AN INVESTIGATION OF THE LOW EQUATION AND THE CHEW-MANDELSTAM EQUATIONS

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ABSTRACT

We show that for scalar theories without a cutoff the asymptotic form for large energies ω of the perturbation expansion of the Low equation in the one-meson approximation is a double power series in the coupling constant g^2 and $\ln\omega$. The method applied by Gell-Mann and Low to the photon propagator in electrodynamics is used to show that if the crossing matrix has only one negative eigenvalue this power series reduces to a series in a single variable

$$y = \frac{g_1}{1 - \frac{g_1 \theta}{\pi} \ln\omega}$$

where $g_1 = g_1(g^2)$, and θ is a constant. The series in y is evaluated for several interactions and for some crossing matrices with no physical interpretation; for the former the series is a simple algebraic function, while for the latter it usually diverges for all values of $y \neq o$. We obtain the exact solution of the one-meson approximation for the symmetric scalar pion-nucleon interaction; it is a multiple-valued function of g^2. We compare perturbation approximation to the determinantal function of Baker and the cotangent of the phase shift, with the numerical solution of Salzman, for the symmetric pseudoscalar theory with a cutoff; they are found to be often accurate to a few percent. We show that Chew and Mandelstam's approximate equations for pion-pion scattering have no solution for positive coupling λ, and that the perturbation expansion of the solution of their equations for isotopic spin o pions diverges for $\lambda \neq o$. For pions with I = 1 we present calculations of the perturbation expansion to sixth order in λ.

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I. INTRODUCTION

The next few chapters of this thesis are devoted to working out various analytic and numerical features of the Low equation in the one-meson approximation. In this chapter we shall review the basis for the Low equation; we shall then review the previous work on the one-meson approximation, and discuss the questions that we shall study in the following chapters.

The Low equation is an equation satisfied by the scattering amplitudes for elastic (and charge-exchange) pion-nucleon scattering in the static model. It was derived by Low[(1)], but better references are by Wick[(2)], and by Chew and Low[(3)]. We shall not derive the Low equation here; we shall simply summarize the static model theory of the pion-nucleon interaction, and list the properties of the pion-nucleon scattering amplitudes discovered by Low and others which combine to give the Low equation.

In the static model theory of the pion-nucleon interaction, the pions are treated by the usual methods of relativistic field theory, but the nucleon field is replaced by a fixed source [with a finite number of spin and charge states] which can emit or absorb pions. The source is required to emit or absorb pions one at a time; the form of the Hamiltonian then depends only on the spin and number of charge states of the nucleon, and the parity and number of charge states of the pion. In practice the pion is pseudoscalar and has three charge states, the nucleon has spin 1/2 and two charge states, and isotopic spin is conserved. In this case the interaction is called the symmetric pseudoscalar theory. For mathematical interest, and for possible application to strange particle interactions, one wishes to consider other choices for the spin and isotopic spin of the nucleon, and the number of charge states and parity of the pion. In particular, we consider the symmetric scalar theory, in which the pion has isotopic spin 1 but is scalar, and the nucleon has isotopic spin 1/2: the ordinary spin of the nucleon now does not enter into the Hamiltonian. There is also the charged scalar theory, which has only charged pions, but otherwise is the same as the symmetric scalar theory.

The Hamiltonian for the symmetric pseudoscalar theory is

$$H = \sum_{i=1}^{3} \frac{1}{2} \int \left[\pi_i^2(x) + \nabla\phi_i(x) \cdot \nabla\phi_i(x) + m^2 \phi_i^2(x) \right] d^3x + \sum_{i=1}^{3} \frac{g_0}{m} \int \left[\tau_i \, \sigma \cdot \nabla\phi_i(x) \right] \rho(x) \, d^3x \quad . \tag{1.1}$$

Here m is the pion mass, and $\phi_i(x)$ is the field operator for the i[th] meson field: the three mesons are represented by three real scalar fields [the charged mesons correspond to the field operators $\frac{\phi_1 + i\phi_2}{\sqrt{2}}$ and $\frac{\phi_1 - i\phi_2}{\sqrt{2}}$]. The operator $\pi_i(x)$ is the operator conjugate to $\phi_i(x)$ in the Hamiltonian formalism. The σ and τ matrices are the spin and isotopic spin matrices of the nucleon source, and $\rho(x)$ is the spatial distribution of the source, which we shall assume to be spherically symmetric about x = o. The field operators $\phi_i(x)$ and $\pi_i(x)$ and the matrices σ and τ are all operators in the Hilbert space of the pion-nucleon system. They all commute except as follows:*

$$\left[\phi_i(x), \pi_j(x') \right] = i\,\delta_{ij}\,\delta(x - x') \quad , \tag{1.2}$$

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* We assume $\hbar = c = 1$.

$$[\sigma_i, \sigma_j] = \sum_{k=1}^{3} 2i\, e(ijk)\, \sigma_k \quad , \tag{1.3}$$

$$[\tau_i, \tau_j] = \sum_{k=1}^{3} 2i\, e(ijk)\, \tau_k \tag{1.4}$$

(where $e(ijk) = 1$ if ijk is an even permutation of 123, -1 if ijk is an odd permutation of 123, and zero otherwise). The constant g_0 is the unrenormalized coupling constant.

For the symmetric scalar theory the interaction part of the Hamiltonian is

$$H_I = \sum_{i=1}^{3} g_0 \int \tau_i\, \phi_i(x)\, \rho(x)\, d^3x \quad . \tag{1.5}$$

For the charged scalar theory the interaction Hamiltonian is

$$H_I = \sum_{i=1}^{2} g_0 \int \tau_i\, \phi_i(x)\, \rho(x)\, d^3x \quad . \tag{1.6}$$

In the symmetric scalar theory, isotopic spin is conserved; in the charged scalar theory one does not have isotopic spin, but there is symmetry with respect to the simultaneous interchange of π^+ for π^- and proton for neutron: for example, the phase shift for π^+-proton scattering is the same as the phase shift for π^--neutron scattering.

In these static model theories only mesons in one partial wave interact with the nucleon; for scalar theories only s-wave mesons interact, while for pseudoscalar theories only p-wave mesons interact. Thus the elastic and charge exchange pion-nucleon scattering is completely characterised by a set of phase shifts for the possible states of total angular momentum and isotopic spin.

