S T A B I L I T Y   D E R I V A T I V E S

O F

H E L I C O P T E R   R O T O R S

by

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In Partial Fulfilment
of the
Requirements for the Master of Science
Degree.

California Institute of Technology
Pasadena, Calif.

1938.

On my arrival in the United States, Dr. von Karman was asked by the Commonwealth Fund to be my adviser during my stay in this country. It is difficult for me, in this short space, to express adequately my deep sense of gratitude to him. He has stood to me more in loco parentis than as a supervisor, and I have always felt that I could turn to him for helpful advice, when needed, not only on academic subjects, but on questions of outside interest.

One of the most pleasant memories of my twenty months stay in America, will be the extreme friendliness of all those connected with the Institute. I was made very welcome by the staff of the Guggenheim Aeronautical Laboratory, and I should like to thank Drs. Millikan, Klein, and Sechler for the way in which they made me feel at home. I am well aware that I shall return to England having received more than an education in aeronautical engineering; having learnt a great deal about those who live and work in this country. I should like to take this opportunity, the only public one afforded to me, of thanking those responsible.

To Dr. von Karman's faith in the future of Helicopters, I owe the inspiration for this study, and I thank him for the interest he has shown in the whole work. Mr. George Schairer, of the Consolidated Aircraft Corporation at San Diego, has spent some of his valuable time in discussing several of the problems with me, for which I am properly grateful.

I should also like to thank Mrs. Vorkink for typing this thesis; the responsibility for typographical mistakes is entirely mine.

PRÉCIS.

At present there are only three published papers dealing with the stability of helicopters, all based on helicopters with rigid rotors. Since all rotors are, in practice, constructed with the blades free to "flap", these analyses are not sufficiently general.

Below, the blade motion equation has been written in terms of small incremental accelerations and velocities; for which an approximate solution is obtained. This is then used to calculate the actual forces and moments that are produced on the rotor; by taking suitable mean values, it has been found possible to express these forces and moments without reference to the actual position of the blades at that instant, but only as functions of the position of the whole helicopter. The aerodynamic mechanism of the rotor has therefore been expressed ~~as~~ in a series of equations, and the stability of the helicopter now becomes solely a dynamical problem.

It appears that a helicopter is most likely to become unstable when hovering; this analysis is confined to this case.

Calculated values for a helicopter of the same size as the "C.30" autogyro, are given to show the dimensions of the various forces and moments.

C O N T E N T S .

INTRODUCTION.

For some time it has been known that it is theoretically possible to design rotary wing aircraft that would have superior performance to the conventional airplane at both the high and low speed ends of the scale. The success of de la Cierva, and his Autogyro, has bridged the gap between the airplane and the helicopter; but, as yet, there have been few actual helicopters built. However, it seems that in a very short time we shall see many of these machines flying: considerable research is now being carried on in all countries, and great strides have been made in the last two or three years towards a successful helicopter. The recent flight of Fraulein Reitsh in an indoor hall, with the Focke-Wolfe F.W.61 has impressed many people; and, certainly, the machine seemed to be under perfect control.

The stability of such a machine presents a very complex mathematical problem, which accounts for the lack of published work on the subject. The Kellett Autogyro Corporation calculate the stability of their autogyros by means of a rather inaccurate method, based on the experimental relations obtained by Wheatley in the N.A.C.A. tunnels on the tilt of the blades of autogyro rotors at various forward speeds. This method breaks down entirely for hovering helicopters, and we are forced to use some theoretical analysis. All three of the papers that have so far been published on this subject, have treated only rotors with rigidly attached blades. In practice, all rotors are made with several degrees of freedom for the blades, so that they are free to "flap": the analyses do not seem to be sufficiently general.

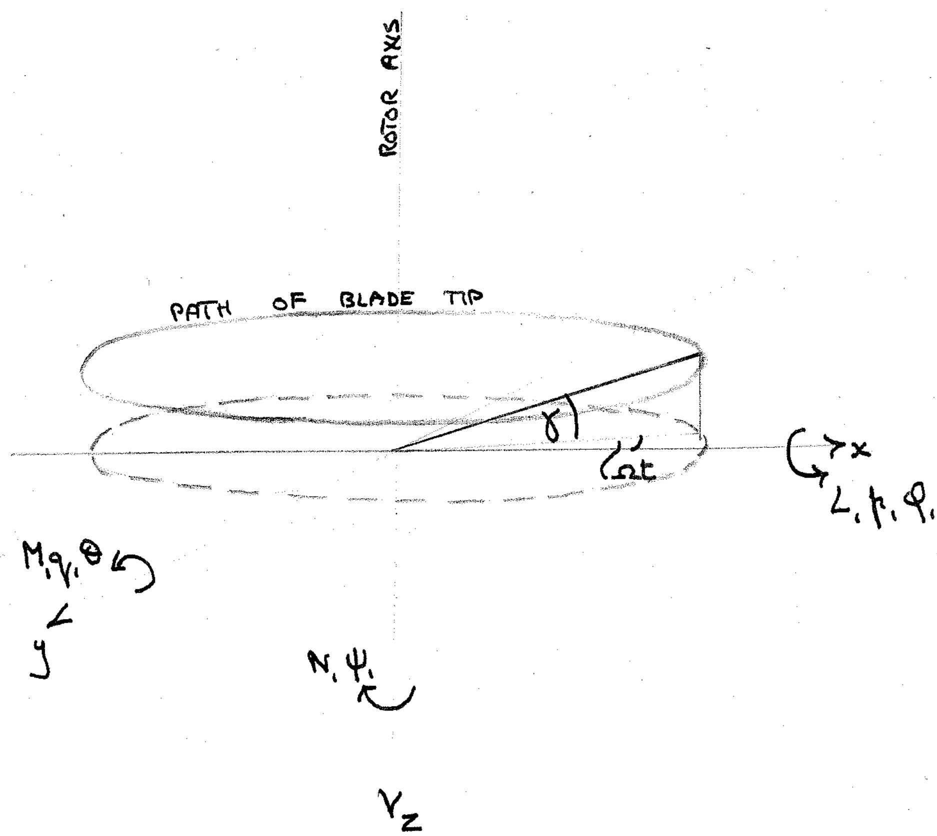
This analysis attempts to solve the case of hinged blades, and so to pave the way for a more general treatment of the whole problem.

The blades of the usual type of helicopter rotor are hinged about two sets of pins: about pins parallel to the rotor axis, to relieve the blade of bending stresses; and about pins perpendicular to this axis, to reduce the pitching and rolling moments. For the case of the autogyro, Wheatley has shown, both theoretically and experimentally, that the motion about the "vertical" pin is very small, and that it can be neglected. Breguet states that motion in the plane of the rotor disc is important in reducing the total drag of the helicopter; this is possibly true, but such motion has negligible effect on the behaviour of the whole helicopter, and we shall neglect it in this case. If the horizontal pins were exactly on the rotor axis, there would be no rolling and pitching moments transmitted to the helicopter during steady flight; but, due to mechanical difficulties, these pins are displaced slightly; this will be taken into account below.

Due to the action of the centrifugal, gravity, and thrust forces, the blades take up positions about the horizontal hinge so as to lie on the surface of a flat cone, the axis of which lies along the rotor axis during steady hovering. Lock, Schrenk, Wheatley, and others have shown that this cone distorts slightly during forward flight of an autogyro, but this change can be neglected in a first order theory. During small oscillations, the inertia forces acting on the blades tend to keep the blade cone axis vertical; however, if the rotor axis tilts, the angle of attack of the blade alters, and aerodynamic forces are produced, whose sense is to tilt this cone so as to make the two axes coincide again. It will be shown later, that in the absence of downwash, the blades would remain at the same angle to the rotor axis during any tilting of the axis: during forward motion, the blade cone axis tilts backwards very slightly, due to forward velocity; and very slightly sideways, due to forward acceleration. The cone becomes flatter from upward motion.

It is convenient to take the blade coning angle γ as measured from the horizontal plane through the centre of the rotor; therefore γ does not correspond with the angle β used by Lock and Wheatley, which is measured from the plane perpendicular to the rotor axis. The two angles only differ during tilting of the rotor axis.

The equation of motion of the whole helicopter will eventually be written down in terms of velocities and accelerations along Cartesian axes, and so we shall evaluate the resistance and moment derivatives in terms of the usual velocities $\dot{x}$, $\dot{y}$, $\dot{z}$, p , q . However, the actual blade motion equation will be written down in relation to a system of horizontal and vertical axes through the actual centre of the rotor, that are momentarily fixed in space, but they are revolving with uniform angular velocity Ω . Relative to these axes the blade moves only in a stationary plane, and the x , y , z , axes are cyclic. The final forces will not be based on these axes, so the whole analysis appears to be one of Ignoration of Co-ordinates.



NOTATION.

a	length of each blade
A_n	$\frac{n\rho}{2}\int_0^a C_L c r^n dr$.
b	distance of C.G. of blade from rotor axis measured along blade.
B_n	$\frac{n\rho}{2}\int_0^a \left(C_D + \frac{dC_L}{d\alpha}\right) c r^n dr$.
c	chord of blade element.
C_L C_D	lift and drag coefficients of blade element measured from hinge line.
C_n	$\frac{n\rho}{2}\int_0^a \left(C_L - \frac{dC_D}{d\alpha}\right) c r^n dr$.
$(D + \Delta D)'$	aerodynamic force per unit length of blade parallel to blade hinge.
D'	aerodynamic force per unit length of blade parallel to resultant wind velocity.
D_n	$\frac{n\rho}{2}\int_0^a C_D c r^n dr$.
I_b	total moment of inertia of all the blades about hinges.
$(L + \Delta L)'$, $(M + \Delta M)'$, $(N + \Delta N)'$,	moments acting on the helicopter per unit length of blade about the three fixed axes.
$\bar{L}$, $\bar{M}$, $\bar{N}$,	mean values of the total moments acting on the helicopter.
$L_u L_v L_w M_u M_v M_w N_u N_v N_w$	moment derivatives, based on the rate of change of moment with disturbance velocity per <u>unit</u> mass of helicopter.

m mass of blade per unit length.

M mass of helicopter.

M_b total mass of all blades.

n number of blades.

p , q , angular velocities about X , Y , axes.

P , P' , mean centre of rotation, and actual centre of rotor.

Q torque of rotor.

r radius of blade element.

t time.

T thrust of rotor.

$(T + \Delta T)'$ aerodynamic force per unit length of blade acting perpendicular to blade axis and perpendicular to hinge.

T' aerodynamic force per unit length of blade acting perpendicular to resultant wind velocity.

$(u + \Delta u)$ horizontal velocity of element perpendicular to blade axis.

w vertical downwash velocity from rotor.

$(w + \Delta w)$ vertical velocity of wind relative to blade element.

x , y , z , co-ordinates of P'.

$(X + \Delta X)'$, $(Y + \Delta Y)'$, $(Z + \Delta Z)'$, aerodynamic forces per unit length of blade in the directions of the x , y , z , axes.

$\overline{X}$, $\overline{Y}$, $\overline{Z}$, mean values of the total forces acting on the helicopter.

$X_u X_v X_w Y_u Y_v Y_w Z_u Z_v Z_w$

resistance derivatives, based on the rate of change of force with disturbance velocity per unit mass of helicopter.

$(\alpha - \alpha_0)$ angle of attack of blade element measured from the line of zero lift to the hinge line.

β phase angle of blade flapping motion relative to position.

γ blade angle measured from the blade axis to the horizontal.

Γ_u, Γ_ω, Γ_θ, Γ'_u, Γ'_ω

"Gamma Derivatives" giving change of blade angle with disturbances.

ε percentage of blade length of the distance between hinge and axis.

ρ air density in slugs/cubic foot.

θ, φ, ψ, angular displacements of rotor axis.

Ω angular velocity of rotor about axis.

ASSUMPTIONS

(1) That the helicopter is hovering.

(2) That the strip theory can be applied without corrections for the tip losses and blade interference.

(3) That we can neglect the radial component of velocity in computing the forces acting on a blade element.

(4) That the motion of the blade about the "vertical" hinge pin may be neglected.

(5) That the coning angle, γ, is small, and that we can replace its sine by the value of the angle, and its cosine by unity.

(6) That the vertical downwash velocity w , is constant over the disc, and does not vary during small disturbances.

