A STUDY OF THE

PHYSICAL AND CHEMICAL COMPOSITION OF HOMOGENEOUS

AND INHOMOGENEOUS MODELS OF THE SUN

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— ABSTRACT —

In part I a radiative envelope, convective core model of the sun has been constructed on the assumption that the radiative opacity is due entirely to a Russell Mixture of heavy elements. The effect of scattering opacity is taken into account. Following the method of Schwarzschild and using the latest available data on the cross-section of the N^{14} reaction, an estimate of the central temperature and composition of the sun has been made. It is found that the central temperature is approximately 17.5×10^6 °K, and that hydrogen, helium and heavy element abundances are .66, .31 and .03 respectively. A comparison is made with the results obtained from other recent investigations, and with the results of spectroscopic analysis of the solar atmosphere.

In part II the requirements of stability, energy generation and age of the sun are used to construct models of the sun in which the mean molecular weight of the envelope differs from that of the core. The physical and chemical properties of three such models are listed. A comparison with the results of spectroscopic analysis is made, and the conclusion is drawn that the latter are more in accord with the assumption of chemical homogeneity in the sun. A note is appended upon the results of a recent similar investigation by Ledoux.

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CHAPTER I

INTRODUCTION

In this chapter we shall give a brief outline of the method of Schwarzschild, and a discussion of those factors through which the chemical composition enters the equations of stability and energy generation. This will serve to facilitate the more detailed treatment of the problem which follows in Chapters II and III.

§1. Outline of Method[1]. Spectroscopic observations show that the elements hydrogen and helium go to make up the main bulk of the sun, while other elements are present in comparatively small amounts. The proportions are approximately[2] .70, .28 and .02 for hydrogen, helium and combined heavy elements respectively, where the ratio H/He is uncertain by at least a factor of 2. Let us then, describe the composition of the sun by the three parameters X (Hydrogen), Y (Helium) and $\mathcal{U}$ (Heavy elements). Since we have the relation:

$$X + Y + \mathcal{U} = 1, \tag{1 - 1}$$

only two of these parameters are independent. The two additional relations which serve to completely determine the composition are (i) the mass-luminosity relation; (ii) the energy production equation.

(i) Mass-Luminosity Relation. For a star with given mass (M), radius (R) and luminosity (L) the first relation is provided by obtaining that solution of the equations of stellar equilibrium (hydrostatic and radiative) which satisfies the boundary conditions. As will be shown later, the equations of equilibrium when written in terms of suitable dimensionless variables are seen to have a one-parameter family of solutions. The eigen-value parameter, which we shall denote by C , depends upon the mean molecular weight (μ) of the star and the opacity constant (κ') both of which are functions of the hydrogen (X) and heavy element ($\mathcal{U}$) abundances. Hence if the value (C_0) of the parameter C is chosen so as to give a solution of the equations of equilibrium which satisfy the boundary conditions for a particular star - in this case the sun - we have the first relation connecting X and $\mathcal{U}$, viz.

$$C(X, \mathcal{U}) = C_0, \tag{2-1}$$

(ii) The second relation required for the complete determination of the composition is provided by the Energy Production Equation which equates the total flux of luminous energy leaving the star per unit time to the amount of energy generated per unit time within the body of the star:

$$L = \int_0^R \varepsilon \, 4\pi r^2 \rho \, dr \quad \text{ergs/sec}, \tag{2-2}$$

where L is the total luminosity at the surface ($r = R$) of the star.

ρ is the density at the point r.

ε is the rate at which energy is generated per gr. per sec. at the point r.

For a particular star the value of L is known from observation. The quantity ε , besides depending upon the density and temperature, is a function of hydrogen (X) and heavy element ($\mathcal{U}$) abundances. The density and temperature throughout the star are known as functions of r from the integration of the equations of equilibrium, thus enabling us to evaluate the integral (2-2) and obtain the required relation connecting X and $\mathcal{U}$:

$$L = \int_0^R \varepsilon \, 4\pi r^2 \rho \, dr = \mathcal{L}(X, \mathcal{U}). \qquad (3\text{-}1)$$

Solving (2-1), (3-1) and (4-1) we obtain X, Y and $\mathcal{U}$. The mean molecular weight (μ) in its approximate form depends simply upon X, Y and $\mathcal{U}$, and from a knowledge of μ we can calculate the value of the central temperature (T_c).

NOTE: As will be seen in § 3 it is not possible to write down an expression for the opacity—as we must do in order to integrate the equations of equilibrium and so obtain relation (3-1)—without a knowledge of the hydrogen abundance (X) and the composition of the mixture of heavy elements ($\mathcal{U}$). Similarly, equation (4-1) cannot be expressed in terms of X, $\mathcal{U}$ and numerical constants without first assuming an approximate value for the central temperature (T_c). Hence this method of calculating X, Y, $\mathcal{U}$ and T_c is essentially one of repeated approximations. We assume approximate values for X, $\mathcal{U}$ and T_c so as to obtain the relations (3-1) and (4-1). Then solve these for X, $\mathcal{U}$ and T_c and use the new values for a second approximation, continuing in this manner until consistent values are obtained.

The parameters X and $\mathcal{U}$ enter into the equations of equilibrium—and hence into the mass-luminosity relation (3-1)—by way of the mean molecular weight (μ) and the total mean opacity ($\bar{\kappa}$). These two quantities are discussed in the following sections (§§ 2, 3).

§ 2. Mean Molecular Weight (μ). If we neglect the effect of degeneracy, the pressure, density and temperature in the sun are related by the equation of state for a mixture of perfect gases

$$P = \frac{\mathcal{R}}{\mu} \rho T,$$

where P is the sum of the hydrostatic pressure (p) and the radiation pressure (p_R),

μ is the mean molecular weight of the mixture,

$\mathcal{R}$ is the gas constant for one mole of a perfect gas.

If we neglect the contribution of the radiation pressure (p_R) to the total pressure (P), we may write

$$p = \frac{\mathcal{R}}{\mu} \rho T.$$

The mean molecular weight μ for a mixture of hydrogen, helium and heavy elements at high temperatures is given by the relation*

$$\mu = \frac{4}{5X - U + 3}, \tag{4-1}$$

in which complete ionization of the elements is assumed. If this were strictly true there could be no absorptive opacity due to photo-ionization, but the approximation is sufficiently good for present purposes.

§ 3. Total Mean Opacity ($\bar{\kappa}$). The three processes that are of importance in creating opacity of hot gas-mixtures in stars are, the photo-electric absorption (bound-free transitions), collision absorption (free-free transitions), and free-electron scattering. The important quantity in stellar calculations is the Rosseland mean opacity(3). In the low temperature regions the mean scattering opacity ($\bar{\kappa}_s$) is negligible compared with the mean absorptive opacity ($\bar{\kappa}_a$). It has been shown(4) that the mean absorptive opacity is given by the expression

$$\bar{\kappa}_a = 7.23 \times 10^{24} \Gamma \rho (1+X)(U) T^{-3.5} \frac{\bar{g}}{t} = \kappa_0 (1+X)(U) \rho T^{-3.5} \frac{\bar{g}}{t}, \tag{4-2}$$

where Γ is the 'composition factor' of the mixture of heavy elements

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* vd. Appendix I

which compose $\mathcal{U}$, and is defined by $\Gamma = \sum_{z} \frac{c_z Z^2}{M_z}$, in which c_z is the proportion by weight of the element of atomic number Z and atomic Mass M_z in the mixture of heavy elements. $\bar{g}$ is the average "Gaunt factor" and t the "guillotine factor" $\bar{g}/t$ is a function of the hydrogen concentration (X), the heavy element mixture ($\mathcal{U}$), the density (ρ) and temperature (T). No explicit form exists for the functional dependence, but tables of log $(t/\bar{g})$ as a function of log $\rho(1+X)$ and log T (corrected for screening of filled shells and effects of pressure ionization) have been constructed(5) for various mixtures of heavy elements. It is necessary to obtain an empirical expression for the factor $t/\bar{g}$. To do this we must first choose a particular **mixture** of heavy elements, then assume a likely value for X, and if we know the values of ρ and T which occur throughout the sun, we can represent Morse's values of $t/\bar{g}$ along the path of the points $(\rho(1+X), T)$ by an expression of the form

$$\frac{t}{\bar{g}} = \tau_0 \frac{\rho^{\alpha}(1+X)^{\alpha}}{T^{\beta}}, \tag{5-1}$$

where τ_0, α, β are constants chosen to give the best fitting. Substituting for $t/\bar{g}$ from (5-1) in the relation (4-2) we obtain for the mean absorptive opacity

$$\bar{\kappa}_a = \frac{\kappa_0}{\tau_0}(1+X)^{1-\alpha}(\mathcal{U})\rho^{1-\alpha}T^{-3.5+\beta}. \tag{5-2}$$

In high temperature regions the absorptive opacity is negligible compared with the scattering opacity. The mean scattering opacity is given(6) by

$$\bar{\kappa}_s = .19(1+X)N.$$

The factor N is a slowly varying function of temperature. At solar temperatures we have $N = 1$, giving

$$\bar{\kappa}_s = .19(1+X). \tag{6-1}$$

To obtain the total mean opacity ($\bar{\kappa}$), (i.e. the Rosseland mean of the combined absorptive and scattering opacity) in the temperature regions where $\bar{\kappa}_a$ and $\bar{\kappa}_s$ are of the same order of magnitude, we make use of another table of Morse which gives log ($\bar{\kappa}/\bar{\kappa}_a$) as a function of log ($\bar{\kappa}_s/\bar{\kappa}_a$). A curve constructed from this table is shown in fig. (1). As the slope of the curve does not vary rapidly in the region $-2.5 < \log(\bar{\kappa}_s/\bar{\kappa}_a) < -.5$, we may replace it in that region by a straight line passing through the point at which $\log(\bar{\kappa}/\bar{\kappa}_a) = 0$. Denote the slope of this line by m. We then have as an approximate relation in this region

$$\log \frac{\bar{\kappa}}{\bar{\kappa}_a} = m \log \frac{\bar{\kappa}_s}{\bar{\kappa}_a} + \text{const.}$$

$$\text{or} \quad \bar{\kappa} = (\bar{\kappa}_a)^{1-m} (\bar{\kappa}_s)^m \times \text{const.} \tag{6-2}$$

Substituting for $\bar{\kappa}_a$ and $\bar{\kappa}_s$ from (5-2) and (6-1) we obtain as an approximate expression for the total mean opacity

$$\bar{\kappa} = (.19)^m \left(\frac{\kappa_0}{T_0}\right)^{1-m} (1+X)^{(1-\alpha)(1-m)+m} (U)^{1-m} \rho^{(1-\alpha)(1-m)} T^{(-3.5+\beta)(1-m)} \times \text{const.}$$

$$= \kappa'(1+X)^{(1-\alpha)(1-m)+m} (U)^{1-m} \rho^{(1-\alpha)(1-m)} T^{(-3.5+\beta)(1-m)}. \tag{6-3}$$

This reduces to $\bar{\kappa}_a$ for $m = 0$. The values of κ' and m are chosen so as to give the best agreement between the values of $\bar{\kappa}$ as calculated from (6-3) and as obtained from Morse's tables.

In the present work it was assumed that the heavy elements are combined in the proportions of a Russell mixture, and Morse's tables of the guillotine factor for this mixture were used to obtain suitable values of the exponents α and β in (5-1). For this purpose it was assumed that the hydrogen content (X) was .70, the heavy element content ($\mathcal{U}$) .03, and that the values of ρ and T are those given by Schwarzschild's model (in which the effect of scattering is neglected). $\alpha = \beta = \frac{1}{2}$ was found to give good agreement with Morse's table. Values of $\log(\bar{\kappa}_s/\bar{\kappa}_a)$ were then computed from (4-2) and (6-1) along the same path of $\rho(1+X)$ and T, and from an examination of the slope of the curve in fig. (1) for these values of log $(\bar{\kappa}_s/\bar{\kappa}_a)$ an average slope of $m = \cdot 2$ was arrived at. Putting $\alpha = \beta = \frac{1}{2}$, $m = \cdot 2$ in (6-3) we obtain

$$\bar{\kappa} = \kappa'(1+X)^{\cdot 6}(\mathcal{U})^{\cdot 8}\rho^{\cdot 4}T^{-2\cdot 4}. \tag{7-1}$$

Finally the constant κ' was chosen so as to give the best agreement with the values of $\bar{\kappa}$ computed from Morse's tables. $\kappa' = 4\cdot 78 \times 10^{17}$ was found suitable, giving

$$\bar{\kappa} = 4\cdot 78 \times 10^{17}(1+X)^{\cdot 6}(\mathcal{U})^{\cdot 8}\rho^{\cdot 4}T^{-2\cdot 4} \tag{7-2}$$

for the total mean opacity.

The parameters X and $\mathcal{U}$ enter into the energy integral by way of the quantity ε - the rate of energy production. A brief discussion of this quantity is given in the following section.

§ 4. Rate of energy production. It has been shown[7] that the most likely source of energy in main-sequence stars is the Carbon-Nitrogen cycle. If we denote by p the number of cycles taking place per gramme of material per second and by E the energy released in one cycle, then ε - the energy produced per gr. per sec. is clearly given by

$$\varepsilon = E\, p. \tag{8-1}$$

When the carbon and nitrogen nuclei partaking in the reaction have attained an equilibrium distribution, the number of cycles per second (p) is equal to the number per second of any of the individual reactions composing the cycle. In the present work the N^{14} reaction is considered. Hence we have[8]

$$p = p_N = \frac{4}{3^{5/2}} \frac{X X_N}{m_1 m_N} \frac{\Gamma_N}{\hbar} a R^2 e^{(32R/a)^{\frac{1}{2}}} \rho\, \tau^2 e^{-\tau} \text{ cycles/gr./sec.}, \tag{8-2}$$

where X is the concentration by weight of hydrogen,

X_N is the concentration by weight of N^{14},

m_1, m_N are the masses of the H^1 and N^{14} nuclei.

a is the 'Bohr radius' for the system $(= \hbar^2/m e^2 Z_1 Z_N$ where m is the reduced mass of the system).

R is the combined radius of the nuclei $(= 1{\cdot}6 \times 10^{-13}(A_1 + A_N)^{1/3})$.

$\tau = B T^{-\frac{1}{3}}$ where B is defined by $B = 3(\pi^2 m e^4 Z_1^2 Z_N^2)^{1/3}/(2\hbar^2 k)^{1/3}$

Γ_N is the "effective width" of the N^{14} reaction defined by the expression for the cross-section

$$\sigma = \frac{\pi R^2}{2E} \frac{(A_1 + A_N)}{A_N} \Gamma_N \exp\left[(32R/a)^{\frac{1}{2}} - 2\pi e^2 Z_1 Z_N/\hbar v \right], \tag{8-3}$$

where v is the proton velocity, E the proton energy.

(i) We can replace X_N in (8-2) by $h_N U$ where h_N is the fraction of N^{14} present in the mixture of heavy elements in the central regions of the sun. The value of h_N must be obtained from the results of spectroscopic analysis. The latter, however, gives us information about the abundances of elements in the solar atmosphere only. The following data is taken from a quantitative analysis due to Unsöld[9].

Element	Abundance by Weight	h
H	25,100	
He	18,200	
C	75.0	.052
N	190	.130
O	389	.206
Ne	575	.394
Na	1.32	
Mg	38	
Al	2.57	
Si	39.8	
S	12.0	
Ca	3.02	
Fe	132.	

We see that the values of h for carbon and nitrogen in the solar atmosphere are .052 and .130. We would like to know the value of h for Nitrogen at the center of the Sun. It seems reasonable to take this as being equal to the sum of the values for carbon and nitrogen at the surface of the sun. This can be seen in the following way. In the solar atmosphere the ratio of nitrogen to carbon abundance from the above data is 2.5. At the center, however, the ratio of abundance is approximately 50. This figure is based on the most recent measurements on the C^{12} and N^{14} reactions (See (iii) Table (2)). We may explain the difference by supposing that there is imperfect mixing of elements between center and surface, so that while the ratio 2.5/1 of nitrogen to carbon at the surface represents the primitive distribution of these elements, the ratio

50/1 in the hot central regions represents the equilibrium distribution at the high temperatures prevailing there. This means that most of the carbon nuclei in the central region have been transformed into nitrogen nuclei so that we are justified in assuming that the nitrogen abundance in the central regions is equal to the sum of carbon and nitrogen abundance at the surface, i.e. h_N has the approximate value .18.

(ii) The value of the 'effective width' (Γ_N) of the reaction is obtained from the formula (8-3) for the cross-section (σ) using the experimental values of σ at specified bombarding energies (E). Recent measurements of the cross-sections of the C^{12} and N^{14} reactions at energies of the order of 100 K V have been made by Hall and Fowler(10). The results obtained indicate that the behavior of the cross-section at low energies is reasonably well represented by the formula (8-3), but the values of Γ are somewhat greater than those obtained from earlier measurements(11) - 75 and 300 e.v. for C^{12} and N^{14} as against .6 and 60 e.v. In the present work the effective width for N^{14} was taken as 288 e.v. .