Define $S_\alpha(\omega)$ to be the S matrix element for the scattering of a pion of energy ω in the state α, and let $\delta_\alpha(\omega)$ be the corresponding phase shift:

$$S_\alpha(\omega) = e^{2i\delta_\alpha(\omega)} \tag{1.7}$$

In the theories mentioned above we assign the numbers α as follows (let j be the total angular momentum, I the total isotopic spin):

Theory	α	State
Symmetric Pseudoscalar	1	$j = \frac{1}{2}$, $I = \frac{1}{2}$
	2	$j = \frac{1}{2}$, $I = \frac{3}{2}$ (or vice versa)
	3	$j = \frac{3}{2}$, $I = \frac{3}{2}$
Symmetric Scalar	1	$I = \frac{1}{2}$
	2	$I = \frac{3}{2}$
Charged Scalar	1	$\pi^+ p$
	2	$\pi^- p$

Because of the symmetry between σ and τ the $j = \frac{1}{2}$, $I = \frac{3}{2}$ state has the same properties as the $j = \frac{3}{2}$, $I = \frac{1}{2}$ state, in the symmetric pseudoscalar theory.

Define the "cutoff function" v(k) as the Fourier transform of the source function:

$$v(k) = \int \rho(x)\, e^{-ik\cdot x}\, d^3x \tag{1.8}$$

Because ρ is spherically symmetric, v(k) is a real function depending only on the magnitude of k. We now define the "scattering amplitude" $Q_\alpha(\omega)$ by*

$$S_\alpha(\omega) = 1 + 2i\,\frac{k v^2(k)}{\omega}\, Q_\alpha(\omega) \tag{1.9}$$

where k is now the meson momentum. We shall now measure energy and momentum in units of the meson mass; thus

$$k = \sqrt{\omega^2 - 1} \quad . \tag{1.10}$$

We actually use equation 1.9 to define Q_α only for scalar theories; for pseudoscalar theories we use instead the definition

$$S_\alpha(\omega) = 1 + 2i \, \frac{k^3 v^2(k)}{\omega} Q_\alpha(\omega) \quad . \tag{1.11}$$

In either case the definition of $Q_\alpha(\omega)$ has been chosen so that in Born approximation Q_α is a constant independent of ω.

The theorem of Low is that $Q_\alpha(\omega)$ is the boundary value of an analytic function of the complex variable ω; precisely, that $Q_\alpha(\omega)$ is the limit of an analytic function of ω as ω approaches the real axis from above, and that this function is analytic in the entire complex plane except for two branch cuts on the real axis, one for $\omega > 1$ and the other for $\omega < -1$ (see figure 1). We shall now define $Q_\alpha(\omega)$ to be this analytic function, but in order to be able to use equations 1.9 or 1.11 we shall also define $Q_\alpha(\omega)$ for $\omega > 1$ to be the limit from above.

The other properties of the analytic function $Q_\alpha(\omega)$ are as follows:

A.) $\lim_{\omega \to \infty} \frac{Q_\alpha(\omega)}{\omega} = 0$.

B.) $Q_\alpha(\omega)$ is real for $-1 < \omega < 1$.

C.) $Q_\alpha(0) = g^2 a_\alpha$,

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* Our function Q_α differs from any of the functions defined by Wick[2] or Chew and Low[3], but we find this definition more convenient for our purposes.

FIGURE 1

The region of analyticity of the scattering amplitude $Q_\alpha(\omega)$, showing the two cuts $\omega > 1$ and $\omega < -1$. On the positive branch cut $Q_\alpha(\omega)$ is defined as the limit from above, as shown by the arrows.

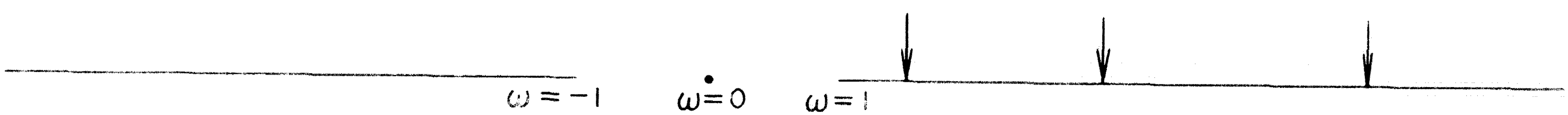

FIGURE I

D.) $Q_\alpha(-\omega) = -\sum_\beta A_{\alpha\beta}\, Q_\beta(\omega)$ (ω not on a branch cut) .

E.) $\mathrm{Im}\, Q_\alpha(\omega) = \frac{k\, v^2(k)}{\omega}\, |Q_\alpha(\omega)|^2$ for $1 < \omega < 2$, $k = \sqrt{\omega^2 - 1}$

(replace k by k^3 in E if the theory is a pseudoscalar theory).

In C. the constant g^2 is the renormalized coupling constant, and the numbers a_α are numbers which may be determined from the Born approximation to Q_α. Property D. is the Gell-Mann-Goldberger crossing theorem[4] expressed in terms of the amplitudes of total spin and isotopic spin: the crossing matrix $A_{\alpha\beta}$ may be determined from the relation of the total spin and isotopic spin states to the individual pion-nucleon states. Property E. is the unitarity condition below the threshold for meson production; for $\omega > 2$ this condition involves the amplitudes for meson production. Statements A. and B. are simple results which can be deduced from perturbation theory, except that to obtain A. easily requires that the cutoff $v^2(k)$ go to zero sufficiently rapidly for large k.