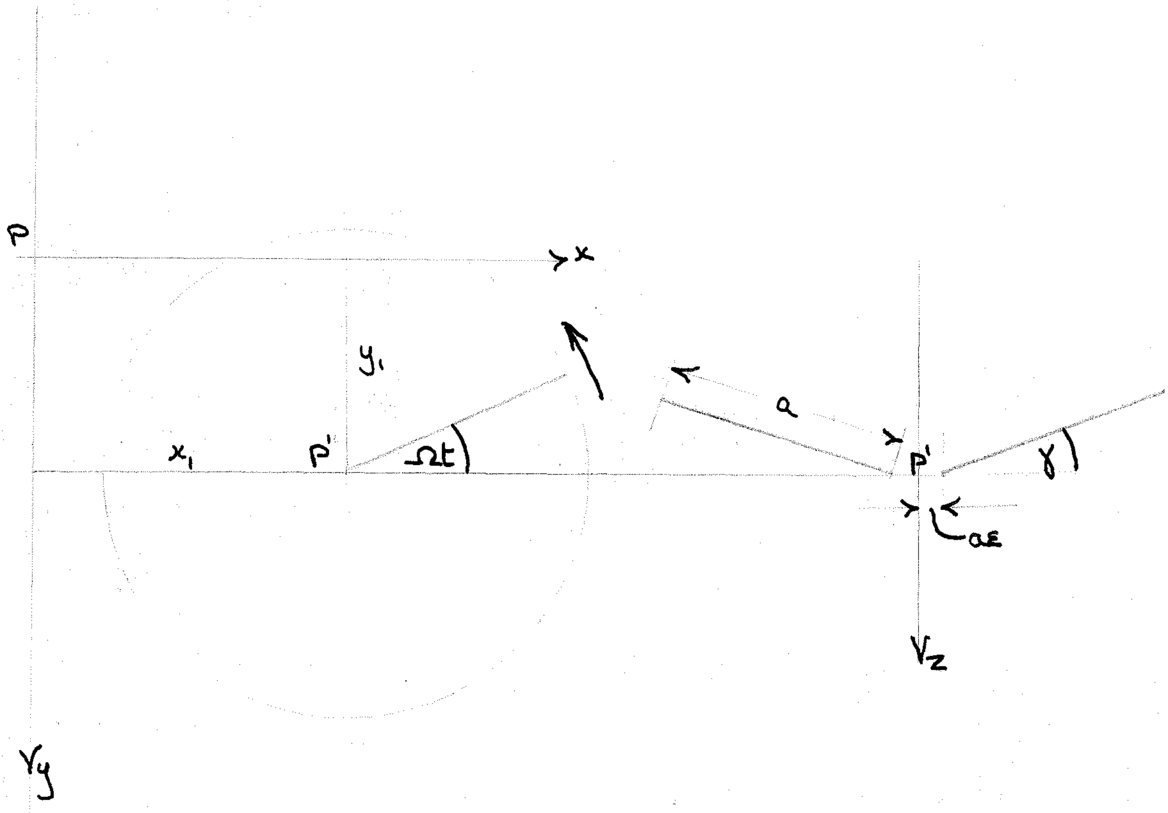
(7) That the inertia of the rotor and its driving mechanism is so great that the rotor does not alter its rate of rotation relative to the helicopter body during small oscillations.

(8) That the rate of rotation, Ω, is so much greater than the rates of angular rotation of the helicopter body, p, q, that we may take the mean values of the aerodynamic forces acting on the blade element throughout a cycle. This is equivalent to there being an infinite number of blades.

GEOMETRY OF SYSTEM

Since the helicopter is hovering, there is no forward direction, and we take the x , y , z, axes arbitrarily fixed in space through P, the mean centre of rotation. The z axis projects downwards.

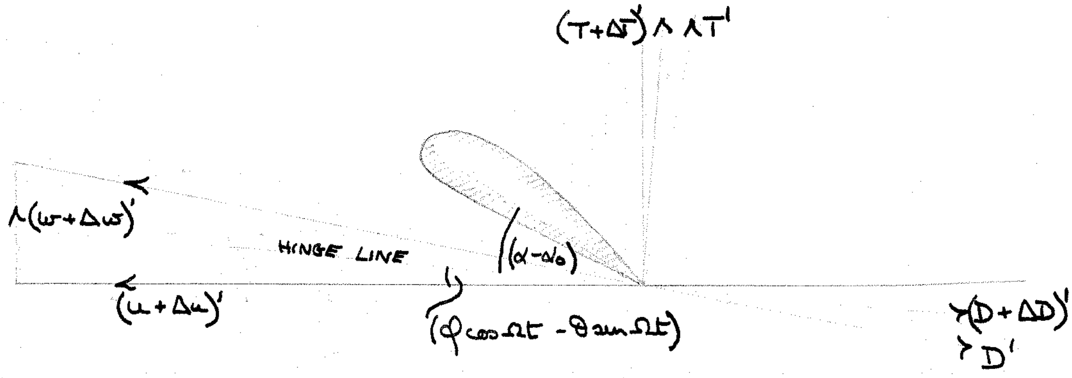
The usual notation for ϕ, θ, ψ, p, q, is adopted.
For all this work we assume a rotor that is rotating steadily with an angular velocity Ω in a positive direction from the y to the x axis.
The azimuth angle, Ωt, is measured from the positive x axis, on a plane perpendicular to the rotor axis.



The blade coning angle γ is measured from a horizontal plane through P'. The blades are pivoted about hinges that are not exactly on the axis, but are displaced by a distance $a\varepsilon$ from it. We assume ε to be so small that the resultant motion of the blades is unaffected, though rolling and pitching moments are transmitted through the small lever arm $a\varepsilon$.

We define C_L as the lift coefficient corresponding to the geometrical angle of attack of the blade element, which is that angle made by the line of zero lift with the hinge. In the same way C_D corresponds to the geometrical angle of attack.

The chord of the blade element is c.



The x, y, z, co-ordinates of the typical element are, referred to P,

$$x = x_1 + r\cos\Omega t$$

$$y = y_1 - r\sin\Omega t$$

$$z = z_1 - r\gamma$$

Let u be the horizontal velocity of the element perpendicular to the blade axis in steady hovering, and $(u+\Delta u)$ be the same velocity during small oscillations, then

$$u = \Omega r$$

u will contain terms in $\dot{x}_1$, $\dot{y}_1$, and will also have small terms due to the rate of tilt of the rotor axis.

$$(u+\Delta u) = \Omega r - \dot{x}_1\sin\Omega t - \dot{y}_1\cos\Omega t + r\gamma(\dot{\theta}\sin^2\Omega t - \dot{\phi}\cos^2\Omega t)$$

Now these last two terms are of a lower order than the terms due to horizontal velocity, and we can therefore neglect them, and any other terms which are introduced from axis tilt, for they are all of double frequency.

Let w be the steady downwash velocity, and $(w+\Delta w)$ the relative wind velocity perpendicular to the blade axis in a vertical plane during small oscillations; then, if $(w+\Delta w)$ is measured upwards,

$$(w+\Delta w) = w + r\dot{\gamma} - \dot{z},$$

The component of velocity parallel to the blade axis is of second order, and we shall neglect it.

In equilibrium the blade hinge is horizontal; but during small oscillations the main body of the helicopter tilts, and hence the angle of attack varies. If α is the angle that the element makes with the hinge line, measured from the line of zero lift, then during a small oscillation the "geometrical" angle of attack of the element becomes

$$\left(\alpha + \phi \cos \Omega t - \theta \sin \Omega t\right)$$

The "effective" angle of attack has therefore changed from

$$\left(\alpha - \frac{w}{u}\right) \text{ to } \left(\alpha + \phi \cos \Omega t - \theta \sin \Omega t - \frac{(w+\Delta w)}{(u+\Delta u)}\right)$$

The effective lift coefficient is

$$\left[C_L + \frac{dC_L}{d\alpha}\left(\phi \cos \Omega t - \theta \sin \Omega t - \frac{(w+\Delta w)}{(u+\Delta u)}\right)\right]$$

And effective drag coefficient is

$$\left[C_D + \frac{dC_D}{d\alpha}\left(\phi \cos \Omega t - \theta \sin \Omega t - \frac{(w+\Delta w)}{(u+\Delta u)}\right)\right]$$

AERODYNAMIC BLADE FORCES

Let D' be the force acting on the blade in the direction of the resultant wind velocity, per unit length of the blades;

let T' be the force per unit length acting perpendicular to the blade axis and the resultant wind velocity;

These forces are not perpendicular and parallel to the hinge, but are rotated through an angle

$$\left[\phi \cos \Omega t \quad - \theta \sin \Omega t \quad - \frac{(w + \Delta w)}{(u + \Delta u)}\right]$$

Let $(D + \Delta D)'$ be the force per unit length perpendicular to the blade axis and parallel to the hinge;

let $(T + \Delta T)'$ be the force per unit length perpendicular to the blade axis and perpendicular to the hinge;

We can resolve the forces T', D', into $(T + \Delta T)'$, $(D + \Delta D)'$.

Since the angle turned through is small, we can write its value for its sine, and unity for its cosine;

$$(T + \Delta T)' = T' - D' \left(\frac{(w + \Delta w)}{(u + \Delta u)} - \phi \cos \Omega t + \theta \sin \Omega t\right)$$

$$(D + \Delta D)' = D' + T' \left(\frac{(w + \Delta w)}{(u + \Delta u)} - \phi \cos \quad t + \theta \sin \quad t\right)$$

T', D', are given by the expressions-

$$T' = \frac{\rho c}{2} \left[C_L + \frac{dC_L}{d\alpha} \left(\phi \cos \Omega t - \theta \sin \Omega t - \frac{(w + \Delta w)}{(u + \Delta u)}\right)\right] \left[(u + \Delta u)^2 + (w + \Delta w)^2\right]$$

$$D' = \frac{\rho c}{2} \left[C_D + \frac{dC_D}{d\alpha} \left(\phi \cos \Omega t - \theta \sin \Omega t - \frac{(w + \Delta w)}{(u + \Delta u)}\right)\right] \left[(u + \Delta u)^2 + (w + \Delta w)^2\right]$$

Inserting these values, and neglecting terms of a high order of small quantities, i.e. in $\phi^2, \theta^2, \frac{w}{u}\phi, \frac{w}{u}\theta$;

$$(T + \Delta T)' = \frac{\rho c}{2} \left[C_L + \left(C_D + \frac{dC_L}{d\alpha}\right)\left(\phi \cos\Omega t - \theta \sin \Omega t\right)\right] \left[(u + \Delta u)^2 + (w + \Delta w)^2\right]$$

$$- \frac{\rho c}{2} \left(C_D + \frac{dC_L}{d\alpha}\right)\left((u + \Delta u)(w + \Delta w) + \frac{(w + \Delta w)^3}{(u + \Delta u)}\right)$$

$$(D + \Delta D)' = \frac{\rho c}{2} \left[C_D - \left(C_L - \frac{dC_D}{d\alpha}\right)\left(\phi \cos\Omega t - \theta \sin \Omega t\right)\right] \left[(u + \Delta u)^2 + (w + \Delta w)^2\right]$$

$$+ \frac{\rho c}{2} \left(C_L - \frac{dC_D}{d\alpha}\right)\left((u + \Delta u)(w + \Delta w) + \left(\frac{w + \Delta w}{u + \Delta u}\right)^3\right)$$

The expansion of the terms containing $(u + \Delta u)$ presents certain difficulties, as u goes to zero at the origin, and Δu is constant over the disc. This is due to there being a point at which the forward velocity is zero, and hence the angle of attack becomes 90°. This singularity has not much physical significance, because it occurs very near the rotor axis, and at a point where the steady forward velocity is small, and the forces and moments smaller. In all usual plan forms of helicopter rotors, it is also a point where the blade chord is small, and the section poor aerodynamically. We can, therefore, treat Δu as being of a lower order than u over the effective part of the rotor disc area.

Expanding, and neglecting high orders of small quantities,

$$\left[(u+\Delta u)^2 + (w+\Delta w)^2\right] = u^2 + w^2 + 2u\Delta w + 2w\Delta u$$

$$= \left(\Omega^2 r^2 + w^2\right) - 2\Omega r\left(\dot{x}_1 \sin\Omega t + \dot{y}_1 \cos\Omega t\right) + 2w\left(r\dot{\gamma} - \dot{z}_1\right)$$

and

$$\left[(u + u)(w + \Delta w) + \left(\frac{w+\Delta w}{u+\Delta u}\right)^3\right]$$

$$= \frac{w}{\Omega r}\left(\Omega^2 r^2 + w^2\right) + \Omega r\left(1 + \frac{3w^2}{\Omega^2 r^2}\right)\left(r\dot{\gamma} - \dot{z}_1\right) - w\left(1 - \frac{w^2}{\Omega^2 r^2}\right)\left(\dot{x}_1 \sin\Omega t + \dot{y}_1 \cos\Omega t\right)$$

In all conventional helicopter rotors the rotational velocity is of the same order as w, and hence we can neglect the terms in $\frac{w^2}{\Omega^2 r^2}$, since they will be negligible when integrated over the rotor. In all subsequent work they will be omitted, therefore.

