neglected, we then have a simple linear first order partial differential equation

$$\frac{\partial \omega}{\partial t} + \mathcal{U} \frac{\partial \omega}{\partial z} = 0 \tag{41}$$

The characteristic of this equation can immediately be seen to be $z = \mathcal{U} t$ + constant. Suppose that the boundary values of ω at $z = 0$ are given as $F(x, y, t)$, then this condition determines the solution uniquely since $z = 0$ is a time-like axis. The solution can be written as

$$\omega = F(x, y, \zeta) \tag{42}$$

where

$$\zeta = \left(t - \frac{z}{\mathcal{U}}\right)$$

If we know the probability distribution for ω at $z = 0$, the corresponding statistical information at any $z > 0$ can be obtained quite easily by Eq. (42). In a more general case $\mathcal{U}$ will be a function of ω. Eq. (42) then provides a non-linear functional relationship between ω. It is the non-linear character that produces the steep-front of vorticity and finally leads to discontinuity.* In the region around the pseudo discontinuity the viscous terms are important, no matter how small the viscosity is.

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*More properly the regrouping of vorticity.

6b. General Solution to the Linearized Equation

We have the general linearized equation

$$\frac{\partial \omega}{\partial t} + U \frac{\partial \omega}{\partial z} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} \right) \tag{40}$$

In the course of linearization we have neglected (1) the vorticity transport due to the fluctuating velocity component and (2) the increase of vorticity due to the deformation of vortex tubes (including both stretching and shear). These limitations should be kept in mind in order to not over emphasize the linearized results.

The problem now is to solve Eq. (40) under the following boundary conditions:

Given at $z = 0$; $\omega = \omega_0 (x, y, t)$

$z \to \infty$; $\omega \to 0$ damping conditions

We assume ω_0 can be expanded as a triple Fourier series in x , y and t . The Fourier coefficients at each fixed time are random variables. (This obviously can be extended to those boundary conditions which can be represented by a Fourier integral or almost periodic function.)

One may easily convince oneself that the solution for this problem with a definite boundary condition at $z = 0$ and a damping at infinity is unique. No uniqueness proof will be given here.

We assume the fundamental solution $\widetilde{\omega}$ is of the following form

$$\tilde{\omega}(\alpha,\beta,\mu) = e^{az}\left\{ f(\alpha,\beta,\mu)\sin(\alpha x+\beta y+\gamma z+\mu t) + g(\alpha,\beta,\mu)\cos(\alpha x+\beta y+\gamma z+\mu t)\right\} \quad (41)^{*}$$

where a and γ can be expressed in terms of α , β and μ (see Appendix VI) and a , α , β , γ and μ are real numbers

$$\begin{aligned} a &= \frac{U}{2\nu}\left\{ 1 - \left[1 + \frac{4\nu^2}{U^2}(\alpha^2+\beta^2+\gamma^2)\right]^{1/2}\right\} \\ \gamma &= -\left\{ -\frac{1}{2}\left[\frac{U^2}{4\nu^2} + (\alpha^2+\beta^2)\right] + \frac{1}{2}\left[\left\{\frac{U^2}{4\nu^2} + (\alpha^2+\beta^2)\right\}^2 + \frac{\mu^2}{\nu^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \end{aligned} \qquad (42)$$

Since Eq. (40) is linear we may by principle of superposition write down the general solution as follows

$$\omega = \sum_{\alpha}\sum_{\beta}\sum_{\mu} \tilde{\omega}(\alpha,\beta,\mu) \qquad (43)$$

We assume the vorticity ω_0 at $Z = 0$ can be expanded into a triple Fourier series. Let us denote the expansion of ω_0 as follows

$$\omega_0(x,y,t) = \sum_{\alpha}\sum_{\beta}\sum_{\mu} f_0(\alpha,\beta,\mu)\sin(\alpha x+\beta y + \mu t) + g_0(\alpha,\beta,\mu)\cos(\alpha x+\beta y+\mu t)$$

For simplicity we assume $\omega_0(x,y,t) = \omega_0(x+2\pi, y+2\pi, t+2\pi)$ by a proper normalization, then both α , β and μ are integers and

$$\begin{aligned} f_0(\alpha,\beta,\mu) &= \frac{1}{\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \omega_0(x,y,t)\sin(\alpha x+\beta y+\mu t)\,dx\,dy\,dt \\ g_0(0,0,0) &= \frac{1}{8\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \omega_0(x,y,t)\,dx\,dy\,dt \end{aligned} \qquad (44)$$

*The corresponding solution for the linearized Navier-Stokes equation can be similarly obtained by considering the effects of random pressure and random velocities at the grid plane separately.

and

$$g_0(\alpha,\beta,\mu) = \frac{1}{\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \omega_0(x,y,t)\cos(\alpha x+\beta y+\mu t)\,dx\,dy\,dt$$

when α , β and μ are not zeros.

The complete solution to the present problem is

$$\omega = \sum_{\alpha,\beta,\mu} e^{az}\left\{ f_0(\alpha,\beta,\mu)\sin(\alpha x+\beta y+\gamma z+\mu t) + g_0(\alpha,\beta,\mu)\cos(\alpha x+\beta y+\gamma z+\mu t)\right\}$$

where f_0 and g_0 are given by Eq. (44), a and γ by Eq. (42). It should again be emphasized that $f_0(\alpha,\beta,\mu)$ and $g_0(\alpha,\beta,\mu)$ are random variables given by the boundary condition. If we assume that $f_0(\alpha,\beta,\mu)$ and $g_0(\alpha,\beta,\mu)$ are statistically independent and that they have the dispersion $\Delta^2(\alpha,\beta,\mu)$, it is very easy to show that the normalized vorticity time correlation function can be expressed as

$$K(t,\tau) = \frac{\overline{\omega(t+\tau)\,\omega(t)}}{\overline{\omega^2(t)}} = \frac{\sum_{\alpha,\beta,\mu} e^{2a(\alpha,\beta,\mu)z}\Delta^2(\alpha,\beta,\mu)\cos\mu\tau}{\sum_{\alpha,\beta,\mu} e^{2a(\alpha,\beta,\mu)z}\Delta^2(\alpha,\beta,\mu)} \qquad (45)\,*$$

We see that $K(t,\tau)$ does not depend on t , hence the process is stationary in the sense we defined before (cf. section 3 Part I).

In the linearized treatment the vorticity is simply transported downstream from the grid with an exponential decay and with a proper amount of phase shift. The rate of decay and the actual phase shifts are in general different for different space wave numbers and time frequencies. Eq. (42) is too complicated for general discussion. In the following different limiting cases will be investigated. At the

*Compare this equation with Eq. (32).

same time the question of whether it is allowable to replace $\frac{\partial}{\partial t}$ by $-\mathcal{U}\frac{\partial}{\partial z}$ for the evaluation of certain statistical averages is also investigated. This replacement is a standard technique and sometimes is necessary for a comparison of theory and experiment. The implicit assumption involved is that "locally we have a uniform statistical pattern which flows with uniform velocity $\mathcal{U}$ in the z direction." We see that this is a rather severe assumption.

The following cases are investigated.

(1) When μ is small, i.e. $\mu \ll \frac{\mathcal{U}^2}{\nu}$, two cases are distinguished.

(a) α, β are small compared with $\frac{\mathcal{U}}{\nu}$

(b) α, β are large compared with $\frac{\mathcal{U}}{\nu}$

(2) When μ is large, i.e. $\mu \gg \frac{\mathcal{U}^2}{\nu}$ and $\alpha, \beta \ll \frac{\mathcal{U}}{\nu}$.