For future reference, we give the values of a_α and $A_{\alpha\beta}$ for the three interactions we have considered:

Symmetric Pseudoscalar

$$a_\alpha = \begin{vmatrix} -4 \\ -1 \\ +2 \end{vmatrix} \qquad A_{\alpha\beta} = \frac{1}{9}\begin{vmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{vmatrix}$$

Symmetric Scalar

$$a_\alpha = \begin{vmatrix} -2 \\ 1 \end{vmatrix} \qquad A_{\alpha\beta} = \frac{1}{3}\begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix}$$

Charged Scalar

$$a_\alpha = \begin{vmatrix} 1 \\ -1 \end{vmatrix} \qquad A_{\alpha\beta} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

Low[1] did not express his theorem in the form in which we have stated it; rather, he proved that $Q_\alpha(\omega)$ satisfies an equation (known as a dispersion equation), which we may derive from the fact that Q_α is analytic, using conditions A, B, C, and D. Conversely, his equation implies that Q_α is analytic in the cut ω plane (i.e. except on the branch cuts $\omega \geq 1$ and $\omega \leq -1$) and satisfies conditions B, C, D, and unless Q_α is extremely pathological, A. To obtain Low's equation, we apply Cauchy's theorem to $\frac{Q_\alpha(\omega)}{\omega}$, using a path C as shown in figure 2:

$$\frac{Q_\alpha(\omega)}{\omega} = \frac{1}{2\pi i} \int_C \frac{1}{\omega_1 - \omega} \frac{Q_\alpha(\omega_1)}{\omega_1} d\omega_1 \tag{1.12}$$

(note that the path C does not contain the point $\omega_1 = 0$, for which $\frac{Q_\alpha(\omega_1)}{\omega_1}$ is singular). We now expand the path C until it consists of two large semicircles, two paths along the two branch lines, and a little circle about $\omega_1 = 0$. As the semicircles increase in radius, the integrals along them will vanish, by condition A; we are left with integrals involving the discontinuity of Q_α across the branch cut, and the integral over the circle about $\omega_1 = 0$. Since Q_α is real for real values of ω not on the branch cut, the discontinuity of Q_α will be twice its imaginary part as the branch cut is approached from one side. Thus we obtain (in the limit of $\epsilon \to 0$)

$$\begin{aligned} \frac{Q_\alpha(\omega)}{\omega} = {} & \frac{1}{\pi} \int_{-\infty}^{-1} \frac{1}{\omega_1 - \omega} \operatorname{Im} Q_\alpha(\omega_1 + i\epsilon) \frac{d\omega_1}{\omega_1} \\ & + \frac{Q_\alpha(0)}{\omega} + \frac{1}{\pi} \int_1^\infty \frac{1}{\omega_1 - \omega} \operatorname{Im} Q_\alpha(\omega_1 + i\epsilon) \frac{d\omega_1}{\omega_1} . \end{aligned} \tag{1.13}$$

By our definition of Q_α we may omit the $i\epsilon$ in the second term. From crossing symmetry, for $\omega > 1$ we have

$$\begin{aligned} \operatorname{Im} Q_\alpha(-\omega + i\epsilon) &= -\sum_\beta A_{\alpha\beta} \operatorname{Im} Q_\beta(\omega - i\epsilon) \\ &= +\sum_\beta A_{\alpha\beta} \operatorname{Im} Q_\beta(\omega) , \end{aligned} \tag{1.14}$$

FIGURE 2

Path of integration c for Cauchy's Integral (note that it consists of two parts).

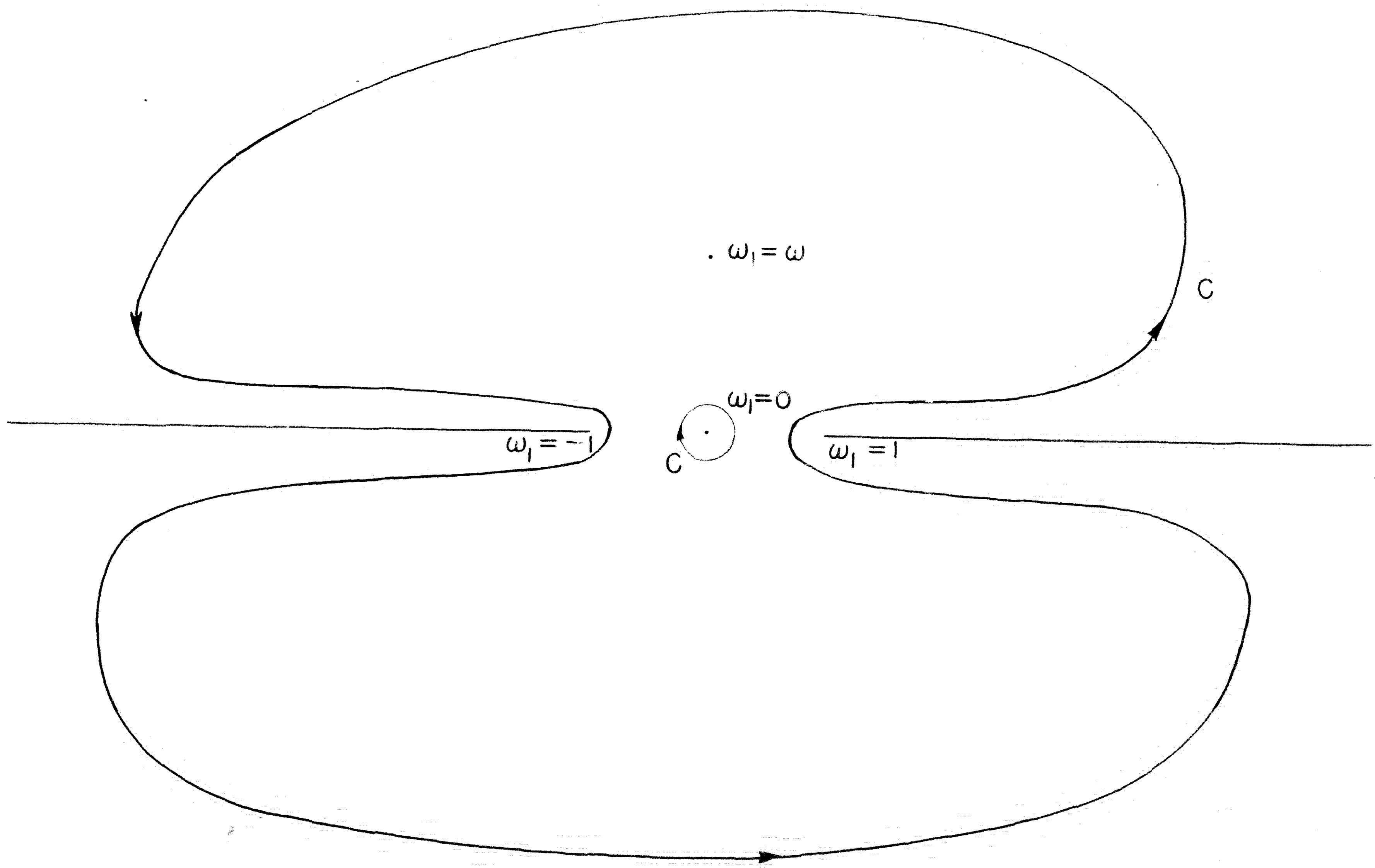

FIGURE 2

and hence (for values of ω not on a branch cut)

