EFFECTS OF PLANFORM CURVATURE
IN SUPERSONIC WINGS

Thesis by

Hendrik Willem Marie Hoeijmakers

In Partial Fulfillment of the Requirements
for the Degree of
Aeronautical Engineer

California Institute of Technology
Pasadena, California

1974
(Submitted May 31, 1974)

ACKNOWLEDGMENTS

I wish to express my appreciation to Dr. H. J. Stewart for the many helpful suggestions and ideas in the course of the work described in this thesis.

I am also indebted to Dr. R. Coene of the Delft University of Technology, who initiated my interest in the subject.

This research also has been made possible by the National Aeronautics and Space Administration International University Fellowship for the year 1972-1973 and by the Netherlands Ministry of Education and Sciences for the years 1972-1974.

Thanks are due to my wife for her patience, to Mrs. Virginia Conner for the typing of the manuscript and to Mrs. Betty Wood for the drawing of the figures.

This thesis is dedicated to my parents, who made all this possible for me.

ABSTRACT

In this thesis a method is developed for calculating supersonic wings with curved subsonic leading edges. The linearized theory is used throughout the thesis.

The wing with the curved subsonic leading edges is transformed into a wing with straight subsonic leading edges by means of a transformation as used by Coene for quasi-homogeneous approximations to the solution of this problem. The Mach cone is invariant under the transformation.

The solution of the transformed Prandtl-Glauert equation is expressed in terms of Fenain's solutions for the delta wing. In general the solution is an infinite sum of terms, each term related to a solution for the delta wing. However, a condition is formulated under which certain families of wings with curved leading edges possess solutions in closed form. It is shown that any boundary value problem for such wings can be solved by the superposition of these exact solutions of the Prandtl-Glauert equation. The problem is thus reduced to determining the number of terms necessary to approximate the given boundary values within satisfactory bounds, and within a satisfactory region of the wing.

One family of wings with curved leading edges that has a solution in closed form is found. The flat plate with these leading edges is studied in detail. In order to find a reasonable approximation to the flat plate, in a satisfactory region of the wing, up to five solutions are superposed. It has been found that the curvature has a considerable effect on the perturbation velocity and the leading

ABSTRACT (Continued)

edge suction force. The leading edge suction force thus found is compared with that calculated by some other approximate methods.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF SYMBOLS

Symbol	Definition
a	Parameter in hyperbolic leading edge, introduced in (4-6)
a_0	Speed of sound in undisturbed flow
$A^*(\xi)$, $B^*(\xi)$, $C^*(\xi)$	Functions defined by (B-10)
C_D	Drag coefficient; Drag/dynamic pressure
C_L	Lift coefficient; Lift/dynamic pressure
C_p	Pressure coefficient; Pressure difference/dynamic pressure
$D^*(\xi)$	Function defined by (B-10)
$E(r)$	Truncation function
$E'(k)$	Complete elliptic integral of the second kind; (3-13)
$f(x)$	Function defining leading edge; $\|y\| = f(x)$
$F(x, y, z)$, $F(x, \rho)$	Function occurring in transformation (4-2) and (5-1)
$F^*_{np}(\xi)$	Function introduced in (3-7), see Appendix B
g_1, g_2, g_3	Expressions occurring by the transformation of PG-equations, defined by (5-4)
$G(x_1, \rho)$	Function used in Chapter IV
$G_i(x_1, \rho)$	Function used in Chapter V
$G^*_{np}(\xi)$	Function introduced in (3-7), see Appendix B
k	Parameter in leading edge; $k = \tau\beta$
k'	Parameter defined by (4-35)

LIST OF SYMBOLS (Continued)

k^*	Parameter defined by (4-36)
k^{**}	Parameter defined by (4-37a)
$K'(k)$	Complete elliptic integral of first kind; (3-13)
$K^*_{np}(\xi)$	Function defined by (B-1)
m	Degree of homogeneity in Chapters IV and V
$M_{2p}(k)$	Function defined by (3-12a)
$M(x_1, x_2, x_3)$	Solution from the homogeneous flow theory, homogeneous of degree m
n	Degree of homogeneity in Chapter III
$N_{2p}(k)$	Function defined by (3-12b)
p	Pressure; $p = p_0 + p'$
p_0	Pressure in the undisturbed flow
p'	Perturbation pressure
t	Time
T	Leading edge suction force
$\underset{\sim}{u}$	Velocity vector; $\underset{\sim}{u} = (U + u', v', w')$
$\underset{\sim}{u}'$	Perturbation velocity vector, $\underset{\sim}{u}' = (u', v', w')$
u_n, v_n, w_n	Components of velocity distribution for a flow, homogeneous of degree n, over the delta wing
U	Velocity of the undisturbed flow
w_0	Vertical velocity on flat plate (= constant)
$\bar{w}_0^{(N)}$	Approximation of vertical velocity to w_0, N^{th} order
(x, y, z)	Cartesian coordinate system in which the wing with the curved leading edges is stationary

(x_1, x_2, x_3)	Cartesian coordinate system in which the delta wing is stationary
$x_{LE}(y)$	The value of x at the leading edge
$x_{TE}(y)$	The value of x at the trailing edge
X	Scaled x-coordinate; $X = ax$
$y_{LE}(x)$	The value of y at the leading edge
Y	Scaled y-coordinate; $Y = a\beta y$
$\alpha(x, y)$	Angle of attack
α_p^s	Coefficient defined by (3-11)
β	Coefficient in the PG-equation; $\beta = (\frac{U^2}{a_0^2} - 1)^{\frac{1}{2}}$
γ	Specific heat coefficient
λ_{np}^*	Coefficient introduced in (3-7)
ρ	Density; $\rho = \rho_0 + \rho'$
ρ_0	Density in undisturbed flow
ρ'	Perturbation density
ρ	Variable used in Chapters IV and V; $\rho = \beta^2(y^2+z^2)$
I	Inverse of the sweep of the delta wing
φ	Perturbation velocity potential
φ_n	Perturbation velocity potential, homogeneous of degree n
Φ	Velocity potential $\Phi = U_x + \varphi$
ξ	Variable $\xi = \frac{x_2}{\tau x_1}$

<u>Superscripts</u>:

$(^+)$	Expression evaluated at the upper surface of the wing
$(^-)$	Expression evaluated at the lower surface of the wing
$(\tilde{\ })$	Expression transformed from (x_1, x_2, x_3) to (x, y, z)

LIST OF ABBREVIATIONS

PG	Prandtl-Glauert
LE	Leading edge
TE	Trailing edge
D. L. P.	Direct lifting problem
D. T. P.	Direct thickness problem
I. L. P.	Inverse lifting problem
I. T. P.	Inverse thickness problem

LIST OF FIGURES

LIST OF FIGURES (Continued)

I. INTRODUCTION

In the initial stages of the development of aircraft, it is often required to gain qualitative insight in the characteristics of possible configurations. This is needed in order to choose the optimal configuration which meets the requirements of the design. In this process, the wing plays obviously a very important role, i.e. what can be gained by changing the wing planform, the shape of the leading edges, sweep, etc.

It is in this stage of the development that the linearized theory, with its powerful property of the superposition principle, in general is a very useful concept.

In this thesis, we study the linearized theory for supersonic wings with curved subsonic leading edges, that is, wings where the pressure distribution on the upper and lower surface of the wing are influenced by each other. We want to study the influence of the curvature of the leading edges, so that for simplicity we assume that the trailing edge is supersonic and that the wing is symmetric.

Insofar as the linearized theory is concerned, Germain, [8] Fenain, [7] Stewart [13] and others give the solution for straight leading edges (delta wings) Evvard,[5] Etkin and Woodward [4] and Stewartson, [15] give implicit solutions in terms of integrals over singularity distributions, which can be used to obtain approximations to the solution for curved leading edges by an iterative process. Coene, [1], [2] and [3] gives approximations to the solution for curved leading edges in terms of the solutions of the homogeneous flow theory of Germain and Fenain. This method is quite straightforward, but gives no guarantee that the successive approximations converge rapidly enough to be useful. However, this method is systematic enough that it can be used to calculate the properties of wings on the computer.

In this thesis we try to find the solution of this problem as a superposition of exact solutions to the Prandtl-Glauert equation. This implies that we don't consider the boundary conditions in the first place, but satisfy these by the superposition of exact solutions. We will employ the same methods as Coene, that is, expressing the solution in terms of the solutions of the homogeneous flow theory.

Another possible way to investigate this problem is to find transformations which make the Prandtl-Glauert equation separable in its variables and which are such that the curved leading edges become simple in the new coordinate system. Robinson [12] considered in this context the socalled hyperboloido-conal coordinates and solved the problem of the lifting flat delta wing in this way (see also Stewart [14]). Miles [11] finds a number of such transformations to the unsteady case by modifying the classical transformations which make La Place's equation separable. Neither these transformations, nor similar modified ones for the steady case, seem to yield coordinate systems in which some family of curved leading edges is represented by a simpler curve.

In Chapter II we derive the Prandtl-Glauert equation and discuss the boundary conditions. In Chapter III we give a review of the results of the homogeneous flow theory. Chapter IV is devoted to a special family of curved leading edges, for which exact solutions are constructed. Chapter V gives a possible generalization to other families of leading edges with exact solutions.

II. THE EQUATIONS AND BOUNDARY CONDITIONS

II. 1 The governing equation

In supersonic wing theory, it is customary to make the following assumptions:

-- The influence of body forces (gravity) is negligible.

-- Viscosity and heat conduction are negligible. One can account indirectly for the effect of viscosity by applying the Joukowski condition at the trailing edges and by adjusting the wing thickness for the displacement thickness of the boundary layer on the wing surfaces.

-- Air is considered to be an ideal gas.

-- The flow is isentropic; i.e. no shocks are present.

With these assumptions the equations of motion reduce to

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \underset{\sim}{u} = 0 \qquad \text{(continuity)} \qquad (2\text{-}1)$$

$$\rho \frac{D\underset{\sim}{u}}{Dt} + \operatorname{grad} p = 0 \qquad \text{(momentum)} \qquad (2\text{-}2)$$

$$p \sim \rho^{\gamma}; \quad a_0^2 = \left(\frac{dp}{d\rho}\right)_0 = \gamma \frac{p_0}{\rho_0} \qquad \text{(isentropic relation)} \qquad (2\text{-}3)$$

In the linearized theory it is assumed that the velocities, induced by the wing at small angle of attack, are small compared with the velocity of the undisturbed flow.

Consider therefore perturbations on the undisturbed flow:

$$\begin{aligned} u &= U + u' \qquad & p &= p_0 + p' \\ v &= v' & \rho &= \rho_0 + \rho' \\ w &= w' \end{aligned} \qquad (2\text{-}4)$$

Substitution into the equations of motion gives then:

$$\frac{\partial \rho'}{\partial t} + U \frac{\partial \rho'}{\partial x} + \rho_0 \operatorname{div} \underset{\sim}{u}' = 0 \qquad (2\text{-}5)$$

$$\frac{\partial \underset{\sim}{u}'}{\partial t} + U \frac{\partial \underset{\sim}{u}'}{\partial x} = - \frac{a_0^2}{\rho_0} \operatorname{grad} \rho'$$

where terms of second order in the small quantities are neglected. Since we are considering inviscid, irrotational flow, we may introduce a velocity potential, defined as

$$\underset{\sim}{u} = \operatorname{grad} \Phi \qquad (2\text{-}7)$$

This can be written as the sum of the velocity potential for the undisturbed flow (in the x-direction) and the perturbation potential φ, so that

$$\Phi = Ux + \varphi(x, y, z)$$

Substitution into (2-6) gives us then:

$$\operatorname{grad} \left\{ \frac{\partial \varphi}{\partial t} + U \frac{\partial \varphi}{\partial x} + \frac{a_0^2}{\rho_0} \rho' \right\} = 0$$

From this we can express ρ' as a function of φ:

$$\rho' = - \frac{\rho_0}{a_0^2} \left\{ \varphi_t + U \varphi_x \right\} \qquad (2\text{-}8)$$

Substitution of (2-8) into (2-5) leads to the governing equation for the linearized supersonic wing theory:

$$\beta^2 \varphi_{xx} - \varphi_{yy} - \varphi_{zz} + \frac{1}{a_0^2} \varphi_{tt} + 2 \frac{U}{a_0^2} \varphi_{xt} = 0 \qquad (2\text{-}9)$$

where $\beta^2 = \dfrac{U^2}{a_0^2} - 1 > 0$

For the steady case, we obtain the Prandtl-Glauert equation

$$\beta^2 \varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \tag{2-10}$$

This is a hyperbolic second order linear partial differential equation for the perturbation velocity potential. From (2-8) we obtain the expression for the pressure perturbation p':

$$p' = -\rho_0(U\varphi_x + \varphi_t) \tag{2-11}$$

II. 2 The boundary conditions

In cartesian coordinates (x, y, z), with the origin at the apex of the wing, the envelope of the disturbances is given by the equation of the Mach cone,

$$x^2 - \beta^2(y^2+z^2) = 0, \quad x \geq 0 \tag{2-12}$$

For wings with subsonic landing edges only, ahead of the Mach cone no perturbations are present, so that on the Mach cone $\varphi = 0$. For the boundary conditions on the wing, we have, in the absence of blowing or suction,

$$\underset{\sim}{u} \cdot \underset{\sim}{n} = 0 \quad \text{on } S \tag{2-13}$$

where $\underset{\sim}{n}$ is the normal to the wing surface S (see Fig. 1). Suppose that the wing surface is given by

$$\left.\begin{aligned} z^+ &= f^+(x,y) \\ z^- &= f^-(x,y) \end{aligned}\right\} \quad \text{on } S$$

where the + and - denote the upper and lower surface respectively. With $\underset{\sim}{n} = (n_x, n_y, n_z)$ and $\underset{\sim}{u} = (U + u, v, w)$ (2-13) becomes

$$(U + u)n_x + vn_y + wn_z = 0 \tag{2-14}$$

The linearized theory will give reasonable results for plane wings at small angle of attack; for such wings we may assume that n_x and n_y $\ll n_z$, so that $n_z \approx 1$

$$\frac{n_x}{n_z} = -\alpha(x, y) \qquad (\alpha = \text{angle of attack})$$

Then we can write (2-14) as

$$\alpha(x, y) = \frac{w}{U + u} \approx \frac{w}{U} = \frac{\varphi_z}{U} \tag{2-15}$$

In the linear theory this condition is applied at $z = \pm 0$.

In order to solve the boundary value problem for a given wing or for a given pressure distribution, we make a distinction into four possible cases. By the superposition principle we are allowed to do so and thus simplify the problem considerably.

Write the equations for the wing surface as the sum of a symmetric and an asymmetric part:

$$z^{\pm} = \tfrac{1}{2}(f^{+} + f^{-}) \pm \tfrac{1}{2}(f^{+} - f^{-}) = k \pm \delta$$

where k is the asymmetric and δ the symmetric part.

i) <u>The direct lifting problem</u> (D. L. P).

For this it is given that $\delta = 0$; $k \neq 0$ $\frac{\partial k}{\partial x} \neq 0$ so that

$$z^{\pm} = k(x, y) \text{ and } \alpha^{\pm} = \frac{\partial k}{\partial x}$$

and with (2-15)

$$\varphi_z^{\pm} = U\frac{\partial k}{\partial x} \tag{2-16}$$

From this it may be seen that

$$\begin{cases} \varphi_z & \text{is even in } z \\ \varphi, \varphi_x & \text{are odd in } z\text{: e.g. } \varphi^- = -\varphi^+ \end{cases}$$

Since outside the wing the pressure must be continuous, we have

$$\varphi = \varphi_x = 0 \text{ outside the wing for } z = 0$$

There exists then a discontinuity in the φ across the wing, which strength is related to the circulation around the lifting surface and gives rise to the lift.

Summarizing: Given $\varphi_z^+ = \varphi_z^-$ on the wing and $\varphi = \varphi_x = 0$ outside the wing for $z = 0$, calculate φ and φ_x on the wing.

ii) <u>The direct thickness problem</u> (D. T. P.)

For this it is given that $k = 0$; $\delta \neq 0$; $\frac{\partial \delta}{\partial x} \neq 0$ so that

$$z^{\pm} = \pm \delta(x, y) \text{ and } \alpha^{\pm} = \frac{\partial \delta}{\partial x}$$

and with (2-15):

$$\varphi_z^{\pm} = \pm U \frac{\partial \delta}{\partial x} \tag{2-17}$$

From this it may be seen that

$$\begin{cases} \varphi_z & \text{is odd in } z \\ \varphi, \varphi_x & \text{are even in } z\text{; e.g. } \varphi^- = \varphi^+ \end{cases}$$

so that outside the wing for $z = 0$ $\varphi_z = 0$. We now have a jump in the φ_z across the wing of strength

$$\Delta \varphi_z = U(\alpha^+ - \alpha^-) = 2\, U \alpha.$$

Summarizing: Given $\varphi_z^+ = -\varphi_z^-$ on the wing and $\varphi_z = 0$ outside the wing for $z = 0$, calculate φ_x and φ on the wing.

iii) The inverse thickness problem (I. T. P.)

For this it is given: $\varphi_x^+ = \varphi_x^-$ on the wing

$\varphi_z = 0$ outside the wing for $z = 0$

Calculate: φ_z on the wing; $\varphi_z^- = -\varphi_z^+$

iv) The inverse lifting problem (I. L. P)

Given: $\varphi_x^+ = -\varphi_x^-$ on the wing

$\varphi = \varphi_x = 0$ outside the wing for $z = 0$

Calculate: φ_z on the wing; $\varphi_z^+ = \varphi_z^-$

II. 3 Additional conditions

In addition to the boundary conditions on the Mach cone on the wing and outside the wing in the plane $z = 0$, we have the following conditions:

-- At subsonic trailing edges the Joukowsky condition must be applied.

-- Near rounded leading edges the angle between the normal to the wing and the z-direction is no longer small, so that (2-15) is no longer valid. It is assumed that this region of nonlinearity is relatively small and does not influence the linear solution further away from the leading edge.

-- Near sharp leading edges, the velocity will go to infinity in case the leading edge is subsonic. The assumptions of small perturbations is locally violated there. It turns out that one can account for this by admitting a square root singularity at the leading edge and taking into account a suction force.

II. 4 Methods to solve the problem of linearized theory

There are four methods to solve above sketched boundary value problems for the Prandtl-Glauert equation, with which there are found solutions

i) Analogy with the 2-dimensional wave equation

$$\varphi_{xx} + \varphi_{yy} = \frac{1}{a^2}\varphi_{tt}$$

This was studied by Volterra and Hadamard; applications of their theory lead to the acoustic analogy

ii) La Place transform

Most applications are found in the "slender body" theory

iii) Conical and homogeneous flow

The concepts of homogeneous flow are mostly applied to wings with straight leading edges and to a lesser extent to wings with slightly curved subsonic leading edges.

iv) Distribution of singularities

In this method one makes use of the analogy of the Prandtl-Glauert equation with La Place's equation,

$(x' = x, y' = i\beta y, z' = i\beta z)$

$$\varphi_{x'x'} + \varphi_{y'y'} + \varphi_{z'z'} = 0$$

and makes use of the fundamental source and doublet solutions of this equation to define their supersonic counterparts.

In general this will lead to integral equations for the perturbation velocity potential. For wings with supersonic leading edges these equations can easily be solved, but for wings with purely subsonic leading edge this is impossible in the D. L. P. and I. T. P.

Since in this thesis the results of the homogeneous flow theory are used quite extensively, we will study the theory of this approach in the next chapter.

III. RESULTS OF THE HOMOGENEOUS FLOW THEORY

III. 1 Introduction

Germain [8] generalized Busemann's conical flow theory into the theory of homogeneous flow. The solutions to the boundary value problems, as described in the previous chapter, are found by constructing analytic functions. Fenain [7] carried out many systematic calculations with Germain's theory, and found that the solution can be expressed in terms of functions which are independent of the boundary values. The problem is thus reduced to an algebraic one.

In this chapter we give the results for the direct lifting problem. For the other problems, the reader is referred to [7] or [3].

A flow homogeneous of degree n is defined as a flow for which the perturbation velocity potential is a homogeneous function of degree n in the cartesian coordinates x_1, x_2 and x_3.

$$f(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda^n f(x_1, x_2, x_3) \tag{3-1}$$

Upon differentiation of (3-1) with respect to λ and putting $\lambda = 1$, we obtain the Euler relation

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} = nf \tag{3-2}$$

From (3-2) we conclude that, since f is homogeneous of degree n, the first derivatives of f are homogeneous of degree n-1. Differentiation of (3-2) with respect to x_1 gives us the Euler relation

$$x_1 \frac{x^2 f}{\partial x_1^2} + x_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + x_3 \frac{\partial^2 f}{\partial x_1 \partial x_3} = (n-1) \frac{\partial f}{\partial x_1} \tag{3-3}$$

In general, we obtain from (3-2):

$$x_1 \frac{\partial^{i+1} f}{\partial x_1^{i+1-p-q} \partial x_2^p \partial x_3^q} + x_2 \frac{\partial^{i+1} f}{\partial x_1^{i-p-q} \partial x_2^{p+1} \partial x_3^q} + x_3 \frac{\partial^{i+1} f}{\partial x_1^{i-p-q} \partial x_2^p \partial x_3^{q+1}} = (n-i) \frac{\partial^i f}{\partial x_1^{i-p-q} \partial x_2^p \partial x_3^q} \qquad (3\text{-}4)$$

For a wing with a general planform, it is not expected that the perturbation velocity potential satisfies (3-1). However, the "delta-like" wing with proper boundary values may exhibit the character of the homogeneous flow (see Fig. 2).

In the next section we give the solution, as evaluated by Fenain,[7] for the delta wing with leading edges

$$\begin{cases} |x_2| = \tau x_1 \\ x_3 = 0 \end{cases} \qquad (3\text{-}5)$$

It is assumed that the wing lies in the $x_3 = 0$ plane, and that φ satisfies the PG-equation in x_1, x_2 and x_3. We treat only elementary problems, that is, problems for which the boundary values are given as a homogeneous polynomial in x_1, x_2 and x_3.

III. 2 The solution for the D. L. P.

For the D. L. P. we have given as the boundary values

$$w_n^+ = w_n^- = \sum_{s=0}^{n-1} c^*_{n-1-s,\,s}\, x_1^{n-1-s} \left|\frac{x_2}{\tau}\right|^s \quad \text{for } x_3 = 0 \text{ and } |x_2| \le \tau x_1 \qquad (3\text{-}6)$$

$$u_n = \varphi_n = 0 \quad \text{for } x_3 = 0 \text{ and } \tau x_1 < |x_2| \le \frac{1}{\beta} x_1$$

The solution is then expressed as below:

$$\begin{cases} u_n^+ = -u_n^- = -\frac{2}{\pi}\tau x_1^{n-1}\sum_{p=1}^{n}\lambda_{np}^* F_{np}^*(\xi) \\ v_n^+ = -v_n^- = +\frac{2}{\pi} x_1^{n-1}\sum_{p=1}^{n}\lambda_{np}^* G_{np}^*(\xi) \quad ; \text{ for } \begin{cases} x_3 = 0 \\ |x_2| \le \tau x_1 \end{cases} \\ \varphi_n^+ = -\varphi_n^- = -\frac{2}{\pi}\tau x_1^{n}\sum_{p=1}^{n}\lambda_{np}^* F_{n+1,p}^*(\xi) \end{cases} \quad (3\text{-}7)$$

The function $F_{np}^*(\xi)$ satisfies the differential equation

$$\frac{d^n F_{np}^*}{d\xi^n} = (-1)^{n+p} \frac{\xi^{2-n}}{(1-\xi^2)^{p+\frac{1}{2}}} \quad (3\text{-}8)$$

and G_{np}^* satisfies

$$\boxed{\xi = \frac{x_2}{\tau x_1}}$$

$$\frac{d^n G_{np}}{d\xi^n} = (-1)^{n+p} \frac{\xi^{1-n}}{(1-\xi^2)^{p+\frac{1}{2}}} \quad (3\text{-}9)$$

In Appendix B we evaluate F_{np}^* and G_{np}^* in more detail.
The λ_{np}^*'s in (3-7) are found by solving a system of n equations for n unknowns:

$$\sum_{p=1}^{n} (-1)^{p-1}\lambda_{np}^*\alpha_p^s = (n-1-s)!\,s!\,c_{n-1-s,\,s}^*; \; s = 0, 1, \ldots, n-1 \quad (3\text{-}10)$$

where α_p^s is given as

$$\begin{cases} \alpha_p^0 = \frac{2}{\pi} M_{2p-2} \\ \alpha_p^{2m} = \frac{2}{\pi}\left[\sum_{j=0}^{p-1}\binom{p+m-j-2}{m-1} M_{2j} + \sum_{j=0}^{m-1}\binom{p+m-j-2}{p-1} N_{2j}\right]; \; m \neq 0 \\ \alpha_p^{2m+1} = \frac{1}{2^{2m}}\sum_{t=0}^{m}\frac{\binom{2m-2t}{m-t}\binom{2t}{t}\binom{2p+2m-2t}{2p}}{(2t-1)\binom{p+m-t}{p}} k^{2t}; \; m \ge 0 \end{cases} \quad (3\text{-}11)$$

where $k = \tau\beta$.