(3) When α, β, μ are large. $\mu \gg \frac{\mathcal{U}^2}{\nu}$; $\alpha, \beta, \gg \frac{\mathcal{U}}{\nu}$

The following table gives the approximate values of a and γ for each of the above cases.

μ \ α, β	Small	large
Small	$\gamma \cong -\frac{\mu}{\mathcal{U}}$ $a \cong -\frac{\nu}{\mathcal{U}}(\alpha^2+\beta^2+\gamma^2)$	$\gamma \cong -\frac{\mu}{2\nu\sqrt{\alpha^2+\beta^2}}$ * $a \cong -\sqrt{\alpha^2+\beta^2} + O\left(\frac{\mathcal{U}}{\nu}\right)$
large	$\gamma \cong -\sqrt{\frac{\mu}{2\nu}}$ $a \cong -\sqrt{\frac{\mu}{2\nu}}$	$\gamma \cong -\left\{-\frac{1}{2}(\alpha^2+\beta^2) + \frac{1}{2}\left[(\alpha^2+\beta^2)^2+\frac{\mu^2}{\nu^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$ $a \cong -\left\{\frac{1}{2}(\alpha^2+\beta^2) + \frac{1}{2}\left[(\alpha^2+\beta^2)^2+\frac{\mu^2}{\nu^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$

In case α , β and μ are small, the solution (Eq. (43)) can be written in the following form

$$\omega \cong \sum_{\alpha,\beta,\mu} e^{-\frac{\nu}{U}(\alpha^2+\beta^2+\gamma^2)} \left\{ f_0 \sin\left[\alpha x+\beta y+\mu\left(t-\frac{z}{U}\right)\right] + g_0 \cos\left[\alpha x+\beta y+\mu\left(t-\frac{z}{U}\right)\right]\right\} \quad (46)$$

It is immediately seen that the replacement of $\frac{\partial}{\partial t}$ by $-U\frac{\partial}{\partial z}$ is justified in this particular case. In all other cases the replacement cannot be justified. Since α and β are generally of the order $\frac{1}{M}$ hence the interchanging of $\frac{\partial}{\partial t}$ and $-U\frac{\partial}{\partial z}$ is permissible if

(i) $$R = \frac{UM}{\nu} \gg 1$$

and

(ii) $$\mu \ll \frac{U^2}{\nu}$$

The first condition requires that the Reynolds number in terms of mesh size be large compared to one. Again we should be aware that this conclusion is reached by the linearized treatment.

The present conclusion is that if the contribution to a given statistical mean value is mostly from the turbulence elements with low frequencies ("low" compared to $\frac{U^2}{\nu}$), then the replacement of $\frac{\partial}{\partial t}$ by $-U\frac{\partial}{\partial z}$ is a good approximation. The approximation gets worse if we try to replace the higher order derivatives $\frac{\partial^n}{\partial t^n}$ by $(-U)^n\frac{\partial^n}{\partial z^n}$, $n=2,3,\cdots$ for the same quantity, because in each differentiation process we introduce more weight on those turbulence elements

*In this range, the decay is independent of viscosity to the first approximation.

with higher frequencies.

7. Study of the Non-Linear Effects

At present we are still quite ignorant of the complete solution to the non-linear Navier-Stokes equation. In order that we shall be able to discuss some of the non-linear effects we are forced to simplify drastically the equations of motion. The simplified equations may not have any physical reality. More properly we should call the simplified equations a mathematical model. The use of mathematical models in studying turbulence was first made by Burgers with some success (Ref. 55). The use of the model is to elucidate some principal features such as the non-stationary character, the non-linear effects and the singular perturbations. The simplest one which will suit our present purpose is the following

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{47}$$

This equation has also been used to some extent by Burgers in his models (see also Cole Ref. (56)). In Burger's paper no exact solution has been given. In this section an exact solution is found and consequently is of more interest. In Eq. (47) u can be interpreted as either of the velocity components in the x , y or z direction. It is a well-known fact that the equation $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$ does permit similarity solutions of the type

$$\begin{aligned} u &= \sqrt{\frac{\nu}{t}} \; f(\eta) \\ \eta &= \frac{x}{2\sqrt{\nu t}} \end{aligned} \tag{48}$$

i.e., the velocity at any point x_1 at time t_1 is equal to the velocity at point x_0 at time t_0 multiplied by $\sqrt{\frac{t_0}{t_1}}$ where

$$\frac{x_0}{\sqrt{t_0}} = \frac{x_1}{\sqrt{t_1}} \tag{49}$$

It is appropriate to ask ourselves "do we still have the similarity solutions if the non-linear term $u\frac{\partial u}{\partial x}$ is present." The answer is in the affirmative. Using the transformation Eq. (48), Eq. (47) then becomes

$$f'' + 2\eta f' - 2f'f + 2f = 0 \tag{50}$$ *

where the prime denotes the differentiation with respect to η.
Eq. (50) can be integrated once and we have

$$f' = (-2\eta f + f^2) - 2\Lambda \tag{51}$$

Λ is an integration constant. One immediately recognizes that this is a type of Riccati equation and can be transformed to a linearized second order equation in S (Ref. 57) by the following transformation

$$S = e^{-\int^{\eta} f\, d\eta} \qquad \text{or} \qquad f = -\frac{d}{d\eta}(\log S) \tag{52}$$

The resulting equation is

$$\frac{d^2S}{d\eta^2} + 2\eta\frac{dS}{d\eta} - 2\Lambda S = 0 \tag{53}$$

The general solution to Eq. (53) is

$$S = A\left\{ {}_1F_1\left(-\frac{\Lambda}{2}; \frac{1}{2}; -\eta^2\right) + k\eta\ {}_1F_1\left(\frac{-\Lambda+1}{2}; \frac{3}{2}; -\eta^2\right)\right\}$$

---

*A particular solution of this equation can be easily written down by inspection namely $f = 2\eta$ or $u = \frac{x}{t}$ which is a non-viscous solution.

Let us denote

$$\mathcal{F} = {}_1F_1\left(-\frac{\Lambda}{2};\ \frac{1}{2};\ -\eta^2\right)$$

$$G = \eta\ {}_1F_1\left(\frac{-\Lambda+1}{2};\ \frac{3}{2};\ -\eta^2\right)$$

and then

$$S = A\left\{\mathcal{F} + \kappa G\right\} \tag{54}$$

κ and A are integration constants. Hence the general similarity solution for u is

$$u = \sqrt{\frac{\nu}{t}}\left\{\frac{\mathcal{F}' + \kappa G'}{\mathcal{F} + \kappa G}\right\} \tag{55}$$

The integration constant A cancels out, and we have two integration constants κ and Λ remaining. These are supposed to be determined from boundary and initial conditions.