$$Q_\alpha(\omega) = \frac{\omega}{\pi}\int_1^\infty \frac{1}{\omega_1-\omega}\,\frac{\operatorname{Im} Q_\alpha(\omega_1)}{\omega_1}\,d\omega_1 + g^2 a_\alpha + \sum_\beta A_{\alpha\beta}\,\frac{\omega}{\pi}\int_1^\infty \frac{\operatorname{Im} Q_\beta(\omega_1)}{\omega_1(\omega_1+\omega)}\,d\omega_1 \quad (1.15)$$

Equation 1.15 is the Low equation, except that $\operatorname{Im}Q_\alpha$ is usually expressed as a quadratic form in Q_α and the multiple-meson production amplitudes, by means of the unitarity condition.

The one-meson approximation is obtained by neglecting multiple meson production amplitudes in the unitarity condition, thus requiring

$$E'.)\quad \operatorname{Im} Q_\alpha = \frac{k\,v^2(k)}{\omega}\,|Q_\alpha|^2 \quad \text{for all} \quad \omega > 1 .$$

This is not a very satisfactory approximation, from a formal point of view, because the only limit in which this approximation becomes exact (i.e. the only limit for which the ratio of any meson production amplitude to Q_α goes to zero) is the weak coupling limit. This approximation can be justified only for low energies ω, in which case the imaginary part of Q_α for low energies might dominate the dispersion equation 1.15. In particular, one supposes that the $\frac{3}{2}, \frac{3}{2}$ resonance in the observed pion-nucleon scattering dominates the dispersion integrals for the symmetric pseudoscalar theory, and since this resonance occurs below the energies for which inelastic scattering is important, this resonance should appear in the solution of the one-meson approximation. Chew and Low[3] and Salzman and Salzman[5] have shown that the solution of the one-meson approximation for the symmetric pseudoscalar theory does contain a resonance in the $\frac{3}{2}, \frac{3}{2}$ state at the observed energy, when the coupling constant g^2 has its observed value of .08 and the cutoff $v^2(k)$ drops to zero for k at about the nucleon mass. Furthermore, in the calculations of Salzman and Salzman, the

$\frac{1}{2}, \frac{1}{2}$ and $\frac{1}{2}, \frac{3}{2}$ states have small phase shifts for low energies. Thus the one meson approximation describes the general features of low energy p-wave pion-nucleon scattering.

Besides the numerical calculations of Salzman and Salzman, the other investigation of the one-meson approximation of importance to us is the work of Castillejo, Dalitz, and Dyson[6]. This will be reported later in this chapter.

We have seen that in the one-meson approximation $Q_\alpha(\omega)$ is required to have a number of properties: to repeat, $Q_\alpha(\omega)$ is to be analytic in the cut ω plane (excluding the branch lines $\omega \geq 1$ and $\omega \leq -1$), satisfy properties A to D and the simplified unitarity condition E'. We now observe that these properties are all <u>simplified forms of properties of the scattering amplitudes for fully relativistic theories</u>. In particular the following features are common to the one-meson approximation and to relativistic field theories:

(1). The unitarity condition relates the imaginary part of the scattering amplitude to a quadratic form in the amplitudes for various processes.

(2). The crossing relation is linear in the scattering amplitudes.

(3). Some form of analyticity requirement on the scattering amplitude as a function of one or several complex variables.

I do not wish to make a detailed comparison here, since in a later chapter we shall study the approximate equations of Mandelstam and Chew[7] for the pion-pion scattering amplitudes, when we shall see more precisely the analogy between the one-meson approximation to the Low equation and the properties of some fully covariant scattering amplitudes. But since every property used to define the scattering amplitude in the one-meson approximation has some (more complicated) analogue in covariant field theory, I think it is interesting to discover any further properties of the solution of the one-meson approximation, whether or not this solution really approximates the exact solution of the static model.

Since in field theory, apart from the Low equation all calculations have been based on perturbation theory, we shall study in the next few chapters the perturbation expansion (in powers of g^2) of the solution of the one-meson approximation. If we assume that integration and summation may be interchanged, equation 1.15 and the unitarity condition E' uniquely define a power series expansion of Q_α if the lowest order term is required to be $g^2 a_\alpha$; it is this power series that we shall investigate.

Our original purpose for studying the perturbation expansion of the one-meson approximation was to see whether one could obtain a useful approximation from the first few terms when the coupling constant g^2 is too large for the first term to be a good approximation. An ideal example for investigating this problem is the symmetric pseudoscalar theory, using the cutoff used by Salzman and Salzman, for then we have their numerical solution to compare with any proposal for a perturbation approximation. There are two proposals for using perturbation theory that we shall examine, both of which are obtained by analogy with the Schrödinger equation. One is the determinantal method as extended to field theory by Marshall Baker(8); this method derives from the result that for the Schrödinger equation for a particle in a central potential, the scattering amplitude can be expressed as the ratio of two entire functions of the strength of the potential.

The other method is derived from one of the Lippman-Schwinger[9] variational principles, with a trial function taken from perturbation theory[10]. Both these methods are formally valid only for the Schrödinger equation, but they both lead to formulas involving only the scattering matrix: with some modification these formulas can be calculated for the scattering matrix of field theory, if inelastic scattering is neglected. We shall not consider whether these formulas can be extended to cases where inelastic scattering is important.

Considering that neither of these methods have any basis in field theory, we shall find that they can be surprisingly effective. We shall examine them as applied to the symmetric pseudoscalar theory with the cutoff of Salzman and Salzman. We shall also give a brief resume of the basis for these methods in the Schrödinger theory, using the square-well potential as an example.