The M_{2j}'s and N_{2j}'s in (3-11) can be expressed in terms of complete elliptic integrals of the first and second kind, by means of the following recurrence relations:

$$\begin{cases} (2j+1)(1-k^2)M_{2j} = [(2j-1)(1-2k^2)+1]M_{2j-2}+(2j-3)k^2 M_{2j-4};\ j \geq 2 \\ M_0 = E' \ ;\ M_2' = \dfrac{(2-k^2)E' - k^2K'}{3(1-k^2)} \end{cases} \qquad (3\text{-}12a)$$

$$\begin{cases} (2j+1)N_{2j} = [(2j-1)(1+k^2)+(1-k^2)]N_{2j-2}-(2j-3)k^2N_{2j-4};\ j \geq 2 \\ N_0 = E' - k^2K' \ ; \quad N_2 = (1-k^2)M_2 \end{cases} \qquad (3\text{-}12b)$$

In (3-12a) and (3-12b), E' and K' are the complete elliptic integrals of the second and first kind respectively, with modulus $(1-k^2)^{\frac{1}{2}}$

$$\begin{cases} E' = \int_0^{\pi/2} [1 - (1-k^2)\sin^2\theta]^{\frac{1}{2}}\, d\theta \\ K' = \int_0^{\pi/2} [1 - (1-k^2)\sin^2\theta]^{-\frac{1}{2}}\, d\theta \end{cases} \qquad 0 \leq k \leq 1 \qquad (3\text{-}13)$$

A table of E' and K' can be found in any book of transcendental functions.

In the following section we apply the above results to the problem of the lifting flat delta wing.

In the calculations in Chapter IV we use the results of this chapter quite extensively. The results can be simplified further for numerical calculations. These are not given here, but can be easily derived from the formulae (3-10) and (3-11).

III. 3 Application

The flat delta wing

This is the simplest direct lifting problem; the boundary conditions on the wing are

$$w_1^+ = w_1^- = w_0 = c_{00}^* \tag{3-14}$$

(3-14) implies that the perturbation velocity potential is homogeneous of degree one. From (3-7) it follows that the solution can be expressed as

$$\begin{cases} u_1^+ = -u_1^- = -\frac{2}{\pi}\tau\lambda_{11}^* F_{11}^* \\ \varphi_1^+ = -\varphi_1^- = -\frac{2}{\pi}\tau\lambda_{11}^* F_{21}^* x_1 \end{cases} \tag{3-15}$$

From (3-10):

$$\lambda_{11}^* \alpha_1^s = (-s)!s!\, c_{-s,s}^* \quad ; \quad \text{for } s = 0 \tag{3-16}$$

With (3-11) and (3-12a) we find from (3-16):

$$\lambda_{11}^* = \frac{\pi}{2}\frac{w_0}{E'} \tag{3-17}$$

From Appendix B1:
$$\begin{cases} F_{11}^* = A^* = \dfrac{\tau x_1}{\sqrt{\tau^2 x_1^2 - x_2^2}} \\ F_{21}^* = C^* = \dfrac{\sqrt{\tau^2 x_1^2 - x_2^2}}{\tau x_1} \end{cases}$$

So that the solution for the flat delta wing with leading edges $|x_2| = \tau x_1$ is:

$$\begin{cases} u_1^+ = -\dfrac{\tau^2 x_1 w_0}{E'\sqrt{\tau^2 x_1^2 - x_2^2}} \\ \varphi_1^+ = -\dfrac{w_0}{E'}\sqrt{\tau^2 x_1^2 - x_2^2} \end{cases} \qquad |x_2| \leq \tau x_1 \qquad (3\text{-}18)$$

From Appendix A it is easily calculated that

$$C_L = \frac{2\pi\tau}{E'} \frac{w_0}{U}$$

$$C_D = \frac{2\pi\tau}{E'} \left(\frac{w_0}{U}\right)^2$$

The drag coefficient is reduced by the leading edge suction force as given in (A-10), so that

$$C_{D_{tot.}} = \frac{2\pi\tau}{E'}\left(\frac{w_0}{U}\right)^2 - \frac{\pi\sqrt{1-k^2}\,\tau}{E'^2}\left(\frac{w_0}{U}\right)^2$$

$$= \frac{2\pi\tau}{E'}\left(\frac{w_0}{U}\right)^2\left[1 - \frac{\sqrt{1-k^2}}{2E'}\right]$$

For $k = 0$ ("slender body"): C_D is reduced by 50% by the suction force.

For $k = 1.0$ ("sonic" leading edges): suction force is zero.

IV. THE SOLUTION FOR A SPECIAL FAMILY OF CURVED LEADING EDGES

IV.1 Introduction

In the scope of the linearized theory, no exact solution has yet been found for wings with curved subsonic leading edges. However, for slightly curved leading edges, Coene [3] attacked the problem by constructing transformations which transform the leading edge into a straight one.

It is shown in [3] that in order to straighten the leading edge

$$\begin{cases} |y| = f(x) \\ z = 0 \end{cases} \quad \text{into} \quad \begin{cases} |x_2| = \tau x_1 \\ x_3 = 0 \end{cases} \tag{4-1}$$

it is sufficient to stretch the x-axis only. This implies that the transformation has the form

$$\begin{cases} x_1 = x + \{x^2 - \beta^2(y^2+z^2)\}F(x, y, z) \\ x_2 = y \\ x_3 = z \end{cases} \tag{4-2}$$

where $F(x, y, z)$ satisfies

$$f(x) - \tau x = \tau\{x^2 - \beta^2 f^2(x)\} F(x, f(x), 0) \tag{4-3}$$

It is assumed that the Jacobian of (4-1) is nonvanishing in the region under consideration.

The Mach cone

$$x^2 - \beta^2(y^2+z^2) = 0 \qquad \text{transforms into}$$

$$x_1^2 - \beta^2(x_2^2+x_3^2) = 0 .$$

In the (x_1, x_2, x_3) space we have now the same geometry as in the

case of the straight leading edges. If it is possible to express the perturbation potential φ in the physical (x, y, z) space, which satisfies the PG-equation (2-10), into terms of solutions of the homogeneous flow theory $M(x_1, x_2, x_3)$, which satisfies the PG-equation in the (x_1, x_2, x_3) space, the solution can be determined.
Therefore, we try as solution

$$\varphi = \sum_{q=0} \frac{\partial^q M}{\partial x_1^q} G_q(x_1, x_2, x_3) . \tag{4-4}$$

For slightly curved leading edges, it is shown in [3] that upon expansion of G_q in (4-4) and $F(x, y, z)$ in (4-2) and substitution into (2-10), a system of equations is found for the coefficients in the power series expansion for G_q. It is an a priori assumption that the so introduced expansions are convergent in some neighborhood of the origin.

In this thesis we circumvent this problem of convergence by finding the differential equations G_q satisfies and solving these for one special case in closed form. This special case corresponds to F = constant in (4-2). Coene [3] noted that for this case the first term in (4-4) is sufficient for finding a solution, which subsequently was obtained as a power series.

IV. 2 The hyperbolic leading edge

In this chapter we study the most simple transformation (4-2):

$$\begin{cases} x_1 = x + \{x^2 - \beta^2(y^2+z^2)\}a \\ x_2 = y \qquad\qquad ; \quad a = \text{constant} \\ x_3 = z \end{cases} \tag{4-6}$$

The leading edge which is straightened by (4-6) follows from (4-3):

$$\begin{aligned} |y| &= \frac{1}{2a\beta k}\left[-1+\sqrt{1+4axk^2(1+ax)}\right] \\ z &= 0 \qquad\qquad (k = \tau\beta) \end{aligned} \tag{4-7}$$

(4-7) corresponds to hyperbolae in the z = 0 plane, going through the origin. It is seen from (4-7), that a is not a real parameter of the problem, but may be regarded as a scaling factor. By introducing

$$\begin{cases} X = ax \quad ; \quad x \geq 0 \\ Y = a\beta y \quad ; \quad y \gtrless 0 \end{cases} \qquad a \lessgtr 0 \tag{4-8}$$

we eliminate a in the problem.

For $X < 0$, the leading edge (4-7) becomes a trailing edge for $X = -\frac{1}{2}$. For $X > 0$, the leading edge becomes at most sonic.* So we have:

$$R \begin{cases} -\frac{1}{2} \leq X < \infty \\ \qquad |Y| \leq X \end{cases} \tag{4-9}$$

as the region where the solution is required.

It turns out that in R the Jacobian is positive, and becomes zero at $X = -\frac{1}{2}$. In Fig. 3a and b the leading edge is shown for different values of k.

The PG-equation in the x_1, x_2 and x_3 coordinates can, with (4-6), be written as:

$$\beta^2\varphi_{x_1x_1} - \varphi_{x_2x_2} - \varphi_{x_3x_3} + 6a\beta^2\varphi_{x_1} + 4a\beta^2(x_1\varphi_{x_1x_1} + x_2\varphi_{x_1x_2} + x_3\varphi_{x_1x_3}) = 0 \tag{4-10}$$

In the x_1, x_2, x_3 space we have now the same geometry as in the case of the homogeneous flow. As solution to (4-10) we try therefore:

$$\varphi(x_1, x_2, x_3) = M(x_1, x_2, x_3)G(x_1, x_2, x_3) \tag{4-11}$$

* for $X \rightarrow +\infty$

where M is homogeneous of degree m in x_1, x_2 and x_3 and assumed to be a solution of the homogeneous flow theory, so that M satisfies the PG-equation

$$\beta^2 M_{x_1x_1} - M_{x_2x_2} - M_{x_3x_3} = 0 \qquad (4\text{-}12)$$

Also, we may utilize formulae (3-2), (3-3) and (3-4). With this we obtain for (4-10):

$$\frac{\partial M}{\partial x_1}\left[2(2m+1)a\beta^2 G+2(1+2ax_1)\beta^2\frac{\partial G}{\partial x_1}+4(x_1+2a\rho)\beta^2\frac{\partial G}{\partial \rho}\right]+$$
$$+M\left[2(2m+3)a\beta^2\frac{\partial G}{\partial x_1}+(1+4ax_1)\beta^2\frac{\partial^2 G}{\partial x_1^2}-4(m+1)\beta^2\frac{\partial G}{\partial \rho}-4\rho\beta^2\frac{\partial^2 G}{\partial \rho^2}+8a\rho\beta^2\frac{\partial^2 G}{\partial x_1\partial\rho}\right]$$
$$= 0 \qquad (4\text{-}13)$$

Sufficient conditions for (4-13) to hold are:

$$\begin{cases} 2(2m+3)a\dfrac{\partial G}{\partial x_1}-4(m+1)\dfrac{\partial G}{\partial \rho}+(1+4ax_1)\dfrac{\partial^2 G}{\partial x_1^2}+8a\rho\dfrac{\partial^2 G}{\partial x_1\partial\rho}-4\rho\dfrac{\partial^2 G}{\partial \rho^2}=0 \\ (2m+1)a\,G+(1+2ax_1)\dfrac{\partial G}{\partial x_1}+2(x_1+2a\rho)\dfrac{\partial G}{\partial \rho} = 0 \end{cases} \qquad (4\text{-}14)$$

In (4-13) and (4-14) we have defined ρ as

$$\rho = \beta^2(x_2^2+x_3^2) = \beta^2(y^2+z^2) \qquad (4\text{-}15)$$

With

$$\begin{cases} \xi = 1+2ax_1 \\ \eta = [1+4a\,x_1+4a^2\rho]^{\frac{1}{2}} \end{cases} \qquad (4\text{-}16)$$

(4-14) becomes:

$$\begin{cases} (2m+3)\dfrac{\partial G}{\partial \xi}+(2\xi-1)\dfrac{\partial^2 G}{\partial \xi^2}+2\eta\dfrac{\partial^2 G}{\partial\xi\,\partial\eta}+\dfrac{\partial^2 G}{\partial \eta^2} = 0 \\ (m+\frac{1}{2})\,G+\xi\dfrac{\partial G}{\partial \xi}+\eta\dfrac{\partial G}{\partial \eta} = 0 \end{cases} \qquad (4\text{-}17)$$

The second equation in (4-17) has as general solution any function homogeneous of the degree $-(m+\frac{1}{2})$ in ξ and η. This implies that (Euler relation):

$$-(m+\tfrac{3}{2})\frac{\partial G}{\partial \xi} = \xi\frac{\partial^2 G}{\partial \xi^2} + \eta\frac{\partial^2 G}{\partial \xi\,\partial \eta} \tag{4-18}$$

Using (4-18) in (4-17) yields the one-dimensional wave equation:

$$\frac{\partial^2 G}{\partial \eta^2} - \frac{\partial^2 G}{\partial \xi^2} = 0 \tag{4-19}$$

So that the general solution of (4-17) is:

$$G = C_1\left[1 + 2ax_1 + \sqrt{1+4ax_1+4a^2\rho}\right]^{-(m+\frac{1}{2})} + C_2\left[(1+2ax_1)-\sqrt{1+4ax_1+4a^2\rho}\right]^{-(m+\frac{1}{2})} \tag{4-20}$$

The solution as represented in (4-20) contains two arbitrary constants C_1 and C_2, which will be determined below.

In the (x, y, z) space (4-11) is now:

$$\begin{aligned}\varphi(x,y,z) = {} & \tilde{C}_1\cdot \tilde{M}\{x_1(x,\rho),y,z\}\cdot [(1+2ax) + a^2(x^2-\beta^2(y^2+z^2))]^{-(m+\frac{1}{2})} \\ & + \tilde{C}_2\,\tilde{M}\{x_1(x,\rho),y,z\}\cdot [x^2-\beta^2(y^2+z^2)]^{-(m+\frac{1}{2})}\end{aligned} \tag{4-21}$$

Since M and all its derivatives up to m^{th} derivatives are zero at the Mach cone, the two terms in (4-21) correspond to a supersonic source at $(-\frac{1}{a}, 0, 0)$ and (0, 0, 0) respectively. In the following we choose

$$\begin{aligned}\tilde{C}_1 &= 1 \\ \tilde{C}_2 &= 0\end{aligned} \tag{4-22}$$

By this choice there are no singularities in the function G in the region (4-9). Furthermore φ becomes M for the limit a = 0.

The solution is now:

$$\varphi(x, y, z) = \widetilde{M}\{x_1(x, \rho), y, z\} \cdot [1+2ax+a^2(x^2-\rho)]^{-(m+\frac{1}{2})} \quad (4\text{-}23a)$$

$$w(x, y) = \varphi_z(x, y, 0^+) = \widetilde{w}_m \cdot [1+2ax+a^2(x^2-\beta^2 y^2)]^{-(m+\frac{1}{2})} \quad (4\text{-}23b)$$

$$u(x, y) = \varphi_x(x, y, 0^+) = \widetilde{u}_m \cdot [1+2ax+a^2(x^2-\beta^2 y^2)]^{-(m+\frac{1}{2})}$$
$$-a[\widetilde{M}]_{z=0}(2m+1)(1+ax)[1+2ax+a^2(x^2-\beta^2 y^2)]^{-(m+\frac{3}{2})} \quad (4\text{-}23c)$$

In this section we obtained an exact solution to the PG-equation, for a wing with hyperbolic leading edges. The boundary value problem is reduced to a boundary value problem of the homogeneous flow theory.

For a direct lifting problem, we have given

$$\varphi_z(x, y, 0^+) = f(x, y) \quad ; \quad \text{on the wing} \quad (4\text{-}24a)$$

From (4-23c) we have, that f(x, y) should be expanded as

$$f(x, y) = \sum_{m=1}^{\infty} \frac{\sum_{s=0}^{m-1} c^*_{m-1-s,\, s}[x+a(x^2-\beta^2 y^2)]^{m-1-s} \left|\frac{y}{T}\right|^s}{[1+2ax+a^2(x^2-\beta^2 y^2)]^{m+\frac{1}{2}}} \quad (4\text{-}24b)$$

so that the solution can be written as

$$\varphi(x, y, 0^+) = \sum_{m=1}^{\infty} \frac{\widetilde{\varphi}_m[x+a(x^2-\beta^2 y^2), y, 0^+]}{[1+2ax+a^2(x^2-\beta^2 y^2)]^{m+\frac{1}{2}}} \quad ; \quad \text{on the wing} \quad (4\text{-}24c)$$

where $\varphi_m(x_1, x_2, 0^+)$ is given in (3-7). The perturbation velocity follows from (4-24c) or from (4-23c) by summation over m.

Note that each term in (4-24c), say for m = N, is an exact solution of the PG-equation, which satisfies boundary conditions given in (4-24b) for m = N. This implies that for any choice of $c^*_{m-1-s,\, s}$ in (4-24b) we have an exact solution.

In the neighborhood of the origin, that is for

$$|2ax + a^2(x^2 - \beta^2 y^2)| < 1 \qquad (4\text{-}24d)$$

the denominator in (4-24b) can be expanded in a convergent power series. By this we can write (4-24b) as

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{s=0}^{\infty} b^*_{m,s}\, x^m |y|^s$$

where $b^*_{m,s}$ is a function of $c^*_{m-1-s,s}$

Expanding f(x, y) in the same way, we determine the $c^*_{m-1-s,s}$ for any m and s. In the range (4-24d), (4-24e) will be a convergent series, so that by analytic continuation (4-24e) is valid everywhere in the aft Mach cone for $a > 0$ and for $x < -\frac{1}{2a}$ for $a < 0$.

This implies that any boundary condition f(x, y) can be written as in (4-24b), so that the solution is known for any f(x, y).

IV.3 Flat plate with hyperbolic leading edges

The flat plate has as boundary condition on the wing:

$$w(x, y) = w_0 = \text{constant} \qquad (4\text{-}25)$$

From (4-24b) we obtain therefore

$$w_0 = \sum_{i=1}^{N} \frac{\sum_{s=0}^{i-1} c^*_{i-1-s,s}\,[x+a(x^2-\beta^2 y^2)]^{i-1-s}\,|\frac{y}{T}|^s}{[1+2ax+a^2(x^2-\beta^2 y^2)]^{i+\frac{1}{2}}} \; ; \; |y| < f(x) \qquad (4\text{-}26)$$

In (4-26) there are $\frac{1}{2}N(N+1)$ coefficients to be determined, where N denotes the order of approximation

<u>N=1:</u> $$\bar{w}_0^{(1)} = \frac{c^*_{00}}{[1+2ax+a^2(x^2-\beta^2 y^2)]^{\frac{3}{2}}}$$

It is convenient to choose c^*_{00} such that $\overline{w}_0^{(1)} = w_0$ at the origin, so that

$$c^*_{00} = w_0 \tag{4-27a}$$

N=2: $$\overline{w}_0^{(2)} = \overline{w}_0^{(1)} + \frac{c^*_{10}[x+a(x^2-\beta^2y^2)]+c^*_{01}\left|\frac{y}{\tau}\right|}{[1+2ax+a^2(x^2-\beta^2y^2)]^{\frac{5}{2}}}$$

We have 3 c^*'s to determine. This can be done in many ways, for instance $\overline{w}_0^{(2)} = w_0$ at 3 points, etc. We choose here a method that corresponds with the Taylor series expansion of the right-hand side of (4-26). For N = 2 this means that:

$$\left.\begin{array}{l} \overline{w}_0^{(2)} = w_0 \\ \frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial y} \text{ of } \overline{w}_0^{(2)} \text{ are zero} \end{array}\right\} \text{ for } (0, 0, 0)$$

N=3, etc. Similar conditions for $\overline{w}_0^{(N)}$:

$$\overline{w}_0^{(N)} = w_0, \ \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = \frac{\partial^2}{\partial x^2} = --- = \frac{\partial^{N-1}}{\partial y^{N-1}} = 0 \text{ at origin.}$$

The result is given below:

N = 1	N = 2	N = 3	N = 4
$c^*_{00} = w_0$	$c^*_{10} = 3aw_0$	$c^*_{20} = 6a^2w_0$	$c^*_{30} = 10a^3w_0$
	$c^*_{01} = 0$	$c^*_{11} = 0$	$c^*_{21} = 0$
		$c^*_{02} = \frac{3}{2}k^2a^2w_0$	$c^*_{12} = \frac{15}{2}k^2a^3w_0$
			$c^*_{03} = 0$

(4-27b)

(continued)

(4-27b) continued

$$N = 5$$

$$\begin{cases} c^*_{40} = 15\, a^4 w_0 \\ c^*_{31} = 0 \\ c^*_{22} = 22.5\, k^2 a^4 w_0 \qquad\qquad k = \tau\beta \\ c^*_{13} = 0 \\ c^*_{04} = 1.875\, k^4 a^4 w_0 \end{cases}$$

For N = 5 for example we have 15 coefficients, but it turns out that in the method we use here $c^*_{30}, \ldots, c^*_{03}$, $c^*_{20}, \ldots, c^*_{02}, c^*_{10}, c^*_{01}$ and c^*_{∞} are the same as calculated at N = 4, 3, 2 and 1. In Fig. 4a and b we have plotted the successive approximations to the flat plate, up to N = 5. From this we see that for ax small, the flat plate is quite well approximated, but that for larger positive ax $w \to 0$, and for $ax \to -\frac{1}{2}$ w diverges to values of $w \gg w_0$. The flat plate is well approximated within the range

$$-0.15 < ax < 0.3 \tag{4-28}$$

In the following we denote wings with inward curved leading edges ($a < 0$) by gothic, and wings with outward curved leading edges ($a > 0$) by ogee.

We can conclude from Fig. 4 that the employed method is more promising for the ogee wing than for the gothic one.

The perturbation velocity potential is given by

$$\varphi(x, y, 0^+) = -\frac{2}{\pi}\tau \sum_{i=1}^{N} \frac{[x+a(x^2-\beta^2y^2)]^i}{[1+2ax+a^2(x^2-\beta^2y^2)]^{i+\frac{1}{2}}} \left[\sum_{p=1}^{i} \lambda^*_{ip}\, F^*_{i+1,p}(\xi)\right] \tag{4-29}$$

where $\xi = \frac{y}{\tau[x+a(x^2-\beta^2 y^2)]}$, λ^*_{ip} satisfies (3-10) and $F^*_{i+1,p}(\xi)$ is found in Appendix B.

In Fig. 5a-d we have plotted the perturbation velocity potential (4-29) for two values of the parameter k. In this we use the c^*'s as given in (4-27b), up to N = 5. For the ogee wing there is a rapid convergence, even beyond the range as given in (4-28). Due to the alternating behavior of the successive approximations the question of convergence is more difficult for the gothic wing.

In Fig. 5e, we compare the results of the approximation for N = 5, with the result for the flat delta wing, as a function of k. From this we see that the potential for the ogee wing is greater and for the gothic wing is smaller than the potential for the delta wing, all at the same point on the wing, and at the same k.

The perturbation velocity u is given below:

$$u(x,y,0^+) = -\frac{2}{\pi}\tau\sum_{i=1}^{N}\frac{(1+2ax)[x+a(x^2-\beta^2y^2)]^{i-1}}{[1+2ax+a^2(x^2-\beta^2y^2)]^{i+\frac{1}{2}}}\left[\sum_{p=1}^{i}\lambda^*_{ip}F^*_{ip}(\xi)\right] + \frac{2}{\pi}\tau\sum_{i=1}^{N}\frac{(2i+1)a(1+ax)[x+a(x^2-\beta^2y^2)]^{i}}{[1+2ax+a^2(x^2-\beta^2y^2)]^{i+\frac{3}{2}}}\left[\sum_{p=1}^{i}\lambda^*_{ip}F^*_{i+1,p}(\xi)\right] \quad (4\text{-}30)$$

In Fig. 6a-d we have plotted the perturbation velocity for the successive approximations, up to N = 5. The same remarks can be made about the velocity as about the potential above. In Fig. 6e we compare u^+ for N = 5 with u^+ for the flat delta wing (= const. along the centerline of the wing) for different values of k.