When η i.e. $\frac{x}{2\sqrt{\nu t}}$ is small, Eq. (55) can be expanded into power series, we have

$$\mathcal{F} = 1 + \Lambda\eta^2 + \frac{\Lambda(\Lambda-2)}{6}\eta^4 + \cdots$$

$$G = \eta + \frac{(\Lambda-1)}{3}\eta^3 + \frac{(\Lambda-3)(\Lambda-1)}{30}\eta^5 + \cdots$$

hence

$$u = \left(\frac{\nu}{t}\right)^{1/2}\kappa + (2\Lambda - \kappa^2)\frac{x}{2t} + \cdots \qquad \text{for } \eta \ll 1 \tag{56}$$

When η is large the solution of the differential Eq. (53) can be expressed by two semi-convergent series (Ref. 58)

$$S = A_1\left\{\widetilde{\mathcal{F}}(\Lambda) + \kappa_1 \widetilde{G}(\Lambda)\right\}$$

where

$$\tilde{F} = (-\eta^2)^{-\frac{\Lambda}{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{-\Lambda}{2}\right)_n \left(\frac{1-\Lambda}{2}\right)_n}{n!} (\eta^2)^{-n}$$

$$\tilde{G} = e^{-\eta^2} (-\eta^2)^{-\frac{\Lambda+1}{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{\Lambda+1}{2}\right)_n \left(1+\frac{\Lambda}{2}\right)_n}{n!} (-\eta^2)^{-n} \tag{57}$$

hence if $\Lambda \neq 0$ *, we have

$$u \approx \frac{2\nu\Lambda}{x} + O\left(\frac{\nu^{\frac{3}{2}} t^{\frac{1}{2}}}{x^2}\right) \qquad \text{for} \quad \eta \gg 1 \tag{58}$$

The general properties of the solution are:

1. For a fixed finite x, the time history of velocity u is as follows: it starts off with a velocity equal to $\frac{2\nu\Lambda}{x}$, and then at large t it falls off as $1/t^{\frac{1}{2}}$. The curve asymptotically approaches the curve $u t^{\frac{1}{2}} = \nu^{\frac{1}{2}} \kappa$.

2. For a fixed finite t the space distribution of u is as follows: it begins at $x=0$ with a velocity equal to $\left(\frac{\nu}{t}\right)^{\frac{1}{2}} \kappa$ and then falls off as $\frac{1}{x}$ as x approaches infinity i.e., the curve asymptotically approaches the curve $ux = 2\nu\Lambda$. It is clear that the behaviors of the space and time decays are different.

We have seen in Eq. (52) that the logarithmic derivative transformation throws the non-linear Riccati equation into a second order linear differential equation. A natural attempt is then to apply a similar transformation directly to the partial differential Eq. (47) to see whether we shall be able to reduce it to a linear partial

*Two particular values of Λ are of interest. If $\Lambda = 0$ the solution represents the response to a pulse in the non-linear case, and if $\Lambda = 1$ the solution is stationary.

differential equation. Fortunately this works and we may further explore some interesting solutions to help us with the understanding of the non-linear effects.

Introducing the transformation $u = -2\nu \frac{\partial}{\partial x} \log f$, then f satisfies the following equation

$$\frac{\partial f}{\partial t} = \nu \frac{\partial^2 f}{\partial x^2} \tag{59}$$

Eq. (59) is the simplest one-dimensional heat equation. The solutions for initial value problems and radiation problems are well known and can be used in the present discussion.

(A) The Initial Value Problem

Given an initial velocity distribution u_0 at $t = 0$, we are asked to find the velocity field at any later time. Since only first order derivatives in t occur, we need only one initial condition to determine the complete solution. The initial condition in terms of f is

$$\left[-2\nu \frac{\partial}{\partial x} (\log f(x, t)\right]_{t=0} = u_0$$

The derivative on the left hand side is evaluated at $t = 0$. Since the differentiation is in respect to x only we may write

$$u_0 = -2\nu \frac{\partial}{\partial x} \log [f(x, 0)]$$

then the initial condition in the heat equation (59) is $t = 0$, $f = f_0(x)$

$$f_0(x) = e^{-\int^x \frac{u_0(\xi)}{2\nu} d\xi} \tag{60}$$

The general solution of the initial value problem for one-dimensional heat equation can be written as

$$f = \frac{1}{2\sqrt{\pi \nu t}} \int_{-\infty}^{\infty} f_0(\eta)\, e^{-\frac{(x-\eta)^2}{4\nu t}}\, d\eta \qquad (61)$$

Hence the general solution of the initial value problem for the non-linear equation is

$$u = \frac{\int_{-\infty}^{\infty} (x-\eta) \exp\left[-\left(\int^{\eta} \frac{u_0(\xi)}{2\nu} d\xi + \frac{(x-\eta)^2}{4\nu t}\right)\right] d\eta}{t \int_{-\infty}^{\infty} \exp\left[-\int^{\eta} \frac{u_0(\xi)}{2\nu} d\xi - \frac{(x-\eta)^2}{4\nu t}\right] d\eta} \qquad (62)$$

The general properties of solution (62) can be best illustrated by the following two examples:

(A1) Non-Linear Response to a Pulse at the Origin

In this case the initial condition in the physical plane is

$$u = u_0\, \delta(x) \qquad \text{at} \qquad t = 0 \qquad (63)$$

$\delta(x)$ is a Dirac function and $\int_{-\infty}^{\infty} \delta(x)\, dx = 2\ell$, ℓ may be interpreted as the initial spread of disturbance. Hence

$$\begin{aligned} f_0(\eta) &= 1 & \eta &\leq 0 \\ &= e^{-\frac{u_0 \ell}{\nu}} & \eta &> 0 \end{aligned} \qquad (64)$$

$\frac{u_0 \ell}{\nu}$ denotes the Reynolds number in the present case and will be denoted by R. The solution is obtained by substituting $f_0(\eta)$ in Eq. (62)

$$u = 2\sqrt{\frac{\nu}{\pi t}} \; \frac{\exp\left(-\frac{x^2}{4\nu t}\right)}{\coth\left(\frac{R}{2}\right) - \operatorname{erf}\left(\frac{x}{2\sqrt{\nu t}}\right)} \qquad (65)$$

This is a kind of similarity solution and indeed it is one of the similarity solutions we found before having $\Lambda = 0$. The linearized solution in the present case, by neglecting the $u \frac{\partial u}{\partial x}$ term, is u_L

$$u_L = \sqrt{\frac{\nu}{\pi t}} \quad R \quad \exp\left(-\frac{x^2}{4\nu t}\right) \tag{66}$$

The linearized solution is symmetric with respect to $x = 0$, however in the non-linearized case the solution is not symmetric with respect to $x = 0$, and the location for the maximum velocity is displaced toward the positive x axis. We know in the linearized heat equation that the total amount of heat introduced is preserved, or in the present case we should interpret that the disturbance moment D is preserved. This fact can be immediately verified. If we integrate in Eq. (66) with respect to x from minus infinity to plus infinity and keeping time fixed we have

$$D = \sqrt{\frac{\nu}{\pi t}} \; R \int_{-\infty}^{\infty} e^{-\frac{x^2}{4\nu t}} \, dx = 2\nu R = 2 u_0 \ell \tag{67}$$

In the non-linearized case the same statement holds true. We have

$$D_n = 2\sqrt{\frac{\nu}{\pi t}} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{x^2}{4\nu t}\right)}{\coth\left(\frac{R}{2}\right) - \operatorname{erf}\left(\frac{x}{2\sqrt{\nu t}}\right)} \, dx$$

$$= 2\nu \log \left| \frac{\coth\left(\frac{R}{2}\right) + 1}{\coth\left(\frac{R}{2}\right) - 1} \right| = 2\nu R = 2 u_0 \ell \tag{68}$$

Hence we know there is a certain "disturbance moment" which is invariant even in the non-linear case. This concept is an accordance with the Loitsiansky invariant.

We expect that for the small Reynolds number, the linearized solution should give a good approximation. Indeed this is true in the present case. As $R \to 0$; $\coth \frac{R}{2} \sim \frac{2}{R}$, since $|\operatorname{erf} \frac{x}{2\sqrt{\nu t}}| \leq 1$ the denominator in Eq. (66) simply reduces to $\frac{2}{R}$, and the non-linearized solution coincides exactly with the linearized solution.