(iii) <u>Mean Life of Elements</u>. <u>Relative abundance</u>.

The mean life of nucleus 2 is given by the relation

Mean Life (t_2) = no. of seconds per reaction per nucleus 2

$$= \frac{\text{no. of nuclei 2 per gr. } (X_2/m_2)}{\text{no. of reactions 2 per gr. per sec. } (p_2)}$$

$$= \frac{X_2}{m_2 p_2} \quad \text{seconds.} \tag{10-1}$$

Let us rewrite (8-2)

$$p_2 = \varepsilon_2' X X_2 \Gamma_2 \rho \quad \text{reactions /gr/sec.} \tag{10-2}$$

where $$\varepsilon_2' = \frac{4}{3^{5/2}} \frac{1}{m_1 m_2} \frac{1}{\hbar} a R^2 e^{\left(32\frac{R}{a}\right)^{\frac{1}{2}}} B^2 T^{-\frac{2}{3}} e^{-BT^{-\frac{1}{3}}}$$

and substitute for p_2 in (10-1) the expression (10-2)

$$t_2 = \frac{1}{\varepsilon_2' m_2 \Gamma_2 \rho X} \text{ seconds}$$

$$= \frac{C_2}{\Gamma_2 \rho X} \text{ years.} \qquad (11\text{-}1)$$

A table of values of the constant C_2 for C^{12} C^{13} N^{14} N^{15} at temperatures T = 16, 17, 18, 19, 20 x 10^6 is given below (Table 1)

TABLE (1)

		C_2		6
T	C^{12}	C^{13}	N^{14}	N^{15}
16×10^6	1.61×10^9	1.58×10^9	3.94×10^{11}	3.44×10^{11}
17×10^6	5.71×10^8	5.52×10^8	1.23×10^{11}	1.08×10^{11}
18×10^6	2.18×10^8	2.11×10^8	4.20×10^{10}	3.66×10^{10}
19×10^6	8.94×10^7	8.70×10^7	1.55×10^{10}	1.35×10^{10}
20×10^6	3.88×10^7	3.74×10^7	6.15×10^9	5.32×10^9

Relative Abundance.

When equilibrium has been established between the elements taking part in the cycle we have

$$p_2 = p_3 \qquad \text{for elements 2 and 3.}$$

Hence from (10-1)

$$\frac{t_2 m_2}{X_2} = \frac{t_3 m_3}{X_3}$$

or

$$\frac{X_2}{X_3} = \frac{m_2 t_2}{m_3 t_3} \tag{12-1}$$

$$= \frac{m_2}{m_3} \frac{C_2}{C_3} \frac{\Gamma_3}{\Gamma_2} \text{ by } (11-1). \tag{12-2}$$

If we take the effective widths of the C^{12} and N^{14} reactions to be 75 and 300 e.v. respectively, then we have from (12-2) - using the values of C_2 given in Table (1) - the following values for the ratio of nitrogen to carbon abundance at the center of the sun.

TABLE (2)

$T \times 10^{-6}$	X_n/X_c
16	71.5
17	63
18	56
19	50.5
20	46.2

We see that the ratio of abundances in the center of the sun ($T = 18 - 20 \times 10^6$ degrees) is approximately 50/1.

CHAPTER II

THE MASS-LUMINOSITY RELATION

The Mass-Luminosity relation results as a product of the integration of the equations of radiative and hydrostatic equilibrium. In this chapter we describe these equations and their boundary conditions, and discuss the method of integration and its results. We suppose in accordance with accepted theory that the sun consists of an outer radiative envelope surrounding an inner convective core.

§ 1. Differential equations of equilibrium in radiative envelope.

The equation of hydrostatic equilibrium equates the gradient of total pressure (hydrostatic and radiative) to the gravitational attraction,

$$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho,$$

where P is the total pressure,

G is the constant of gravitation.

If we neglect the contributions of the radiative pressure, we can replace P by the hydrostatic pressure (p) and writing $p = \frac{R}{\mu}\rho T$ we obtain

$$\frac{R}{\mu}\frac{d(\rho T)}{dr} = -G\frac{M(r)}{r^2}\rho.$$

The mass (M) and the density (ρ) are connected by the relation

$$\frac{dM(r)}{dr} = 4\pi r^2\rho.$$

The equation of radiative equilibrium[12] relates the gradient of the radiative pressure to the flow of radiation —

$$\frac{dp_R}{dr} = \frac{d}{dr}\left(\tfrac{1}{3}aT^4\right) = -\frac{\bar{\kappa}\rho}{c}\,\frac{L(r)}{4\pi r^2},$$

where a is Stefan's constant.

$L(r)$ is the flux of radiation across the surface of a sphere of radius r.

$\bar{\kappa}$ is the total mean opacity.

Finally, the luminosity $L(r)$ is given by

$$L(r) = \int_0^r \varepsilon\, 4\pi r^2 \rho\, dr.$$

Due to the exceedingly steep increase in the rate of energy generation (ε) with temperature ($\varepsilon \propto T^{19}$ for the C - N cycle) we may assume that all the significant energy generation takes place in the core, so that the luminosity (L) in the radiative envelope has the constant value

$$L = \int_0^{r_i} \varepsilon\, 4\pi r^2 \rho\, dr, \tag{14-1}$$

where r_i is the radius of the core.

Hence the behavior of the mass, density and temperature in the radiative envelope ($r_i < r < R$) is described by the equations

$$\frac{\mathcal{R}}{\mu}\frac{d(\rho T)}{dr} = -G\frac{M(r)}{r^2}\rho, \tag{14-2}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho, \tag{14-3}$$

$$\frac{d}{dr}\left(\tfrac{1}{3}aT^4\right) = -\frac{\bar{\kappa}\rho}{c}\ \frac{L}{4\pi r^2}. \tag{15-1}$$

Introducing the dimensionless variables t, u, σ, τ which are defined[13] as follows:

$$r = \frac{R}{t}, \tag{15-2}$$

$$M(r) = uM, \qquad M = \text{total mass} \tag{15-3}$$

$$\rho = \sigma\frac{M}{4\pi R^3} = \sigma\rho_0, \qquad \left(\rho_0 = .470\frac{M}{R^3}.\ M, R \text{ in solar units}\right) \tag{15-4}$$

$$T = \tau\ \frac{\mu GM}{\mathcal{R}R} = \tau T_0, \qquad \left(T_0 = 23.11 \times 10^6\mu\frac{M}{R}.\ M, R \text{ in solar units}\right) \tag{15-5}$$

and replacing $\bar{\kappa}$ by the expression (6-3) developed in the previous chapter, the equations of equilibrium (14-2), (14-3), (15-1) become

$$\begin{aligned} \sigma' &= (u - \tau')\frac{\sigma}{\tau}, \\ u' &= -\frac{\sigma}{t^4}, \\ \tau' &= C\ \frac{\sigma^{(1-\alpha)(1-m)+1}}{\tau^{(3.5-\beta)(1-m)+3}}, \end{aligned} \tag{15-6}$$

where primes denote differentiation with respect to t and the eigenvalue parameter C is given by

$$C = \frac{3\kappa' L}{16\pi a c R}\ \frac{\rho_0^{(1-\alpha)(1-m)+1}}{T_0^{(3.5-\beta)(1-m)+4}}(1+X)^{(1-\alpha)(1-m)+m}(U)^{1-m}L\frac{R^{(1-m)(3\alpha-\beta+.5)}}{M^{(\alpha-\beta+2.5)(1-m)+3}} \tag{15-7}$$

Setting $\alpha = \beta = \frac{1}{2}$, $m = .2$, $\kappa' = 4.78 \times 10^{17}$ (Chap. I, § 3), substituting for ρ_0 and T_0 from (15-4) and (15-5), and writing $\mu = 4/(5X - U + 3)$

we have for the equations of equilibrium

$$\sigma' = (u - \tau')\frac{\sigma}{\tau}, \tag{16-1}$$

$$u' = -\frac{\sigma}{\tau^4}, \tag{16-2}$$

$$\tau' = C\,\frac{\sigma^{1.4}}{\tau^{5.4}}, \tag{16-3}$$

and for C,

$$C = [\bar{9}.3951](1+X)^{.6}(U)^{.8}(5X-U+3)^{6.4}\frac{LR^{1.2}}{M^5} \quad (L, R, M \text{ in solar units}). \tag{16-4}$$

§ 2. Differential equations of equilibrium in convective core.

It has been shown(14) that the transport of energy through a gaseous star by radiation results in a stable condition as long as the density and temperature satisfy the relation

$$\frac{d\log\rho}{d\log T} > \frac{1}{\gamma - 1},$$

where γ is the ratio of specific heats (here assumed to have the same value, 5/3, as for the monatomic gases). When a point in the star is reached at which

$$\frac{d\log\rho}{d\log T} = \frac{1}{\gamma - 1}, \tag{16-5}$$

the radiative equilibrium becomes unstable, equation (15-1) ceases to apply, and the transport of energy is by means of convection currents.

In the convective region we have an adiabatic temperature gradient set up with the relation

$$p \propto \rho^{\gamma} \propto T^{\gamma/(\gamma-1)}, \quad \text{where} \quad \gamma = 5/3,$$

and the equations of equilibrium are

$$\frac{dp}{dr} = -G\frac{M(r)}{r^2}\rho,$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho,$$

$$p = b\rho^{5/3},$$

where b is a constant of proportionality.

Combining the first two equations we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho.$$

Substituting for p in terms of ρ and introducing the Lane-Emden variables, ξ and θ defined by

$\rho = \rho_c\,\theta^{3/2}$, where ρ_c is the central density, (17-1)

$T = T_c\,\theta$, where T_c is the central temperature, (17-2)

$r = a\,\xi$, where $a = \left(\frac{5b}{8\pi G}\,\frac{1}{\rho_c^{1/3}}\right)^{\frac{1}{2}}$,

the equations of equilibrium take the form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^{3/2}. \qquad (17\text{-}3)$$

In terms of these variables we also have the following useful relations

$$M(r) = -4\pi a^3 \rho_c \xi^2 \theta', \tag{18-1}$$

$$r = \left(\frac{5}{8\pi G}\,\frac{R}{\mu}\,\frac{T_c}{\rho_c}\right)^{\frac{1}{2}} \xi, \tag{18-2}$$

$$\frac{\rho r^3}{M(r)} = -\left(\frac{\xi\,\theta^{3/2}}{\theta'}\right) = \mathcal{U}(\xi), \tag{18-3}$$

$$\frac{M(r)}{T r} = -\frac{5}{2}\left(\frac{\xi\,\theta'}{\theta}\right) = V(\xi). \tag{18-4}$$

Tables of θ, $\mathcal{U}$ and V as functions of ξ ($\theta = 1$ and $\theta' = 0$ at $\xi = 0$) exist(15).

§ 3. Boundary Conditions. At the outer boundary of the sun ($t = 1$) we must have

$$\sigma = \tau = 0\,, \qquad u = 1. \tag{18-5}$$

At the interface of the radiative envelope and the convective core - defined as that value of t for which $d(\log\sigma)/d(\log\tau) = 3/2$ - we must require continuity of mass density and temperature. Hence, denoting values at the interface by the subscript i, we have

$$\frac{R}{t_i} = a\,\xi_i\,, \qquad (\alpha) \tag{18-6}$$

$$u_i M = -4\pi a^3 \rho_c (\xi^2\theta')_i\,, \qquad (\beta)$$

$$\sigma_i \frac{M}{4\pi R^3} = \rho_c\, \theta_i^{3/2}, \qquad (\gamma)$$

$$\tau_i\, \mu \frac{GM}{\mathcal{R} R} = T_c\, \theta_i, \qquad (\delta)$$

or $$\frac{\sigma_i}{u_i t_i^3} = -\left(\frac{\xi\,\theta^{3/2}}{\theta'}\right)_i = \mathcal{U}(\xi_i), \qquad \left(\frac{\gamma}{\beta\alpha^3}\right) \qquad (19\text{-}1)$$

$$\frac{u_i t_i}{\tau_i} = -\frac{5}{2}\left(\frac{\xi\,\theta'}{\theta}\right)_i = \mathcal{V}(\xi_i). \qquad \left(\frac{\alpha\beta}{\delta}\right) \qquad (19\text{-}2)$$

The equations (16-1), (16-2), (16-3), (17-3) and the boundary conditions (18-5), (18-6), (19-1), (19-2) completely determine the physical state of the star. We note that the boundary condition (18-6) α, γ, δ, may be written in the form

$$a = \frac{R}{t_i\, \xi_i}, \qquad (19\text{-}3)$$

$$\rho_c = \frac{\sigma_i}{\theta_i^{3/2}} \frac{M}{4\pi R^3} = \frac{\sigma_i}{\theta_i^{3/2}}\, \rho_0, \qquad (19\text{-}4)$$

$$T_c = \frac{\tau_i}{\theta_i}\, \mu \frac{GM}{\mathcal{R} R} = \frac{\tau_i}{\theta_i}\, T_0, \qquad (19\text{-}5)$$

which enables us to write down the central temperature and density once we know the values of σ_i, τ_i, θ_i and μ.

We note that the solutions of (16-1), (16-2), (16-3) form a one parameter family. In general, the boundary conditions will not be satisfied for arbitrary choice of the parameter C. The requirement that the boundary conditions be satisfied serves to determine a characteristic value of C which we denote by C_0 and the relation

$$C_0 = C = [\bar{9}.3951](1+X)^{.6}(U)^{.8}(5X\cdot U+3)^{6.4}\frac{LR^{1.2}}{M^5} \qquad (20\text{-}1)$$

is the so-called Mass-Luminosity Relation.

§ 4. Integration of the equations of equilibrium.

The standard procedure for integrating the equations of equilibrium is, having given C some arbitrary value, to start from the outer boundary ($t=1$) with an approximate analytical solution of equations (16-1), (16-2), (16-3), satisfying the conditions

$$\sigma = \tau = 0 \;, \; u = 1 \quad \text{at} \quad t = 1.$$

The integration is continued by numerical methods up to the point (t_i) where $d\log\sigma / d\log\tau = 3/2$. $\mathcal{U}$ and $\mathcal{V}$ are then computed from (19-1) and (19-2), and from tables of $\mathcal{U}$ and $\mathcal{V}$ the corresponding values ($\xi(\mathcal{U})$ and $\xi(\mathcal{V})$ of ξ are obtained. If $\xi(\mathcal{U}) = \xi(\mathcal{V})$, then, the interfacial boundary conditions are satisfied and the value assigned to C at the beginning is the required eigen-value. Otherwise, the integration must be repeated using different values for C until $\xi(\mathcal{U}) = \xi(\mathcal{V})$ is obtained. To obtain an analytic solution the usual procedure of letting $u=1$ as a first approximation was employed. Equations (16-1), (16-2), (16-3) then reduce to

$$\sigma' = (1-\tau)\frac{\sigma}{\tau} \,,$$

$$\tau' = C\,\frac{\sigma^{1.4}}{\tau^{5.4}}.$$

Assume a relation of the form $\sigma = A\tau^n$ where A and n are constants. Substituting into the above equations we find

$$A = \frac{1}{C(1+n)^{1/1.4}},$$

$$n = \frac{5.4}{1.4},$$

and $$\tau' = \frac{1}{1+n}.$$

Hence $\tau = (t-1)/(1+n)$ and $\sigma = A(t-1)^n/(1+n)^n$ satisfy the approximate equations and the boundary conditions $\sigma = \tau = 0$ at $t = 1$. Substituting for σ in (16-2) we get

$$u' = A\frac{(t-1)^n}{(1+n)^n}\frac{1}{t^4}.$$

Integrating with respect to t from $t = 1$ to $t = t$, we obtain

$$u = 1 - \frac{A}{(n+1)^n}\int_1^t \frac{(t-1)^n}{t^4}\,dt.$$

We also have $$\tau = \frac{(t-1)}{(1+n)} \qquad (21\text{-}1)$$

and $$\sigma = A\frac{(t-1)^n}{(1+n)^n}.$$

From (21-1) starting values for u, σ, τ were calculated for $t = 1.05, 1.10, 1.15$. C was assumed to have the value $C = 2.18 \times 10^{-5}$ Five significant **figures** in the independent variables were carried throughout. The integration was continued numerically using the method of Milne. The variables u, σ, τ for the nth step of **the** integration were predicted from the formula

$$y_n = \frac{4}{3}h\left(2y'_{n-3} - y'_{n-2} + 2y'_{n-1}\right) + y_{n-4},$$

and corrected by Simpson's formula

$$y_n = \frac{h}{3}\left(y'_{n-2} + 4\,y'_{n-1} + y'_n\right) + y_{n-2}.$$

The interval h was increased progressively from .05 to .10, to .20- the point t_i being reached in sixty steps approximately. After four integrations had been completed a fitting of the boundary conditions was obtained by interpolation for log C = $\bar{5}$.38125. The values of the variables at the surface are

$t_i = 8.270,$ $u_i = .135,$ $\sigma_i = 197,$ $\tau_i = .944,$

$\xi_i = 1.180,$ $\theta_i = .790.$

Substituting log C = $\bar{5}$. 38125 in (20-1) we obtain as the result of the integration of the equation of equilibrium the mass-luminosity relation for the sun (M = R = L = 1)

$$(1+X)^{.6}(U)^{.8}(5X-U+3)^{6.4} = [3.9862]. \qquad (22\text{-}1)$$

CHAPTER III

THE ENERGY PRODUCTION EQUATION

§ 1. Introduction. If we assume that all significant energy generation takes place in the convective core ($0 \leq r \leq r_i$) we have (14-1)

$$L = \int_0^{r_i} \varepsilon\, 4\pi r^2 \rho\, dr. \qquad (22\text{-}2)$$

The value of the luminosity (L) is known from observation, $r_i (= R/t_i)$ is known from the results of the integration of the equations of equilibrium. The dependence of ε on density and temperature is known (8-2), and the run of temperature and density throughout the core is known in terms of the Emden variable θ (17-1) and (17-2). We can therefore evaluate the integral on the right-hand side and obtain the energy production equation. Our object in this chapter is to discuss some problems which arise in connection with this equation.