Substituting these expressions into the expansion for $(T+\Delta T)'$, and neglecting terms of a high order of small quantities,

$$\begin{aligned}(T+\Delta T)' = {} & \frac{\rho c}{2}\Omega^2 r^2\left(C_L - \frac{w}{\Omega r}\left(C_D + \frac{dC_L}{d\alpha}\right)\right) \\ & + \frac{\rho c}{2}\left(\varphi\cos\Omega t - \theta\sin\Omega t\right)\left(C_D + \frac{dC_L}{d\alpha}\right)\Omega^2 r^2 \\ & - \frac{\rho c}{2}\left(\dot{x}_1 \sin\Omega t + \dot{y}_1 \cos\Omega t\right)\left(2\Omega r C_L - w\left(C_D + \frac{dC_L}{d\alpha}\right)\right) \\ & + \frac{\rho c}{2}\left(r\dot{\gamma} - \dot{z}_1\right)\left(2wC_L - \Omega r\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\end{aligned}$$

In considering the resultant forces on the helicopter we shall require the vertical and two horizontal components of the forces T', D',.

Vertical force per unit length. $= \left[T' - D' \dfrac{(w + \Delta w)}{(u + \Delta u)} \right]$

$$= \frac{\rho c}{2} \Omega^2 r^2 \left(C_L - \frac{w}{\Omega r} \left(C_D + \frac{dC_L}{d\alpha} \right) \right)$$

$$+ \frac{\rho c}{2} \frac{dC_L}{d\alpha} \left(\phi \cos \Omega t - \theta \sin \Omega t \right) \Omega^2 r^2$$

$$- \frac{\rho c}{2} \left(\dot{x}_1 \sin \Omega t + \dot{y}_1 \cos \Omega t \right) \left(2 \Omega r C_L - w \left(C_D + \frac{dC_L}{d\alpha} \right) \right)$$

$$+ \frac{\rho c}{2} \left(r \dot{\gamma} - \dot{z}_1 \right) \left(2 w C_L - \Omega r \left(C_D + \frac{dC_L}{d\alpha} \right) \right)$$

Horizontal force in plane of blade axis per unit length.

$$= \left[T' - D' \frac{(w + \Delta w)}{(u + \Delta u)} \right] \gamma$$

Horizontal force perpendicular to blade axis per unit length.

$$= \left[D' + T' \frac{(w + \Delta w)}{(u + \Delta u)} \right]$$

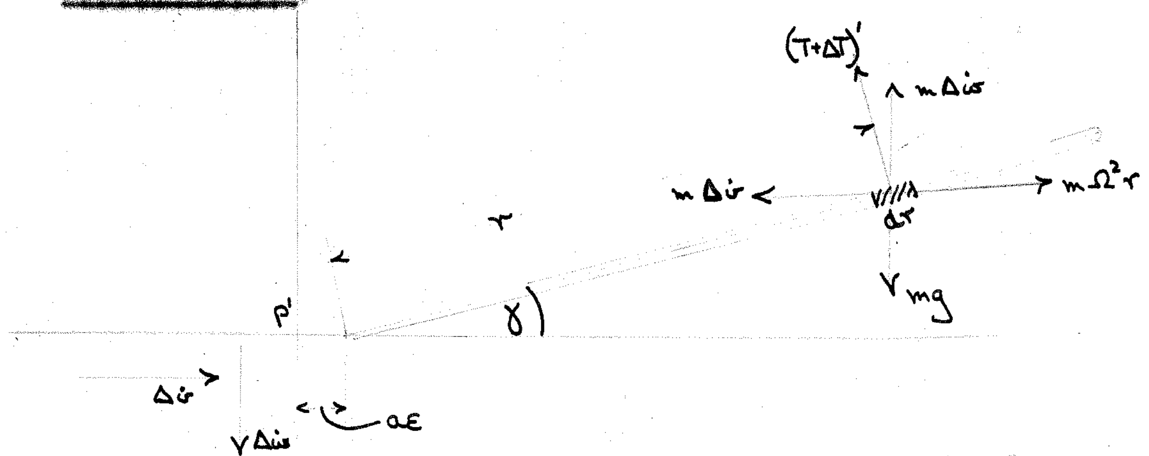
$$= \frac{\rho c}{2} \Omega^2 r^2 \left(C_D + \frac{w}{\Omega r} \left(C_L - \frac{dC_D}{d\alpha} \right) \right)$$

$$+ \frac{\rho c}{2} \frac{dC_D}{d\alpha} \left(\phi \cos \Omega t - \theta \sin \Omega t \right) \Omega^2 r^2$$

$$- \frac{\rho c}{2} \left(\dot{x}_1 \sin \Omega t + \dot{y}_1 \cos \Omega t \right) \left(2 \Omega r C_D + w \left(C_L - \frac{dC_D}{d\alpha} \right) \right)$$

$$+ \frac{\rho c}{2} \left(r \dot{\gamma} - \dot{z}_1 \right) \left(2 w C_D + \Omega r \left(C_L - \frac{dC_D}{d\alpha} \right) \right)$$

BLADE MOTION.



When the helicopter is hovering steadily, the blade angle γ assumes a constant value γ_0 that does not change round the cycle. It will change during small oscillations.

There are several definite integrals connected with the physical dimensions of the blades that will be met with; these are written lower case when they are summed over one blade only, and upper case if summed over all the n blades.

Let m be the mass per unit length of blade.

$$A_0 = n\frac{\rho}{2}\int_0^a C_L c\,dr \qquad D_0 = n\frac{\rho}{2}\int_0^a C_D c\,dr \qquad C_0 = n\frac{\rho}{2}\int_0^a\left(C_L - \frac{dC_D}{d\alpha}\right)c\,dr$$

$$A_1 = n\frac{\rho}{2}\int_0^a C_L cr\,dr \qquad D_1 = n\frac{\rho}{2}\int_0^a C_D cr\,dr \qquad C_1 = n\frac{\rho}{2}\int_0^a\left(C_L - \frac{dC_D}{d\alpha}\right)cr\,dr$$

$$A_2 = n\frac{\rho}{2}\int_0^a C_L cr^2\,dr \qquad D_2 = n\frac{\rho}{2}\int_0^a C_D cr^2\,dr \qquad C_2 = n\frac{\rho}{2}\int_0^a\left(C_L - \frac{dC_D}{d\alpha}\right)cr^2\,dr$$

$$A_3 = n\frac{\rho}{2}\int_0^a C_L cr^3\,dr \qquad D_3 = n\frac{\rho}{2}\int_0^a C_D cr^3\,dr \qquad C_3 = n\frac{\rho}{2}\int_0^a\left(C_L - \frac{dC_D}{d\alpha}\right)cr^3\,dr$$

$$B_{-1} = n\frac{\rho}{2}\int_0^a\left(C_D + \frac{dC_L}{d\alpha}\right)c\frac{dr}{r} \qquad B_2 = n\frac{\rho}{2}\int_0^a\left(C_D + \frac{dC_L}{d\alpha}\right)cr^2\,dr \qquad M_b = n\int_0^a m\,dr$$

$$B_0 = n\frac{\rho}{2}\int_0^a\left(C_D + \frac{dC_L}{d\alpha}\right)c\,dr \qquad B_3 = n\frac{\rho}{2}\int_0^a\left(C_D + \frac{dC_L}{d\alpha}\right)cr^3\,dr \qquad M_b b = n\int_0^a m\,r\,dr$$

$$B_1 = n\frac{\rho}{2}\int_0^a\left(C_D + \frac{dC_L}{d\alpha}\right)cr\,dr \qquad I_b = n\int_0^a mr^2\,dr$$

The dimensions of the above integrals are, respectively,

$M L^{-2}$		B_{-1}			
$M L^{-1}$	A_0	B_0	C_0	D_0	
M	A_1	B_1	C_1	D_1	M_b
$M L$	A_2	B_2	C_2	D_2	
$M L^2$	A_3	B_3	C_3	D_3	I_b

In setting up the blade motion equation, we have to consider both the velocities and accelerations of the rotor hub. These introduce effective forces.

We use a rectangular system of axes through the blade hinge that are parallel and perpendicular to the hinge line, and that are rotating steadily with angular velocity Ω, so that the blade moves only in a plane that is stationary with respect to this system of axes. Relative to this new system, the x, y, axes are cyclic.

Δw is the downward velocity of the rotor hub, and hence the downward acceleration of the hub, $\Delta \dot{w}$, is $\ddot{z}_1$.

Δv is the outward radial velocity, then

$$\Delta v = \left(\dot{x}_1 \cos \Omega t - \dot{y}_1 \sin \Omega t\right)$$

Differentiating, we get the acceleration in this direction,

$$\Delta \dot{v} = \left(\ddot{x}_1 \cos \Omega t - \ddot{y}_1 \sin \Omega t\right) - \Omega\left(\dot{x}_1 \sin \Omega t + \dot{y}_1 \cos \Omega t\right)$$

There will be effective accelerational forces due to these two accelerations, and there will also be small Coriolis forces due to the cross product of the velocities and the rate of tilt of the rotor axis. However, all these accelerations are of second order in $\dot{\phi}$, $\dot{\theta}$, $\ddot{x}_1$, $\ddot{y}_1$, and in a first order theory can be neglected.

The complete blade motion equation is

$$\frac{I_b}{n}\ddot{\gamma} = \int_0^a \left[\begin{array}{l} (T+\Delta T)'r - mgr\cos\gamma - m\Omega^2 r^2 \sin\gamma\ \cos\gamma \\ \qquad + m\Delta\dot{v}r\sin\gamma + m\Delta\dot{w}r\cos\gamma \end{array} \right] dr$$

As before, we can write γ for $\sin\gamma$, and can also put $\cos\gamma = 1$. However, it will simplify the following discussion if we continue to include $\cos\gamma_0$ in the centrifugal term. We can integrate the equation above in terms of the defined integrals and the expression for $(T+\Delta T)'$.

$$\begin{aligned} I_b\ddot{\gamma} + \Omega^2\cos\gamma_0 I_b + \left(M_b g b - A_3\Omega^2 + B_2\Omega w\right) - \dot{\gamma}\left(2A_2 w - B_3\Omega\right) \\ = (2A_2\Omega - B_1 w + M_b b\Omega\gamma)(\dot{x}_1\sin\Omega t + \dot{y}_1\cos\Omega t) \\ + M_b b(\ddot{x}_1\cos\Omega t - \ddot{y}_1\sin\Omega t) \\ - (2A_1 w - B_2\Omega)\dot{z}_1 + M_b b\ddot{z}_1 \\ + B_3\Omega^2(\phi\cos\Omega t - \theta\sin\Omega t) \end{aligned}$$

The terms on the Left Hand Side give the Particular Integral γ_0 hence-

$$\gamma_0 = \frac{\left(A_3\Omega^2 - B_2\Omega w - M_b bg\right)}{\Omega^2\cos\gamma_0 I_b}$$

The terms on the Right Hand side give the change in γ due to various disturbances of the helicopter, and they can be treated separately. Since we are specifically assuming that the rate of rotation of the rotor is much greater than any of the angular velocities of the rotor axis, p, q, we can obtain a solution of the above equation as if $\dot{x}_1$, $\dot{y}_1$, ϕ, θ, and the accelerations were constant throughout a cycle. It is obvious that, with this simplification, we shall find solutions for γ that are periodic in (Ωt).

As it stands above, the equation is not homogeneous, but contains product terms on the Right Hand side. We shall find a solution of this inhomogeneous equation in terms of a series, and shall find the conditions under which this expansion is valid. We therefore treat a generalized equation of this form.

SOLUTION OF THE INHOMOGENEOUS TERMS IN THE BLADE MOTION EQUATION.

Consider the equation,

$$I(\ddot{\gamma}-\gamma_0) + k(\dot{\gamma}-\gamma_0) + I\Omega^2(\gamma-\gamma_0) = (M_1\sin\Omega t + M_2\cos\Omega t)\gamma$$

where M_1, M_2 are functions independent of time. This is an equation that can be solved in a series of periodic terms in (Ωt). The solution tends to be cumbersome unless a method of successive approximation is used, and the procedure adopted here is an unusual one whose merit lies in the fact that the conditions for convergence of the series are immediately apparent.

Rearrange the equation above, and drop the terms in which are obviously zero-, then

$$k(\dot{\gamma}-\gamma_0) = -\Omega^2 I(\gamma-\gamma_0) - I\ddot{\gamma} + (M_1\sin\Omega t + M_2\cos\Omega t)\gamma_0$$
$$+ (M_1\sin\Omega t + M_2\cos\Omega t)(\gamma-\gamma_0)$$

First solution.