Next, we expect that the solution will approach the linearized solution at very large t . Let us consider when x is finite. Then as t approaches infinity, the term $\frac{x}{2\sqrt{\nu t}}$ can be neglected in comparison with $\coth \frac{R}{2}$. Hence the solution becomes

$$u \cong 2\sqrt{\frac{\nu}{\pi t}} \tanh\left(\frac{R}{2}\right) \exp\left(-\frac{x^2}{4\nu t}\right) \tag{69}$$

Eq. (69) almost agrees with Eq. (66). However, the linearized solution fails to predict the correct amplitude of the solution.

An asymptotic solution for very large Reynolds number may be written as

$$u \cong 2\sqrt{\frac{\nu}{\pi t}} \frac{\exp\left(R - \frac{x^2}{4\nu t}\right)}{e^R \operatorname{erf} \frac{x}{2\sqrt{\nu t}} - 2} \tag{70}$$

When R is large, then for any finite x , "2" in the denominator can be neglected. The solution than approaches the limit for infinitely large Reynolds number. It is of interest to study the motion of the location of maximum velocity X_{max}. From a dimensional reasoning $\sqrt{\frac{\nu}{t}}$ defines an intrinsic velocity in the field, we have

$$\dot{X}_{max} = \eta_0 \sqrt{\frac{\nu}{t}}$$

η_0 is a function of Reynolds number. $\eta_0 = 0$ when $R = 0$ and $\eta_0 = \infty$ when $R = \infty$.

The δ function as initial condition is typical in an infinite domain of fluid. We shall now treat another typical example in a bounded domain.

(A2) Response to a Sinusoidal Disturbance

Let us consider x lies within a closed interval $-\frac{\pi}{k} \leq x \leq \frac{\pi}{k}$. The initial condition is

$$u_0(x) = 2u_0 \sin(kx) \tag{71}$$

and the solution should satisfy the boundary conditions $u(x,t) = 0$ at $x = \frac{\pi}{k}$ and $-\frac{\pi}{k}$. The characteristic Reynolds number R in the present case is $\frac{u_0}{\nu k}$.

We may reduce the problem to an unbounded domain $-\infty < x < \infty$, and the corresponding initial condition is

$$u_0(x) = 2u_0 \sin(kx) \qquad -\infty < x < \infty$$

The general formula (62) can now be applied, the solution is

$$u = \frac{\int_{-\infty}^{\infty} (x-\eta) \exp\left[-\frac{(x-\eta)^2}{4\nu t} + R\cos(\eta k)\right] d\eta}{t \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\eta)^2}{4\nu t} + R\cos(\eta k)\right] d\eta} \tag{72}$$

Using the known expansion formula

$$e^{z\cos\theta} = I_0(z) + 2\sum_{n=1}^{\infty} I_n(z) \cos n\theta \tag{73}$$

$I_n(z)$ is the modified Bessel function of the first kind of order n. The integrals in Eq. (72) can be integrated out and the final solution is

$$u = \frac{4\nu \sum_{n=1}^{\infty} \left\{ nk\, I_n(R)\, e^{-\nu k^2 t n^2} \sin(nkx) \right\}}{I_0(R) + 2\sum_{n=1}^{\infty} I_n(R)\, e^{-\nu n^2 k^2 t} \cos(nkx)} \tag{74}$$

It is not difficult to verify that the boundary conditions and initial conditions are satisfied. The solution for the initial condition $u_0(x) = \bar{U} + 2u_1 \sin Kx$ where $\bar{U}$ is a constant velocity is

$$u = \bar{U} + \frac{4\nu \sum_{n=1}^{\infty} \left\{ nK\, I_n(R)\, e^{-\nu K^2 n^2 t} \sin\left[nK(x - \bar{U}t)\right] \right\}}{I_0(R) + 2\sum_{n=1}^{\infty} I_n(R)\, e^{-\nu n^2 K^2 t} \cos\left[nK(x - \bar{U}t)\right]} \tag{75}$$

The effect of uniform flow $\bar{U}$ is to produce a corresponding phase shift for each Fourier component.

When $R \ll 1$, solution (74) becomes

$$u \cong 2u_0 \sin(Kx)\, e^{-\nu K^2 t} \tag{76}$$

Eq. (76) coincides with the linearized solution. In this case the linearized solution again is a good approximation when $R \ll 1$. In the non-linear solution all the higher harmonics are excited. This characteristic feature due to the inertia terms is absent in all linearized theory. Since all higher harmonics damp faster than the fundamental harmonic when t is sufficiently large, Eq. (74) can be approximated as follows

$$u \cong \frac{4\nu K\, I_1(R)}{I_0(R)}\, e^{-\nu K^2 t} \sin(Kx) \tag{77}$$

This is the form of a linearized solution. As in the previous case, the linearized solution fails to predict the correct amplitude. It is believed that the effect of non-linear terms is to produce steeper fronts where the viscous dissipation is more effective. Hence the amplitude given by Eq. (77) should be smaller than that given by linearized theory. Let us denote the amplitude given by linearized theory by A_L and that given by non-linear theory A_N, then the amplitude ratio $A_L / A_N = \psi(R)$

$$\psi(R) = \frac{R \; I_0(R)}{2 \; I_1(R)} \tag{78}$$

Some numerical values are given in the following:

R	0.1	1.0	10.0	16.0
$\psi(R)$	1.0125	1.1204	5.25	8.21

We see when R = 16 the amplitude given by linearized theory is about 8 times larger. A next higher approximation for the spectrum distribution at large t in the non-linear case can be written as

$$u \cong \sum_{n=1}^{\infty} (-1)^{n-1} 4\nu K \left(\frac{I_1(R)}{I_0(R)} \right)^n e^{-\nu K^2 t n} \sin(nKx) \tag{79}$$

We have seen in the previous discussions on spectrum and correlation functions the Kolmogoroff theory applies very well in the case of indefinitely large Reynolds number. It is interesting to see what the corresponding limiting solution is in the present case. The asymptotic series for $I_n(R)$ when R is large is

$$I_n(R) \approx \frac{e^R}{\sqrt{2\pi R}} \left[1 - \frac{4n^2 - 1^2}{1!\, 8R} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\, (8R)^2} - \cdots \right] \tag{80}$$

The first terms in the asymptotic expansions of I_n are the same for all finite n, hence as $R \to \infty$ the limiting solution is u_∞

$$u_\infty = \frac{4\nu \sum_{n=1}^{\infty} \left\{ nK \, e^{-\nu n^2 K^2 t} \sin(nKx) \right\}}{1 + 2 \sum_{n=1}^{\infty} e^{-\nu n^2 K^2 t} \cos(nKx)} \tag{81}$$

Eq. (81) can be rewritten in terms of Θ function (Ref. 59)

$$u_\infty = \kappa\nu \frac{\partial}{\partial z} \log \left[\Theta_3 (z ; q) \right]$$

$$q = e^{-\nu\kappa^2 t} \tag{82}$$

$$z = \frac{\kappa x}{2}$$

Eq. (82) can be further simplified (Ref. 59)

$$u_\infty = 4\kappa\nu \sum_{n=1}^{\infty} \frac{e^{-n\nu\kappa^2 t} (-)^n \sin(n\kappa x)}{1 - e^{-2n\nu\kappa^2 t}}$$

$$= 2\kappa\nu \sum_{n=1}^{\infty} \frac{(-)^n \sin(n\kappa x)}{\sinh(n\nu\kappa^2 t)} = \sum_{n=1}^{\infty} \xi_n \tag{83}$$

It is seen that when $\nu\kappa^2 t \ll 1$, then most of the Fourier components ξ_n are proportional to $\frac{1}{n}$, or the energy per each component $\left(E_n = \frac{1}{2} |\xi_n|^2 \right.$ is proportional to $1/n^2$. However, when n is very large, such that $n\nu\kappa^2 t \gg 1$, the amplitude falls off exponentially. This fact was interpreted by Burgers, that there exists a practical limit for the range of turbulent spectrum. It is obvious that the low wave number components contribute the most energy. The dissipation per each Fourier component is proportional to $n^2 E_n$, hence in this case we have an equi-dissipation of energy for low wave number components.