From (8-2) we have

$$\varepsilon = \varepsilon_0 \rho T^{-2/3} e^{-BT^{-1/3}}, \tag{23-1}$$

where

$$\varepsilon_0 = E \frac{4}{3^{5/2}} \frac{X U}{m_1 m_N} \frac{h_N \Gamma_N}{\hbar} a R^2 e^{\left(3 \frac{z R}{a}\right)^{\frac{1}{2}}} B^2. \tag{23-2}$$

In order that the equation (22-2) may yield a relation of X and U which is readily solvable it is necessary to replace (23-1) by an approximate formula of the form

$$\varepsilon = \varepsilon_0' \rho T^{\eta}, \tag{23-3}$$

where ε_0' and η are suitably chosen constants.
Substituting (23-3) in (22-2) and expressing r, ρ and T in terms of ξ and θ from (17-1) and (17-2), we have

$$\begin{aligned} L &= 4\pi \varepsilon_0' \rho_c^2 T_c^{\eta} a^3 \int_0^{\xi_i} \theta^{\eta+3} \xi^2 d\xi \\ &= 4\pi \varepsilon_0' \rho_0^2 T_0^{\eta} R^3 \left(\frac{\sigma_i}{\theta_i^{3/2}}\right)^2 \left(\frac{\tau_i}{\theta_i}\right)^{\eta} \left(\frac{1}{t_i \xi_i}\right)^3 \int_0^{\xi_i} \theta^{\eta+3} \xi^2 d\xi. \end{aligned} \tag{23-4}$$

The unknown quantities X and U are contained in the factors ε_0' and T_0. All other quantities are known, so that (23-4) yields a relation between X, U and known constants - the energy production equation.

§ 2. <u>The constants ε_0' and η.</u> The constants ε_0' and η are to be chosen in such a way that the two expressions $\varepsilon_0 \rho T^{-2/3} e^{-BT^{-1/3}}$ and $\varepsilon_0' \rho T^{\eta}$ are equal at some temperature (T), let us say, the central temperature (T_c). We must also require that the integral $L(r_i) = \int_0^{r_i} 4\pi \varepsilon_0 \rho^2 T^{-\frac{2}{3}} e^{-BT^{-\frac{1}{3}}} r^2 dr$ and $L'(r_i) = \int_0^{r_i} 4\pi \varepsilon_0' \rho^2 T^{\eta} r^2 dr$

be equal for the value of r_i obtained from the integration of the equations of equilibrium. In short, ε_0' and η must satisfy the relations

$$\varepsilon_0 T_c^{-\frac{2}{3}} e^{-BT_c^{-\frac{1}{3}}} = \varepsilon_0' T_c^{\eta},$$

or $$\varepsilon_0' = \frac{\varepsilon_0}{T_c^{\eta+\frac{2}{3}} e^{BT_c^{-\frac{1}{3}}}}, \qquad (24\text{-}1)$$

and $$L(r_i) = L'(r_i).$$

In terms of the Emden variables, ξ and θ, the last relation becomes

$$\frac{L(\xi_i)}{L'(\xi_i)} = 1 = \frac{\varepsilon_0 T_c^{-\frac{2}{3}} \int_0^{\xi_i} \theta^{\frac{2}{3}} e^{-B(T_c\theta)^{-\frac{1}{3}}} \xi^2 d\xi}{\varepsilon_0' T_c^{\eta} \int_0^{\xi_i} \theta^{\eta+3} \xi^2 d\xi} \qquad (24\text{-}2)$$

$$= \frac{e^{BT_c^{-\frac{1}{3}}} \int_0^{\xi_i} \theta^{\frac{2}{3}} e^{-B(T_c\theta)^{-\frac{1}{3}}} \xi^2 d\xi}{\int_0^{\xi_i} \theta^{\eta+3} \xi^2 d\xi}$$

by (24-1).

Hence for a given value of ξ_i and T_c, the constant η must be chosen so as to satisfy equation (24-2) and the corresponding value of ε_o' is then given by (24-1) in which ε_o is replaced by its expression in (23-2). Table (3) gives the values of the ratio $L(\xi_i) / L'(\xi_i)$ for ξ_i = .8, 1.0, 1.2, 1.4 and for T_c = 16, 17, 18, 19, 20 million degrees. From this table, for a given value of ξ_i, we can make a plot of those values of η which make $L/L' = 1$ for various values of T_c.

TABLE (3)

$\eta \backslash \xi$	.8	1.0	1.2	1.4		
18	.934	.910	.896	.891		
18.5	.956	.937	.928	.924		
19.0	.972	.960	.955	.952	T_c	16.10^6
19.5	1.002	.995	.990	.988		
20.0	1.017	1.017	1.016	1.014		
18	.951	.932	.919	.915		
18.5	.974	.960	.952	.948		
19.0	.990	.983	.980	.978	T_c	$17\text{x}10^6$
19.5	1.021	1.019	1.016	1.015		
20.0	1.036	1.036	1.042	1.042		
18	.968	.952	.942	.944		
18.5	.991	.981	.975	.968		
19.0	1.007	1.004	1.004	1.008	T_c	$18\text{x}10^6$
19.5	1.039	1.041	1.041	1.047		
20.0	1.054	1.064	1.069	1.075		
18	.984	.972	.963	.959		
18.5	1.008	1.002	.997	.994		
19.0	1.024	1.025	1.026	1.025	T_c	$19\text{x}10^6$
19.5	1.056	1.063	1.064	1.064		
20.0	1.072	1.086	1.091	1.092		
18	.998	.990	.984	.981		
18.5	1.022	1.021	1.018	1.017		
19.0	1.039	1.044	1.048	1.048	T_c	$20\text{x}10^6$
19.5	1.071	1.083	1.087	1.088		
20	1.087	1.106	1.115	1.117		

The ratio $L(\xi_i)/L'(\xi_i)$ is plotted as a function of η for $\xi_i = 1\cdot 2$ and $T_c = 16, 17, 18, 19$ and 20×10^6 ^oK in fig. 2. From the curves we can read off the values of η for which the ratio is unity for each value of T_c. A plot of these values of η against corresponding values of T_c is given in fig. 2.

It is clear from (24-1) that in order to evaluate ε_o' in terms of X, U and numerical constants, we must assign an approximate value for T_c together with the corresponding value of η as obtained from the curves in fig. 2. Inserting these values of ε_o' and η in (23-4) we can solve (22-1) and (23-4) for X and U, obtain T_c from (19-5), and compare this value with the assumed approximation for T_c. The effect of errors in the choice of T_c and η upon ε_o' is seen from Table 4 in which the quantity $\left(T_c^{\eta+\frac{2}{3}} e^{BT_c^{-\frac{1}{3}}}\right)^{-1}$ is plotted for $\eta = 17, 18, 19$ and $T_c = 17, 18, 19$ (T_c is expressed in units of 10^6 degrees).

TABLE 4

$T_c \diagdown \eta$	17	18	19
17	1.32×10^{-23}	2.14×10^{-23}	3.46×10^{-23}
18	1.48×10^{-23}	2.25×10^{-23}	3.44×10^{-23}
19	1.59×10^{-23}	2.30×10^{-23}	3.33×10^{-23}

From the table we see that a change of T_c from 17 to 19 alters the value of ε_o' by a factor of $1\cdot 2$. More important is the correct initial choice of η in evaluating ε_o' since a change in η from 17 to 19 introduces an error by a factor of $2\cdot 5$ in ε_o'. Hence the desirability of using the (η , T_c) curve (Fig. 2) to obtain η . On the other hand the choice of η does not greatly influence the value of the integral in (23-4), as can be seen from Table 5 where the integral is plotted for several values of η and ξ_i .

TABLE 5

$\eta \backslash \xi_i$	.8	1.0	1.2	1.4
18.0	52.640	62.066	65.620	66.519
18.5	51.404	60.226	63.386	64.169
19.0	50.587	58.846	61.578	62.261
19.5	49.047	56.776	59.372	59.968
20.0	48.316	55.551	57.904	58.423

Table 5 also shows that for a given value of η the integral rapidly approaches a limit with increasing ξ_i . This justifies our earlier assumption that we may ignore the energy production outside the convective core.

In the present work a central temperature of 17.5×10^6 °K (η=19) was assumed in evaluating ε_0' as given by (24-1). In evaluating ε_0 , Γ_N = 288 e.v. for N^{14} and h_N = .18 were used.

Substituting for ρ_0 and T_0 in (23-4) from (15-4) and (15-5) and putting in the numerical value for $\sigma_i, \tau_i, \theta_i, t_i, \xi_i$, the energy production equation in X and U was obtained in the form

$$\left(\frac{5X-U+3}{XU}\right)^{19} = [16.9070] \frac{M^{21}}{R^{22}L} \quad M, R, L \text{ in solar units}$$
$$= [16.9070] \quad \text{for } M = R = L = 1 . \qquad (27\text{-}1)$$

CHAPTER IV

THE PHYSICAL AND CHEMICAL PROPERTIES OF THE MODEL

§ 1. Survey of earlier work. Before discussing the properties of the solar model constructed in the present work, it will be convenient to give here a brief summary of the results of some previous investigations made by Schwarzschild(16), Keller(17) and Harrison(18). All the models to be described have certain features in common.

(a) The sun is assumed to consist of a convective core (in which $\gamma = 5/3$) surrounded by a radiative envelope. The chemical composition is taken to be uniform throughout, and the approximate expression for the mean molecular weight ($\mu = 4 / (5X - U - 3)$ is used.

(b) The effects of radiation pressure and of scattering opacity are neglected.

(c) The source of energy is the carbon-nitrogen cycle, and all energy generation is assumed to take place within the core.

They differ, however, in the following respects:

(i) The choice of the mixture of heavy elements — either Russell Mixture or Russell Mixture with the addition of oxygen is used.

(ii) The treatment of the "guillotine factor" in the absorptive opacity. In some of the models Strömgren's formulae have been used to evaluate the guillotine factor — in others, the table of Morse which add to Strömgren's values a correction for sreening of filled electron shells and effects of pressure ionization.

(iii) The choice of the empirical constants Γ, η and h in the energy-production equation (23-4).

We shall summarize the distinguishing features of these models under the above headings (i), (ii) and (iii).

Schwarzschild:

Properties:

$$X = 47\%$$
$$Y = 41\%$$
$$U = 12\%$$
$$\mu = .76$$
$$T_c = 19.8 \times 10^6 \ {}^{\circ}K.$$

(i) Russel Mixture of heavy elements.

(ii) Morse's tables of the guillotine factor represented by the empiracle formula log $t/g = .4 \log \rho^{\,.25}$, leading to an expression for the opacity $\bar{\kappa}_a = \kappa_0 \rho^{.75} T^{-3.5}$

(iii) C^{12} reaction with $\Gamma_c = .6$ e.v., $\eta = 17$, $h_c = .02$

Keller:

A model consisting of hydrogen, helium and pure oxygen was constructed, and by an interpolation (based upon a study of aspidal motions) between this oxygen model and the R. M. model of Schwarzschild, an estimate was made of the properties of an intermediate model.

Oxygen Model:

Properties:

$X = 97\%$

$Y = 0$

$U = 3\%$

$\mu = .51$

$T_c = 18.0 \times 10^6 \ {}^oK.$

(i) 100% Oxygen in heavy element group

(ii) Guillotine factor $(t/\bar{g})$ calculated from Strömgren's formulae, and represented by Empirical formula

$$t/\bar{g} = 1.7 \times 10^{-6}(1 + X)^{.2} \rho^{.2} \ T.$$

$$\bar{K}_a = K_o \rho^{.8} T^{-4.5} \qquad K_o = 1.7 \times 10^{31}(1+X)^{.8} U.$$

(iii) C^{12} reaction $\Gamma_c = .6$ e.v., $\eta = 17$, $h_c = .05$

Intermediate Model:

Properties:

$X = 67\%$

$Y = 29\%$

$U = 4\%$

$\mu = .63$

$T_c = 18.9 \times 10^6 \ {}^oK.$

(i) 70% Oxygen, 30% Russell Mixture.

(ii) $t/\bar{g} = 3.8 \rho (1 + X).^{.35}$

(iii) Same as oxygen model.

Harrison:

A number of models were constructed for various combinations of Russell Mixture and Oxygen. Of these, two are described here:

R. M. Model:

Properties:

$$X = 59.8\%$$
$$Y = 33.8\%$$
$$U = 6.4\%$$
$$\mu = .67$$
$$T_c = 18.9 \times 10^6 \quad {}^oK.$$

(i) Russell Mixture of heavy elements

(ii) Strömgren's values of guillotine factor were represented by two expressions, one for the outer part of the radiative envelope ($1.3735\,\rho^{-.0404}$) and another for the inner part ($1.7274\,\rho^{.5045}$).

(iii) N^{14} reaction $\Gamma_N = 240$ e.v., $\eta = 17$, $h_{c+N} = .02$.

Oxygen and R. M. Model:

Properties:

$$X = 59.7\%$$
$$Y = 34.3\%$$
$$U = 6.0\%$$
$$\mu = .67\%$$
$$T_c = 19 \times 10^6 \quad {}^oK.$$

(i) 60% Oxygen - 40% Russell Mixture

(ii) $t/\bar{g} = 1.7266\,\rho^{-.0126}$ $\qquad \log\rho < .0261$.

$t/\bar{g} = 1.6757\,\rho^{.4849}$ $\qquad \log\rho > .0261$.

Calculated from Strömgren's formulae.

(iii) Same as R. M. Model.

§ 2. Discussion of the results and methods of present investigation.

The hydrogen and heavy element abundance were obtained by the graphical solution (see Fig. 3) of the mass-luminosity relation (22-1) and the energy-production equation (27-1). μ can be written down immediately from (4-1), and ρ_c , T_c are given by (19-4), (19-5), using the values of σ_i, τ_i, θ_i given on page 22.

Properties:

$$X = 66\%$$

$$Y = 31\%$$

$$U = 3\%$$

$$\mu = .64$$

$$T_c = 17.5 \times 10^6$$

$$\rho_c = 131.7$$

Fortunately, the values of X, U and T_c obtained are very close to the values (.70, .03, 17.5) assumed when deriving a formula for the guillotine factor (page 7) and evaluating ε_0' (page 27). Hence it was not necessary to make a second approximation.

The mass distribution and the density and temperature variation in the model, in terms of the variables u, σ, τ, are plotted in Figure 4.

(i) Choice of heavy element mixture. It was decided to represent the heavy elements by the Russell Mixture

$$(K, Ca : Si : Fe : Na, Mg : O = 1 : 1 : 2 : 4 : 8)$$

in order to evaluate the guillotine factor. It is true that the more recent results of spectroscopic analysis of stellar atmosphere do not support this choice, since they indicate that the elements of the oxygen group (O, C, N, S, Ne and A) make up 70 - 80% of the heavy elements (cf. 50% for Russell Mixture). However, the addition of oxygen to the Russell Mixture does not greatly alter the final results as can be seen from the properties of the two models of Harrison described on pages 31-32. In addition, the choice of Russell Mixture makes available the tables of the guillotine factor as corrected by Morse.