It is obvious that a first solution is obtained by assuming γ to have a constant value, and from inspection we see that the actual first solution is $\gamma = \gamma_0$. We now assume this solution, and evaluate the Right Hand Side of the equation above, and then solve by integrating both sides of the equation.

$$k(\dot{\gamma}-\gamma_0) = (M_1\sin\Omega t + M_2\cos\Omega t)\gamma_0$$

Integrating, we get the

__Second solution.__

$$\gamma = \gamma_0 - \frac{\gamma_0}{\Omega k}(M_1 \cos \Omega t - M_2 \sin \Omega t)$$

We now assume this second solution, and evaluate the Right Hand Side again, and integrate,

$$k(\dot{\gamma} - \gamma_0) = \cancel{\frac{\Omega I \gamma_0}{k}(M_1 \cos \Omega t - M_2 \sin \Omega t)} - \cancel{\frac{\Omega I \gamma_0}{k}(M_1 \cos \Omega t - M_2 \sin \Omega t)}$$

$$+ \gamma_0 (M_1 \sin \Omega t + M_2 \cos \Omega t)$$

$$- \frac{\gamma_0}{\Omega k}(M_1 \sin \Omega t + M_2 \cos \Omega t)(M_1 \cos \Omega t - M_2 \sin \Omega t)$$

Rearranging the product terms as sums and differences,

$$k(\dot{\gamma} - \gamma_0) = \gamma_0 (M_1 \sin \Omega t + M_2 \cos \Omega t) - \frac{\gamma_0}{2\Omega k}\left((M_1^2 - M_2^2)\sin 2\Omega t + M_1 M_2 \cos 2\Omega t\right)$$

Integrating, we obtain the

__Third solution.__

$$\gamma = \gamma_0 - \frac{\gamma_0}{\Omega k}(M_1 \cos \Omega t - M_2 \sin \Omega t) + \frac{\gamma_0}{4\Omega^2 k^2}\left((M_1^2 - M_2^2)\cos 2\Omega t - M_1 M_2 \sin 2\Omega t\right)$$

In order to see clearly the conditions for convergence, it is necessary to obtain the fourth solution, so we apply the same process again, and obtain an equation, which can be simplified to,

$$k(\dot{\gamma} - \gamma_0) = \gamma_0 M_1 \left(1 - \frac{M_1^2}{8\Omega^2 k^2}\right) \sin \Omega t + \gamma_0 M_2 \left(1 - \frac{M_2^2}{8\Omega^2 k^2}\right) \cos \Omega t$$

$$- \frac{\gamma_0}{2\Omega k} \sin 2\Omega t \left((M_1^2 - M_2^2) + \frac{3}{2}\,\frac{M_1 M_2 I \Omega}{k}\right)$$

$$- \frac{\gamma_0}{2\Omega k} \cos 2\Omega t \left(M_1 M_2 - \frac{3}{2}\,\frac{I\Omega}{k}(M_1^2 - M_2^2)\right)$$

$$+ \frac{\gamma_0}{8\Omega^2 k^2} \sin 3\Omega t \; M_1 (M_1^2 - 2M_2^2) + \frac{\gamma_0}{8\Omega^2 k^2} \cos 3\Omega t \; M_2 (2M_1^2 - M_2^2)$$

Integrating, we obtain the

<u>Fourth solution</u>.

$$\gamma = \gamma_0 - \frac{\gamma_0}{\Omega k}\left[M_1 \cos\Omega t \left(1 - \frac{M_1^2}{8\Omega^2 k^2}\right) - M_2 \sin\Omega t \left(1 - \frac{M_2^2}{8\Omega^2 k^2}\right)\right]$$

$$+ \frac{\gamma_0}{4\Omega^2 k^2}\left[\begin{array}{l}\left(\left(M_1^2 - M_2^2\right) + \frac{3}{2}\frac{M_1 M_2 I \Omega}{k}\right)\cos 2\Omega t \\ \qquad - \left(M_1 M_2 - \frac{3}{2}\frac{I\Omega}{k}\left(M_1^2 - M_2^2\right)\right)\sin 2\Omega t\end{array}\right]$$

$$- \frac{\gamma_0}{24\Omega^3 k^3}\left[M_1\left(M_1^2 - 2M_2^2\right)\cos 3\Omega t + M_2\left(M_2^2 - 2M_1^2\right)\sin 3\Omega t\right]$$

It is obvious from the analysis that any furthur approximation will not change the value of the coefficients of the "fundamental". The terms in this series will decrease with increase in the order of the "harmonic" if two conditions are obeyed; since $|M_1| \simeq |M_2|$, we can see, without a formal discussion, that these conditions are that both $\left|\frac{M}{\Omega k}\right|$, $\left|\frac{I\Omega}{k}\right|$ are smaller than unity. I have since found that this equation is a modified Mathieu equation the conditions for convergence of which will be found in standard mathematical books.

In actual practice, the first of these factors is of the order 10^{-3}, but the second is quite large. However, this factor is always found as a factor of terms of the order $|M^2|$, and this enables us to use this approximation method, for $|M|$ is a function of the disturbing velocities, and can be made as small as we please. For a first order theory, therefore, we can use this expansion. Moreover, since we are only interested in terms of the first order in $\dot{x}_1$, $\dot{y}_1$, etc., we can neglect all terms of the second and higher "harmonics".

SOLUTION OF THE BLADE MOTION EQUATION.

Using the result of the preceding section, we can obtain solutions for the blade motion equation for the different types of disturbances. We shall consider the effect of horizontal, vertical, and tilting disturbances separately.

Effect of horizontal motion.

Consider first, the motion in the x direction. Selecting pertinent terms in the equation,

$$I_b\ddot{\gamma} + (B_3\Omega - 2A_2 w)\dot{\gamma} + \cos\gamma_0\,\Omega^2 I_b(\gamma - \gamma_0) = (2A_2\Omega - B_1 w + M_b b\,\Omega\gamma_0)(\dot{x}_1\sin\Omega t) + M_b b\,\ddot{x}_1\cos\Omega t$$

Since we are assuming that the rate of rotation of the rotor is much greater than p, q; we can assume that $\dot{x}_1$, $\ddot{x}_1$ are effectively constant throughout the cycle, and hence write the solution down thus,

$$\gamma = \gamma_0 + \frac{M_b b\ddot{x}_1\cos(\Omega t - \beta) - (2A_2\Omega - B_1 w + M_b b\,\Omega\gamma_0)\dot{x}_1\sin(\Omega t - \beta)}{\sqrt{I_b^2\Omega^4\,(\cos\gamma_0 - 1)^2 + \Omega^2(B_3\Omega - 2A_2 w)^2}}$$

where the phase angle β is given by,

$$\cot\beta = \frac{I_b\Omega(\cos\gamma_0 - 1)}{(B_3\Omega - 2A_2 w)}$$

In all usual rotors the steady value of the coning angle is about 10° , and hence the value of the factor $(\cos\gamma_0 - 1)$ is very small indeed. Hence the phase angle β is very nearly $-\frac{\pi}{2}$, and the blade motion is so nearly in quadrature with the increase in the forward velocity of the blade that we can neglect the in-phase component. Hence we obtain the final result,

$$\gamma = \gamma_0 + \frac{M_b b\ddot{x}_1 \sin\Omega t + (2A_2\Omega - B_1 w + M_b b\Omega\gamma_0)\dot{x}_1 \cos\Omega t}{\Omega(B_3\Omega - 2A_2 w)}$$

It is obvious that there is an analogous expression for motion in the y direction,

$$\gamma = \gamma_0 + \frac{M_b b\ddot{y}_1 \cos\Omega t - (2A_2\Omega - B_1 w + M_b b\Omega\gamma_0)\dot{y}_1 \sin\Omega t}{\Omega(B_3\Omega - A_2 w)}$$

Effect of axis tilt.

In the same manner we select those terms in ϕ giving the motion about the x axis.

$$I_b\ddot{\gamma} + (B_3\Omega - 2A_2 w)\dot{\gamma} + \Omega^2 I_b \cos\gamma_0(\gamma - \gamma_0) = B_3\Omega^2\phi\cos\Omega t$$

In exactly the same way we neglect the in-phase component, and obtain the quadrature solution,

$$\gamma = \gamma_0 + \frac{B_3\Omega^2}{\Omega(B_3\Omega - 2A_2 w)}\phi\sin\Omega t$$

and similarly for θ ; ξ

$$\gamma = \gamma_0 + \frac{B_3\Omega^2}{\Omega(B_3\Omega - 2A_2 w)}\theta\cos\Omega t$$

It will be noticed that if the rotor axis tilts slowly so that the terms in $\dot{\phi}$, $\dot{\theta}$, are small, then the cone axis keeps perfectly in step; and, in the absence of downwash, would actually coincide.

Effect of vertical movement.

The terms involving $\dot{z}_1$, $\ddot{z}_1$, are not periodic, and the solution is of a rather different form,

$$I_b\ddot{\gamma} + (B_3\Omega - 2A_2 w)\dot{\gamma} + I_b\Omega^2\cos\gamma_0(\gamma - \gamma_0) = -(2A_1 w - B_2\Omega)\dot{z}_1 + M_b b\ddot{z}_1$$

and the solution

$$\gamma = \gamma_0 + \frac{(B_2\Omega - 2A_1 w)\dot{z}_1 + M_b b\,\ddot{z}_1}{\Omega^2 I_b}$$

We have now obtained a complete approximate solution to the blade motion equation. It is convenient to express these results in such a way that we may use the theoretical values of the relations between the helicopter velocities and the changes in the blade angle, or that we may use experimental values which may be obtained at some future date in the laboratory or in the field. We therefore introduce the so-called "Gamma Derivatives".

GAMMA DERIVATIVES.

We can write the solution for the blade angle thus,

$$\gamma = \gamma_0 + \Gamma_u(\dot{x}_1\cos\Omega t - \dot{y}_1\sin\Omega t) + \Gamma_w\dot{z}_1 + \Gamma_\theta(\phi\sin\Omega t + \theta\cos\Omega t) + \Gamma_u'(\dot{x}_1\sin\Omega t + \dot{y}_1\cos\Omega t) + \Gamma_w'\ddot{z}_1$$

where the "Gamma Derivatives", Γ_u, Γ_w, Γ_θ, Γ_u', Γ_θ', have the following values,

$$\Gamma_u = \frac{(2A_2\Omega - B_1 w + M_b b\,\Omega\gamma_0)}{\Omega(B_3\Omega - 2A_2 w)} \qquad 1.12 \times 10^{-4}$$

$$\Gamma_w = \frac{(B_2\Omega - 2A_1 w)}{\Omega^2 I_b} \qquad 2.36 \times 10^{-3}$$

$$\Gamma_\theta = \frac{\Omega B_3}{(B_3\Omega - 2A_2 w)} \qquad 1.01$$

$$\Gamma_u' = \frac{M_b b}{\Omega(B_3\Omega - 2A_2 w)} \qquad 1.9 \times 10^{-4}$$

$$\Gamma_w' = \frac{M_b b}{\Omega^2 I_b} \qquad 1.43 \times 10^{-4}$$

The values given in the last column are those calculated for a helicopter of the same size as the "C 30" autogyro, that would have four thin but fairly heavy blades. It is to be noticed that

$$\frac{\Gamma_w}{\Gamma'_w} \simeq 16 \qquad \text{but} \qquad \frac{\Gamma_u}{\Gamma'_u} \simeq \frac{1}{2}$$

The signs of the changes in γ should also be noticed, they indicate that the cone axis tilts backwards for any forward horizontal velocity, and tilts with a right handed motion for any horizontal acceleration: the cone becomes flatter for upward velocity and acceleration: and tends to keep its axis coincident with the rotor axis during tilt.

