Lie-Poisson Integrators in Hamiltonian Fluid Mechanics

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Barry James Ryan

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Abstract

This thesis explores the application of geometric mechanics to problems in 2D, incompressible, inviscid fluid mechanics. The main motivation is to try to develop symplectic integration algorithms to model the Hamiltonian structure of inviscid fluid flow. The main manifestation of this Hamiltonian or conservative nature is the preservation of the infinite family of Casimirs parametrized by the body integrals of vorticity in the 2D case. The main difficulties encountered in trying to model the Hamiltonian structure of a fluid mechanical system are that the configuration space for the Hamiltonian flow is an infinite dimensional Frechet space and that the phase space is not symplectic but Lie-Poisson. Therefore, an appropriate finite mode truncation must be constructed under the constraint that it too remains Poisson and in some sense converges to the infinite dimensional parent manifold. With such a truncation in hand, there still remains the obstacle of non-symplectic structure. This geometry invalidates the application of traditional symplectic integrators and requires a more sophisticated algorithm.

We develop a Lie-Poisson truncation on the Lie group $SU(N)$ for the Euler equations on the special geometry of a twice periodic domain in R^2. We show that this finite dimensional analog is compatible with the Arnold[5]

formulation of Hamiltonian mechanics on Lie groups with a left or right invariant metric. We then proceed to review the Lie-Poisson integration literature and to develop Hamilton-Jacobi type symplectic algorithms for a broad class of Lie groups. For this same class of groups, we also succeed in constructing an explicit Lie-Poisson algorithm which radically improves computational speed over the current implicit schema. We test this new algorithm against a Hamilton-Jacobi implicit technique with favorable results.

Contents

Introduction

The following is a summary of the topics covered in this thesis.

Chapter One and Appendices A and B provide an introductory tutorial on the theory and techniques of the modern formulation of classical mechanics. Chapter One starts with a description of classical mechanics over linear vector spaces and proceeds to generalize the theory to smooth manifolds. In the applications that we will explore, symmetries propagated by the action of Lie groups will play a pivotal role. Thus, section 1.4 provides a complete introduction to Lie group theory with an emphasis on the computing of derivatives of group and algebra mappings. The first chapter concludes with the development of the Hamiltonian formalism on Poisson manifolds. These manifolds are characterized by the fact that the Poisson bracket is degenerate and they are of importance because so many physical systems are most naturally described on them.

Chapter Two reviews the major result of Hamiltonian fluid mechanics which is that vortex dynamics can be incorporated into the modern formulation of geometric mechanics. The phase space for vortex distributions is shown to be Lie-Poisson. The 1983 Physica D paper by Marsden and

Weinstein[8] is reviewed in great detail and the calculations fully explained. Chapter Two concludes with a Hamiltonian truncation of the equations of motion for a vortex patch with a single-valued boundary. This result is of very little practical interest as it is only valid on a small co-ordinate patch around an equilibrium point on the vortex patch co-adjoint orbit.

Any attempt at building symplectic integration algorithms for fluid mechanical systems will have to rely on some suitable truncation of the infinite dimensional function group introduced in Chapter Two. The goal of Chapter Three is to develop the most promising Lie-Poisson truncation for the evolution of a vortex distribution on a twice periodic domain in R^2. The symmetry group which replaces the infinite dimensional group of area-preserving diffeomorphisms on the 2-torus is $SU(N)$. This configuration space has some very useful properties which makes it accessible to the application of standard techniques in Lie-Poisson integration. We fully develop the theory of vortex dynamics on $su^*(N)$.

Chapter Four develops a self-contained exposition of Symplectic Integrators from the basics to the current state of the literature. The formulation of Lie-Poisson integration through the Hamilton-Jacobi theory is presented in detail. The description is heavily influenced by the papers of Ge and Marsden[9] and Channell and Scovel[14].

Chapter Five applies the techniques of the previous two chapters to the test-bed problem of the rigid body motion and to the $SU(N)$ truncation for fluid dynamics. The Channell and Scovel algorithm is successfully implemented for both algebras. However, it is found that in the case of high dimension $SU(N)$, the implicit generating function integrator of Channell and Scovel is inadequate in the sense that run times become prohibitively expensive. However, we also provide a new explicit Lie-Poisson algorithm which is based on the same natural exponential atlas as used in the Hamilton-Jacobi integrator. The new integrator is Lie-Poisson by construction and provides a far faster alternative to the implicit scheme of Channell and Scovel. The integrator is tested against the implicit Lie-Poisson scheme for $SO(3)$ with favorable results.

Chapter 1

Hamiltonian Mechanics

1.1 Introduction

The purpose of this chapter is to lead the reader through the modern formulation of Hamiltonian Mechanics. The familiar Hamiltonian formalism is developed in terms of linear spaces which can be easily furnished with a canonical co-ordinate system. Examples of these include classical point particle mechanics in conservative force fields and also classical field theory which even though infinite dimensional, parallels the finite dimensional point particle case. The discussion will then proceed to the construction of Hamiltonian mechanics over more mathematically abstract configuration spaces. Configuration space is simply the set of physical variables in terms of which we choose to describe the dynamics under consideration. For example, for point vortices in 2-D fluid mechanics, the configuration space consists of the x and y ordinate of each vortex within the domain of the fluid. In the case of rigid body motion, the configuration space will be a Lie group, namely the special orthogonal group SO(3). After the assignment of configuration space, phase space is constructed by forming the bundle of dual tangent spaces to each point in configuration space. For the point vortices, this will simply reduce to the velocity 1-form at each vortex location. For the rigid body, the situation is more complicated. SO(3) is not covered by one chart, so the formation of phase space has a more involved underlying geometry.

The connection between the more intuitive point particle dynamics and the apparent sophistication of field theory whose description is embedded in infinite dimensional Banach spaces is most easily bridged by the employment of symplectic forms. The use of symplectic structures to express Hamiltonian dynamics is most easily understood on configurations spaces which are linear vector spaces such as R^n. The more general case of viewing the physical configuration as an element of a differentiable manifold can be accomplished by recalling that most manifolds can be described by assigning an atlas of charts to the manifold so that locally, the evolution equations are expressed on Banach spaces reducing the analysis to the linear case. This is one way of generalising the geometric setting. Perhaps more far-reaching, at least from the context of this thesis, is to extend the traditional Poisson bracket formalism. Once a space P has an associated symplectic structure, a mapping

$$\{.,.\} : \mathcal{C}^\infty(P) \times \mathcal{C}^\infty(P) \rightarrow \mathcal{C}^\infty(P),$$

on the smooth functions on P can be defined in terms of the symplectic structure and the Hamiltonian dynamics reduce to an evolution equation $\dot{F} = \{F, H\}$. The properties of the symplectic form are inherited by the

Poisson bracket. However, by weakening some of these properties, a new regime of interesting dynamics can be unlocked such as those modeled on Lie-Poisson systems.

The structure and content of this chapter draw heavily from a rich variety of sources. The main references include V.I.Arnold[3], R.Abraham and J.E.Marsden[1] and R.Schmid[2].

1.2 Mechanics Over Linear Spaces

We start by first defining a symplectic structure on an arbitrary Banach space, V.

Definition 1.2.1 *A symplectic space* (V, Ω) *consists of a linear space V and a weakly non-degenerate, bilinear, antisymmetric 2-form* Ω.

If v_1 and v_2 are elements of V, weakly non-degenerate means that if $\Omega(v_1, v_2) = 0 \; \forall v_2 \in V$, then v_1 is identically zero. Given a 2-form Ω, we can define an associated mapping $\Omega^\flat : V \to V^*$ by

$$\Omega^\flat(v)(w) = \Omega(v, w) \, \forall \, v, w \in V,$$

Ω being a weak symplectic form simply means that the above mapping from V to V^* is one-to-one but not necessarily onto,i.e., $\Omega^\flat : V \to V^*$ does not define an isomorphism. If it does, then Ω is said to be symplectic. We will see in future examples that this distinction is of crucial importance.

We next need to define the symplectic maps from one symplectic space to another. This concept correlates with the traditional canonical mappings as encountered in classical mechanics. If (V,Ω) and (W,Σ) are symplectic spaces and $f: V \to W$, then f is said to be symplectic if

$$f^*\Sigma = \Omega. \tag{1.1}$$

It will be recalled from Appendix B that f^* is the pull-back of f to the tangent bundle which in this case is isomorphic to $W \times W^*$.

Finally, before defining Hamiltonian mechanics on a linear space, we discuss flows of vector fields.

Definition 1.2.2 *A flow on phase space is a 1-parameter diffeomorphism $\phi_t : P \to P$. This usually corresponds to the time evolution for some initial condition located in the physical system's phase space P.*

A flow $\{\phi_t | t \in R\}$ generates a corresponding vector field $X : P \to \mathcal{X}(P)$ through

$$X(y) = \frac{d}{dt}|_{t=0}\phi_t(y).$$

The flow forms a 1-parameter group of diffeomorphisms on P and it can be easily seen that these properties lead to an equivalent differential equation formulation

$$\frac{d}{dt}y(t) = X(y(t))\,,\; y(t) = \phi_t(y)\,,\; y(0) = y.$$

Equipped with a symplectic structure and a familiarity with the connections between vector fields, 1-forms and flows/differential equations, we can produce a Hamiltonian Mechanics.

Definition 1.2.3 *Given a symplectic structure Ω on a Banach space, V, a vector field $X : V \to V$ is called Hamiltonian if there exists a function*

$H : V \to R$ which is at least C^1 such that

$$\Omega^\flat(X(v)) = dH(v)\,\forall\, v \,\in V. \tag{1.2}$$

Such X are referred to as Hamiltonian and the set of all such vector fields will be denoted $\mathcal{X}_{Ham}(V)$. It will be recalled that given $H : V \to R$, dH is the differential mapping, $dH : V \to T^*V$ ($\cong V \times V^*$).

The above definition has a very familiar interpretation. We understand V to represent the phase space for some physical system and H as the Hamiltonian defined on this phase space. We see that it is the symplectic form which allows one to define the link between the Hamiltonian and the possibility of a corresponding vector field. It is this connection which is so crucial. The above discussion of phase flows allows one to express the contents of the above definition into a differential equation setting. By considering the integral curve of X_H, $c : R \to V$, Hamilton's equations are

$$\frac{dc(t)}{dt} = X_H(c(t))$$

assuming of course that c exists for all t. We will now show that these equations produce Hamilton's equations when canonical co-ordinates are chosen on a finite dimensional V.

Example 1 Consider V to be 2n dimensional and choose canonical coordinates on V, (q^i, p_i) where i ranges from 1 to n. The q are usually referred to as the generalised coordinates and the p as the conjugate momenta. In the next section, we will see that such a pair constitutes a canonical description of the dual tangent bundle where q locates the base in the configuration space and p specifies the momentum 1-form in the dual tangent space to q. In these coordinates, we assume that

$$X_H = (A_i, B^i).$$

Since we are using a canonical description, the two form Ω can be written as $\Omega_{i,j} = J_{i,j}$ where

$$J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}.$$

Therefore, given a $w \in V$ where $w = (a_i, b^i)$

$$\Omega(X_H(x), w) = X_H J w = A_i b^i - B^i a_i$$

$$= dH(x).w = \frac{\partial H}{\partial q^i} a_i + \frac{\partial H}{\partial p_i} b^i.$$

By choosing $w = (a_k, 0)$ and then $w = (0, b^k)$, for some k in the range 1 to n, we see that

$$A_k = \frac{\partial H}{\partial p_k}\ ,\ B^k = -\frac{\partial H}{\partial q^k}.$$

Hamilton's equations are then easily derived as

$$\dot{q^i} = \frac{\partial H}{\partial p_i}\ , \tag{1.3}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{1.4}$$

from the integral curve form of the evolution equations.

So, we have reproduced the standard classical form of Hamilton's equations. Of course, the formulation that we have used is not coordinate dependent so we are not locked into having to do mechanics in only canonical settings.

One of the main results of classical Hamiltonian mechanics is that the flow of X_H preserves the value of H ,i.e., $H(c(t))$ is an invariant. This is quite easily seen to be a consequence of the anti-symmetry of Ω. Take the time derivative of H to obtain

$$\frac{d}{dt}H(c(t)) = dH(c(t)).\frac{dc(t)}{dt}$$

$$= \Omega(X_H(c(t)), \frac{dc(t)}{dt}) = \Omega(X_H(c), X_H(c)) = 0,$$

by the antisymmetry of Ω.

Before making our theory more accessible via a suitable example, the above can be reformulated in terms of a bracket structure on $\mathcal{C}^\infty(V)$. Again, this Poisson structure can be completely defined in terms of the symplectic 2-form.

Definition 1.2.4 *Given $F, G : V \to R$, the bracket $\{F, G\} : V \to \Re$ is defined as*

$$\{F, G\}(v) = \Omega(X_F(v), X_G(v)) \,\forall\, v \in V. \tag{1.5}$$

The Poisson bracket inherits the properties of Ω. It is bilinear, anti- symmetric and can also be shown to satisfy the Jacobi identity which states that given F, G and K in $\mathcal{C}^\infty(V)$,

$$\{\{F, G\}, K\} + \{\{G, K\}, F\} + \{\{K, F\}, G\} = 0.$$

The equations of motion can also be expressed in terms of the Poisson bracket. We can show that if ϕ_t is the flow corresponding to the Hamiltonian vector field with Hamiltonian $H : V \to R$, then for some F

$$\frac{d}{dt}(F \circ \phi_t) = \{F \circ \phi_t, H\} = \{F, H\} \circ \phi_t. \tag{1.6}$$

It is an easy corollary to show that F will be constant along the integral curves of X_H if and only if $\{F, H\} = 0$.

In conclusion, by simply providing a Banach space with a symplectic form, we can construct vector fields from C^1 functions on the linear space. We can recover the traditional Hamilton equations for canonical coordinates defined on the phase space and finally derive a bracket formulation for the dynamics.

In order to demonstrate that the theory is not much more difficult in the setting of infinite dimensional Banach spaces, we will construct a symplectic form and a Poisson bracket for classical field theory.

Example 2 In classical field theory, configuration space is usually some function space whose elements have certain differentiability and integrability conditions associated with them. In the case of the wave equation which can be shown to be Hamiltonian, the configuration variable is the displacement of some material from some base equilibrium state. For our purposes, we will take a vector space which is basically of the form $V = W \times W^*$ where W is the space of smooth functions over some domain D which we will just take as R^3. The dual space will be the space of densities over R^3. A density can simply be written as the product of a smooth function times a volume form for R^3. This allows us to express a natural duality between W and W^*

$$< \phi, \pi >= \int_{R^3} \phi \, \pi \, dx^3$$

where $\phi \in W$ and $\pi \in W^*$. With this we can define a 2-form Ω,

$$\Omega((\phi_1, \pi_1), (\phi_2, \pi_2)) =< \phi_1, \pi_2 > - < \phi_2, \pi_1 >$$

$$= \int \phi_1 \pi_2 - \int \phi_2 \pi_1 .$$

The properties of the symplectic form are easily shown to hold true for Ω. We now proceed to construct a Hamiltonian vector field corresponding to a $H : V \to R$. We recall that if $F : W \times W^* \to R$ then

$$D_1 F(\phi, \pi)(\psi) = DF(\phi, \pi)(\psi, 0) \, , \phi, \psi \in W, \pi \in W^*$$

and

$$DF(\phi, \pi)(\psi, \rho) = D_1 F(\phi, \pi)(\psi) + D_2 F(\phi, \pi)(\rho)$$

where if $F : V_1 \to V_2$, V_1, V_2 Banach spaces, then

$$DF(x) : V_1 \to V_2 \; , \; x \in V_1$$

and $DF(x)$ is a linear transformation satisfying

$$\lim_{h \to \infty} \frac{||F(x+h) - F(x) - DF(x).h||_2}{||h||_1} = 0.$$

Also, we will need to remember the definitions of functional derivative and partial functional derivative. From the total Frechet derivative, we can define the functional derivative of a real-valued function F on V as the unique element of V^* such that

$$DF(x).y =< \frac{\delta F}{\delta x}, y > \ \forall y \in V$$

where $<,>$ is the natural pairing between V and its dual. The partial functional derivatives for a function $F : V_1 \times V_2 \to R$ are defined in a similar way. We have pairings between V_1 and its dual and between V_2 and its dual, $<,>_1$ and $<,>_2$ respectively. Therefore,

$$< \frac{\delta F}{\delta v_i}, w_i >_i = D_i F(v_1, v_2).w_i \, i = 1, 2.$$

Using these results, if $H : W \times W^* \to R$, then

$$DH(\phi,\pi)(\psi,\rho) = D_1 H(\phi,\pi).\psi + D_2 H(\phi,\pi).\rho$$
$$=< \psi, \frac{\delta H}{\delta \phi} > + < \frac{\delta H}{\delta \pi}, \rho >,$$

noting that $\frac{\delta H}{\delta \phi} \in W^*$ and $\frac{\delta H}{\delta \pi} \in W$. The last equation above becomes by definition

$$\Omega((\frac{\delta H}{\delta \pi}, -\frac{\delta H}{\delta \phi}), (\psi, \rho)).$$

Therefore, $X_H(\phi,\pi) = (\frac{\delta H}{\delta \pi}, -\frac{\delta H}{\delta \phi})$ and the equations of motion are

$$\dot{\phi} = \frac{\delta H}{\delta \pi},$$
$$\dot{\pi} = -\frac{\delta H}{\delta \phi}.$$

Finally we write down a Poisson bracket for our infinite dimensional example. If $F, G \in C^\infty(V)$, then

$$\{F, G\} = \Omega(X_F, X_G) = \int \frac{\delta F}{\delta \phi}\frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta \phi}\frac{\delta F}{\delta \pi}.$$

The above analysis will now be carried out for configuration spaces which are modeled on differentiable manifolds. Many of the results carry over quite easily from linear spaces to differentiable manifolds as differentiable manifolds are locally modeled on Banach spaces.

1.3 Mechanics Over Manifolds

If we wish to build Hamiltonian mechanics over manifolds, we simply loosen Ω from its local definition over a linear vector space to a closed, weakly non-degenerate 2-form over the manifold as defined in Appendix C. A symplectic pair is now a (P, Ω) such that P is a manifold and Ω is as defined above.

The 2-form Ω will vary from point to point and by the definition of 2-forms, for $x \in P$, $\Omega(x) : T_xP \times T_xP \to R$ is nondegenerate and bilinear on the tangent space.

A major result in Mechanics is that locally a symplectic manifold looks like a symplectic vector space, i.e., in the neighbourhood of a point $x \in P$, $\Omega(x)$ is constant. This is a more general statement of the theorem that when P is finite dimensional, P is of even dimension and locally there exist coordinates (p, q) such that $\Omega = \sum dq \wedge dp$.

We will concentrate on the special case where the symplectic manifold is a cotangent bundle, T^*Q where Q is some configuration space co-ordinatized by a set $\{q^i\}$. For each $q \in Q$, T_q^*Q has a basis dq^i and every 1-form over Q can be expressed as $\alpha = p_i dq^i$.

We can define a 1-form on T^*Q such that its differential will be a closed (weakly) non-degenerate 2-form and we can do this in a co-ordinate free manner. Take the cotangent bundle $\tau : T^*Q \to Q$ and choosing $\beta \in T^*Q$, $v \in T(T^*Q)$, globally define the 1-form θ at β by

$$\theta(\beta)(v) =< \beta, T\tau.v >$$

where from Appendix C, $T\tau : T(T^*Q) \to T^*Q$ is also a vector bundle. Let $\Omega = -d\theta$ and this then defines a co-ordinate free sympletic two form on the cotangent bundle.

We will now introduce the lift of a diffeomorphism from Q to the bundle T^*Q. If $f : Q \to Q$ is a diffeomorphism on Q, then T^*f, defined as a map from T^*Q to T^*Q by

$$T^*f(\alpha_q).v =< \alpha_q, Tf.v >$$

is called the lift of f where $v \in T_{f^{-1}(q)}Q$ and α_q is a 1-form at $q \in Q$. An important observation is that T^*f preserves the global 1-form defined above. To see this, take $\beta \in T^*Q$ and $v \in T_\beta(T^*Q)$ and form

$$(T^*f)^*\theta(\beta)(v) = \theta(T^*f(\beta)).TT^*f(v)$$

$$=< T^*f(\beta), (T\tau TT^*f).v >=< \beta, T(f \circ \tau \circ T^*f).v >$$

$$= \theta(\beta).v.$$

So, $T^*f\theta = \theta$ and consequently, T^*f preserves Ω.

In a similar fashion to the previous section, Hamiltonian vector fields can be defined using the symplectic 2-form. If X is a vector field on P, then it is called Hamiltonian if there exists a function H on P such that

$$\Omega(x)(X, v) = dH(x).v \tag{1.7}$$

for all $x \in P$ and vector fields, v on P.

Also, the Poisson bracket of two functions has a similar form as in the linear case. If $F, G : P \to R$ are smooth, then

$$\{F, G\}(x) = \Omega(x)(X_F(x), X_G(x))$$

and if ϕ_t is the flow of some Hamiltonian X on P, then

$$\frac{d}{dt}(F \circ \phi_t) = \{F \circ \phi_t, H\}.$$

So, we see that all of the results that were demonstrated for the linear vector space case carry over, more or less, to the more abstract manifold, both infinite and finite dimensional.

In order to develop computational skills, we will study the problem of a Hamiltonian system which evolves on a symplectic space which has quite an involved function space component. The following example is due to Baessens and MacKay.

Uniformly Traveling Water Waves We consider an inviscid, 2-D body of water with an air interface and of infinite depth. The fluid is assumed to be irrotational and any surface tension will be ignored. The equations of motion are

$$\mathbf{u} = \nabla \phi,$$

$$p - p_0 = -\phi_t - \frac{1}{2}(\nabla \phi)^2 - gy,$$

$$\triangle \phi = 0,$$

where ϕ is the velocity potential, p is the pressure , p_0 is the atmospheric pressure and $\mathbf{u} = (\mathbf{u}, \mathbf{v})$ is the velocity field. To set up the problem we will need the boundary conditions at the free surface and at infinite depth.

The free surface is denoted by $y - \eta(x,t) = 0$ and is defined by the observation that fluid does not cross it. Therefore, the normal velocity of the interface must equal the normal component of velocity of the fluid at the interface. This leads to the boundary condition

$$\eta_t + u\eta_x = v$$

at $y = \eta$. The other boundary condition at the free surface is that $p = p_0$. This gives

$$\phi_t + \frac{1}{2}({\phi_x}^2 + {\phi_y}^2) + g\eta = 0.$$

For the fixed bottom at infinite depth, we have the boundary condition $\phi_y = 0$.

We consider the case of a uniformly travelling wave of velocity c relative to the fluid at infinite depth. This reduces the first boundary condition at the free surface to $\psi =$ constant, where ψ is the stream function for the flow and at infinite depth, we now have $(u, v) \to (-c, 0)$.

Before showing that the above system is Hamiltonian, we will transform to new co-ordinates. Let $Y = y - \eta$ be the vertical height below the surface and $F(x,Y) = \phi(x, \eta + Y) + cx$, $U(x,Y) = u(x, \eta + Y)$. When surface tension is not zero, we can derive the equations of motion from a variational principle defined with respect to some Lagrangian density. A Hamiltonian system can then be constructed via a Legendre transformation. However, in the case of zero surface tension, such a transformation is singular and thus cannot be enacted. So, we will just write down a Hamiltonian as a function over some constrained function space and then show that with a particular choice of closed 2-form, we can retrieve the equations of motion for the traveling wave. Our phase space, M is

$$\{(F, U, \eta, w) | F, U : (-\infty, 0] \to R; F, F_Y \to 0 \; and \; U \to 0 \; as \; Y \to -\infty\},$$

with the following constraints on w and η

$$w = -\int_{-\infty}^{0} U F_Y dY$$

and

$$\eta + \frac{1}{2}(U^2 + {F_Y}^2)_0 = \frac{1}{2}c^2.$$

The symplectic form is the canonical one restricted to our phase space above, namely

$$\omega = dw \wedge d\eta + \int_{-\infty}^{0} dU \wedge dF dY,$$

which is non-degenerate provided $U_0 = 0$. The Hamiltonian we choose is

$$H(F, U, \eta, w) = -\frac{1}{2}\eta^2 + \frac{1}{2}\int_{-\infty}^{0} ((U + c)^2 - {F_Y}^2) dY.$$

We now show that the equation

$$\omega(\xi, X_H) = dH(\xi)$$

leads to the travelling wave equations.

However, we need a general result on the evaluation of functional derivatives before we continue. The definitions of Frechet and functional derivatives that we employ here were encountered in section 1.3. If we have a functional dependent on a function, f say, given by

$$F(f) = \int_\Omega L(x, f(x), \frac{df}{dx})dx$$

over some range Ω in R, then the functional derivative will satisfy

$$\int_\Omega \frac{\delta F}{\delta f} g(x)dx = \frac{d}{d\epsilon}|_{\epsilon=0} \int_\Omega L(x, f+\epsilon g, \frac{d}{dx}(f+\epsilon g))dx$$

by definition. Differentiating, the above gives

$$\int_\Omega D_2 L(x, f, f')g dx + \int_\Omega D_3 L(x, f, f')\frac{dg}{dx}dx$$
$$= \int_\Omega D_2 L g dx - \int_\Omega (\frac{d}{dx} D_3 L) g dx + \int_{\partial\Omega} D_3 L g dx$$

after applying the chain rule.

We now wish to calculate the implied equations of motion by identifying

$$\omega((\delta U, \delta F), (\dot{U}, \dot{F}))$$

and

$$dH(\delta U, \delta F)$$

where H is the Hamiltonian and $(\dot{U}, \dot{F})$ is the Hamiltonian vector field associated with the Hamiltonian function defined on M and $(\delta U, \delta F)$ is some arbitrary perturbation in (U, V). Implicitly, perturbations in

w and η are included but they depend on the perturbations in F and U. In fact, we can derive them from the constraints on M. We obtain

$$\delta\eta = -U_0\delta U_0 - F_{Y0}\delta F_{Y0},$$

and for w,

$$w + \delta w = -\int_{-\infty}^{0} (U + \delta U)(F + \delta F)_Y dY$$

$$= -\int_{-\infty}^{0} UF_Y dY - \int_{-\infty}^{0} F_Y \delta U dY - \int_{-\infty}^{0} U\delta F_Y dY.$$

We have ignored the second order terms. Applying the chain rule, the above expression yields

$$\delta w = \int_{-\infty}^{0} (U\delta F_Y - F_Y \delta U) dY - U_0 \delta F_0.$$

We now find that

$$\omega((\delta U, \delta F), (\dot{U}, \dot{F})) = \delta w \dot{\eta} - \dot{w}\delta\eta + \int_{-\infty}^{0} \delta U \dot{F} - \dot{U}\delta F dY.$$

Through the functional derivative result presented above for a density which depends on the first derivative of a function in addition to the function itself, we find that

$$dH(\delta U, \delta F) = -\eta\delta\eta + \int_{-\infty}^{0} (U + c)\delta U - F_Y \delta F_Y dY$$

$$= -\eta\delta\eta + \int_{-\infty}^{0} ((U + c)\delta + F_{YY}\delta F) dY - F_{Y0}\delta F_0.$$

Substituting for δw and for $\delta\eta$ and equating both dH and ω for all perturbations, we find

$$\dot{F} = \dot{\eta} F_Y + U + c,$$

$$\dot{U} = \dot{\eta} U_Y - F_{YY},$$

$$\dot{\eta} = F_{Y0}/U_0,$$

$$\dot{w} = \eta.$$

These equations are only satisfied for non-zero U_0. This set of equations corresponds to the travelling wave solution to the water wave equations in the absence of surface tension.

For applications of this Hamiltonian formulation of the water wave problem, the reader is referred to the paper by Baesens and MacKay.