There are a number of questions that can be asked concerning the solution of the one-meson approximation, apart from its numerical value for physically useful parameters. The first is whether a solution exists, and if so whether it is unique. Castillejo, Dalitz, and Dyson[6] investigated this problem for the charged scalar theory, and showed that for the charged scalar case there are an infinite number of solutions. This result has been obtained for the symmetric pseudoscalar theory also; it arises because the same set of equations may be derived from other Hamiltonians in which the nucleon in the absence of π mesons has a number of excited states with energies greater than the rest mass of the meson. When the coupling is turned on these states disappear as such, but they still affect meson-nucleon scattering[11].

In this work we shall not be concerned with the multiplicity of solutions of the one-meson approximation, because for reasons stated earlier we wish to examine the perturbation series defined by the one-meson approximation, and this is unique.

With respect to the perturbation series, several questions arise (among others); they are

1) Does the series converge for some value of g^2?

2) If the radius of convergence of the series is finite, can the sum of the series be analytically continued beyond the radius of convergence?

3) If so, is the analytic continuation still a solution of the equations of the one-meson approximation?

4) Is the analytic continuation a single-valued function of g^2?

These are very general questions, such that their answers for the simple problem of the one-meson approximation could easily be relevant to the far more difficult problem of covariant field theory. In field theory it has been shown that perturbation theory is totally divergent (i.e. diverges for all values of g except g = o) for a scalar field interacting with itself[12-15]; otherwise nothing is known about any of these questions.

We shall be able to investigate these questions only for special choices of the cutoff function $v^2(k)$; in particular we shall consider only scalar theories with the cutoff equal to 1 for all k. For some parts of our investigation we actually require that $v^2(k)$ be equal to one only for large values of k, but it seems pointless to retain this freedom. This special choice of the cutoff has the advantage that the perturbation series depends logarithmically on k for large k, and this gives the one-meson approximation an added similarity to field theory, where perturbation theory also contains logarithms. Specifically, we shall see that for large energies ω the perturbation series for the one-meson approximation becomes asymptotically*

$$Q_\alpha(\omega) \simeq g^2 a_\alpha + g^4 c_{\alpha 1} \ln\omega + g^6 c_{\alpha 2} \ln^2\omega + \cdots . \qquad (1.16)$$

We shall be able to calculate the coefficients $c_{\alpha n}$, and find that the result is essentially the same result as had previously been obtained for the <u>photon propagator</u> in quantum electrodynamics[16,17]; we shall then see that both results can be derived by the same method.

This choice of the cutoff also has the consequence that even for very small values of g^2, the higher orders of the perturbation series are important when ω is large, as is evident from equation 1.16. Thus our questions 2-4 are relevant even to small values of g^2. They are easier to investigate for this case because the power series simplifies when ω is large, although it is necessary to include more terms than those included in equation 1.16; we must in fact include all terms in any order which do not vanish in the limit $\omega\to\infty$ (in particular, lower powers of the logarithm of the energy--i.e. terms varying as g^4, $g^6 \ln\omega$, etc.). Only a few of the terms not included in equation 1.16 can be evaluated, for the photon propagator, but for the one-meson approximation we shall obtain them all.

For other than large values of the energy we can examine only the charged and symmetric scalar theories, for I have been unable to solve any others. The power series expansion for the charged scalar

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* For the exceptional case of the charged scalar theory, all the $c_{\alpha n}$ are zero.

theory was obtained by Lee and Serber[6]; it has no unusual properties. We shall present the solution of the symmetric scalar theory and discuss its properties (it may have been obtained previously by W. K. Hayman[18], but no published account of it exists).

This summarizes the problems we shall investigate concerning the one-meson approximation. The specific arrangement of the material into chapters is as follows: In chapter II we develop the method for obtaining the asymptotic form of the perturbation expansion of $Q_\alpha(\omega)$ when ω is large, and give the result for the various theories (symmetric scalar, symmetric pseudoscalar*, etc.). In chapter III we give the asymptotic form of the perturbation series for a general class of mathematically constructed crossing matrices $A_{\alpha\beta}$ (i.e. not deriving from any choice for the nuclear and mesonic isotopic spin). In chapter IV we give the solution of the symmetric scalar theory. In chapter V we review the determinantal and variational methods for improving perturbation theory for the Schrodinger equation. In chapter VI we present calculations on how well these methods work for the Low equation in the one-meson approximation.

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* i.e. using a_α and $A_{\alpha\beta}$ from the symmetric pseudoscalar theory, but replacing $k^3v^2(k)$ by k.

II. ASYMPTOTIC FORM OF THE LOW EQUATION FOR LARGE ENERGIES

In this chapter we develop a method for studying the asymptotic form for large energies of the perturbation expansion of the solution of the Low equation in the one-meson approximation. We treat only scalar theories without a cutoff, as stated in the introduction. We shall show that for large values of the energy ω, any term in the perturbation expansion of the amplitude Q_α is a polynomial in $\ln \omega$, neglecting terms of order $\frac{1}{\omega}$. Our method then consists in reducing the calculation of these polynomials to the calculation of a single power series: for simple cases (such as the three interactions mentioned in the previous chapter) these single power series can be obtained by inspection from a knowledge of the first few terms. The reduction of the asymptotic form to a single power series requires that the crossing matrix $A_{\alpha\beta}$ have only one eigenvalue -1 (the rest being +1; see below). At the end of this chapter we shall show how the arguments which Gell-Mann and Low(16) used to investigate the photon propagator can be applied to this problem; this only gives an alternate proof that the asymptotic form can be reduced to a single power series when $A_{\alpha\beta}$ has only one eigenvalue -1, but it serves to strengthen the analogy between the Low equation and field theory.

In section B of this chapter we derive a set of equations for the asymptotic form of the perturbation series. In section C we present the method for reducing the solution of these equations to the calculation of a single power series; a series for which an hour's time on a desk calculator yields twelve or so terms (for a simple interaction). In section D we give the asymptotic forms for the symmetric scalar and symmetric pseudoscalar theories (and two others) (for the pseudoscalar theory $k^3 v^2(k)$ is replaced by k), and show that they satisfy the equations. In section E we discuss the results. In section F we apply the method of Gell-Mann and Low to this problem.

B.

We begin by restating the properties of the amplitudes Q_α which lead to the Low equation, in the one-meson approximation. These properties are

A.) $\lim_{\omega \to \infty} \frac{Q_\alpha(\omega)}{\omega} = 0$.