In calculating the forces actually acting on the helicopter we shall need the expressions for $\dot\gamma$, $\ddot\gamma$; differentiating,

$$\begin{aligned}
\dot\gamma = {} & \Gamma_u(\ddot x_1 \cos\Omega t - \ddot y_1 \sin\Omega t) + \Gamma_w \ddot z_1 + \Gamma_\theta(\dot\phi \sin\Omega t + \dot\theta \cos\Omega t) \\
& - \Omega\Gamma_u(\dot x_1 \sin\Omega t + \dot y_1 \cos\Omega t) + \Omega\Gamma_\theta(\phi \cos\Omega t - \theta \sin\Omega t) \\
& + \Gamma'_u(\dddot x_1 \sin\Omega t + \dddot y_1 \cos\Omega t) + \Gamma'_w \dddot z_1 + \Omega\Gamma'_u(\ddot x_1 \cos\Omega t + \ddot y_1 \sin\Omega t)
\end{aligned}$$

$$\begin{aligned}
\ddot\gamma = {} & \Gamma_u(\dddot x_1 \cos\Omega t - \dddot y_1 \sin\Omega t) + \Gamma_w \dddot z_1 + \Gamma_\theta(\ddot\phi \sin\Omega t + \ddot\theta \cos\Omega t) \\
& - 2\Omega\Gamma_u(\ddot x_1 \sin\Omega t + \ddot y_1 \cos\Omega t) + 2\Omega\Gamma_\theta(\dot\phi \cos\Omega t - \dot\theta \sin\Omega t) \\
& - \Omega^2\Gamma_u(\dot x_1 \cos\Omega t - \dot y_1 \sin\Omega t) - \Omega^2\Gamma_\theta(\phi \sin\Omega t + \theta \cos\Omega t) \\
& + \Gamma'_u(\ddddot x_1 \sin\Omega t + \ddddot y_1 \cos\Omega t) + \Gamma'_w \ddddot z_1 + 2\Omega\Gamma'_u(\dddot x_1 \cos\Omega t - \dddot y_1 \sin\Omega t) \\
& - \Omega^2\Gamma'_u(\ddot x_1 \sin\Omega t + \ddot y_1 \cos\Omega t)
\end{aligned}$$

FORCES AND MOMENTS ACTING ON THE HELICOPTER DUE TO ROTOR.

The forces and moments that are due to the rotor can be divided into two parts; those due to the aerodynamic forces, and those due to the inertia of the blades. We shall treat these separately.

Let $(X+\Delta X)'$, $(Y+\Delta Y)'$, $(Z+\Delta Z)'$, be the forces acting on the helicopter per unit length of blade in the directions of the x, y, z, axes, and

let $(L+\Delta L)'$, $(M+\Delta M)'$, $(N+\Delta N)'$, be the moments acting on the helicopter per unit length of blade about the x, y, z, axes; then

$$(X+\Delta X)' = \left[D' + T'\frac{(w+\Delta w)}{(u+\Delta u)}\right]\sin\Omega t - \gamma\left[T' - D'\frac{(w+\Delta w)}{(u+\Delta u)}\right]\cos\Omega t$$

$$(Y+\Delta Y)' = \left[D' + T'\frac{(w+\Delta w)}{(u+\Delta u)}\right]\cos\Omega t + \gamma\left[T' - D'\frac{(w+\Delta w)}{(u+\Delta u)}\right]\sin\Omega t$$

$$(Z+\Delta Z)' = \left[T' - D'\frac{(w+\Delta w)}{(u+\Delta u)}\right]$$

$$(L+\Delta L)' = \left[T' - D'\frac{(w+\Delta w)}{(u+\Delta u)}\right] r\sin\Omega t + \gamma\left[D' + T'\frac{(w+\Delta w)}{(u+\Delta u)}\right] r\cos\Omega t$$

$$(M+\Delta M)' = \left[T' - D'\frac{(w+\Delta w)}{(u+\Delta u)}\right] r\cos\Omega t - \gamma\left[D' + T'\frac{(w+\Delta w)}{(u+\Delta u)}\right] r\sin\Omega t$$

$$(N+\Delta N)' = \left[D' + T'\frac{(w+\Delta w)}{(u+\Delta u)}\right] r$$

By integrating the forces above over all the blades, we can obtain the forces acting on the helicopter. Most of these forces have alternating components, and we can take the mean values of these. Let $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{L}$, $\bar{M}$, $\bar{N}$, be the mean values of these total forces and moments acting on the helicopter.

There are only two steady forces and moments acting, the thrust T, and the torque Q. These are given by

$$(Z+\Delta Z)' = -\frac{\rho c}{2}\Omega^2 r^2\left(C_L - \frac{w}{\Omega r}\left(C_D + \frac{dC_L}{d\alpha}\right)\right)$$

$$(N+\Delta N)' = \frac{\rho c}{2}\Omega^2 r^3\left(C_D + \frac{w}{\Omega r}\left(C_L - \frac{dC_D}{d\alpha}\right)\right)$$

Integrating in terms of the integrals defined above,

Thrust $T = -\bar{Z} = (A_2\Omega^2 - B_1\Omega w)$

Torque $Q = \bar{N} = (D_3\Omega^2 + C_2\Omega w)$

__Forces and moments due to horizontal motion.__

Substitute for γ, $\dot{\gamma}$, in the expressions for the forces due to the blade element and select only those terms in $\dot{x}_1$, $\ddot{x}_1$, etc.

We obtain the following series of equations,

$$(X+\Delta X)' = -\frac{\rho c}{2}\dot{x}_1\sin\Omega t\left[\left(2\Omega r C_D + w\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\sin\Omega t - \left(2\Omega r C_L - w\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\cos\Omega t\,\gamma_0\right]$$

$$-\frac{\rho c}{2}\Omega^2 r^2\left(\Gamma_u\dot{x}_1\cos\Omega t + \Gamma_u'\ddot{x}_1\sin\Omega t\right)\left(C_L - \frac{w}{\Omega r}\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\cos\Omega t$$

$$+\frac{\rho c}{2} r\left(\Gamma_u\ddot{x}_1\cos\Omega t - \Omega\Gamma_u\dot{x}_1\sin\Omega t + \Gamma_u'\dddot{x}_1\sin\Omega t + \Omega\Gamma_u'\ddot{x}_1\cos\Omega t\right)$$

$$\left[\left(2wC_D + \Omega r\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\sin\Omega t - \left(2wC_L - \Omega r\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\cos\Omega t\,\gamma_0\right]$$

$$(Y+\Delta Y)' = -\frac{\rho c}{2}\dot{x}_1\sin\Omega t\left[\left(2\Omega r C_D + w\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\cos\Omega t + \left(2\Omega r C_L - w\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\sin\Omega t\,\gamma_0\right]$$

$$+\frac{\rho c}{2}\Omega^2 r^2\left(\Gamma_u\dot{x}_1\cos\Omega t + \Gamma_u\ddot{x}_1\sin\Omega t\right)\left(C_L - \frac{w}{\Omega r}\left(C_D + \frac{dC}{d\alpha}\right)\right)\sin\Omega t$$

equation continued below

$$+\frac{\rho c}{2} r\left(\Gamma_u \ddot{x}_1 \cos\Omega t - \Omega\Gamma_u \dot{x}_1 \sin\Omega t + \Gamma_u' \dddot{x}_1 \sin\Omega t + \Omega\Gamma_u' \ddot{x}_1 \cos\Omega t\right)$$

$$\left[\left(2wC_D + \Omega r\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\cos\Omega t + \left(2wC_L - \Omega r\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\sin\Omega t\, \gamma_0\right]$$

$$(Z+\Delta Z)' = +\frac{\rho c}{2}\dot{x}_1 \sin\Omega t\left(2\Omega r C_L - w\left(C_D + \frac{dC_L}{d\alpha}\right)\right)$$

$$-\frac{\rho c}{2} r\left(\Gamma_u \ddot{x}_1 \cos\Omega t - \Omega\Gamma_u \dot{x}_1 \sin\Omega t + \Gamma_u' \dddot{x}_1 \sin\Omega t + \Omega\Gamma_u' \ddot{x}_1 \cos\Omega t\right)$$

$$\left(2wC_L - \Omega r\left(C_D + \frac{dC_L}{d\alpha}\right)\right)$$

and similarly for the moments,

$$(L+\Delta L)' = -\frac{\rho c}{2} r\,\dot{x}_1 \sin\Omega t\left[\left(2\Omega r C_L - w\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\sin\Omega t + \left(2\Omega r C_D + w\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\cos\Omega t\, \gamma_0\right]$$

$$+\frac{\rho c \Omega^2 r^3}{2}\left(\Gamma_u \dot{x}_1 \cos\Omega t + \Gamma_u' \ddot{x}_1 \sin\Omega t\right)\left(C_D + \frac{w}{\Omega r}\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\cos\Omega t$$

$$+\frac{\rho c}{2} r^2\left(\Gamma_u \ddot{x}_1 \cos\Omega t - \Omega\Gamma_u \dot{x}_1 \sin\Omega t + \Gamma_u' \dddot{x}_1 \sin\Omega t - \Omega\Gamma_u' \ddot{x}_1 \cos\Omega t\right)$$

$$\left[\left(2wC_L - \Omega r\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\sin\Omega t + \left(2wC_D + \Omega r\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\cos\Omega t\, \gamma_0\right]$$

$$(M+\Delta M)' = -\frac{\rho c}{2} r\,\dot{x}_1 \sin\Omega t\left[\left(2\Omega r C_L - w\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\cos\Omega t - \left(2\Omega r C_D + w\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\sin\Omega t\, \gamma_0\right]$$

$$-\frac{\rho c \Omega^2 r^3}{2}\left(\Gamma_u \dot{x}_1 \cos\Omega t + \Gamma_u' \ddot{x}_1 \sin\Omega t\right)\left(C_D + \frac{w}{\Omega r}\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\sin\Omega t$$

$$+\frac{\rho c}{2} r^2\left(\Gamma_u \ddot{x}_1 \cos\Omega t - \Omega\Gamma_u \dot{x}_1 \sin\Omega t + \Gamma_u' \dddot{x}_1 \sin\Omega t + \Omega\Gamma_u' \ddot{x}_1 \cos\Omega t\right)$$

$$\left[\left(2wC_L - \Omega r\left(C_D + \frac{dC_L}{d\alpha}\right)\right)\cos\Omega t - \left(2wC_D + \Omega r\left(C_L - \frac{dC_D}{d\alpha}\right)\right)\sin\Omega t\, \gamma_0\right]$$

$$(N+\Delta N)' = -\frac{\rho c}{2} r\,\dot{x}_1 \sin\Omega t\left(2\Omega r C_D + w\left(C_L - \frac{dC_D}{d\alpha}\right)\right)$$

$$+\frac{\rho c}{2} r^2\left(\Gamma_u \ddot{x}_1 \cos\Omega t - \Omega\Gamma_u \dot{x}_1 \sin\Omega t + \Gamma_u' \dddot{x}_1 \sin\Omega t + \Omega\Gamma_u \ddot{x}_1 \cos\Omega t\right)$$

$$\left(2wC_D + \Omega r\left(C_L - \frac{dC_D}{d\alpha}\right)\right)$$

In considering the actual forces acting on the helicopter to determine the stability conditions, we are , in general, only interested in the mean values of the forces and moments; for the alternating components are of such high frequency that they have little or no effect on these conditions. We therefore take the mean values of the forces above on the assumption that $\dot{x}_1$, $\ddot{x}_1$, $\dddot{x}_1$, $\ddddot{x}_1$, are effectively constant round the cycle, or vary so little that the final result is not affected.

Thus, integrating in terms of the defined integrals,

$$\begin{aligned}
\overline{X} = &-\frac{\dot{x}_1}{2}\Big[(2D_1\Omega + C_0 w) + \Gamma_u(A_2\Omega^2 - B_1\Omega w + 2D_1\Omega w + C_2\Omega^2)\\
&-\frac{\ddot{x}_1}{2}\gamma_0(2A_1\Omega w - B_2\Omega^2)\left(\Gamma_u' + \frac{\Gamma_u}{\Omega}\right) + \frac{\dddot{x}_1}{2}(2D_1\Omega w + C_2\Omega^2)\frac{\Gamma_u'}{\Omega}
\end{aligned}$$

$$\begin{aligned}
\overline{Y} = &-\frac{\dot{x}_1}{2}\gamma_0\Big[(2A_1\Omega - B_0 w) + \Gamma_u(2A_1\Omega w - B_2\Omega^2)\Big]\\
&+\frac{\ddot{x}_1}{2}\Big[(2D_1\Omega w + C_2\Omega^2)\left(\Gamma_u' + \frac{\Gamma_u}{\Omega}\right) + (A_2\Omega^2 - B_1\Omega w)\Gamma_u'\Big] + \frac{\dddot{x}_1}{2}\gamma_0(2A_1\Omega w - B_2\Omega^2)\frac{\Gamma_u'}{\Omega}
\end{aligned}$$

$$\overline{Z} = 0$$

and for the moments,

$$\begin{aligned}
\overline{L} = &-\frac{\dot{x}_1}{2}\Big[(2A_2\Omega - B_1 w) - \Gamma_u(D_3\Omega^2 + C_2\Omega w - 2A_2\Omega w + B_3\Omega^2)\Big]\\
&+\frac{\ddot{x}_1}{2}\gamma_0(2D_2\Omega w + \Omega^2 C_3)\left(\Gamma_u' + \frac{\Gamma_u}{\Omega}\right) + \frac{\dddot{x}_1}{2}(2A_2\Omega w - B_3\Omega^2)\frac{\Gamma_u'}{\Omega}
\end{aligned}$$

$$\begin{aligned}
\overline{M} = &\ \frac{\dot{x}_1}{2}\gamma_0\Big[(2D_2\Omega + C_1 w) + \Gamma_u(2D_2\Omega w + C_3\Omega^2)\Big]\\
&+\frac{\ddot{x}_1}{2}\Big[(2A_2\Omega w - B_3\Omega^2)\left(\Gamma_u' + \frac{\Gamma_u}{\Omega}\right) - \Gamma_u'(D_3\Omega^2 + C_2\Omega w)\Big] - \frac{\dddot{x}_1}{2}\gamma_0(2D_2\Omega w + C_3\Omega^2)\frac{\Gamma_u'}{\Omega}
\end{aligned}$$

$$\overline{N} = 0$$

Exactly analogous terms will be found from analysing the motion in the y direction. However, it will be found that all those terms that contain γ_0 to the first power will be reversed in sign. This is an example of a general rule, and all resistance and moment derivatives that are functions of the odd powers of γ_0 have different signs when evaluated in the different directions. It is therefore apparent that the coning angle should be reduced as much as possible in the interests of stability.