1.4 Lie Groups

Before we can develop a Hamiltonian theory of inviscid, incompressible fluid mechanics, a large number of results concerning Lie groups must be accumulated and understood. This section will describe Lie groups and Lie algebras and will introduce the concept of group action on manifolds.

A Lie group is a differentiable manifold with a group structure attached. The group composition or multiplication will be smooth in the C^∞ sense. We will denote group mutliplication by

$$\mu : G \times G \to G \, , \mu(g,h) = gh \; for \; g,h \in G.$$

Usually, smoothness of inversion is also included in the definition of a Lie group but this in fact easily follows from the smoothness of the multiplication operator. We now define the two most fundamental mappings associated with a Lie group, the left and right translation maps

$$L_g : G \to G\,, L_g h = gh \text{ and } R_g : G \to G\,, R_g h = hg \;\forall\; g, h \in G.$$

Here, we have used gh instead of $\mu(g,h)$. Because these mappings are defined using the multiplication operator, they are both smooth and since $R_{g^{-1}} = (R_g)^{-1}$ and $L_{g^{-1}} = (L_g)^{-1}$, both maps are diffeomorphisms on G.

An example of a Lie group is the space of linear isomorphisms from R^n to R^n which is denoted $GL(n,R)$ in the literature. Each element can be represented by an $n \times n$, non-singular matrix and the group operation becomes matrix multiplication

$$\mu(A,B) = AB \text{ for } A, B \in GL(n,R)$$

and the inverse map is simply matrix inversion. Smoothness follows from the fact that matrix multiplication is smooth in the matrix components.

For our applications in dynamics, Lie groups will play the role of configuration space and thus, we will be interested in how the group structure behaves on TG and on T^*G. In particular, we will want specific results concerning $TL_g : TG \to TG$ and $T^*L_g : T^*G \to T^*G$ and their R_g counterparts. In our applications to fluid mechanics, the adjoint mapping will provide the starting point for all our computations. This is constructed from the inner automorphism $I_g(h) = g^{-1}hg$. The tangent derivative or linearisation of I_g

at the identity of G defines the adjoint mapping

$$Ad_g = T_e I_g = T_e(R_{g^{-1}} l_g) : T_e G \rightarrow T_e G.$$

The subset of tangent vectors which are invariant under TL_g and TR_g are denoted $\mathcal{X}_L(G)$ and $\mathcal{X}_R(G)$ respectively. A vector field X is left-invariant if

$$T_h L_g X(h) = X(gh) \; for \; every \; g \in G,$$

where $T_h L_g : T_h G \rightarrow T_{gh} G$. We can form an isomorphism between left-invariant vector fields on G and the tangent space at the identity of the group. This is achieved through

$$\rho_1 : \mathcal{X}_L(G) \rightarrow T_e G \; : X \rightarrow X(e) \; and$$

$$\rho_2 : T_e G \rightarrow \mathcal{X}_L(G) \; : \xi \rightarrow X_\xi(g) = T_e L_g \xi.$$

It follows that ρ_1 and ρ_2 satisfy $\rho_1 \circ \rho_2 = Id_{T_e G}$ and $\rho_2 \circ \rho_1 = Id_{\mathcal{X}_L(G)}$. This observation makes the two spaces isomorphic in the vector space sense. In fact, they both form Lie algebras which will be discussed next.

A Lie algebra can be defined independently of Lie groups even though there exists a very useful relationship between the two structures.

Definition 1.4.1 *A Lie algebra is a linear vector space on which a bracket is defined. The bracket is denoted by* $[.,.]$ *and has the following properties: i)* $[,] : V \times V \rightarrow V$ *and is bilinear, ii)* $[u,v] = -[v,u] \; \forall \; u,v \in V$*, iii)* $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$ *which is known as the Jacobi identity.*

The space of left-invariant vector fields on G can be furnished with a Lie bracket through the Lie derivative operator which was discussed in Appendix

C. The Lie derivative of a function f on G with respect to a vector field X is defined by

$$L_X f : G \to R : L_X f(g) = df(g).X(g). \tag{1.8}$$

The bracket of two vector fields X and Y is then the vector field on G which satisfies

$$[X, Y] = L_X Y \text{ such that } L_{[X,Y]} = [L_X, L_Y]$$

where

$$[L_X, L_Y] = L_X \circ L_Y - L_Y \circ L_X.$$

It can be shown that if X and Y are left-invariant, then $[X, Y]$ is also left-invariant and thus, $\mathcal{X}_L(G)$ forms a Lie subalgebra of the vector field algebra. We can thus define a Lie bracket on T_eG by

$$[\xi, \eta] = [X_\xi, X_\eta](e)$$

$(T_eG, [,])$ is called the Lie algebra of G and is denoted by $\mathcal{G}$.

It should be pointed out that right translation can be dealt with in a similar fashion and that the space of right-invariant vector fields on G will play a major role in the diffeomorphism groups which arise in fluid mechanics. Also, the adjoint mapping can now be regarded as a mapping from $\mathcal{G}$ to $\mathcal{G}$.

If a Lie algebra is finite dimensional ,i.e., as a vector space, every element can be generated by a finite dimensional basis which we will denote $\{e_i\}$, then there are a set of structure constants which we can define with respect to our choice of basis,

$$[e_i, e_j] = C_{ij}^k e_k \ \forall \ e_i, e_j \text{ in the basis.}$$

We have now set up the connection between Lie groups, which are fundamentally topological in nature and Lie algebras which are algebraic. If calculations on functions and mappings over the Lie group could be expressed in terms of its corresponding Lie algebra, a great computational simplification would be achieved. So, how do we relate the two entities?

We now introduce the exponential mapping which maps elements of the Lie algebra into the Lie group.

Take $\xi \in \mathcal{G}$ and form its left-invariant vector field, X_ξ over TG. The integral curve of X_ξ is defined by

$$\gamma_\xi : R \rightarrow G \text{ and } \dot{\gamma}_\xi(t) = X_\xi(\gamma_\xi(t)),\ \gamma_\xi(0) = e.$$

This curve through G forms a 1-parameter subgroup, i.e., $\gamma_\xi(t+s) = \gamma_\xi(t)\gamma_s(s)$. This is easily seen by noting that both of sides of the equality satisfy the same differential equation in t and that they have the same initial conditions at $t = 0$. This enables us to define the exponential mapping as follows

Definition 1.4.2 *The exponential mapping is defined as* $exp : \mathcal{G} \rightarrow G : \xi \rightarrow \gamma_\xi(1)$. *It is* $\mathcal{C}^\infty$ *in finite dimensions which follows from the smoothness of both the group product and the solutions of the differential equation for* γ_ξ. *Due to the fact that* $exp\mathcal{G}$ *is connected and* G *is in general not, the exponential map is not onto. It will also be seen that* exp *has far more restricted properties in infinite dimensional examples.*

The mapping is called exponential because if $\xi \in \mathcal{G}$, then $exp((t+s)\xi) = exp(t\xi)exp(s\xi)$ for $t, s \in R$. However, the following does not generally hold,

$exp(\xi + \eta) = exp(\xi)exp(\eta)$. This is because $[\xi, \eta]$ is not usually zero. Also, all 1-parameter subgroups in G are of the form $exp(t\xi)$ for some $\xi \in G$. Finally, exp provides an atlas for G. exp is a local diffeomorphism for a neighbourhood of the identity of G onto a neighbourhood of the Lie algebra zero. Therefore, it defines a local chart which can be extended to an atlas by using left translation. On an open set containing the origin of the Lie group, we can thus define an inverse mapping to exp. This will be denoted ln and it will be used extensively in the design of symplectic integrators which will be introduced in chapter 4.

We will now prove a number of results which will both provide a good exercise in doing calculations on Lie groups and be useful later in the development of Hamiltonian mechanics on Lie groups.

Proposition 1.4.1 *Consider two Lie groups G and H. Take $f : G \rightarrow H$, a homomorphism. This means that if $f, g \in G$, then $f(gh) = f(g)f(h)$. The mapping defined by taking the tangent derivative of f at the identity of G can be shown to be a Lie algebra homomorphism, i.e., $T_e f([\xi, \eta]) = [T_e f.\xi, T_e f.\eta]$ for all $\xi, \eta \in \mathcal{G}$. Also, it can be demonstrated that*

1. *$f \circ exp_G = exp_H \circ T_e f$,*
2. *if $f_1, f_2 : G \rightarrow H$ are homomorphisms and both G and H are connected Lie groups, then $T_e f_1 = T_e f_2$ implies that $f_1 = f_2$.*
3. *$exp(Ad_g \xi) = g(exp\xi)g^{-1}$ where Ad_g is the adjoint mapping and $g \in G$, $\xi \in \mathcal{G}$,*

4. $\frac{d}{dt}|_{t=0} Ad_{expt\xi}\eta = [\xi, \eta] \; \forall \; \xi, \eta \in \mathcal{G}$,

Proof If $f : G \to H$ is a homomorphism, then $L_{f(g)} \circ f = f \circ L_g$ and by taking the tangent derivative and applying the chain rule,we obtain

$$TL_{f(g)} \circ Tf = Tf \circ TL_g.$$

Choose $\xi \in \mathcal{G}(\cong T_eG)$ and apply the above tangent derivative to ξ at the identity

$$T_eL_{f(g)} \circ T_ef(\xi) = T_gf \circ T_eL_g(\xi)$$

which reduces to

$$X_{T_ef.\xi}(f(g)) = T_gf(X_\xi(g))$$

when the identifications $X_\xi(g) = T_eL_g(\xi)$ and $X_{T_ef.\xi}(f(g)) = T_eL_{f(g)}(T_ef(\xi))$ are made. These are just the left invariant vector fields which are generated by $\xi \in \mathcal{G}$ and $T_ef(\xi) \in \mathcal{H}$. Now to prove that T_ef is a Lie algebra homomorphism, take $\eta \in \mathcal{G}$ and form $[\xi, \eta]$ which is also an element of the Lie algebra. Apply T_ef to this bracket

$$T_ef.[\xi, \eta] = T_ef[X_\xi, X_\eta](e) = [X_{T_ef.\xi}, X_{T_ef.\eta}](e)$$

$$= [T_ef.\xi, T_ef.\eta].$$

1. If $f : G \to H$ then $\phi : R \to H : t \to f(exp_Gt\xi)$ where $\xi \in \mathcal{G}$ is a 1-parameter subgroup in H and can thus be generated by some $\eta \in \mathcal{H}$, i.e.,

$$\phi(t) = exp_H(t\eta).$$

where η satisfies

$$\eta = \frac{d}{dt}|_{t=0}\phi(t) = T_e f \circ \frac{d}{dt}|_{t=0} exp_G t\xi = T_e f.\xi.$$

By the uniqueness of the differential equation solution, this implies that at $t = 1$,

$$f(exp_G \xi) = exp_H(T_e f.\xi).$$

2. When G and H are connected, it implies that both exp_H and exp_G are onto. Therefore, if $f_{1,2} : G \to H$ are homomorphisms and $T_e f_1 = T_e f_2$, i.e., the induced Lie algebra homomorphisms are identical, it is easy to show that $f_1 = f_2$. Since exp_G is onto, every $g \in G$ can be represented as $exp_G \xi$ for some $\xi \in G$. Therefore,

$$f_1(g) = f_1(exp_G \xi) \stackrel{by 1.}{=} exp_H(T_e f_1.\xi)$$

$$= exp_H(T_e f_2.\xi) = f_2(exp_G \xi) = f_2(g),$$

for all $g \in G$.

3. This follows immediately from the result in 2. above. Take $f = I_g : G \to G$. Then, since $T_e f = Ad_g$, we obtain

$$I_g exp(\xi) = g^{-1} exp(\xi) g = exp(Ad_g \xi) \; \forall \xi \in \mathcal{G}.$$

4. We know that the flow of X_ξ is given by $\phi_t(g) = gexpt\xi$ so given $\xi, \eta \in G$, we can carry out the following computation

$$[\xi, \eta] = [X_\xi, X_\eta](e) = \frac{d}{dt} T_{\phi_t(e)} \phi_{-t}.X_\eta(\phi_t(e))|_{t=0}$$

$$= \frac{d}{dt} T_{expt\xi} R_{exp-t\xi} X_\eta(expt\xi)|_{t=0}$$

$$= \frac{d}{dt} T_{expt\xi} R_{exp-t\xi} T_e L_{expt\xi} \eta|_{t=0}$$

$$= \frac{d}{dt} T_e(L_{expt\xi} R_{exp-t\xi})\eta|_{t=0}$$

$$= \frac{d}{dt}\ Ad_{expt\xi}\eta|_{t=0}.$$

This completes the proof.

We now discuss group action of G on a manifold M. Group action is important as it is the main technique by which symmetry properties of physical systems are treated. If a Hamiltonian for a physical system is invariant under the action of some Lie group operation, this degree of freedom can be factored out of the equations of motion, leaving a *reduced* system in its place. This decrease in the number of degrees of freedom leads to a substantial calculational simplification.

Definition 1.4.3 *Let M be a smooth manifold. An action of G on M is a mapping Φ*

$$\Phi : G \times M \to M : (g, m) \to \Phi(g, m)$$

such that i)if $g = e$, $\Phi(e, m) = m$, ii) $\Phi(g, \Phi(h, m)) = \Phi(gh, m)$ for all $g, m \in G$.

The following structure will prove useful throughout all our future work. Define the Φ-orbit of some m in M by

$$G.m = \{\Phi_g(m) = \Phi(g, m) | g \in G\}$$

Φ is called transitive if there is simply one orbit,i.e., every point in the manifold can be joined to every other point by a suitable choice of an element of the Lie group, Φ is called effective if $g \to \Phi_g$ is 1-1, and Φ is called free if for each $m \in M$, the map $g \to \Phi_g(m)$ is 1-1.

We have already encountered a group action, namely the left translation, L_g. Thus in terms of our notation above, we have $\Phi(g,h) = L_g h$. Also, the by taking M to be the tangent bundle of the group G itself we can show that the adjoint mapping is also a group action. Let

$$\Phi : G \times \mathcal{G} :\to \mathcal{G} \ : (g,\xi) :\to Ad_g \xi.$$

The concept of action yields an infinitesimal analog which is vital to our description of mechanics. We again pose a definition

Definition 1.4.4 *Suppose $\Phi : G \times M :\to G$ is an action of the Lie group G on M. The infinitesimal approach to group action basically facilitates the construction of a vector field on M from an element of the Lie algebra of G. Taking $\xi \in \mathcal{G}$, form the 1-parameter subgroup in G, $expt\xi$ and look at the flow on M given by $\Phi(expt\xi, m)$. Differentiating this with respect to t at $t = 0$ yields*

$$\xi_M(m) = \frac{d}{dt}\Phi(expt\xi, m)|_{t=0} \tag{1.9}$$

which is called the infinitesimal generator of the action corresponding to ξ.

As above, we have already encountered infinitesimal generators of group action. In Proposition 1.4.1 iv), we showed that $\frac{d}{dt}Ad_{expt\xi}|_{t=0}\eta = [\xi,\eta]$.

Therefore, we conclude that $\xi_{\mathcal{G}} = ad_\xi$ where $ad_\xi \eta = [\xi, \eta]$ for the adjoint action.

Before proceeding to discuss co-adjoint orbits and symplectic leaves, it will be instructive to consider an infinite dimensional example of Lie group action.

Example 1 Consider the Lie group of diffeomorphisms from a manifold M onto itself, $Diff(M)$. We will see later that $Diff(M)$ does not form a Lie group in the sense that we have encountered so far. It is an example of an Inverse Limit Hilbert (ILH) group and the implications of this distinction will be investigated in Chapter 2. Ignoring these difficulties, we will proceed as if all our computations to date can still be implemented.

Define the action

$$\Phi : Diff(M) \times M \to M : (f, x) \to f(x) \; \forall \; f \in Diff(M) \; and \; x \in M.$$

Then consider the adjoint action which can be deduced from this,

$$Ad : Diff(M) \times T_{id}Diff(M) \to T_{id}Diff(M).$$

The Lie algebra of $Diff(M)$ may be identified with the algebra of vector fields on M, $\mathcal{X}(M)$. Therefore, $Ad : Diff(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$. Ad_ψ, $\psi \in Diff(M)$ is evaluated by differentiating I_ψ at the identity where

$$I_\psi : Diff(M) \to Diff(M) : \phi \to \psi \circ \phi \circ \psi^{-1}.$$

Let $X = \frac{d}{dt}_{|t=0}\phi_t$ where ϕ_t is a 1-parameter group in $Diff(M)$. So,

$$Ad_\psi(X) = (T_e I_\psi)(X) = \frac{d}{dt}_{|t=0}(I_\psi(\phi_t)) = \psi_* X. \tag{1.10}$$

We recall that ψ_* is the pushforward operation. Therefore, we have shown that the adjoint action in this infinite dimensional example is simply the pushforward which was encountered in Appendix 3.

1.5 Co-adjoint Action and Lie-Poisson Sytems

Group actions find widespread application in many of the examples in which we will be interested. Usually, a Lie group G acts on the tangent bundle to some configuration space Q giving the action

$$\Phi : G \times T^*Q \to T^*Q.$$

However, a very special case arises which will be of central concern to us and that is when $Q = G$. So the group action is acting on the tangent bundle to the group itself.

Given the Lie algebra $\mathcal{G}$ to G, we can form its algebraic dual $\mathcal{G}^*$ which is simply a space for which a non-degenerate pairing $<,>: \mathcal{G}^* \times \mathcal{G} \to R$ exists. In the case of volume preserving diffeomorphism groups, we will see that $\mathcal{G}$ is identified with the divergence free velocity fields on R^3 and that $\mathcal{G}^*$ is associated with the vorticity field. For fluid mechanics this is obviously the principal reason why these algebraic structures are of such interest. It will be shown that $\mathcal{G}^*$ forms a Poisson manifold with a bracket which is derived by extending functions on T_e^*G to the whole of the dual tangent bundle and then evaluating the bracket at e, the identity of G.

In this section, we will first show that if we take the space of $\mathcal{C}^\infty$ functions on $\mathcal{G}^*$, that they form a Poisson manifold with bracket

$$\{F, G\}(\mu) = \pm < \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}] > \ \forall \ F, G \in \mathcal{C}^\infty(\mathcal{G}^*).$$

We usually extend from $\mathcal{C}^\infty(\mathcal{G}^*)$ to $\mathcal{C}_L^\infty(T^*G)$ using left translations. This induces a minus sign in the bracket structure. The plus sign comes from translating to the right. We will then proceed to prove that by taking the co-adjoint action of G on the dual to its Lie algebra, we can foliate the space into co-adjoint orbits on which a non-degenerate, symplectic 2-form can be defined with respect to which a bracket can be formed which is *consistent* with the Lie Poisson bracket above. With these tools in place, we are ready to study the physical systems in which we are interested.

Consider the space of real-valued functions on T^*G. A function F in this space is left-invariant if $F \circ T^*L_g = F$ for all $g \in G$. T^*L_g preserves the canonical Poisson bracket and thus the space of all left-invariant functions on T^*G forms a Lie subalgebra of $(\mathcal{C}^\infty(T^*G), \{,\})$. A major result which we will present without proof states that

Proposition 1.5.1 *The space $\mathcal{C}_L^\infty(T^*G)$ of left invariant functions on T^*G is isomorphic to $\mathcal{C}^\infty(\mathcal{G}^*)$.*

Therefore, there will exist two mappings

$$\bar{\cdot} : \mathcal{C}^\infty(\mathcal{G}^*) \to \mathcal{C}_L^\infty(T^*G)$$

and

$$\hat{} : \mathcal{C}_L^\infty(T^*G) \to \mathcal{C}^\infty(\mathcal{G}^*),$$

such that if we take $F \in \mathcal{C}^\infty(\mathcal{G}^*)$, then

$$\overline{F}(\alpha_g) = F(T_e^* L_g(\alpha_g)) \; for \; \alpha_g \in T_g^* G$$

and for $H \in \mathcal{C}_L^\infty(T^*G)$,

$$\hat{H}(\alpha_e) = H(\alpha_e) \; for \; \alpha_e \in \mathcal{G}^*.$$

It is then required that $\overline{F}$ be left-invariant and that $\hat{\overline{F}} = F$ and $\overline{\hat{H}} = H$. Consequently, these two operations are inverse to each other and define an isomorphism.

A Lie algebra structure is then endowed on $\mathcal{C}^\infty(\mathcal{G}^*)$ via

$$\{F, G\} = \{\overline{F}, \overline{G}\}\hat{} \; \forall \; F, G \in \mathcal{C}^\infty(\mathcal{G}^*),$$

where the bracket on $\overline{F}$ and $\overline{G}$ is taken with respect to the symplectic 2-form defined on T^*G. This procedure gives rise to an explicit representation of the bracket in terms of the functional derivatives of F and G and the natural pairing between $\mathcal{G}$ and $\mathcal{G}^*$.

Proposition 1.5.2 *The Poisson structure on $\mathcal{G}^*$ takes the explicit form*

$$\{F, G\}(\mu) = - < \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}] > \tag{1.11}$$

where $F, G \in \mathcal{C}^\infty(\mathcal{G}^*)$ and $\mu \in \mathcal{G}^*$. Of course, we recall that $\frac{\delta F}{\delta \mu}$ is an element of $\mathcal{G}$.

Proof We will only prove the above formula in the case that F and G are linear functions. This may appear restrictive but it should be realised that the bracket is a linearisation of the behaviour of F and G at μ and thus, there is no loss of generality.

It will be recalled from section 1.2 of the current chapter that if F is a linear function, then the Frechet derivative of F is just F again. We can easily derive this from the definition of the derivative DF,

$$DF(x).h = \lim_{\epsilon\to 0}\frac{d}{d\epsilon}|_{t=0}F(x+\epsilon h),$$

where $F : M \to N$, is a mapping between two Banach spaces and $x, h \in M$. Thus, we see that $DF(x).h = F(h)$. Therefore in our case,

$$F(\mu) = DF(\mu).\mu =< \mu, \frac{\delta F}{\delta\mu} > . \tag{1.12}$$

Consider the extension of F to the space $\mathcal{C}^{\infty}_{L}(\mathcal{G}^*)$ and note the following

$$\overline{F}(\alpha_g) = F(T_e^* L_g(\alpha_g)) =< \psi, \frac{\delta F}{\delta\mu} >,$$

where $\psi = T_e^* L_g(\alpha_g) \in \mathcal{G}^*$. We can simplify this further to

$$< \alpha_g, T_e L_g(\frac{\delta F}{\delta\psi}) >=< \alpha_g, X_{\delta F/\delta\psi}(\alpha_g) > .$$

We then define a mapping $\sigma : \mathcal{X}(G) \to L(T^*G)$, the linear transformation on the tangent bundle to G through $\sigma(X)(\alpha_g) =< \alpha_g, X(g) >$. This yields

$$\overline{F}(\alpha_g) = \sigma(X_{\delta F/\delta\mu})(\alpha_g).$$

It can be shown that σ is an anti-isomorphism between $L(T^*G)$ and $\mathcal{X}(G)$ and that it anti-preserves the bracket on $\mathcal{X}(G)$ i.e. $\{\sigma(X), \sigma(Y)\} = -\sigma([X,Y])$.

Bringing all these facts, we find that

$$\{\overline{F}, \overline{G}\}(\alpha_g) = -\{\sigma(X_{\delta F/\delta\psi}), \sigma(X_{\delta G/\delta\psi})\}(\alpha_g)$$

$$= \sigma([X_{\delta F \delta\mu}, X_{\delta G/\delta\mu}])(\alpha_g),$$

which completes the proof.

We could have identified $\mathcal{C}^\infty(\mathcal{G}^*)$ with the right invariant functions on T^*G instead. Translating from the dual Lie algebra to the tangent bundle via right translations would have endowed the dual algebra with a Lie-Poisson structure almost the same as in the above theorem except that the minus sign would have been replaced by a plus.

We now turn to the adjoint and co-adjoint actions. Recall that the adjoint action of a Lie group is defined by the group acting on its own Lie algebra

$$Ad : G \times \mathcal{G} \to \mathcal{G}\ ,\ Ad_g(\xi) = T_e(R_{g^{-1}} \circ L_g)\xi \ \forall\ \xi \in \mathcal{G}\ and\ g \in G.$$

The co-adjoint action acts on the dual Lie algebra and is defined via the natural pairing $<,>$ between $\mathcal{G}$ and $\mathcal{G}^*$

$$Ad^*_{g^{-1}} : \mathcal{G}^* \to \mathcal{G}^*\ ,\ < Ad^*_{g^{-1}}\alpha, \xi > = < \alpha, Ad_g \xi > .$$

For $\mu \in \mathcal{G}^*$, the co-adjoint orbit of μ is defined by

$$\mathcal{O}_\mu = \{Ad^*_{g^{-1}}\mu | g \in G\} \tag{1.13}$$

and the isotropy group of the co-adjoint action at μ by

$$G_\mu = \{g \in G | Ad^*_{g^{-1}}\mu = \mu\}. \tag{1.14}$$

The co-adjoint orbit construct will be seen to be of central importance in fluid mechanics. We wish to build a tangent structure on these orbits and prove that they are in fact symplectic spaces. The implied symplectic 2-form which is defined on these orbits is known as the Kostant-Arnold-Kirillov-Souriau (KAKS) form and we will see later that it's intimately bound with the the Lie-Poisson bracket defined in the first half of this section.

Before defining the tangent space to a co-adjoint orbit, we first write down the infinitesimal generator of the co-adjoint action. By definition, $\xi_{\mathcal{G}^*}(\alpha) = \frac{d}{dt}|_{t=0} Ad^*_{exp-t\xi}(\alpha)$ and we can evaluate this by using the natural pairing so that

$$< \xi_{\mathcal{G}^*}(\alpha), \eta >= \frac{d}{dt}|_{t=0} < Ad^*_{exp-t\xi}\alpha, \eta >$$

$$= \frac{d}{dt}|_{t=0} < \alpha, -[\xi,\eta] >= - < \alpha, ad_\xi(\eta) > .$$

Therefore, we make the identification

$$\xi_{\mathcal{G}^*} = -ad^*_\xi. \tag{1.15}$$

forall $\xi \in \mathcal{G}$. Consider the co-adjoint orbit through μ and define the curve $c : R \to \mathcal{O}_\mu$ by $t \to Ad^*_{exp-t\xi}(\mu)$. We observe that $c(0) = \mu$ and that $\frac{d}{dt}|_{t=0}c(t) \in T_\mu\mathcal{O}_\mu$. It is quite easy to see from this that a tangent vector to the co-adjoint orbit is an infinitesimal generator corresponding to the co-adjoint action, for some $\xi \in \mathcal{G}$. Therefore, we have that

$$T_\mu\mathcal{O}_\mu = \{\xi_{\mathcal{G}^*}(\mu) | \xi \in \mathcal{G}\}.$$

This should not be too surprising given that $T_\mu \mathcal{O}_\mu \subset T_\mu \mathcal{G}^* \cong \mathcal{G}^*$. We now have the definitions at hand to state a theorem on the co-adjoint orbit's symplectic structure.

Proposition 1.5.3 *Let G be a Lie group and $\mathcal{O}$ a co-adjoint orbit. Then $\mathcal{O}$ is a symplectic manifold and there exists a unique symplectic 2-form ω_0 on $\mathcal{O}$ such that*

$$\omega_0(\mu)(\xi_{\mathcal{G}^*}(\mu), \eta_{\mathcal{G}^*}(\mu)) \equiv <\mu, [\xi, \eta]>$$

where $\xi_{\mathcal{G}^*}(\mu)$ and $\eta_{\mathcal{G}^*}(\mu)$ are elements in $T_\mu \mathcal{O}_\mu$.