Note that only the first term in (4-30) contains a singularity at the leading edge, the second term vanishes at the leading edge.

IV.4 The leading edge suction force

From Appendix A, we have for the leading edge suction force per unit length in the chordwise direction:

$$\frac{dT}{dx} = \pi\rho\sqrt{1-m^2}\; C_x^2 \tag{4-31}$$

where $m = \beta \times$ local slope of the leading edge, i.e.

$$m = \frac{k(1+2ax)}{\sqrt{1+4k^2 ax(1+ax)}} \tag{4.32}$$

and $C_x = \lim\limits_{x \to x_{LE}} u\sqrt{x - x_{LE}}$

For $x \to x_{LE}$ (4-30) behaves like

$$u(x,y,0^+) \sim -\frac{2}{\pi}\tau^2 \sum_{i=1}^{N} \frac{(1+2ax)[x+a(x^2-\beta^2y^2)]^i}{[1+2ax+a^2(x^2-\beta^2y^2)]^{i+\frac{1}{2}}} \frac{\frac{\lambda_{ii}^*}{(2i-1)!!}}{[\tau^2\{x+a(x^2-\beta^2y^2)\}-y^2]^{\frac{1}{2}}}$$

so that with $x_{LE} = \frac{1}{2a}\left[-1+\sqrt{1+4(\frac{a\beta y}{k}+a^2\beta^2y^2)}\right]$

we find for C_x: $(a > 0)$

$$C_x = \frac{k^3}{\beta}\sqrt{\frac{2}{a}}\,(1+2ax)^{\frac{1}{2}} \sum_{i=1}^{N} \frac{\left[-1+\sqrt{1+4k^2ax(1+ax)}\right]^{i-\frac{1}{2}}\left[-\frac{2}{\pi}\,\frac{\lambda_{ii}^*}{(2i-1)!!}\right]}{[2k^2(1+ax)-1+\sqrt{1+4k^2ax(1+ax)}\,]^{i+\frac{1}{2}}\, a^{i-1}} \tag{4-33}$$

The leading edge suction force per unit length in the streamwise direction is obtained by substitution of (4-32) and (4-33) into (4-31). For $a < 0$, a similar expression as in (4-33) can be obtained for C_x. Upon integration of the so found expressions with respect to x (from 0 to x) and multiplication by 2 (for both leading edges), we obtain for any a:

$$T = \frac{\pi\rho k^2}{a^2\beta^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\left(-\frac{2}{\pi}\frac{\lambda^*_{ii}}{(2i-1)!!}\right)\left(\frac{2}{\pi}\frac{\lambda^*_{jj}}{(2j-i)!!}\right)}{(1-k^2)^{i+j+\frac{1}{2}}\, a^{i+j-2}} \left[-\frac{\tilde{v}^{i+j+2}}{i+j+2} + \frac{(1-k^2)\tilde{v}^{i+j}}{i+j}\right] \tag{4-34}$$

where $$\tilde{v} = \frac{-(1+2k^2ax)+\sqrt{1+4k^2ax(1+ax)}}{\left[-1+\sqrt{1+4k^2ax(1+ax)}\,\right]}$$

(4-34) gives the thrust or suction force due to the square root singularity at the leading edges. For $a \to 0$, (4-34) yields the thrust for the delta wing, as given in Appendix A.

It is of practical interest to compare the above results for $\frac{dT}{dx}$ and T with some intuitive approximations (Fig. 7). The approximations we consider here are:

i) Delta wing, with a sweep, such as the initial sweep of the wing with the hyperbolic leading edges.

$$\tau' = \tau \; ; \quad k' = \tau'\beta = k \, . \tag{4-35}$$

ii) Delta wing, with a local span, such as the local span of the wing with the hyperbolic leading edges.

$$\tau^* = \frac{1}{2ax\beta k}\left[-1+\sqrt{1+4k^2ax(1+ax)}\,\right]$$

$$k^* = \tau^*\beta = k^*(ax) \tag{4-36}$$

iii) Delta wing, with a sweep, such as the local sweep of the ogee and gothic wing, but a translated apex.

$$\tau^{**} = \frac{\tau(1+2ax)}{\sqrt{1+4k^2ax(1+ax)}}$$

$$k^{**} = \tau^{**}\beta = k^{**}(ax) \tag{4-37a}$$

The $\left(\frac{dT}{dx}\right)$ is calculated at

$$x^{**} = \frac{\left[-1+\sqrt{1+4k^2ax(1+ax)}\right]\sqrt{1+4k^2ax(1+ax)}}{2ak^2(1+2ax)} \qquad (4\text{-}37b)$$

The leading edge suction force per unit length in chordwise direction for a flat delta wing, as in i), ii) and iii) may be calculated using the formulae of Appendix A. For T itself we have to use numerical integration techniques in case ii) and iii). For i) T is obtained from Appendix A immediately.

In Fig. 8a-h, the results are shown for the successive orders of approximations N = 1, . . . , 5 and for the cases k', k^* and k^{**}. From these figures it is seen once more that the approximation is good for the ogee wing, but less satisfactory for the gothic wing. We may conclude that in the range (4-28) both the k^* and k^{**} methods are in good agreement with the results for N = 5 for $k < 0.6$, but fail for $k > 0.7$. For greater x than in (4-28) it may be predicted from Fig. 8a that the k', k^* and k^{**} methods all fail to approximate the solution for $N \to \infty$.

IV. 5 Other matching procedures

The calculations in Sections 3 and 4 were done with the coefficients in the boundary conditions for M chosen in such a way that the upwash on the wing is matched with that for a flat plate at the origin. There are other ways to match the right side of (4-26) with its left hand side.

As an example we consider matching at several points on the centerline of the wing. In (4-26) for $a > 0$, we determine the c^*'s by imposing the following conditions:

<u>N = 1:</u> $w(0, 0) = w_0$

<u>N = 2:</u>
$$\begin{cases} w(0, 0) = w(a, 0) = w_0 \\ \frac{\partial w}{\partial y} = 0 \quad \text{for } y = 0 \end{cases}$$

<u>N = 3:</u>
$$\begin{cases} w(0, 0) = w(\frac{a}{2}, 0) = w(a, 0) = w_0 \\ \frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{for } y = 0 \end{cases}$$

<u>N = 4:</u>
$$\begin{cases} w(0, 0) = w(\frac{a}{3}, 0) = w(\frac{2a}{3}, 0) = w(a, 0) = w_0 \\ \frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y^2} = \frac{\partial^3 w}{\partial x \partial y^2} = \frac{\partial^3 w}{\partial y^3} = 0 \quad \text{for } y = 0 \end{cases}$$

<u>N = 5:</u>
$$\begin{cases} w(0, 0) = w(\frac{a}{4}, 0) = w(\frac{a}{2}, 0) = w(\frac{3a}{4}) = w(a, 0) = w_0 \\ \frac{\partial w}{\partial y} = \text{--------} = \frac{\partial^3 w}{\partial y^3} = \frac{\partial^4 w}{\partial x^3 \partial y} = \frac{\partial^4 w}{\partial x \partial y^3} = \frac{\partial^4 w}{\partial x \partial y^3} = \frac{\partial^4 w}{\partial y^4} = 0 \\ \text{for } y = 0 . \end{cases}$$

For $a < 0$ similar conditions, give different values for the c^*'s in (4-26), i.e. on the centerline:

N = 1: $w = w_0$ at $ax = 0$

N = 2: $w = w_0$ at $ax = 0$ and $-\frac{1}{2}$

N = 3: $w = w_0$ at $ax = 0$, $-\frac{1}{4}$ and $-\frac{1}{2}$

N = 4: $w = w_0$ at $ax = 0$, $-\frac{1}{6}$, $-\frac{1}{3}$ and $-\frac{1}{2}$

N = 5) $w = w_0$ at $ax = 0$, $-\frac{1}{6}$, $-\frac{1}{4}$, $-\frac{3}{8}$, and $-\frac{1}{2}$

The result of this method is shown in Fig. 9a and b. It looks quite promising for the gothic wing, however the cross-section is still far from flat.

IV. 6 Concluding remarks

In this chapter we derived the exact solution to the PG-equation for wings with "hyperbolic" leading edges. The solution satisfies non-trivial boundary conditions on the wing. It is expected that by superposition of a sufficient number of solutions any boundary conditions can be satisfied up to a satisfactory order of approximation. For the boundary value problem, obtained by any order of approximation, the solution is exact.

For the flat wing, the perturbation potential and velocity and the leading edge suction force were calculated up to the 4th order in the parameter a, which denotes the deviation from straight. It was shown that the flat plate can be well-approximated in a certain region.

The sign of the curvature (ogee or gothic) has a significant effect on the leading edge suction force. None of the "intuitive" approximations for finding the leading edge suction force give good results for wings with "near-sonic" leading edges.

The solutions derived in this chapter clearly have advantages over the solution in powerseries as given in [3]. In the next chapter we derive the differential equations from which the solutions in [3] may be derived. Furthermore, we formulate the (sufficient) condition for finding exact solutions similar to those for the "hyperbolic" leading edges.

V. A POSSIBLE GENERALIZATION

In Chapter IV we obtained an exact solution to the PG-equation for one family of curved leading edges. In this chapter we derive (sufficient) conditions for leading edges that lead to exact solutions. We employ the same method as in the previous chapter.

Consider the transformation

$$\begin{cases} x_1 = x + \{x^2 - \beta^2(y^2+z^2)\}\, F\{x,\ \beta^2(y^2+z^2)\} \\ x_2 = y \\ x_3 = z \end{cases} \tag{5-1}$$

(5-1) has the same properties as the transformation (4-2). The leading edge that is straightened by (5-1) is:

$$|y| = f(x)\,, \tag{5-2a}$$

where f(x) satisfies

$$f(x) - \tau x = \tau[x^2 - \beta^2 f^2(x)]\, F\{x, \beta^2 f^2(x)\} \tag{5-2b}$$

The PG-equation in the (x, y, z) space becomes in the new coordinates:

$$\beta^2 \varphi_{x_1x_1} - \varphi_{x_2x_2} - \varphi_{x_3x_3} + g_1\varphi_{x_1x_1} + g_2\varphi_{x_1} + g_3(x_1\varphi_{x_1x_1} + x_2\varphi_{x_1x_2} + x_3\varphi_{x_1x_3}) = 0 \tag{5-3}$$

where:

$$\begin{cases} g_1 = -\beta^2 + \beta^2\left(\dfrac{\partial x_1}{\partial x}\right)^2 - \left(\dfrac{\partial x_1}{xy}\right)^2 - \left(\dfrac{\partial x_1}{\partial z}\right)^2 + \dfrac{2x_1}{x_2}\,\dfrac{\partial x_1}{\partial y} \\ g_2 = \beta^2\,\dfrac{\partial^2 x_1}{\partial x^2} - \dfrac{\partial^2 x_1}{\partial y^2} - \dfrac{\partial^2 x_1}{\partial z^2} \\ g_3 = -\dfrac{2}{x_2}\,\dfrac{\partial x_1}{\partial y} = -\dfrac{2}{x_3}\,\dfrac{\partial x_1}{\partial z} \end{cases} \tag{5-4}$$

(5-3) is a linear partial differential equation of the second order. With (5-1):

$$\begin{cases} x_1 = x + (x^2 - \rho)F(x, \rho) = x_1(x, \rho) \\ x_2 = y \\ x_3 = z \qquad\qquad \rho = \beta^2(y^2+z^2) \end{cases} \tag{5-5}$$

(5-4) can be written as:

$$\begin{cases} g_1 = \beta^2\left[-1 + \left(\frac{\partial x_1}{\partial x}\right)^2 - 4\rho\left(\frac{\partial x_1}{\partial \rho}\right)^2 + 4x_1\left(\frac{\partial x_1}{\partial \rho}\right)\right] \\ g_2 = \beta^2\left[\frac{\partial^2 x_1}{\partial x^2} - 4\rho\frac{\partial^2 x_1}{\partial \rho^2} - 4\frac{\partial x_1}{\partial \rho}\right] \\ g_3 = -4\beta^2\frac{\partial x_1}{\partial \rho} \end{cases} \tag{5-6}$$

From (5-6) it can be seen that the only transformation under which (5-3) is the PG-equation again, is the transformation

$$\begin{cases} x_1 = \pm x + \text{const.} \\ x_2 = y \\ x_3 = z \end{cases}$$

Now we have to relate φ with the solutions of the homogeneous flow theory. We set:

$$\varphi(x_1, x_2, x_3) = \sum_{j=0} \frac{\partial^j M}{\partial x_1^j} G_j(x_1, \rho) \tag{5-7}$$

In (5-7) M is homogeneous of degree m, and a solution of the homogeneous flow theory, i.e.

$$\beta^2 M_{x_1x_1} - M_{x_2x_2} - M_{x_3x_3} = 0 \tag{5-8}$$

M also satisfies the Euler relation:

$$(m - \ell)\frac{\partial^{\ell} M}{\partial x_1^{\ell}} = x_1 \frac{\partial^{\ell+1} M}{\partial x_1^{\ell+1}} + x_2 \frac{\partial^{\ell+1} M}{\partial x_1^{\ell}\partial x_2} + x_3 \frac{\partial^{\ell+1} M}{\partial x_1^{\ell}\partial x_3} \qquad (\ell \geq 0)$$

In (5-7) $G_j(x_1, \rho)$ is unknown for the moment. It is our purpose to find those transformations for which (5-7) is a series with a finite number of terms. Substitution of (5-7) into (5-3) gives with (5-8) and (5-9):

$$\sum_{j=0}\left[\frac{\partial^{j+2} M}{\partial x_1^{j+2}}(g_1 G_j)\right]$$
$$+\sum_{j=0}\left[\frac{\partial^{j+1} M}{\partial x_1^{j+1}}\left\{G_j(g_2+(m-j-1)g_3)+\frac{\partial G_j}{\partial x_1}(2\beta^2+2g_1+x_1 g_3)+\frac{\partial G_j}{\partial \rho}(4\beta^2+2\rho g_3)\right\}\right]$$
$$+\sum_{j=0}\left[\frac{\partial^{j} M}{\partial x_1^{j}}\left\{\frac{\partial G_j}{\partial x_1}(g_2+(m-j)g_3)-\frac{\partial G_j}{\partial \rho}4\beta^2(m-j+1)+\frac{\partial^2 G_j}{\partial x_1^2}(\beta^2+g_1+x_1 g_3)\right.\right.$$
$$\left.\left.+\frac{\partial^2 G_j}{\partial x_1 \partial \rho}2\rho g_3-4\beta^2\rho\frac{\partial^2 G_j}{\partial \rho^2}\right\}\right] = 0 \qquad (5\text{-}10)$$

Define now three partial differential operators as:

$$\begin{cases} L_q^{(1)} = (\beta^2+g_1+x_1 g_3)\dfrac{\partial^2}{\partial x_1^2} + 2\rho g_3 \dfrac{\partial^2}{\partial x_1 \partial \rho} - 4\beta^2\rho\dfrac{\partial^2}{\partial \rho^2} \\ \qquad\qquad + (g_2 + (m-q)g_3)\dfrac{\partial}{\partial x_1} + (\beta^2+g_1+x_1 g_3)\dfrac{\partial}{\partial \rho} \\ L_q^{(2)} = (2\beta^2+2g_1+x_1 g_3)\dfrac{\partial}{\partial x_1} + (4\beta^2 x_1 + 2\rho g_3)\dfrac{\partial}{\partial \rho} + (g_2+(m-q-1)g_3) \\ L_q^{(3)} = g_1 \end{cases} \qquad (5\text{-}11)$$

The sufficient conditions for satisfying (5-10) are that the expressions which multiply

$$M,\ M_{x_1},\ M_{x_1 x_1},\ \text{etc.}$$

vanish identically.

The system of partial differential equations we obtain from this, can with (5-11) be written as:

$$
\begin{aligned}
L_0^{(1)} G_0 &= 0 \\
L_1^{(1)} G_1 &= -L_0^{(2)} G_0 \\
L_2^{(1)} G_2 &= -L_1^{(2)} G_1 - L_0^{(3)} G_0 \\
&\vdots \\
L_q^{(1)} G_q &= - L_{q-1}^{(2)} G_{q-1} - L_{q-2}^{(3)} G_{q-2} \; ; \quad q = 2, 3, ----
\end{aligned}
\tag{5-12}
$$

(5-12) is an uncoupled system of linear partial differential equations. Each can be solved with the solution of the two foregoing ones. The coefficients that appear in the differential operators are functions of m and of the transformation. It can be shown that $L_q^{(1)}$ is equivalent to:

$$
L_q^{(1)} = \beta^2 \left[\frac{\partial^2}{\partial x^2} - 4\rho \frac{\partial^2}{\partial \rho^2} - 4(m+1-q) \frac{\partial}{\partial \rho} \right] \tag{5-13a}
$$

and also that $L_q^{(2)}$ in the (x, y, z) coordinates is:

$$
L_q^{(2)} = \beta^2 \left[\left\{ \frac{\partial^2 x_1}{\partial x^2} - 4\rho \frac{\partial^2 x_1}{\partial \rho^2} - 4(m-q) \frac{\partial x_1}{\partial \rho} \right\} + \right.
$$
$$
\left. + 2 \frac{\partial x_1}{\partial x} \frac{\partial}{\partial x} + 4(x_1 - 2\rho \frac{\partial x_1}{\partial \rho}) \frac{\partial}{\partial \rho} \right] \tag{5-13b}
$$

The first equation in (5-12) is, according to (5-13a), independent of the transformation (5-5). Since there are no boundary conditions specified for the G's, the system (5-12) will have an infinite number of solutions.

We are interested in the case that (5-7) is a finite series, i.e.

$$\varphi(x_1, x_2, x_3) = \sum_{q=0}^{p} \frac{\partial^q M}{\partial x_1^q} G_q(x_1, \rho) \tag{5-14}$$

This implies that the system (5-12) consists out of p+3 equations for (p+1) functions G_q... The last three equations are:

$$\begin{aligned} L_p^{(1)} G_p &= - L_{p-1}^{(2)} G_{p-1} - L_{p-2}^{(3)} G_{p-2} \\ 0 &= - L_p^{(2)} G_p - L_{p-1}^{(3)} G_{p-1} \\ 0 &= - L_p^{(3)} G_p \end{aligned} \tag{5-15}$$

From (5-15) it follows that a necessary condition for having a finite sum as solution (5-14), is that

$$\beta^2\left[-1 + \left(\frac{\partial x_1}{\partial x}\right)^2 - 4\rho\left(\frac{\partial x_1}{\partial \rho}\right)^2 + 4x_1\left(\frac{\partial x_1}{\partial \rho}\right)\right] G_p = 0$$

This means that unless $G_p \equiv 0$, the transformation (5-5) has to satisfy.

$$-1 + \left(\frac{\partial x_1}{\partial x}\right)^2 - 4\rho\left(\frac{\partial x_1}{\partial \rho}\right)^2 + 4\, x_1\left(\frac{\partial x_1}{\partial \rho}\right) = 0 \tag{5-16a}$$

As "initial" condition we have the invariance of the Mach cone, which gives:

$$x_1 = x \qquad \text{for} \qquad \rho = x^2 . \tag{5-16b}$$

(5-16a) is a first order nonlinear partial differential equation, which is considered in more detail in Appendix C.

Note that the transformation in the previous chapter (4-6) indeed satisfies (5-16a).

If we are able to construct solutions to (5-16), the system (5-12), (5-15) consists of (p+2) equations for (p+1) unknowns:

$$\begin{aligned} L_0^{(1)} G_0 &= 0 \\ L_1^{(1)} G_1 &= L_0^{(2)} G_0 \\ L_2^{(1)} G_2 &= L_1^{(2)} G_1 \qquad \text{(p+2) equations, (p+1) } G_q\text{'s} \\ &\vdots \\ L_p^{(1)} G_p &= L_{p-1}^{(2)} G_{p-1} \\ 0 &= L_p^{(2)} G_p \end{aligned} \tag{5-17}$$

The difficulty of the over-determined (5-17) can be resolved by breaking the system up as indicated below:

$$\left\{ \begin{aligned} L_q^{(1)} G_q &= 0 \\ L_q^{(2)} G_q &= 0 \end{aligned} \right. \qquad q = 0, \ldots, p \tag{5-18}$$

(5-18) implies then that now each term in the summation (5-14) is a solution of the PG-equation, and that as most simple exact solution we have

$$\varphi(x_1, x_2, x_3) = M \cdot G_0(x_1, \rho) \tag{5-19}$$

where G_0 satisfies (5-18) for q = 0.

In this chapter we showed that it is possible to construct finite exact solutions of the PG-equation for the case of curved subsonic leading edges. The actual construction of such solution depends on the ability to construct solutions of one nonlinear 1st order partial differential equation.

We showed also that for a given leading edge, it is possible to construct a solution, but in general this solution is the infinite sum (5-7).

In conclusion we give below a few possible solutions of the system (5-12):

$$G_q = \sum_{i=0}^{q} A_i \frac{(x-x_1)^{q-i}}{(q-i)!} \quad ; \quad A_i = \text{constant}, \; q = 0, ---- \qquad (5\text{-}20)$$

It is easily checked that this is a solution of (5-12). With the condition that (5-7) yields

$$\varphi = M$$

for $x_1 = x$, we find: $G_q = \dfrac{(x-x_1)^q}{q!}$

So that (5-7) is now:

$$\varphi(x_1, x_2, x_3) = \sum_{j=0}^{\infty} \frac{\partial^j M}{\partial x_1^j} \frac{(x-x_1)^j}{j!} \qquad (5\text{-}21)$$

(5-21) is the Taylor series expansion Coene [3] uses for the quasi-homogeneous flow theory.

The first equation of (5-12) has as solution also

$$G_0 = \frac{c}{[(x-\alpha)^2 - \rho]^{m+\frac{1}{2}}} \qquad (5\text{-}22)$$

(5-22) is in fact the G from the previous chapter for $\alpha = 0$ and $\alpha = \frac{-1}{a}$, $c = a^{-(2m+1)}$.

Using (5-22) it can be shown that (5-12) also has as solution:

$$G_q = \frac{c(x-x_1)^q}{q!\,[x^2-\rho]^{m+\frac{1}{2}}} + \frac{\tilde{c}(x-\alpha-x_1)^q}{q!\,[(x-\alpha)^2-\rho]^{m+\frac{1}{2}}} \qquad (5\text{-}23)$$

Coene [1], [2] and [3] uses combinations of powerseries expansions of (5-22) and (5-23) to construct solutions in the quasi-homogeneous flow theory.

APPENDIX A

FORCES ON A WING WITH SUBSONIC LEADING EDGES

The pressure distribution on the wing is given by (2-11). For the steady case we write

$$p' = -\rho_0 U \varphi_x = p - p_0 \tag{A-1}$$

For a lifting problem there exists a jump in the pressure across the plane $z = 0$, so that there $p^+ \neq p^-$

$$C_p^+ = \frac{p - p_0}{\frac{1}{2}\rho U^2} = -\frac{2u}{U} \tag{A-2}$$

For a planar wing with a supersonic trailing edge we obtain the lift coefficient C_L by integration over the area of the wing, with $u^- = -u^+$ we find:

$$\begin{aligned} C_L &= -\frac{2}{S}\iint_S C_p \, dxdy \quad ; \quad S = \text{wing area} \\ &= -\frac{8}{US}\int_0^{y_{LE}} \int_{x_{LE}(y)}^{x_{TE}} \varphi_x d_x \quad \text{(for a symmetric wing)} \end{aligned} \tag{A-3}$$

Since for a lifting problem $\varphi = 0$ at the leading edge, we find:

$$C_L = -\frac{8}{US}\int_0^{y_{LE}} \varphi(x_{TE}, y, 0^+) dy \tag{A-4}$$

(A-4) shows that to find the lift coefficient, we only have to integrate the perturbation potential along the trailing edge of the wing.