It will be noticed that the torque about the rotor axis is independent of the forward motion, and this enables us to separate out the lateral motion of a helicopter from the vertical and rotatatory motion for some of the simple cases of helicopters, though not in the case of a helicopter with two rotors on two parallel axes rotating in opposite directions, i.e. as in the Focke-Wolfe F.W. 61.

Forces and moments due to vertical motion.

In an exactly similar manner we find the effect of $\dot{z}_1$, $\ddot{z}_1$,

$$(X+\Delta X)' = -\frac{\rho c}{2}\dot{z}_1\left[\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\sin\Omega t-\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\cos\Omega t\,\gamma_0\right]$$

$$+\frac{\rho c}{2}\Omega^2 r^2\left(\Gamma_w\dot{z}_1+\Gamma'_w\ddot{z}_1\right)\left(C_L-\frac{w}{\Omega r}\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\cos\Omega t$$

$$+\frac{\rho c}{2} r\left(\Gamma_w\ddot{z}_1+\Gamma'_w\dddot{z}_1\right)$$

$$\left[\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\sin\Omega t-\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\cos\Omega t\,\gamma_0\right]$$

$$(Y+\Delta Y)' = -\frac{\rho c}{2}\,\dot{z}_1\left[\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\cos\Omega t-\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\sin\Omega t\,\gamma_0\right]$$

$$+\frac{\rho c}{2}\Omega^2 r^2\left(\Gamma_w\dot{z}_1+\Gamma'_w\ddot{z}_1\right)\left(C_L-\frac{w}{\Omega r}\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\sin\Omega t$$

$$+\frac{\rho c}{2}\,r\,\left(\Gamma_w\ddot{z}_1+\Gamma'_w\dddot{z}_1\right)$$

$$\left[\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\cos\Omega t-\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\sin\Omega t\,\gamma_0\right]$$

$$(Z+\Delta Z)' = \frac{\rho c}{2}\,\dot{z}_1\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)$$

$$-\frac{\rho c}{2}\left(\Gamma_w\ddot{z}_1+\Gamma'_w\dddot{z}_1\right)\quad r\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)$$

and for the moments,

$$(L+\Delta L)' = -\frac{\rho c}{2}\,r\,\dot{z}_1\left[\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\sin\Omega t+\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\cos\Omega t\,\gamma\right.$$

$$+\frac{\rho c}{2}\,r^3\Omega^2\left(\Gamma_w\dot{z}_1+\Gamma'_w\ddot{z}_1\right)\left(C_D+\frac{w}{\Omega r}\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\cos\Omega t$$

$$+\frac{\rho c}{2}\,r^2\quad\left(\Gamma_w\ddot{z}_1+\Gamma'_w\dddot{z}_1\right)$$

$$\left[\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\sin\Omega t+\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\cos\Omega t\,\gamma_0\right]$$

$$(M+\Delta M)' = -\frac{\rho c}{2}\,r\,\dot{z}_1\left[\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\cos\Omega t-\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\sin\Omega t\,\gamma_0\right]$$

$$-\frac{\rho c}{2}\Omega^2 r^3\left(\Gamma_w\dot{z}_1+\Gamma'_w\ddot{z}_1\right)\left(C_D+\frac{w}{\Omega r}\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\sin\Omega t$$

$$+\frac{\rho c}{2}\,r^2\left(\Gamma_w\ddot{z}_1+\Gamma'_w\dddot{z}_1\right)$$

$$\left[\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\cos\Omega t-\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\sin\Omega t\,\gamma_0\right]$$

$$(N+\Delta N)' = -\frac{\rho c}{2}\,r\,\dot{z}_1\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)$$

$$+\frac{\rho c}{2}\,r^2\left(\Gamma_w\ddot{z}_1+\Gamma'_w\dddot{z}_1\right)\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)$$

As before, we integrate in terms of the defined integrals, and then take the mean value round the cycle and so obtain the mean forces and moments,

$$\bar{L} = \bar{M} = \bar{X} = \bar{Y} = 0$$

$$\bar{Z} = \dot{z}_1 (2A_0 w - B_1 \Omega) - (\Gamma_{ws} \ddot{z}_1 + \Gamma'_{ws} \dddot{z}_1)(2A_1 w - B_2 \Omega)$$

$$\bar{N} = -\dot{z}_1 (2D_1 w + C_2 \Omega) + (\Gamma_{ws} \ddot{z}_1 + \Gamma'_{ws} \dddot{z}_1)(2D_2 w + C_3 \Omega)$$

If these results are expressed in the conventional resistance and moment derivative form, it will be noticed that

$$X_w = Y_w = Z_u = Z_v = 0$$

$$X_u = Y_v$$

$$X_v = -Y_u$$

<u>Forces and moments due to rotor axis tilt.</u>

In calculating the forces and moments due to the rotor axis tilt we follow the same procedure as above. The equations are,

$$\begin{aligned}(X + \Delta X)' = {} & \frac{\rho c}{2} \Omega^2 r^2 \phi \cos\Omega t \left(\frac{dC_D}{d\alpha} \sin\Omega t - \frac{dC_L}{d\alpha} \cos\Omega t \, \gamma_0\right) \\ & - \frac{\rho c}{2} \Omega^2 r^2 \Gamma_\theta \phi \sin\Omega t \left(C_L - \frac{w}{\Omega r} \left(C_D + \frac{dC_L}{d\alpha}\right)\right) \cos\Omega t \\ & + \frac{\rho c}{2} r \Gamma_\theta (\dot{\phi} \sin\Omega t + \Omega \phi \cos\Omega t) \\ & \left[\left(2wC_D + \Omega r \left(C_L - \frac{dC_D}{d\alpha}\right)\right) \sin\Omega t - \left(2wC_L - \Omega r \left(C_D + \frac{dC_L}{d\alpha}\right)\right) \cos\Omega t \, \gamma_0\right]\end{aligned}$$

$$\begin{aligned}(Y + \Delta Y)' = {} & \frac{\rho c}{2} \Omega^2 r^2 \phi \cos\Omega t \left(\frac{dC_D}{d\alpha} \cos\Omega t + \frac{dC_L}{d\alpha} \sin\Omega t \, \gamma_0\right) \\ & + \frac{\rho c}{2} \Omega^2 r^2 \Gamma_\theta \phi \sin\Omega t \left(C_L - \frac{w}{\Omega r} \left(C_D + \frac{dC_L}{d\alpha}\right)\right) \sin\Omega t \\ & + \frac{\rho c}{2} r \Gamma_\theta (\dot{\phi} \sin\Omega t + \Omega \phi \cos\Omega t) \\ & \left[\left(2wC_L + \Omega r \left(C_L - \frac{dC_D}{d\alpha}\right)\right) \cos\Omega t + \left(2wC_L - \Omega r \left(C_D + \frac{dC_L}{d\alpha}\right)\right) \gamma_0 \sin\Omega t\right.\end{aligned}$$

$$(Z+\Delta Z)' = -\frac{\rho c}{2}\Omega^2 r^2 \frac{dC_L}{d\alpha}\varphi\cos\Omega t$$

$$-\frac{\rho c}{2} r\,\Gamma_\theta\left(\dot{\varphi}\sin\Omega t+\Omega\varphi\cos\Omega t\right)\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)$$

and for the moments,

$$(L+\Delta L)' = +\frac{\rho c}{2}\Omega^2 r^3\ \varphi\cos\Omega t\left(\frac{dC_L}{d\alpha}\sin\Omega t+\frac{dC_D}{d\alpha}\cos\Omega t\,\gamma_0\right)$$

$$+\frac{\rho c}{2}\Omega^2 r^3\Gamma_\theta\varphi\sin\Omega t\left(C_D+\frac{w}{\Omega r}\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\cos\Omega t$$

$$+\frac{\rho c}{2} r^2\,\Gamma_\theta(\dot{\varphi}\sin\Omega t+\Omega\varphi\cos\Omega t)$$

$$\left[\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\sin\Omega t+\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\gamma_0\cos\Omega t\right]$$

$$(M+\Delta M)' = \frac{\rho c}{2}\Omega^2 r^3\ \varphi\cos\Omega t\left(\frac{dC_L}{d\alpha}\cos\Omega t-\frac{dC_D}{d\alpha}\sin\Omega t\,\gamma_0\right)$$

$$-\frac{\rho c}{2}\Omega^2 r^3\Gamma_\theta\varphi\sin\Omega t\left(C_D+\frac{w}{\Omega r}\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\sin\Omega t$$

$$+\frac{\rho c}{2} r^2\,\Gamma_\theta(\dot{\varphi}\sin\Omega t+\Omega\varphi\cos\Omega t)$$

$$\left[\left(2wC_L-\Omega r\left(C_D+\frac{dC_L}{d\alpha}\right)\right)\cos\Omega t-\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)\sin\Omega t\,\gamma_0\right]$$

$$(N+\Delta N)' = \frac{\rho c}{2}\Omega^2 r^3\frac{dC_D}{d\alpha}\varphi\cos\Omega t$$

$$+\frac{\rho c}{2} r^2\,\Gamma_\theta\left(\dot{\varphi}\sin\Omega t+\Omega\varphi\cos\Omega t\right)\left(2wC_D+\Omega r\left(C_L-\frac{dC_D}{d\alpha}\right)\right)$$

Taking the mean values and integrating in terms of the defined integrals, we obtain the mean forces and moments acting on the helicopter,

$$\bar{X} = -\frac{\varphi}{2}\gamma_0\left[(B_2-D_2)\Omega^2+\Gamma_\theta(2A_1\Omega w-B_2\Omega^2)\right]+\frac{\dot{\varphi}}{2}\Gamma_\theta(2D_1w+C_2\Omega)$$

$$\bar{Y} = \frac{\varphi}{2}\left[(A_2-C_2)\Omega^2+\Gamma_\theta(A_2\Omega^2-B_1\Omega w+2D_1\Omega w+C_2\Omega^2)\right]+\frac{\dot{\varphi}}{2}\gamma_0\Gamma_\theta(2A_1w-B_2\Omega)$$

$$\bar{Z} = 0$$

$$\bar{L} = \frac{\varphi}{2}\gamma_0\left[(A_3-C_3)\Omega^2+\Gamma_\theta(2D_2\Omega w+C_3\Omega^2)\right]+\frac{\dot{\varphi}}{2}\Gamma_\theta(2A_2w-B_3\Omega)$$

$$\overline{M} = \frac{\phi}{2}\left[\Omega^2(B_3 - D_3) - \Gamma_\delta(D_3\Omega^2 + C_2\Omega w - 2A_2\Omega w + B_3\Omega^2)\right] - \frac{\dot{\phi}}{2}\gamma_0\Gamma_\delta(2D_2 w - C_3\Omega)$$

$$\overline{N} = 0$$

On analysing the forces and moments due to θ, $\dot{\theta}$, we shall find they have the same values, but are of opposite sign, as those due to ϕ, $\dot{\phi}$, unless they are of first order in γ_0, when they are of the same sign. It will be noticed that the torque and thrust are independent of axis tilt.

Inertia forces and moments.