The infinite Fourier series in Eq. (83) can be approximately summed if $\nu\kappa^2 t \ll 1$, the result is

$$u_\infty = \frac{x}{t} \left\{ \left(1 - \frac{\pi}{\kappa x} \right) + \frac{\pi}{\kappa x} \tanh \left[\frac{\pi}{2\nu\kappa t} \left(\frac{\pi}{\kappa} - x \right) \right] \right\} \tag{84}$$

in the region $0 \leq x \leq \frac{\pi}{k}$. Solution in this form shows that there is a steep front developed by the non-linear term in the neighborhood of $x = \frac{\pi}{k}$.

We could go on to discuss the space correlations in the respective cases. However, the mathematical expression gets complicated and hardly any definite physical significance can be drawn. In the next section we shall discuss another class of problem known as the radiation problem.

(B) Radiation Problem

In this class of problem we restrict ourselves to the domain $0 \leq t \leq \infty$, $0 \leq x \leq \infty$. Velocity $u_0(t)$ is prescribed at $x = 0$ for $t > 0$. For simplicity we assume that there is no disturbance initially, i.e. at $t = 0$, $u = 0$ for all $x > 0$. Then the problem is to find the velocity field for all x at any time $t > 0$. The fundamental equation used for this problem is again Eq. (47)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{47}$$

This class of problem is difficult and no general solution such as Eq. (62) in the last section has yet been obtained. The uniqueness of this problem is still questionable. The following gives a method of approach.

First we have to rephrase our boundary conditions in terms of f and the corresponding heat equation.

(1) $x = 0$, $u = u_0(t)$ implies $-2\nu \frac{\partial}{\partial x} \log f = u_0(t)$. Let $h(t) = \frac{u_0(t)}{2\nu}$ then

$$\left(\frac{\partial f}{\partial x}\right)_{x=0} + h(t) f = 0 \qquad \text{at} \quad x = 0 \tag{a}$$ *

*First differentiate with respect to x and then set $x = 0$.

(ii) $t=0$, $u=0$ implies $f=\alpha$ at $t=0$ (b)

α is certain constant. The result which we want to get should be independent of α , since α does not come into the actual physical problem.

(iii) u is bounded and integrable with respect to X implies that f is bounded. (c)

The equation for f is the one-dimensional heat equation

$$\frac{\partial f}{\partial t} = \nu \frac{\partial^2 f}{\partial X^2} \tag{59}$$

The following analysis will be based on the theory of Laplace transform. Reference should be made to standard treatise on Laplace transform (for example Ref. (60)).

Define the Laplace transform of f by $\bar{f}(\sigma)$

$$\bar{f}(\sigma) = \int_0^\infty f\, e^{-\sigma t}\, dt \tag{85}$$

Eq. (59) becomes

$$\frac{\partial^2 \bar{f}}{\partial X^2} - \frac{\sigma}{\nu}\bar{f} = -\frac{\alpha}{\nu} \tag{a$'$}$$

Boundary conditions (a), (b) and (c) become

(i) $\frac{\partial \bar{f}}{\partial X} + \frac{1}{2\pi i}\int_{\circlearrowleft} \bar{h}(\sigma - z)\,\bar{f}(z)\, dz = 0$ at $X=0$ (b$'$)

(ii) $X \to \infty$; $\bar{f}$ is finite (c$'$)

where $\bar{h}(\sigma)$ is the Laplace transform of $h(t)$ and the contour integration (indicated by $\circlearrowleft$) is carried out around a suitable contour.

The appropriate solution to Eq. (a$'$) is

$$\bar{f} = B(\sigma)\, e^{-\sqrt{\frac{\sigma}{\nu}}\,x} + \frac{\alpha}{\sigma} \tag{86}$$

where B in general is a function of σ. $B(\sigma)$ can be determined from the boundary condition (b$'$). Substituting Eq. (86) in (b$'$) we have an integral equation for $B(\sigma)$, namely

$$-\sqrt{\frac{\sigma}{\nu}}\; B(\sigma) + \frac{1}{2\pi i}\int_{\sigma} \bar{h}\,(\sigma - z)\left[\, B(z) + \frac{\alpha}{z}\right] dz = 0 \tag{87}$$

Having determined $B(\sigma)$ from a given boundary condition, we then by substituting the result in Eq. (86) and transforming Eq. (86) back, arrive at the required solution for the given problem.

Only a simple example is worked out here. The problem and its connection with statistical theory need further investigations.

Let us suppose at $x = 0$, $u = u_0$ for all $t > 0$; u_0 may be interpreted as stochastic constant. Then $\bar{h} = \frac{u_0}{2\nu\sigma}$ and the resulting integral equation is

$$-\sqrt{\frac{\sigma}{\nu}}\; B(\sigma) + \frac{1}{2\pi i}\int_{\sigma} \frac{u_0}{2\nu}\left\{ \frac{B(z)}{\sigma - z} + \frac{\alpha}{z(\sigma - z)}\right\} dz = 0 \tag{88}$$

Fortunately, this integral equation can be solved and the result is

$$B(\sigma) = \frac{\alpha\,\frac{u_0}{2\nu}}{\sigma\left[\sqrt{\frac{\sigma}{\nu}} - \frac{u_0}{2\nu}\right]}$$

Hence

$$f = \alpha\left\{ \operatorname{erf}\left(\frac{x}{2\sqrt{\nu t}}\right) + e^{-\frac{u_0 x}{2\nu} + \frac{u_0^2 t}{4\nu}} \operatorname{erfc}\left(\frac{x}{2\sqrt{\nu t}} - \frac{u_0}{2}\sqrt{\frac{t}{\nu}}\right)\right\}$$

The solution for u is

$$u = u_0 \left\{ \frac{e^{-\frac{u_0 x}{2\nu} + \frac{u_0^2}{4\nu} t} \operatorname{erfc}\left(\frac{x}{2\sqrt{\nu t}} - \frac{u_0}{2}\sqrt{\frac{t}{\nu}}\right)}{\operatorname{erf} \frac{x}{2\sqrt{\nu t}} + e^{-\frac{u_0 x}{2\nu} + \frac{u_0^2}{4\nu} t} \operatorname{erfc}\left(\frac{x}{2\sqrt{\nu t}} - \frac{u_0}{2}\sqrt{\frac{t}{\nu}}\right)} \right\} \tag{89}$$

It is easy to verify that this satisfies the required boundary conditions. The linearized solution u_L for this problem is well-known

$$u_L = u_0 \operatorname{erfc} \frac{x}{2\sqrt{\nu t}} \tag{90}$$

Again when $\frac{u_0 x}{2\nu} \ll 1$ and $\frac{u_0^2}{4\nu} t \ll 1$, the non-linear solution approaches the linearized solution. When $x \gg u_0 t$ we have

$$u \cong u_0 e^{-\frac{u_0 x}{2\nu}} \operatorname{erfc} \frac{x}{2\sqrt{\nu t}} \tag{91}$$

The amplitude in Eq. (91) is much smaller than that predicted by the linearized theory.

APPENDIX I

A. Notion of Time Averages

With the definition given by Eq. (4), the postulates (i) and (iv) can be easily satisfied. However, (b) and (c) are not in general true. This can be seen from the examples below.

Assume

$$A = \sin t$$

then

$$\bar{A} = \frac{1}{\tau}\left[\cos t - \cos(t+\tau)\right]$$

$$\bar{\bar{A}} = \frac{1}{\tau^2}\left[2\sin(t+\tau) - \sin t - \sin(t+2\tau)\right]$$

hence $\bar{A} \neq \bar{\bar{A}}$ in general.