(ii) Treatment of Opacity. In the solar models described in § 1 the effects of scattering opacity were neglected. Its relative importance can be seen from the value of the logarithm of the ratio of the mean scattering to the mean absorptive opacity - $\log(\bar{K}_s/\bar{K}_a)$—at any point in the sun. For Schwarzschild's model the value of this quantity at the interface between the core and envelope is - .8. From Morse's table (Fig. 1) we see that for $\log(\bar{K}_s/\bar{K}_a) = -.8$ we have $\log(\bar{K}/\bar{K}_a) = .11$ or $\bar{K} = 1.29\ \bar{K}_a$. Thus the inclusion of scattering opacity would increase the total mean opacity by 20 - 30%. For this reason it seemed desirable to construct the present model in which account has been taken of scattering opacity. The formula used for total mean opacity has been given (7.2). It is clear that (7.2) is only a rough approximation to the numerical values of $\bar{K}$ computed from Morse's tables since the guillotine factor t/$\bar{g}$ cannot be adequately represented by an expression such as (5-1), and the curve ($\log \bar{K}/\bar{K}_a$, $\log \bar{K}_s/\bar{K}_a$) has been approximated by a straight line. To check the accuracy of (7-2) $\bar{K}$ was computed from (7-2) and from Morse's tables for values of $\log \rho(1 + X)$ and log T occurring throughout the present model.

TABLE 6

$\log \rho (1+X)$	log T	$\log \bar{K}$ (Morse)	$\log \bar{K}$ (9-2)	Difference
-1.5	6.1	.964	1.229	- .265
-1.0	6.2	1.123	1.189	- .066
- .5	6.3	1.180	1.149	.031
0	6.5	1.038	.869	.169
.5	6.6	1.011	.829	.182
1.0	6.7	.946	.789	.157
1.5	6.9	.514	.509	.005
2.0	7.0	.381	.468	.087
2.0	7.1	.146	.228	- .082
2.19	7.16	.062	.160	- .098

The error is greater in the low temperature region. However, it is well known that the solutions of the equations of equilibrium are quite insensitive to conditions in the outer part of the star. The agreement between (7-2) and Morse's values of $\bar{\kappa}$ was regarded as sufficiently close so that it was not judged necessary to adjust the value of the constant κ' and so obtain a second approximation mass-luminosity relation.

(iii) The choice of Γ, η and h has been adequately discussed in Chapters I and III. Here we may note that in the models constructed by Harrison (page 31) $\eta = 17$ and $h_{C+N} = .02$ have been used for the N^{14} reactions. The discussion in Chapters I and III, however, shows that $\eta = 19$ and $h_N = .18$ are probably more correct. This would introduce a change by a factor of 16 in the right-hand side of the energy production equation.

In order to compare the results of the present work with the other models which differ from it in the treatment of the opacity, the mass luminosity relations obtained by Schwarzschild and Harrison were solved in conjunction with the energy production equation (23-4) with $\Gamma_N = 288$ e.v., $h_N = .18$ and the appropriate boundary values of the variables σ, τ, θ, t, ς for each case. The results are tabulated on page 36:

Model	Opacity	log C_o	log ϵ	X Y U	μ	T_c ρ_c
<u>SCHWARZSCHILD</u>		$\bar{6}.48326$	16.3178	60%	.67	17.4×10^6
Russell Mixture				36%		
Absorptive Opacity	$\bar{\kappa}_a = \kappa_0(1+X)(U)\rho^{.75}T^{-3.5}$			4%		111.6
<u>HARRISON</u>		$\bar{5}.0032$	17.2517	71.5%	.61	17.3×10^6
Russell Mixture				26.0%		
Absorptive Opacity	$\bar{\kappa}_a = \kappa_0(1+X)(U)\rho^{1.04}T^{-3.5}$ $\bar{\kappa}_a = \kappa_0(1+X)(U)\rho^{.49}T^{-3.5}$			2.5%		163.2
<u>PRESENT MODEL</u>		$\bar{5}.38125$	16.9070	66%	.64	17.5×10^6
Russell Mixture				31%		
Total Opacity	$\bar{\kappa} = \kappa'(1+X)^{.6}(U)^{.8}\rho^{.4}T^{-2.4}$			3%		131.7
<u>HARRISON</u>		$\bar{6}.8690$	16.9308	68%	.63	17.3×10^6
60% O. 40% R.M.				29%		
Absorptive Opacity	$\bar{\kappa}_a = \kappa_0(1+X)(U)\rho^{1.01}T^{-3.5}$ $\bar{\kappa}_a = \kappa_0(1+X)(U)\rho^{.51}T^{-3.5}$			3%		157.

NOTE: $\log \epsilon$ is the constant on the right of the energy production equation (27-1).

Some conclusions may be drawn from these results.

(a) If Γ_N = 288 e.v. and h_N = .18 are taken as reliable values for the N^{14} 'effective width' and abundance, the central temperature for a homogeneous model of the sun is approximately 17.5×10^6 degrees, and the hydrogen helium and heavy element abundances are 67%, 30%, 3%. These figures agree well with the results of spectroscope observation. (b) Neither the inclusion of scattering opacity, nor the addition of oxygen to the Russell Mixture has any marked effect upon the hydrogen, helium or heavy element abundance, or the central temperature. (c) The central density is quite sensitive to changes in the expression used to represent the opacity.

CHAPTER 5

INHOMOGENEOUS MODELS. INTRODUCTION.

In the following chapters we shall consider how the preceding results are modified when we discard the assumption of uniform chemical composition throughout the sun, and in particular, when we assume that the mean molecular weight is uniform in the convective core and in the radiative envelope but has different values in the two regions. That the assumption is not unreasonable is seen when we examine the agencies which tend to promote or remove differences in chemical composition. On the one hand, the accretion of interstellar matter at the surface, diffusion of elements in the interior, and nuclear transformations at the center will act to bring about differences in chemical composition. On the other hand, convection currents in the core and circulatory motions set up as a result of the rotation of the sun will tend to preserve uniformity of composition.

In regard to the former, Atkinson(19) has advanced reasons for maintaining that the accretion of interstellar matter would not give rise to any appreciable change in composition in a period of less than 10^{10} years. Similarly, no marked separation of elements by diffusion could take place in less than 10^{13} years.(20) However, the conversion of hydrogen into helium by nuclear transformation is capable, as we shall see, of giving rise to considerable differences in composition, but because of the strong temperature dependence of the nuclear reactions the changes will take place only in the central region.

In regard to those factors which tend to promote mixing in the interior of the sun — the convective currents resulting from instability of the radiative gradient in the core, will ensure that those changes taking place at the center are transmitted throughout the whole of the core. As to whether any mixing takes place between the core and the radiative envelope, it is impossible to speak with certainty. The axial rotation of the sun will bring about a flattening of the core as well as tending to stabilize the convective currents in certain directions[(21)]. The resultant unequal heating of the poles and equator will favor the creation of large scale currents circulating in meridional planes. These might, perhaps, produce mixing between core and envelope, but it seems more likely that such currents will not pass from the core to the envelope because of the different conditions prevailing in these two regions but will rather set themselves up as independent circulating currents in each region. Little can be said with certainty on this matter, but at least, there are no compelling reasons for rejecting the possibility of unequal chemical composition in core and envelope.

Stellar models in which the mean molecular weight (μ) takes on different values in different regions of the star have been considered on a number of occasions in recent years.[(22),(23)] The most important physical context in which the problem has arisen is in connection with the expected increase in the mean molecular weight in the central regions as a result of the conversion of hydrogen into helium under the continued operation of the carbon-nitrogen cycle. The object of these earlier investigations has generally been to determine what stellar configurations are compatible with a given degree of inhomogeneity and with the requirements of stability. By 'configuration' is here meant the mass, radius and luminosity of the star, the radius and mass of the core, and the physical state of the core (convective, isothermal, partially degenerate, etc.). The degree of inhomogeneity is measured by the ratio (μ_c/μ_e) of the mean molecular weight in the core to its value in the radiative envelope. A study of stable configurations for various values of μ_c/μ_e was expected to furnish a picture of the evolution undergone by the star according as its supply of hydrogen is used up and the value of μ in the core is increased.

In the present work our purpose is not so much to discuss the configurations assumed by the sun during its past history as to try to determine what its present physical and chemical composition is, supposing that the chemical composition of the core differs from that of the envelope. If we describe the composition of the core by the parameters X_c, Y_c, U_c, and that of the radiative envelope by the parameters X_e, Y_e, U_e, then the problem before us is to determine the values of these six parameters which are consistent with all the known requirements and data.

Inasmuch as the composition of the inhomogeneous model is described by six parameters, it will be necessary to find six relations connecting them in order to completely determine the solution. The first two are clearly provided by the relations

$$X_c + Y_c + U_c = 1, \tag{39-1}$$

$$X_e + Y_e + U_e = 1. \tag{39-2}$$

Again, since we are taking into account only those differences of chemical composition which result from the operation of the carbon-nitrogen cycle - the net effect of which is to convert hydrogen into helium - we may set down a third relation, viz.,

$$U_c = U_e . \tag{40-1}$$

We are here assuming that the sun started out as a chemically homogeneous mass. The requirements of stability and energy production will impose two further restrictions on the chemical composition, as in the case of the homogeneous model.

$$C(X_e , U_e) = C_o , \tag{40-2}$$

$$L = \int_0^{r_i} \varepsilon \, 4\pi r^2 \rho \, dr = \mathcal{L}(X_e , X_c , U_e , U_c). \tag{40-3}$$

It remains to find one further relation. Unfortunately, there is no <u>exact</u> requirement which can be imposed to completely determine the problem. We can, however, derive an <u>approximate</u> relation which connects the difference in hydrogen abundance in the core and the envelope ($X_e - X_c$) with the estimated age of the sun. For, if we assume, as is reasonable, that all changes in chemical composition due to thermonuclear reactions are confined to the core, then the present abundance of hydrogen in the outer envelope (X_{e1}) is the same as the original abundance of hydrogen in the core (X_{co}) so that ($X_{e1} - X_{co}$) is a measure of the amount of hydrogen consumed in the core during the lifetime of the sun ($t_1 - t_0$). Since we know the rate of conversion of hydrogen into helium it is clearly possible to establish a connection between ($X_e - X_{co}$) and ($t - t_0$)

$$(t - t_0) = f(X_e - X_{co}). \qquad (41\text{-}1)$$

This provides us with the sixth relation, so that the solution to the problem is completely determined, at least to the extent that an <u>approximate</u> <u>estimate</u> can be made of the degree of inhomogeneity which is possible in the sun.

As the relations (39-1), (39-2) and (40-1) present no difficulty we shall proceed in Chapters 6 and 7 to treat in turn the requirements of stability, energy generation and rate of change of hydrogen concentration (40-2), (40-3) and (41-1). A discussion of the results will follow in Chapter 8.

CHAPTER 6

STABILITY AND ENERGY PRODUCTION OF INHOMOGENEOUS MODELS.

§ 1. Stability. As in the case of the chemically homogeneous model, we must require that the non-homogeneous model satisfy all the requirements for stability, that is to say, the equations of Hydrostatic and radiative equilibrium and the equation of conservation of mass, (14-2), (14-3), (15-1) and the boundary conditions. The form of the equations remains unchanged--except that μ in (14-2) is replaced by μ_e. The boundary conditions at the outer surface ($r = R$) are the same as before, viz., $M(R) = M$, $\rho = T = 0$. At the interface between the core and the envelope some modification of the boundary conditions has to be made in order to take account of the discontinuity in the mean molecular weight: Clearly we must have continuity of pressure (p) and temperature (T) at the interface.

$$(p)_{r \to r_{i+}} = (p)_{r \to r_{i-}} ,$$

$$(T)_{r \to r_{i+}} = (T)_{r \to r_{i-}} ,$$

where r_i is the radius of the core and $r \to r_{i+}$, $r \to r_{i-}$ denote approach of r to the value r_i from the envelope side and the core side respectively.

But since

$$p = \frac{R}{\mu} \rho T,$$

we must also require the continuity of ρ/μ

$$\left(\frac{\rho}{\mu_e}\right)_{r \to r_{i+}} = \left(\frac{\rho}{\mu_c}\right)_{r \to r_{i-}}. \tag{43-1}$$

The remaining boundary condition in the case where μ had the same value in the core and the envelope was provided by the requirement that $d(\log \rho)/d(\log T)$ be continuous across the interface and that it have the value 3/2 in the core

$$\left(\frac{d \log \rho}{d \log T}\right)_{r \to r_{i+}} = \left(\frac{d \log \rho}{d \log T}\right)_{r \to r_{i-}} = \frac{3}{2}.$$

In the earlier investigations [24] into the stable configurations of non-homogeneous models, this form of the boundary condition was retained. It was pointed out by Hoyle and Lyttleton[25] that if μ is not continuous, this condition on $d(\log \rho)/d(\log T)$ must be altered. The new form of the boundary condition in the present problem is easily deduced. Since we have continuity of mass (M) and radial distance (r) it follows from the equation of hydrostatic equilibrium

$$\frac{dp}{dr} = - G \frac{M(r)}{r^2} \rho,$$

that we must have continuity of $(dp/dr)(\frac{1}{p})$

$$\left(\frac{1}{\rho} \frac{dp}{dr}\right)_{r \to r_{i+}} = \left(\frac{1}{\rho} \frac{dp}{dr}\right)_{r \to r_{i-}},$$

or equivalently

$$\left(\frac{1}{\rho p}\,\frac{dp}{dr}\right)_{r\to r_{i+}} = \left(\frac{1}{\rho p}\frac{dp}{dr}\right)_{r\to r_{i-}}. \qquad (44\text{-}1)$$

From the equation of radiative equilibrium

$$\frac{d}{dr}\left(\tfrac{1}{3}aT^4\right) = \frac{\bar{K}}{c}\rho\,\frac{L}{4\pi r^2},$$

with $\bar{K}$ replaced by its expression (7-1)

$$\bar{K} = K'(1+X)^{(1-\alpha)(1-m)+m}(\mathcal{U})^{1-m}\rho^{(1-\alpha)(1-m)}\Big/T^{(3.5-\beta)(1-m)},$$

we get

$$\frac{1}{(\mathcal{U})^{1-m}(1+X)^{(1-\alpha)(1-m)+m}\rho^{(1-\alpha)(1-m)+1}}\;\frac{1}{T}\;\frac{dT}{dr} = -\frac{3K'L}{16\pi ac}\;\frac{1}{r^2}\;\frac{1}{T^{(3.5-\beta)(1-m)+4}}.$$

On the right-hand side of this equation r and T are continuous. L, the rate of transfer of energy, is also continuous because in the radiative envelope, $L = L$(radiative), and within the core, $L = L$(radiative) + L(convective) but at the boundary of the core L(convective) = 0 – by definition, and hence, $L = L$(radiative) is continuous across the interface. We conclude that the left-hand side of the equation is continuous.

$$\left[\frac{1}{(1+X_e)^{(1-\alpha)(1-m)+m}(\mathcal{U}_e)^{1-m}\rho^{(1-\alpha)(1-m)+1}}\;\frac{1}{T}\;\frac{dT}{dr}\right]_{r\to r_{i+}}$$

$$= \left[\frac{1}{(1+X_c)^{(1-\alpha)(1-m)+m}(\mathcal{U}_c)^{1-m}\rho^{(1-\alpha)(1-m)+1}}\;\frac{1}{T}\;\frac{dT}{dr}\right]_{r\to r_{i-}}. \qquad (44\text{-}2)$$

Combining (44-1) and (44-2)

$$\left[(1+X_e)^{(1-\alpha)(1-m)+m}(\mathcal{U}_e)^{1-m}\rho^{(1-\alpha)(1-m)}\frac{d\log p}{d\log T}\right]_{r\to r_{i+}}$$

$$=\left[(1+X_c)^{(1-\alpha)(1-m)+m}(\mathcal{U}_c)^{1-m}\rho^{(1-\alpha)(1-m)}\frac{d\log p}{d\log T}\right]_{r\to r_{i-}}. \qquad (45\text{-}1)$$

Noting that we have

$$\frac{d\log p}{d\log T}=\frac{d\log \rho}{d\log T}+1,$$

denoting $d(\log\rho)/d(\log T)$ by the symbol n (the polytropic index) and recalling that ρ/μ is continuous (43-1) we can rewrite (45-1)

$$(1+X_e)^{(1-\alpha)(1-m)+m}(\mathcal{U}_e)^{(1-m)}\mu_e^{(1-\alpha)(1-m)}(n_e+1)$$

$$=(1+X_e)^{(1-\alpha)(1-m)+m}(\mathcal{U}_c)^{1-m}(\mu_c)^{(1-\alpha)(1-m)}(n_c+1), \qquad (45\text{-}2)$$

where n_c has the value 3/2 and n_e is the value of $d(\log\rho)/d(\log T)$ on the envelope side of the interface.

$$n_e=\left(\frac{d\log\rho}{d\log T}\right)_{r\to r_{i+}}.$$

When the chemical composition is uniform (45-2) reduces to the former condition

$$n_e=n_c=3/2.$$

For $\alpha = \beta = m = 0$ (45-2) reduces to the form discussed by Hoyle and Lyttleton[26]. In the present case we have (page 7) $\alpha = \beta = .5$ and $m = .2$ (40-1). Hence the boundary condition (45-2) reduces to

$$(1+X_e)^{.6}(\mu_e)^{.4}(n_e+1) = \frac{5}{2}(1+X_c)^{.6}(\mu_c)^{.4}, \tag{46-1}$$

where n_c has been replaced by its value 3/2.