We cannot directly write down the forces due to the inertia of the rotor, because the blade motion is a complex function of the motion of the helicopter. For small disturbances, the blade cone axis lags behind the rotor axis, which sets up small reversed forces. To calculate these accurately, we should have to express the accelerations as an infinite series of powers of the blade angle. This is not practicable, and not necessary to the order of accuracy needed; there is, moreover, another source of inaccuracy. We have so far, neglected the effect of the "vertical" hinge, whose sole purpose is to allow the blade to flap vertically even when the axis is tilted, and so to neutralise all sideways bending stresses on the blades. If the full theory is worked out, there will be found terms in $(\gamma\dot{\gamma})$; these terms are small, and in any case are absorbed in practice by the "vertical" hinges. It therefore seems useless to extend this part of the analysis to the case where γ is not small unless the effect of these hinges is taken into account. With these assumptions, we can differentiate the expressions for the coordinates of the element twice with respect to time, and so obtain the accelerations,

$$\ddot{x} = \ddot{x}_1 - \Omega^2 r\cos\Omega t$$

$$\ddot{y} = \ddot{y}_1 + \Omega^2 r\sin\Omega t$$

$$\ddot{z} = \ddot{z}_1 - r\ddot{\gamma}$$

Taking the mean values of these accelerations over the cycle and integrating, we obtain the <u>Effective</u> forces that act on the helicopter,

$$\overline{X} = -M_b\ddot{x}_1$$

$$\overline{Y} = -M_b\ddot{y}_1$$

$$\overline{Z} = -M_b\ (\ddot{z}_1 - b(\Gamma_w\dddot{z}_1 + \Gamma'_w\ddddot{z}_1))$$

We obtain the effective moments in the same way; and, neglecting the Coriolis forces as being of small order,

$$\overline{L} = -r\gamma(\ddot{y}_1 + \Omega^2 r\sin\Omega t) + r\sin\Omega t\ (\ddot{z}_1 - r\ddot{\gamma})$$

$$\overline{M} = +r\gamma(\ddot{x}_1 - \Omega^2 r\cos\Omega t) + r\cos\Omega t\ (\ddot{z}_1 - r\ddot{\gamma})$$

$$\overline{N} = 0$$

Since we have postulated an unvarying rate of rotation about the rotor axis, $\overline{N}$ 0.

Substituting for γ and $\ddot{\gamma}$, we obtain the final moments,

$$\overline{L} = -M_b b\ \gamma_0\ddot{y}_1 - \frac{I_b}{2}\ (\Gamma_\delta\ddot{\phi} - \Gamma_u\dddot{y}_1 + \Gamma'_u\ddddot{x}_1) + \Omega I_b(\Gamma_\delta\dot{\theta} + \Gamma_u\ddot{x}_1 + \Gamma'_u\dddot{y}_1)$$

$$\overline{M} = +M_b b\ \gamma_0\ddot{x}_1 - \frac{I_b}{2}\ (\Gamma_\delta\ddot{\theta} + \Gamma_u\dddot{x}_1 + \Gamma_w\ddddot{y}_1) - \Omega I_b(\Gamma_\delta\dot{\phi} - \Gamma_u\ddot{y}_1 + \Gamma'_u\dddot{x}_1)$$

The terms in the last brackets are gyroscopic moments, those in the second brackets are due to the direct moment involved in rotating the blade cone. It will be noticed that all terms due to the centripetal acceleration have cancelled out.

Though it may seem that no great accuracy has been obtained in the results of this action, it will be realized that small errors in calculating the inertia forces and moments have but little effect on the final stability equations, as these forces and moments are but a small part of those due to the rest of the helicopter. The aerodynamic forces are of a higher order, since the helicopter body contributes comparatively little to these.

<u>Moments due to the finite distance between the rotor axis and blade hinge.</u>

The horizontal hinge in all usual helicopter rotors is displaced a small distance $a\varepsilon$ from the rotor axis. This has no effect on the resultant forces due to the rotor, but produces small rolling and pitching moments due to the product of the lift forces and this small lever arm. In steady hovering flight, these moments cancel out.

Neglecting the effect of the horizontal forces as being of a low order, to a first approximation the moments per unit length of blade are,

$$(L+\Delta L)' = a\varepsilon \sin\Omega t \left[T' - D' \frac{(w + \Delta w)}{(u + \Delta u)} \right]$$

$$(M+\Delta M)' = a\varepsilon \cos\Omega t \left[T' - D' \frac{(w + \Delta w)}{(u + \Delta u)} \right]$$

Substituting from above, neglecting all terms of a low order, taking the mean value and integrating, we get the final moments,

$$\bar{L} = -a\varepsilon\left[A_1\Omega\dot{x}_1 + \frac{(B_2 - D_2)}{2}\Omega^2\theta + B_1\Omega\left(\frac{\Gamma_\theta\dot{\phi} - \Gamma_u\ddot{y}_1 + \Gamma_u'\dddot{x}_1}{2}\right) - B_1\Omega^2\left(\frac{\Gamma_\theta\theta + \Gamma_u\dot{x}_1 + \Gamma_u'\ddot{y}_1}{2}\right)\right]$$

$$\bar{M} = -a\varepsilon\left[A_1\Omega\dot{y}_1 - \frac{(B_2 - D_2)}{2}\Omega^2\phi + B_1\Omega\left(\frac{\Gamma_\theta\dot{\theta} + \Gamma_u\ddot{x}_1 + \Gamma_u'\dddot{x}_1}{2}\right) + B_1\Omega^2\left(\frac{\Gamma_\theta\phi - \Gamma_u\dot{y}_1 + \Gamma_u\ddot{x}_1}{2}\right)\right]$$

SUMMARY

The calculated values of the rotor constants are summarised in this section with some discussion of their origin. The figures on the Right Hand Side give the calculated values for a helicopter similar in size to the C.30 Autogyro, where the forces are expressed in lbs., and the moments in lbs-ft.

If the blade angle, γ, is measured from a horizontal plane through P' , then we can express γ in terms of the disturbances by means of an approximate solution, thus,

$$\gamma = \gamma_0 + \Gamma_u(\dot{x}_1 \cos\Omega t - \dot{y}_1 \sin\Omega t) + \Gamma_w \dot{z}_1 + \Gamma_\theta(\phi \sin\Omega t + \theta\cos\Omega t) + \Gamma'_u(\ddot{x}_1 \sin\Omega t + \ddot{y}_1 \cos\Omega t) + \Gamma'_w \ddot{z}_1$$

where the "Gamma Derivatives" have the following values,

$$\Gamma_u = \frac{(2A_2\Omega - B_1 w + M_b b\gamma_0\Omega)}{(B_3\Omega - 2A_2 w)} \qquad 1.12 \times 10^{-4}$$

$$\Gamma_w = \frac{(B_2\Omega - 2A_1 w)}{\Omega^2 I_b} \qquad 2.36 \times 10^{-3}$$

$$\Gamma_\theta = \frac{\Omega B_3}{(B_3\Omega - 2A_2 w)} \qquad 1.01$$

$$\Gamma'_u = \frac{M_b b}{\Omega(B_3\Omega - 2A_2 w)} \qquad 1.9 \times 10^{-4}$$

$$\Gamma'_w = \frac{M_b b}{\Omega^2 I_b} \qquad 1.43 \times 10^{-4}$$

Substituting for γ by means of this solution, we can find the forces that act on the helicopter directly in terms of the disturbances, without involving the blade azimuth angle.

In the discussion that follows, those forces and moments that are due to the coning angle, or "dihedral", are separated out.

(1) Aerodynamic forces that would be produced on a similar rigid rotor,

(a) due to horizontal motion,

$$MX_u = MY_v = \frac{(2D_1\Omega + C_0\omega)}{2} \qquad .47$$

(b) same, due to "dihedral",

$$-MX_v = MY_u = \gamma_0\frac{(2A_1\Omega - B_0 w)}{2} \qquad .09$$

(c) due to vertical motion,

$$MZ_w = \frac{(B_1\Omega - 2A_0 w)}{2} \qquad 31.6$$

(d) due to change of the mean angle of attack from axis tilt,

$$MX_\theta = -MY_\varphi = \frac{(A_2 - C_2)\Omega^2}{2} \qquad 810.$$

(e) same, but due to "dihedral",

$$MX_\varphi = MY_\theta = \gamma_0\frac{(B_2 - D_2)\Omega^2}{2} \qquad 620.$$

(2) Aerodynamic forces due to the tilt of the blade cone axis,

(a) due to horizontal motion,

$$MX_u = MY_v = \Gamma_u\left[\frac{(A_2 + C_2)\Omega^2 + (2D_1 - B_1)\Omega w}{2}\right] \qquad .16$$

(b) same, but due to "dihedral",

$$MX_v = -MY_u = \gamma_0 \Gamma_u \frac{\left(B_2\Omega^2 - 2A_1\Omega w\right)}{2} \qquad .067$$

(c) due to tilt of rotor axis,

$$MX_\theta = -MY_\varphi = \Gamma_\theta \left[\frac{\left(A_2 + C_2\right)\Omega^2 + \left(2D_1 - B_1\right)\Omega w}{2}\right] \qquad 1505.$$

(d) same, but due to "dihedral",

$$-MX_\varphi = -MY_\theta = \gamma_0 \Gamma_\theta \frac{\left(B_2\Omega^2 - 2A_1\Omega w\right)}{2} \qquad 605.$$

(3) Aerodynamic "inertia" forces.

(a) due to tilt of the blade cone axis due to horizontal acceleration,

$$MX_{\dot v} = MY_{\dot u} = \Gamma_u' \left[\frac{\left(A_2 + C_2\right)\Omega^2 + \left(2D_1 - B_1\right)\Omega w}{2}\right] \qquad .234$$

(b) same as last, but due to the "dihedral",

$$-MX_{\dot u} = -MY_{\dot v} = \gamma_0 \Gamma_u' \frac{\left(B_2\Omega - 2A_1 w\right)}{2}\Omega \qquad .115$$

(c) due to change in the blade coning angle due to vertical acceleration,

$$-MZ_{\dot w} = \Gamma_\omega\left(B_2\Omega - 2A_1 w\right) \qquad 1.8$$

(d) due to change in the blade coning angle due to horizontal acceleration,

$$MX_{\dot v} = -MY_{\dot u} = \Gamma_u \frac{\left(C_2\Omega + 2D_1 w\right)}{2} \qquad .0033$$

(e) same as last, but due to the "dihedral",

$$-MX_{\dot u} = -MY_{\dot v} = \gamma_0 \Gamma_u \frac{\left(B_2\Omega - 2A_1 w\right)}{2} \qquad .0031$$

(f) due to rate of tilt of blade cone axis due to horizontal motion,

$$MX_{\ddot{u}} = -MY_{\ddot{v}} = \Gamma_u' \frac{(C_2\Omega + 2D_1 w)}{2} \qquad .0059$$

(g) same as last, but due to the "dihedral",

$$MX_{\ddot{v}} = MY_{\ddot{u}} = \gamma_0 \Gamma_u' \frac{(B_2\Omega - 2A_1 w)}{2} \qquad .0051$$

(h) due to rate of change of blade coning angle due to vertical motion,

$$-MZ_{\ddot{w}} = \Gamma_w'(B_2\Omega - 2A_1 w) \qquad .110$$

(i) due to rate of tilt of blade cone axis due to tilt of rotor axis,

$$-MX_p = -MY_q = \Gamma_\theta \frac{(C_2\Omega + 2D_1 w)}{2} \qquad 28.9$$

(j) same as last, but due to the "dihedral",

$$-MX_q = MY_p = \gamma_0 \Gamma_\theta \frac{(B_2\Omega - 2A_1 w)}{2} \qquad 27.2$$

(4) Mass inertia forces.

$$MX = -M_b \ddot{x}_1 \qquad 156.$$

$$MY = -M_b \ddot{y}_1 \qquad 156.$$

$$MZ = -M_b \ddot{z}_1 \qquad 156.$$

due to the change in blade coning angle with vertical motion,

$$MZ = M_b b(\Gamma_w \dddot{z}_1 + \Gamma_w' \ddddot{z}_1) \qquad 3.58;\ .22;$$

It is instructive to evaluate the total stability derivatives for the helicopter similar to the "C.30" Autogyro, as by so doing we are able to see which forces can be entirely neglected.