Next assume

$$B = \cos t$$

then

$$\bar{B} = \frac{1}{\tau}\left[\sin(t+\tau) - \sin t\right]$$

and

$$\overline{\bar{A}\,B} = \frac{1}{\tau^2}\Big[\left\{\frac{\tau}{2} + \frac{1}{4}\left(\sin(t+\tau) - \sin t\right)\right\}(1-\cos\tau) + \frac{\sin\tau}{2}\left\{\sin^2(t+\tau) - \sin^2 t\right\}\Big]$$

hence

$$\overline{\bar{A}\,B} \neq \bar{A}\,\bar{B}$$

The conclusion is that the iteration property and the associativity condition cannot in general be satisfied. However, for this simple example, we could have chosen $\tau = 2n\pi$, then the postulates

(b) and (c) are satisfied. Again this reduces to the theoretical problem of how to determine an appropriate time interval τ in general.

B. Notion of Space Averages

The discussions on the first three postulates are similar to those for time averages. The postulate (d) needs a further consideration. The following discussion is limited to one-dimensional (the generalization to cases for more than one-dimension is obvious).

Let us suppose we have the same notion of neighborhoods for every point in the one-dimensional space. Denote a point in the space. We define specifically the neighborhood as follows, namely

$$x \in N_a(x_0) \iff |x - x_0| < a$$

where a is a given positive real constant. The space average is then defined as

$$\bar{A}(x) = \frac{1}{2a}\int_{x-a}^{x+a} A(\xi)\, d\xi$$

By definition

$$\bar{A}(x+\Delta x) = \frac{1}{2a}\int_{x+\Delta x-a}^{x+\Delta x+a} A(\xi)\, d\xi = \frac{1}{2a}\int_{x-a}^{x+a} A(\xi+\Delta x)\, d\xi$$

Hence the derivative of $\bar{A}(x)$ is

$$\frac{d\bar{A}(x)}{dx} = \lim_{\Delta x \to 0} \frac{\bar{A}(x+\Delta x) - \bar{A}(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{2a}\int_{x-a}^{x+a} \frac{A(\xi+\Delta x) - A(\xi)}{\Delta x}\, d\xi$$

$$= \frac{1}{2a}\int_{x-a}^{x+a} A'(\xi)\, d\xi$$

where prime denotes differentiation with respect to ξ .

APPENDIX II

A Simple Cascade Process

The starting point of the following analysis is the Helmholtz vorticity equation, i.e.

$$\frac{\partial \omega_i}{\partial t} + u_k \frac{\partial \omega_i}{\partial x_k} - \omega_k \frac{\partial u_i}{\partial x_k} = \nu \nabla^2 \omega_i \tag{a}$$

We assume without proving the convergence that ω_k and u_i can be expanded in Taylor series for sufficiently small values of t as follows

$$\begin{aligned} \omega_i &= \omega_i^{(0)} + \frac{\nu t}{L^2} \omega_i^{(1)} + \left(\frac{\nu t}{L^2}\right)^2 \frac{\omega_i^{(2)}}{2!} + \cdots\cdots \\ u_i &= u_i^{(0)} + \left(\frac{\nu t}{L^2}\right) u_i^{(1)} + \left(\frac{\nu t}{L^2}\right)^2 \frac{u_i^{(2)}}{2!} + \cdots\cdots \end{aligned} \tag{b}$$

L represents a characteristic length in the flow field. $\omega_i^{(n)}$ are functions of position only. $\omega_i^{(0)}$ and $u_i^{(0)}$ are the initial velocity and vorticity distributions which are supposed to be given in the present case. Substituting (b) into (a) we have

$$\begin{aligned} \omega_i^{(1)} &= L^2 \nabla^2 \omega_i^{(0)} + \frac{L^2}{\nu} \left[\omega_k^{(0)} \frac{\partial u_i^{(0)}}{\partial x_k} - u_k^{(0)} \frac{\partial \omega_k^{(0)}}{\partial x_k} \right] \\ \omega_i^{(2)} &= L^2 \nabla^2 \omega_i^{(1)} + \frac{L^2}{\nu} \left[\omega_k^{(0)} \frac{\partial u_i^{(1)}}{\partial x_k} + \omega_k^{(1)} \frac{\partial u_i^{(0)}}{\partial x_k} - u_k^{(1)} \frac{\partial \omega_i^{(0)}}{\partial x_k} - u_k^{(0)} \frac{\partial \omega_i^{(1)}}{\partial x_k} \right] \\ &\cdots\cdots \end{aligned} \tag{c}$$

We see in principle that if we know the initial velocity distribution then we can calculate the initial vorticity distribution, hence $\omega_i^{(1)}$ the first approximation can be calculated directly. Having known we may by introducing the vector potential* evaluate the first approximation of the velocity and then proceed to calculate the second vorticity approximation. The method is tedious and only the first approximation will be given here. In this particular example we consider the initial velocity distribution as follows:

$$
\begin{aligned}
u_1 &= A_1 \sin a_1 x_1 \cos a_2 x_2 \cos a_3 x_3 \\
u_2 &= A_2 \cos a_1 x_1 \sin a_2 x_2 \cos a_3 x_3 \\
u_3 &= A_3 \cos a_1 x_1 \cos a_2 x_2 \sin a_3 x_3
\end{aligned}
\tag{d}
$$

Velocity distribution in (d) represents the flow inside a rectangular parallelepiped $\left(\frac{\pi}{a_1} \times \frac{\pi}{b_1} \times \frac{\pi}{c_1}\right)$ inside the fluid.

On account of the equation of continuity we must restrict ourselves to

$$\sum_{i=1}^{3} A_i a_i = 0$$

The computation is straight forward and will not be given here. The first approximation is

Let $$K = a_1^2 + a_2^2 + a_3^2$$

$$\omega_1^{(1)} = -L^2 K^2 \omega_1^{(0)} - \frac{L^2}{2\nu} A_1 a_1 (A_3 a_2 - a_3 A_2) \sin(2 a_2 x_2) \sin(2 a_3 x_3)$$

$$P_1^{(1)} = -L^2 K^2 P_1^{(0)} - \frac{L^2}{\nu} \frac{A_1 a_1 (A_3 a_2 - A_2 a_3) \sin(2 a_2 x_2) \sin(2 a_3 x_3)}{8 (a_2^2 + a_3^2)}$$

*The vector potential $\vec{P}$ must also satisfy $\nabla \cdot \vec{P} = 0$.

$$u_1^{(1)} = -L^2 K^2 u_1^{(0)} - \frac{L^2}{\nu} \left\{ \frac{A_3 a_3 (A_2 a_1 - A_1 a_2) a_2 \sin(2 a_1 x_1) \cos(2 a_2 x_2)}{4 (a_1^2 + a_2^2)} \right.$$

$$\left. - \frac{A_2 a_2 (A_1 a_3 - A_3 a_1) a_3 \sin(2 a_1 x_1) \cos(2 a_3 x_3)}{4 (a_1^2 + a_3^2)} \right\} \qquad \text{(f)}$$

The other components can be obtained by cyclic permutations of the above formulas. It is not difficult to see that the first terms represent the viscous dissipation of the primary components $u_i^{(0)}$, $\omega_i^{(0)}$. The second terms represent the superposed flow on the primary components. They represent the flow inside a new rectangular parallelepiped of 1/8 of the original volume.