Summing up the boundary conditions at the interface, and expressing quantities relating to the envelope in terms of the variables u, σ, τ defined on page 15, and those relating to the core in terms of the Lane-Emden variables ξ and θ, we have the following: (cf. (18-6) (19-1) (19-2)).

At $t = t_i$ defined as the point where $d(\log\sigma)/d(\log\tau)$ attains some value n_e,

radial distance $$\frac{R}{t_i} = a\,\xi_i\,, \tag{α}$$

mass $$u_i M = -4\pi a^3 \rho_c (\xi^2\theta')_i\,, \;{}^* \tag{β}$$

density $$\frac{1}{\mu_e}\sigma_i \frac{M}{4\pi R^3} = \frac{1}{\mu_c}\rho_c\,\theta_i^{3/2}, \;{}^* \tag{γ}$$

temperature $$\tau_i\,\mu_e \frac{GM}{\mathcal{R}R} = T_c\,\theta_i\,, \;{}^* \tag{δ}$$

polytropic index $$(1+X_e)^{.6}(\mu_e)^{.4}(n_e+1) = 5/2\,(1+X_c)^{.6}(\mu_c)^{.4}. \tag{46-2}$$

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* The subscript c on ρ and T refers to central values--not to be confused with subscript c on μ, X, U, denoting core values.

As on page 19 we combine these boundary conditions into the more useful form

$$\frac{\mu_c}{\mu_e}\,\frac{\sigma_i}{u_i t_i^3} = \left(\frac{\xi\,\theta^{3/2}}{\theta'}\right)_i = \mathcal{U}(\xi_i), \tag{47-1}$$

$$\frac{\mu_c}{\mu_e}\,\frac{u_i t_i}{\tau_i} = -\frac{5}{2}\left(\frac{\xi\,\theta'}{\theta}\right)_i = \mathcal{V}(\xi_i), \tag{47-2}$$

and

$$(1+X_e)^{.c}(\mu_e)^{.4}(n_e+1) = \frac{5}{2}(1+X_c)^{.c}(\mu_c)^{.4}. \tag{47-3}$$

There are two questions which naturally arise in connection with the boundary conditions (47-1), (47-2), (47-3).

(i) Do they represent a physically possible model?

(ii) How are the conditions to be applied in practice?

(i) It seems antecedently unlikely that in the actual physical case a sharp discontinuity of the molecular weight will occur. "Physical intuition" would lead us to expect a smooth transition from one value of μ to another. The question has been investigated by Ledoux[27]. He examines the effect of a discontinuity of μ in the deep interior of the radiative envelope and concludes that if the jump in the value of μ is such that the boundary condition of Hoyle and Lyttleton ((45-2) above with $\alpha = \beta = m = 0$) is not satisfied, then the discontinuity in μ produces a condition of instability in a small zone past the place where the discontinuity occurs. The resulting turbulence causes mixing and the star will rapidly adjust itself to a neighboring stable state, in which the region of higher μ and the exterior region of lower μ are separated by a transition region in which μ varies according to the law $\mu \propto M(r)\, p^{7/5}$. On the assumption that the helium content of the star is negligible, Ledoux has constructed stellar models which consist of convective cores and radiative envenope with assigned mean molecular weights μ_c and μ_e respectively, separated by transition zones of variable μ. It is shown that these models satisfy all the conditions for stability, and further, that they do not differ appreciably in their physical properties from earlier models constructed by Schönberg, Harrison and Chandrasekhar(28) in which μ suffers a sharp discontinuity and in which the fitting of core to envelope was made under he condition $n_e = n_c = 3/2$, provided that μ_c/μ_e is small. Ledoux concludes that while a true physical model should include a transition zone of variable μ , a satisfactory first approximation is obtained by keeping μ discontinuous, and using the condition $n_e = n_c = 3/2$ at the interface. As has been previously mentioned, Hoyle and Lyttleton have pointed out that this condition should be modified to allow for the change in chemical composition so as to assume the form indicated in (45-2). If then, the incorrect condition $n_e = 3/2$ gives a good first approximation to Ledoux's model with transition zone of variable μ , it is legitimate to expect that the approximation would be further improved by the use of the correct boundary condition (45-2). This was the procedure adopted in the present work, but as we shall see, it turns out that the two conditions are almost identical numerically--since in all the models constructed the ratio $(1+X_e)^{.6}(\mu_e)^{.4}/(1+X_c)^{.6}(\mu_c)^{.4}$ assumes a value very close to unity.

(ii) It remains to discuss how the boundary conditions (47-1), (47-2) and (47-3) are to be applied in practice. A difficulty arises here in as much as the boundary conditions involve the as yet unknown quantities μ_e, μ_c, X_e, X_c. In order to overcome this difficulty it is necessary to resort to a method of trial and error as follows. Firstly, an arbitrary value (C_o) is assigned to the eigenvalue parameter C and the equations of equilibrium (page 14) are integrated for this value into a point where $n_e \ (= (\sigma'\tau)/(\sigma\tau'))$ has fallen to the value 3/2. Next, the quantities n_e , $\sigma/(u t^3)$ and ut/τ are evaluated for the last three or four steps of the integration. A particular value of n_e is selected--say $n_e = 1{\cdot}6$, and the value of μ_c/μ_e is so chosen ($= \beta$, say) that the point ($\mathcal{U}$, $\mathcal{V}$) (vd. (47-1), (47-2)) lies on the Lane-Emden polytrope of index 3/2. (B. A. Tables). This permits the evaluation of ξ and θ from the tables. Knowing C_o we can write down the mass-luminosity equation

$$C_o = [\bar{9}{\cdot}3951]\,(1+X_e)^{.6}(\mathcal{U}_e)^{.8}(5X_e-\mathcal{U}_e+3)^{6{\cdot}4}\frac{L R^{1{\cdot}2}}{M^5}, \tag{49-1}$$

which gives us a relation between X_e and $\mathcal{U}_e$. Knowing the values of σ, τ, θ, t, ξ and μ_c/μ_e at the point where $n_e = 1{\cdot}6$, we can write down the energy production equation (**40-3**)

$$L = \mathcal{L}(X_e, \mathcal{U}_e), \tag{49-2}$$

giving us a second equation in X_e, U_e. In addition we have

$$X_e + Y_e + U_e = 1, \tag{50-1}$$

$$X_c + Y_c + U_c = 1, \tag{50-2}$$

$$U_c = U_e , \tag{50-3}$$

and $$\mu_c/\mu_e = \beta. \tag{50-4}$$

Solving (49-1)-(50-4) for X_e, X_c, μ_e, μ_c we can now test the remaining boundary condition (47-3). If it is not satisfied, we then choose another value of n_e in the integration and repeat the above procedure. In fact, however, it was found that for every c_0 chosen, all three boundary conditions (47-1), (47-2) and (47-3) were able to be satisfied at $n_e = 1.5$ to a sufficient degree of accuracy. The fact that condition (47-3) can be satisfied for the cases studied (μ_c/μ_e = 1.098, 1.2, 1.32) shows, following the argument of Ledoux[29], that in these cases no transition zone will appear at all, since it is only in cases where μ_c/μ_e is of such a value that condition (47-3) cannot be satisfied that it becomes necessary to introduce a zone of variable μ.

§ 2. Energy Production. For the case of the homogeneous model the energy production equation has the form (23-4),

$$L = 4\pi\, \varepsilon_0'\, \rho_c^{\,2}\, T_c^{\,\eta}\, a^3 \int_0^{\xi_i} \theta^{\eta+3}\, \xi^2\, d\xi. \tag{51-1}$$

To adapt it to the inhomogeneous case we have only to replace a, ρ_c, T_c by their appropriate expressions derived from the boundary conditions α, γ, δ, (page 46) and note that the ε_0' depends now upon the values of X and U in the core, i.e., X_c, U_c. Thus, we have

$$\rho_c = \frac{\mu_c}{\mu_e}\,\frac{\sigma_i}{\theta_i^{3/2}}\,\frac{M}{4\pi R^3} = \beta\,\frac{\sigma_i}{\theta_i^{3/2}}\,\rho_0,$$

where β is the value of μ_c/μ_e which gives the proper values to the fitting functions, $\mathcal{U}$ and $\mathcal{V}$ - (47-1) and (47-3),

$$T_c = \mu_e\,\frac{\tau_i}{\theta_i}\,\frac{GM}{\mathcal{R}R} = T_0\,\frac{\tau_i}{\theta_i},$$

where $T_c = 23.11 \times 10^6\, \mu_e,$

$$a = \frac{R}{\iota_i\,\xi_i},$$

and $\varepsilon_0' = \dfrac{\varepsilon_0}{T_c^{\eta+\frac{2}{3}}\, e^{B T_c^{-1/3}}}$

$$= \frac{1}{T_c^{\eta+\frac{2}{3}}\, e^{B T_c^{-1/3}}}\; E\, \frac{4}{3^{5/2}}\, \frac{X_c U_c}{m_1 m_N}\, \frac{\Gamma_N}{\hbar}\, h_{N\,a}\, R^2\, e^{(32\frac{R}{a})^{\frac{1}{2}}}\, B^2.$$

where the symbols are as defined on page 8.

Substituting for a , ρ_c , T_c in (51-1) we have:

$$L = 4\pi \varepsilon_0' \left(\frac{\mu_c}{\mu_e}\right)^2 \rho_0^2 T_0^\eta R^3 \left(\frac{\sigma_i}{\theta_i^{3/2}}\right)^2 \left(\frac{\tau_i}{\theta_i}\right)^\eta \left(\frac{1}{t_i \xi_i}\right)^3 \int_0^{\xi_i} \theta^{\eta+3} \xi^2 \, d\xi . \tag{52-1}$$

μ_c/μ_e , σ_i , τ_i , θ_i , ξ_i , t_i are found from the fitting of the boundary conditions (47-1), (47-2), (47-3) for whatever value of C_0 is chosen. η is taken as before to correspond with the estimated value of T_c. Upon inserting numerical values of all the constants and solving in terms of X_c, U_c, X_e, U_e, we get

$$\frac{(5X_e - U_e + 3)^{19}}{X_c \, U_c} = \text{const} \times \frac{M^{21}}{R^{22} L} .$$

Using $\mu_c/\mu_e = \beta$ and $U_c = U_e$ this becomes

$$\frac{(5X_e - U_e + 3)^{19}}{\frac{U_e}{\beta}\left[X_e + \frac{\beta-1}{5}(U_e - 3)\right]} = \text{Const.} \frac{M^{21}}{L R^{22}} . \tag{52-2}$$

(52-2) represents the final form of the energy production equation which in conjunction with the mass-luminosity equation (49-1) determines X_e and U_e.

CHAPTER 7

THE AGE OF INHOMOGENEOUS MODELS

In this chapter we shall investigate the connection between the age of the sun and the difference between the hydrogen concentration in the core and in the radiative envelope. The question of the rate of consumption of hydrogen has been treated by P. ten Bruggencate[30], on the assumption that there is complete mixing of elements throughout the star, so that the composition remains uniform. The methods and results of ten Bruggencate will be briefly reviewed here, after which we shall apply the method with suitable modifications to the case of an inhomogeneous model in which all changes are confined to the core.

The conversion of one gram of hydrogen into helium is accompanied by the release of an amount of energy $\Delta m\, c^2$ where Δm is the mass loss per gram of hydrogen. The rate of conversion of hydrogen into helium is $-M\, dX/dt$ where M is the total mass and X, the hydrogen concentration, is assumed uniform everywhere within the star. Assuming that this is the only source of energy in the star, we can write

$$-M\,\Delta m\, c^2 \frac{dX}{dt} = L, \qquad (53\text{-}1)$$

where L is the luminosity at time t.

Now, if L can be expressed as a function of X we can integrate this equation to obtain X as a function of the time t. The desired expression for L can be obtained by eliminating R, the radius of the star, from the mass-luminosity and energy production equations. For the mass-luminosity equation ten Bruggencate takes

$$L = \text{const}\ \frac{ac}{3K_0}\left(\frac{G\mu}{R}\right)^{5/2}\frac{M^{11/2}}{R^{1/2}},$$

where $K_0 = 3.9 \times 10^{25} \cdot \frac{\bar{g}}{t}(1+X)(U)$.

This is seen to be identical with (15-7) when we put $\alpha = \beta = m = 0$ Since U is constant in time we have, writing $\mu \doteq 4/(5X+3)$, (since $U \ll X$)

$$L = \text{const}\ \frac{G^{7.5}M^{5.5}}{(1+X)(1+\frac{5}{3}X)^{7.5}R^{.5}}. \tag{54-1}$$

The energy production equation has the form

$$L \sim X M \rho_c T_c^{\eta},$$

with $\rho_c \sim \frac{M}{R^3}$, $T_c \sim \mu\frac{GM}{R}$,

whence $$L \sim \frac{X}{(5X+3)^{\eta}}\ \frac{M^{\eta+2}G^{\eta}}{R^{3+\eta}}. \tag{54-2}$$

In (53-1) and (54-1) M is approximately constant; and if the gravitational "constant" G is not a function of time (ten Bruggencate considers also the case where $G \sim 1/t$) the only variables are L, X and the radius R. Eliminating R between (53-1) and (54-1)

$$L = \text{const}\,\frac{1}{X^{\frac{1}{2\eta+5}}(1+X)^{\frac{2\eta+6}{2\eta+5}}\left(1+\frac{5}{3}X\right)^{\frac{14\eta+45}{2\eta+5}}}.$$

For $\eta = 18$ (ten Bruggencate)

$$L = \frac{1}{X^{.02}(1+X)^{1.02}\left(1+\frac{5}{3}X\right)^{7.25}}. \tag{55-1}$$

We see that (55-1) differs little from the mass-luminosity relation (54-1), so that we may use (54-1) instead of (55-1). Substituting the expression (54-1) for L in (53-1) and integrate from $t = t_0$ (t_0 is the time at which the sun was formed) to $t = t_1$, (the present instant, at which $L = L_1$, and $X = X_1$, say) we get

$$(t_1 - t_0) = \frac{M\,\Delta m\,c^2}{L_1}\,\frac{(\mathcal{U}_0 - \mathcal{U}_1)}{u_1^{15}(1+X_1)}, \tag{55-2}$$

where $\quad \mathcal{U}(X) = \frac{18}{19\times 25}\,u^{19}(X) + \frac{12}{17\times 25}\,u^{17}(X),$

and $\quad u^2(X) = \left(1 + \frac{5}{3}X\right).$

Assuming that the present concentration of hydrogen in the sun is .60 ($X_1 = .60$) ten Bruggencate evaluates the age of the sun ($t_1 - t_0$) for various values of X_0, the original hydrogen concentration

X_0	$(t-t_0)$
1.0	187 x 10^9 yrs.
.9	96 x 10^9
.8	44 x 10^9
.7	15.4 x 10^9
.66	8.0 x 10^9
.63	3.6 x 10^9
.61	1.08 x 10^9

A knowledge of the age of the sun could enable us to determine X_0. It has been shown(31) that an examination of all the available evidence leads to an estimated 3.6 x 10^9 yrs. as the age of the sun. ten Bruggencate concludes that $X_0 = .63$ so that the decrease in the hydrogen concentration during the life time of the sun amounts to a mere 3%.