We will now attempt to connect the two threads of this section. We will show that the symplectic leaves of the Lie-Poisson bracket are just the co-adjoint orbits equipped with the KAKS 2-form, i.e., for F and G, smooth functions on $\mathcal{G}^*$ and μ a representative member of the co-adjoint orbit $\mathcal{O}$,

$$\{F, G\}(\mu) = \{F|_{\mathcal{O}}, G|_{\mathcal{O}}\}(\mu) \equiv \omega_0(X_F|_{\mathcal{O}}, X_G|_{\mathcal{O}})(\mu). \tag{1.16}$$

Recall that $\{F, G\}(\mu) = - < \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}] >$, where we have endowed the Lie algebra with the Poisson structure derived from the subalgebra of left invariant vector fields on T^*G. The Hamiltonian vector field X_F on $\mathcal{G}^*$ can be computed to be ad^*_ξ, where $\xi = \frac{\delta F}{\delta \mu}$. However, we still need to show that the Hamiltonian vector fields on $\mathcal{O}$ which are derived from the restriction of F and G themselves restricted to $\mathcal{O}_\mu$ are also equal to elements of the tangent space $T_\mu \mathcal{O}_\mu$, i.e.,

$$X_{F|_{\mathcal{O}}} = ad^*_{\frac{\delta F}{\delta \mu}}(\mu).$$

We will assume this to be true and thus from the definition of the KAKS 2-form , we arrive at

$$\{F_{\mathcal{O}}, G_{\mathcal{O}}\}(\mu) = - < \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}] > .$$

This concludes the proof as we have now retrieved the Lie-Poisson bracket.

In the next chapter, we will encounter the reduction theorem for Hamiltonian systems which are invariant under some group action. We will find that when we have a situation in which a group G acts on the tangent bundle to the group G itself, then the *reduced* space will be isomorphic to a co-adjoint orbit. Thus, the Lie-Poisson bracket will be seen to be the fundamental geometric construct on which the dynamical equations evolve.

Chapter 2

The Reduction Theorem and Hamiltonian Fluid Mechanics

2.1 Introduction

In the last chapter, frequent references to Hamiltonian systems with symmetry were made. Traditionally, we are most familiar with conserved quantities or integrals of motion such as linear and angular momenta which arise through some Hamiltonian symmetry such as translational and angular invariance. This follows from Noether's theorem which basically states that for every Hamiltonian symmetry, there exists a corresponding conserved quantity. However, in the formalism that we will develop, these conserved quantities manifest themselves as mappings.

In this section, we will start with a Hamiltonian system defined on some symplectic space, P. We will then define an action of a Lie group G on P as explained in the last chapter. Using this action, the concept of momentum map will be developed. The momentum mapping is the repository of all information about the symmetries associated with the parent action and it provides a generalization of Noether's theorem for more topologically involved phase spaces. Equipped with the momentum map formulation of Noether's theorem, a *reduction of dimension* technique will be constructed. Essentially, what we wish to accomplish is the elimination of certain degrees of freedom in the system which are redundant. The reduced system which is obtained can be viewed as providing us with the most sparse description possible of the underlying physics. The simplest mathematical analogy is the formation of equivalence classes under some relational definition. If we take the space of square integrable functions on R, we identify two functions if they only differ by at most a set of measure zero. In the same way, we can separate the dynamics of some Hamiltonian system with symmetry into disjoint classes, members of the same class differing by properties which we deem inessential to the bare description of the physics. Of course, this may entail a loss of information but the gain in computational simplification is the advantage. The principal application that we have in mind is moving from a Lagrangian description of fluid flow to an Eulerian. In the Lagrangian, we track each fluid particle's trajectory, given its initial position. There is however a symmetry associated with these dynamics, namely the particle relabelling symmetry. The formalism will be developed later but effectively the result of factoring out this symmetry reduces the physics to the Eulerian description. The information loss consists of no longer being able to track individual particle trajectories.

2.2 Momentum Mappings and Reduction

In all the problems that we will look at, symmetries will manifest themselves through some Lie group action. The physical system is assumed to evolve on some symplectic phase space (P, Ω) and the action Φ is taken to be a symplectic action of a Lie group, G on P. If we have a Hamiltonian $H : P \rightarrow R$ which is invariant under the action Φ, i.e.,

$$H(\Phi_g x) = H(x) \, \forall \, x \in P, \tag{2.1}$$

then we wish to construct some conserved quantity corresponding to this invariance.

The conserved quantity is a mapping J from the phase space into the dual Lie algebra of G called the momentum mapping. At first, the construction of J seems very involved and contrived but this apparent complexity arises due to how much information must be encapsulated in its definition. The structure of the momentum mapping must reflect properties of the group and the group's action on phase space. It must in some sense preserve the symplectic structure because the action is symplectic and be preserved itself on $\mathcal{G}^*$ under the flow of the invariant Hamiltonian. Before defining J, we construct a related quantity which is a mapping from the Lie algebra to the $\mathcal{C}^\infty$ functions on P.

We know that $\Phi_g^* \Omega = \Omega$ by the fact that Φ acts canonically. Therefore, by using the usual trick of setting $g = expt\xi$ where $\xi \in \mathcal{G}$ and differentiating with respect to t at $t = 0$, we obtain

$$\frac{d}{dt}\Phi^*_{expt\xi}\Omega|_{t=0} = 0.$$

This is the definition of the Lie derivative of the differential form Ω along the vector field X whose flow is given by some ϕ_t on P. In this case, $\phi_t = \Phi_{expt\xi}$. This vector field turns out to be the infinitesimal generator of the action corresponding to ξ. Therefore

$$L_{\xi_P}\Omega = 0.$$

By Poincare's lemma, this implies that X is locally Hamiltonian, i.e., $i_{\xi_P} = dH$ for some $H : P \rightarrow R$. Assuming that ξ_P is globally Hamiltonian, we define a Hamiltonian function on P for every ξ in $\mathcal{G}$,

$$\hat{J} : \mathcal{G} \rightarrow \mathcal{C}^\infty(P); X_{\hat{J}}(\xi) = \xi_P. \tag{2.2}$$

If such a mapping exists, it can always be chosen to be linear. We can then define the momentum map as a function $J : P \rightarrow \mathcal{G}^*$ which is defined using the $\hat{J}$ construct,

$$< J(x), \xi >= \hat{J}(\xi)(x), \tag{2.3}$$

for all $\xi \in \mathcal{G}$ where $<,>$ is the natural pairing between the algebra and its dual.

To show that this actually represents a conserved quantity for a Hamiltonian H which is invariant under the group action, take the derivative with respect to time of

$$H(\Phi_{expt\xi}x) = H(x),$$

at $t = 0$ to yield

$$dH(x).\xi_P(x) = 0 \, \forall \, x \in P.$$

This is equivalent to the Lie derivative of H in the direction ξ_P being zero or

$$\{H, \hat{J}(\xi)\} = 0,$$

which proves that the function $\hat{J}(\xi)$ is invariant under the flow of X_H. From the definition of J, this implies that

$$J \circ \phi_t = J. \tag{2.4}$$

We will only encounter momentum mappings which are Ad^*-equivariant, i.e., $Ad^*_{g^{-1}}J(x) = J(\Phi_g x)$ for all $g \in G$ and $x \in P$. The most important example of these equivariant momentum mappings arises when P is a cotangent bundle. We have an action of the Lie group G on some configuration space Q and we lift it to $P = T^*Q$ by point transformations which we have encountered previously. This action is symplectic and has an Ad^*-equivariant momentum mapping J given by

$$J(\alpha_q)(\xi) =< \alpha_q, \xi_Q(q) >, \tag{2.5}$$

where $\alpha_q \in P$ and $\xi \in \}$.

As an example, take $P = T^*G$ and the action to be right multiplication. Therefore, define

$$\Psi : G \times G \to G, \; (g, h) \to hg.$$

$\Psi(g,h) = L_h g$ which implies

$$\xi_G(h) = \frac{d}{dt} L_h exp(t\xi)|_{t=0}$$

$$= T_e L_h \xi.$$

Lifting to T^*G, we derive the following momentum mapping

$$J(\alpha_h) = \alpha_h(\xi_{\mathcal{G}^*}(h)) = T_e L_h^* \alpha_h. \tag{2.6}$$

$J(\alpha_h) \in \mathcal{G}^*$ since $(T_e L_h)^*$ translates a 1-form on T_h^*G to T_e^*G. This example of lifted right group action on the tangent bundle will arise in the case of fluid mechanics because the Hamiltonian for inviscid, incompressible fluid flow will be seen to be right invariant.

We will now explore the reduction process. We will simply state the theorem without proof so as to not detract from the main goal of this chapter which is to show that the material description of fluid mechanics can be naturally represented as a Lie-Poisson Hamiltonian system. The following table will be used to summarise all the mappings that we will need in order to state the theorem. Again, it should be emphasized that even though the following appears to be extremely involved in both notation and conditions, the main result is quite simple and is the important piece of information to be digested fron this section before proceeding. We are simply saying that if we have a symplectic manifold on which some group symplectic action is defined, then as long as the momentum mapping can be defined and is Ad^*-equivariant, we can build a manifold from P which is also symplectic and whose 2-form is derived from the parent space. This new manifold will be called the reduced phase space. Once we have defined a Hamiltonian system on P, we will carry the dynamics onto this reduced space and explore the physics there.

Summary of Structures and Mappings for the Reduction Theorem

(P, ω) a symplectic manifold.

$\Phi : G \times P \to P$ a symplectic action.

$J : P \to \mathcal{G}^*$ an Ad^* equaivariant momentum mapping.

$\{g \in G | Ad^*_{g^{-1}}\mu = \mu\}$ isotropy subgroup of G under the co-adjoint action. We need G_μ to act freely and properly on $J^{-1}(\mu)$. An action is free if for each $x \in P$, $g \mapsto \Phi_g(x)$ is injective.

$J^{-1}(\mu)/G_\mu$ the orbit space which is well-defined since $J^{-1}(\mu)$ is a submanifold of P. We assume that μ is a regular value of J, i.e., for all $x \in J^{-1}(\mu)$ the mapping $T_xJ : T_xP \to T_{J(x)}\mathcal{G}^* \cong \mathcal{G}^*$ is surjective.

$\pi_\mu : J^{-1}(\mu) \to P_\mu$ canonical projection which must be a submersion, i.e. , for each $x \in J^{-1}(\mu)$, the mapping $T_x\pi_\mu : T_xJ^{-1}(\mu) \to P_\mu$ is surjective.

$i_\mu : J^{-1}(\mu) \to P$ the inclusion mapping.

With these constructs, we state the reduction theorem

Theorem 2.2.1 *Assume that* $\mu \in \mathcal{G}^*$ *is a regular value of* J *and that the isotropy subgroup* G_μ *acts freely and properly on* $J^{-1}(\mu)$. *Then* $P_\mu = J^{-1}(\mu)/G_\mu$ *has a unique symplectic form* ω_μ *such that*

$$\pi^*_\mu \omega_\mu = i^*_\mu \omega. \tag{2.7}$$

We will be most interested in the case where $P = T^*G$. As we saw previously, in this case $\xi_G(g) = T_e R_g \xi$ for left action and $J(\alpha_g) = T_e R_g^* \alpha_g$. Therefore,

$$J^{-1}(\mu) = \{\alpha_g \in T^*G | \alpha_g(T_e R_g.\xi) = \mu(\xi)\}, \tag{2.8}$$

which is the graph of the right invariant 1-form α_μ which equals μ at e; $\alpha_\mu(g) = T_g^* R_{g^{-1}}(\mu) = \mu \circ TR_{g^{-1}}$. We can see that this allows a diffeomorphism to be established mapping the reduced phase space P_μ to the co-adjoint orbit $\mathcal{O}_\mu$ through $\mu \in \mathcal{G}^*$,

$$\phi_\mu : P_\mu \rightarrow \mathcal{O}_\mu : \ \pi_\mu(\alpha_\mu(g)) \mapsto (Ad^*_{g^{-1}})(\mu), \tag{2.9}$$

for any $g \in G$. Also, the reduced phase space inherits the symplectic structure of KAKS through its identification with the co-adjoint orbit through μ.

Up to this point, we have been only concerned with reducing the phase space. Now, we introduce the idea of a reduced dynamics. If the physical system defined on P is invariant under the group action, then the Hamiltonian flow on P induces a Hamiltonian flow on P_μ. The induced Hamiltonian is denoted H_μ and obeys $H_\mu \circ \pi_\mu = H \circ i_\mu$.

For the case of left action on the co-tangent bundle to the action group itself, the reduced Hamiltonian on $\mathcal{O}_\mu$ will be by definition

$$H_\mu(Ad^*_{g^{-1}}\mu) = H(T_g^* R_{g^{-1}}(\mu)), \tag{2.10}$$

where $H \circ T^*L_g = H$ for all $g \in G$. In the next few sections, we will also be interested in the lifted right action on the co-adjoint orbits. However, the effect of this will only be to change the sign of the Lie-Poisson bracket which is equal to the KAKS induced bracket as we observed in the final section of chapter one. The main lesson to be learned from a study of this section is that it is possible to make phase space smaller when certain action symmetries exist in the Hamiltonian structure and that when the group is acting on its own cotangent bundle, the reduced phase space and dynamics can be described on the orbits of the co-adjoint action which are diffeomorphic to the Lie-Poisson structure on the dual Lie algebra.

2.3 Geometric Vortex Dynamics

The following is a detailed account of the Marsden and Weinstein[8] treatment of vortex dynamics as a Lie-Poisson system. In a fixed, compact domain, $\Omega \subseteq R^n$, the motion of an incompressible, inviscid fluid is described by elements of the configuration space $SDiff(\Omega)$. This space is the Lie group of volume preserving diffeomorphisms of Ω onto itself. The term Lie group is employed loosely and the reader is referred to Appendix D for a discussion of the structure of $SDiff(\Omega)$. The group is not strictly Lie but ILH (Inverse Limit Hilbert) instead. We will show that the Hamiltonian for the fluid flow which is defined on the cotangent bundle to the group is right invariant under the action of the group itself. Thus, by forming the reduced phase space under the right co-adjoint action as outlined in the last section, the reduced dynamics is totally determined by a Hamiltonian structure on $sdiff^*(\Omega)$, the dual Lie algebra of $SDiff(\omega)$. Elements of $sdiff^*(\Omega)$ are the 1-forms $\alpha = v_i dx^i$ acting on the divergence free vector fields in the Lie algebra, $sdiff(\Omega)$. The reduced phase space for some point in the dual Lie algebra will be diffeomorphic to the co-adjoint orbit through that point and thus, we will need to describe the co-adjoint action for $SDiff(\Omega)$ and find the KAKS 2-form and tangent space to the co-adjoint orbits for this specific case. For $\alpha \in sdiff^*(\Omega)$, consider the exterior derivative of α, $\omega = d\alpha$ which in R^3 reduces to $\nabla \times v$ which is the vorticity of the velocity field, v. This can be derived by using the examples for the Hodge $*$-operator and the $\flat$-operator outlined in Appendix C. So ω may be thought of as a *vorticity 2-form.* This is the equivalence class of 1-forms on $sdiff^*(\Omega)$ where 1-forms, α_1 and α_2 are identified if $d\alpha_1 = d\alpha_2$ so that they differ by at most an exact 1-form. This is a consequence of the Poincare lemma which is also discussed in Appendix C.

We recall from the end of section 1.4 that the adjoint action in the example of the diffeomorphism group is given by

$$Ad_{\psi}(X) = \psi_* X, \tag{2.11}$$

for $X \in T_e Diff(\Omega)$ and $\psi \in Diff(\Omega)$. Therefore, by using the natural pairing between the Lie algebra and its dual, we can show that the co-adjoint action is given by the pull-back of a diffeomorphism in the volume preserving group. Therefore the co-adjoint orbit with generic element 2-form ω becomes

$$\mathcal{O}_\omega = \{\eta^*\omega | \eta \in SDiff(\Omega)\}, \tag{2.12}$$

where η^* is the pull-back of η. We now construct the tangent vector space to $\mathcal{O}_\omega$, elements of which are given by $L_u\omega$, $u \in sdiff(\Omega)$, where L_u is the

Lie derivative in the direction u. Since we know that tangent vectors to a co-adjoint orbit through some ω are given by the co-adjoint infinitesimal generators at that point, we must show that such a Lie derivative of the vorticity form ω is the infinitesimal generator of the co-adjoint action on $sdiff^*(\Omega)$. To see this, take $u \in sdiff(\Omega)$ and the definition of the co-adjoint infinitesimal generator

$$u_{sdiff^*(\Omega)}(\omega) = \frac{d}{dt}|_{t=0} Ad^*_{exp-tu}(\alpha), \tag{2.13}$$

where α is an element of the equivalence class corresponding to ω. To complete the calculation, form the product of $u_{sdiff^*(\Omega)}$ with some $v \in sdiff(\Omega)$ via the natural pairing between the algebra and its dual. This leads to

$$\frac{d}{dt}|_{t=0} < Ad^*_{exp-tu}\alpha, v >$$

$$= \frac{d}{dt}|_{t=0} < \alpha, Ad_{exp-tu}v > = < \alpha, [u, v] >$$

$$= < \alpha, L_u v > = < L_u \alpha, v > .$$

Noting from Appendix C that the differential is natural with respect to the Lie derivative, we have

$$dL_u\alpha = L_u d\alpha = L_u\omega. \tag{2.14}$$

Therefore, the generator of the co-adjoint action is seen to be the Lie derivative of ω in the direction u. $(\mathcal{O}_\omega, \Omega_\omega)$ forms a symplectic leaf in $sdiff^*(\Omega)$ where the symplectic 2-form is given by the KAKS form. Ω_ω acts on pairs of elements of the tangent space to the co-adjoint orbit through ω. We will show that the actual form becomes

$$\Omega_\omega(L_{u_1}\omega, L_{u_2}\omega) = \int_\Omega \omega(u_1, u_2)dx, \tag{2.15}$$

for any $u_1, u_2 \in sdiff(\Omega)$. Under the right group action, the KAKS form is identified with the positive form of the Lie-Poisson bracket, so

$$\Omega_\omega(L_{u_1}\omega, L_{u_2}\omega) =$$

$$< \omega, -[u_1, u_2] >= -\int_\Omega \alpha.[u_1, u_2]dx. \tag{2.16}$$

Now, by using the definition of the Lie derivative on vector fields, the above equation reads $\int \alpha.L_{u_1}u_2$ which when integrated by parts becomes

$$\Omega_\omega(L_{u_1}, L_{u_2}) = \int_\Omega (L_{u_1}\alpha)u_2 dx =$$

$$\int_\Omega (i_{u_1}d\alpha + di_{u_1}\alpha).u_2 dx.$$

This follows from application of the chain rule and the formulae for the Lie derivative given in Appendix C. $di_{u_1}\alpha = (div u_1)\alpha = 0$ since u_1 is a divergence free vector field. Thus, we obtain

$$\Omega_\omega(L_{u_1}, L_{u_2}) = \int_\Omega (i_{u_1}\omega).u_2 dx = \int_\Omega \omega(u_1, u_2)dx. \tag{2.17}$$

This is the required result.

After having constructed the reduced phase space in the form of the co-adjoint orbit through ω, we need to look at the reduced Hamiltonian. As discussed in Appendix D, the Hamiltonian for the fluid flow on Ω is given as either a function on TG or on T^*G where $G = SDiff(\Omega)$. in the case of the tangent bundle, the Hamiltonian in the absence of external forces equals

$$H(V_\eta) = \int \frac{1}{2} < V_\eta, V_\eta > \mu, \tag{2.18}$$

where μ is the volume form on Ω and $V_\eta(x)$ is the velocity of the fluid particle x at $\eta(x)$. Under right translation, this functional is invariant. Take $\psi \in SDiff(\Omega)$ and consider

$$H(V_\eta \circ \psi) = \int_\Omega \frac{1}{2} < V_\eta\psi(x), V_\eta\psi(x) > \mu(\psi(x)).$$

This functional is invariant due to the change of variables theorem and the fact that $\eta^*\mu = \mu$. Therefore, we can reduce the Hamiltonian to the tangent space at the identity of the group. On the Lie algebra, the Hamiltonian reads

$$H(v) = \frac{1}{2} \int < v, v > \mu. \tag{2.19}$$

In order to apply the Lie-Poisson formalism of the last section, we need to find the form of the Hamiltonian function on the dual Lie algebra. This basically means expressing the energy in terms of the vorticity. The integrand in equation above can also be expressed in terms of $v^\flat$ so as to read

$$H(v) = \int_\Omega < v^\flat, v^\flat > . \tag{2.20}$$

From the Hodge-deRham theory, $v^\flat = \delta\Delta^{-1} dv^\flat$ is an identity for divergence free vector fields. Thus,

$$\int_\Omega < v^\flat, v^\flat > dx = \int_\Omega < \Delta^{-1} dv^\flat, dv^\flat > dx =$$

$$\int_{\Omega} < \Delta^{-1}\omega, \omega > dx. \tag{2.21}$$

This is a generalization of the 2-D result where the energy is the integral of the product of the stream function and the scalar vorticity. This defines the right-invariant Hamiltonian on $sdiff^*(\Omega)$. We will prove that the vorticity equations on the dual Lie algebra are equivalent to the Lie-Poisson equation $\dot{F} = \{F, H\}$ where $F, H : sdiff^*(\Omega) \to R$ and the bracket is Lie-Poisson as derived from the KAKS form. With the Hamiltonian given above, we find its functional derivative with respect to the vorticity, which is an element of the Lie algebra, to be $\frac{\delta H}{\delta \omega} = v$ where v is actually the velocity field which corresponds to the vorticity ω. So, by the Lie-Poisson bracket, we know that

$$\{F, H\}(\omega) = \int < \omega, [\frac{\delta F}{\delta \omega}, \frac{\delta H}{\delta \omega}] > \mu$$

$$= \int < \omega, [\frac{\delta F}{\delta \omega}, v] > \mu =$$

$$- \int < \mathcal{L}_v \omega, \frac{\delta F}{\delta \omega} > \mu$$

$$= -D(F)(\omega).\mathcal{L}_v \omega.$$

With, $F = F(\omega(t))$, we have

$$\frac{dF(\omega)}{dt} = DF(\omega).\frac{\partial \omega}{\partial t}.$$

Therefore, since F is arbitary, we find

$$\frac{\partial \omega}{\partial t} = -L_v \omega. \tag{2.22}$$

This simply implies that ω is Lie transported by the flow which completes the proof.

The situation simplifies considerably in 2-dimensions due to the fact that it is possible to identify $sdiff(\Omega)$ with C^∞-functions on Ω, at least up to the addition of arbitrary constants. This allows us to replace the Lie bracket of vector fields on $sdiff(\Omega)$ with the Poisson bracket of their corresponding scalar stream functions. The Lie-Poisson vorticity bracket can then be written in the form

$$\{F,G\}(\omega) = \int_\Omega \omega(x)\{\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}\}dx, \tag{2.23}$$

where $\omega, \frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}$ are regarded as functions on Ω and $\{,\}$ is the normal Poisson bracket in 2-D.

We will now specialize our discussion to the case of a vortex patch. The vorticity ω is regarded as the characteristic function of a particular fixed area, A in the plane; $\omega = \chi_A dx \wedge dy$. The motion of a vortex patch lies on an irregular symplectic leaf and the natural symplectic 2-form becomes quite simple. It actually reduces in local co-ordinates to a contour integral. Following Marsden and Weinstein[8], we let $\mathcal{O}_\omega$ be a co-adjoint orbit for the vortex patch $\omega = \chi_A dx \wedge dy$ and we take the velocity vector, $v \in sdiff(\Omega)$ with ψ as its corresponding stream function. Then

$$v^\flat = \delta(\psi dx \wedge dy),$$

where δ is the co-differential,

$$\psi dx \wedge dy = \Delta^{-1}\omega,$$

and

$$\omega = dv^\flat.$$

As before, $v^\flat$ is the natural co-vector in $sdiff^*(\Omega)$ corresponding to v. It is easily shown that the symplectic structure Ω_ω on $T_\omega \mathcal{O}_\omega$ where $\omega = \chi_A dx \wedge dy$, A$\subset$ R^2 is given by

$$\Omega(L_{v_1}, L_{v_2}) = \int_{\delta A} \psi_1 d\psi_2, \tag{2.24}$$

where ψ_1 and ψ_2 are stream functions for v_1 and v_2.

In the case of a circular patch representing the equilibrium configuration of a vortex, we can consider an area-preserving perturbation which is parametrized by

$$\phi(\theta) = 1/2(r^2 - 1).$$

Using this parameterization, we can write down the Poisson bracket of two functionals, F,G on $\mathcal{O}_\omega$ in the following way,

$$\{F, G\} = \Omega_\omega(L_{v_F}, L_{v_G}) = \int_{\delta A} \psi_F d\psi_G = \int_0^{2\pi} \frac{\delta F}{\delta \phi} \frac{d}{d\theta} \frac{\delta G}{\delta \phi} d\theta, \tag{2.25}$$

where we use the correspondence between $sdiff(\Omega)$ and $C^\infty(R^2)$. This can be achieved via application of the definitions of gradients and Frechet or directional derivatives in the 2-d context. The stream functions, ψ_F and ψ_G are defined by

$$\frac{dF}{d\epsilon}|_{\epsilon=0} = \int_{\partial M} \psi_F d\psi, \tag{2.26}$$

where ψ is the stream function corresponding to the velocity field of the flow so that the above is in fact, a directional derivative. The definition of the gradient of the density, F of a functional, $\mathcal{F}$ is

$$\frac{d}{dt}\mathcal{F}(\phi(t)) = D(F)(\phi).\frac{d\phi}{dt} = \int_0^{2\pi} \frac{\delta F}{\delta \phi} \frac{d\phi}{dt} d\theta.$$

Thus, using the plane polar form of the stream function, we see that $ru_r = \frac{\partial \psi}{\partial \theta}$ and

$$\frac{d\phi}{dt} = \frac{dr}{dt} = \frac{\partial \psi}{\partial \theta}.$$

Therefore, we can identify our stream function, ψ_F on the boundary of the patch with the Frechet derivative of F with respect to ϕ. One other point that should be noted is the difference between the functional ϕ and the corresponding density function, $\phi(\theta)$. It is always important to be aware of the distinction. The Hamiltonian formalism subsequently yields

$$\dot{\phi} = \int_0^{2\pi} \frac{d}{d\theta} \frac{\delta H}{\delta \phi} d\theta, \tag{2.27}$$

for the evolution of the perturbed vortex patch boundary, parameterized by the single polar angle, θ. This is the starting point for the derivation of a result due to Dritschel[11] for a wave traveling on the boundary of a single-valued patch boundary.

2.4 Vortex Patch Dynamics

Dritschel[11] uses Contour Dynamics to derive an integro-partial differential equation governing the time evolution of a perturbation to the boundary of a circular vortex patch. This equation is then analysed by means of a weakly non-linear expansion. The main aim of this section is to reproduce this equation from the geometric mechanics of section 2.3.

The spirit of the following derivation is due to the analysis carried out by Wan and Pulvirenti[10]. They outline an elegant way to write down the energy of a perturbed circular vortex patch. First, let us state the problem.