In the same way we find for the lift induced drag:

$$C_D = -\frac{4}{US}\iint_S \varphi_z \varphi_x \, dxdy \tag{A-5}$$

The drag is reduced by the suction force on the leading edges. According to Jones and Cohen [10] :

$$\frac{dT}{dx} = \pi\rho\sqrt{1-m^2}\; C_x^2 \quad ; \quad T = \text{thrust due to suction} \tag{A-6}$$

where $m = \beta$/local sweep of the leading edge and

$$C_x = \lim_{x \to x_{LE}} u\sqrt{x-x_{LE}} \tag{A-7}$$

For the delta wing we have then:

$$\begin{cases} |y| = \tau x \\ z = 0 \end{cases}$$

so that $m = \tau\beta = k$

The terms that contribute in (A-7) are the singular terms only, from Chapter III: (for a D. L. P.)

$$u(x, y, 0^+) \sim -\frac{2}{\pi}\tau x^{n-1} \frac{\lambda^*_{nn}}{(2n-1)!!} \frac{\tau x}{\sqrt{\tau^2 x^2 - y^2}}$$

So that:

$$C_x^2 = \frac{2}{\pi^2}\tau^2 \frac{\lambda^{*2}_{nn}\; x^{2n-1}}{(2n-1)!!(2n-1)!!}$$

So now from (A-6): (for the two leading edges)

$$\frac{dT}{dx} = \frac{4}{\pi}\rho\sqrt{1-k^2}\;\frac{\tau^2\;\lambda^{*2}_{nn}\;x^{2n-1}}{(2n-1)!!(2n-1)!!} \tag{A-8}$$

Integrate from 0 to x:

$$T = \frac{4}{\pi}\rho\sqrt{1-k^2}\;\frac{\tau^2\;\lambda^{*2}_{nn}\;x^{2n}}{(2n)!!\;(2n-1)!!} \tag{A-9}$$

For a flat plate: $\lambda_{11}^{*} = \frac{\pi}{2} \frac{w_0}{E'}$ (see 3-17). So that

$$T = \frac{\pi}{2} \rho \sqrt{1-k^2} \; \frac{\tau^2 w_0^2}{E'^2} \; x^2 \qquad \text{(A-10)}$$

APPENDIX B

THE FUNCTIONS F^*_{np} AND G^*_{np}

Define the function K^r_{np} as:

$$\frac{d^n K^*_{np}}{d\xi^n} = (-1)^{n+p} \frac{\xi^{r-n}}{(1-\xi^2)^{p+\frac{1}{2}}} \; ; \; \xi = \frac{x_2}{\tau x_1} \; ; \begin{cases} n \geq 1 \\ n \geq p \geq 1 \end{cases} \tag{B-1}$$

F^*_{np} is obtained by putting r = 2 and G_{np} by putting r = 1.

From (B-1) we derive the identity

$$\frac{d^n K^r_{np}}{d\xi^n} = -\xi \frac{d^{n+1} K^r_{n+1,p}}{d\xi^{n+1}} \tag{B-2}$$

and also

$$K^r_{np} = n K^r_{n+1,p} - \xi \frac{d}{d\xi} K^r_{n+1,p} = -\xi^{n+1} \frac{d}{d\xi}\left(\frac{K^r_{n+1,p}}{\xi^n}\right) \tag{B-3}$$

Since $\xi = \dfrac{x_2}{\tau x_1}$ we derive from (B-3), keeping x_2 constant:

$$\frac{\partial}{\partial x_1}\left(x_1^n K^r_{n+1,p}\right) = x_1^{n-1} K^r_{np} \tag{B-4}$$

(B-4) could also be derived from (3-7), by

$$\frac{\partial \varphi_n}{\partial x_1} = u_n$$

Differentiation of (B-4) with respect to x_1, keeping x_2 = constant, using (B-4) yields

$$\frac{d^2}{dx_1^2}\left(x_1^n K^r_{n+1,p}\right) = x_1^{n-2} K^r_{n-1,p}$$

Proceeding in this fashion gives:

$$\frac{d^m}{dx_1^m}\left(x_1^n K^r_{n+1,p}\right) = x_1^{n-m} K^r_{n-m+1,p}\,; \; n > m \qquad \text{(B-5)}$$

With (B-4) it is now very easy to calculate any derivative of φ_n.

In order to reduce the number of integrations that have to be carried out to calculate K^r_{np} from (B-1), we derive two recurrence relations for K^r_{np}.

From (B-1) we see that

$$\frac{d^n}{d\xi^n} K^r_{n_3 p-1} = (-1)^{n+p} \frac{\xi^{r-n}}{(1-\xi^2)^{p+\frac{1}{2}}} [-1 + \xi^2]$$

$$\frac{d^n}{d\xi^n} K^r_{n-1,p-1} = (-1)^{n+p} \frac{\xi^{r-n}}{(1-\xi^2)^{p+\frac{1}{2}}} [(r-n+1)+(n+2p-r-2)\xi^2]$$

Combining this gives us then

$$(2p-1)K^r_{np} = -(n+2p-r-2)K^r_{n,p-1} + K^r_{n-1,p-1}$$

for $n \geq 2$ and $p \geq 2$ (B-6)

and also for $p = 1$, it is easily verified that for $n \geq 4$

$$\begin{cases} (n-1)(n-3)F^*_{n1} = (2n-5)F^*_{n-1,1} - (1-\xi^2)F^*_{n-2,1}\,; \; r=2, \; n \geq 4 \\ (n-2)^2 \; G^*_{n1} = (2n-5)G^*_{n-1,1} - (1-\xi^2)G^*_{n-2,1}\,; \; r=1, \; n \geq 3 \end{cases} \qquad \text{(B-7)}$$

Put together we have

$$(n+r-3)(n-r-1)K^r_{n1} = (2n-5)K^r_{n-1,1} - (1-\xi^2)K^r_{n-2,1}\,; \text{ for } r = 1 \text{ or } 2$$

(B-8)

(B-6) and (B-7) or (B-8) don't include F^*_{11}, F^*_{21}, F^*_{31}, G^*_{11} and G^*_{21}.

These functions are easily calculated from (B-1):

$$\begin{cases} F^*_{11} = A^* \\ F^*_{21} = C^* \\ F^*_{31} = \frac{1}{2} C^* - \frac{1}{2}\xi^2 D^* \end{cases} \qquad \begin{cases} G^*_{11} = B^* \\ G^*_{21} = \xi D^* \end{cases} \qquad \text{(B-9)}$$

The A^*, B^*, C^* and D^* are defined as:

$$\begin{cases} A^* = \dfrac{1}{\sqrt{1-\xi^2}} \\ B^* = \dfrac{\xi}{\sqrt{1-\xi^2}} \\ C^* = \sqrt{1-\xi^2} \\ D^* = \tanh^{-1}\sqrt{1-\xi^2} \end{cases} \qquad |\xi| \leq 1 \qquad \text{(B-10)}$$

For F^*_{np} we derive now the explicit form. From (B-7) and (B-9, 6) it can be seen that F^*_{np} may be written as

$$F^*_{np} = a_{np} A^* + \sum_{t=0}^{E(\frac{n-2}{2})} \xi^{2t} C^{2t}_{np} \; C^* + \sum_{s=0}^{E(\frac{n-3}{2})} \xi^{2(s+1)} d^{2(s+1)}_{np} \; D^*$$

(B-11)

From (B-7) it may be seen that the A^* only appear in F^*_{nn}. The coefficients a_{nn} follow from

$$(2n-1)a_{nn} = a_{n-1,n-1} \; ; \; a_{11} = 1$$

so that: $a_{np} = \dfrac{\delta^p_n}{(2n-1)!!}$; δ^p_n is the Kronecker delta (B-12a)

The C^{2t}_{np}'s satisfy:

$$\begin{cases} (2p-1)C^{2t}_{np} = -(2p+n-4)C^{2t}_{n,p-1} + C^{2t}_{n-1,p-1} & n\geq 2 \\ (n-1)(n-3)C^{2t}_{n1} = (2n-5)C^{2t}_{n-1,1} - C^{2t}_{n-2,1} + C^{2t-2}_{n-2,1} & n\geq 4 \end{cases} \qquad \text{(B-12b)}$$

$$C_{11}^{2t} = 0 \quad \text{for all } t$$

$$C_{21}^{0} = 1; \; C_{21}^{2t} = 0 \quad \text{for } t = 1, ---$$

$$C_{31}^{0} = \tfrac{1}{2}; \; C_{31}^{2t} = 0 \quad \text{for } t = 1, ---$$

and in general $C_{np}^{2t} = 0$ for $t > E(\frac{n-2}{2})$

The d_{np}^{2t}'s satisfy:

$$\begin{cases} (2p-1)d_{np}^{2r} = -(2p+n-4)d_{n,p-1}^{2r} + d_{n-1,p-1}^{2r} & n \geq 2 \\ (n-1)(n-3)d_{n1}^{2r} = (2n-5)d_{n-1,1}^{2r} - d_{n-2,1}^{2r} + d_{n-2,1}^{2(r-1)} & n \geq 4 \end{cases} \tag{B-12c}$$

$$d_{np}^{0} = 0 \quad \text{all } n \text{ and } p$$

$$d_{11}^{2r} = d_{21}^{2r} = 0$$

$$d_{31}^{2} = -\tfrac{1}{2}; \; d_{31}^{2r} = 0 \quad \text{for } r = 2, ----$$

and in general $d_{np}^{2r} = 0$ for $r > E(\frac{n-1}{2})$.

It turns out that the recurrence relations (B-12c) can be solved explicitly:

$$d_{np}^{2r} = (-1)^p \frac{(2p+2r-3)!!}{(2p-1)!!} \frac{1}{(n-2r-1)!\,(2r)!\,2^{r-1}(r-1)!} \tag{B-13}$$

$$\text{for } r = 1, 2, \ldots, E(\tfrac{n-1}{2}) \; ; \; n \geq 3$$

$$d_{np}^{2r} = 0 \quad \text{for } n < 3 \tag{B-14}$$

The results derived in this appendix are very useful for the numerical evaluation of φ_n and u_n in Chapter IV.

APPENDIX C

THE DIFFERENTIAL EQUATION $g_1 = 0$

In (5-16a) we give the condition $g_1 = 0$, which corresponds to the condition of a finite sum as solution of the PG-equation for curved leading edges.

$$-1 + \left(\frac{\partial x_1}{\partial x}\right)^2 - 4\rho\left(\frac{\partial x_1}{\partial \rho}\right)^2 + 4x_1 \frac{\partial x_1}{\partial \rho} = 0 \tag{C-1}$$

$x_1 = x$ for $\rho = x^2$.

In Chapter IV we discussed one solution of (C-1):

$$x_1 = x + a(x^2 - \rho) \tag{C-2}$$

We did not succeed in finding another solution than (C-2), nor did we succeed in proving that (C-2) is the only possible solution. In the following we give some of the calculations we carried out.

It is possible to simplify (C-1) considerably by a change in variables.

1. $\begin{cases} \xi = x \\ \eta = x^2 - \rho \end{cases}$

The equation and "initial" condition become:

$$-1 + \left(\frac{\partial x_1}{\partial \xi}\right)^2 + 4\xi \frac{\partial x_1}{\partial \xi}\frac{\partial x_1}{\partial \eta} + 4\eta\left(\frac{\partial x_1}{\partial \eta}\right)^2 - 4x_1 \frac{\partial x_1}{\partial \eta} = 0 \tag{C-3}$$

$x_1 = \xi$ for $\eta = 0$

2. Change dependent and independent variables:

$x_1 = x_1(\xi, \eta)$ to $\eta = \eta(x_1, \xi)$

$$\eta = x_1 \frac{\partial\eta}{\partial x_1} + \xi \frac{\partial\eta}{\partial\xi} + \frac{1}{4}\left[\left(\frac{\partial\eta}{\partial x_1}\right)^2 - \left(\frac{\partial\eta}{\partial\xi}\right)^2\right] \qquad \text{(C-4)}$$

$$\eta = 0 \quad \text{for} \quad x_1 = \xi$$

3. $$\begin{cases} u = x_1 + \xi \\ v = x_1 - \xi \end{cases}$$

$$\eta = u\frac{\partial\eta}{\partial u} + v\frac{\partial\eta}{\partial v} + \left(\frac{\partial\eta}{\partial u}\right)\left(\frac{\partial\eta}{\partial v}\right) \qquad \text{(C-5)}$$

$$\eta = 0 \quad \text{for} \quad v = 0$$

4. Put $\eta = -uv + h(u, v)$

$$\frac{\partial h}{\partial u}\frac{\partial h}{\partial v} = h \qquad \text{(C-6)}$$

$$h = 0 \quad \text{for} \quad v = 0$$

Note: -- (C-4) and (C-5) are both in the form of the Clairault-equation.

-- The Jacobian of the transformation for (C-3) and (C-5 never vanishes.

-- The initial curve always lies on the singular integral of each equation. This also implies that the initial curve is a characteristic of the equation.

-- No uniquely determined solution of the equation.

In the following we demonstrate the method of characteristics for (C-6):

Write (C-6) as

$$u_x u_y = u \, ; \quad u = 0 \quad \text{for} \quad y = 0 \qquad \text{(C-7)}$$

With $p = u_x$, $q = u_y$ and s a parameter, the characteristic equations of (C-7) are:

$$\begin{aligned} \frac{dx}{ds} &= -q \\ \frac{dy}{ds} &= -p \\ \frac{du}{ds} &= -2pq \\ \frac{dp}{ds} &= -p \; ; \quad \frac{dq}{ds} = -q \end{aligned} \qquad \text{(C-8)}$$

(C-8) can be integrated quite easily:

$$\begin{aligned} x &= (x_0 - q_0) + q_0\, e^{-s} \\ y &= (y_0 - p_0) + p_0\, e^{-s} \\ u &= u_0\, e^{-2s} \\ p &= p_0\, e^{-s} \\ q &= q_0\, e^{-s} \end{aligned} \qquad \text{(C-9)}$$

In (C-9) $u_0 - p_0 q_0 = 0$

The characteristics are:

$$\begin{cases} (y-y_0) = \dfrac{p_0}{q_0}(x-x_0) \\ (u-u_0) = \dfrac{p_0}{q_0}(x-x_0)\left[(x-x_0) + 2q_0\right] \end{cases} \qquad \text{(C-10)}$$

The integral conoid, that is the surface on which all the characteristics through the point (x_0, y_0, u_0) lie, is the surface

$$4\, u_0(x-x_0)(y-y_0) - \left[(u-u_0)-(x-x_0)(y-y_0)\right]^2 = 0 \qquad \text{(C-11)}$$

Near the origin this is a hyperbolic paraboloid. From (C-10) it is seen that the initial curve is a characteristic. From (C-7) it is seen that u = 0 is the singular solution of the equation. This means that the initial curve u = 0; y = 0 is a characteristic and lies on the singular surface. According to Forsyth [6] this problem is singular and the method of characteristics fails in this case. The only non-trivial solution of (C-7) we have found is the solution $u = y(x+a^*)$, which corresponds to the transformation

$$x_1 = x + a(x^2 - \rho) \qquad \text{(c-12)}$$

and this is the transformation for the hyperbolic leading edge.

Expansion of x_1 as a powerseries in x and ρ in (C-1) or in a powerseries in ξ and η in (C-2) indicates, however, that there are more solutions than that given in (C-12).

REFERENCES

[1] Coene, R.: On a class of quasi-conical supersonic wings with curved subsonic leading edge. Journal of Engineering Mathematics, Vol. 4, No. 2, April 1970, pp. 155-167.

[2] Coene, R.: Quasi-homogeneous approximations for wings with curved subsonic leading edges at supersonic speeds. ICAS paper 72-54. 8th ICAS Congress (1972).

[3] Coene, R.: Quasi-homogeneous approximations for the calculation of wings with curved subsonic leading edges flying at supersonic speeds. Ph.D. Thesis, Delft University of Technology (1973).

[4] Etkin, B. and Woodward, F.A.: Lift distribution on supersonic wings with subsonic leading edges and arbitrary angle of attack distribution. Journal of Aeronautical Sciences, 21 (1954), pp. 783-785.

[5] Evvard, J. C.: Use of source distribution for evaluating theoretical aerodynamics of thin finite wings at supersonic speeds. NACA report 951 (1950).

[6] Forsyth, A. R.: Theory of differential equations. Vol. 5 and 6. Dover publications (1959).

[7] Fenain, M.: La théorie des mouvements homogènes et son application au calcul des ailes en régime supersonique. Progress in Aeronautical Sciences, Vol. I, Pergamon Press (1961), pp. 26-103

[8] Germain, P.: General theory of conical flow and its application to supersonic aerodynamics. NACA Technical Memorandum 1354 (1955).

[9] Germain, P.: La théorie des mouvements homogènes et son application au calcul de certaines ailes delta en régime supersonique. Recherche Aéronautique No. 7 (1949).

[10] Jones, R. T. and Cohen, D.: High speed wing theory. Princeton Aeronautical Paperback Vol. 6.

[11] Miles, J. W.: On the transformation of the linearized equation for unsteady supersonic flow. Quarterly of Applied Mathematics Vol. XII, No. 1 (1954), pp. 1-12.

[12] Robinson, A.: Aerofoil theory of a flat delta wing at supersonic speed. R & M No. 2548 (1946).

REFERENCES (Continued)

[13] Stewart, H. J.: The lift of delta wings of supersonic speeds. Quarterly of Applied Mathematics. Vol. IV, No. 3 (1946).

[14] Stewart, H. J.: A review of source superposition and conical flow methods in supersonic wing theory. Journal of Aeronautical Sciences (1954), pp. 507-516.

[15] Stewartson, K.: On the linearized potential theory of unsteady motion II. Quarterly Journal of Mechanics and Applied Mathematics, Vol. V, No. 2 (1952), pp. 137-154.