B.) $Q_\alpha(\omega)$ is real for $-1 < \omega < 1$.

C.) $Q_\alpha(0) = g^2 a_\alpha$.

D.) $Q_\alpha(-\omega) = -\sum_\beta A_{\alpha\beta} Q_\beta(\omega)$ (ω not on a branch cut) .

E.) $\mathrm{Im}\, Q_\alpha(\omega) = \frac{\sqrt{\omega^2-1}}{\omega} |Q_\alpha(\omega)|^2 \qquad (\omega > 1)$.

F.) $Q_\alpha(\omega)$ is analytic in ω except for branch points at $\omega = \pm 1$, and branch lines on the real axis for $|\omega| \geq 1$; we define $Q_\alpha(\omega)$ on the branch line $\omega \geqslant 1$ to be the limit from above:

$$Q_\alpha(\omega) = \lim_{\epsilon \to 0^+} Q_\alpha(\omega + i\epsilon) \qquad \text{for} \quad \omega \geqslant 1 \ .$$

For property E. we have put $v^2(k) = 1$. From these properties we obtained the Low equation:

$$Q_\alpha(\omega) = g^2 a_\alpha + \frac{\omega}{\pi} \int_1^\infty \frac{1}{\omega_1(\omega_1 - \omega)} \mathrm{Im}\, Q_\alpha(\omega_1)\, d\omega_1$$
$$+ \sum_\beta A_{\alpha\beta} \frac{\omega}{\pi} \int_1^\infty \frac{1}{\omega_1(\omega_1+\omega)} \mathrm{Im}\, Q_\beta(\omega_1)\, d\omega_1 \ . \qquad (2.1)$$

The purpose of this section is to derive a modified set of conditions A.-F. and a modified Low equation, whose perturbation expansion will give precisely the asymptotic form we seek.

First we note that the crossing matrix must have two properties as a direct consequence of the crossing condition D. They state that $A^2 = 1$ and that a is an eigenvector of A with eigenvalue -1:

$$\sum_\beta A_{\alpha\beta} A_{\beta\gamma} = \delta_{\alpha\gamma} \ , \qquad (2.2)$$

$$\sum_{\beta} A_{\alpha\beta}\, a_{\beta} = -a_{\alpha} \tag{2.3}$$

(the second condition must hold if crossing symmetry is to hold separately for each order in perturbation theory). For our methods to be useful we must also require that A have only one negative eigenvalue; since $A^2 = 1$, A can have only + 1 and - 1 as eigenvalues, and so this condition means that the matrix A will appear to be the unit matrix to an arbitrary vector except for its component along the vector a--that is, A will have the form

$$A_{\alpha\beta} = \delta_{\alpha\beta} + a_{\alpha} K_{\beta} \quad , \tag{2.4}$$

where the K_β are a set of numbers that determine the component of any vector along the vector a. It can be verified that the crossing matrices reported in the previous chapter all have this form. This form is not valid for all theories that one can construct; in particular the crossing matrix for the pseudoscalar interaction of Σ particles (nucleons with isotopic spin one) with π mesons has three negative eigenvalues. Because of equation 2.3, the K's must satisfy

$$\sum_{\beta} K_{\beta}\, a_{\beta} = -2 \quad . \tag{2.5}$$

For the three theories we have considered, the K_β are as follows:

Symmetric Pseudoscalar $\qquad K_\beta = \frac{1}{9}\begin{vmatrix} 2 \\ 2 \\ -4 \end{vmatrix}$

Symmetric Scalar $\qquad K_\beta = \frac{1}{3}\begin{vmatrix} 2 \\ -2 \end{vmatrix}$

Charged Scalar $\qquad K_\beta = \begin{vmatrix} -1 \\ +1 \end{vmatrix}$

For the purpose of what follows it is convenient to restate the crossing condition D. Condition B. implies the Schwarz reflection principle holds:

$$Q_\alpha^*(\omega^*) = Q_\alpha(\omega) \tag{2.6}$$

for ω not on a branch cut. This means we can rewrite D. as

$$\text{D}'.)\quad Q_\alpha^*(-\omega^*) = -\sum_\beta A_{\alpha\beta}\, Q_\beta(\omega) \quad , \tag{2.7}$$

which has the advantage of involving $Q_\alpha(\omega)$ only for values of ω in the upper half plane.

Now for real values of $\omega > 1$ define

$$Q_\alpha(\omega) = P_\alpha(\omega) + i\, F_\alpha(\omega) \quad . \tag{2.8}$$

By letting ω approach the real axis from above we obtain from equation 2.1 the result

$$\begin{aligned} P_\alpha(\omega) = g^2 a_\alpha &+ \frac{\omega}{\pi}\,\text{P.V.}\int_1^\infty \frac{F_\alpha(\omega_1)}{\omega_1(\omega_1-\omega)}\, d\omega_1 \\ &+ \sum_\beta A_{\alpha\beta}\, \frac{\omega}{\pi}\int_1^\infty \frac{F_\beta(\omega_1)}{\omega_1(\omega_1+\omega)}\, d\omega_1 \quad , \end{aligned} \tag{2.9}$$

where P.V. stands for Cauchy Principal Value. The unitarity condition E becomes

$$F_\alpha(\omega) = \frac{\sqrt{\omega^2-1}}{\omega}\left\{ P_\alpha^2(\omega) + F_\alpha^2(\omega)\right\} \quad . \tag{2.10}$$

The perturbation series for $P_\alpha(\omega)$ and $F_\alpha(\omega)$ are defined by assuming expansions of the form

$$P_\alpha(\omega) = \sum_{n=1}^\infty g^{2n}\, P_{n\alpha}(\omega) \quad , \tag{2.11}$$

$$F_\alpha(\omega) = \sum_{n=1}^\infty F_{n\alpha}(\omega)\, g^{2n} \quad , \tag{2.12}$$

substituting in equations 2.9 and 2.10, and requiring the coefficients of each power of g to be equal. We obtain immediately

$$P_{1\alpha} = a_\alpha \quad , \tag{2.13}$$

$$F_{1\alpha} = 0 \quad , \tag{2.14}$$

and evidently if we know the first n terms of P and F, the equations allow us to calculate F_{n+1} and then P_{n+1}; thus any term in the series can be obtained by performing a finite number of integrations.