As might be expected these results of ten Bruggencate are altered notably if we assume that changes in chemical composition are confined to the convective core. In this case, since X_e, Y_e, μ_e are constants, the mass-luminosity relation becomes simply((15-7) with $\alpha = \beta = .5 \quad m = .2$)

$$L = \text{const.}\ \frac{1}{R^{1.2}}. \qquad (56\text{-}1)$$

The energy production equation (52-1) may be reduced to the form

$$\frac{\mu_c^2\, \mu_e^{17}}{X_c\, \mu_c} = \text{const.}\ \frac{M^{21}}{L R^{22}},$$

or, since U_c, μ_e are constants and $\mu_c \doteq 4/(5X_c+3)$,

$$L = \text{const.}\ \frac{(5X_c+3)}{X_c}\ \frac{1}{R^{22}}. \tag{57-1}$$

Eliminating R from (56-1), (57-1)

$$L = C\ \frac{(5X_c+3)^{6/52}}{X_c^{\ 3/52}}. \tag{57-2}$$

The constant C in (57-2) may be evaluated by replacing L and X_c by their present values, L_1, X_{c1}. L_1 is known from observation ($L_1 = 3.78 \times 10^{33}$ ergs / sec.), and X_{c1} is the hydrogen abundance in the core as computed by the method of Chapter 6. Replacing C by its expression in terms of L_1, X_{c1}, we have

$$\frac{L}{L_1} = \left(\frac{5X_c+3}{5X_{c1}+3}\right)^{6/52}\left(\frac{X_{c1}}{X_c}\right)^{3/52}. \tag{57-3}$$

We can now substitute for L from (57-3) in (53-1) - noting that the total mass M in (53-1) must now be replaced by the mass of the core $u_i M$.

$$-\frac{u_i M \Delta m c^2}{L_1}\ \frac{dX_c}{dt} = \frac{L}{L_1}$$

$$= \left(\frac{5X_c+3}{5X_{c1}+3}\right)^{6/52}\left(\frac{X_{c1}}{X_c}\right)^{3/52}.$$

Integrating between the limits $t = t_0$, $X_c = X_{c0}$, and $t = t_1$, $X_c = X_{c1}$ --where, as before, t_0 and t_1, mark the beginning and the present moment of the sun's life-- we get,

$$(t_1 - t_0) = \frac{u_i M \Delta m c^2}{L_1} \frac{(5X_{c1}+3)^{6/52}}{(X_{c1})^{3/52}} \int_{X_{c1}}^{X_{c0}} \frac{(X_c)^{3/52}}{(5X_c+3)^{6/52}} \, dX_c . \tag{58-1}$$

Since no change has taken place in the hydrogen and helium abundance in the envelope, we may write $X_{c0} = X_{e1}$, where X_{e1} is the hydrogen abundance in the envelope as calculated by the method of Chapter 6. X_{e1} is simply the present abundance of hydrogen in the core, also obtained by the method of the previous chapter. We can simplify (54-1) by noting that the integrand is a very slowly varying function of X_c in the range $.6 < X_c < .9$ in which we are interested. We can therefore evaluate the integrand at the lower limit and obtain finally

$$\begin{aligned} (t_1 - t_0) &= \frac{u_i M \Delta m c^2}{L_1} (X_{e1} - X_{c1}) \\ &= 10^{10} u_i (X_{e1} - X_{c1}) \text{ years.} \end{aligned} \tag{58-2}$$

The relation (58-2) provides the final check (see Chapter 5) upon the permissible degree of inhomogeneity in the sun. Admittedly it is not a precise check since estimates of the age of the sun, ($t_1 - t_0$) vary within wide limits, and ($X_{e1} - X_{c1}$) is linear in ($t_1 - t_0$). This latter feature of (58-2) is in marked contrast to the relation (55-2) obtained on the assumption of perfect mixing throughout the sun, where ($t_1 - t_0$) depends quite sensitively on ($X_1 - X_0$). (See page 56).

CHAPTER 8

PROPERTIES OF INHOMOGENEOUS MODELS. DISCUSSION OF RESULTS.

The procedure described in Chapters 6 and 7 was applied to the construction of three models, the properties of which will be outlined in this chapter.

(1)

$$\mathrm{Log}\, C_0 = \bar{5}.43846.$$

Mass-Luminosity Relation (49-1)

$$(1+X_e)^{.6}(U_e)^{.8}(5X_e - U_e + 3)^{6.4} = [4.0430].$$

At $n_e = 3/2$ fitting obtained for $\mu_c/\mu_e = 1.089$ (47-1) and (47-2).

$$t_i = 9.075$$
$$u_i = .127$$
$$\tau_i = 1.011$$
$$\sigma_i = 225.7$$
$$\xi_i = 1.2$$
$$\theta_i = .784$$

Energy Production Equation (52-2)

$$\frac{(5X_e - U_e + 3)^{19}}{U_e/1.089\left[X_e + .017\,(U_e - 3)\right]} = [17.6065].$$

Chemical Composition:

X_e	= 74%	X_c	= 63%
Y_e	= 24%	Y_c	= 35%
U_e	= 2%	U_c	= 2%
μ_e	= .60	μ_c	= .65

Central temperature and density

$$T_c = 17.9 \times 10^6 \quad {}^{\circ}K,$$
$$\rho_c = 166 .$$

Test of fitting condition (47-3)

$$(n_c + 1) = \left(\frac{1+X_e}{1+X_c}\right)^{.6}\left(\frac{\mu_e}{\mu_c}\right)^{.4}(2.5)$$
$$= 1.007 \times 2.5 = 2.52$$
$$n_c = 1.52 .$$

Age (58-2)

$$(t_1 - t_0) = 1.4 \times 10^9 \text{ yrs.}$$

(2)

$$\text{Log } C_0 = \bar{5}.5044 .$$

Mass-Luminosity Relation (49-1)

$$(1+X_e)^{.6}(U_e)^{.8}(5X_e - U_e + 3)^{6.4} = [4.1085].$$

At $n_e = 3/2$, fitting obtained for $\mu_c/\mu_e = 1.18$ (47-1) and (47-2).

$$t_i = 10.24$$
$$u_i = .110$$
$$\tau_i = 1.104$$
$$\sigma_i = 271.0$$
$$\xi_i = 1.22$$
$$\theta_i = .776$$

Energy Production Equation (52-2)

$$\frac{(5X_e - U_e + 3)^{19}}{U_e/1.2\,[X_e + .04(U_e - 3)]} = [18.4924].$$

Chemical Composition:

X_e	= 85.3%	X_c	61.2%
Y_e	= 13.5%	Y_c	37.6%
U_e	= 1.2%	U_c	1.2%
μ_e	= .55	μ_c	.66

Central temperature and density

$$T_c = 18 \times 10^6\ {}^{\circ}K,$$

$$\rho_c = 186.$$

Test of fitting condition (47-3)

$$(n_c + 1) = \left(\frac{1 + X_e}{1 + X_c}\right)^{.6} \left(\frac{\mu_e}{\mu_c}\right)^{.4} 2.5$$

$$= 1.01 \times 2.5 = 2.52.$$

$$n_c = 1.52.$$

Age (58-2)

$$t_1 - t_0 = 2.6 \times 10^9 \text{ years.}$$

(3) $\quad \text{Log } C_0 = \bar{5}.54695.$

Mass-Luminosity Relation (49-1)

$$(1+X_e)^{.6}(U_e)^{.8}(5X_e - U_e+3)^{6.4} = [4.1515].$$

At $n_e = 3/2$ fitting obtained for $\mu_c/\mu_e = 1.32$ (47-1) and (47-2).

$$t_i = 11.44$$
$$u_i = .108$$
$$\tau_i = 1.187$$
$$\sigma_i = 310.2$$
$$\xi_i = 1.25$$
$$\theta_i = .768$$

Energy Production Equation (52-2)

$$\frac{(5X_e - U_e+3)^{19}}{U_e/1.32\left[X_e + .064(U_e-3)\right]} = [19.2111].$$

Chemical Composition:

X_e =	94%	X_c =	57%
Y_e =	5%	Y_c =	42%
U_e =	1%	U_c =	1%
μ_e =	.52	μ_c =	.68

Central temperature and density

$$T_c = 18.4 \times 10^6 \ {}^{o}K,$$
$$\rho_c = 240.$$

Test of fitting condition (47-3)

$$(n_c+1) = \left(\frac{1+X_e}{1+X_c}\right)^{.6}\left(\frac{\mu_e}{\mu_c}\right)^{.4}\left(\frac{5}{2}\right)$$

$$=1.012 \times 2.5 = 2.53.$$

$$n_c = 1.53.$$

Age (58-2)

$$(t_1 - t_0) = 4 \times 10^9 \text{ years.}$$

Discussion of Results.

(a) Fitting of boundary condition (47-3). The fitting of the envelope to the core was made in each case at the point where $n_e = 1.5$ leading to calculated values of $n_c = 1.52$, 1.52 and 1.53 (cf. condition (47-3)). For exact fitting to a convective core we should, of course, have $n_c = 1.5$. The error in fitting is reflected in the values of the variables u, τ, σ, t, θ, ξ at the interface and hence appears in the energy production equation, but in view of the uncertainty which already attaches to the value of the "effective width" (Γ) of the nuclear reaction, the additional error in the equation resulting from the imperfect fitting is of no account.

(b) Physical characteristics. The most noteworthy features of the models constructed are the progressive shrinking of the size and mass of the core, the rapid increase in central density and the slow increase in central temperature as we go from $\mu_c/\mu_e = 1$ to $\mu_c/\mu_e = 1.32$. The decrease in the mean molecular weight of the radiative envelope, which would be expected to reduce the central temperature, is more than compensated for by the increase in the ratio τ_i / θ_i at the interface.

(c) Chemical Composition. The difficulties and uncertainties which beset the spectroscopic analysis of stellar atmospheres are such that we must avoid attaching too great significance to a comparison between the results of spectroscopic analysis and the abundance of chemical elements as deduced from the internal constitution and energy production of stars. At most, such a comparison can be expected to yield conclusions of a very general nature. Professor J. L. Greenstein* suggests the following relative abundances (by weight) as being in good agreement with the most recent spectroscopic data: H 70% He 28%, Oxygen group (O, C, N, S, Ne and A) 1.5%, metals (Si and Mg) .5%. The ratio H/He is uncertain by at least a factor of 2. While the first model described at the beginning of this chapter ($\mu_c/\mu_e = 1.089$) has a composition (H 74%, He 24%, other elements 2%) which agrees closely with these figures, it would seem to be excluded on the grounds of age-- 1.4×10^9 years, as compared with estimated age $3 - 5 \times 10^9$ years. On the other hand, the third model ($\mu_c/\mu_e = 1.32$), whose age (4×10^9 years) agrees well with the estimated age of the sun, has a chemical composition (H 94%, He 5%, other elements 1%) which is less easy to reconcile with the results of spectroscopic analysis. The uncertainty of the latter, however, leaves open the question of the

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* Private communication to Professor R. F. Christy

suitability of this model. We note that the abundance of heavy elements decreases with increasing inhomogeneity and that the requirement $X<1$ places an upper limit on the value of μ_c/μ_e. This limit is somewhere in the neighborhood of 1.35 (a fourth model constructed for $\log C_0 = \bar{5}.56496$ was fitted at $\mu_c/\mu_e = 1.38$ but gave a value of X_e greater than unity. The ratio H/He differs appreciably in the three models -- 3.1, 6.3, 18.8.

A comparison of the above results with those derived (Part I) from the study of a homogeneous model leads us to the following conclusions.

(a) Present estimates, based on the spectroscopic data, of the abundances of chemical elements in the solar atmosphere, while not decisive, seem to favor the supposition that the convective core and radiative envelope have the same chemical composition.

(b) Should future investigation alter the estimated spectroscopic abundances in the direction of great hydrogen content and lower helium and heavy element content, it will always be possible to construct an inhomogeneous model in agreement with such results. No marked difference in central temperature will result.

NOTE: While the present work was in progress, a paper by P. Ledoux(32) was published in which the chemical composition of an inhomogeneous solar model is investigated. The treatment of the problem differs from the foregoing in the following respects.

(i) The variation of the "guillotine factor" with temperature and density is not taken into account. The opacity law $\bar{\kappa} = \kappa_0 (1+X)(u)\rho T^{-3.5}$ is used.

(ii) The effect of scattering opacity is neglected,

(iii) In fitting the envelope to the core, allowance is made for the possible appearance of a transition zone of variable μ.

(iv) Method. The distinguishing feature of the method - as contrasted with the above - is that of the six variables X_c, Y_c, U_c, X_e, Y_e, U_e which describe the composition of the model, the last three are given values in agreement with the spectroscopic analysis of the solar atmosphere by Unsöld[33]. The remaining variables X_c, Y_c, U_c are obtained from the relations

$$u_c = u_e,$$

and $\mu_c/\mu_e = \beta$ (obtained by fitting).

The energy production integral (51-1) and age relation (58-2) are then used as checks on the result. Thus, giving R, L and M their solar values, taking $X_e = .56$, $U_e = .04$ and $K_o = 10^{25}$ the constant C_o in the mass-luminosity relation

$$C_o = \frac{3 K_o L}{16\pi a c R} \frac{\rho_o^2}{T_o^{7.5}} (1 + X_e)(u_e),$$

(cf. (15-7) with $\alpha = \beta = m = 0$).

is evaluated. By interpolation between a number of models (constructed for different values of C_o and fitted to convective cores, with a transition zone where necessary) a fitting of the boundary conditions is made for the value of C_o obtained, leading to the following results*:

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* The notation adopted in the present work is used.

$$\mu_c/\mu_e = 1.352$$
$$\mu_{ts}/\mu_e = 1.345$$
$$u_i = .151$$
$$u_{ts} = .155$$
$$\tau_i = .877$$
$$\theta_i = .713$$

$$X_c = 26\%$$
$$Y_c = 70\%$$
$$T_c = 19.7 \times 10^6$$
$$\rho_c = 125.$$

where μ_{ts} is the mean molecular weight in the transition zone at its boundary with the radiative envelope,

u_i is the fractional mass of the core,

u_{ts} is the fractional mass of the core plus transition zone.

The energy production integral is evaluated as a check—using $X_c = .26$, X_N (nitrogen abundance) = .0053—and the result $L = 1.1\,L_\odot$ obtained. The estimated age of the model is 5×10^9 years. The conclusion is drawn that an inhomogeneous model constructed so as to agree in the chemical composition of its envelope with the results of spectroscopic analysis, satisfies all the conditions of the problem for $\mu_c/\mu_e = 1.352$, and has an age which is of the correct order of magnitude.

The discrepancy between the results obtained in the present work and those of Ledoux may be attributed to:

(a) The different treatment of the opacity law in the two cases-leading to a different mass-luminosity relation

$$C_o = \frac{3K'L}{16\pi acR} \frac{\rho_o^{(1-\alpha)(1-m)+1}}{T_o^{(3.5-\beta)(1-m)+4}} (1+X_e)^{(1-\alpha)(1-m)+m} (\mathcal{U}_e)^{1-m},$$

with $\alpha = \beta = .5 \quad m = .2 \quad K' = 4.78 \times 10^{17}$ in the present work,

$\alpha = \beta = 0 \quad m = 0 \quad K' = 10^{25}$ in Ledoux's treatment.

(b) The values assigned to X, Y and $\mathcal{U}$ in the radiative envelope $X_e = .56$, $Y_e = .40$, $\mathcal{U}_e = .04$, while in agreement with Unsöld's earlier estimates(33), differ from the later estimates $X_e = .70 \quad Y_e = .28 \quad \mathcal{U}_e = .02$ which were used for purposes of comparison in the present work. (See footnote on page 64).

(c) The value of the "effective width" (Γ) of the N^{14} reaction used in the present work (288 e.v. as against 60 e.v.).

The differences mentioned under (b) and (c) arise out of the more recent investigations of the spectroscopic data on the sun and of the cross-sections of the nuclear reactions, and need no further justification at this time. It remains only to discuss the importance of taking into account the variation of the guillotine factor with temperature and density, and the effect of scattering opacity mentioned in (a). Two questions have to be answered in this respect. What is the ratio between total mean opacity (absorptive plus scattering) and the absorptive mean opacity for the temperatures and densities which occur in solar models? Does the empirical formula (7-2) for the total mean opacity used in the present work give reasonably good representation of the total mean opacity as calculated from Morse's tables? To answer these questions the two quantities $\bar{\kappa}/\bar{\kappa}_a$ (Morse) and $\bar{\kappa}$(Morse)$/\bar{\kappa}$ (7-2) have been computed for a number of points ($\log T$, $\log \rho(1+X)$) in each of the four models constructed (μ_c/μ_e = 1.0, 1.089, 1.18, 1.32). The values are listed below (Table 7).

TABLE 7

	$\log T$	$\log \rho(1+X_e)$	$\bar{\kappa}/\bar{\kappa}_a$ (Morse)	$\bar{\kappa}$(Morse)$/\bar{\kappa}$ (7-2)
$\mu_c/\mu_e = 1$	6.6	.5	1.11	1.52
	6.7	1.0	1.12	1.44
	6.9	1.5	1.22	1.01
	7.0	2.0	1.31	.82
	7.1	2.0	1.51	.83
Interface	7.16	2.19	1.64	.80
$\mu_c/\mu_e = 1.089$	6.8	1.5	1.22	1.15
	6.9	1.5	1.30	1.00
	6.9	2.0	1.24	.89
	7.0	2.0	1.38	.81
	7.1	2.0	1.70	.86
	7.1	2.5	1.51	.67
	7.2	2.5	1.91	.81
Interface	7.15	2.25	1.82	.84

$\mu_c/\mu_e = 1.18$	6.8	1.5	1.27	1.07
	6.9	1.5	1.46	1.00
	6.9	2.0	1.35	.86
	7.0	2.0	1.62	.84
	7.1	2.0	2.22	1.03
	7.2	2.5	1.89	.75
	7.2	2.5	2.63	1.01
Interface	7.5	2.4	2.24	1.16
$\mu_c/\mu_e = 1.32$	6.8	1.5	1.32	1.07
	6.9	1.5	1.56	1.03
	6.9	2.0	1.41	.89
	7.0	2.0	1.79	.89
	7.1	2.0	2.51	1.09
	7.1	2.5	2.11	.80
Interface	7.2	2.5	3.72	1.17

An examination of the above figures shows that the error due to neglect of scattering opacity, already considerable in the homogeneous model (64% at the interface), becomes quite serious in non-homogeneous models (82%, 124%, 372% at the interfaces). The necessity of removing this source of error as far as possible is clear. The extent to which the present empirical formula (7-2) does so can be seen from the figures in the fourth column which show that the errors introduced by the formula vary from 52% to 33%. These extreme errors, however, occur in the cooler regions of the envelopes where their effect on the final results is less serious (cf. remarks on page 35). It is satisfactory to note that the errors at the interfaces, the significant part of the envelope as far as final results are concerned, nowhere exceed 20%. It is of interest to consider the effect of a 20% error in $\overline{K}$ on the chemical composition of the models. This effect was studied by correcting the constant K' in (7-2) in accordance with the results listed in Table 7 and using the new value of K' to compute the chemical composition.