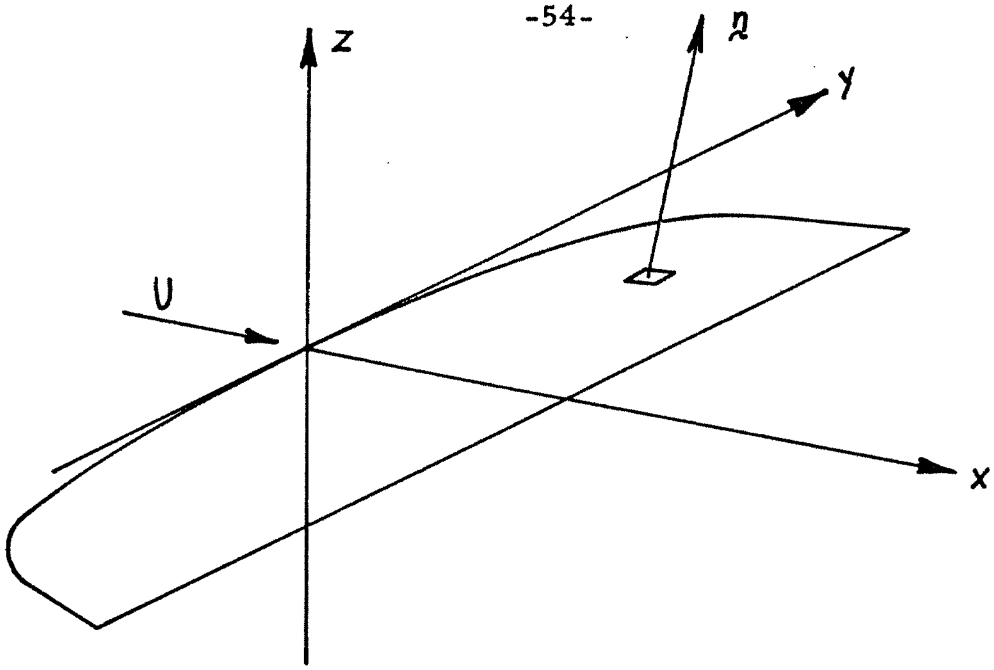


Fig. 1 The Coordinate System

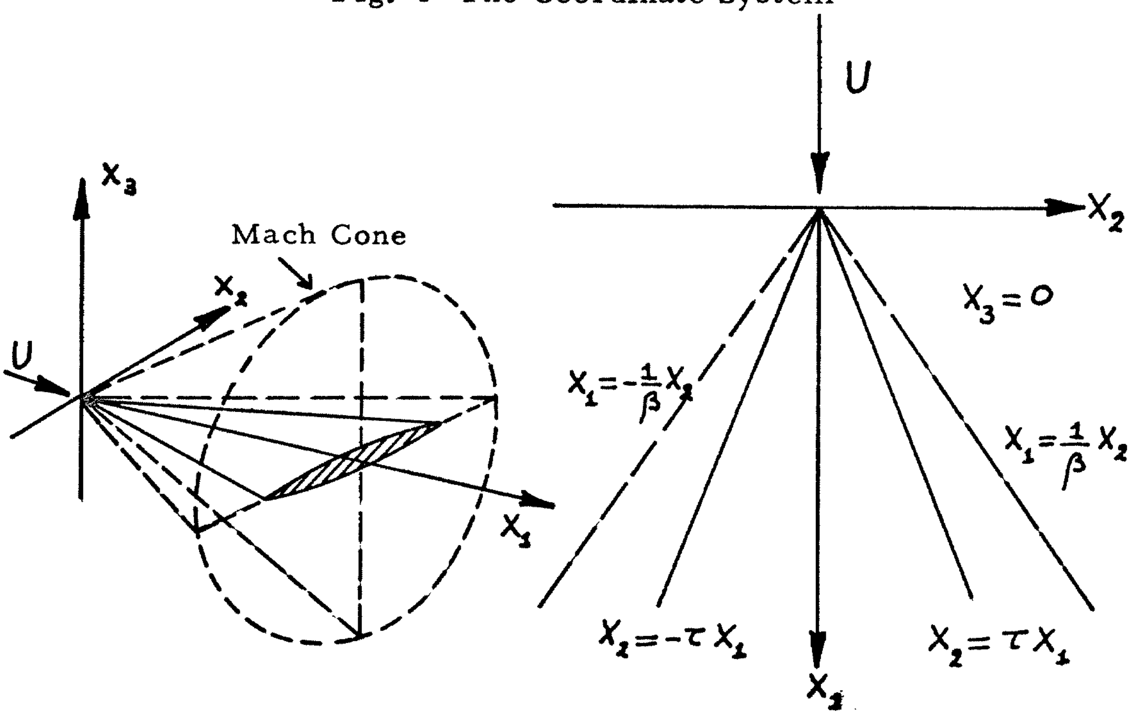


Fig. 2 Delta wing with subsonic leading edges

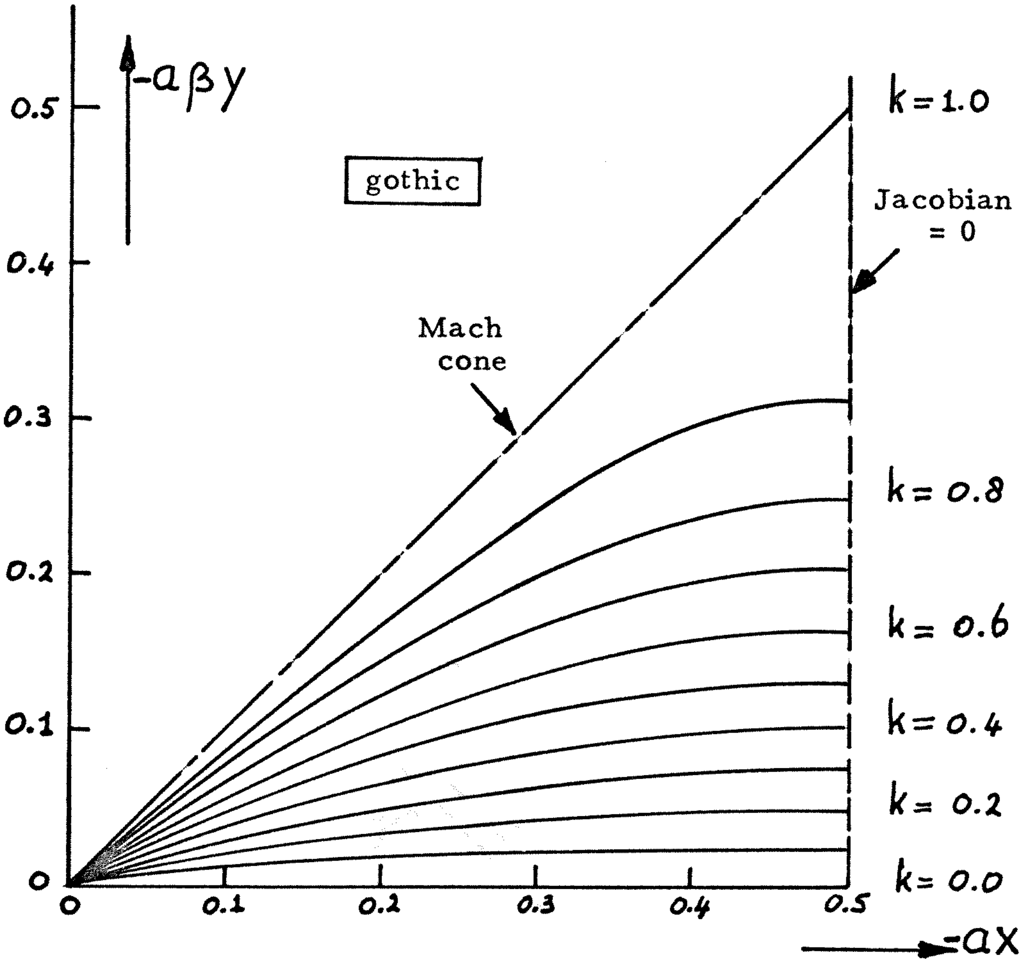


Fig. 3a Hyperbolic leading edge, a < 0

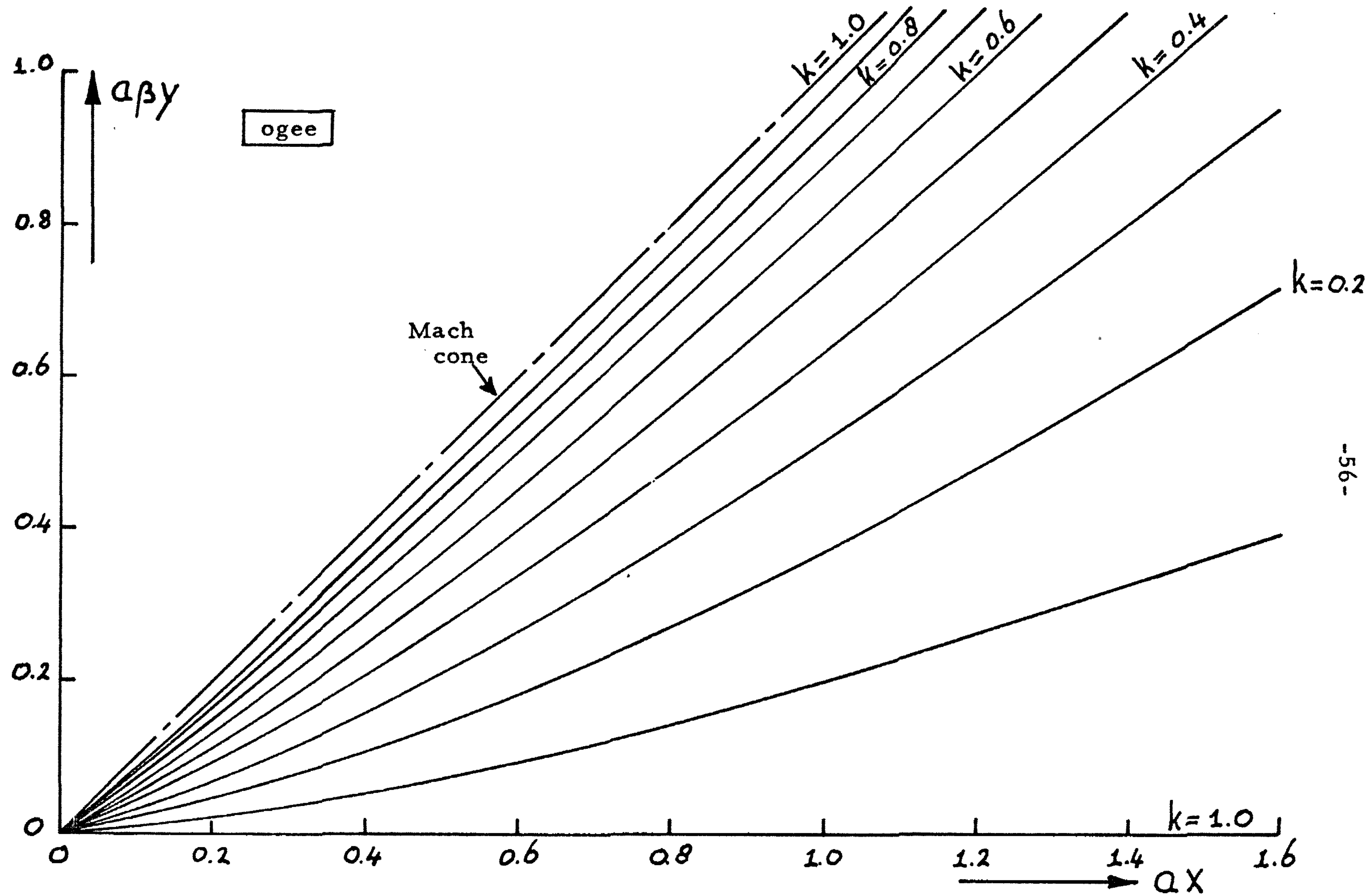


Fig. 3b Hyperbolic leading edge, a > 0

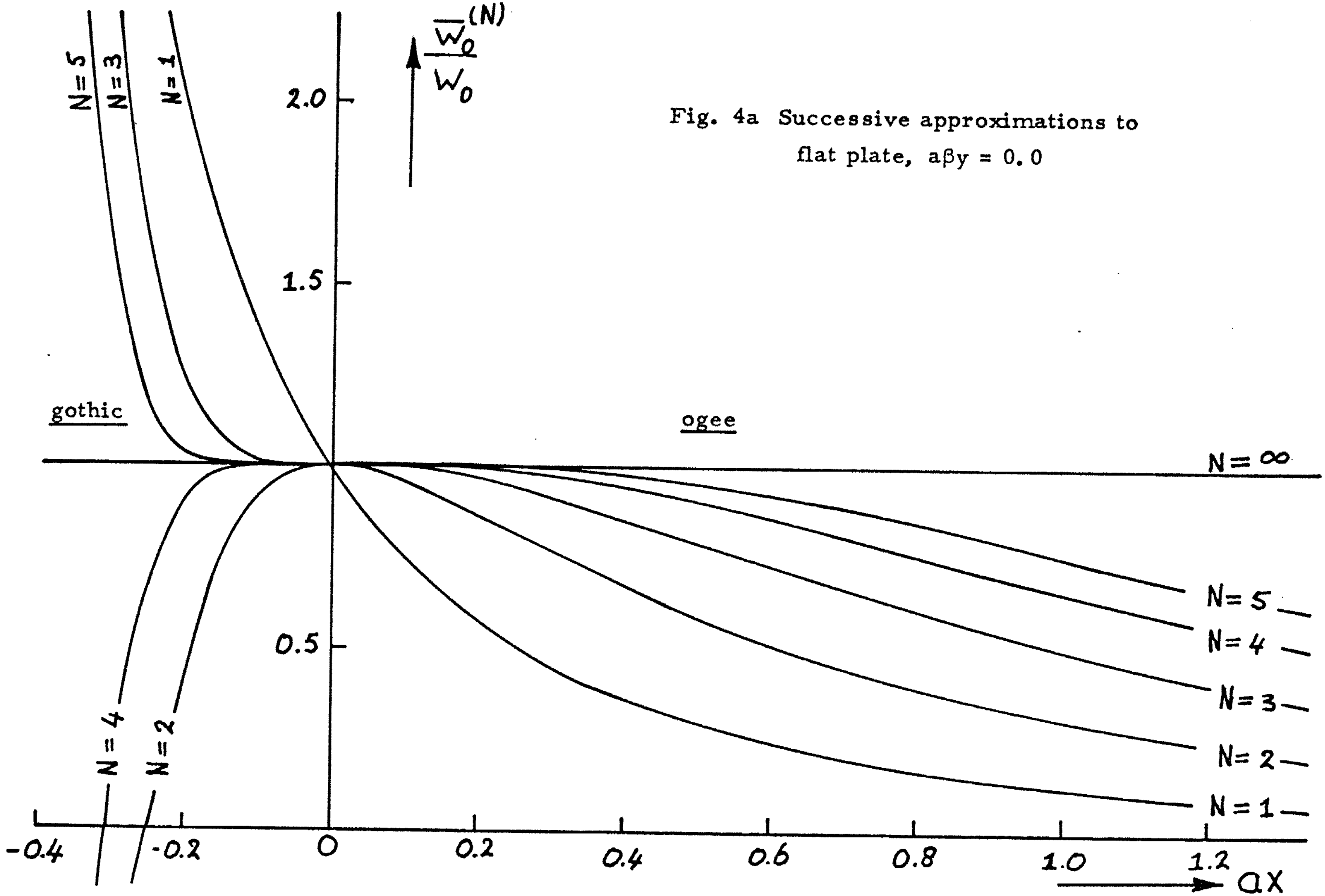


Fig. 4a Successive approximations to flat plate, $a\beta y = 0.0$

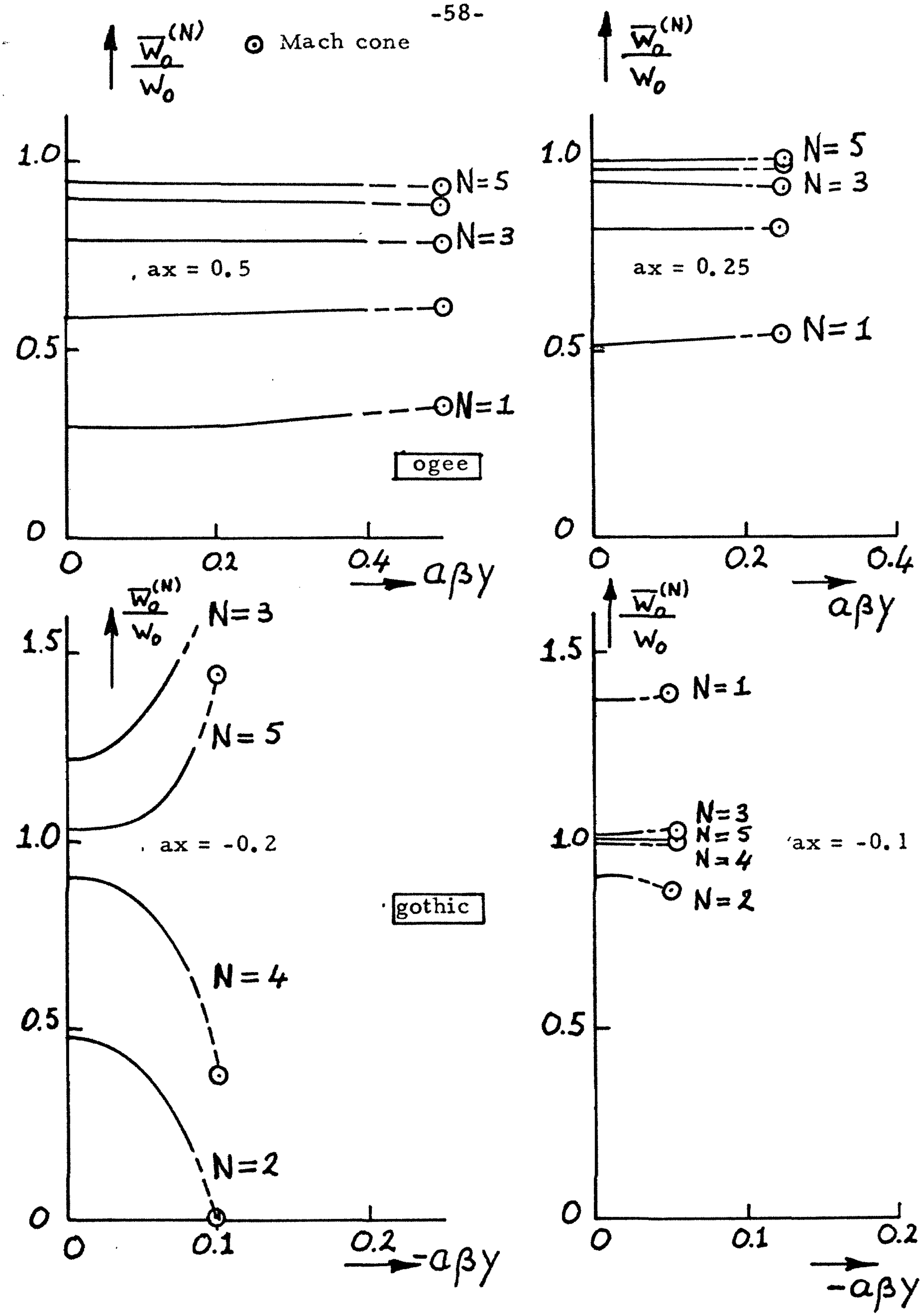
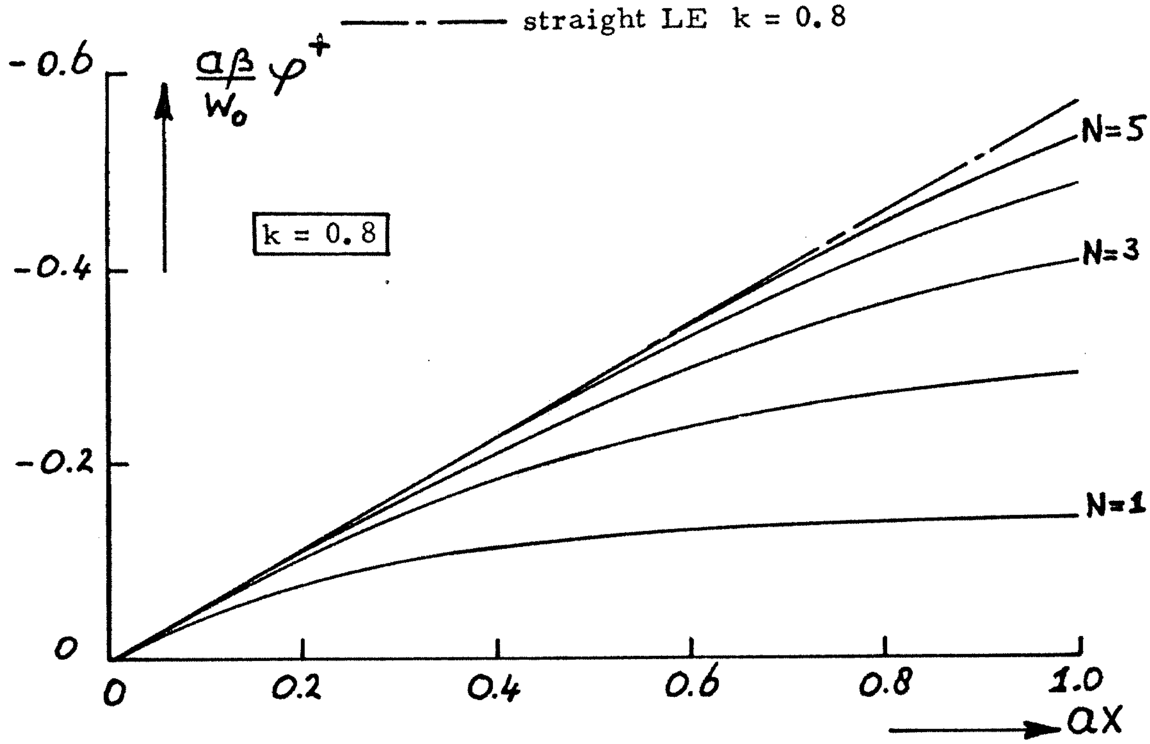


Fig. 4b Successive approximations to flat plate,

ax = const.



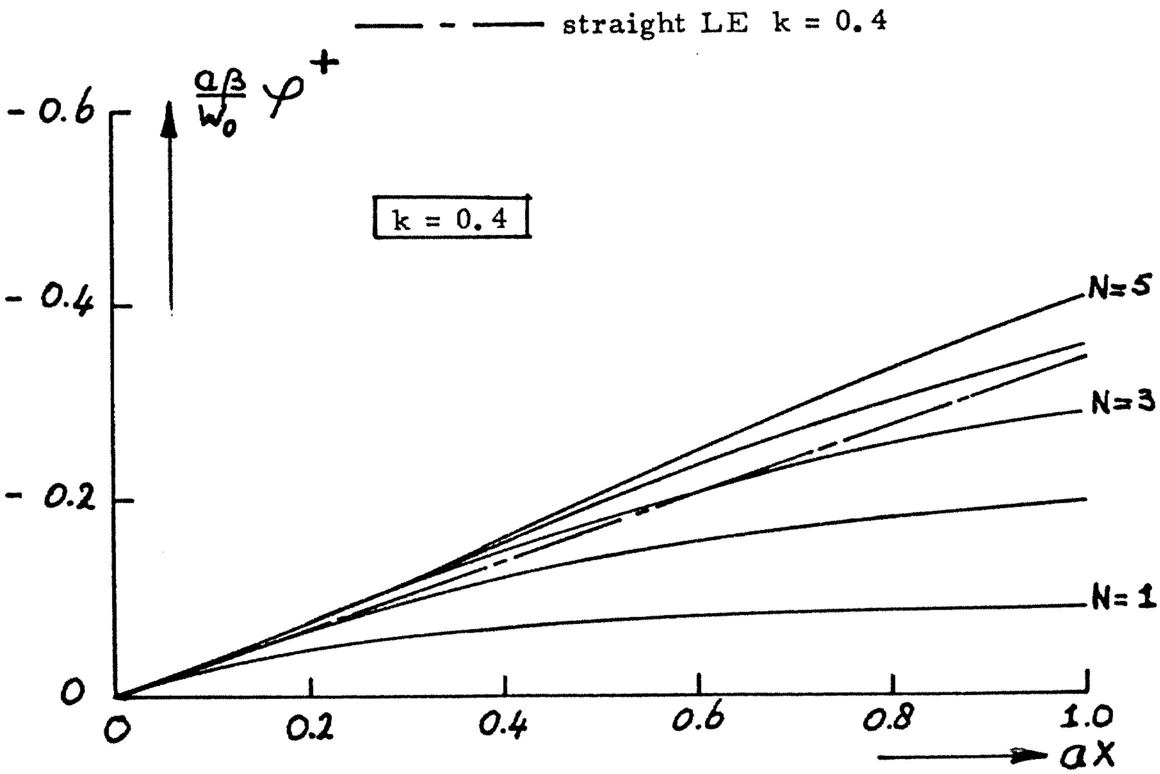


Fig. 5a Successive approximations to potential, ogee wing, $a\beta y = 0.0$

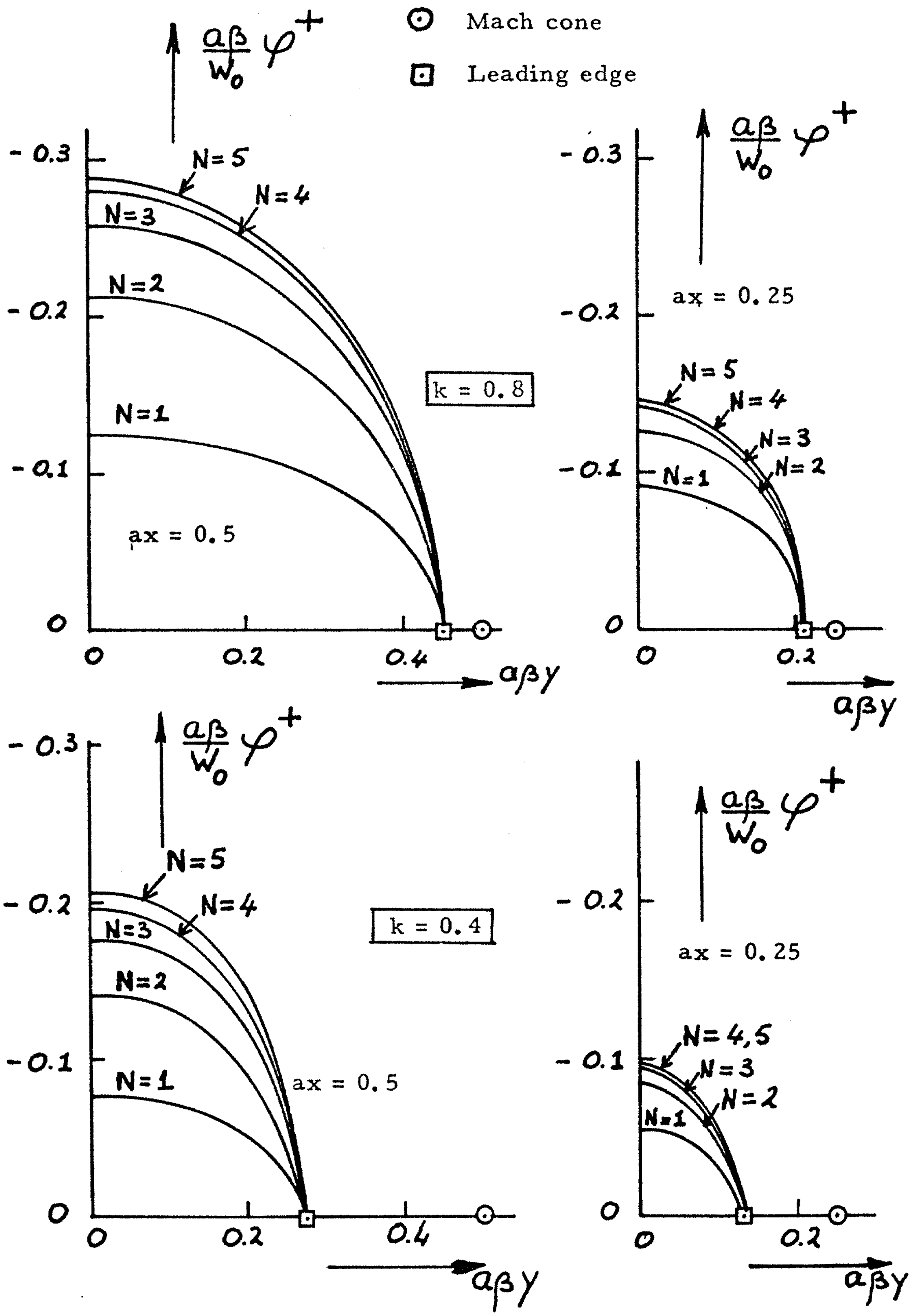


Fig. 5b Successive approximations to potential, ogee wing, ax = const.