APPENDIX III

The asseration that "Taylor's spectrum $E(\kappa_1)\, d\kappa_1$ represents the contribution to $\overline{u_1^2}$ from those Fourier components having in x_1 direction the wave number between κ_1 and $\kappa_1 + d\kappa_1$ " can be made clearer if we make a direct calculation.

From Eq. (79) we have (assume the flow is isotropic)

$$\overline{u_1^2} = \iiint_{-\infty}^{\infty} \Phi_{11}(\vec{\kappa})\, d\tau(\vec{\kappa})$$

$$= 2 \int_0^{\infty} d\kappa_1 \iint dA(\kappa)\, \Phi_{11}(\kappa) \tag{a}$$

where dA_κ is the element of surface area of the plane perpendicular to κ_1 axis at κ_1 and the double integration is carried out over the entire plane. Hence our previous assertion corresponds to

$$E(\kappa_1) = 2 \iint \Phi_{11}(\kappa)\, dA(\kappa) \tag{b}$$

Using Eq. (77) we have

$$\Phi_{11}(\kappa) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} dx_1 \iint dA(r)\, R_{11}^{*}(x_1, x_2, x_3)\, e^{i(\kappa x)} \tag{c}$$

where $dA(r)$ bears the same significance as $dA(\kappa)$ in the physical space. Substituting (c) into (b) we obtain

$$E(\kappa_1) = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} e^{ik_1 x} \iiiint R_{11}^{*}(x_1, x_2, x_3)\, e^{i(k_3 x_3 + k_2 x_2)}\, dA(\kappa)\, dA(r)\, dx_1$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{ik_1 x_1} \left[\iint R_{11}^{*}(x_1, x_2, x_3)\, \delta(x_2)\, \delta(x_3)\, dA(r) \right] dx_1$$

---

* δ is the Dirac function.

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i k_1 x_1} R_{11}^{*}(x_1, 0, 0) \, dx_1$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} u^2 f(x_1) \, e^{i k_1 x_1} \, dx_1$$

$$= \frac{2 \overline{u^2}}{\pi} \int_{0}^{\infty} f(x_1) \cos(k_1 x_1) \, dx_1 \qquad \text{(d)}$$

Q.E.D.

APPENDIX IV

In this appendix the solution for the spectrum with Heisenberg's assumption will be derived, together with a more detailed investigation of von Karman's assumption.

von Karman suggested that W_k be represented in the following form

$$W_k = -C\left[\mathcal{F}^{\alpha} k^{\beta}\int_0^k \mathcal{F}^{3/2-\alpha}(k')\, k'^{1/2-\beta}\, dk' - \mathcal{F}^{3/2-\alpha} k^{1/2-\beta}\int_k^\infty \mathcal{F}^{\alpha}(k')\, k'^{\beta}\, dk'\right] \quad \text{(a)}$$

where C is a constant. The condition for low wave number range of the equilibrium part of the spectrum is exactly $W_k = 0$. Physically this means the amount of energy transferred into that wave number interval is equal to the amount of energy transferred out from that interval. If we set (a) $W_k = 0$ and let

$$\begin{aligned} \psi &= \int_0^k \mathcal{F}^{3/2-\alpha}(k')\, k'^{1/2-\beta}\, dk' \\ \eta &= \int_\infty^k \mathcal{F}^{\alpha}(k')\, k'^{\beta}\, dk' \end{aligned} \quad \text{(b)}$$

then we have

$$\begin{aligned} \mathcal{F} &= \psi'^{\frac{2}{3-2\alpha}}\, k^{\frac{2\beta-1}{3-2\alpha}} \\ \eta' &= \psi'^{\frac{2\alpha}{3-2\alpha}}\, k^{\frac{3\beta-\alpha}{3-2\alpha}} \end{aligned} \quad \text{(c)}$$

Equation $W_k = 0$ then becomes

$$k^{\frac{3\beta-\alpha}{3-2\alpha}}\, \psi^2\, \psi'^{\frac{4\alpha-3}{3-2\alpha}} = C_1 \quad \text{(d)}$$

where C_1 is an integration constant. Prime denotes differentiation with respect to κ. If $4\alpha - 3 \neq 0$ i.e. $\alpha \neq 0.75$, the general solution can be expressed as

$$\psi = \left[A' \kappa^{\frac{5\alpha - 3\beta - 3}{4\alpha - 3}} + B' \right]^{\frac{4\alpha - 3}{3}}$$

and

$$\mathcal{F} = \left[A \kappa^{\frac{5\alpha - 3\beta - 3}{4\alpha - 3}} + B \right]^{-4/3} \kappa^{\frac{1 - 4\beta/3}{4\alpha - 3}} \qquad \text{(e)}^{*}$$

It is easy to verify if $B = 0$ in (e) then

$$\mathcal{F} \simeq \kappa^{-5/3} \qquad \text{(f)}$$

This explains why any dimensionally correct assumption leads to the well-known -5/3 power law.

If we put Heisenberg's value of α and β in solution (e) we have $B \neq 0$

$$\mathcal{F} \sim \frac{\kappa^{-5/3}}{(1 + \gamma\kappa^{4})^{4/3}} \qquad \text{(g)}$$

This is exactly the form of Eq. (103). One would be surprised at the first instant because this derivation does not involve the viscous terms. However, a closer investigation shows that with the Heisenberg special choice of α and β, the presence of viscous terms does not affect the reduction to the differential equation (d). The details will not be reproduced here.

---

*This solution does not make sense if the convergence conditions of the integral W_κ are not satisfied.

If $4\alpha - 3 = 0$ i.e. $\alpha = 0.75$, the equation $W_k = 0$ has a unique solution $\mathcal{F} \sim k^{-5/3}$ provided $\beta \neq 0.25$.

However, when $\alpha = 0.75$; $\beta = 0.25$ the solution for the equilibrium spectrum can be solved explicitely. We have in this case

$$C\left(\int_0^k \mathcal{F}^{3/4} k'^{1/4} dk' - \int_k^\infty \mathcal{F}^{3/4} k'^{1/4} dk'\right) + 2\nu k^{\frac{7}{4}} \mathcal{F}^{\frac{1}{4}} = 0 \qquad \text{(h)}$$

Differentiating

$$C\, \mathcal{F}^{3/4} k^{1/4} + \nu\left(\frac{7}{4} k^{\frac{3}{4}} \mathcal{F}^{\frac{1}{4}} + \frac{1}{4} k^{\frac{7}{4}} \mathcal{F}^{-\frac{3}{4}} \mathcal{F}'\right) = 0$$

or

$$C\, \mathcal{F}^{3/2} k^{-3/2} + \nu\left(\frac{7}{4} k^{-1} \mathcal{F} + \frac{1}{4} \mathcal{F}'\right) = 0$$

Let

$$\mathcal{F} = \sqrt{\frac{\nu}{c}}\, G$$

then

$$G^{3/2} k^{-3/2} + \left(\frac{7}{4} k^{-1} G + \frac{1}{4} G'\right) = 0 \qquad \text{(i)}$$

Solving (i)

$$G = \frac{k}{(1 + A k^4)^2}$$

A is an integration constant. Or

$$\mathcal{F} = \sqrt{\frac{\nu}{c}}\; \frac{k}{(1 + A k^4)^2} \qquad \text{(j)}$$

The solution (j) behaves linearly when k is small and behaves like the k^{-7} law when $k \to \infty$, the same as Heisenberg's prediction.

APPENDIX V

In this appendix we prove the following theorem (the proof is originally due to J. L. Doob).