The results obtained were as follows:

$\mu_c/\mu_e = 1$ $\quad K'(\text{new}) = .79\ K'(\text{old})$.

$X(\text{new}) = 67.8\%$. $\quad U(\text{new}) = 3.3\%$.

$X(\text{old}) = 66.2\%$. $\quad U(\text{old}) = 2.7\%$.

$\mu_c/\mu_e = 1.089$ $\quad K'(\text{new}) = .79\ K'(\text{old})$.

$X_e(\text{new}) = 75.4\%$. $\quad U(\text{new}) = 2.3\%$.

$X_e(\text{old}) = 75.0\%$. $\quad U(\text{old}) = 2.1\%$.

$X_c(\text{new}) = 64.5\%$.

$X_c(\text{old}) = 63.0\%$.

$\mu_c/\mu_e = 1.18$ $\quad K'(\text{new}) = 1.26\ K'(\text{old})$

$X_e(\text{new}) = 83.3\%$. $\quad U(\text{new}) = 1.5\%$.

$X_e(\text{old}) = 85.3\%$. $\quad U(\text{old}) = 1.2\%$.

$X_c(\text{new} = 59.7\%$.

$X_c(\text{old}) = 61.2\%$.

$\mu_c/\mu_e = 1.2$. K' (new) $= 1.26$ K' (old),

X_e(new) $= 92.9\%$. $\mathcal{U}$(new) $= .6\%$.

X_e(old) $= 94.2\%$. $\mathcal{U}$(old) $= .8\%$.

X_c(new) $= 56.7\%$.

X_c(old) $= 57.2\%$.

It is seen that the errors in X_e, X_c do not amount to more than $\pm 2\%$ and the errors in $\mathcal{U}_e$ to more than $\pm .3\%$ for non-homogeneous, $+.6\%$ for homogeneous models.

The foregoing remarks indicate that the mass-luminosity relation (22-1) based upon the expression (7-2) for $\bar{K}$ is reasonably accurate. This means that it is now possible to estimate the permissible variation in the effective width (Γ) of the N^{14} reaction. For Γ ought to have such a value that the value of X_e computed from the mass luminosity relation and the energy production equation should not exceed unity. It is of interest to investigate the limits imposed on Γ by these two requirements, viz., mass-luminosity relation as in (22-1) and $X_e' < 1$. The following procedure was adopted. Several values were assigned in turn to $\mathcal{U}$, ranging from .34% to 4%. Using the mass-luminosity relation (22-1) the corresponding values of X were obtained. These values of X and $\mathcal{U}$ were used to evaluate the constant on the right-hand side of the energy generation equation (27-1). The factor by which this constant was increased is a measure of the permissible increase in Γ (the only empirical constant in the right-hand side of (27-1), excepting h_N, the ratio of nitrogen to heavy element abundance) for the selected value of $\mathcal{U}$. The results are tabulated below.

μ	X	Y	Fractional increase in Γ
.34%	99.66%	0. %	501
.5%	93.1%	6.4%	164
.6%	90. %	9.4%	95.5
.7%	87.2%	12.1%	58.8
1.0%	81.3%	17.7%	20.1
1.2%	78.4%	20.4%	11.67
2.0%	70.5%	27.5%	2.69
3.0%	64.8%	32.2%	.74
4.0%	60.7%	35.3%	.306

It is seen that a change in the value of μ from 3.0% to 1.0% is effected by a change in Γ of from 20 to .74 times the value adopted in the present work. Since spectroscopic estimates of μ might easily lie anywhere between 1.0% and 3.0%, the restriction placed on possible values of Γ is not very severe. In other words, as far as the above theoretical considerations and the spectroscopic evidence go, the experimental value of Γ could vary by a factor of 20. To this extent the consideration raised in (c) (page 68) is not decisive as between the results of the present work and those of Ledoux.

An examination of Ledoux's results shows that a fitting of core to envelope without the necessity for a transition zone of variable μ can be made for values of μ_c/μ_e less than 1.374. At $\mu_c/\mu_e = 1.374$ the mass and extent of such a zone are still so small that its introduction does not alter the boundary values of the variables of integration at the interface appreciably, and hence cannot affect the values of X_c, Y_c, T_c, ρ_c to any noticeable extent. This is a further justification of the adoption in Part II of the present work of models in which a sharp discontinuity of μ exists.

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APPENDIX I

THE MEAN MOLECULAR WEIGHT OF HIGHLY IONIZED STELLAR MATERIAL.

We wish to determine the appropriate molecular weight (μ) that has to be used in the equation of state adopted:

$$p = \frac{\mathcal{R}}{\mu} \rho T.$$

In a gas of molecular weight μ and density ρ , the number of particles per unit volume is clearly given by

$$n = \frac{\rho}{\mu m_H}, \tag{1}$$

where m_H is the mass of a hydrogen atom.

Suppose we have a mixture of elements and that an element of atomic number Z occurs with an abundance factor x_z --in other words, 1 gram of the material contains x_z grams of the element. Let us suppose further that each atom of the element contributes, on the average, $\bar{n}_z$ free particles per unit atomic weight, i.e., if A is the atomic weight, then each atom contributes $A\,\bar{n}_z$ free particles. We then have

$$n = \frac{\rho}{m_H} \sum x_z \bar{n}_z , \tag{2}$$

where the summation is extended over all the elements.

Comparing (1) and (2), we find

$$\mu = \frac{1}{\sum x_z \bar{n}_z}.$$

The determination of μ involves, therefore, a specification of the state of ionization of the stellar material at a prescribed temperature and density. Let us suppose that the conditions are such that the ionization is complete, so that an element of atomic number Z and atomic weight A gives rise to $Z+1$ particles. Then

$$\bar{n}_z = \frac{Z+1}{A}. \tag{3}$$

It is well known that except for the lightest elements (hydrogen and helium), the ratio $\bar{n}_z$ as defined in (3), is approximately 1 : 2. Hence, if we assume that in 1 gram of stellar material there are X grams of hydrogen, U grams of the heavy elements and ($1-X-U$) grams of helium, we can write

$$\bar{n}_1 = 2 \ , \ \bar{n}_2 = \frac{3}{4}, \quad \bar{n}_z = \frac{1}{2} \quad Z > 2.$$

The expression for μ becomes

$$\mu = \frac{1}{2X + \frac{3}{4}(1-X-U) + \frac{1}{2}U}$$

$$= \frac{4}{(5X - U + 3)}.$$

APPENDIX II

Integration of equations (16-1), (16-2), (16-3) for log C = $\bar{5}\cdot37935$, $\bar{5}\cdot43846$, $\bar{5}\cdot54696$, $\bar{5}\cdot56496$.

log C = $\bar{5}\cdot37935$

t	u	τ	σ
1.05	.99999	.01029	.00001
1.10	.99999	.02059	.00020
1.15	.99998	.03088	.00096
1.20	.99993	.04118	.00293
1.25	.99983	.05147	.00693
1.30	.99964	.06177	.01399
1.35	.99933	.07206	.02535
1.40	.99885	.08234	.04241
1.45	.99821	.09264	.06680
1.50	.99732	.10291	.10016
1.55	.99622	.11320	.14457
1.60	.99481	.12345	.20183
1.65	.99313	.13372	.27437
1.70	.99110	.14395	.36434
1.75	.98873	.15418	.47423
1.80	.98591	.16438	.60643
1.85	.98291	.17458	.76357
1.90	.97939	.18473	.94817
1.95	.97564	.19488	1.1629
2.00	.97134	.20498	1.4104
2.05	.96682	.21507	1.6932
2.10	.96176	.22510	2.0140
2.15	.95647	.23512	2.3751
2.20	.95064	.24507	2.7792
2.30	.93803	.26487	3.7251
2.40	.92395	.28447	4.8687
2.50	.90809	.30385	6.2251
2.60	.89212	.32300	7.8060
2.70	.87457	.34187	9.6206
2.80	.85596	.36049	11.674
2.90	.83663	.37879	13.969
3.00	.81650	.39081	16.503
3.20	.77480	.43180	22.272

t	u	τ	σ	n
3.40	.73186	.46556	28.909	
3.60	.68854	.49788	36.309	
3.80	.64562	.52879	44.335	
4.00	.60367	.55829	52.846	
4.20	.56318	.58642	61.688	
4.40	.52446	.61318	70.725	
4.60	.48776	.63865	79.821	
4.80	.45318	.66285	88.869	
5.00	.42080	.68587	97.766	
5.20	.39060	.70774	106.44	
5.40	.36255	.72857	114.83	
5.60	.33655	.74833	122.89	
5.80	.31253	.76719	130.59	
6.00	.29035	.78510	137.91	
6.20	.26993	.80222	144.84	
6.40	.25110	.81850	151.38	
6.60	.23380	.83409	157.53	
6.80	.21785	.84894	163.30	
7.00	.20322	.86319	168.70	
7.20	.18971	.87678	173/75	1.847
7.40	.17733	.88985	178.46	1.777
7.60	.16588	.90234	182.86	1.707
7.80	.15538	.91437	186.95	1.641
8.00	.14566	.92589	190.76	1.575
8.20	.13673	.93701	194.29	1.512
8.40	.12845	.94767	197.59	1.450
8.60	.12084	.95798	200.64	1.392

t_i = 8.217 $\qquad U_i$ = 2.598

u_i = .1353

τ_i = .9384 $\qquad V_i$ = 1.185

σ_i = 195.0

n_i = 1.5

$\log C = \bar{5}.43846$

t	u	τ	σ
1.05	.99999	.01029	.00001
1.10	.99999	.02059	.00018
1.15	.99998	.03088	.00088
1.20	.99994	.04118	.00266
1.25	.99985	.05147	.00628
1.30	.99968	.06176	.01269
1.35	.99939	.07206	.02299
1.40	.99896	.08233	.03847
1.45	.99837	.09264	.06053
1.50	.99758	.10288	.09086
1.55	.99657	.11319	.13106
1.60	.99530	.12342	.18313
1.65	.99377	.13372	.24894
1.70	.99193	.14393	.33071
1.75	.98977	.15417	.43052
1.80	.98722	.16439	.55073
1.85	.98449	.17458	.69366
1.90	.98129	.18477	.86175
1.95	.97788	.19491	1.0573
2.00	.97398	.20506	1.2830
2.05	.96986	.21514	1.5410
2.10	.96525	.22522	1.8340
2.15	.96043	.23524	2.1641
2.20	.95513	.24525	2.5340
2.30	.94366	.26511	3.4014
2.40	.93076	.28480	4.4526
2.50	.91683	.30428	5.7040
2.60	.90159	.32356	7.1669
2.70	.88550	.34257	8.8529
2.80	.86931	.36137	10.768
2.90	.85050	.37985	12.918
3.00	.83182	.39811	15.303
3.20	.79305	.43365	20.770
3.40	.75289	.46797	27.128
3.60	.71210	.50096	34.293
3.80	.67143	.53265	42.158
4.00	.63140	.56299	50.597
4.20	.59251	.59202	59.475
4.40	.55504	.61975	68.661
4.60	.51930	.64622	78.028
4.80	.48537	.67148	87.461
5.00	.45341	.69556	96.860

t	u	τ	σ	n
5.20	.42338	.71854	106.14	
5.40	.39533	.74044	115.23	
5.60	.36916	.76136	124.07	
5.80	.34484	.78130	132.62	
6.00	.32225	.80037	140.86	
6.20	.30133	.81856	148.75	
6.40	.28195	.83599	156.29	
6.60	.26404	.85263	163.47	
6.80	.24746	.86860	170.29	
7.00	.23215	.88387	176.75	
7.20	.21799	.89856	182.87	
7.40	.20490	.91261	188.65	
7.60	.19280	.92617	194.112	
7.80	.18160	.93914	199.26	
8.00	.17124	.95170	204.11	1.787
8.20	.16165	.96372	208.69	1.730
8.40	.15276	.97539	212.99	1.675
8.60	.14452	.98657	217.05	1.622
8.80	.13687	.99744	220.86	1.571
9.00	.12977	1.0079	224.46	1.521
9.20	.12317	1.0180	227.84	1.473
9.40	.11704	1.0278	231.03	1.427
9.60	.11132	1.0373	234.03	1.383

$t_i = 9.075$ $\qquad$ $\mathcal{U}_i = 2.369$

$u_i = .1275$

$\tau_i = 1.0113$ $\qquad$ $\mathcal{V}_i = 1.144$

$\sigma_i = 225.75$

$n_i = 1.5$

$\log C = \bar{5}\cdot 54696$

t	u	τ	σ
1.05	.99999	.01029	.00001
1.10	.99999	.02059	.00015
1.15	.99999	.03088	.00073
1.20	.99995	.04118	.00222
1.25	.99987	.05147	.00526
1.30	.99973	.06177	.01062
1.35	.99949	.07206	.01924
1.40	.99914	.08235	.03220
1.45	.99865	.09264	.05069
1.50	.99798	.10292	.07605
1.55	.99714	.11320	.10975
1.60	.99607	.12347	.15335
1.65	.99479	.13373	.20854
1.70	.99325	.14399	.27709
1.75	.99147	.15423	.36087
1.80	.98940	.16446	.46183
1.85	.98706	.17467	.58199
1.90	.98443	.18487	.72341
1.95	.98151	.19504	.88820
2.00	.97829	.20520	1.0785
2.05	.97477	.21533	1.2965
2.10	.97095	.22543	1.5443
2.15	.96683	.23551	1.8241
2.20	.96241	.24555	2.1380
2.30	.95271	.26553	2.8766
2.40	.94187	.28537	3.7759
2.60	.91706	.32451	6.1155
2.70	.90323	.34378	7.5808
2.80	.88856	.36284	9.2564
2.90	.87314	.38167	11.149
3.00	.85706	.40025	13.266
3.20	.82329	.43664	18.176
3.40	.78795	.47195	23.982
3.60	.75171	.50610	30.644
3.80	.71515	.53908	38.096
4.00	.67878	.57086	46.251
4.20	.64301	.60144	55.006
4.40	.60817	.63084	64.252
4.60	.57451	.65906	73.879
4.80	.54222	.68615	83.778

t	u	τ	σ	n
5.00	.51141	.71213	93.850	
5.20	.48217	.73705	104.00	
5.40	.45452	.76094	114.15	
5.60	.42847	.78386	124.23	
5.80	.40398	.80584	134.18	
6.00	.38102	.82693	143.95	
6.20	.35953	.84718	153.51	
6.40	.33944	.86662	162.81	
6.60	.32069	.88531	171.85	
6.80	.30319	.90327	180.59	
7.00	.28688	.92056	189.05	
7.20	.27166	.93719	197.19	
7.40	.25750	.95323	205.04	
7.60	.24428	.96867	212.58	
7.80	.23199	.98359	219.82	
8.00	.22050	.99797	226.77	
8.20	.20983	1.0119	233.44	
8.40	.19983	1.0253	239.83	
8.60	.19054	1.0383	245.96	
8.80	.18183	1.0509	251.82	
9.00	.17373	1.0631	257.45	
9.20	.16612	1.0749	262.83	
9.40	.15904	1.0864	268.01	
9.60	.15238	1.0975	272.95	
9.80	.14617	1.1083	277.71	
10.00	.14033	1.1188	282.26	
10.20	.13487	1.1290	286.64	1.674
10.40	.12973	1.1390	290.84	1.644
10.60	.12492	1.1486	294.88	1.613
10.80	.12038	1.1581	298.76	1.584
11.00	.11613	1.1672	302.50	1.557
11.20	.11211	1.1763	306.09	1.531
11.40	.10834	1.1849	309.56	1.504
11.60	.10477	1.1936	312.89	1.481