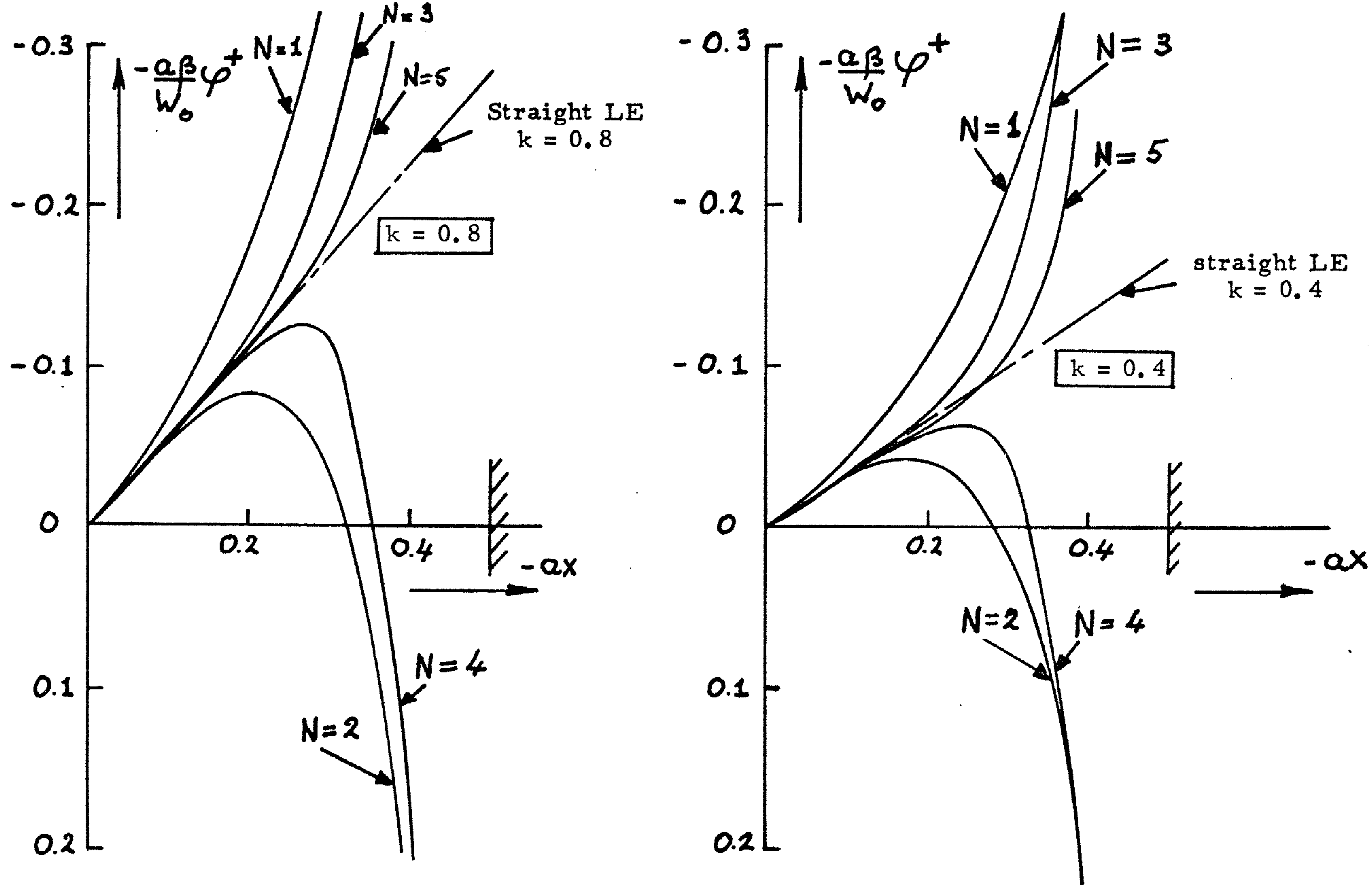


Fig. 5c Successive approximations to potential, gothic wing, $a\beta y = 0.0$

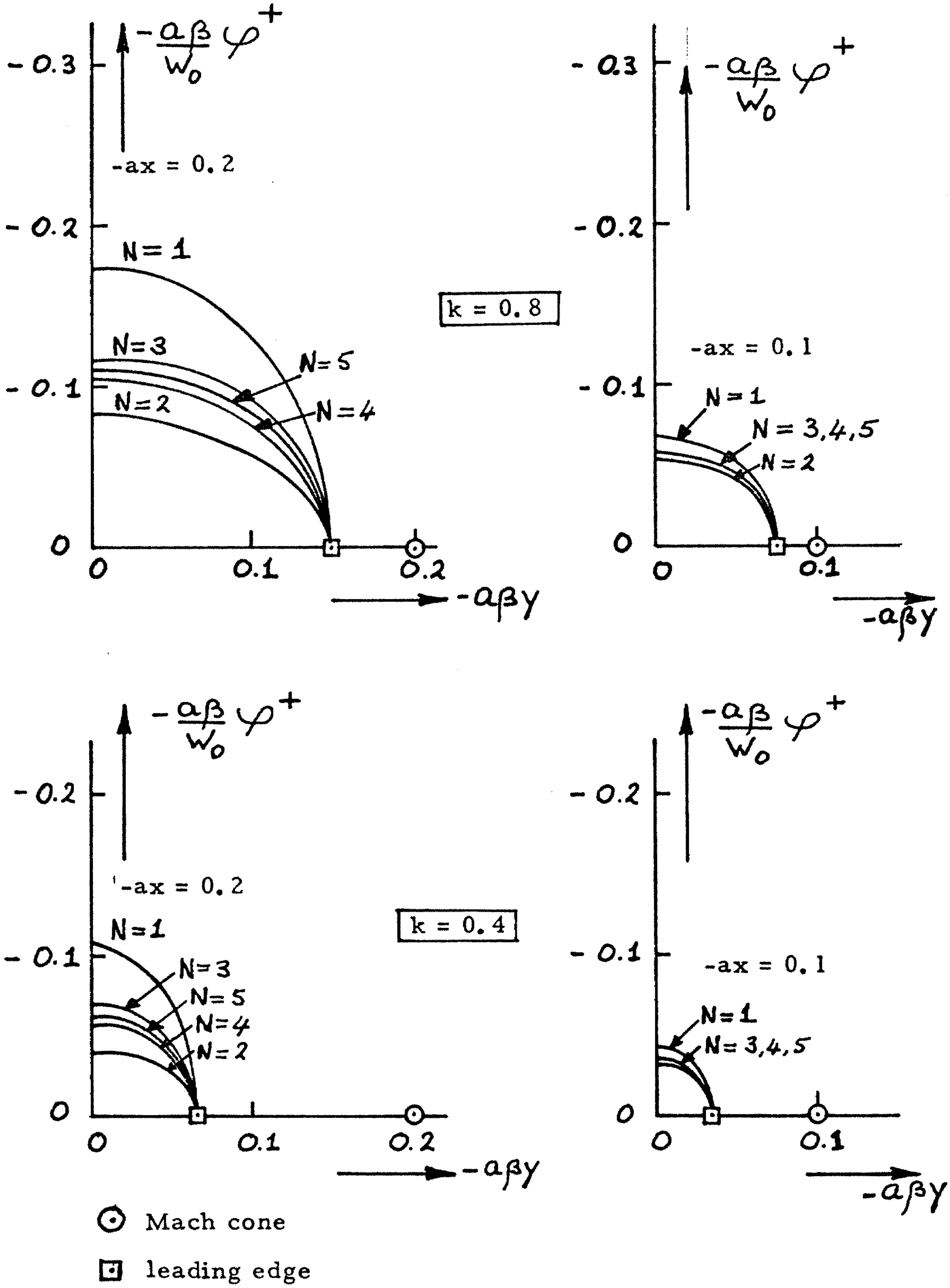


Fig. 5d Successive approximations to potential, gothic wing, ax = const.

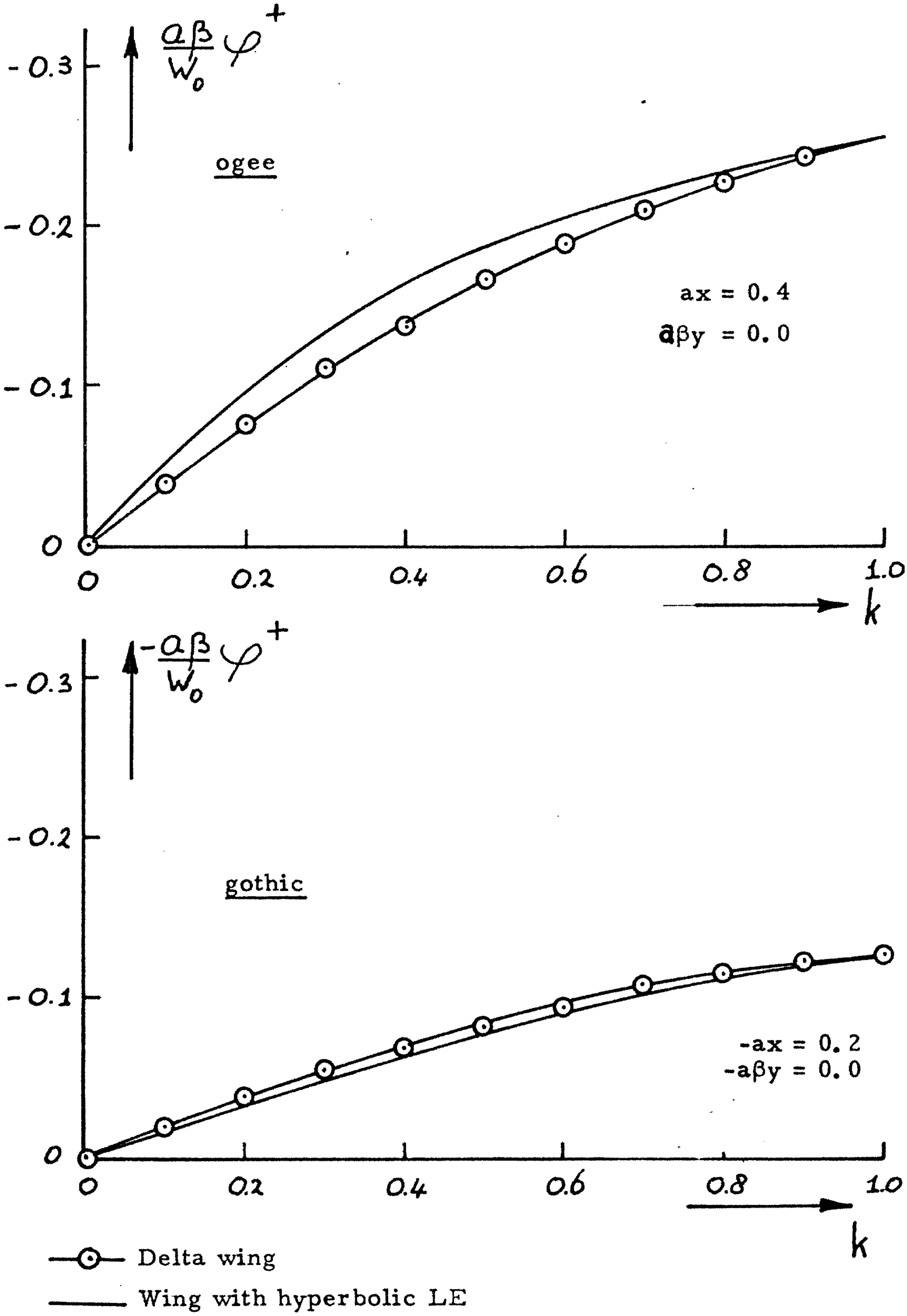
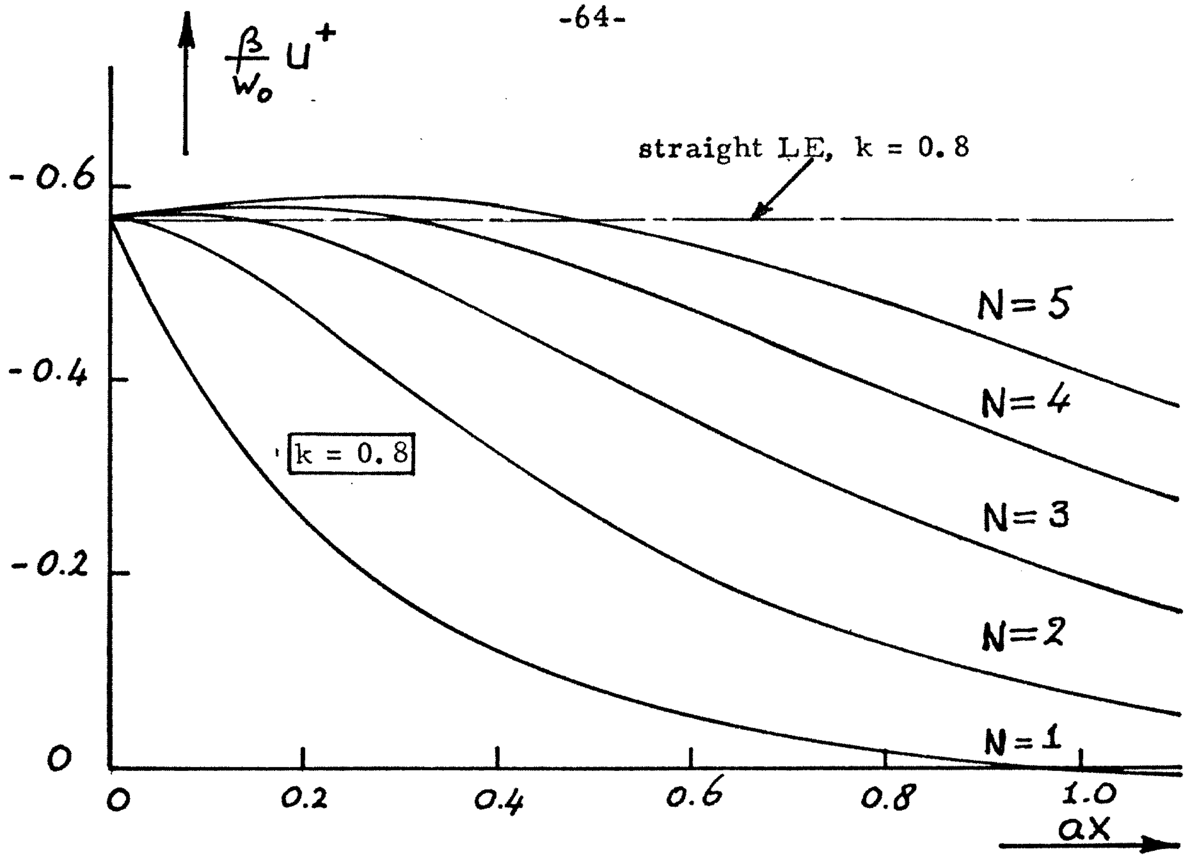


Fig. 5e Comparison of potentials for flat plate with straight and hyperbolic LE



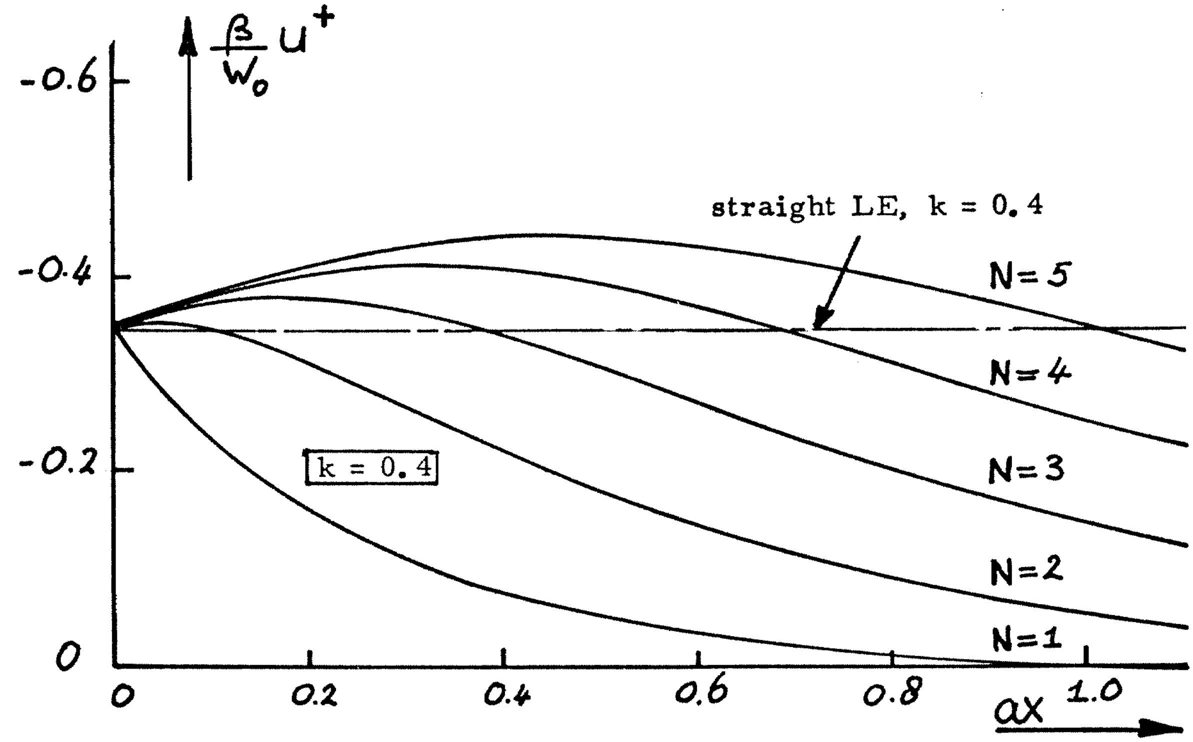


Fig. 6a Successive approximations to perturbation -- velocity, ogee wing, $a\beta y = 0.0$

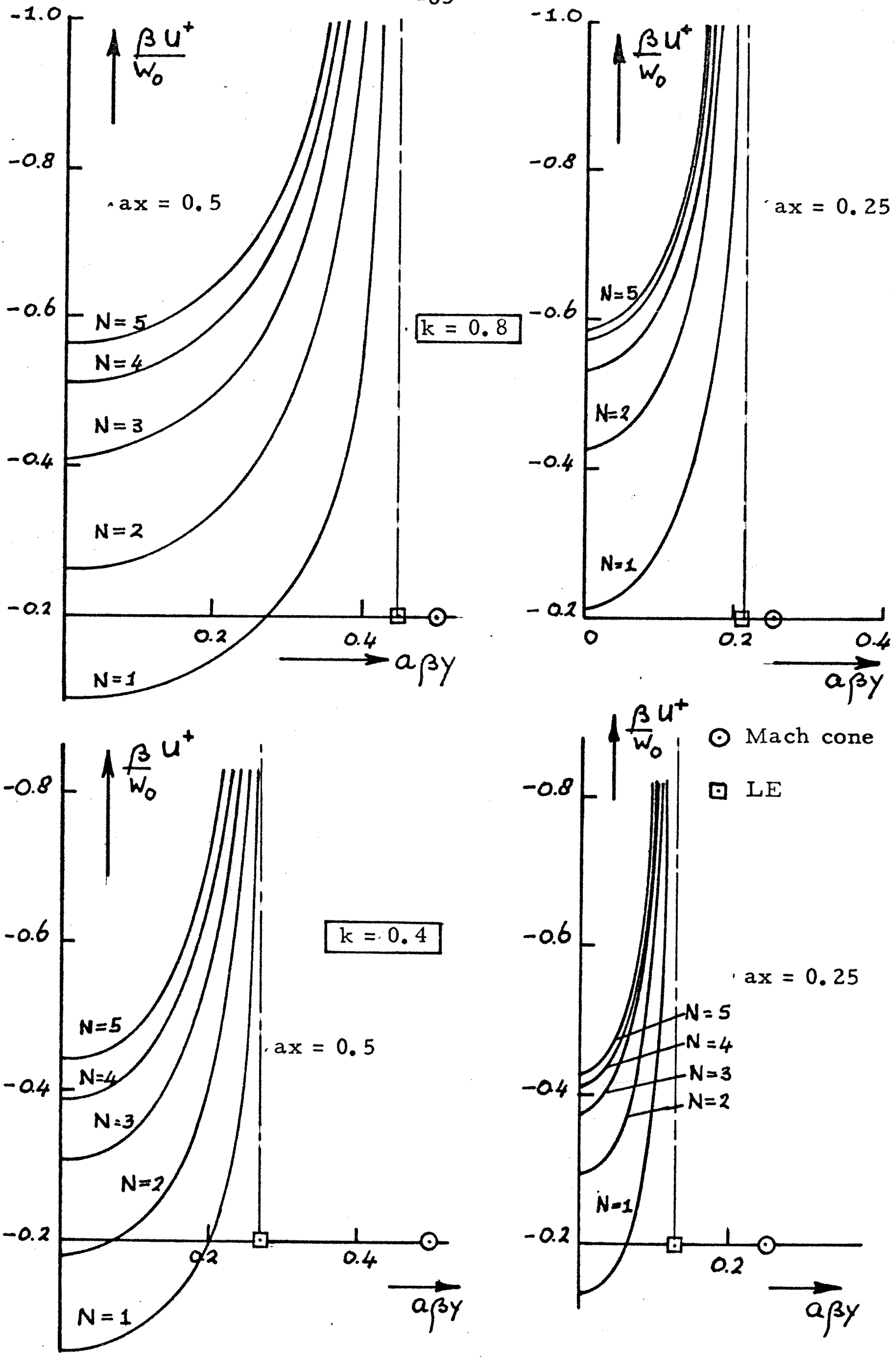


Fig. 6b Perturbation velocity, ogee wing, ax = const.

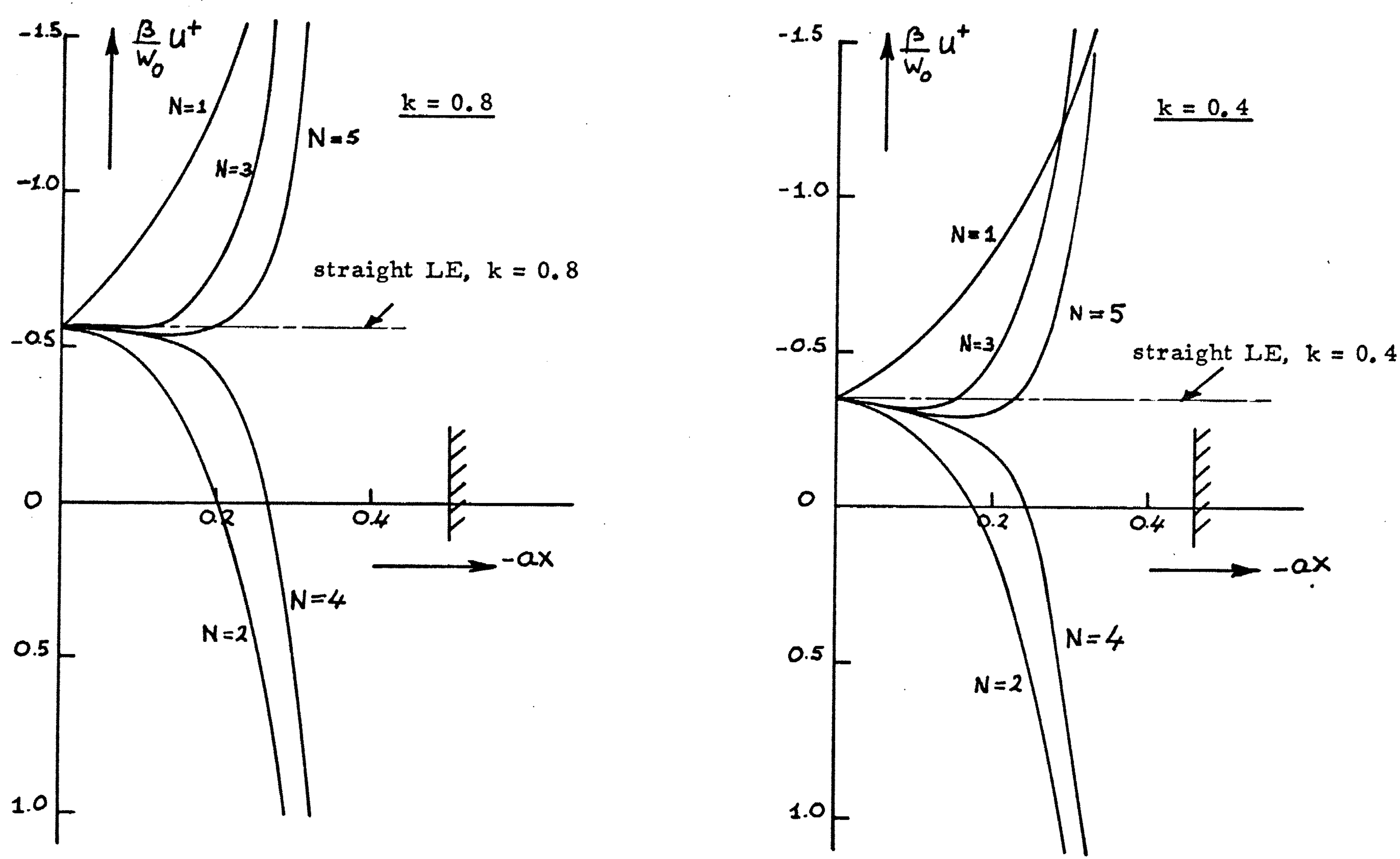


Fig. 6c Perturbation velocity, gothic wing, $a\beta y = 0.0$

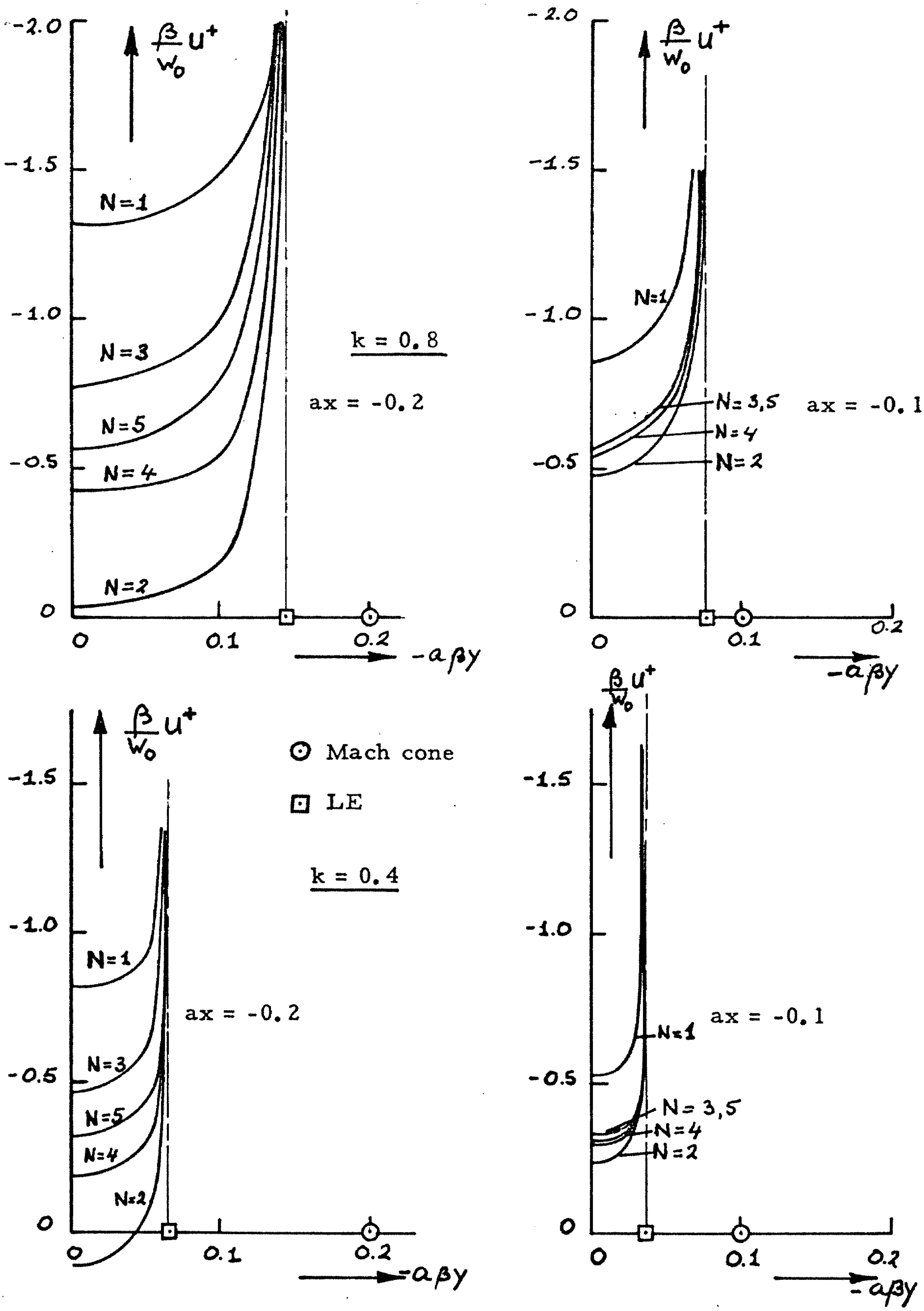


Fig. 6d Perturbation velocity, gothic wing, ax = const.