$MX_u = .63$	$MX_\theta = 2315.$
$MX_v = -.023$	$MX_\phi = 15.$
$MZ_w = 32.$	$MX_{\dot\theta} = -27.2$
$MX_{\dot u} = +156.$	$MX_{\dot\phi} = -26.8$
$MX_{\dot v} = .23$	
$MZ_{\dot w} = +156.$	
$MX_{\ddot u} = .005$	
$MX_{\ddot v} = .005$	
$MZ_{\ddot w} = -.11$	
$MZ_{\dddot w} = -3.6$	
$MZ_{\ddddot w} = -.22$	

These values are incorrect by several hundred percent
Chris [illegible]

From this rough tabulation it is seen that the important forces are those in section (1), that would be produced on a similar rigid rotor; those in section (2), due to the tilt of the blade cone axis; and those in section (4), due to the mass inertia terms. The terms in section (3), due to aerodynamic inertia terms, are nearly all unimportant, as they are very much smaller than those due to the mass inertia terms, except those in section (3, a, i, j,). The forces in section (3, h) is the only one of its kind, in $Z_{\ddot w}$, but is small in comparison with those in $Z_{\dddot w}$, $Z_{\ddddot w}$, and can be neglected. The forces in section (3, a,) are due to blade cone axis tilt, due to horizontal acceleration, and we should not expect them to be small.

(5) Aerodynamic moments that would be produced on a similar rigid rotor,

(a) due to horizontal motion,

$$\mathbb{ML}_u = \mathbb{MM}_v = \frac{(2A_2\Omega - B_1 w)}{2} \qquad 102.$$

(b) same as last, but due to the "dihedral",

$$\mathbb{ML}_v = -\mathbb{MM}_u = \gamma_0 \frac{(2D_2\Omega + C_1 w)}{2} \qquad .56$$

(c) due to vertical motion,

$$\mathbb{MN}_w = (C_2\Omega + 2D_1 w) \qquad 51.$$

(d) due to change of the mean angle of attack from axis tilt,

$$\mathbb{ML}_\theta = -\mathbb{MM}_\varphi = \frac{(B_3 - D_3)\Omega^2}{2} \qquad 124,000.$$

(e) same as last, but due to the "dihedral",

$$-\mathbb{ML}_\varphi = -\mathbb{MM}_\theta = \gamma_0 \frac{(A_3 - C_3)\Omega^2}{2} \qquad 11,200.$$

(6) Aerodynamic moments due to the tilt of the blade cone axis.

(a) due to horizontal motion,

$$-\mathbb{ML}_u = -\mathbb{MM}_v = \Gamma_u \left[\frac{(B_3 + D_3)\Omega^2 - (2A_2 - C_2)\Omega w}{2}\right] \qquad 13.9$$

(b) same as last, but due to the "dihedral",

$$\mathbb{ML}_v = -\mathbb{MM}_u = \gamma_0 \Gamma_u \frac{(C_3\Omega + 2D_2 w)\Omega}{2} \qquad .071$$

(c) due to tilt of the rotor axis,

$$-\mathbb{ML}_\theta = +\mathbb{MM}_\varphi = \Gamma_\theta \left[\frac{(B_3 + D_3)\Omega^2 - (2A_2 - C_2)\Omega w}{2}\right] \qquad 105,000.$$

(d) same as last, but due to the "dihedral",

$$-ML_{\phi} = -MM_{\theta} = \gamma_0 \Gamma_0 \frac{(C_3\Omega + 2D_2 w)\Omega}{2} \qquad 632.$$

(7) Aerodynamic "inertia" moments,

(a) due to tilt of the blade cone due to horizontal acceleration,

$$-ML_{\dot{v}} = MM_{\dot{u}} = \Gamma_u' \left[\frac{(B_3 + D_3)\Omega^2 - (2A_2 - C_2)\Omega w}{2}\right] \qquad 23.6$$

(b) same as last, but due to the "dihedral",

$$-ML_{\dot{u}} = -MM_{\dot{v}} = \gamma_0 \Gamma_u' \frac{(C_3\Omega + 2D_2 w)\Omega}{2} \qquad .12$$

(c) due to change in the blade coning angle due to vertical acceleration,

$$-MN_{\dot{w}} = \Gamma_w (C_3\Omega + 2D_2 w) \qquad 1.89$$

(d) due to change in the blade coning angle due to horizontal acceleration,

$$-ML_{\dot{v}} = MM_{\dot{u}} = \Gamma_u \left(\frac{B_3\Omega - 2A_2 w}{2}\right) \qquad .58$$

(e) same as last, but due to the "dihedral",

$$-ML_{\dot{u}} = -MM_{\dot{v}} = \gamma_0 \Gamma_u \left(\frac{C_3\Omega + 2D_2 w}{2}\right) \qquad .072$$

(f) due to rate of tilt of blade cone axis due to horizontal motion,

$$ML_{\ddot{u}} = -MM_{\ddot{v}} = \Gamma_u' \left(\frac{B_3\Omega - 2A_2 w}{2}\right) \qquad 1.03$$

(g) same as last, but due to the "dihedral",

$$ML_{\ddot{v}} = MM_{\ddot{u}} = \gamma_0 \Gamma_u' \left(\frac{C_3\Omega + 2D_2 w}{2}\right) \qquad .0056$$

(h) due to rate of change of blade coning angle due to vertical motion,

$$-MN_{\ddot{w}} = \Gamma_w'(C_3\Omega + 2D_2 w) \qquad .115$$

(i) due to rate of tilt of blade cone axis due to tilt of rotor axis,

$$ML_p = MM_q = \Gamma_\theta \frac{(B_3\Omega - 2A_2 w)}{2} \qquad 5420.$$

(j) same as last, but due to the "dihedral",

$$-ML_q = MM_p = \gamma_0\Gamma_\theta \frac{(C_2\Omega + 2D_2 w)}{2} \qquad 28.$$

(8) Moments due to the hinges not being coincident with the rotor axis.

$$\bar{L} = -a\varepsilon\left[A_1\Omega\dot{x}_1 + \frac{(B_2 - D_2)}{2}\Omega^2\theta + B_1\Omega\frac{(\Gamma_\theta\dot{\phi} - \Gamma_u\ddot{y}_1 + \Gamma_u'\dddot{x}_1)}{2} - B_1\Omega^2\frac{(\Gamma_\theta\theta + \Gamma_u\dot{x}_1 + \Gamma_u'\ddot{y}_1)}{2}\right]$$

$$\bar{M} = -a\varepsilon\left[A_1\Omega\dot{y}_1 - \frac{(B_2 - D_2)}{2}\Omega^2\phi + B_1\Omega\frac{(\Gamma_\theta\dot{\theta} + \Gamma_u\ddot{x}_1 + \Gamma_u'\dddot{y}_1)}{2} + B_1\Omega^2\frac{(\Gamma_\theta\phi - \Gamma_u\dot{y}_1 + \Gamma_u'\ddot{x}_1)}{2}\right]$$

(9) Mass inertia moments.

(a) Moment due to rotation of blade cone axis,

$$\bar{L} = -\frac{I_b}{2}(\Gamma_\theta\ddot{\phi} - \Gamma_u\dddot{y}_1 + \Gamma_u'\ddddot{x}_1)$$

10,400 1.16 1.94

$$\bar{M} = -\frac{I_b}{2}(\Gamma_\theta\ddot{\theta} + \Gamma_u\dddot{x}_1 + \Gamma_u'\ddddot{y}_1)$$

(b) blade inertia moments, due to the "dihedral"

$$\bar{L} = -\gamma_0 M_b b \ \ddot{y}_1 \qquad 106.$$

$$\bar{M} = +\gamma_0 M_b b \ \ddot{x}_1$$

(c) gyroscopic moments,

$$\bar{L} = \Omega \, I_b (\Gamma_\theta \dot{\theta} + \Gamma_u \ddot{x}_1 + \Gamma_u' \dddot{y}_1)$$

470,000 89.1 52.4

$$\bar{M} = -\Omega \, I_b (\Gamma_\theta \dot{\phi} - \Gamma_u \dot{y}_1 + \Gamma_u' \ddot{x}_1)$$

Again we tabulate the results that have been worked out for the "C.30" Autogyro.

$ML_u = 88.$	$ML_\theta = +19000.$
$ML_v = .63$	$ML_\phi = -11800.$
$MM_w = 51.$	$ML_{\dot{\theta}} = -470000.$
$ML_{\dot{u}} = 89.$	$ML_{\dot{\phi}} = 5400.$
$ML_{\dot{v}} - 106.$	$ML_{\ddot{\theta}} = 0.$
$MM_{\dot{w}} = -1.9$	$ML_{\ddot{\phi}} = 10,400$
$ML_{\ddot{u}} = +1.0$	
$ML_{\ddot{v}} = 52.$	
$MM_{\ddot{w}} = -.11$	

Those moments that are in section (5), that would be produced on a similar rigid rotor; in section (6), due to tilt of the blade cone axis; and in section (9), due to inertia, are the most significant. Of the aerodynamic inertia moments, only those in section (7, a, c, f, i,) are at all large. The terms due to the finite distance between the blade hinge and the rotor axis are fairly large, but can be reduced by careful mechanical design.

Since the thrust and moment should, to a first approximation, remain unaltered during any tilting of the blade cone axis, it is obvious that the forces in section (2, a, c,) should give the projection of the thrust along the axes due to this blade cone axis tilt. The actual value is nearly a quarter more, and this error must be counted one of the main deficiencies of this analysis. The same error occurs in section (6, a, c,) where the moment should be the projected torque of the rotor. The error is a little difficult to place, but appears to be due to small inaccuracies in the assumption that the blade coning angle is small, and hence the angles through which the blade turns are not strictly correct.

With these limitations, this analysis does seem to give a satisfactory basis for the estimation of the stability of a helicopter. Since the solution of the blade angle equation has been carried out to include terms in Γ_u', Γ_w', and the forces accruing from them are all small, it seems certain that this solution of the blade motion equation does in fact represent the motion of the blade to a sufficient degree of accuracy. This in turn, proves the approximate validity of our assumptions as to the cyclic motion of the blades.

APPENDIX.

Dimensions of "C.30" helicopter used in calculations.

The main dimensions are similar to the "C.30" autogyro, made in England.

Thrust $\quad T = 1300$ lbs.

rotor diameter $\quad 2a = 37$ ft.

angular velocity $\quad \Omega = 22.6$ radians per sec.

downwash velocity $\quad w = 18.5$ feet per sec.

four blades, of N.A.C.A. section 23012, constant and chord.

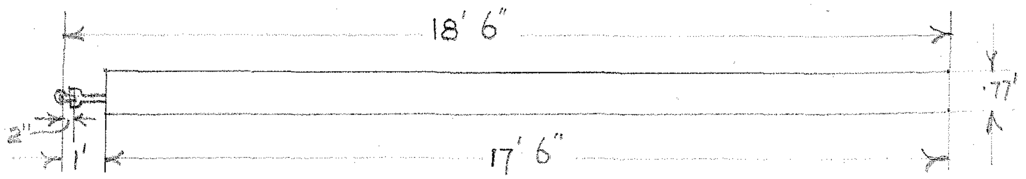
area of each blade $= 13.4$ sq. ft.

blade volume $= .32$ cu. ft.

blade mass $\quad m_b = 39$ lbs. $= 1.22$ slugs

radius of C.G. $\quad b = 9.7$ ft.

M.I. of whole rotor $I_b = 2.03 \times 10^4$ lbs.-ft.^{2} $= 650$ slugs-ft.^{2}



Scale $1/48^{th}$

Evaluating the integrals, expressed in slugs,

$$\int_1^{18.5} \frac{\rho nc}{2}\, dr = .064$$

$$\int_1^{18.5} \frac{\rho nc}{2}\, r\,dr = .62$$

$$\int_1^{18.5} \frac{\rho nc}{2}\, r^2 dr = 7.65$$

$$\int_1^{18.5} \frac{\rho nc}{2}\, r^3 dr = 106$$

From the Polar curve of this section, $\dfrac{dC_L}{d\alpha} = 4.5$,

hence $\quad B_1 \triangleq 2.8$

Solving for A_2 from the equation for T,

$$A_2 = 5.6$$

hence $C_L = .73$

from the Polar curve we can now get the working conditions.

$\alpha = 3.5^{\circ}$

$C_D = .04$	$\frac{dC_D}{d\alpha} = .40$	$\left(C_D + \frac{dC_L}{d\alpha}\right) = 4.54$
$\frac{dC_L}{d\alpha} = 4.5$	$C_L = .73$	$\left(C_L - \frac{dC_D}{d\alpha}\right) = .33$

from these values we get the set of integrals, expressed in lbs.,

$A_0 = .047$	$B_0 = .29$	$C_0 = .021$	$D_0 = .0026$
$A_1 = .45$	$B_1 = 2.8$	$C_1 = .20$	$D_1 = .025$
$A_2 = 5.6$	$B_2 = 34.6$	$C_2 = 2.5$	$D_2 = .30$
$A_3 = 77.3$	$B_3 = 431$	$C_3 = 35$	$D_3 = 4.2$

solving for γ_0, we get the coning angle 4.1°, and hence the "Gamma Derivatives" shown above.