Theorem: Given a one-dimensional stochastic process $u(t)$ of continuous time variable t. Assume the following are true:

A1. The process is stationary, i.e. temporally homogeneous

A2. The process is Markhoffian

A3. For any given t_1 and t_2 the joint probability distribution of the random variables $u(t_1)$ and $u(t_2)$ is double Gaussian; then we obtain

R1. For $t_1 < t_2 \cdots < t_n$, the joint probability distribution of random variables $u(t_1), u(t_2), \cdots, u(t_n)$ is n-variate Gaussian

R2. The correlation function $\dfrac{\overline{u(t)\,u(t+s)}}{\overline{u^2(t)}} = e^{-\beta|s|}$

where the bars here may be interpreted as time average.

Proof: For simplicity we shall assume the mean of $u(t) = 0$ and the dispersion $\overline{u^2(t)} = 1$ (normalization).

First we note that since the process is stationary, we can identify the time average as stochastic average. By A1 the correlation function $r(t,s) = \overline{u(t)\;u(t+s)}$ is independent of t. Hence we may write

$$r(s) = \overline{u(t)\;u(t+s)} = \mathcal{E}\{u(t)\;u(t+s)\} \tag{a}$$

By A3 we can write the joint probability distribution P_2 for $u(t)$ and $u(t+s)$ assuming $s > 0$

$$P_2 = \frac{1}{2\pi\,(1-r^2(s))^{1/2}} \exp\left\{ - \frac{u^2(t) - 2r\,u(t)\,u(t+s) + u^2(t+s)}{2\,(1-r^2(s))} \right\} \tag{b}$$

r is only a function of s . By the method of desent the probability distribution for $u(t)$ is one-dimensional Gaussian

$$P_1 = \frac{1}{(2\pi)^{1/2}} \exp\left(- \frac{u^2(t)}{2}\right) \tag{c}$$

But

$$P_2 = P_1\,T \tag{d}$$

T is the transition probability, i.e. the probability that $u(t+s)$ at a certain value knowing $u(t)$ at a definite value. By A2 the transition probability T together with the probability distribution of one initial random variable specifies the Markhoff process completely. Combining Eqs. (b), (c) and (d) we have

$$T = \frac{1}{(2\pi)^{1/2}(1-r^2(s))^{1/2}} \exp\left\{ - \frac{[u(t+s) - r\,u(t)]^2}{1 - r^2(s)} \right\} \tag{e}$$

where r is a function of s , the time interval between the transition.

Introducing the notation for the correlation function

$$r_i = \overline{u(t_i)\;u(t_{i+1})} \qquad r_i \text{ a function of } t_{i+1} - t_i \text{ only} \tag{f}$$

where $t_{i+1} \geq t_i$. The the n-dimensional probability distribution for $u(t_1), \cdots\cdots, u(t_n)$ can be expressed as

$$P_n = \frac{1}{(2\pi)^{\frac{n}{2}} \prod_1^{n-1} (1-r_j^2)^{1/2}} \exp\left(- \frac{1}{2}u_1^2 - \frac{1}{2}\sum_1^{n-1} \frac{(u_{j+1} - r_j\,u_j)^2}{1 - r_j^2} \right) \tag{g}$$

By expanding out the quadratics in Eq. (g), R1 is easily verified. Let us now take the trivariate Gaussian distribution for $u(t_1)$, $u(t_2)$ and $u(t_3)$ assuming $t_1 \leq t_2 \leq t_3$ and then calculate the expectation value of $u(t_3)\,u(t_1)$. This value is the correlation function $r(s)$ evaluated at $s = t_3 - t_1$. After integration we get the following functional equation

$$r(t_3 - t_1) = r(t_3 - t_2)\, r(t_2 - t_1) \tag{h}$$

for any $t_1 \leq t_2 \leq t_3$. The unique non-trivial non-singular solution of the above equation is

$$r(t_3 - t_1) = e^{-\beta(t_3 - t_1)} \qquad \beta > 0$$

Hence R2 is proved.

APPENDIX VI

If we substitute $\tilde{\omega}$ in Eq. (40) we get

$$[(\mu + \mathcal{U}\gamma - 2a\nu\gamma) f(\alpha,\beta,\mu) + (\mathcal{U}a + \nu\gamma^2 - \nu a^2 + \nu(\alpha^2+\beta^2))\, g(\alpha,\beta,\mu)] \cos(\alpha x + \beta y + \gamma z + \mu t)$$

$$+ [(-\mu - \mathcal{U}\gamma + 2a\nu\gamma)\, g(\alpha,\beta,\mu) + (\mathcal{U}a + \nu\gamma^2 - \nu a^2 + \nu(\alpha^2+\beta^2))\, f(\alpha,\beta,\mu)] \sin(\alpha x + \beta y + \gamma z + \mu t)$$

$$= 0 \qquad \text{(a)}$$

The Eq. (a) must be valid for all t , x , y and z , hence Eq. (a) can be split into two equations

$$\begin{aligned} &(\mu + \mathcal{U}\gamma - 2a\nu\gamma) f + [\mathcal{U}a + \nu\gamma^2 - \nu a^2 + \nu(\alpha^2+\beta^2)]\, g = 0 \\ &[\mathcal{U}a + \nu\gamma^2 - \nu a^2 + \nu(\alpha^2+\beta^2)] f + (-\mu - \mathcal{U}\gamma + 2a\nu\gamma)\, g = 0 \end{aligned} \qquad \text{(b)}$$

The simultaneous equations admit non-trivial solution of f and g only if the following determinant is equal to zero, i.e.

$$\begin{vmatrix} \mu + \mathcal{U}\gamma - 2a\nu\gamma \; ; & \mathcal{U}a + \nu\gamma^2 - \nu a^2 + \nu(\alpha^2+\beta^2) \\ \mathcal{U}a + \nu\gamma^2 - \nu a^2 + \nu(\alpha^2+\beta^2) \; ; & -\mu - \mathcal{U}\gamma + 2a\nu\gamma \end{vmatrix} = 0$$

A necessary and sufficient condition for the determinant to vanish for real α , β , γ , μ and a is

$$\mu + \mathcal{U}\gamma - 2a\nu\gamma = 0 \qquad \text{(c1)}$$

$$\mathcal{U}a + \nu\gamma^2 - \nu a^2 + \nu(\alpha^2+\beta^2) = 0 \qquad \text{(c2)}$$

Eq. (2) is a quadratic equation in a . The solutions are

$$a_{1,2} = \frac{\mathcal{U}}{2\nu}\left\{ 1 \pm \left(1 + \frac{4\nu^2}{\mathcal{U}^2}(\alpha^2+\beta^2+\gamma^2)\right)^{1/2} \right\}$$

The positive root of a is ruled out by the damping condition at infinity, hence

$$a = \frac{U}{2\nu}\left\{ 1 - \left(1 + \frac{4\nu^2}{U^2}\left(\alpha^2+\beta^2+\gamma^2\right)\right)^{1/2}\right\} \tag{c3}$$

From (c1) we have

$$a = \frac{\mu + U\gamma}{2\nu\gamma} = \frac{\mu}{2\nu\gamma} + \frac{U}{2\nu} \tag{c4}$$

Combining (c2) and (c4) we have

$$\frac{\mu}{\gamma} = - U\left(1 + \frac{4\nu^2}{U^2}\left(\alpha^2+\beta^2+\gamma^2\right)\right)^{1/2} \tag{c5}$$

μ is the time frequency and is in general taken as positive, so γ must be negative. Solving (c5) explicitely we have

$$\gamma = -\left\{ -\frac{1}{2}\left[\frac{U^2}{4\nu^2} + (\alpha^2+\beta^2)\right] + \frac{1}{2}\left[\left\{\frac{U^2}{4\nu^2} + (\alpha^2+\beta^2)\right\}^2 + \frac{\mu^2}{\nu^2}\right]^{1/2}\right\}^{1/2}$$

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