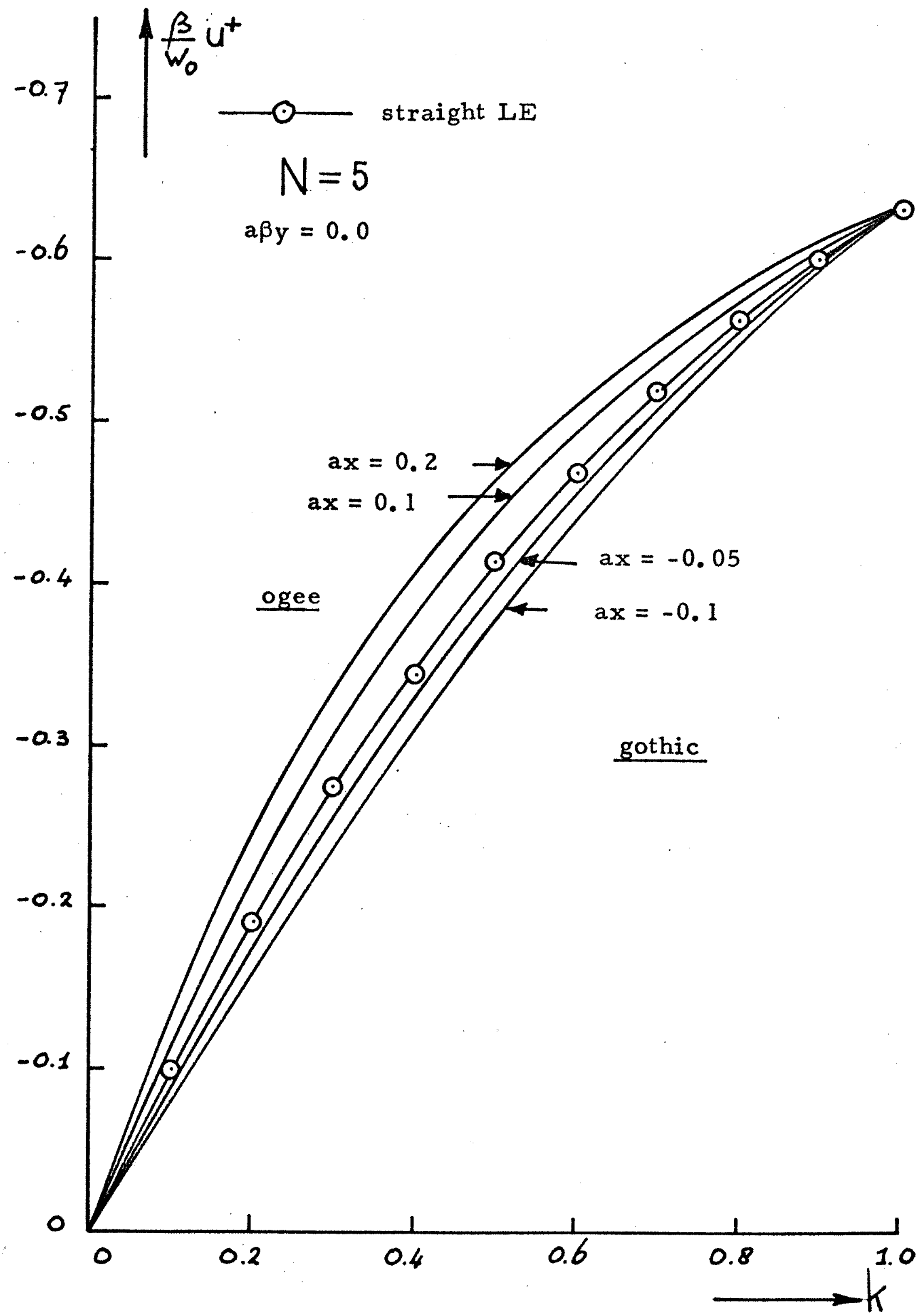


Fig. 6e Comparison perturbation velocity for wing with hyperbolic and straight LE