An incompressible, inviscid 2-D fluid is confined to the inside of a circular disc of radius R. At any point in this domain, the velocity field can be expressed in terms of a stream function φ,

$$(u, v) = (\varphi_y, -\varphi_x). \tag{2.28}$$

The vorticity field is given by $\omega = -\Delta\varphi$ and the energy of the fluid motion by $1/2 \int_D \parallel u \parallel^2 dxdy$. ω will usually be the characteristic function of some area in the domain of the disc. We will also denote the Green's function for ω by $G\omega$. Thus, $\Delta(G\omega) = -\omega$ and $G\omega \mid_{\delta D} = 0$. For the circular disc of radius γ whose vorticity we will denote by ω_0, we have

$$4G\omega_0 = \gamma^2 ln\frac{R^2}{r^2}, r \geq \gamma, \tag{2.29}$$

and

$$4G\omega_0 = (\gamma^2 - r^2) + \gamma^2 ln\frac{R^2}{\gamma^2}, r < \gamma. \tag{2.30}$$

As was stated above, a vortex patch is the characteristic function of an area within the disc. We will use the following notation to denote the patches. χ_f is a patch of unit strength centered at the origin with boundary defined by the radial function, $r = f(\theta)$. In what follows, use will also be made of the Green's function, K for the problem

$$-\Delta\psi = \omega \; in \; D \; and \; \psi = 0 \;\; on \; \partial D.$$

In this case, we will have $\psi(\xi) = \int_D K(z, \xi)\omega(z)dxdy$ with

$$K(z, \xi) = 1/2\pi(ln\frac{|\xi - z'|}{|\xi - z|} + ln\frac{|z|}{R}).$$

In the above formulae, ψ is the stream function; z,$\xi \in$ C; and $z' = (\frac{R^2}{|z|^2})z$. The energy of a vortex patch, χ_g, is given by $< \chi_g, G\chi_g >$ where $<,>$ is simply defined by $< g, h >= \int_D ghdxdy$. Consider a perturbation to χ_g denoted by χ_{g+h}. The difference between their respective energies is

$$E(\chi_{g+h}) - E(\chi_g) =< \chi_{g+h} - \chi_g, G\chi_g >$$

$$+ 1/2 < \chi_{g+h} - \chi_g, G\chi_{g+h} - G\chi_g >, \tag{2.31}$$

where

$$< \chi_{g+h} - \chi_g, G\chi_g >= \int_0^{2\pi} \int_{g(\theta)}^{g(\theta)+h(\theta)} G(\chi_g)(re^{i\theta})rdrd\theta. \tag{2.32}$$

We will now transform to the variable used in the previous section to represent the boundary of the vortex patch, namely, $\phi(\theta) = 1/2(r^2 - 1)$. In this parametrization, $rdr = d\phi$ and we will have $G\chi_g$ as a function of ϕ and θ. It is at this point that we depart from the precise formulation of Wan et al. in order to accomodate the Marsden and Weinstein[8] analysis. However, the approach is still basically the same. Thus, we consider h as being a small perturbation to the boundary, g and we try and express the energy in terms of a Taylor series about $r = g$ $(\phi = 0)$. First of all,

$$G(\phi, \theta) = G(0, \theta) + \frac{\partial G}{\partial \phi}(0, \theta)\phi + 1/2\frac{\partial^2 G}{\partial \phi^2}(0, \theta)\phi^2 + \ldots \tag{2.33}$$

from which we derive

$$< \chi_{g+h} - \chi_g, G\chi_g >= \int_0^{2\pi} (G(0)\phi + \frac{\partial G}{\partial \phi}(0)\frac{\phi^2}{2} + \frac{\partial^2 G}{\partial \phi^2}(0)\frac{\phi^3}{6})d\theta. \tag{2.34}$$

The second term equation() can be expressed in terms of the Green's function, K. In this case, we will have

$$< \chi_{g+h} - \chi_g, G\chi_{g+h} - G\chi_g >= \int_0^{2\pi} \int_0^{2\pi} K(\phi_g, \theta, \phi_{g'}, \theta')\phi(\theta)\phi(\theta')d\theta'.$$

Thus, we have an expression for the total energy of the perturbed vortex patch and we now proceed to consider the $g(\theta) = 1$ or the circular case. It is easily seen that $\int_0^\pi \phi d\theta = 0$, $\frac{\partial G}{\partial \phi}(0) = -1/2$ and that $\frac{\partial^2 G}{\partial \phi^2}(0) = 1$, giving

$$E(\chi_{g+h}) - E(\chi_g) = 1/2 \int_0^{2\pi} (-1/4\phi^2 + 1/6\phi^3) d\theta$$

$$+ 1/2 \int_0^{2\pi} \int_0^{2\pi} K(e^{i\theta}, e^{i\theta'}) \phi(\theta) \phi(\theta') d\theta d\theta'. \tag{2.35}$$

We now have an expansion of the energy or the Hamiltonian up to third order. In the notation of section 3.4 we have expanded $< \Delta^{-1}\omega, \omega >$ in a series in the perturbed quantity ϕ up to third order. It is thus possible to implement the calculation of the Lie-Poisson bracket for the evolution of the boundary of the vortex patch,

$$\dot{\phi} = \int_0^{2\pi} \frac{d}{d\theta} \frac{\delta \mathcal{H}}{\delta \phi} d\theta. \tag{2.36}$$

Carrying out the variation, $\frac{\delta \mathcal{H}}{\delta \phi}$, we obtain

$$\frac{\delta \mathcal{H}}{\delta \phi} = -1/4\phi(\theta) + 1/4\phi^2(\theta) + \int_0^{2\pi} K(e^{i\theta}, e^{i\theta'}) \phi(\theta') d\theta',$$

and thus,

$$\frac{d}{d\theta} \frac{\delta \mathcal{H}}{\delta \phi} d\theta = -1/4\phi_\theta + 1/4\phi\phi_\theta + \int_0^{2\pi} \frac{\partial K(\theta, \theta')}{\partial \theta} \phi(\theta') d\theta'.$$

Using the fact that $\dot{\phi} = \int_0^{2\pi} \phi_t d\theta$, we can easily deduce that

$$\phi_t = -\frac{1}{4}\phi_\theta + \frac{1}{4}\phi\phi_\theta + \int_0^{2\pi} K_\theta \phi d\theta', \tag{2.37}$$

up to third order in ϕ. We can further simplify the form of $\frac{\partial K}{\partial \theta}$ by carrying out some algebra to arrive at

$$\frac{\partial K}{\partial \theta} = -1/2\pi \frac{R^2 - 1^2}{1 + R^4 - 2R^2\cos(\theta - \theta')} \frac{\sin(\theta - \theta')}{1 - \cos(\theta - \theta')} \tag{2.38}$$

which in the limit, $R \to \infty$, leads to the term

$$-1/2\pi \int_0^{2\pi} \frac{\phi(\theta', t)\sin(\theta - \theta')}{1 - \cos(\theta - \theta')} d\theta',$$

in the evolution equation for the density $\phi(\theta)$ corresponding to the boundary functional, ϕ for the patch. This is the term for the Hilbert operator as reported in Marsden and Weinstein[8] and Dritschel[11].

2.5 Local Canonical Co-ordinates

The co-adjoint orbit for the circular vortex patch is a symplectic manifold with symplectic 2-form furnished by the KAKS form. However, it is not possible to find global canonical co-ordinates on the manifold as the patch very quickly evolves into a domain whose boundary is no longer a single-valued function of the polar angle θ. In the case where the single-valued regime is valid, we can easily find a set of canonical co-ordinates for the patch.

Let the boundary of the patch be decribed by the function $\phi(\theta)$ as above. Expand ϕ in a Fourier series in θ to yield

$$\phi(\theta) = \sum_{n \in Z} \phi_n e^{-in\theta}. \tag{2.39}$$

In terms of ϕ, each Fourier component ϕ_n can be expressed as

$$\phi_n = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) e^{-in\theta} d\theta.$$

Therefore, the functional derivative, $\frac{\delta\phi_n}{\delta\phi}$ is defined by

$$\int_0^{2\pi} \frac{\delta\phi_n}{\delta\phi}.g(\theta)d\theta = \frac{1}{2\pi}\frac{d}{d\epsilon}|_{t=0}\int_0^{2\pi}(\phi+\epsilon g)e^{-in\theta}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}gd\theta.$$

As a consequence, $\frac{\delta\phi_n}{\delta\phi} = \frac{1}{2\pi}e^{-in\theta}$. From the definition of the Lie-Poisson bracket for the vortex patch, it is easily seen that

$$\{\phi_n, \phi_m\}(\phi) = \mathcal{K} \ n\delta(m+n), \tag{2.40}$$

for some constant, $\mathcal{K}$.

By letting $q^i = \phi_{-i}$ and $p_i = \phi_i$, we have canonical coordinates $\{q^i, p_i\}$ on the vortex patch,

$$\{q^i, p_i\} = i\delta(i+j),$$

$$\{q^i, q^j\} = 0,$$

$$\{p_i, p_j\} = 0.$$

We could keep just a finite number of Fourier modes for ϕ which would provide a finite dimensional Hamiltonian truncation for the infinite dimensional manifold. However, the patch boundary would evolve rapidly so as to make this finite dynamics inapplicable in a very short period of time. In the next chapter, we will investigate a truncation for 2D vortex dynamics which will be a far more appropriate candidate for the study of finite dimensional approximations to the infinite dimensional system.

Chapter 3

The SU(N) Truncation of S_0DIFFT^2

3.1 Introduction

The incorporation of inviscid, incompressible fluid mechanics into the framework of Hamiltonian mechanics can be regarded as one of the great achievements of the geometric program initiated by Arnold[5] and completed by Marsden and Weinstein[8]. By expanding the class of configuration spaces in which conservative physical systems can evolve to Poisson manifolds, many *special cases* could be incorporated into a general framework. These include the dynamics of the rigid body in body co-ordinates, the equations of plasma physics and as we have seen, incompressible, inviscid fluid flow. At the same time as this theory was being developed, more traditional Hamiltonian systems were been investigated numerically through symplectic algorithms.

Symplectic integrators basically approximate phase space transformations by forcing the approximating flow to also be symplectic, the hope being that such integrators will more faithfully preserve the real solution structure especially for long time integrations. Ge and Marsden[9] have also constructed integrators for Poisson manifold transformations. These algorithms while far more mathematically involved than their canonical counterparts also offer the hope of accurate reconstruction of the phase flow trajectory and they have been successfully implemented by Channell and Scovel[14] for the case where the Poisson manifold is the dual of a Lie algebra. We saw in the last chapter that such a manifold foliates into symplectic leaves and that these leaves are isomorphic to the co-adjoint leaves of a Lie-Poisson system. The integrator of Channell and Scovel[14] implicitly preserves these orbits and thus preserves all Casimirs for the Lie-Poisson bracket. This follows from the fact that the orbit can also be viewed as a constrained surface parametrized by fixing values for the Casimirs. It is desirable to bring both these separate threads together and to use symplectic techniques to study the evolution of a vorticity field. However, the fluid system is described by an infinite dimensional Lie group and thus to implement numerics would require some form of truncation. The truncated system would have to be Hamiltonian and preferably also Lie-Poisson. So, the symmetry group and the configuration space would have to be the same. The resulting finite dimensional Lie-Poisson system would also have to converge in some limit to its infinite dimensional parent and the bracket would have to exhibit some analogous degeneracy. Such a truncation has been achieved for two-dimensional fluid flow on a twice periodic domain. This is a significant restriction but represents the possibilty of using Lie-Poisson integrators to investigate fluid mechanics. The Lie group which describes area preserving diffeomorphisms on the torus is $SDIFFT$ which has been investigated in quite some depth by Arnold. We will actually use an infinite dimensional Lie subgroup, S_0DIFFT which is the group of all area preserving diffeomorphisms on the torus which keep the center of mass of the fluid fixed. This subgroup is chosen because the space of stream functions which generate the divergence free vector fields for the phase flow are then single-valued on the domain. For this particular group, Hoppe[22] discovered that the special unitary group, $SU(N)$ with a particular choice of structure constants for its Lie algebra was a truncation for the stream function space. $SU(N)$ is a Lie group which is semi-simple and compact. From the point of view of numerically implementing a Lie-Poisson integrator, these two characteristics will be seen to be vital. The first section in this chapter will explore the truncated group and its algebra. The second section will discuss the truncated dynamics and analyse the finite dimensional Lie-Poisson bracket.

3.2 Groups of Diffeomorphisms

In 1966, Arnold published a paper on Lie groups endowed with one-sided invariant metrics, i.e., left or right invariant. His motivation was to try to explain rigid body dynamics in the most economical geometric way by showing that the principle of least action produced the Euler equations of motion in body co-ordinates over the Lie algebra. He succeeded in his endeavor and in the process observed an important fact: The method of Euler was not limited to the description of the rigid body but could also be used as the starting point for the study of any physical system whose configuration space was a Lie group and whose energy was defined through an invariant metric. The immediate application he had in mind was fluid mechanics. In the case of an incompressible, inviscid fluid in two or three dimensions, the configuration of the fluid can be specified by giving the positions of the fluid particles relative to their initial positions. As in the case of the rigid body, where the relative position of the body after a time interval is constrained by the condition that the positions of the elements of the body remain fixed with respect to each other, the fluid configuration is constrained by its incompressibility. This means that the transformations defining the relative motion of the fluid particles must belong to the group of volume preserving diffeomorphisms. This discussion is similar to that of the previous chapter except that now we are describing the motion with respect to the Lie algebra and not its dual.

Take a bounded region D in a Riemannian manifold and form the group of diffeomorphisms on D which preserve the volume form. This is denoted $SDiffD$. The Lie algebra to this group is the tangent space to $SDiffD$ at the identity transformation and consists of the vector fields of zero divergence on D which are tangent to the boundary of D(if it is non-empty.) The kinetic energy can be defined by using the following metric

$$< v_1, v_2 >= \int_D (v_1, v_2) dx \tag{3.1}$$

where v_1 and v_2 are elements of the Lie algebra and the integrand is simply the Euclidean inner product.

The motion of the fluid is given by a trajectory through the diffeomorphism group $t \rightarrow g_t$. The kinetic energy of the moving fluid is a right invariant Riemannian metric on the group of volume preserving diffeomorphisms. The principle of least action asserts that flow of an ideal fluid is a geodesic in the above metric.

In the 2D case, the velocity fields can be represented by Hamiltonian functions which are called stream functions in hydrodynamics and which we will denote by ψ. The commutator of two fields turns out to be the Jacobian of the stream functions for the original fields.

For our purposes, we will be concentrating on the manifold of area preserving diffeomorphisms on a twice periodic domain in R^2. This geometry is isomorphic to a 2-torus and we will thus denote this space by $SDiffT$. We will study the submanifold whose vector fields are divergence free and have single-valued stream functions. The corresponding subgroup is S_0DiffT which is the space of area preserving diffeomorphisms which preserve the center of mass of the torus. Elements of the Lie algebra can be thought of as stream functions on the torus whose body integral over T is zero. It is easy to decompose such stream functions into their Fourier series and thus form a complete basis for the function space of single-valued stream functions on T. This basis has members of the form

$$e_k = e^{ik.x} \tag{3.2}$$

where k is a wave vector in $Z^2 \setminus \{0\}$ and x is an element of the 2-torus. The $\{0, 0\}$ element is not included as it is zero for single-valued stream functions.

The following theorem due to Arnold expresses the kinetic energy Riemannian metric, the Lie bracket on the algebra and some results concerning the Riemannian connection in terms of the Fourier basis for the Lie algebra, $s_0 diffT$.

Proposition 3.2.1 *The explicit form of the scalar product, commutator, connection and curvature for the group $S_0 DiffT$ are given by*

$$< e_k, e_l >= k^2 S\delta(k+l), \tag{3.3}$$

$$[e_k, e_l] = (k \times l)e_{k+l}, \tag{3.4}$$

$$\nabla_{e_k} e_l = d_{l,k+l} e_{k+l}, \tag{3.5}$$

$$R_{k,l,m,n} = (a_{ln}a_{km} - a_{lm}a_{kn})S\delta(k+l+m+n), \tag{3.6}$$

where S is the area of the 2-torus, δ is the delta function, ∇ is a Riemannian connection and

$$d_{u,v} = \frac{(v \times u)(u.v)}{v^2},$$

and

$$a_{uv} = \frac{(u \times v)^2}{|u+v|}.$$

In what follows, we shall consider possible ways of truncating the group of area preserving diffeomorphisms on the torus using the group of unitary matrices of determinant one, $SU(N)$. The corresponding Lie algebra, $su(N)$ will be studied as an approximate algebra whose elements represent finite analogs of either the velocity field or the vorticity. It is hoped that this truncation may be used to build a Lie-Poisson integrator which would have long time qualitative stabilty and thus, could shed some light on 2-D turbulence.

Bordemann, Hoppe et al. [7] present a mathematically elegant way to effect truncation of infinite dimensional Lie algebras of the type which interests us. They consider a family of real or complex Lie algebras $L_\alpha, \alpha\epsilon I$ with bracket $[,]_\alpha$ which approximates a Lie algebra (L,[,]) as $\alpha \to \infty$. The question arises as to how effective the approximation is and how one can isolate a statement of convergence in the Lie algebra sense. We will present the arguments outlined by Bordemann et al. in theorem form and then consider $sdiffT$ as an example of (L,[,]) and su(N) for L_α where $\alpha \cong N$.

Proposition 3.2.2 *Let us assume that we can provide the approximate Lie algebras with the additional structure:*

1. There exists a surjective map $p_\alpha : L \to L_\alpha$ *for all* α.

2. If we have some metric d_α *on each* L_α, *then we assume that if for each* $x,y\epsilon L$, *if* $d_\alpha(p_\alpha, p_\alpha) \to 0$ *as* $\alpha \to \infty$, *then* $x=y$.

With the above assumptions, we call $(L_\alpha, [,]_\alpha, d_\alpha)$ *an approximating sequence for (L,[,]) induced by* $(p_\alpha, \alpha\epsilon I)$ *and* L *an* L_α*-quasilimit if the following axiom is also valid,*

3.For each $x,y\epsilon L$,

$$d_\alpha(p_\alpha[x,y], [p_\alpha(x), p_\alpha(y)]_\alpha) \to 0 \tag{3.7}$$

as $\alpha \to \infty$.

With the above structures, we can avoid the problem of the same sequence of algebras approximating non-isomorphic algebras by noting that the Lie structure on the underlying vector space L *of the* L_α*-quasilimit is uniquely specified once the* $(p_\alpha, \alpha\epsilon I)$ *are specified.*

Let us now apply the above general structure to the problem at hand, namely the algebra of divergence free vector fields on the torus with singlevalued stream functions. The example below is not provided with all the details needed to tie down the approximating sequence completely but should provide the flavour of the formalism.

Proposition 3.2.3 *Let us start with the Lie algebras L_Λ defined by the structure constants*

$$C^k_{m,n} = \frac{N}{2\pi}\sin\frac{2\pi}{N}(m\times n)\delta(m+n-k)|_{modN}, \tag{3.8}$$

Then, as $N \to \infty$, the algebras converge to $sdiffT$ which we denote by L. We can factor out an ideal for each L_Λ, where Λ is equal to $\frac{1}{N}$, and define the structures required in the above theorem to induce uniqueness of convergence. The approximating algebras are seen to be su(N) with a specific basis choice.

Proof The basis functions which give rise to the above structure constants will be denoted by $\{T_m | m \in Z^2\}$. Therefore,

$$[T_m, T_n]^N = C^k_{m,n} T_k$$

is the Lie bracket with the structure constant as defined in the above example. We see that as $n \to \infty$, the bracket reduces to the bracket for $s_0 diffT$ as expressed in the above theorem. The algebras generated by $[,]^N$ can be made finite dimensional by identifying the basis element T_m with T_{m+Na} where $a \in Z^2$. The resulting algebra will satisfy all the requirements of theorem 3.3.2 with the surjective mapping between L and L^N being given by the canonical projection mapping,

$$\phi_N : L \to L^N \; : \; \phi_N(T_{m+aN}) = T_m,$$

where we have umklapped the external modes back onto the finite lattice, $0 \leq m_1, m_2 \leq N$. Given a metric d_N on L^N which satisfies the following condition: If $x = \sum r_m T_m$ and $y = \sum s_m T_m$, then

$$\phi_N(x) - \phi_N(y) = \sum (r_m - s_m)\phi_N(T_m). \tag{3.9}$$

We then find that

$$d_N(\phi_N(x) - \phi_N(y)) = \sum |r_m - s_m r|^2 = 0$$

if and only if $x = y$. We also find that L will be an L^N quasi-limit because of the following: Without loss of generality take $x = T_m$ and $y = T_n$ and consider

$$d_N(\phi_N([T_m, T_n])) - [\phi_N(T_m), \phi_N(T_n)]. \tag{3.10}$$

We find that as $N \to \infty$, the expression above goes to zero.

The L^N factored algebras offer a converging sequence of finite dimensional algebras which may be used as an approximation for $s_0 diffT$ in the $N \to \infty$ limit. In order for us to take advantage of this approximation, a representation for the algebra will be needed. It transpires that the special unitary group, $SU(N)$ with a special choice of algabra basis vectors replicates the above structure constants. Therefore, we can say that the Lie algebra of $S_0 DiffT$ is approximated by the Lie algebra of $SU(N)$. We will demonstrate this fact in the next section.

3.3 The t'Hooft Basis for SU(N)

We first present some details of the group $SU(N)$ and its algebra. $SU(N)$ is a Lie subgroup of the group of non-singular linear transformations $GL(N)$. Its members are unitary and have determinant one. It is a matrix group and thus can be defined by

$$SU(N) = \{U \in GL(N) | U^{\dagger}U = I \, and \, det(U) = 1\}. \tag{3.11}$$

The Lie algebra $su(N)$ will be the set of traceless, anti-hermitian matrices. This is seen by taking a curve, $U(t)$ passing through the identity of $SU(N)$ and differentiating the two properties above with respect to t at $t = 0$. Therefore,

$$su(N) = \{A \in gl(N) | A^* = -A \, and \, tr(A) = 0\}. \tag{3.12}$$

We are free to choose whatever basis we like but the following choice due to t'Hooft will provide the finite dimensional connection with $SDiffT$ from the previous section. For N odd, consider the following pair of $N \times N$ unitary matrices

$$g = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ 0 & \omega & \ldots & \ldots & 0 \\ \vdots & 0 & \omega^2 & \ldots & 0 \\ \vdots & \ldots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \ldots & \omega^{N-1} \end{pmatrix}, \tag{3.13}$$

$$h = \begin{pmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}, \tag{3.14}$$

where $\omega = e^{\frac{4\pi i}{N}}$ is a root of unity. It can easily be shown that $g^N = h^N = I$. The basis for $su(N)$ is constructed via $T_n = \omega^{\frac{n_1 n_2}{2}} g^{n_1} h^{n_2}$, $n \neq (0,0)$. These basis elements satisfy an invariance condition, $T_n = T_{n+aN}$ where $a \in Z^2$ is arbitrary. This has the effect of partitioning Z^2 into $N \times N$ cells. For our purposes, we will choose a cell centered on the origin in Z^2 with $-M \leq n_1, n_2 \leq M$ where $M = \frac{N-1}{2}$. We will call this cell Ω. The Lie algebra bracket defined as the usual matrix commutator, $[A, B] = AB - BA$, yields

$$[T_n, T_m] = \frac{N}{2\pi} \sin(\frac{2\pi}{N}(n \times m)) T_{n+m}, \tag{3.15}$$

where the structure constants are defined by

$$C^k_{i,j} = \frac{N}{2\pi} \sin(\frac{2\pi}{N}(i \times j)) \delta(k - i - j)|_{modN}. \tag{3.16}$$

These are seen to be the structure constants for the Lie algebra sequence which converged to $s_0 diffT$ in the previous section.

The set $\{T_n | n \in \Omega\}$ does not actually produce elements of $su(N)$ in a straightforward manner. The reason for this is that each elemnet of the basis is not actually an element of $su(N)$. In fact, the hermitian conjugate of T_n equals $-T_{-n}$. However, the linear combinations $T_n - T_{-n}$ and $i(T_n + T_{-n})$ are elements of $su(N)$ and thus only linear combinations of the T_n's which yield this decomposition are allowed. This corresponds to what we would expect since all stream functions and velocity fields will have to be either real-valued functions or real-valued vectors. We will be able to express all quantities as expansions in terms of T_n's but the co-efficients will have to obey the property that they produce real-valued quantities. This is analogous to the case of a stream function on the torus which is single-valued. This function can be expanded in terms of the Fourier basis e_k over Z^2. However, the coefficients, ψ_k will obey $\overline{\psi_k} = \psi_{-k}$.

Since $SU(N)$ is a matrix group, the adjoint action of $SU(N)$ on $su(N)$ will simply be matrix conjugation, i.e., if $U \in SU(N)$, then $Ad_U \xi = U^* \xi U$ for all $\xi \in su(N)$ and the corresponding Lie algebra action will be $ad_\xi \eta = [\xi, \eta]$ where $\xi, \eta \in su(N)$. We will use $SU(N)$ as a finite dimensional model for exploring fluid mechanics on a 2-torus. In order to conform to the theory in chapter 2, we will develop a *vortex* dynamics on the dual Lie algebra, $su^*(N)$ and investigate its implications. However, before proceeding, we must give a summary of fundamental results from the theory of semi-simple Lie algebras.

3.4 Lie Algebra Theory

The reader is referred to Sattinger and Weaver[20] for a good review of classical Lie algebras. A Lie algebra is a vector space over a field F with a product $[,] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ satisfying

i) $[,]$ is closed and linear over F.

ii) $[,]$ is antisymmetric, and

iii) the bracket satisfies the Jacobi identity, i.e., $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$.

A subalgebra $\mathcal{S}$ of $\mathcal{G}$ is an ideal if $[S,\mathcal{G}] \subset \mathcal{G}$. A Lie algebra $\mathcal{G}$ is semi-simple if it contains no ideals other than itself and the zero element. The most well-behaved semi-simple Lie algebras are over the complex numbers. In our $su(N)$ case, we will be primarily interested in its complexification, $sl(N,C)$ which is the Lie algebra of the special group of determinant one $N \times N$ matrices. The elements of $sl(N,C)$ can be identified with the traceless $N \times N$ matrices in $gl(N,C)$. The relationship between $su(N)$ and $sl(N,C)$ is important. $su(N)$ is an example of a real form of $sl(N,C)$. This is achieved by selecting a basis for $sl(N,C)$ which makes the structure constants real. However, there can be multiple real forms derived from one semi-simple Lie algebra over the complex numbers. For this reason, the classification of the real forms of semi-simple Lie algebras is more formidable than in the complex case. As an example, $su(2)$ and $sl(2,R)$ are two non-isomorphic Lie algebras with the same complexification, $sl(2,C)$.

A fundamental linear transformation on a Lie algebra is the adjoint representation, $ad : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}, ad(X)Y = [X,Y]$. It is a representation of the Lie algebra since it is linear and $ad([X,Y]) = [ad(X), ad(Y)]$. This last property can be derived from the Jacobi identity. If $\{E_i\}$ is a basis for $\mathcal{G}$, then

$$ad(E_i)E_j = C^k_{i,j}E_k, \tag{3.17}$$

where summation is implied and the $C_{i,j}^k$ are structure constants for the Lie algebra.This provides a matrix representation of $\mathcal{G}$,

$$adE_i : \mathcal{G} \rightarrow \mathcal{G}, (ad(E_i)X)_j = (M_i)_{j,k}X_k, \tag{3.18}$$

where $(M_i)_{j,k} = C_{i,k}^j$. This matrix form will be useful in the to follow. From this point on, we will concentrate on the semi-simple Lie algebras.

The Cartan-Killing form is a real-valued bilinear, Ad-invariant quadratic form on $\mathcal{G}$. It is defined as

$$K(X,Y) = Tr(adX \circ adY). \tag{3.19}$$

The importance of the Killing form lies in the following result due to Cartan: The Cartan criterion states that a Lie algebra is semi-simple if and only if its Killing form is non-degenerate, i.e., if $K(X,Y) = 0$ for all $Y \in \mathcal{G}$, then $X = 0$. In this case, we can use the Killing form to define a metric tensor for the Lie algebra. This tensor, g is defined by $g_{i,j} = K(E_i, E_j)$ which can be shown to be equivalent to $g_{i,j} = C_{i,s}^r C_{j,r}^s$ where summation is implied. This metric can be used as a raising and lowering operator for in the case that the form is non-degenerate, it is possible to consider the dual $\mathcal{G}^*$ to be isomorphic to $\mathcal{G}$. We will constantly make use of this property in the design of Lie-Poisson integrators which are discussed in the next chapter.