Now consider the nature of $P_{n\alpha}$ and $F_{n\alpha}$ for large ω. We begin by proving, by induction, that $P_{n\alpha}$ becomes a polynomial in $\ln \omega$ of degree n-1, while $F_{n\alpha}$ becomes a polynomial of degree n-2.

First we note that this statement holds for $P_{1\alpha}$ and $F_{2\alpha}$, and that the unitarity condition (equation 2.10) guarantees the statement for $F_{n+1,\alpha}$ if it holds for up to n^{th} order in P and F. Thus we have only to examine the dispersion equation 2.9.

Let us define

$$P_{n\alpha} = \bar{P}_{n\alpha} + P'_{n\alpha} \quad , \tag{2.15}$$

$$F_{n\alpha} = \bar{F}_{n\alpha} + F'_{n\alpha} \quad , \tag{2.16}$$

where $\bar{P}_{n\alpha}$ and $\bar{F}_{n\alpha}$ are the asymptotic forms of $P_{n\alpha}$ and $F_{n\alpha}$, and $P'_{n\alpha}$ and $F'_{n\alpha}$ are the remainders; by assumption $\bar{P}_{n\alpha}$ and $\bar{F}_{n\alpha}$ are polynomials in $\ln\omega$, and we shall assume $F'_{n\alpha}$ to be of order $\frac{1}{\omega}$ for large ω (neglecting factors of $\ln\omega$). To find the nature of $\bar{P}_{n\alpha}$ we consider equation 2.9 for large ω. The contribution of $F'_{n\alpha}$ to $\bar{P}_{n\alpha}$ is just a constant:

$$d_{n\alpha} = -\frac{1}{\pi}\int_1^\infty \frac{F'_{n\alpha}(\omega_1)}{\omega_1}\,d\omega_1 + \sum_\beta A_{\alpha\beta}\int_1^\infty \frac{F'_{n\beta}(\omega_1)}{\omega_1}\,d\omega_1 \quad , \tag{2.17}$$

which becomes by the identity 2.4:

$$d_{n\alpha} = a_\alpha \sum_\beta \frac{\kappa_\beta}{\pi} \int_1^\infty F'_{n\beta}(\omega_1) \frac{d\omega_1}{\omega_1} \quad . \tag{2.18}$$

Thus for large ω ,

$$\begin{aligned} \bar{F}_{n\alpha}(\omega) = d_{n\alpha} &+ \frac{\omega}{\pi} \, P.V. \int_1^\infty \frac{\bar{F}_{n\alpha}(\omega_1)}{\omega_1(\omega_1 - \omega)} \, d\omega_1 \\ &+ \sum_\beta A_{\alpha\beta} \frac{\omega}{\pi} \int_1^\infty \frac{\bar{F}_{n\beta}(\omega_1)}{\omega_1(\omega_1 + \omega)} \, d\omega_1 \quad . \end{aligned} \tag{2.19}$$

We can obtain the nature of the integrals by considering their effect on a power of $\ln \omega_1$. Consider the crossed integral: define

$$I_m(\omega) = \frac{\omega}{\pi} \int_1^\infty \frac{\ln^m \omega_1}{\omega_1(\omega_1 + \omega)} \, d\omega_1 \quad . \tag{2.20}$$

We evaluate this integral by dividing the range of integration at $\omega_1 = \omega$, and expanding the denominator in partial fractions for $\omega_1 < \omega$:

$$\begin{aligned} I_m(\omega) - \frac{1}{\pi} \int_1^\omega \ln^m \omega_1 \frac{d\omega_1}{\omega_1} &- \frac{1}{\pi} \int_1^\omega \frac{\ln^m \omega_1}{\omega_1 + \omega} \, d\omega_1 \\ &+ \frac{\omega}{\pi} \int_\omega^\infty \frac{\ln^m \omega_1}{\omega_1(\omega_1 + \omega)} \, d\omega_1 \quad . \end{aligned} \tag{2.21}$$

The first integral is elementary; the second and third are evaluated by making a change of variable $\omega_1 = \omega t$. We note that the lower limit of the second integral may be changed to o since we assume ω is large. Thus

$$\begin{aligned} I_m(\omega) = \frac{1}{\pi} \frac{\ln^{m+1}\omega}{m+1} &- \frac{1}{\pi} \int_0^1 (\ln \omega + \ln t)^m \frac{dt}{t+1} \\ &+ \frac{1}{\pi} \int_1^\infty (\ln \omega + \ln t)^m \frac{dt}{t(t+1)} \quad . \end{aligned} \tag{2.22}$$

Using the binomial theorem we find that $I_m(\omega)$ is a polynomial in $\ln \omega$ of degree m+1, the coefficients being integrals over t independent of ω. Evidently the same analysis can be applied to the uncrossed integral (the principal value gives no difficulty) and since $\overline{F}_{n\alpha}$ is of degree n-2 in $\ln\omega$ it follows that $\overline{P}_{n\alpha}$ is of degree n-1. This proves our statement. Also it follows that the errors $P'_{n\alpha}$ and $F'_{n\alpha}$ are of order $\frac{1}{\omega}$ as assumed because our approximations were to replace $\frac{\sqrt{\omega^2-1}}{\omega}$ by 1 in the unitarity condition 2.10 and to replace 1 by 0 as a limit in equation 2.21, both of which cause errors of order $\frac{1}{\omega}$.

We observe that the coefficients of the highest power of $\ln\omega$ in $\overline{P}_{n\alpha}$ and $\overline{F}_{n\alpha}$ can be computed without knowledge of the other terms, e.g. that the coefficient of $\ln^2\omega$ in $\overline{P}_{3\alpha}$ may be computed without knowing the constant term in $\overline{P}_{2\alpha}$. If we wished to compute these terms only, we should need only the first term in $I_m(\omega)$ and the right hand side of equation 2.19 would reduce to

$$-\frac{1}{\pi}\int_1^{\omega} \overline{F}_{n\alpha}(\omega_1)\,\frac{d\omega_1}{\omega_1} + \frac{1}{\pi}\sum_{\beta} A_{\alpha\beta}\int_1^{\omega} \overline{F}_{n\beta}(\omega_1)\,\frac{d\omega_1}{\omega_1}$$

which is a much easier expression to use. Thus it is convenient to obtain the asymptotic forms by first computing the $\ln^{n-1}\omega$ term in $\overline{P}_{n\alpha}$ and the $\ln^{n-2}\omega$ term in $\overline{F}_{n\alpha}$, for all n, then the $\ln^{n-2}\omega$ term in $\overline{P}_{n\alpha}$, and so forth. Formally this amounts to writing $\overline{P}_{\alpha}$ and $\overline{F}_{\alpha}$ as power series in g whose coefficients are power series in ($g^2\ln\omega$).