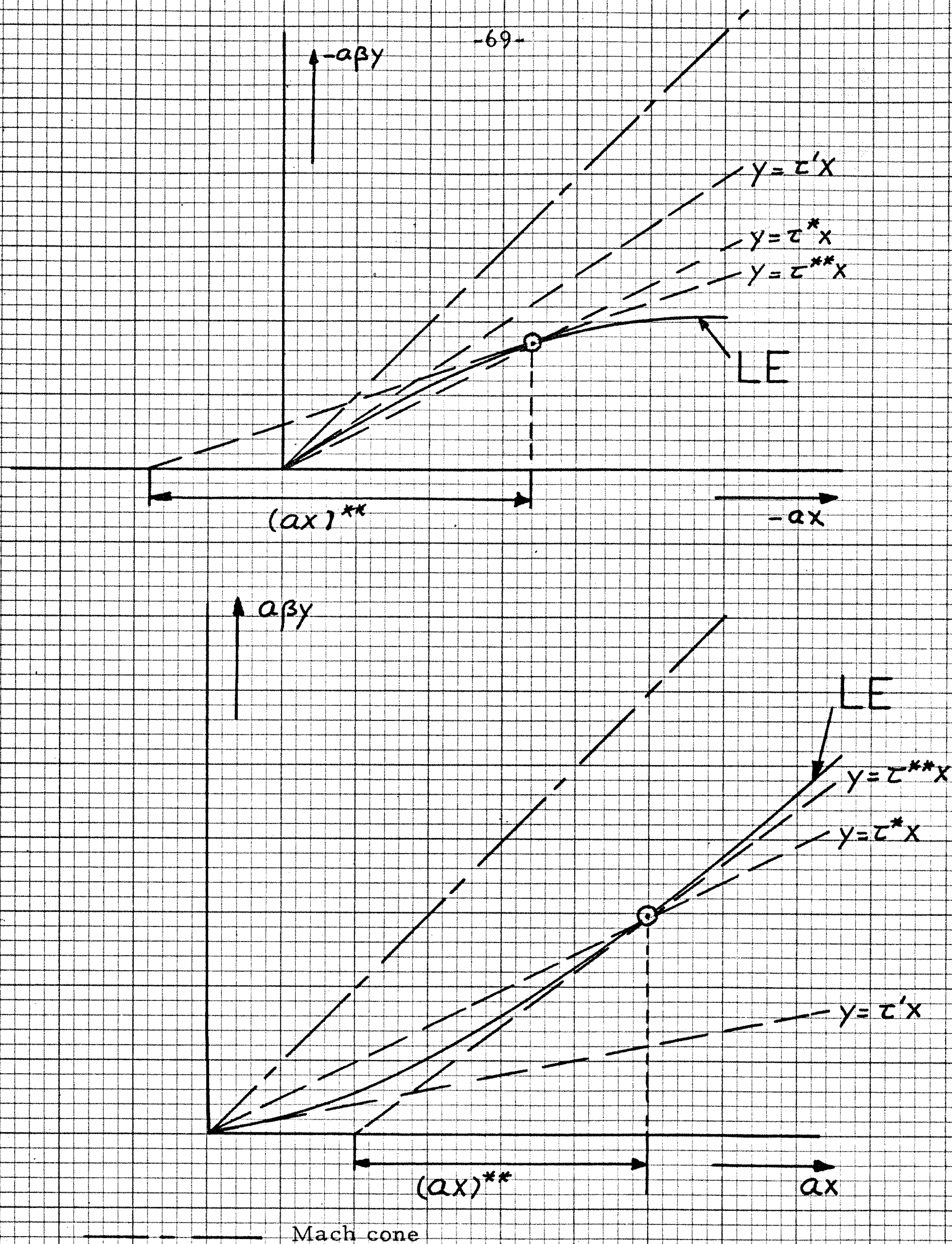


Fig. 7 Possible approximations to hyperbolic leading edge

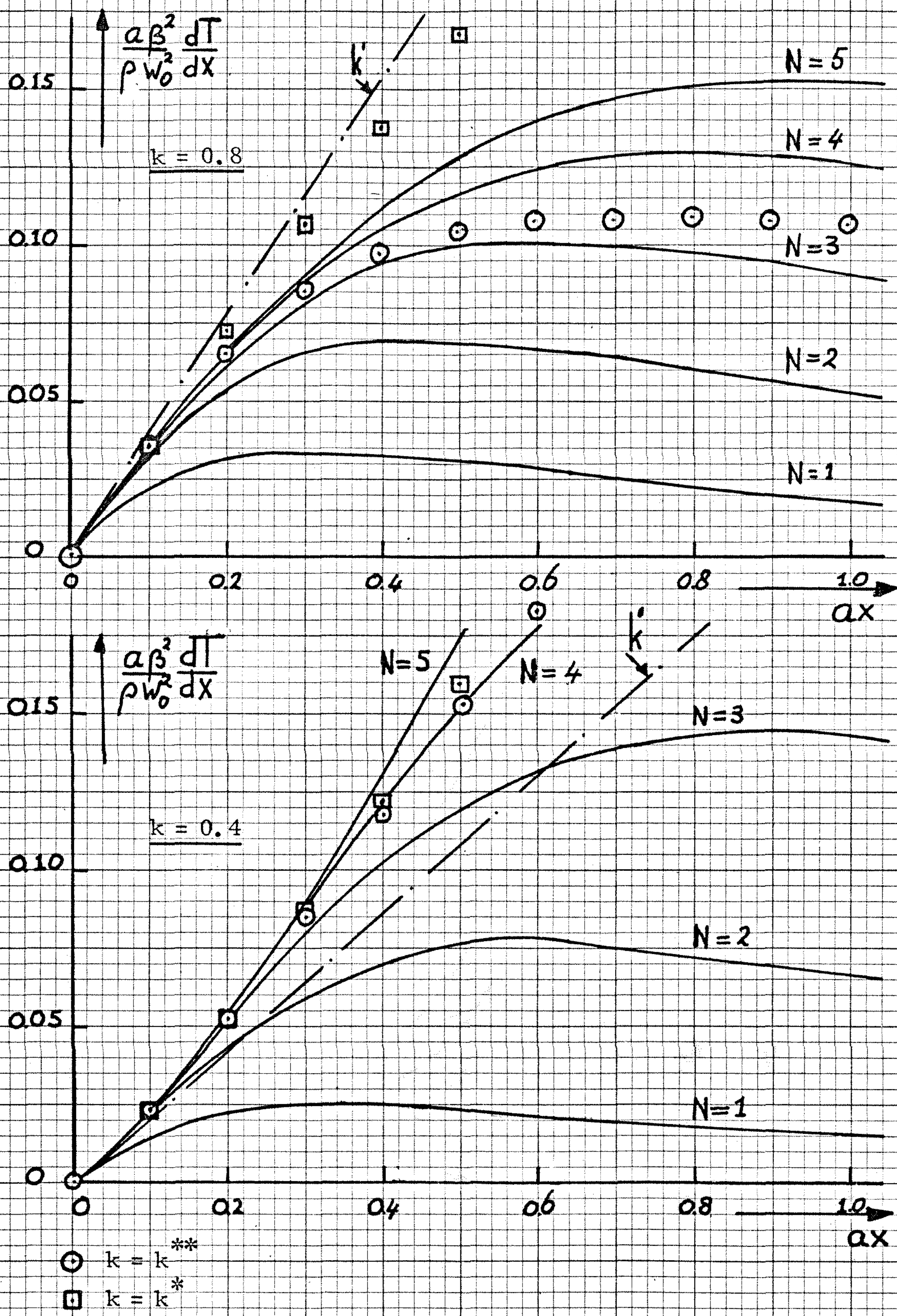


Fig. 8a Leading edge suction force/unit length, ogee wing

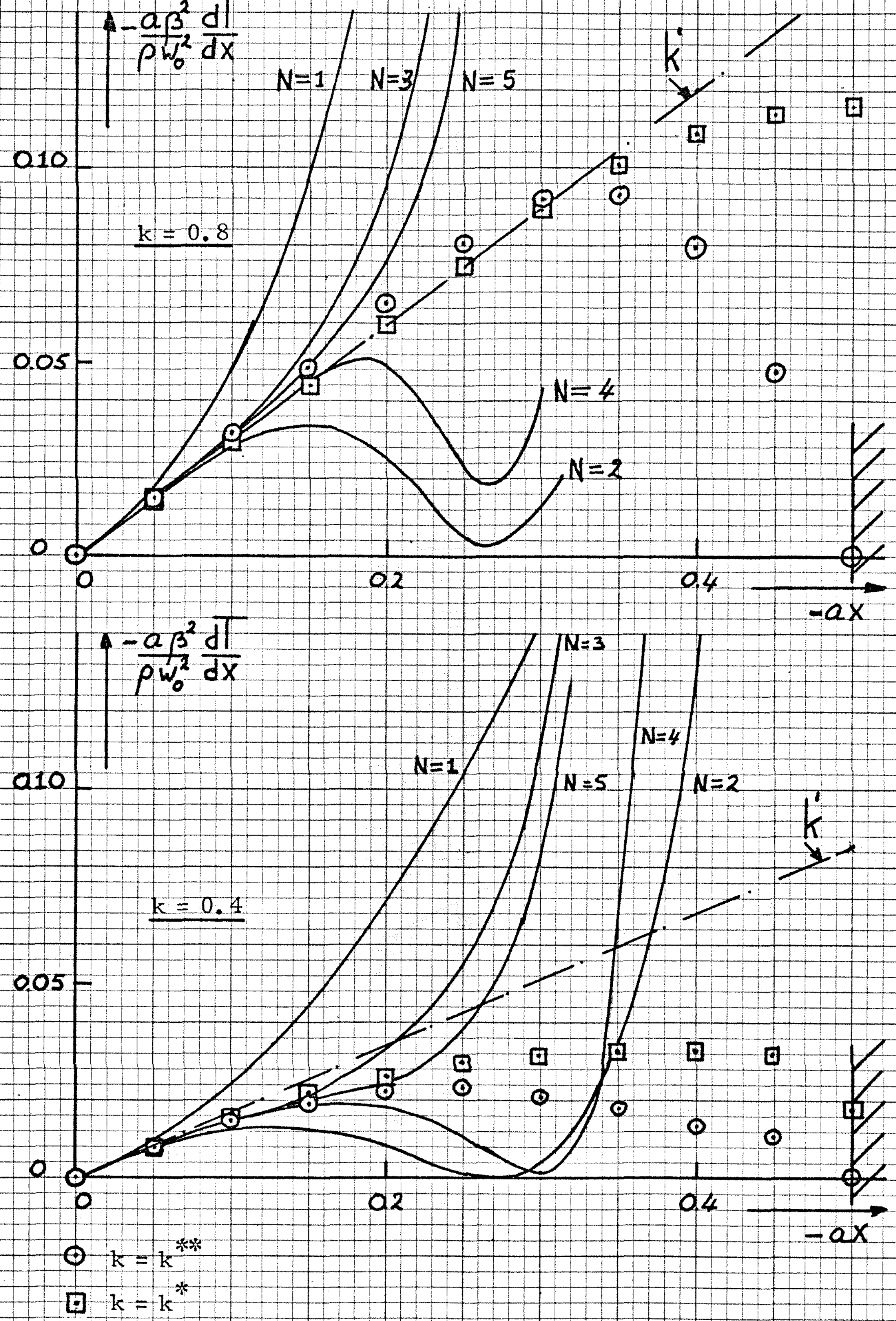


Fig. 8b Leading edge suction force/unit length, gothic wing

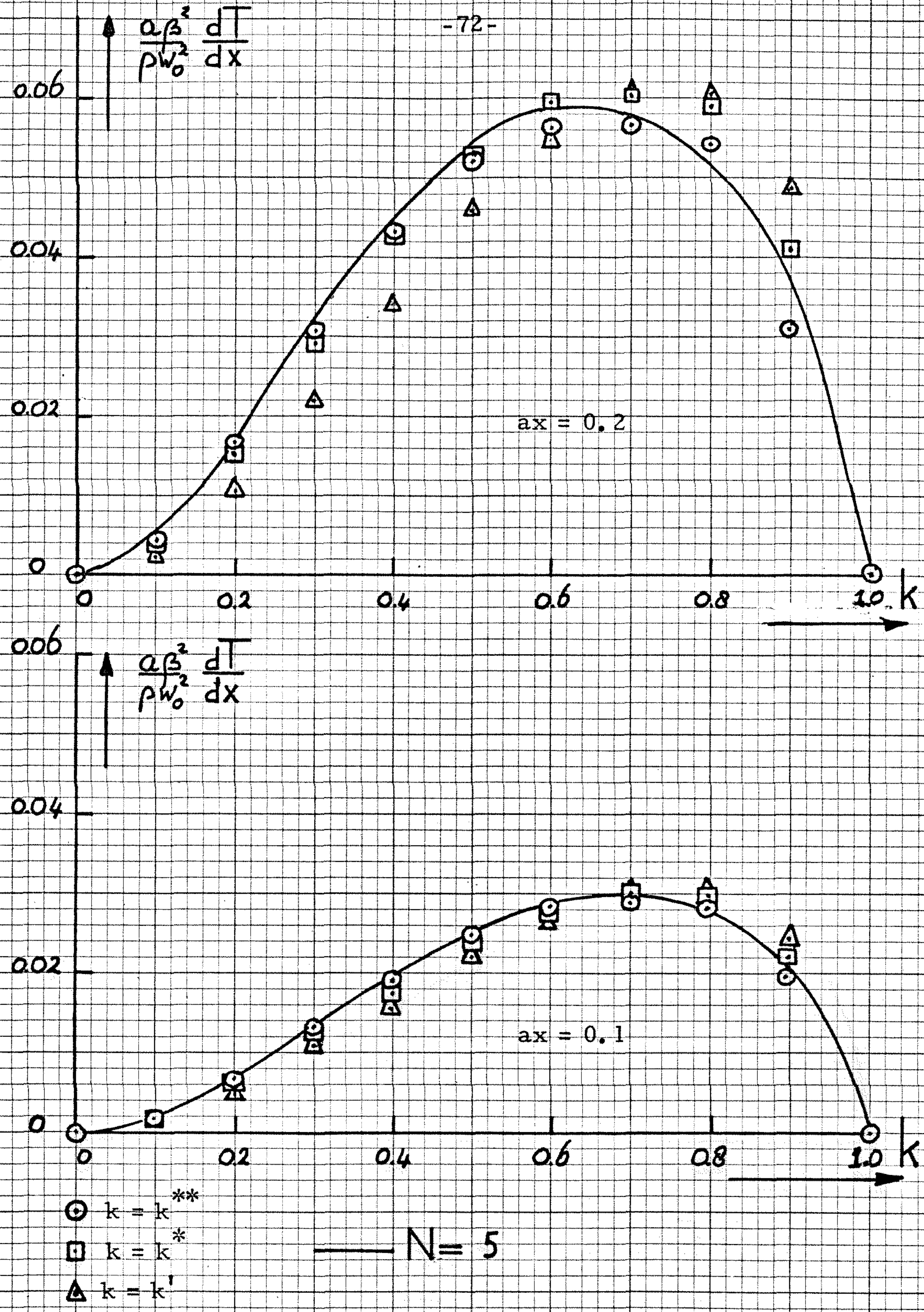


Fig. 8c Comparison of leading edge suction force/unit length, ogee wing

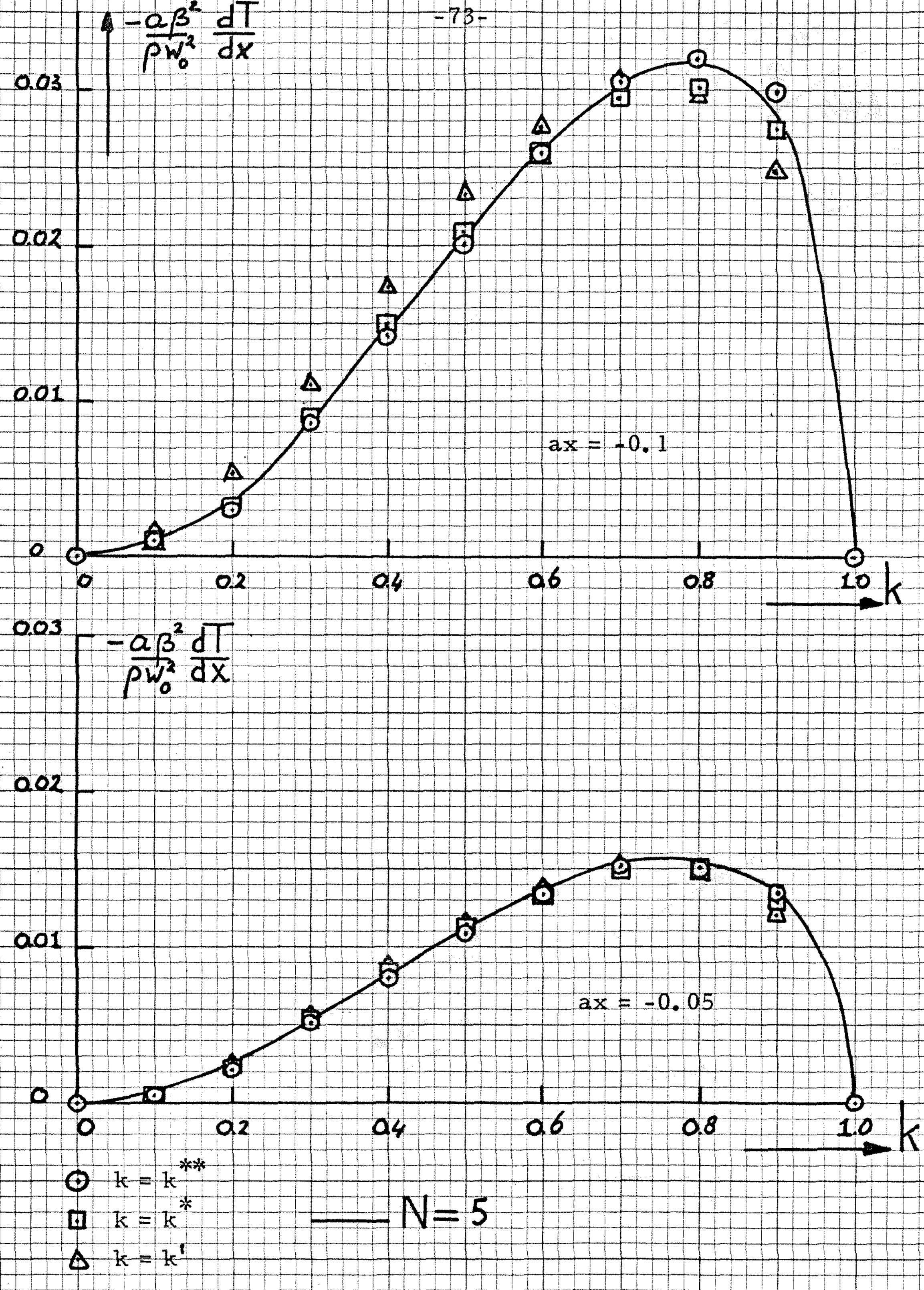


Fig. 8d Comparison of leading edge suction force/unit length, gothic wing

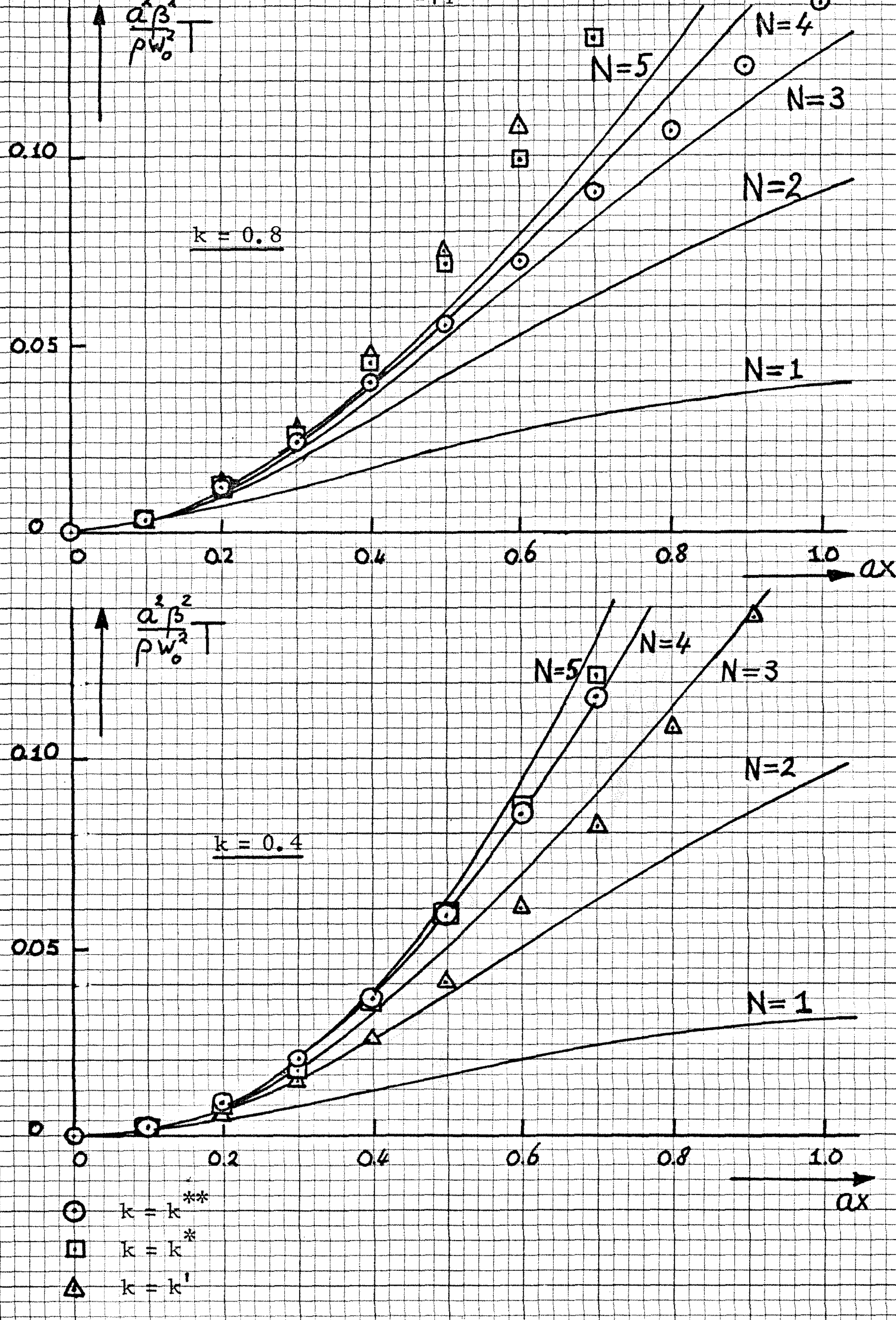


Fig. 8e Leading edge suction force, ogee wing

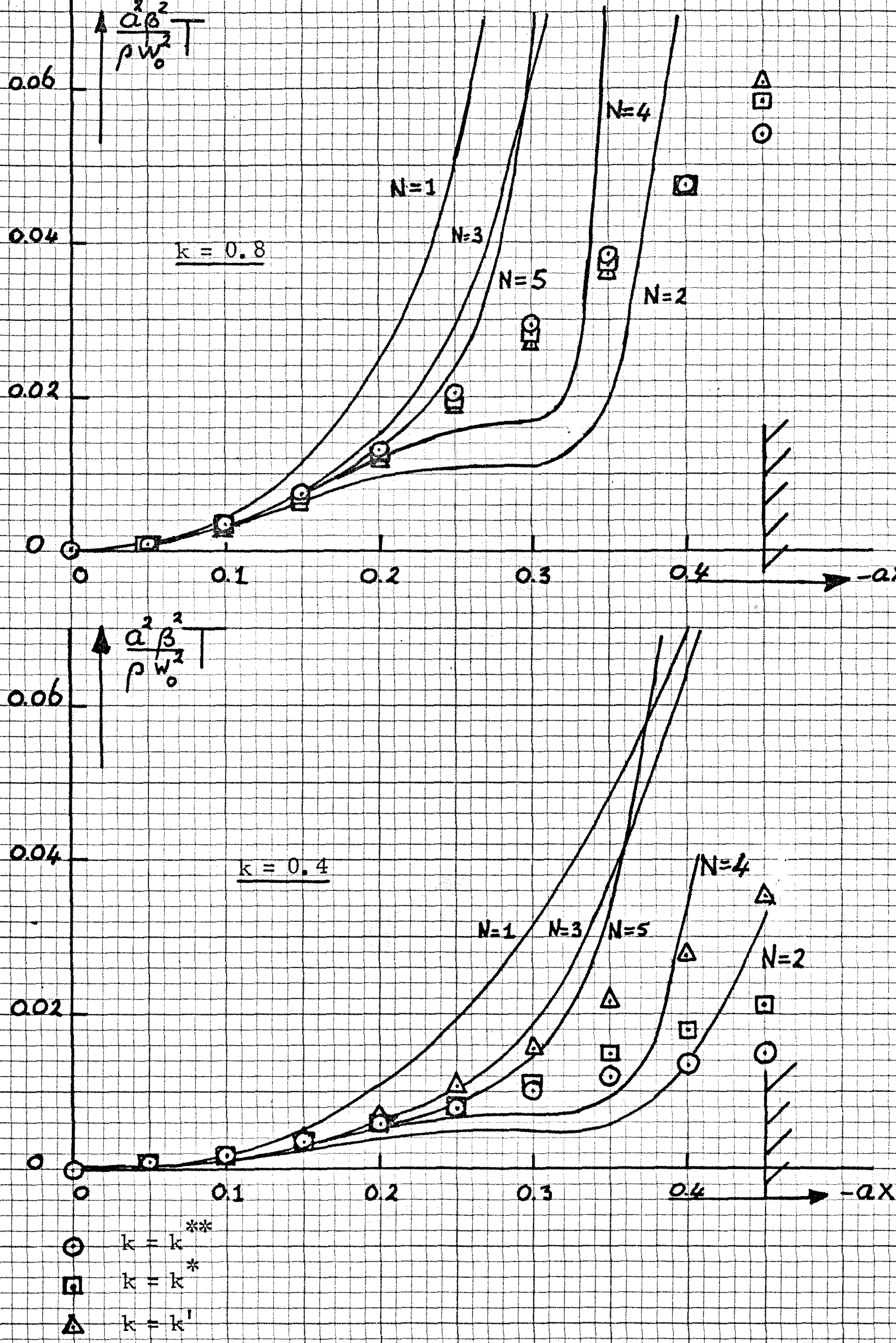


Fig. 8f Leading edge suction force, gothic wing

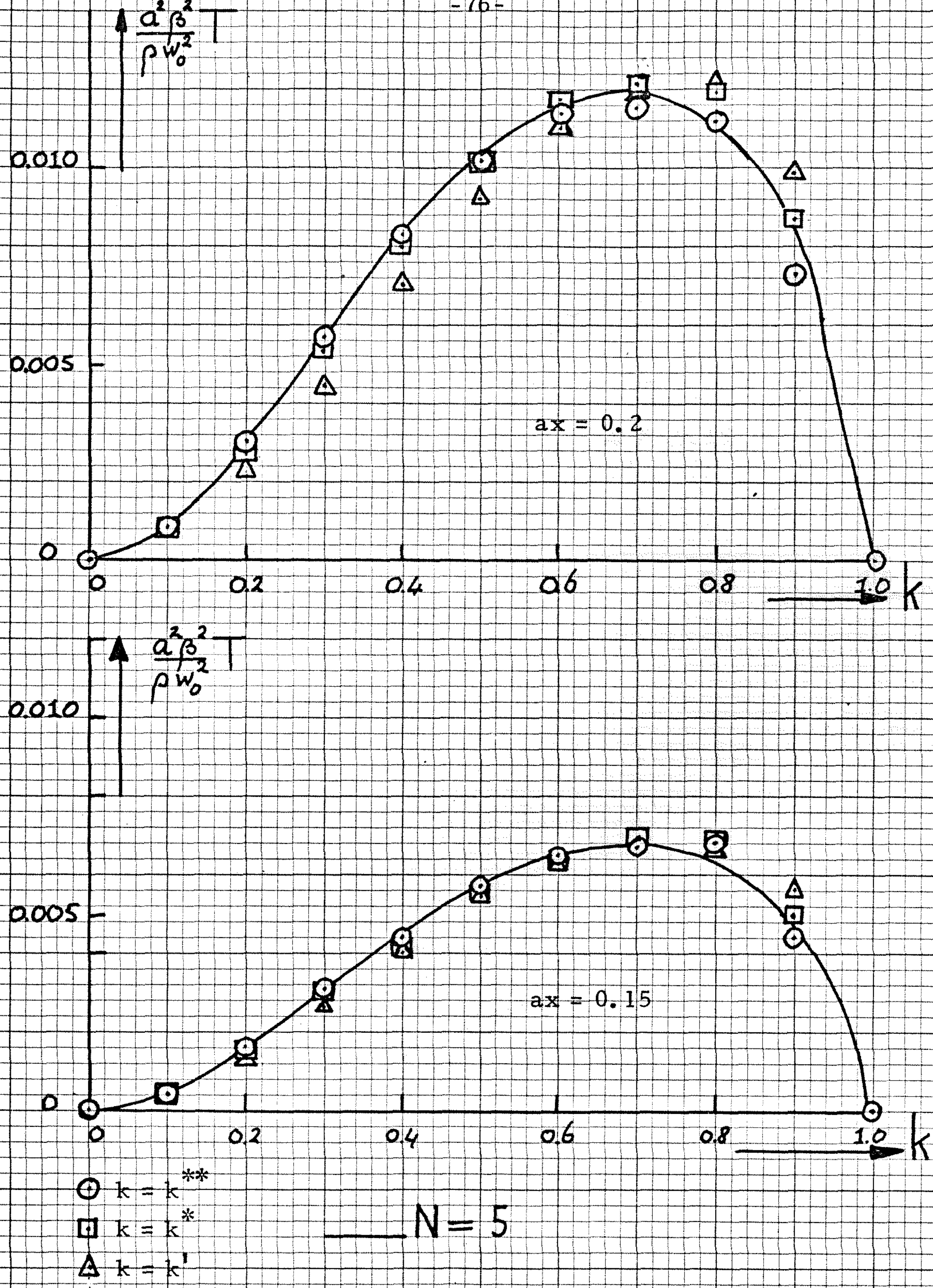


Fig. 8g Comparison of leading edge suction force for ogee wing with other approximations

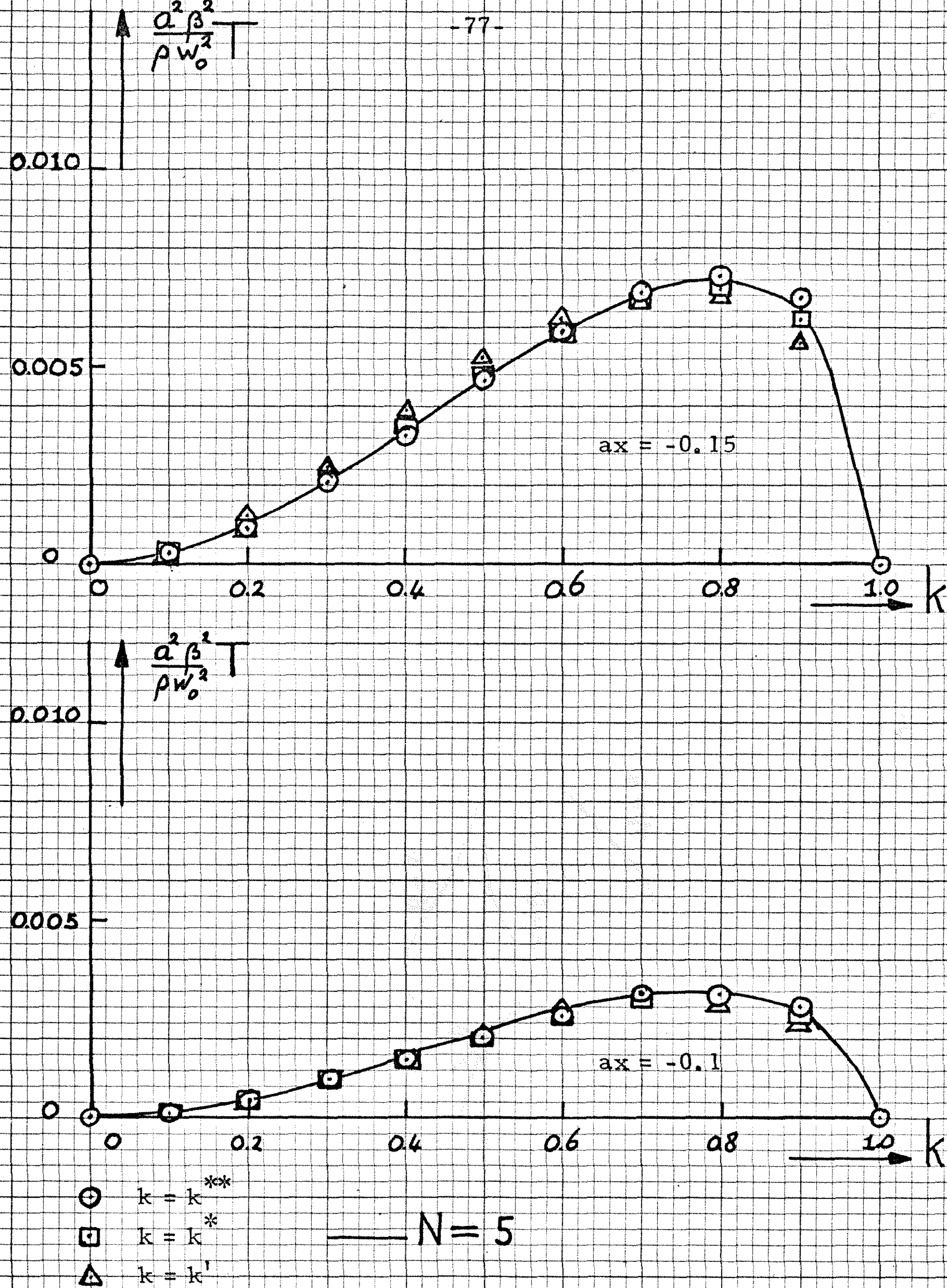


Fig. 8h Comparison of leading edge suction force for gothic wing with other approximations

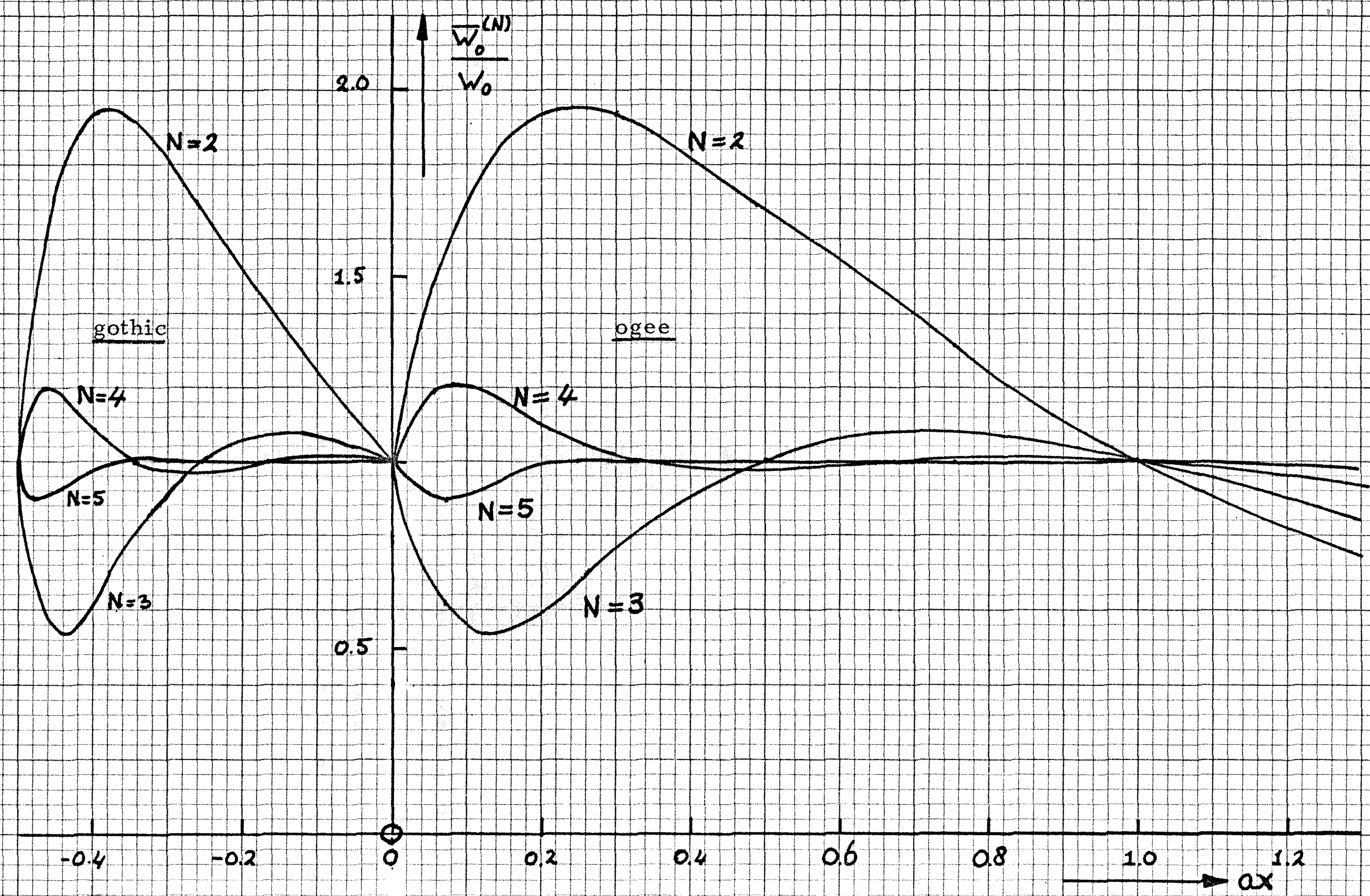
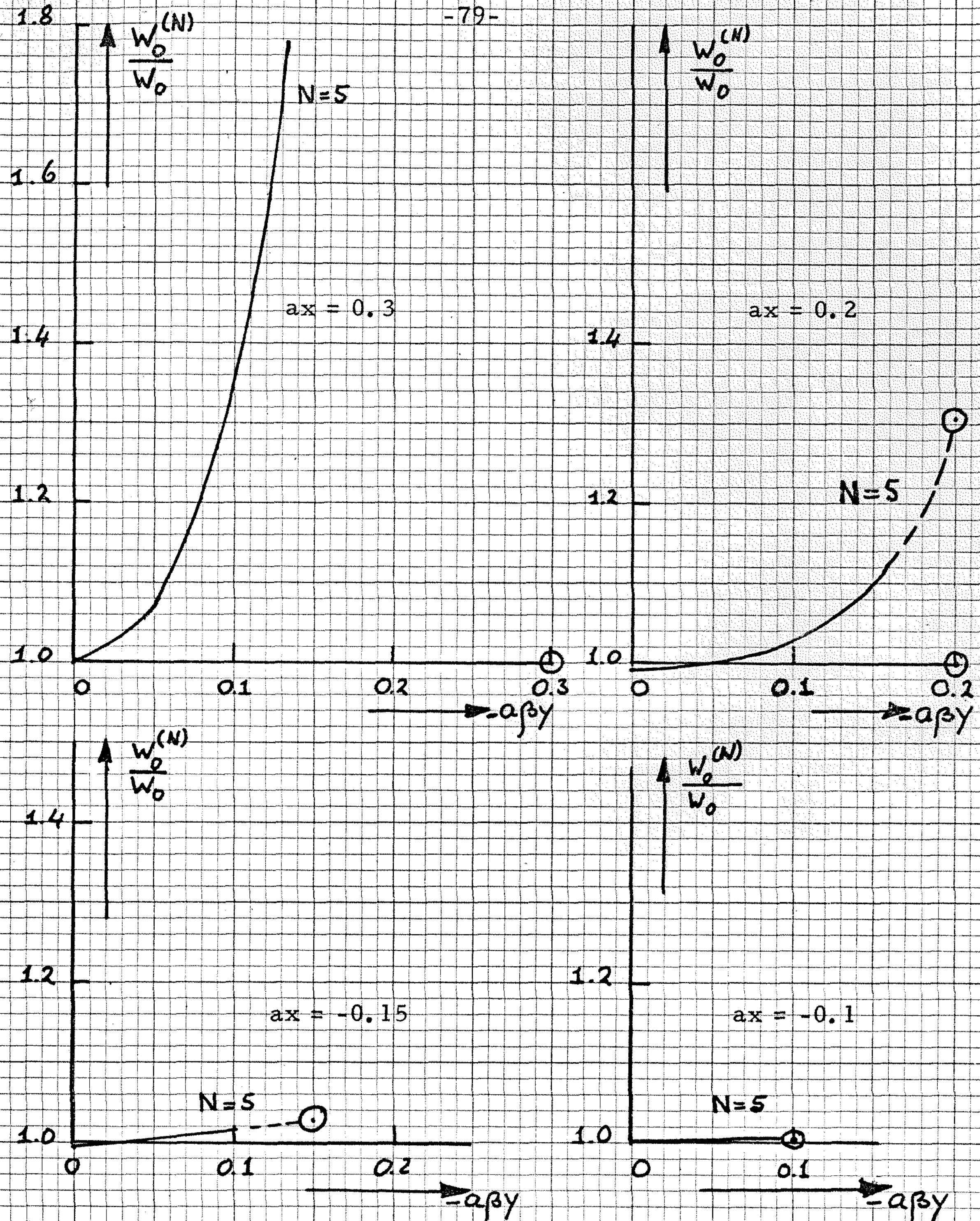


Fig. 9a Successive approximations to flat plate by matching at several points, $a\beta y = 0.0$



⊙ Mach cone

Fig. 9b Successive approximations to flat plate by matching at several points, ax = const.