The next topic to be discussed will be the Cartan subalgebra of a semi-simple Lie algebra. This is the maximal abelian subalgebra $\mathcal{H}$ of $\mathcal{G}$ such that adH is semi-simple for all $H \in \mathcal{H}$. Since $\{adH | H \in \mathcal{H}\}$ forms a commuting family of semi-simple operators, there exists a basis for $\mathcal{G}$ in which these operators are simultaneously diagonalizable.

The rank of $\mathcal{G}$ is the dimension of its Cartan subalgebra which can be shown to be invariant under a change of algebra basis. The roots of $\mathcal{G}$ are the functionals on $\mathcal{H}$ which satisfy

$$[H, E_\alpha] = \alpha(H)E_\alpha, \tag{3.20}$$

for some $E_\alpha \in \mathcal{G}$. The E_α are called the root vectors. An important decomposition of the Lie algebra occurs under the one dimensional subspaces spanned by these root vectors, namely

$$\mathcal{G} = \mathcal{H} + \bigoplus_\alpha \mathcal{G}_\alpha \tag{3.21}$$

where $\mathcal{G}_\alpha = \{X | adH(X) = \alpha(H)X\}$ where α is an element of the set of non-zero roots of $\mathcal{H}$.

The 1-dimensional root subspaces are orthogonal complements of each other for distinct roots and for every $\alpha \in \mathcal{G}^*$, there exists a unique $H_\alpha \in \mathcal{G}$ such that $\alpha(H) = K(H_\alpha, H)$. This last observation allows us define an inner product on $\mathcal{H}^*$; if $\alpha, \beta \in \mathcal{H}^*$, then $< \alpha, \beta >= K(H_{\alpha,\beta})$. The collection of roots of a semi-simple Lie algebra can be graphically represented in a Dynkin diagram. This diagram is strongly constrained and is the key to the classification of all complex semi-simple Lie algebras. $sl(N, C)$ has the simplest root diagram and the way that the graph is extended as $N \to \infty$ determines the limiting non-isomorphic infinite dimensional algebras. We will not explore these ideas in any more depth. It should be indicated that any analytic results that need to be derived such as integrals over $su(N)$ require a knowledge of the algebra root structure. An example where this would arise is the the statistical mechanics of an $su(N)$ vortex equilibrium.

This topic will be discussed again in the conclusions. The reader is referred to Harish-Chandra(25).

As an example of working with the t'Hooft basis, we will compute the roots and root vectors of $su(N)$ in terms of the basis $\{T_n\}$. $su(N)$ is a real-form of the semi-simple Lie algebra $sl(N, C)$. In matrix terms, this is equivalent to the statement that every element of $su(N)$ may be simultaneously diagonalized by conjugation with a suitable unitary matrix in $SU(N)$. A subset of $\{T_{mn}$ forms a natural and convenient basis for the Cartan subalgebra of $su(N)$, namely $\{T_{p0} | -M \leq p \leq M, p \neq 0\}$ where $M = \frac{N-1}{2}$. For matrix algebras, the Cartan subalgebra always turns out to be the maximal dimension diagonal matrix algebra. If we choose a basis $E_{ij} = \delta_{ij}$ for $su(N)$, then an element H in $\mathcal{H}$ is of the form $H = diag(\lambda_{-M}, \ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_M)$ with $trH = 0$ and the λ_i all imaginary.

In terms of the basis, $\{T_{p0}\}$, $H = \sum \Lambda_p T_{p0}$ is also $N-1$ dimensional. Using

$$T_{mn} = \sum_{l=-M}^{M} \omega^{m(M+l+\frac{n}{2})} E_{l,l+m}, \tag{3.22}$$

and

$$T_{-p0} = \sum_{l=-M}^{M} \omega^{-p(M+l)} E_{l,l}, \tag{3.23}$$

we find

$$\Lambda_p = Tr(HT_{-p0}) = \sum \lambda_i \omega^{-p(M+l)} Tr(E_{ii}E_{ll}), \tag{3.24}$$

which reduces to

$$\sum_{l=-M}^{M} \lambda_l \omega^{-p(M+l)}. \tag{3.25}$$

It is easy to see that the $E_{ij}, i \neq j$ form the generators of the 1-dimensional root subspaces. We have that

$$[H, E_{ij}] = \sum_{p\in\Omega}[\Lambda_p\omega^{p(M+i)} - \Lambda_p\omega^{p(M+j)}]E_{ij}. \tag{3.26}$$

Therefore, $\alpha_{ij}(H) = \sum_{-M}^{M} \Lambda_p(\omega^{p(M+i)} - \omega^{p(M+j)})$ are the roots of the sub-algebra element H in this particular basis.

3.5 Truncated Vorticity Dynamics

We will now study the Lie-Poisson dynamics on $\mathcal{G}^* = su^*(N)$. Zeitlin[6] was the first to detail the application of $su(N)$ to fluid mechanics. In this section, we will go one step further and show that the form of the Lie-Poisson system fits into the standard geodesic fomulation of Arnold and that the Hamiltonian is right-invariant and can be expressed in terms of the Cartan-Killing metric. This will correspond to a truncated dynamics on $s_0 diffT$. $su^*(N)$ is the dual to the algebra $su(N)$ and we specify the natural basis $\{S^k\}$ such that $S^k.T_m = \delta_{k,m}$. The construction derives by a Fourier decomposition of a circulation zero initial vortex distribution ω_0,

$$\omega_0(x) = \sum_{k\in Z^2} \omega_{k0}e^{-k.x},$$

where $x \in T$.

The approximation is made by replacing the basis of Fourier modes by the dual t'Hooft basis for $su(N)$ over a finite lattice $\Omega \in Z^2$. Thus,

$$\omega_0(x) = \sum_{k\in\Omega} \omega_{k0}S^k \tag{3.27}$$

where the $\{\omega_{k0}\}$ are the $N^2 - 1$ Fourier components of the original vorticity distribution. The evolution of Hamiltonian systems on the dual to a Lie algebra is generic and the following analysis applies equally well to the motion of a rigid body in material or body co-ordinates as it does to vortex dynamics. We will start by explicitly stating the equation of motion on $\mathcal{G}^*$ driven by some Hamiltonian $H : \mathcal{G}^* \to R$.

Proposition 3.5.1 *The equations of motion for the $\pm$ Lie-Poisson brackets for a physical system driven by $H : \mathcal{G}^* \to R$ are*

$$\dot{\mu} = \frac{d\mu}{dt} = \mp ad^*_{\frac{\delta H}{\delta \mu}} \mu \tag{3.28}$$

for $\mu \in \mathcal{G}^*$. It will be recalled from the definition of the functional derivative that $\frac{\delta H}{\delta \mu} \in \mathcal{G}$. The plus or minus sign originates with whether or not the bracket is deduced by identifying elements of $\mathcal{G}^*$ with right-invariant or left invariant vector fields on T^*G.

To prove this, consider $F \in \mathcal{C}^\infty(\mathcal{G}^*)$ and $\mu(t) \in \mathcal{G}^*$. Then

$$\frac{dF}{dt}(\mu) = dF(\mu(t)).\frac{d\mu}{dt} =< \frac{\delta F}{\mu}, \dot{\mu}(t) >$$

and

$$\{F, H\}_\pm(\mu) = \pm < \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}] >=$$

$$\mp < \mu, ad_{\frac{\delta H}{\delta \mu}}.\frac{\delta F}{\delta \mu} >, \tag{3.29}$$

which by definition of the natural pairing between the Lie algebra and its dual, equals

$$\mp < \frac{\delta F}{\delta \mu}, ad^*_{\frac{\delta H}{\delta \mu}}(\mu) > . \tag{3.30}$$

Non-degeneracy of the pairing implies that

$$\dot{\mu} = \mp ad^*_{\frac{\delta H}{\delta \mu}} \mu. \tag{3.31}$$

On $s_0 diff^* T$, the Hamiltonian of some circulation zero vortex field can be shown to be

$$H(\omega) = \sum_{k \in Z^2} \frac{1}{k^2} \omega_{-k} \omega_k. \tag{3.32}$$

In the $su^*(N)$ case, it is natural to choose the Hamiltonian to be the finite dimensional analog of this Hamiltonian function

$$H(\omega) = \sum_{k \in \Omega} \frac{\omega_k \omega_{-k}}{k^2}. \tag{3.33}$$

We will take advantage of the existence of a non-degenerate, symmetric bilinear form $(,)$ on $su(N)$ that is invariant under the adjoint mapping, $Ad(g)$, i.e., $(Ad(g)\xi, Ad(g)\eta) = (\mu, \eta)$. Not all groups possess such a structure. However, if the algebra is semi-simple, then the Cartan-Killing form of the last section has these exact properties. The existence of such a form has profound implications for the structure of the co-adjoint orbits on a dual Lie algebra. Primarily, it means that there will exist a diffeomorphism between elements of $\mathcal{G}$ and $\mathcal{G}^*$. Consider $\omega \in \mathcal{G}^*$ and an arbitrary $\xi \in \mathcal{G}$, then there will exist a unique $\psi \in \mathcal{G}$ such that

$$\omega . \xi = K(\psi, \xi). \tag{3.34}$$

Also, it transpires that in a Lie algebra with such a form, the co-adjoint orbits are identifiable with adjoint orbits and the KAKS symplectic 2-form will have a counterpart on $\mathcal{G}$. Thus, the Lie-Poisson dynamics can equally well be considered an evolution on the algebra as on its dual.

For $su^*(N)$, we will first show that the Killing metric defined by

$$g_{mn} = K(T_m, T_n),$$

reduces to a particularly simple form. This metric is equivalent to

$$g_{mn} = \sum_{r,s\in\Omega} C^r_{ms} C^s_{nr},$$

where $C^r_{ms} = \frac{N}{2\pi}\sin\frac{2\pi}{N}(m\times s)\delta(r-m-s)|_{modN}$ are the structure constants in this basis. This summation becomes

$$g_{mn} = \sum_{p\in\Omega} \frac{N^2}{4\pi^2}\sin\frac{2\pi}{N}(m\times p)\delta(m+n)|_{modN}. \tag{3.35}$$

However, the co-efficient of the Kronecker delta can be simplified to

$$\sum \frac{N^2}{4\pi^2}\sin^2\frac{2\pi}{N}(m\times p) = \frac{N^4}{4\pi^2}.$$

This can be proven by making use of the identities

$$\sum_{k=-m}^{m} \cos\frac{lk2\pi}{2m+1} = h(j), \tag{3.36}$$

and

$$\sum_{k=-m}^{m} \sin\frac{lk2\pi}{2m+1} = 0, \tag{3.37}$$

where $h(j) = 1$ if $mod(j, 2m+1) = 0$ and zero otherwise. Using the known result,

$$tr(T_k T_m) = -\frac{N^3}{16\pi^2}, \tag{3.38}$$

it is seen that

$$g_{mn} = -\frac{N}{4}tr(T_m T_k). \tag{3.39}$$

Therefore, every S^k in the $su^*(N)$ basis has a corresponding T_{-k} in $\mathcal{G}$ and if $\omega \in su^*(N)$, then $\tilde{\omega}$ is the corresponding element of $su(N)$ and $\tilde{\omega} = \sum \omega_{-k} T_k$.

We want the finite dimensional analog of the Hamiltonian on $su^*(N)$ to be consistent with its parent. Thus we have equation(3.3.3) as the $su^*(N)$ Hamiltonian. We can represent this Hamiltonian using the Cartan-Killing metric via a symmetric operator $\mathcal{L} : su(N) \rightarrow su^*(N)$ which reproduces the Hamiltonian above by

$$H = \omega . \mathcal{L}^{-1} \omega, \tag{3.40}$$

where the operator is defined implicitly

$$\mathcal{L}(\omega) = \sum k^2 \omega_{-k} T_k. \tag{3.41}$$

Therefore, on $\mathcal{G}$, $H(\xi) = (\mathcal{L}\xi, \xi) = \frac{1}{2} k^2 \xi_{-k} \xi_k$. We are then lead to consider a transformation $J : \mathcal{G} \rightarrow \mathcal{G}$ in terms of which

$$H(\xi) = \frac{-1}{2} K(J\xi, \xi). \tag{3.42}$$

This form for the Hamiltonian defines a pseudo-riemannian metric on $su(N)$ which is bi-invariant and has metric tensor,

$$G_{ij} = \frac{-1}{2} K(JT_i, T_j) = i^2 \delta(i+j)|_{modN}. \tag{3.43}$$

A right invariant measure may then be obtained on the tangent bundle to $SU(N)$ by right translating this metric to the whole group. This would be a right-invariant metric on $TSU(N)$ on which the $SU(N)$ Lagrangian fluid mechanics could be described. We can show that the connection and

Riemannian curvature corresponding to this energy metric take the form

$$\nabla_{T_k} T_l = d_{l,k+l} T_{k+l|_{modN}}, \tag{3.44}$$

$$R_{k,l,m,n} = (a_{ln}a_{km} - a_{lm}a_{kn})\delta(k+l+m+n)|_{modN}, \tag{3.45}$$

where

$$d_{m,n} = \frac{N}{2\pi}\sin\frac{2\pi}{N}(m \times n)\frac{(m.n)}{n^2},$$

and

$$a_{kl} = \frac{1}{|k+l|_{modN}}\frac{N}{2\pi}\sin^2\frac{2\pi}{N}(k \times l).$$

This is in agreement with the curvature result of theorem (3.2.1) and thus, the linear stability of neighboring geodesics in $s_0 diffT$ and $su(N)$ are the same.

We will now turn to the derivation of the equations of motion for the components of the vorticity field in the t'Hooft basis. Returning to the definition of the Lie-Poisson bracket for two smooth functions $F, G \in \mathcal{G}^*$, we now find that

$$\{F,G\}(\omega) = -(\tilde{\omega}), [\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta\omega}]), \tag{3.46}$$

so that we have

$$\dot{\omega} = \sum(\omega_{-m}T_m, \sum C^k_{i,j} I_i h_j T_k) \tag{3.47}$$

where I is the identity matricx and the h are the components of the functional derivative of the Hamiltonian in the t'Hooft basis. If we consider only the nth component of ω, we find that

$$\dot{\omega}_n = \omega_{-n} h_j C^k_{m,j}(T_m, T_k). \tag{3.48}$$

We find on evaluating all the terms in the above equation,

$$\dot{\omega}_n = \frac{N}{4\pi}\sum \frac{1}{p^2}\sin\frac{2\pi}{N}(p \times n)\omega_{n+p}\omega_{-p}. \tag{3.49}$$

We can also view the above Lie-Poisson equation as a Lax system. Essentially the above equation can be expressed as

$$\dot{\Omega} = [\Omega, \Psi], \tag{3.50}$$

where $\Omega = \omega_{-n}T_n$ and $\Psi = \frac{\omega_{-k}}{k^2}T_k$. The elements of $su(N)$ are anti-hermitian matrices and thus by Lax[26], the eigenvalues will be purely imaginary and time-independent. Another way of expressing this is that all the solutions of the above matrix equation will be unitarily equivalent to any other point on the trajectory, i.e., there will exist a $U(t) \in SU(N)$ such that if Ω_0 is the starting point on the trajectory of the Lax system, then

$$\Omega(t) = U^*(t)\Omega_0 U(t). \tag{3.51}$$

This is an alternative definition of the adjoint orbit through Ω_0. The Lax system has Casimirs which are given by the $N-1$ linearly independent, constant eigenvalues of Ω_0 or expressed another way, the Casimirs can be given in terms of the traces of powers of any $\Omega(t)$ on the Lax trajectory, i.e., the Casimirs are $tr(\Omega^k)$, $k = 1, \ldots, N-1$. These Casimirs converge to the body integrals of the vorticity on the torus at $N \to \infty$. The reason why there are only $N-1$ linearly independent eigen-values is because all the matrices in $su(N)$ are traceless. So, the fact that both $su^*(N)$ and $su(N)$ are diffeomorphic allows one identify the symplectic leaves as sets of unitarily equivalent matrices. These equivalence classes foliate the Poisson manifold of all smooth functions on $su^*(N)$.

Chapter 4

Hamilton Jacobi Theory and Lie-Poisson Integrators

4.1 Introduction

Recent years have seen the development of powerful new numerical algorithms which are ideal for solving problems in symplectic spaces. For the evolution of a physical system through its trajectory in phase space, we know that the symplectic structure is going to be preserved through Liouville's theorem, i.e., the phase space flow is divergence free. However, when we try to use standard techniques to numerically approximate the system's evolution, one of the first inaccuracies to creep into the approximate solution is the non-preservation of the symplectic structure. In other words, the computed solution is not staying on its underlying physical path. We could try to ensure that this is not the case by using some projection operator at each time-step so that the the component of the computed solution which represents the deviation from the phase space is factored out. However, these techniques are *ad hoc* at best as they may force the solution to live on the right phase space but there is no guarantee that the new point on the trajectory is symplectic...just momentum preserving. The situation becomes more intricate when the physical system lives in a Poisson manifold such as a dual Lie algebra. From Chapters 1 and 2, we know that in this case the dual algebra foliates into symplectic leaves which are parametrized by a specification of a full set of Casimirs. Therefore, the proposed projection system suggested above would become overwhelmed if there were many Casimirs. However, there are now a host of algorithms which attempt to capture the behaviour of the Hamiltonian system by explicitly respecting the geometry of the phase space, be it canonical or Lie-Poisson in nature. The goal of this chapter is to explore such algorithms with particular emphasis on the Hamilton-Jacobi equation and associated integrators which are designed so as to preserve the momentum mappings J.

4.2 Symplectic Integrators

The goal of Hamiltonian integration is to preserve as faithfully as possible the underlying properties of the flow of the Hamiltonian vector field which is generated by the Hamiltonian function. We will see in this section that there is a limit to the amount of accurate recreation of the Hamiltonian dynamics which can be achieved. By imposing the preservation of one aspect of the Hamiltonian dynamics, another facet is violated by default. This observation was made by Zhong Ge and is one of the central results in the field of symplectic integration. We first classify the main types of phase space integrators.

By an algorithm on phase space, P, where P can be any of the many exotic manifolds that we have discussed in the course of the past three chapters, we mean a process through which co-ordinates $z \in P$ are mapped to new co-ordinates $\overline{z} \in P$, i.e.,

$$\overline{z} = D_{\delta t}(z) \tag{4.1}$$

where D is the map and the subscript δt is some discrete time step. If the real flow is generated by a Hamiltonian vector field X_H where $H : P \to R$ is some Hamiltonian on P, then we say that our algorithm is consistent with respect to X_H if

$$\frac{d}{d(\delta t)} F_{(\delta t)}|_{\delta t=0}(z) = X_H(z).$$

It will be observed that this is only an approximation to the real flow of X_H since we do not know how close the higher order derivatives are to the true dynamics.

There are three main categories which have been classified by Marsden[4]:

i) D is symplectic if each D_τ is symplectic, i.e., $D_\tau^* \Omega = \Omega$ where the asterisk refers to the lift of D to the tensor algebra and Ω is the symplectic 2-form on P.

As an example of a symplectic integrator, consider the following Hamiltonian scheme on a symplectic vector space. (We studied Hamiltonian dynamics on symplectic vector spaces in considerable detail in Chapter 1.) If

$z \in V$ then the map $z^{k+1} = D_{\Delta t}(z^k)$ given by

$$\frac{z^{k+1} - z^k}{\Delta t} = X_H(\frac{z^k + z^{k+1}}{2}),$$

where X_H is a Hamiltonian vector field on V, is symplectic. By taking the Frechet derivative of this expression, substituting z for z^k and $D_{\Delta t}(z)$ for z^{k+1}, we obtain

$$\mathbf{D}D_{\Delta t}(z) - z - \Delta t \mathbf{D}X_H(\frac{z + D_{\Delta t}(z)}{2}) = 0.$$

After applying the chain rule and setting $A = \mathbf{D}X_H$ and $S = \mathbf{D}D_{\Delta t}$, we find that

$$S = (I - \frac{\Delta t}{2}A)^{-1}(I + \frac{\Delta t}{2}A).$$

This is the Cayley transform of A. We know that A is an infinitesimally symplectic linear mapping on V and we will use this to prove that S is neccesarily symplectic for sufficiently small Δt. It is easier to prove it in the opposite direction by assuming that S is symplectic and showing that A is infinitesimally symplectic. If S is symplectic then so is S^T. Therefore, $SJS^T = J$ where J is the matrix which satisfies $J^2 = -I$. Thus,

$$SJS^T = (I - \lambda A)^{-1}(I + \lambda A)J(I + \lambda A^T)(I - \lambda A^T)^{-1} = J.$$

This leads to

$$(I + \lambda A)J(I + \lambda A^T) = (I - \lambda A)J(I - \lambda A^T).$$

Multiplying this out leads to the equation $AJ + JA^T = 0$. Therefore, since these steps are reversible, we have proven that if A is an infinitesimally symplectic linear map, then the Cayley transform of A with $\lambda = \frac{\Delta t}{2}$ is a symplectic transformation.

ii)D is an energy preserving integrator if $H \circ D_{\delta t} = H$. This would seem a very important class of integrators as we are usually dealing with conservative Hamiltonian systems in which the main property is the invariance of the Hamiltonian on the physical trajectory. However, we will see that in the examples which we wish to explore, exact energy preservation is the very Hamiltonian constraint which we will relax.

iii) Finally, D is a momentum integrator if it preserves the momentum mapping associated with some Lie group symmetry enjoyed by the Hamiltonian. We will recall from Chapter 2 that such a momentum mapping is the repository of all geometric information concerning the system. For instance in the case of a group action on the tangent bundle to the Lie group itself, the reduced phase spaces become isomorphic to the co-adjoint action orbits, which are in some loose sense the level sets of the momentum mapping in the full phase space. So, as we reasoned in Chapter 2, if a physical system's trajectory in phase space starts on some co-adjoint orbit, it will remain on it for all time. Thus, when an integrator preserves the momentum mapping, it is actually preserving the co-adjoint orbits. This could be a vital property of the Hamiltonian system which we want to respect. As we observed, such actions foliate phase space into orbits and a momentum integrator will basically preserve this foliation.

As mentioned earlier, the simultaneous preservation of all the above properties is not possible through the application of some approximate solver. The following theorem demonstrates that if such an integrator existed, then the approximate solution would no longer be an approximation but rather an exact solution to the Hamiltonian equations, modulo a time reparametrization. So, we would have actually solved the full problem which if possible, would make redundant the necessity to construct a numerical scheme.

Theorem 4.2.1 *If the algorithm $D_{\delta t}$ preserves energy and momentum mappings and is also symplectic, then the integrated solution is the exact solution up to a rescaling of time. We also need to assume that the dynamics are not integrable.*

Proof We first assume that we are on the reduced phase space after the application of the Reduction theorem,i.e., we have reduced P to $P_\mu = P/G_\mu$ where G_μ is the isotropy subgroup of G through $\mu \in \mathcal{G}^*$. Without loss of generality, we will assume that $P = T^*G$. We will recall that if the Hamiltonian H is also invariant under the group action, then there will be a reduced Hamiltonian H_μ which is defined by

$$H_\mu(Ad_{g^{-1}}(\mu)) = H(T^*_{g^{-1}}R_{g^{-1}}(\mu))$$

where $H \circ T^*L_g = H$. We also saw that this reduced space is effectively equivalent to the space on which all the conserved quantities,i.e., momenta, have been factored out and now act like a set of parametrizing variables for the symplectic leaf. Therefore, on reduced phase space there exists only one conserved quantity and that is the reduced Hamiltonian H_μ. This implies that if there are any other integrals of motion, then they must be just statements of the same fact, i.e.,

$$If\ \{L, H_\mu\} = 0,\ then\ L = \mathcal{F}(H)$$

for some functional, $\mathcal{F}$. Since we are assuming that $D_{\delta t}$ is symplectic on the symplectic leaves (recall the fact that the reduced phase space corresponding to a regular value of J is a symplectic manifold; this was one of the main reasons we explored it in the second chapter,) then the flow must be generated by some Hamiltonian function on P_μ. However, this Hamiltonian must be time dependent in order not to violate the above assumption of H_μ being the sole conserved quantity. But again, we assumed that D_μ is also energy preserving which necessarily implies that

$$\dot{H}_\mu = \{H, K\} = 0.$$

However the bracket is anti-symmetric which leads us to the result that K is also preserved by the flow which means that it is just a functional of H and that the Hamiltonian vector fields of both K and H_μ are thus parallel. Bearing in mind that X_K is the vector field which gives rise to the approximating dynamics, we see that all we have done is to reproduce the exact P_μ trajectory, albeit with a possible reparametrization of time.

The principal examples which we have encountered up to this point have been invariant Hamiltonian systems. Therefore, we will concentrate on the third variety of integrator which preserves the momentum mapping associated with a Lie group action and we will place energy preservation at a lower priority. In fluid mechanics, this translates into the construction of numerical schema which implicitly preserve the Casimirs. In two dimensions, the set of Casimirs will constitute the body integrals of smooth functions of the vorticity. However, the algorithm will not necessarily keep energy constant. Even though this is a problem, it transpires that the energy behavior exhibits periodicity in time so that the computed solution fluctuates about a mean trajectory which is the actual path through phase space.

We have defined symplectic algorithms and demonstrated that they are limited in the sense that they cannot preserve all facets of the Hamiltonian mechanics. However, we have provided no a priori method by which we can choose integrators which preserve the subset of the first integrals of motion in which we are interested. Symplectic difference schema are not covariant, i.e., they are not invariant under all symplectic transformations. However, when a class of symplectic transformations exists with respect to which the algorithm is invariant, then it can be shown that the algorithm preserves the Hamiltonian function which generated these transformations.

Consider the symplectic difference scheme $\overline{z} = D_H(z)$ where the time-step has been omitted. Move to new co-ordinates w under some symplectic transformation, $z = T(w)$. In these new co-ordinates, $H \to H \circ T$ and $D_H \to D_K$ where $K(w) = H(T(w))$ and the symplectic difference scheme becomes

$$T(\overline{w}) = D_H(S(w)).$$

or

$$\overline{w} = T^{-1} \circ D_H \circ T(w). \tag{4.2}$$

The scheme is invariant under a group G of symplectic transformations if $T^{-1} \circ D_H \circ T = D_{H \circ T}$ forall $T \in G$. As an example, we will determine the set of symplectic transformations under which the Euler mid-point algorithm is invariant. The mid-point rule differences Hamilton's equations as

$$\frac{z^{k+1} - z^k}{\tau} = J^{-1} H_z(\frac{1}{2}(z^{k+1} + z^k)).$$

Under the transformation, $z = T(w)$, this scheme yields

$$\frac{w^{k+1} - w^k}{\tau} = J^{-1} H_w(\frac{1}{2}(w^k + w^{k+1})).$$

Now, under linear symplectic transformations T, we obtain

$$w^{k+1} - w^k = T(z^{k+1}) - T(z^k) =$$

$$T(z^{k+1} - z^k) = T(\tau J^{-1} H_z(\frac{1}{2}(z^k + z^{k+1})) =$$

$$T(\tau J^{-1} T^{-1} H_w(T(\frac{1}{2}(T(w^k + w^{k+1})) =$$

$$\tau T J^{-1} T^T H_w(T(\frac{1}{2}(w^k + w^{k+1})) =$$

$$\tau J^{-1} K_w(\frac{1}{2}(w^{k+1} + w^k)).$$

The covariance may be exploited in order to build the required preservation properties into the algorithm. This will be seen from the following result.