t_i = 11.437 $\qquad U_i$ = 1.926

u_i = .10765

τ_i = 1.1867 $\qquad V_i$ = 1.0375

σ_i = 310.2

n_i = 1.5

$\log C = \bar{5}.56496$

t	u	τ	σ
1.05	.99999	.01029	.00001
1.10	.99999	.02059	.00015
1.15	.99999	.03088	.00071
1.20	.99995	.04118	.00216
1.25	.99988	.05147	.00510
1.30	.99974	.06177	.01031
1.35	.99950	.07206	.01868
1.40	.99915	.08235	.03126
1.45	.99987	.09264	.04921
1.50	.99803	.10292	.07384
1.55	.99721	.11320	.10656
1.60	.99618	.12347	.14889
1.65	.99493	.13374	.20248
1.70	.99344	.14399	.26905
1.75	.99171	.15424	.35043
1.80	.98970	.16446	.44850
1.85	.98743	.17468	.56523
1.90	.98487	.18488	.70263
1.95	.98204	.19506	.86277
2.00	.97891	.20521	1.0477
2.05	.97549	.21535	1.2596
2.10	.97177	.22546	1.5006
2.15	.96777	.23554	1.7728
2.20	.96348	.24559	2.0782
2.30	.95404	.26559	2.7969
2.40	.94350	.28545	3.6729
2.50	.93192	.30514	4.7208
2.60	.91936	.32465	5.9541
2.70	.90589	.34396	7.3846
2.80	.89159	.36306	9.0221
2.90	.87656	.38193	10.874
3.00	.86087	.40057	12.947
3.20	.82789	.43709	17.764
3.40	.79333	.47254	23.474
3.60	.75783	.50687	30.045
3.80	.72196	.54005	37.416
4.00	.68620	.57205	45.506
4.20	.65098	.60288	54.219
4.40	.61660	.63253	63.449
4.60	.58334	.66104	73.091
4.80	.55136	.68841	83.036
5.00	.52080	.71471	93.189

t	u	τ	σ	n
5.20	.49174	.73993	103.45	
5.40	.46421	.76416	113.75	
5.60	.43823	.78739	124.01	
5.80	.41376	.80972	134.17	
6.00	.39079	.83113	144.17	
6.20	.36925	.85173	153.99	
6.40	.34909	.87150	163.57	
6.60	.33023	.89053	172.91	
6.80	.31262	.90882	181.97	
7.00	.29617	.92645	190.75	
7.20	.28082	.94341	199.24	
7.40	.26649	.95978	207.43	
7.60	.25313	.97555	215.33	
7.80	.24065	.99079	222.94	
8.00	.22902	1.0055	230.25	
8.20	.21815	1.0197	237.30	
8.40	.20801	1.0335	244.05	
8.60	.19853	1.0468	250.56	
8.80	.18967	1.0597	256.79	
9.00	.18138	1.0722	262.80	
9.20	.17364	1.0843	268.55	
9.40	.16639	1.0901	274.09	
9.60	.15959	1.1075	279.40	
9.80	.15322	1.1186	284.52	
10.00	.14724	1.1294	289.43	
10.20	.14163	1.1399	294.16	
10.40	.13636	1.1501	298.71	
10.60	.13141	1.1600	303.09	1.676
10.80	.12675	1.1697	307.31	1.648
11.00	.12237	1.1791	311.38	1.621
11.20	.11824	1.1884	315.31	1.596
11.40	.11435	1.1973	319.10	1.571
11.60	.11068	1.2062	322.76	1.549
11.80	.10722	1.2147	326.30	1.526
12.00	.10394	1.2232	329.72	1.506
12.20	.10086	1.2313	333.04	1.484

$t_i = 12.05$ $\qquad U_i = 1.831$

$u_i = .1032$

$\tau_i = 1.2252$ $\qquad V_i = 1.014$

$\sigma_i = 330.6$

$n_i = 1.5$

FIG. I (SEE PAGE 6)

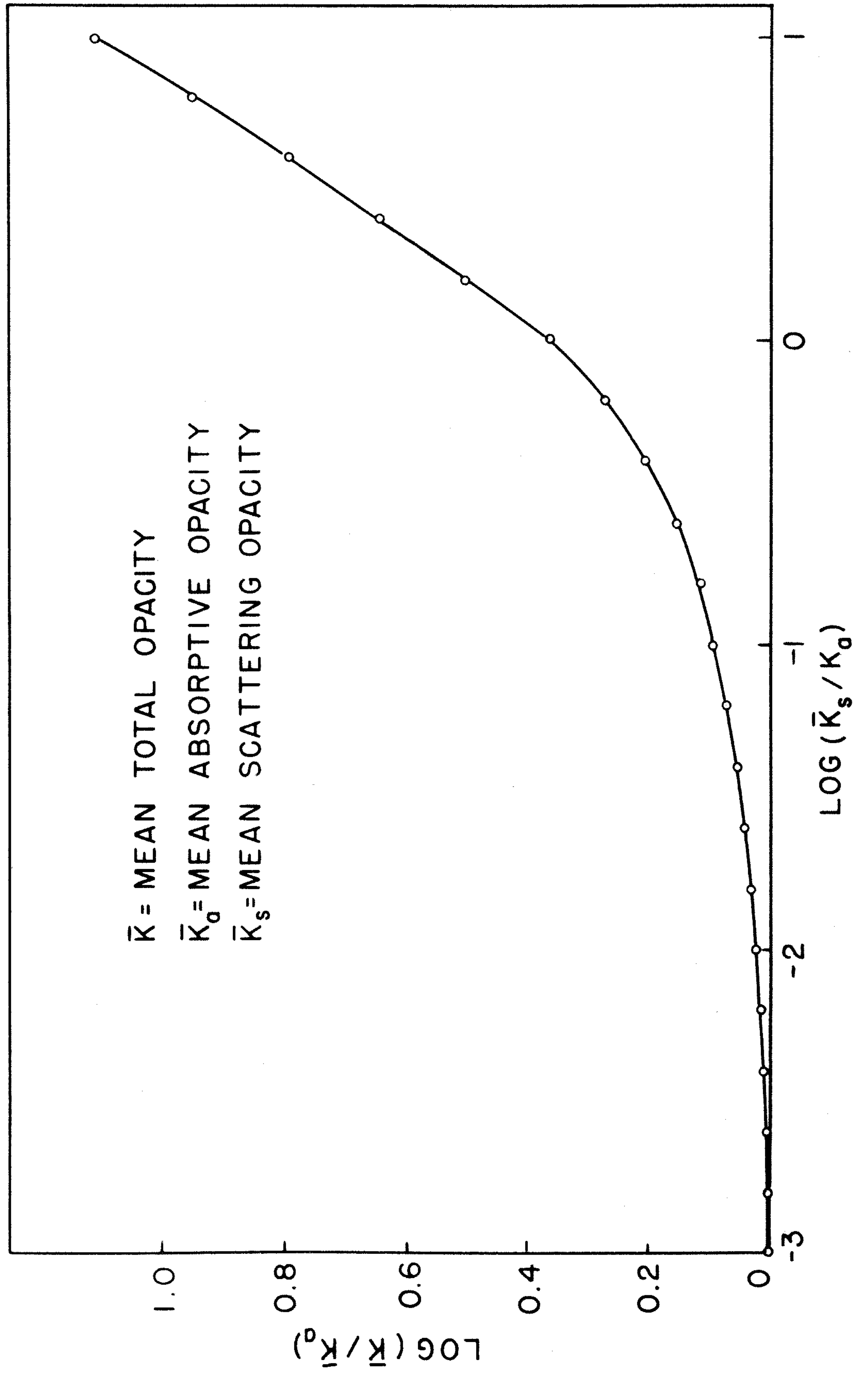

FIG. 2(b) (SEE PAGE 26)

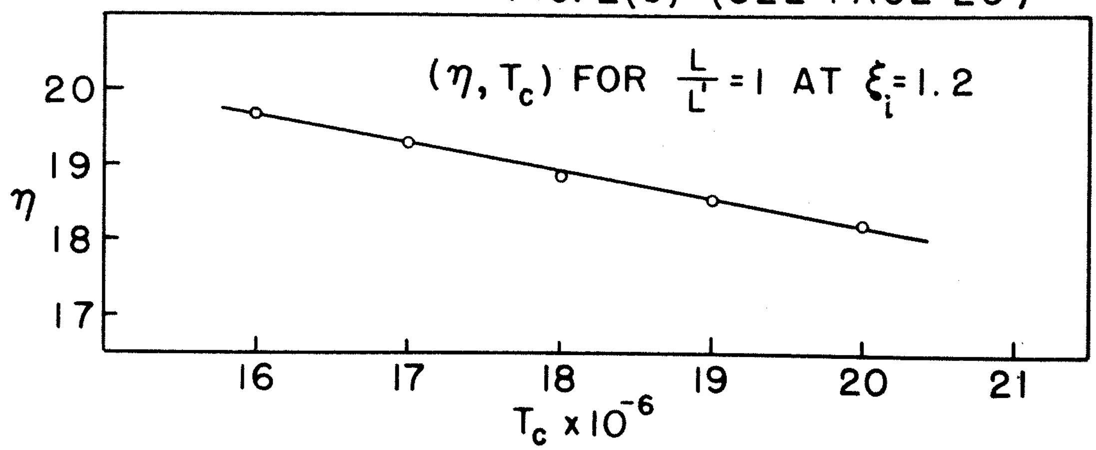

FIG. 2(a)

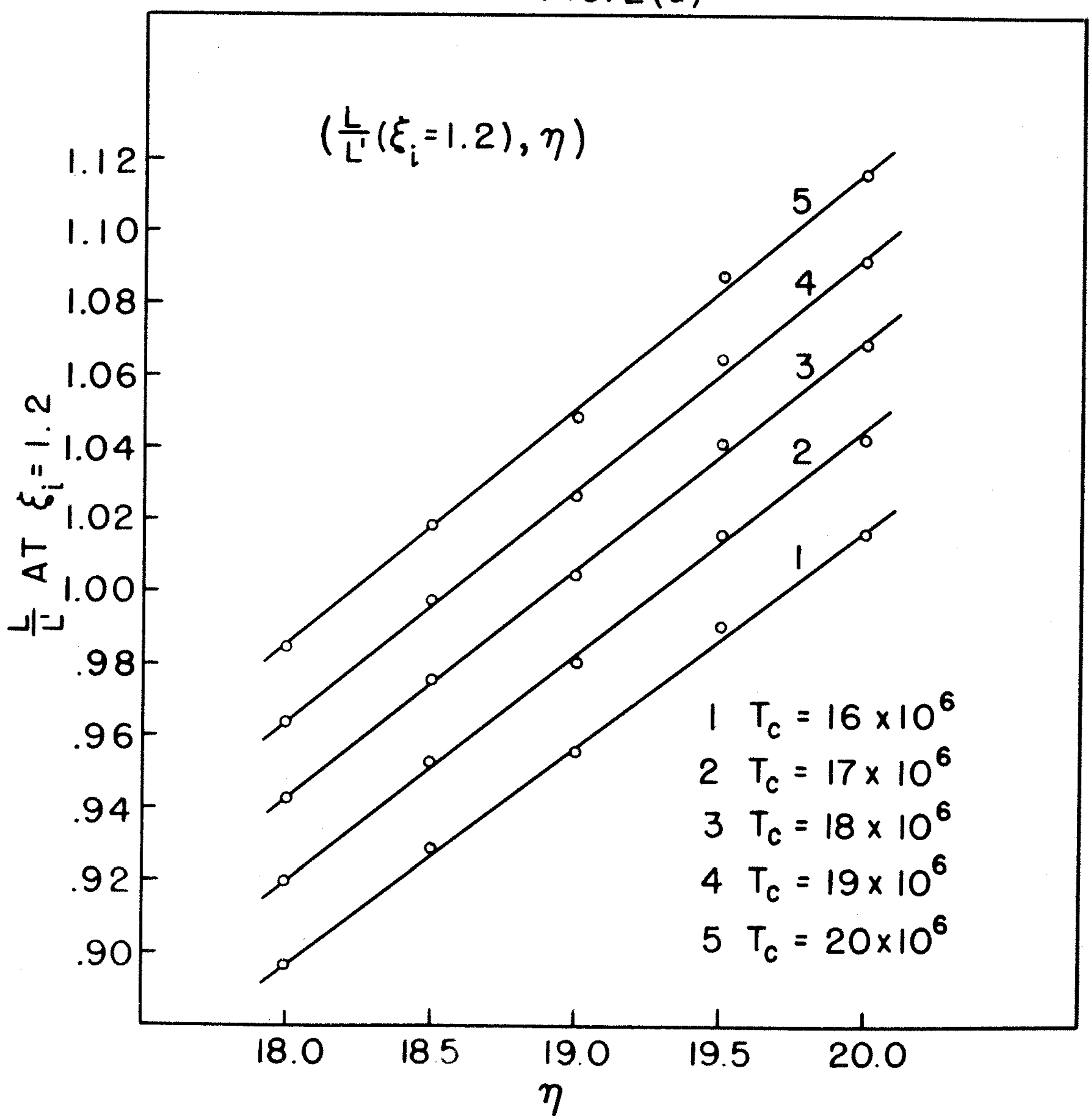

FIG. 3 (SEE PAGE 32)

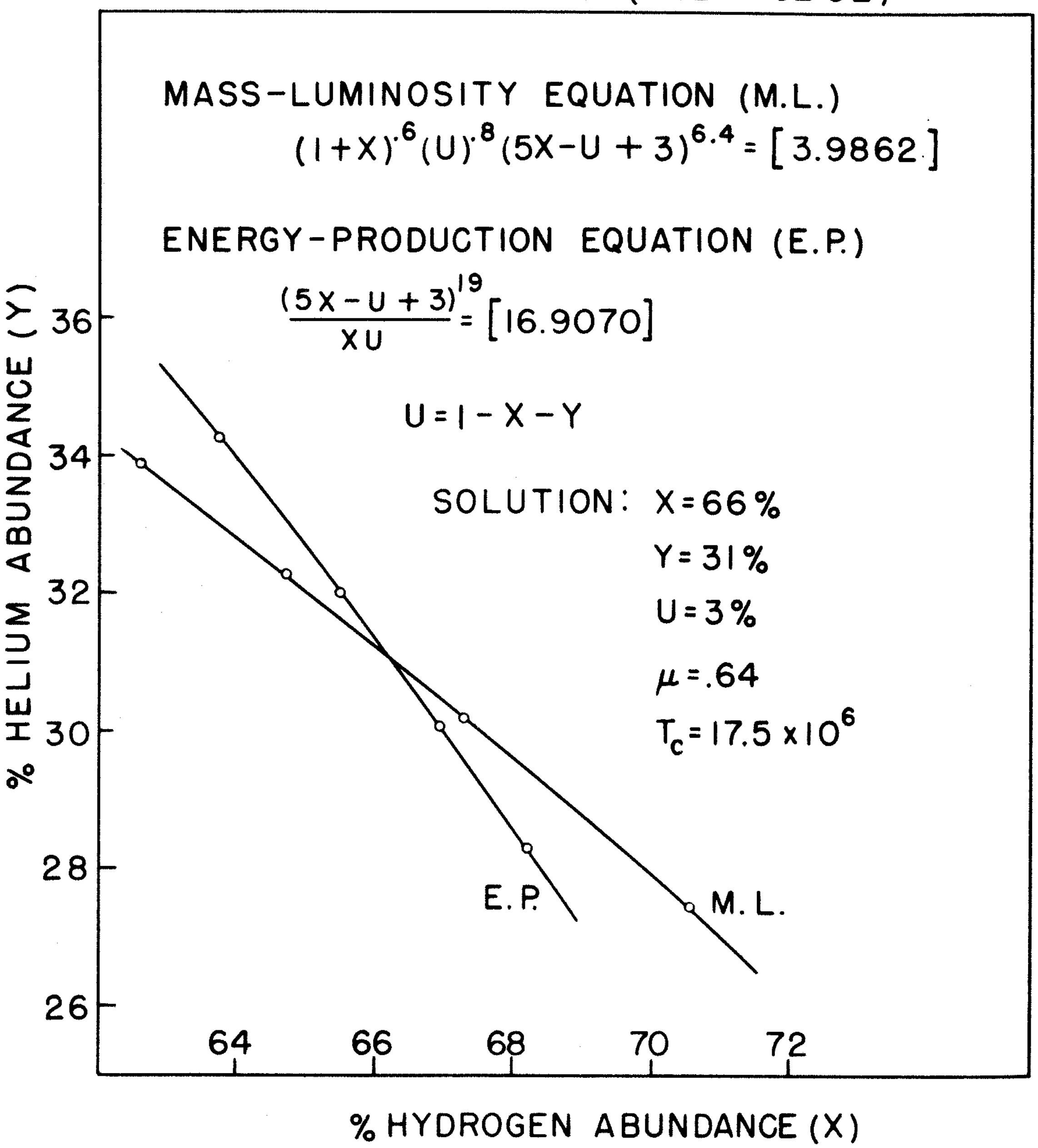

FIG. 4 (SEE PAGE 33)