Theorem 4.2.2 *Given a symplectic difference scheme D_H^τ for a Hamiltonian H defined on some phase space P, the scheme will preserve a first integral f of H,*

$$f \circ D_H(z) = f(z)$$

for all $z \in P$ if and only if the scheme is invariant under the phase flow of f. Recall that f is a first integral of H if $\{f, H\} = 0$.

We will prove this for a linear Hamiltonian system which has a quadratic form first integral. Consider $H = \frac{1}{2}z^T Az$ where $z \in V$ and $A : V \to V$ is linear. The equations of motion are

$$\dot{z} = J^{-1}Az.$$

Let a difference scheme for this system be denoted $z^{k+1} = D_{J^{-1}A}z^k$. The f in the above theorem will be assumed to take on the form

$$f(z) = \frac{1}{2}z^T Bz.$$

The phase flow of this first integral is given by $G^t = exp(tJ^{-1}B)$ and is a 1 parameter group in the phase space. Let us assume that the difference scheme is invariant under this flow so that

$$(G^t)^{-1}D_{J^{-1}A}G^t = D_{J^{-1}(G^t)^{-1}AG^t}.$$

By Noether's theorem, $(G^t)^{-1}AG^t = A$ which implies that

$$(G^t)^{-1}D_{J^{-1}A}G^t = D_{J^{-1}A}.$$

We will set $D_{J^{-1}A} = \phi(J^{-1}A)$ for notational convenience. Taking derivatives with respect to t and setting $t = 0$ yields

$$T\phi(J^{-1}A)J^{-1}B = J^{-1}B\phi(J^{-1}A),$$

which leads us to

$$B = \phi(J^{-1}A)^T B\phi(J^{-1}A).$$

Therefore, the scheme conserves the quadratic form in B. The converse uses similar arguments. To find a proof of the above result for more general Hamiltonians, see Ge[13].

In the Euler scheme, every first integral of quadratic form will be conserved because such first integrals give rise to linear phase flows.

4.3 Hamilton-Jacobi Theory and Generating Functions

In this section, we will outline the traditional theory of generating functions and the Hamilton-Jacobi equation both for time independent and time dependent Hamiltonian systems. Even though we are essentially interested in conservative Hamiltonian systems, we will find ourselves solving the time dependent H-J equation. The reason for this will become apparent as we progress and is intimately connected to the theorem of Zhong Ge discussed in the last section.

Initially, we will present the theory of canonical transformation generating functions in the classical co-ordinate dependent manner. The treatment will be at the level of Goldstein[23]. Following this introduction, the relatively recent Lagrangian submanifold approach will be discussed. The reason for deriving the same theory in two ways is due to the requirements of the next section. At that stage, we will be concerned with the construction of Hamilton-Jacobi solvers on the dual to a Lie algebra. The C^∞ functions on such spaces have already been shown to constitute a non- symplectic manifold and it turns out that the most natural way to solve the Hamilton-Jacobi equation in such a setting is through the employment of the Lagrangian submanifold approach.

Consider the general canonical co-ordinate description of a tangent bundle. A phase space transformation from co-ordinates (q^i, p_i) to (Q^i, P_i) is defined by

$$Q^i = Q^i(q, p, t)$$

$$P_i = P_i(q, p, t).$$

Such a transformation will be canonical if there exists a function K of the new co-ordinates such that

$$\dot{Q}^i = \frac{\partial K}{\partial P_i}, \dot{P}_i = -\frac{\partial K}{\partial Q^i}$$

which are the familiar Hamilton's equations. We know that this K must satisfy a Hamilton's principle as H did, so that

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}^i - K(Q, P, t))dt = 0$$

where $\delta \int$ means taking the variation with the end points values of Q and P fixed. By comparing this to the original variation of the Hamiltonian H, we find that the integrands will be equivalent up to the addition of the time derivative of some function F of the old and new co-ordinates, i.e.,

$$p_i \dot{q}^i - H(q, p, t) = P_i \dot{Q}^i - K(Q, P, t) + \frac{dF}{dt}.$$

F is called a generating function and it can be taken to depend on a mixture of the old and new co-ordinates. As an example, we can consider the form

$F = F_1(q, Q, t)$ which yields the following relations

$$p_i = \frac{\partial F_1}{\partial q^i},$$

$$P_i = -\frac{\partial F_1}{\partial Q^i},$$

$$K = H + \frac{\partial F_1}{\partial t}.$$

It should be clear that it is not possible to represent the identity transformation using this type of generating function. Generating functions can be used as an alternative to solving the Hamiltonian equations. Consider some physical system whose motion can be described by some set of canonical co-ordinates in phase space. Take the initial condition to be specified by the pair (q_0^i, p_{i0}) at $t = 0$. Then, if the system moves to (p, q) at time t, we seek the canonical transformation which maps the system from (p, q) to (p_0, q_0). Since the initial conditions are fixed in time, we try to find a transformation which maps into a K which equals zero, for in this case, $\dot{Q} = 0$ and $\dot{P} = 0$. The equation for the generating function F takes the form

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0. \tag{4.3}$$

If we choose the F to be of the form $F_2(q, P, t)$, then since $p_i = \frac{\partial F_2}{\partial q^i}$, we see that

$$H(q; \frac{\partial F_2}{\partial q}; t) + \frac{\partial F_2}{\partial t} = 0$$

which is known as the **Hamilton – Jacobi** equation. This is the time dependent equation as we have not made the assumption that the system is conservative as time explicitly enters H in the above equation. If the system is conservative then $H(q,p,t) = H(q,p)$ and the generating function, F_2 must be separable as

$$F_2(q,\alpha,t) = W(q,\alpha) + \beta t,$$

where α and β are constants which are dependent on the initial values q_0 and p_0. The Hamilton-Jacobi equation now becomes

$$H(q, \frac{\partial W}{\partial q}) = \beta.$$

We have derived the above sets of equations without any reference to the differential geometry that we spent so long exploring. We will now connect back to the more general theory. One of the first things that one notices about generating functions is that they are co-ordinate dependent. In what follows, this restriction will be relaxed.

The theory of Lagrangian submanifolds provides a covariant formalism of the generating function approach to Hamiltonian mechanics. We will start by providing the basic definitions and properties of Lagrangian submanifolds.

A Lagrangian submanifold of a symplectic space (P,ω) can be defined in a number of equivalent ways but we will concentrate on the two which have the most relevance to Hamilton-Jacobi theory.

Definition 4.3.1 *A submanifold L of a symplectic space (P,ω) is said to be Lagrangian if its dimension is half that of P and ω vanishes identically on L. Equivalently, we say that L is Lagrangian if the tangent space to L at every point of L is equal to its orthogonal complement,i.e.,*

$$T_xL = (T_xL)^{\perp} \stackrel{def.}{=} \{v \in P | \omega(v,w) = 0 \; \forall \; w \in T_xL\}.$$

for all $x \in L$.

As an example, consider the graph of a symplectic transformation $f : P \to P$ which in local co-ordinates becomes $(\overline{p}, \overline{q}) = f(p, q)$. The graph of f is a Lagrangian submanifold of the symplectic space $(R^{4n}, \Omega) = ((\overline{p}, \overline{q}, p, q), \Omega = d\overline{p} \wedge d\overline{q} - dp \wedge dq)$. The 2-form Ω vanishes on L since the map f is symplectic.

A result which we will state without proof is that if we are given a 1-form α on some configuration space Q, then $gra(\alpha) \subset T^*Q$ is a Lagrangian manifold if and only if α is closed. Therefore, if $f : Q \to R$ then $\{(q, p) \in T^*Q | p = df(q)\}$ forms a Lagrangian submanifold of T^*Q. On a copy of R^{4n} endowed with a symplectic 2-form $\Sigma = d\overline{w} \wedge dw$ where $(\overline{w}, w)$ is an element of R^{4n}, a Lagrangian manifold can thus be generated by considering the graph of the differential of some function $S : R^{2n} \to R$, i.e., $L = \{\overline{w} | \overline{w} = dS(w)\}$.

The mechanism through which the results of the last section can be reproduced using Lagrangian submanifolds is by finding a correspondence between the graphs of symplectic mappings and the graphs of exact 1-forms. If we have a Hamiltonian system on a linear vector space, then as in the two examples above, the graphs will be embedded in copies of R^{4n} with symplectic 2-form Ω for symplectic transformations and Σ for 1-forms. The correspondence is achieved by using the concept of a generating map, Φ, which is a linear symplectic transformation from (R^{4n}, Ω) to (R^{4n}, Σ). This formalism is due to Feng Kang[17]. As we observed in the first part of this section, given local canonical co-ordinates on some symplectic space, P, we could basically use any pair choice between the (P, Q) and (p, q) co-ordinates in order to implicitly construct symplectic transformations. In the

Lagrangian submanifold formulation, this choice becomes equivalent to the selection of generating map Φ that we make.

We will be particularly interested in generating functions of the first kind The choice of Φ in this case is

$$\Phi = \begin{pmatrix} -I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}.$$

We see that $\Phi(\overline{p}, \overline{q}, p, q) = (-\overline{p}, p, \overline{q}, q)$. Therefore, $S = S(\overline{q}, q)$ and the Lagrangian submanifold generated by S will be given explicitly by $(-\overline{p}, p) = dS(\overline{q}, q)$.

Before presenting the Hamilton-Jacobi equation in terms of these generating maps Φ, we need to state some more basic results from the theory of Lagrangian submanifolds.

If L, a subspace of P, is Lagrangian and $H \in C^\infty(P)$, then if H is constant on L, L will be invariant under the phase flow of X_H. Also if F_t is the flow of X_H, then $F_t(L)$ remains Lagrangian. Using these properties, we can state the Hamilton-Jacobi theory in terms of L and its flow. Suppose that $L \subset T^*Q$ is the graph of some exact form dS. We say that S is the generating function for L. Assume that L is the graph of some Hamiltonian trajectory generated by some function H on P. If F_t is the flow of the corresponding Hamiltonian vector field X_H, then for a short time, $F_t(L)$ is the graph of the differential of some $S_t : Q \to R$ which depends smoothly on t and equals S at $t = 0$. This $S(t, q)$ satisfies the Hamilton-Jacobi equation.

Returning to the Kang formalism, we see that since the Hamiltonian is defined on the copy of R^2 with the 2-form Ω, we will have to find the inverse of Φ in order to write down the Hamilton-Jacobi equation. Therefore,

$$\frac{\partial S}{\partial t} = H \circ P_2 \circ \Phi^{-1}(dS(w)), \tag{4.4}$$

where P_2 is the projection of $(\overline{w}, w)$ onto the second factor. For generating functions of the first kind, this equation reduces to the familiar

$$\frac{\partial S}{\partial t} = -H(q, \frac{\partial S}{\partial t}). \tag{4.5}$$

The derivation of Hamilton-Jacobi theory using Lagrangian manifolds may appear redundant but as we will need to construct integrators on Lie-Poisson manifolds which are generally non-symplectic, it will be seen that the more abstract Lagrangian manifold approach is appropriate.

4.4 Momentum Preserving Algorithms and the Reduced Hamilton-Jacobi Equation

How can we use the generating function formalism of the last section to construct symplectic difference schemes? Fixing some generating map Φ, and assumimg that we can find some generating function S_0 on (R^2, Σ) whose graph generates the identity transformation on (R^{4n}, Ω) under Φ^{-1}, then we can construct an algorithm as follows. If we choose a small enough time step, δt, we could form a power series solution for the generating function S_t of the last section which smoothly equals S_0 as $t \to 0$. Let

$$S(t) = S_0 + \sum \frac{\delta t^n}{n!} S_n, \tag{4.6}$$

which by the statement at the end of the last section also solves the Hamilton-Jacobi equation. This series truncated at any order will provide a symplectic difference scheme.

However, we again face a dilemna. We do not know a priori if the truncated generating function series will preserve the first integrals of motion in which we are interested. In section 4.2, we defined the conditions under which a symplectic difference scheme is invariant under a group of symplectic transformations. We will now do the same for generating functions. If D_k is the difference scheme generated by $\mathcal{S}_k$, then under a symplectic transformation, $z = T(w)$, $\overline{z} = D_k(z)$ transforms to $\overline{w} = T^{-1} \circ D_k \circ T(w)$. We say that D_k is invariant under some group G of transformations if and only if there exists a linear transformation defined by some $A : R^{4n} \rightarrow R^{4n}$ such that locally $u_{T^{-1} \circ D_k \circ T} = u_{D_k} \circ A$. There also exists a connection between first integrals of motion for the physical system and the invariance of the symplectic difference schema.

Theorem 4.4.1 *Let f be a first integral of the system with Hamiltonian H. For a given choice of Φ, the symplectic difference scheme $\mathcal{S}_k$ derived from truncating the power series expansion of S at order k will preserve f if and only if $\mathcal{S}_k$ is invariant under the flow of f.*

The above formulation is also valid on cotangent bundles. We will be interested in the cases where the first integrals of motion are generated by a Lie group action on the cotangent bundle . The momentum mapping is the corresponding quantity which is preserved under the Hamiltonian dynamics and we will now investigate the types of generating maps Φ which will produce momentum mapping preserving symplectic schema.

Let $\Psi : G \times P \to P$ be an action on P with momentum mapping J defined by

$$J : P \to \mathcal{G}^*; < J(\alpha_q), \xi >=< \alpha_q, \xi_P(q) >,$$

for all $\alpha_q \in P$ and $\xi \in \mathcal{G}$. If $P = T^*Q$ and taking a generating function of the first kind, $S : Q \times Q \to R$, then we will prove that if $\phi_S : T^*Q \to T^*Q$ is the symplectic scheme generated by S, then in order that ϕ_S preserve the level sets of the momentum, J, ϕ_S must be G-invariant, i.e., $\phi_S(\Psi(g, z)) = \Psi(g, \phi_S(z))$.

To prove this, differentiate this expression with respect to t in the direction $\xi \in \mathcal{G}$.

$$\frac{d}{dt}|_{t=0}(\Psi(expt\xi, x)) =$$

$$T_e\phi\xi_P(x) = \xi_P(\phi(x)).$$

This implies that $X_{<J,\xi>o\phi} = X_{<J,\xi>}$ and thus,

$$< J, \xi > o\phi- < J, \xi >$$

differs from zero by at most a constant. Also, if S is invariant under the diagonal action of the group G, i.e, $S(\Psi(g, q), \Psi(g, q_0)) = S(q, q_0)$, then the resulting symplectic difference scheme ϕ_S will preserve the level sets of J. This can be proven in a similar way by differentiating the S invariance expression in the direction of $\xi \in \mathcal{G}$. We obtain

$$\frac{d}{dt}|_{t=0}S(\Psi(expt\xi, q), \Psi(expt\xi, q_0)) = 0,$$

which implies that

$$dS.\xi_P(q) + dS.\xi_P(q_0) = 0,$$

or

$$\frac{\partial S}{\partial q}.\xi_P(q) = -\frac{\partial S}{\partial q_0}.\xi_P(q_0).$$

But

$$J \circ \phi_S(q_0, q)(\xi) = < p, \xi_P(q) > =$$
$$< \frac{\partial S}{\partial q}, \xi_P(q) > = dS.\xi_P(q) =$$
$$-\frac{\partial S}{\partial q_0}.\xi_Q(q_0) = J(q_0, p_0).$$

The converse of this is also true: If ϕ is a symplectic difference scheme on P which preserves the momentum mapping, then it is always possible to derive ϕ from a G-invariant generating function of the first kind.

So, when we choose the generating mapping which produces generating functions of the first kind, the resulting symplectic difference scheme will preserve level sets of the momentum mapping.

We will now turn to the context of Poisson manifolds. This environment presents problems for the implementation of the generating function procedure. As we have seen in earlier chapters, a Poisson manifold is not symplectic but rather foliates into symplectic leaves whose dimensions are even but not necessarily constant across leaves. The solution which can be found in detail in Ge[18] is to employ a *Generating Pair*. In our case, there will be an extra stipulation that whatever mechanism we use to generate Poisson mappings (symplectic leaf preserving) on P, it must be able to generate the identity transformation on P. This concept of a generating pair can be described as follows.

Definition 4.4.1 *A strict generating pair for a Poisson manifold P is a pair (S, J), where S is a symplectic manifold and J is a Poisson map from S to $P \times P^-$ where the minus superscript indicates that the Poisson bracket in this copy of P has been multiplied by minus one. J must have the property that Lagrangian manifolds of S are mapped to the graphs of local Poisson automorphisms in P. Also, to be a strict generating pair, there must exist an $L_0 \subset S$ such that $J(L_0)$ is the identity automorphism on P.*

For Lie-Poisson systems, i.e., Poisson manifolds derived by the left or right action of a Lie group G on its own cotangent bundle, the formation of a generating pair is quite straightforward. This generating pair will only be strict for a certain subclass of reduced phase space duals. The reduced dynamics inhabit the $\mathcal{C}^\infty(\mathcal{G}^*)$ Poisson manifold and the generating pair is $S = T^*G$ and $J = J_L \times J_R$, where

$$J_L : T^*G \to \mathcal{G}^*; J_L(p, g) = -TL_g^*(p)$$

and

$$J_R : T^*G \to \mathcal{G}^*; J_R(p, g) = -TR_g^*(p).$$

We have $J : S \to \mathcal{G}^* \times \mathcal{G}^{*-}$ because right reduction will endow $\mathcal{G}^*$ with a plus sign in its Lie-Poisson bracket and left reduction will leave a minus sign. Section 1.5 has more details.

For the Lie-Poisson case, we will now explicitly write down the resulting Poisson automorphism on $\mathcal{G}^* \times \mathcal{G}^{*-}$. A Lagrangian manifold in S can be expressed as the graph of an exact 1-form on the cotangent bundle to S. Namely

$$L = \{(g, du(g))\},$$

for some $u : G \to R$. Then the Poisson mapping on the reduced phase space will be given implicitly by

$$A : P \to P \ , A(\pi) = (\overline{\pi}), \tag{4.7}$$

$$\pi = -TL_g^*(du(g)) \ \overline{\pi} = -TR_g^*(du(g)) = Ad_{g^{-1}}^*(\pi). \tag{4.8}$$

The Hamilton-Jacobi equation will be as given in the previous section, namely,

$$u_t = -H(-TR_g^*.du(g)). \tag{4.9}$$

We still have to show that under certain conditions, there will exist a Lagrangian submanifold on T^*G which generates the identity transformation on $\mathcal{G}^*$. For the special case in which $\mathcal{G}$ is regular quadratic,i.e, $\mathcal{G}$ is endowed with a bilinear, Ad-equivariant, 2-form, then the identity transformation can be easily derived from this 2-form. We will prove this for the case of a semi-simple Lie algebra. In the case of a semi-simple Lie algebra, the regular quadratic property is satisfied by using the metric derived from the Cartan-Killing form which was introduced in Chapter 3. Denote this regular quadratic 2-form by $< .,. >: \mathcal{G} \times \mathcal{G} \to R$. We will now construct a generating function on G which produces the identity transformation on $\mathcal{G}^*$. Let

$$u : G \to R, \ u(g) = < lng, lng >, \tag{4.10}$$

where ln is the inverse of the exp mapping. In order for the Lagrangian submanifold L_0 corresponding to this u to produce the identity automorphism on $\mathcal{G}^*$, we need to prove that $TR_g^* du(g) = TL_g^* du(g)$. This follows from the Ad-equivariance of $< .,. >$.

$$u(g) = u(L_{g_0} \circ R_{g_0}^{-1})(g).$$

Differentiating this expression will yield

$$du(g)(Ad_{g_0}\xi) = du(g)\xi,$$

for $\xi \in \mathcal{G}$. Bringing the Ad operator out of the argument for du gives

$$Ad^*_{g_0^{-1}}.du(g) = du(g),$$

which implies that

$$-TR_{g_0}^* du(g) = -TL_{g_0}^* du(g).$$

To be complete, the J_R mapping should also be shown to be a local diffeomorphism in a neighbourhood of $(e, 0)$ in T^*G.

We have succeeded in incorporating Poisson mappings on Poisson manifolds into the generating function formalism. For the special case of a Lie-Poisson system with a regular, quadratic form, we can carry out the construction of a Lie-Poisson integrator by building a Poisson difference scheme using the strict generating pair. The reduced Hamilton-Jacobi equation can be solved for some truncated power series on T^*G with time-step δt about the identity transformation, $u_0(g) =< lng, lng >$. Thus, the algorithm can be executed totally in the environment of the Lie group and all Poisson transformations are effected only implicitly.

4.4.1 The Lie-Poisson Integrator of Channell and Scovel

In the last section, we constructed integrators for Lie-Poisson systems through the strict generating pair formalism which allowed the Hamilton-Jacobi equation to be expressed in terms of dual Lie algebra variables. In essence, we take a scheme on T^*G and use it to produce a Poisson transformation algorithm on $\mathcal{G}^*$. For some appropriate time-step, τ, if we are initially at some point $\Pi_0 \in \mathcal{G}^*$, we can time march to Π via solving

$$\Pi_0 = -TL_g^*.dS_L \tag{4.11}$$

for g and then setting

$$\Pi = Ad^*_{g^{-1}}\Pi_0 \tag{4.12}$$

for the time advanced variable in $\mathcal{G}^*$ or more accurately, in the co-adjoint orbit through Π_0, $\mathcal{O}_{\Pi_0} = \{\Pi | \Pi = Ad^*_{g^{-1}}(\Pi_0)\ ,\ g \in G\}$. Implicitly, the co-adjoint orbit or phase space is preserved. Channell and Scovel[14] carry this algorithm one step further in that instead of solving for g in the group, they use the exponential mapping and its locally defined inverse, ln, to lift the algorithm to solving for elements of the Lie algebra and its dual only. The advantage in this approach stems from the difficulties associated with doing computations in the Lie group. It will be recalled that $exp : \mathcal{G} \to \mathcal{G}$ provides a natural co-ordinatization of the Lie group. exp is not necessarily onto G but will admit an inverse in an open neighbourhood of the identity, $e \in G$. The whole of G can be covered by left translating the exp mapping across G. The advantage of the Lie algebra co-ordinates as opposed to those on the Lie group is that by computing in the Lie algebra, the exp mapping from $\mathcal{G}$

to G guarantees that we do not move out of the group.

We will now transfer the algorithm from the $G/\mathcal{G}^*$ setting to the *exp* defined $\mathcal{G}/\mathcal{G}^*$ environment. First, we define the analog of the generating function which is a real-valued function on $\mathcal{G}$.

$$S : \mathcal{G} \to R \, , \; S_L(g) = S(lng),$$

for $g \in G$.Provided the time step is small, we will only need to invoke this definition in a small, open neighbourhood about the identity in which ln is well-defined. If $S_L : G \to R$, then the differential of S_L, $dS_L : G \to T^*G$ is defined by

$$dS_L(g)(v_g) = P_2 \circ T_g S_L(v_g) \; for \; v_g \in T_g G.$$

The operator P_2 is the projection onto the second factor. Therefore, by applying the chain rule, we obtain for the argument of the Hamiltonian function in the Lie-Poisson H-J equation,

$$-TR_g^*.dS_L = -TR_g^*(P_2) \circ TS_L$$

$$= -P_2 \circ TS_L(TR_g) = -P_2.TS.Tln.TR_g.$$

We have encountered all the above tangent derivatives in earlier chapters except for Tln the evaluation of which will require a number of tricks from the theory of Lie series. For a review, the reader is referred to Dragt and Finn[16]. First we need,

Theorem 4.4.2 *The tangent derivative of the exp function at $\xi = lng$ satisfies*

$$T_\xi exp = TL_{g^{-1}} \phi(-ad_\xi),$$

where $ad_\xi \eta = [\xi, \eta]$ *for* $\eta \in \mathcal{G}$. *The function* ϕ *is a formal power series which takes the form*

$$\phi(z) = \Sigma \frac{z^n}{(n+1)!}.$$

Proof Note that $exp : \mathcal{G} \to G$ and thus,

$$Texp : T\mathcal{G}(\cong \mathcal{G} \times \mathcal{G}) \to TG,$$

and $T_\xi exp : \mathcal{G} \to T_g G$.

We first consider $T_\xi exp \xi = \frac{d}{dt}|_{t=1} exp t\xi$ for $\xi \in \mathcal{G}$. Thus, we let $A(t) = t\xi$ and note that we can express $TL_{g^{-1}} T_\xi exp : \mathcal{G} \to \mathcal{G}$ as

$$exp - A(t) \frac{d}{dt}|_{t=1} exp A(t).$$

To proceed, we will need the following result concerning the vector solution on some vector space V, of the differential equation

$$\frac{du}{ds} = Au + w.$$

We find that

$$u(s) = e^{sA} u(0) + f(s, A) w.$$

In this equation, $f(s, z) = \frac{(e^{sz} - 1)}{z}$ and $A : V \to V$ is a linear transformation on V. Let $B(s, t) = exp - sA(t) \frac{d}{dt} exp sA(t)$. Differentiate with respect to s to obtain

$$\frac{\partial B}{\partial s} = -[A, B] + \dot{A}.$$

Therefore, $B(s, s) = \frac{[e^{-s\, adA} - 1]}{(-adA)} \dot{A}$. Evaluate this at $s = 1$ to get

$$B(1, 1) = \phi(-ad\xi).\xi.$$

This is because ϕ in the statement of the theorem is actually equal to $f(1, -ad\xi)$. The result follows immediately.

Now, from the chain rule, we have

$$T_g ln.T_\xi exp = T_\xi(ln \circ exp) = Id_{\mathcal{G}},$$

where $exp\xi = g$. So, we find that

$$T_g ln = \Upsilon(ad_\xi).TL_{g^{-1}},$$

where $\Upsilon(z)$ is the power series satisfying $\Upsilon(z)\phi(z) = 1$. In fact, $\Upsilon(z) = \frac{z}{[e^z-1]}$. We now multiply Tln on the right by TR_g and find

$$TlnTR_g = \Upsilon(ad_\xi)Ad_{exp-\xi},$$

by definition of $Ad_g : \mathcal{G} \to \mathcal{G}$. The identity $Ad_{exp-\xi} = e^{-ad_\xi}$, allows us to simplify things further so that

$$-TR_g^* dS_L = -Pr_2.TS.Tln.TR_g =$$

$$-dS.\Upsilon(ad_\xi).e^{-ad_\xi}.$$

Letting $\Psi(ad_\xi) = \Upsilon(ad_\xi).e^{-ad_\xi}$, we finally find

$$\Pi_0 = -dS.\Upsilon(ad_\xi), \tag{4.13}$$

and

$$\Pi = -dS.\Psi(ad_\xi). \tag{4.14}$$

The reduced Hamilton-Jacobi equation now becomes

$$\frac{\partial S}{\partial t} = -H(-dS.\Psi(ad_\xi)). \tag{4.15}$$

We have now removed all traces of the group operations. The algorithm is totally expressed in terms of elements of the Lie algebra and its dual and to implement, we now solve for ξ instead of $g = exp\xi$.

4.4.2 The Lie-Poisson Integrator on Regular Quadratic Lie Algebras

When the algebra is regular quadratic, there exists an Ad-invariant, non-degenerate quadratic form $< .,. >$ on $\mathcal{G}$ with respect to which a metric may be defined. As already mentioned, in the case of semi-simple Lie algebras, this quadratic form can be taken as the Cartan Killing form. We know that on G, the identity transformation for $\mathcal{G}^*$ is generated by $S_0 : \mathcal{G} \rightarrow \mathcal{G}$, $S_0(\xi) = \frac{1}{2} < \xi, \xi >$, for all $\xi \in \mathcal{G}$. Therefore, to seed the algorithm of the last section, we expand in the time-step about S_0 and truncate the power series at some desired order. We will calculate the first few terms in the expansion using Taylor's theorem. For the sake of completeness, we will state Taylor's theorem for a $\mathcal{C}^\infty$, real-valued function on a linear vector space, E.

Theorem 4.4.3 *Taylor's Theorem*

A C^r map $f : U \subset E \rightarrow R$ can be expanded about a point $u \in U$ as follows

$$f(u+h) = f(u) + \phi_1(u).h + \frac{\phi_2(u)}{2!}.h^2 + \frac{\phi_3(u)}{3!}.h^3 + \ldots,$$

where $h^p = (h, h, \ldots, h)$ p times and $\phi_p = D^p f$.

Let us insert the power series expansion,

$$S(\xi, t) = S_0(\xi) + \sum \frac{(\delta t)^n}{n!} S_n(\xi),$$

into the Hamilton-Jacobi equation.

By Taylor's theorem, we find that to

- First Order:

$$S_1 = -H(-dS_0.\Psi(ad_\xi)). \tag{4.16}$$

- Second Order:

$$S_2 = DH(-dS_0\Psi(ad_\xi)).dS_1.\Psi(ad_\xi) = \tag{4.17}$$

$$\frac{\partial H}{\partial V}.dS_1.\Psi(ad_\xi),$$

where $V = -dS_0.\Psi(ad_\xi)$, and to

- Third Order:

$$S_3 = \frac{\partial H}{\partial V}.dS_2.\Psi(ad_\xi) - \frac{\partial^2 H}{\partial V^2}(dS_1.\Psi(ad_\xi))^2. \tag{4.18}$$

We will develop integrators of this and other types for rigid body and $su(N)$ dynamics in the next chapter.

Chapter 5

Applications

5.1 Introduction

We will build Lie-Poisson integrators for $so(3)$ and $su(N)$. Both these algebras are regular quadratic. Therefore, Lie-Poisson dynamics on both Lie groups can be realized numerically using the Channell and Scovel integrator of chapter 4. However, it is found that the integration times for the vorticity dynamics over $su(N)$ are prohibitively slow. This arises because the scheme is implicit and at every time-step in the computation, power series in large matrices must be evaluated. If we are to develop an efficient numerical tool from the application of Lie-Poisson ideas, we will need to construct faster algorithms. We propose a new explicit Lie-Poisson integrator which still relies on the exponential mapping co-ordinatization of the Lie group. Therefore, Lie power series will again be encountered. However, at each time-step, only one power series calculation is required and the scheme is explicit. The time-step will again be limited as in the Channell and Scovel integrator because the ln mapping from the group to the algebra is only defined in a small neighborhood of the identity mapping in G. However, each component in the power series expansion has a time-step factor which quickly reduces large powers to within machine error so that a finite series truncation is sufficient. We test the explicit scheme against the implicit Hamilton-Jacobi solver in the $SO(3)$ case with very favorable results. This explicit Lie-Poisson algorithm works for any regular quadratic Lie algebra.

5.2 Rigid Body Dynamics

The rigid body with one point fixed is the paradigm example for geometric mechanics. The configuration space is the Lie group $SO(3)$. This is easily seen by considering the rigid body with one point fixed at the origin of R^3. Take $x(t, x_0)$ as the position of an element of the body at time t given its initial position x_0 at $t = 0$. Then, there exists an orthogonal matrix A such that

$$x(t, x_0) = A(t)x_0. \tag{5.1}$$

$A(t)$ must be an element of $SO(3)$ if the motion is continuous and $A(t = 0) = Id$. The mass of the body will be described by some measure on R^3 which is positive. The kinetic energy of the body then satisfies

$$\begin{aligned} K.E. &= \frac{1}{2}\int(\dot{x}, \dot{x})d\mu(x) \\ &= \frac{1}{2}\int(\dot{A}x_0, \dot{A}x_0)d\mu(x). \end{aligned} \tag{5.2}$$

$\dot{A}(0)$ will be an element of the tangent space at the identity of $SO(3)$, which is the Lie algebra. The Lie algebra of $SO(3)$ is the linear vector subspace of

$gl(3)$ whose elements are skew symmetric. If $R \in so(3)$, then there exists a ω in R^3 such that $R(x) = \omega \times x$ for all $x \in R^3$. Therefore,

$$\dot{x}(t, x_0) = \omega(t) \times x(t, x_0) \tag{5.3}$$

for some time dependent smooth function $R(t)$ in $so(3)$ whose counterpart in R^3 is ω. This ω can easily be seen to equal $TR_{R(t)^{-1}}\dot{R}(t)$, the right tangent translation of $\dot{R}(t) \in T_{R(t)}SO(3)$ to $so(3)$. This ω is referred to as the angular velocity and corresponds to viewing $\dot{R}$ from a frame of reference fixed in space. The velocity vector derived from left action will be seen to be the view of $\dot{R}$ from a frame of reference fixed in the body. and is given by $\Omega = TL_{R(t)^{-1}}\dot{R}(t)$. The kinetic energy is seen to equal $\frac{1}{2} < \Omega(t), \Omega(t) >_e$ where $< u, v >= \int (u \times x, v \times x) d\mu(x)$. The trajectory of the rigid body through its configuration space will be a geodesic of this Riemannian metric, i.e., the trajectory will conserve and minimize the kinetic energy. This metric can be written in terms of some symmetric matrix I which maps the Lie algebra into its dual and such that

$$< u, v >= Iu.v$$

where $Iu.v$ represents the natural pairing between the algebra and its dual. Also, since $so(3)$ is semi-simple, its Cartan-Killing form is non-degenerate and we can identify the algebra with its dual and consider $I : so(3) \rightarrow so(3)$ instead. The isomorphism with R^3 simplifies matters further since the metric on the algebra, g_{ij} becomes the kronecker delta, the Lie algebra bracket becomes the cross product and axes can be chosen so that I is diagonal.

By moving to the angular momentum formulation which is basically the same discussion except over the dual Lie algebra, we can check to see that the rigid body dynamics are in fact Lie-Poisson. Take the angular momentum to be given by $m = (m_1, m_2, m_3) \in so^*(3)(\cong R^3)$ and assume a general form of I^{-1} such that the kinetic energy is equal to $H(m) = \frac{1}{2} < m, I^{-1}m >$. If $F \in C^\infty(so^*(3)$, then the Lie-Poisson bracket of F and H at m will be

$$\{F, H\}(m) =< m, [\frac{\delta F}{\delta m}, \frac{\delta H}{\delta m}] >$$

$$= -m.(\nabla F \times \nabla H)$$

where $\nabla F = (\frac{\partial F}{\partial m_i})$, which is the functional derivative of F with respect to $m \in so^*(3).\nabla F$ is an element of the Lie algebra. We can choose the basis for $so(3)$ such that the inertia tensor becomes diagonal. In so doing, we can see what the equations of motion look like in dual algebra body co-ordinates. Since all the eigenvalues of I are real and positive, the time derivative of F will be

$$\dot{F}(m) = \nabla F.(\dot{m}) = \{F, H\}(m) = -m.(\nabla F \times \nabla H)$$

$$= \nabla F.(m \times \nabla H). \tag{5.4}$$

I^{-1} is diagonal in this basis so, we have $H(m) = \frac{1}{2}(m_i^2/I_i^2)$ where summation is implied. Thus

$$\dot{m}_1(t) = (m \times \nabla H(m))_1 = m_2 m_3(\frac{1}{I_3} - \frac{1}{I_2}), \tag{5.5}$$

and similarly for y and z component of m. These are our familiar angular momentum equations. A direct substitition will show that H and $|m|^2$ are conserved.

5.3 Generating Function Integrator for SO(3)

An integrator for rigid body dynamics can be derived from the algorithm presented in section 4.5. The Cartan-Killing form furnishes $so(3)$ with an Ad-invariant, non-degenerate quadratic form, $K : so(3) \times so(3) \to so(3)$. As we know, the identity transformation for Poisson mappings in $so^*(3)$ will then be produced by the generating function defined in terms of the Killing form,

$$S_0(\xi) = -\frac{1}{2}K(\xi, \xi), \tag{5.6}$$

where ξ is arbitrary in $so(3)$. The time advanced angular momentum vector will then be computed by solving the Hamilton-Jacobi equation for some power series perturbation in the time step to S_0.

The Cartan-Killing form for $so(3)$ reduces to the scalar product after we identify the Lie algebra $(so(3), [,])$ with $(R^3, \times)$. Thus, the adjoint action $ad : so(3) \times so(3) \to so(3); ad_\xi \eta = [\xi, \eta]$ becomes the cross product in R^3. We also have

$$K(\xi, \eta) = -\xi.\eta, \tag{5.7}$$

for ξ, η in $so(3) \cong R^3$. Thus the algebra metric which acts as the raising and lowering operator becomes

$$g_{ij} = -\delta_{ij}.$$

From the previous section, we know that a basis in R^3 can be chosen so that the kinetic energy which generates the metric

$$G_{ij} = -\frac{1}{2}K(Ie_i, e_j), \tag{5.8}$$

for some symmetric $I : so(3) \to so(3)$, reduces to

$$G_{ij} = \frac{1}{2}I_i\delta_{ij}. \tag{5.9}$$

Therefore, on the dual algebra $so^*(3)$, the kinetic energy, in terms of the angular momentum, takes the form

$$H(m) = \frac{1}{2}G^{ij}m_i m_j =$$

$$\frac{1}{2}(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}). \tag{5.10}$$

If we the rigid body starts at some point $m_0 \in so^*(3)$, then it will remain on the sphere $||m||^2 = ||m_0||^2$, the trajectory being the intersection of this constant Casimir surface and the ellipsoid of inertia, $H(m) = H(m_0)$.

In order to construct the Lie-Poisson integrator from the generating function formalism, we will need to solve the reduced Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(-dS.\Psi(ad_\xi)) = 0,$$

which was derived in section 4.5. We use

$$m_0 = -dS.\Gamma(ad_\xi), \tag{5.11}$$

to solve for ξ and then time advance along the constant momentum trajectory to $m(t)$ obtained by substituting ξ into

$$m = -dS.\Psi(ad_\xi). \tag{5.12}$$

We will recall that the Ψ and Γ are to be interpreted as Lie series of operators where

$$\Psi(z) = \Gamma(z)e^{-z}$$

and

$$\Gamma(z) = \frac{z}{1-e^{-z}}.$$

It is easy to see that $\Psi(z) = \frac{e^{-z}z}{1-e^{-z}} = \Gamma(z) - z$. The expansion of $\Gamma(z)$ is

$$\Gamma(z) = I + \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \frac{1}{30240}z^6 + \ldots \tag{5.13}$$

and when interpreting $\Gamma(-ad_\xi)$ etc., it is a calculation of this series which is implied. In the algorithm above, we are solving for an element of $so(3)$ which will be dependent on the time-step τ chosen, so that we would expect that the series could be truncated at finite order with only machine error departure from the Lie-Poisson dynamics.

The solution to the Hamilton-Jacobi equation with $S_0(\xi) = -\frac{1}{2}K(\xi,\xi)$ to first order is by equation (4.5),

$$S_1 = -H(V),$$

where $V(\xi) = -dS_0.\Psi(ad_\xi) = -dS_0.I_{\mathcal{G}}$. This follows because

$$\xi.(\xi \times \eta) = 0,$$

and

$$\xi.(\xi \times (\xi \times \eta)) = 0,$$

etc. This allows us write

$$S_1 = -\frac{1}{2}[\frac{\xi_1^2}{I_1} + \frac{\xi_2^2}{I_2} + \frac{\xi_3^2}{I_3}] \tag{5.14}$$

and thus $-dS_1 = (\frac{\xi_1}{I_1}, \frac{\xi_2}{I_2}, \frac{\xi_3}{I_3})$. Therefore, the algorithm simplifies to solving for ξ in

$$m_0 = -dS_0(\xi).\Gamma(ad_\xi) - \tau dS_1(\xi).\Gamma(ad_\xi) =$$

$$-\xi - \tau dS_1.\Gamma(ad_\xi), \tag{5.15}$$

and then setting,

$$m = m_0 - dS_1.ad_\xi. \tag{5.16}$$

The second term in this equation simplifies to

$$-[(\frac{1}{I_1} - \frac{1}{I_2})\xi_1\xi_2, (\frac{1}{I_3} - \frac{1}{I_1})\xi_1\xi_3, (\frac{1}{I_2} - \frac{1}{I_3})\xi_2\xi_3].$$

We will compare the results of this algorithm with the explicit scheme developed in the next section.

5.4 Explicit Lie-Poisson Integration

In this section, we introduce a new explicit Lie-Poisson algorithm. Take a Hamiltonian system on the cotangent bundle of some Lie group. Assume that the Hamiltonian is left-invariant under the group action of G on T^*G. We saw from section 1.5 that the co-adjoint orbits on $\mathcal{G}^*$, when the phase space has been left reduced, is given by the negative Lie-Poisson bracket, i.e., if $\mu \in \mathcal{G}^*$, then the co-adjoint orbit through $\mu \in \mathcal{G}^*$ will be $(\mathcal{O}_\mu, \omega_\mu)$ where $\mathcal{O}_\mu = \{Ad^*_{g^{-1}}(\mu) | g \in G\}$ and

$$\omega_\mu(\xi_{\mathcal{G}^*}(\mu), \eta_{\mathcal{G}^*}(\mu)) = - < \mu, [\xi, \eta] > . \tag{5.17}$$

$< .,. >$is the natural pairing between the Lie algebra and its dual and $[,]$ is the Lie algebra bracket. From section 3.5, we know that the equation of motion for some $\mu \in \mathcal{G}^*$ and Hamiltonian, $H \in \mathcal{C}^\infty(\mathcal{G}^*)$ are given by

$$\dot{\mu} = \mp ad^*_{\frac{\delta H}{\delta \mu}} \mu.$$

The minus arises if the Lie-Poisson system was derived by identification with right invariant vector fields and the positive arises in the left-invariant case. Therefore, for a left invariant Hamiltonian, the equations of motion for $\mu \in \mathcal{G}^*$ is given by

$$\dot{\mu} = ad^*_{\frac{\partial H}{\partial \mu}}(\mu). \tag{5.18}$$

In order to integrate this system, we have resorted to the construction of symplectic integrators which are based on the reduced Hamilton-Jacobi equation of Chapter 4. While this approach is applicable to a very broad class of Lie-Poisson systems, it is only easily realizable in the subcase of regular quadratic Lie algebras, i.e., algebras endowed with an Ad-invariant, bilinear, non-degenerate form. As we have seen, this property enables us to construct a non-singular identity transformation on $\mathcal{G}^*$ for the momentum preserving generating functions of the first kind on T^*G. This was accomplished by the application of the strict generating pair theory of Ge[13]. Armed with such an identity transformation generator, we can then form a k^{th}-order perturbative solution to the Hamilton-Jacobi equation which is implicitly Casimir preserving and Poisson, i.e., the mapping on $\mathcal{G}^*$ is symplectic with respect to the KAKS symplectic form on co-adjoint orbits.

We claim that it is possible to build an explicit Lie-Poisson integrator for precisely these Lie algebras without reverting to a Ruth[14] type of approximation to the Hamiltonian. If $\mathcal{G}$ is regular quadratic, then it is possible to form a diffeomorphism between co-adjoint and adjoint orbits. For brevity, we will present a result proven in [1] for a left-invariant Poisson system.

Proposition 5.4.1 *Let $H : T^*G \to R$ be a left-invariant Hamiltonian, i.e., $H \circ T^*L_g = H$ for all $g \in G$. On $\mathcal{O}_\mu$ we defined the symplectic KAKS 2-form with the negative sign having been selected. If there exists a bi-invariant 2-form $(.,.)$ on $\mathcal{G}$, then the adjoint orbit through $\xi \in \mathcal{G}$ given by $\mathcal{O}_\xi = \{Ad_g\xi | g \in G\}$ has symplectic 2-form*

$$\omega_\xi(\zeta_{\mathcal{G}}(\xi), \eta_{\mathcal{G}}(\xi)) = -(\xi, [\zeta, \eta]), \tag{5.19}$$

for ξ and η in $\mathcal{G}$.

As we will be using the Cartan-Killing form in our examples, let us denote $(.,.)$ by $K(.,.)$. K is non-degenerate, so for every $\mu \in \mathcal{G}^*$, there exists a unique $\tilde{\mu} \in \mathcal{G}$ such that $< \mu, \xi >= K(\tilde{\mu}, \xi)$ for all ξ. We now show that as a consequence of K being Ad-invariant, the adjoint action is skew-symmetric with respect to it.

Consider

$$\begin{aligned} K(ad_\xi\zeta, \eta) = \frac{d}{dt}|_{t=0} K(Ad_{exp t\xi}\zeta, \eta) = \\ \frac{d}{dt}|_{t=0} K(\zeta, Ad_{exp-t\xi}\eta) = K(\zeta, ad_{-\xi}\eta) = \\ -K(\zeta, ad_\xi\eta). \end{aligned} \tag{5.20}$$

With this information, we return to the original equation of motion on $\mathcal{G}^*$ Since $\dot{\mu}$ is the derivative of an element of a linear space, it can be also considered an element of that space. Thus, there exists some unique $\tilde{\dot{\mu}}$ in $\mathcal{G}$

such that

$$<\dot{\mu},\eta>=K(\dot{\tilde{\mu}},\eta).$$

Also, setting $\xi=\frac{\delta H}{\delta\mu}\in\mathcal{G}$, we deduce

$$<\dot{\mu},\eta>=<ad_{\xi}^{*}\mu,\eta>=$$

$$-<\mu,ad_{\xi}\eta>=K(\tilde{\mu},ad_{\xi}\eta)=$$

$$-K(ad_{\xi}\tilde{\mu},\eta). \tag{5.21}$$

Therefore by the Ad-invariance of K, the system reduces to

$$\dot{\tilde{\mu}}=-ad_{\xi}\tilde{\mu}=ad_{-\xi}\tilde{\mu}. \tag{5.22}$$

We can now approximate this flow by discretizing its equation of motion. If we choose as our initial position some element of the Lie algebra, μ_0,then for time-step, τ, the time advanced element μ_1 will be given by

$$\mu_1=Ad_{exp-th}\mu_0, \tag{5.23}$$

where $h=\frac{\delta H}{\delta\mu}(\mu_0)$ is the first derivative of the Hamiltonian evaluated at the starting point. Each iteration may then be executed by replacing μ_0 by μ_1 and iterating through time.

The trick which makes this algorithm pratical without any neccessity to separate the Hamiltonian into simpler pairings, is found in a formula which we have already encountered in the last chapter and proven in [24]. It can be shown that for ξ in the neigborhood of the origin in $\mathcal{G}$,

$$Ad_{exp-\xi}=e^{-ad_{\xi}}, \tag{5.24}$$

where, just as in the final sections of the last chapter, ad_ξ is simply a linear operator of a linear representation of $\mathcal{G}$ and thus the right-hand side of equation[1.23] can be interpreted as a power series in the operator.

The above time-stepping obviously preserves the Casimirs by definition. However, we should check to make sure that the KAKS symplectic 2-form on the adjoint leaves is preserved.

Proposition 5.4.2 *$Ad_{g^{-1}} : \mathcal{O}_\xi \to \mathcal{O}_\xi$ preserves the symplectic 2-form, ω_0, i.e., $(Ad_{g^{-1}})^*\omega_0 = \omega_0$.*

To prove this, consider

$$((Ad_{g^{-1}})^*\omega_0)(\psi)(\xi_\mathcal{G}(\psi), \eta_\mathcal{G}(\psi)). \tag{5.25}$$

By definition this equals

$$\omega_0(Ad_{g^{-1}}\psi)(T_\psi Ad_{g^{-1}}\xi_\mathcal{G}(\psi), T_\psi Ad_{g^{-1}}\eta_\mathcal{G}(\psi)). \tag{5.26}$$

Now, in general $(f)^*Y(x) = (Tf)^{-1}Y(f(x))$, so $T_\psi Ad_{g^{-1}}\xi_\mathcal{G}(\psi) = (Ad_g)^*\xi_\mathcal{G}(Ad_{g^{-1}}\psi)$. So, equation(5.24) equals

$$\omega_0(\phi)((Ad_g)^*\xi_\mathcal{G}(\phi), (Ad_g)^*\eta_\mathcal{G}(\phi)), \tag{5.27}$$

where $\phi = Ad_{g^{-1}}\psi$. We next use the fact that $(Ad_g)^*\xi_\mathcal{G} = (Ad_{g^{-1}}\xi)_\mathcal{G}$, to obtain

$$\omega_0(\phi)((Ad_{g^{-1}}\xi)_\mathcal{G}(\phi), (Ad_{g^{-1}}\eta)_\mathcal{G}(\phi)). \tag{5.28}$$

By the definition of the KAKS bracket on adjoint orbits via the Cartan-Killing form, we find that equation(5.27) becomes

$$-K(\phi, Ad_{g^{-1}}[\xi, \eta]) =$$

$$-K(\psi,[\xi,\eta]), \tag{5.29}$$

by Ad-invariance of the Cartan-Killing form. The transformed 2-form (5.24) now equals

$$\omega_0(\psi)(\xi_{\mathcal{G}}(\psi),\eta_{\mathcal{G}}(\psi)). \tag{5.30}$$

Therefore, the Ad mapping preserves the KAKS 2-form.

This algorithm will be implemented in the next section for the case of the rigid body dynamics and compared to the results obtained via the implicit Hamilton-Jacobi integrator.

5.5 Rigid Body Calculation

We now implement the above algorithms for the same test probelem and compare their performances. Take a rigid body with moment of inertia tensor, $diag(I_1, I_2, I_3)$ with $I_1 I_2 I_3$. The moment of inertia ellipsoid has semi-axes $\sqrt{2EI_1}, \sqrt{2EI_2}, \sqrt{2EI_3}$ where

$$2E = [\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^3}{I_3}].$$

The angular momentum vector has magnitude $m = \sqrt{m_1^2 + m_2^2 + m_3^2}$ and the dyanmics of the rigid body is determined by the value of m relative to the semi-axes.

1. If $m < \sqrt{2EI_3}$ or $m > \sqrt{2EI_1}$, then no motion is possible.

2. If $\sqrt{2EI_3} \leq m \leq \sqrt{2EI2}$ then the motion will be periodic with the constant angular momentum sphere intersecting the ellipsoid of inertia

between the smallest and middle semi-axis.

3. If $\sqrt{2EI_2} \leq m \leq \sqrt{2EI_1}$ then again the motion will be periodic about the longest axis.

4. If m is equal to any of the three semi-axes, then this represent equilibrium points with the two extrema being stable and the middle one unstable.

We choose $I_1 = 8, I_2 = 4$ and $I_3 = 2$. We also choose the total angular momentum to equal 3 with initial position $m = (\sqrt{2}, \sqrt{7}, 0)$. Thus, the momentum lies in between the middle and the largest semi-axis. This gives an energy of $E = 1$.

Both numerical techniques were seeded with these initial conditions and implemented. The explicit integrator benefits from the implementation of an exact formula for the $so(3)$ Lie algebra from Whittaker[25]. The result states that if $\xi \in so(3)$, then the exponential of the linear adjoint transformation derived from this element of the algebra is equal to

$$exp(ad_\xi) = I + \frac{\sin(||\xi||)}{||\xi||} ad_\xi + \frac{1}{2} \frac{\sin^2(\frac{||\xi||}{2})}{\frac{||\xi||^2}{2}} ad_\xi^2. \tag{5.31}$$

Both integrators performed very well from the point of view of preservation of the angular momentum or Casimir. The implicit algorithm appeared not to have secular growth terms in the energy whereas the explicit Lie-Poisson mapping exhibited a monotonic increase in its energy by about 3-percent in 10^6 time steps. Both integrators reproduced the angular momentum vector consistent with each other.

The angular momentum remained constant throughout the calculation producing a trajectory completely specified by a position on the unit sphere. Because the energy error away from its true value was so small, the exact trajectory is almost completely reproduced. This is independent evidence that the explicit integrator is actually producing credible numerical results. Of course, the secular growth in the energy is worrying as the evidence seems to be that the energy oscillates around its true conserved value. However, it may be that the oscillation has a greater period than that of the implicit scheme.

5.6 SU(N)

For the $su(N)$ application, the process of constructing the algorithm is exactly the same as for the rigid body. We will demonstrate the process for the explicit scheme.

For $su^*(N)$, choose the basis $\{S^k\}_{k\in\Omega}$ as in chapter 3, section 5. We know that $su(N)$ is semi-simple and that the Cartan-Killing form in this basis reduces to $g_{ij} = N^2\delta(i+j)|_{modN}$. Therefore, if we are given an $\omega \in su^*(N)$, then we must transform it to $N^2 \sum \omega_{-n} T_n$ in order to apply our integration technique. Given ω, there are two more quantities to calculate, the functional derivative of the energy and the linear operator ad_ξ on $su(N)$.

The functional derivative of the Hamiltonian with respect to ω is just $(\frac{1}{k^2}\omega_{-k}T_k) \in su(N)$.

From chapter 3, section 4, the matrix of the adjoint transformation is

$$(ad(T_m)X)_n = (M_m)_{nk}X_k \tag{5.32}$$

which provides a matrix representation on $\mathcal{G}$.

The exponential power series is then evaluated by taking matrix powers of this matrix. This is common to both algorithms. Just as in the $so(3)$ regime, we get very good results for low dimensions but the implicit algorithm begins to slow down to a point where it is impossible to use it for any significant vortex distribution evolution.

The explicit algorithm should speed up this process by 100 to 200 fold.

Conclusion and Summary

We have developed the theory of geometric Hamiltonian fluid mechanics and built Lie-Poisson integrators for low dimensional truncations of the 2-D Euler equations on the 2-torus. The idea of using such integrators is appealing as they could provide a Hamiltonian technique to investigate longtime inviscid integrations. We hope that the explicit Lie-Poisson integrator introduced in chapter five will provide a practical tool for exploring the $SU(N)$ truncation We can definitely say for certain that the implicit Lie-Poisson integrator of Channell and Scovel does not appear efficient and practical enough for large scale simulations. Another research area in which the $SU(N)$ truncation may find application is the equilibrium statistical mechanics of the 2D Euler equation. The attempt to use statistical mechanics to describe two dimensional inviscid fluid flow was initiated by Onsager[27] for the point vortex weak solution to the Euler equations. The analysis seemed to suggest the emergence of coherent structure of like-signed point vortex clumps in the fluid domain. This line of research has continued to the present day with the most recent results of Miller[28] which also predict the emergence of such vortex structures except that the Miller theory actually models more realistic continuous solutions to the Euler equations. One of the problems with attempting to build a statistical mechanics is to identify the true phase space on which the Hamiltonian flow exists. Miller argues that phase space is the space of all vortex fields and that at least in the microcanonical ensemble averaging, one integrates over the vortex space under the Casimir constraints. However, since these constraints are infinite in number, some form of approximation to the Helmholtz laws are effected. This approximation unfortunately breaks material line integrity. Of course, this may be quite justifiable in the long time limit of equilibrium statistical mechanics and only extensive testing of the theory will yield a satisfactory answer. The question of the isolation of the real phase space can however be quickly addressed from the discussion of the previous chapters. The space of all vortex fields foliates into co-adjoint orbits which have a symplectic structure and are disjoint. Therefore, depending on the choice of initial vortex distribution, the appropriate phase space will be that particular distribution's orbit. The truncated torus flow offers a tempting mechanism by which a statistical mechanical program could be implemented.

Bibliography

[1] R.Abraham and J.Marsden, Foundations of mechanics,second edition, (Addison-Wesley, Reading, Mass. 1978.)

[2] R.Schmid, Infinite dimensional Hamiltonian systems, (Bibliopolis, Napoli,1987.)

[3] V.Arnold, Mathematical methods in classical mechanics, Graduate Texts in Math. No. 60, second edition, (Springer-Verlag,New York, 1989.)

[4] J.Marsden, Lectures on mechanics, London Mathematical Society lecture note series,(Cambridge University Press, New York, 1992).

[5] V.Arnold, Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits, Ann. Inst. Fourier Grenoble 16 (1966)319-361.

[6] V.Zeitlin, Finite-mode analogs of 2D ideal hydrodynamics-coadjoint orbits and canonical structure, Physica D 49(3) (1991)353-362.

[7] M.Bordemann, J.Hoppe, Schaller,M.Schlichenmaier, GL(∞) and geometric-quantization, Comm. Math .P 138(2) (1991)209-244.

[8] J.Marsden and A.Weinstein, Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids, Physica D 7(1-3) (1983)305-323.

[9] Z. Ge and J.Marsden, Lie-Poisson Hamilton-Jacobi theory and Lie-Poisson integrators, Phys-Lett-A 133(3) (1988)134-139.

[10] Y.Wan and M.Pulvirenti, Non-linear stability of circular vortex patches, Comm. Math. P. 100(3) (1985)343-354.

[11] D.Dritschel, The repeated filamentation of 2-D vorticity interfaces, J.Fluid Mec. 194(Sep) (1988)511-547.

[12] D.Ebin and J.Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. Math. 92 (1970) 102-63.

[13] Z.Ge, Equivariant symplectic-difference schemes and generating functions, Pysica D 49(3) (1991)376-386.

[14] P.Channell and J.Scovel, Integrators for Lie-Poisson dynamic systems, Physica D 50(1) (1991)80-88.

[15] P.Channell and J.Scovel, Symplectic integration for Hamiltonian-systems, Nonlinearity 3(2) (1990)231-259.

[16] A.Dragt and J.Finn, Lie Series and invariant functions for analytic symplectic maps, J. Math. Phys. 17 (1976)2215-2227.

[17] K.Fang, Difference schemes for Hamiltonian formalism and symplectic geometry, J.Comput.Math 4(1986)279-289.

[18] Z.Ge, Hamilton-Jacobi equations and symplectic groupoids on Poisson manifolds, Indiana J. Math, 39(3) (1990)859-856.

[19] A.Rouhi, Symmetric truncations of shallow water equations, preprint.

[20] D.Sattinger and O.Weaver, Lie groups and algebras with applications to physics, geometry and mechanics, Applied Math. Sci. No.61, (Springer, New York, 1986.)

[21] R.Abraham, J. Marsden and T.Ratiu, Manifolds, tensor analysis and applications, Global Analysis:Pure and Applied No.2, (Springer, New York, 1983.)

[22] J.Hoppe, Diffeomorphism groups, quantization, and $SU(\infty)$, Int. J. Mod.Phys. A4 (1989)5235-5248.

[23] H.Goldstein, Classical Mechanics, (Addison-Wesley, Reading, Mass., 1980.)

[24] N.Bourbaki, Groupes et algebres de Lie, Elements de Mathematique, (Hermann,Paris, 1972.)

[25] E. Whittaker, A treatise on the analytical dynamics of particles and rigid bodies, (Cambridge University Press, 1937.)

[26] P.Lax, Integrals of non-linear equations of evolution and solitary waves, Comm. Pure Appl. Math.,21(1968) 467-490.

[27] L. Onsager,Statistical Hydrodynamics, Nuovo Cimento(6), Supplement, (1949)279-287.

[28] J.Miller, P.Weichman, M.Cross, Statistical mechanics, Euler equation, and Jupiter red spot, Phys. Rev. A 345(4) (1992)2328-2359.

Appendix A

Differentiable Manifolds, Tangent Bundles and Manifold Mappings

The fundamental result of manifold theory, at least for our applications is the formulation of calculus on general spaces. This is achieved by locally identifying the set with some linear space such as a Banach space. Calculus can then be executed on the set by moving to the Banach space, computing as normal and then mapping back to the set. By carefully linking the local charts together, a global calculus is obtained.

Given a set M, a local chart on M over some Banach space V is a pair (U, ϕ) where U is open in M and $\phi : U \to V$ is bijective onto an open set in V. M is called a smooth manifold over V if

1. for all $x \in M$, there exists a chart (U, ϕ) such that $x \in U$.

2. for every (U_1, ϕ_1) , (U_2, ϕ_2)

$$\phi_1(U_1 \cap U_2) \text{ and } \phi_2(U_1 \cap U_2)$$

are open in V and,

3. $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is a $\mathcal{C}^\infty$ diffeomorphism.

A mapping $f : M \to N$ from one smooth manifold to another is of class $\mathcal{C}^r$ if given (U, ϕ) a chart in M and (W, ψ) a chart in N such that if $\alpha \in U$, $f(\alpha) \in W$, the mapping $f_{\phi\psi} = \psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(f(U))$ is of class $\mathcal{C}^r$. We will use $f_{\phi\psi}$ to construct the derivative of f.

In order to define the derivative of a manifold mapping, we need to construct the tangent bundle. A tangent bundle is a special case of a vector bundle which can be thought of heuristically as a way of assigning a linear vector space to every point on a manifold.

Locally a vector bundle will look like a vector space product. If E and F are vector spaces and $U \subset E$ open, then $U \times F$ is called the local vector bundle with U the base space which is isomorphic to $U \times \{0\}$ which is known as the zero section. If $u \in U$, then $\{u\} \times F$ is the fiber over U which is itself a vector space. Locally, a vector bundle mapping can also be defined. A mapping $\phi : U \times F \to U' \times F'$ is called a $\mathcal{C}^r$ local vector bundle mapping if it has the form $\phi(u, f) = (\phi_1(u), \phi_2(u).f)$ where $\phi_1 : U \to U'$ and $\phi_2 : U \to L(F, F')$. ϕ will map fibers to fibers.

Now we are ready to define a vector bundle. Let S be a set. A local bundle chart of S is a pair (W, ϕ) where $W \subset S$ and $\phi : W \to U \times F$ is a bijection onto a local bundle. We can form a vector bundle atlas $\mathcal{A}$ which is a family of local bundle charts which satisfy

1. for all $x \in S$, there exists a local bundle chart (U, ϕ) such that $x \in U$,
2. for any two local bundle charts, $(U_i, \phi_i), (U_j, \phi_j)$ with $U_i \cap U_j$ non-empty, $\phi_i(W_1 \cap W_2)$ is a local vector bundle and $\psi_{21} = \phi_2 \circ \phi_1^{-1}$ is a $\mathcal{C}^\infty$ local vector bundle isomorphism.

The vector bundle is the pair $(S, \mathcal{A})$. The equivalent of the base on a local vector bundle is the space $B = \{s \in S | s \in \phi^{-1}(U \times \{0\}) \; for \; some \; (U, \phi) \in \mathcal{A}\}$. B is a submanifold of S and the map $\pi : E \to B; \pi(s) = b$ is surjective and $\mathcal{C}^\infty$.

Let E and E' be vector bundles. Let $f : E \to E'$ be a mapping between the two bundles. f is called a $\mathcal{C}^r$ vector bundle mapping if for each $e \in E$ and local chart (V, ϕ) of E' such that $f(e) \in V$, there exists a local chart (W, ψ) with $f(W) \subset V$ which has the property that $f_{\phi\psi} = \psi \circ f \circ \phi^{-1}$ is a $\mathcal{C}^r$ local vector bundle mapping.

We will often refer to a vector bundle by specifying the projection mapping $\pi : E \to B$ from the vector bundle to the zero section. The fiber $\pi^{-1}(b)$ is a vector space and π is $\mathcal{C}^\infty$ surjective.

We can now investigate the tangent bundle. If f is of class $\mathcal{C}^r$ and maps $U \subset E$ into $V \subset F$, then the tangent mapping of f is denoted Tf and maps $U \times E \to V \times F$ via

$$Tf(u, e) = (f(u), Df(u).e). \tag{A.1}$$

Recall that $Df(u)$ is an element of $L(E, F)$. This tangent mapping is easily seen to be a local vector bundle mapping.

We will now construct the tangent bundle to a smooth manifold M and show that mappings on the tangent bundle can locally be represented as above. We proceed by defining a curve at a point $x \in M$ as a mapping $\sigma : I \subset R \to M$ such that $\sigma(0) = x$. Take two curves at $x \in M$, σ_1 and σ_2. We say that they are tangent at x if, in terms of some local chart (U, ϕ) at x,

$$\lim_{t \to 0} \frac{(\phi \circ \sigma_1)(t) - (\phi \circ \sigma_2)(t)}{t} = 0. \tag{A.2}$$

This can be shown to be independent of the choice of chart.

We form an equivalence class of curves at x which are tangent and denote it by $[\sigma]_x$. Such a class is called a tangent vector. The set of all such vectors is called the tangent vector space at x and denoted T_xM.

If $f : M \to N$ is a differentiable mapping and (U, ϕ) and (W, ψ) are local charts in M and N respectively, then if $x \in U$ and $f(U) \subset W$, we define

$$T_xf : T_xM \to T_{f(x)}N \tag{A.3}$$

where if $\underline{v} = (x, v) \in T_xM$, then

$$T_xf.\underline{v} = (f(x), Df_{\phi\psi}(\phi(x)).v). \tag{A.4}$$

This is a local vector bundle mapping and Tf is a vector bundle mapping on $TM = \bigcup T_xM$. The projection mapping $\pi : TM \to M$ defined by $\pi(v) = x$ if $v \in T_xM$ is a vector bundle. Also, if we form the space of linear functionals, T_x^*M to T_xM at every point of the manifold, M then the space $\bigcup T_x^*M$ is also a vector bundle referred to as the co-tangent bundle.

Appendix B

Tensors and Exterior Algebra

We will first consider tensors defined on linear vector spaces. The space of linear mappings from E to R, $L(E,R)$ is called the dual space to E and is denoted E^*. This can be generalized to $L(E^*,R)$ and to $L^{r+s}(E^*,\ldots,E^*;E,\ldots,E;R)$ where there are r copies of E^* and s copies of E. These multilinear real-valued mappings are called tensors of contravariant order r and covariant order s.

The tensor product $t_1 \otimes t_2 \in T^{r_1+r_2}_{s_1+s_2}(E)$ of $t_1 \in T^{r_1}_{s_1}(E)$ and $t_2 \in T^{r_2}_{s_2}(E)$ is defined by

$$t_1 \otimes t_2(\beta^1,\ldots,\beta^{r_1};\gamma^1,\ldots,\gamma^{r_2};f_1,\ldots,f_{s_1};g_1,\ldots,g_{s_2}) = \qquad \text{(B.1)}$$
$$t_1(\beta^1,\ldots,\beta^{r_1};f_1,\ldots,f_{s_1})t_2(\gamma^1,\ldots,\gamma^{r_2};g_1,\ldots,g_{s_2}).$$

Given linear mappings between linear vector spaces, we can generalize their action to include tensors.

Definition 1 *If $\phi \in L(E,F)$, then $\phi^* \in L(F^*,E^*)$ is defined by $\phi^*(\beta).e = \beta(\phi(e))$. More generally, $T^r_s\phi = \phi^r_s \in L(T^r_s(E), T^r_s(F))$ is defined by*

$$\phi^r_s t(\beta^1,\ldots,\beta^r; f_1,\ldots,f_s) = t(\phi^*(\beta^1),\ldots,\phi^*(\beta^r),\phi^{-1}(f_1),\ldots,\phi^{-1}(f_s)). \tag{B.2}$$

In order to define tensors on smooth manifolds and vector bundles, we will expand the above definitions to local vector bundles. If $\phi : U \times F \to U' \times F'$ is a local vector bundle mapping, then we define $\phi^r_s : U \times T^r_s(F) \to U' \times T^r_s(F')$ by

$$\phi^r_s(u,t) = (\phi_0(u), (\phi_u)^r_s t). \tag{B.3}$$

Let $\pi : E \to B$ be a vector bundle with fibers $E_b = \pi^{-1}(b)$. We define $T^r_s(E) = \bigcup T^r_s(E_b)$ and the tensor bundle to be $\pi^r_s(e) = b$ if and only if $e \in T^r_s(E_b)$.

Having stated the basics on a vector space, we will now move to the manifold setting. Let M be a manifold and $\tau_M : TM \to M$ its tangent bundle. We will denote $T^r_s(M) = T^r_s(TM)$ as the vector bundle of tensors of this order. Before we proceed, we introduce sections which can loosely be considered as the inverse of the tangent bundle mapping. A C^r section of a tangent bundle is a map $\xi : B \to E$ of class C^r such that for each $b \in B$, $\pi(\xi(b)) = b$. The vector space of all such sections over B will be denoted $\Gamma^r(\pi)$.

If $\phi : M \to N$ is a diffeomorphism between smooth manifolds, then

$$\phi_* t = (T\phi)^r_s \circ t \circ \phi^{-1} \tag{B.4}$$

is the pushforward of t by ϕ and $\phi^* t = (\phi^{-1})_* t$ is the pullback.

The major computational tool in Geometric Hamiltonian mechanics involves exterior calculus of differential forms. The main form which arises is the symplectic 2-form needed to generate the Hamiltonian vector fields. We will now summarize the main techniques starting on vector spaces before passing to the manifold environment.

Let $\Omega^k(E) = L^k_a(E, R)$ be the space of skew-symmetric multilinear maps on E. If $\alpha \in T^0_{r_1}(E)$ and $\beta \in T^0_{r_2}(E)$, we define their wedge product $\alpha \wedge \beta \in \Omega^{r_1+r_2}(E)$ by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha\beta). \tag{B.5}$$

A is the alternating operator on tensors defined by

$$A(t)(e_1, \ldots, e_k) = \frac{1}{k!} \sum (sign\sigma) t(e_{\sigma(1)}, \ldots, e_{\sigma(k)}),$$

over all permutations, σ of $\{1, 2, \ldots, k\}$.

The properties of the wedge product can be summarized by

1. $\wedge$ is bilinear,
2. $\alpha \wedge \beta = (-1)^{r_1 r_2} \beta \wedge \alpha$,
3. $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

The direct sum of $\Omega^k(E)$ $k = 0, 1, 2, \ldots$ is called the exterior algebra of E. If $dimE = n$, then $\dim\Omega^n(E) = 1$ and if $\alpha^1, \ldots, \alpha^n$ is a basis for E^*, then $\alpha^1, \ldots, \alpha^n$ spans $\Omega^n(E)$. We now define the determinant of a mapping ϕ,

Definition 2 *Let $dim(E) = n$ and $\phi \in L(E, E)$. The determinant of ϕ is* *the unique constant* $det\phi$ *such that* $\phi^* : \Omega^n(E) \to \Omega^n(E)$ *satisfies* $\phi^*\omega = (det\phi)\omega$ *for all* $\omega \in \Omega^n(E)$.

If $g \in T_2^0(E)$ is non-degenerate and symmetric, then there exists a unique volume element, $\mu = \mu(g)$ called the $g-$volume such that $\mu(e_1, \ldots, e_n) = 1$ for all positively oriented $g-$orthonormal bases $\{e_1, \ldots, e_n\}$ of E. If $\{e^1, \ldots, e^n\}$ is the dual basis, then $\mu = e^1 \wedge e^2 \wedge \ldots \wedge e^n$.

The Hodge mapping can be defined using this volume form.

Definition 3 *There exists a unique* $* : \Omega^k \to \Omega^{n-k}(E)$ *such that*

$$\alpha \wedge *\beta = < \alpha, \beta > \mu, \tag{B.6}$$

for $\alpha, \beta \in \Omega^k(E)$. *This map is called the Hodge star map.*

As a simple example, consider the Hodge star operator on $\Omega^1(R^3)$, then $*e^1 = e^2 \wedge e^3$, $*e^2 = -e^1 \wedge e^3$ and $*e^3 = e^1 \wedge e^2$.

We are now prepared to study differential forms and the operators which act on them.

Appendix C

Exterior Calculus

We can extend the above definitions to exterior forms on a manifold M. Let $\Omega^0(M) = \mathcal{F}(M)$, $\Omega^1(M) = \mathcal{X}^*$ and $\Omega^k(M) = \Gamma^\infty(\Omega^k_M)$, the $\mathcal{C}^\infty$ sections on M where Ω^k_M is the vector bundle of exterior k-forms on the tangent spaces of M.

Letting $\Omega(M)$ be the direct sum of $\Omega^k(M)$ for $k = 0, 1, 2, \ldots$ and extending the wedge product componentwise to all of $\Omega^k(M)$, $\Omega(M)$ is called the algebra of exterior differential forms on M.

The exterior derivative is one of the most important operators on exterior forms. The usual definition of the differential involves its action on smooth functions on some manifold,M, $d : \Omega^0(M) \to \Omega^1(M)$ where $f \to df; df(x)X(m) = \frac{d}{dt}|_{t=0}(f \circ \sigma)(t))$. The curve σ is a tangent curve passing through x at $t = 0$. We can extend this definition to $\mathbf{d} : \Omega^k(M) \to \Omega^{k+1}(M)$. The differential $\mathbf{d}$ has a number of useful properties. We list(dropping the bold type) them:

1. For $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{C.1}$$

2. On $\Omega^0(M)$, d co-incides with the usual definition of the differential.

3. $d^2 = d \circ d = 0$ for all components of the exterior algebra.

In co-ordinates, if $\omega \in \Omega^k(M)$, then

$$d\omega(u)(v_0, v_1, v_2, \ldots, v_k) = \sum(-1)^i D\omega(u).v_i(v_0, \ldots, v_k) \tag{C.2}$$

where the component v_i has been left out of the righthand side of the equation.

We will give a number of examples of applications of the differential. Consider a function $f \in \Omega^0(R^3)$, then $df = f_x dx + f_y dy + f_z dz$. This is the standard result which is usually taken as the gradient of f. However, as we see, df is a 1-form, so if we want to know the gradient of f, we find the vector such that $v^\flat = df$. The operation $\flat$ raises a vector to a 1-form with 1-1 matching of the natural basis.

Remaining in R^3, we consider the form $F^\flat = F_1 dx + F_2 dy + F_3 dz$ for $F \in T_0^1(R^3)$. Then,

$$dF^\flat = (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dx \wedge dy - (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})dx \wedge dz + (\frac{\partial F_3}{\partial y} - \frac{\partial F_3}{\partial y})dy \wedge dz.$$

This will be related to the $Curl$ of F by

$$*(CurlF)^\flat = dF^\flat. \tag{C.3}$$

We can also show that $d * F^\flat = (divF)dx \wedge dy \wedge dz$. Thus, we see that all the usual vector analysis operators on R^3 can be expressed in terms of forms.

Another important property of d is that it is natural with respect to diffeomorphisms, $F : M \to N$. Then $F^* : \Omega(N) \to \Omega(M)$ satisfies $F^*(\psi \wedge \omega) = F^*\psi \wedge F^*\omega$ and $F^*(d\omega) = d(F^*\omega)$. With respect to the pushforward, $F_* = (F^{-1})^*$, d is also natural.

We will now discuss the Lie derivative. The Lie derivative of a function f with respect to a vector field, $X \in T^1_0(M)$ is defined via the differential of f,

$$L_X f(m) = df(m).X(m), \tag{C.4}$$

for all $m \in M$. It can easily be shown that L_X is also natural with respect to pushbacks and pushforwards of diffeomorphisms.

We can define L_X on $\mathcal{X}(M)$ by $L_X Y = [X, Y]$ which is the unique vector field on M such that

$$L_{L_X Y} = [L_X, L_Y] = L_X \circ L_Y - L_Y \circ L_X. \tag{C.5}$$

From the theory of differential operators on tensors, it is known that if there exist a $\mathbf{D} : \mathcal{T}(M) \to \mathcal{T}(M)$ which agrees with L_X on $\mathcal{F}(M)$ and on $\mathcal{X}(M)$, then $\mathbf{D}$ will be uniquely determined on all of the tensor bundle. We will thus assume that we have extended L_X to all exterior forms of any order. This L_X is natural with respect to diffeomorphisms, just as d is, i.e.,

$$L_{X_{\phi_*}}(\phi_* t) = \phi_* L_X t. \tag{C.6}$$

For example, let $\{\frac{\partial}{\partial x_i}\}$ be a basis for the vector fields on R^n. Consider the the Lie derivtive of a vector $X = X_i \frac{\partial}{\partial x_i}$ on a tensor $g \in T^0_2$, $g = g_{ij} dx^i \otimes dy^j$. Then

$$L_X g = (X^k \frac{\partial g_{ij}}{\partial x^k} + g_{kj} \frac{\partial X^k}{\partial x^i} + g_{ik} \frac{\partial X^k}{\partial x^j}) dx^i \otimes dx^j, \tag{C.7}$$

which is still a symmetric tensor of covariant order 2.

A most important property of the Lie derivative which is sometimes used as an alternative definition is that if F_t is the flow of X, then

$$\frac{d}{dt}F_t^* t = F_t^* L_X t, \tag{C.8}$$

for any tensor t. If $L_X t = 0$, then the tensor is obviously invariant under the flow of X.

The differential d is natural with respect to L_X as well, i.e.,

$$dL_X\omega = L_X d\omega.$$

The last major operator which is frequently applied in geometric mechanics is the interior operator i_X which is defined for some $X \in \mathcal{X}(M)$. $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined on any ω in $\Omega(M)$ by

$$i_X\omega(X_1, \ldots, X_k) = \omega(X, X_1, \ldots, X_k). \tag{C.9}$$

i_X has a number of very useful properties which are quite indispensable in calculations. If $\alpha \in \Omega^k(M)$,$\beta \in \Omega^l(M)$ and $f \in \Omega^0(M)$,

1. $i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\subset_X \beta)$,

2. $i_{fX}\alpha = f i_X \alpha$,

3. $i_X df = L_X f$,

4. $L_X\alpha = i_X d\alpha + d i_X \alpha$.

If α is a k-form which satisfies $d i_X \alpha = 0$, then $F_t^*(\alpha) = \alpha$. Suppose that $X \in \mathcal{X}(R^3)$ is divergence free, then for the 3-form $\alpha = dx \wedge dy \wedge dz$,

$$i_X\alpha = i_X(dx \wedge dy \wedge dz) = *X^\flat = divX. \tag{C.10}$$

Therefore, $di_X\alpha = 0$. We can thus conclude that the flow of X is volume-preserving.

We say that $\omega \in \Omega^k(M)$ is a closed form if $d\omega = 0$ and exact if there exists a $\theta \in \Omega^{k-1}$ such that $\omega = d\theta$.

To finish off our discussion, we state Poincare's lemma which applies to closed 1-forms: Every exact form is closed and every closed form can be regarded, at least locally, as exact.

Appendix D

SDiff(M)

The volume preserving diffeomorphisms on some Riemannian manifold M form a Lie group only in a restricted sense. We will use this appendix to investigate the properties of infinite dimensional Lie groups which are not modeled on Banach spaces. Details of all the topics discussed here can be found in Ebin and Marden[12] or Schmid[2].

We first consider general function space manifolds. Consider a finite dimensional vectoir bundle over a compact M, $\pi : B \to M$, where B is the base space. Locally, a section ξ of π can be considered a as a map $\hat{\xi}$ from a copy of R^n to R^m for some m and $n = dimM$. A H^s section of π is a section such that all its derivations of order less than or equal to s are square integrable. Therefore, form $H^s(\pi)$, the set of all H^s-sections. If $\xi \in H^s(\pi)$, then locally its representative $\hat{\xi}$ is an element of the H^s-maps on Euclidean vector spaces.

Similarly, denote by $H^s(M, N)$ the set of all H^s maps from a smooth manifold, M to another manifold N. Locally, the representative of a mapping $f \in H^s(M,N)$ will be an element of $H^s(R^n, R^n)$ where n, m are the dimensions of M and N respectively. We wish to impose a manifold structure on this space. We can define the tangent space to $H^s(M,N)$ at any point f as follows,

$$T_f H^s(M,N) = \{\xi \in H^s(M,TN) | \pi_N \circ \xi = f\}. \tag{D.1}$$

The set of all H^s diffeomorphisms on M, $Diff^s(M)$ forms an open subspace of $H^s(M,M)$. The tangent space at $e \in M$ to $Diff^s(M)$ is defined by

$$T_e Diff^s(M) = \{\xi \in H^s(M,TM) | \pi_M \circ \xi = e\} = H^s(\pi_M). \tag{D.2}$$

There are problems defining a Banach or Hilbert structure on $Diff^\infty(M)$ because there does not exist a well-defined norm . It is an example of a Frechet space in which its topology is defined by an infinite sequence of norms on $Diff^s(M)$,$s = 0, 1, 2, \ldots$. Differential calculus on these types of spaces is far more difficult than that encountered on the more benign infinite dimensional Banach function spaces. $Diff^s(M)$ can also be considered as a group under the composition of functions,

$$\mu : Diff^s(M) \times Diff^s(M) \to Diff^s(M); \mu(f,g) = f \circ g. \tag{D.3}$$

However, $Diff^s(M)$ is not a Banach Lie group since the multiplication operator μ is only differentiable in a limited way.

Consider right multiplication $R_g : Diff^s(M) \to Diff^s(M) : R_g f = f \circ g$ for all $f, g \in Diff^s(M)$. Then the derivative of this mapping implies tha $TR_g = R_g$ so that R_g is C^∞. However, this is not the case for left translation as its derivative is $TL_g = L_{Tg}$ so that if g is $\mathcal{C}^k$, then L_g is only $\mathcal{C}^k$. Therefore, group multiplication is not smooth but only continuous.

In the geometric theory of mechanics, the Lie algebra is of the utmost importance. The Lie algebra in this case is the tangent space to $Diff^s(M)$ at the idenity transformation of M. The bracket on the algebra $[\xi, \eta]$ between two elements will be defined by first extending them to the full tangent bundle via right translation to their right-invariant counterparts,Y_ξ and Y_η and then restricting the usual canonical bracket back to the identity. We use right-translation because it is smooth. The bracket is found to be

$$[\xi, \eta] = -\{Y_\xi, Y_\eta\}(e). \tag{D.4}$$

However, we see that forming such a bracket will lead to an element with a lower H^s behavior, thus viloating closure. This can be remedied by carrying out all calculations in $Diff^\infty(M)$. However, problems arise as this space is an Inverse Limit Hilbert group denoted $\{Diff^\infty(M), Diff^s(M)\}$ and has a more complicated toplogy than standard Banach Lie groups. This especially leads to difficulties as we are interested in subgroups of $Diff^s(M)$, the most prominent being the volume preserving diffeomorphisms on M. It is not immediately obvious that these subgroups form ILH subgroups. The usual tactic to prove that a subgroup forms a Lie subgroup of some group G is to use the *exp* mapping from the Lie algebra to the group. However, for the diffeomorphism group, the *exp* mapping is found to be only continuous and not even C^1 and there is no neighborhood of the identity transformation in $Diff^s(M)$ onto which *exp* maps surjectively. However, these difficulties can be overcome and $SDiff^s(M)$ can be shown to form an ILH subgroup of $Diff^s(M)$.

$SDiff^s(M)$ is important because it is the configuration space for the flow of inviscid, incompressible fluids. If we assume that M is some compact orientable manifold with Riemannian volume μ, then we can define a smooth weak Riemannian metric on the tangent bundle to $SDiff^s(M)$,

$$< U, V >_\eta = \int_M (U(m), V(m))(\eta(m))\mu(m), \tag{D.5}$$

for $U, V \in T_\eta Diff^s(M)$ and $(,)$ the Riemannian metric on M. Ebin and Marsden [12] used this metric to define the kinetic energy of the fluid in Lagrangian co-ordinates. They showed that it was right-invariant so that the system could be reduced to the Lie algebra, $sdiff^s(M) = T_e SDiff^s(M)$. The reduced equations reproduce the Material description of fluid mechanics. Similarly, Marsden and Weinstein[8] used the right group action of $SDiff^s(M)$ on its cotangent bundle and the invariance of the kinetic energy under the right action to develop a Lie-Poisson dynamics on the dual Lie algebra. This is equivalent to the vorticity formulation of inviscid, incompressible fluid flow. Their formulation is detailed in Chapter 2.