The first step is to obtain the decomposition of equation 2.19 in the form suggested by equation 2.22, without putting in the explicit form of $\overline{F}_{n\alpha}$. This means expanding $\overline{F}_{n\alpha}(\omega t)$ by Taylor's theorem:

$$\overline{F}_{n\alpha}(\omega t) = \overline{F}_{n\alpha}(\omega) + (\ln t)\,\frac{d\,\overline{F}_{n\alpha}(\omega)}{d\ln\omega} + \frac{(\ln^2 t)}{2}\,\frac{d^2\,\overline{F}_{n\alpha}(\omega)}{(d\ln\omega)^2} + \cdots \tag{2.23}$$

For convenience we shall obtain the decomposition not of equation 2.19 but its analogue for the entire asymptotic series $\overline{P}_\alpha$ and $\overline{F}_\alpha$. We define*

$$g_0 = g^2 + \sum_{n=2}^{\infty} \frac{1}{a_\alpha} d_{n\alpha}\, g^{2n} . \tag{2.24}$$

Then in analogy with equation 2.22 we obtain

$$\begin{aligned}\overline{P}_\alpha(\omega) &= g_0 a_\alpha - \frac{1}{\pi}\int_1^\omega \overline{F}_\alpha(\omega_1)\frac{d\omega_1}{\omega_1} \\ &+ \lim_{\epsilon \to 0}\frac{1}{\pi}\left\{ \int_0^{1-\epsilon}\left[\sum_{n=0}^{\infty}\frac{\ln^n t}{n!}\frac{d^n \overline{F}_\alpha(\omega)}{(d\ln\omega)^n}\right]\frac{dt}{t-1} + \int_{1+\epsilon}^{\infty}\overline{F}_\alpha(\omega t)\frac{dt}{t(t-1)}\right\} \\ &+ \sum_\beta A_{\alpha\beta}\frac{1}{\pi}\int_1^\omega \overline{F}_\beta(\omega_1)\frac{d\omega_1}{\omega_1} \\ &+ \sum_\beta A_{\alpha\beta}\frac{1}{\pi}\left\{-\int_0^1 \overline{F}_\beta(\omega t)\frac{dt}{t+1} + \int_1^\infty \overline{F}_\beta(\omega t)\frac{dt}{t(t+1)}\right\} .\end{aligned} \tag{2.25}$$

The second, third, and fourth terms (the latter inside the parentheses) come from the uncrossed integral; $\overline{F}_\alpha(\omega t)$ and $\overline{F}_\beta(\omega t)$ are to be expanded as in the third term. Now define

$$e_n = \lim_{\epsilon \to 0}\frac{1}{\pi^{n+1}}\frac{1}{n!}\left\{\int_0^{1-\epsilon}\frac{\ln^n t}{t-1}\,dt + \int_{1+\epsilon}^{\infty}\frac{\ln^n t}{t(t-1)}\,dt\right\} , \tag{2.26}$$

$$c_n = \frac{1}{\pi^{n+1}}\frac{1}{n!}\left\{-\int_0^1 \frac{\ln^n t}{t+1}\,dt + \int_1^\infty \frac{\ln^n t\, dt}{t(t+1)}\right\} , \tag{2.27}$$

$$b_n = e_n + c_n , \tag{2.28}$$

and

$$z = \frac{g_0 \ln\omega}{\pi} \tag{2.29}$$

(and similarly for z_1); using the identity 2.4 we obtain

$$\begin{aligned}\overline{P}_\alpha(z) &= g_0 a_\alpha + a_\alpha\sum_\beta K_\beta \frac{1}{g_0}\int_0^z \overline{F}_\beta(z_1)\,dz_1 \\ &+ \sum_{n=0}^{\infty} b_n g_0^n \frac{d^n \overline{F}_\alpha(z)}{dz^n} + a_\alpha \sum_{n=0}^{\infty} c_n g_0^n \sum_\beta K_\beta \frac{d^n \overline{F}_\beta(z)}{dz^n} .\end{aligned} \tag{2.30}$$

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* The constant g_0, and the constant g_1 to be introduced subsequently, are to be thought of as power series in g introduced only to obtain compactness in the following expressions.

We include for future reference the form of equation 2.25 when $\bar{F}_\alpha(\omega_1)$ is not expanded:

$$\bar{P}_\alpha(\omega) = g_0 a_\alpha + \frac{\omega}{\pi} \text{ P.V.} \int_1^\infty \frac{\bar{F}_\alpha(\omega_1)\, d\omega_1}{\omega_1(\omega_1-\omega)} + \frac{1}{\pi} \text{ P.V.} \int_0^1 \frac{\bar{F}_\alpha(\omega_1)\, d\omega_1}{\omega_1-\omega}$$

$$+ \sum_\beta A_{\alpha\beta} \frac{1}{\pi} \left\{ \omega \int_1^\infty \frac{\bar{F}_\beta(\omega_1)\, d\omega_1}{\omega_1(\omega_1+\omega)} - \int_0^1 \frac{\bar{F}_\beta(\omega_1)\, d\omega_1}{\omega_1+\omega} \right\}, \tag{2.31}$$

and the unitarity condition

$$\bar{F}_\alpha = \bar{P}_\alpha^2 + \bar{F}_\alpha^2 \quad . \tag{2.32}$$

Equations 2.31 and 2.32 can now be used to define $\bar{F}_\alpha$ and $\bar{P}_\alpha$.

The constants b_n and c_n may be expressed in terms of the Bernoulli numbers:*

$$b_{2n-1} = \frac{B_n}{(2n)!}\left[2^{2n+1} - 2\right], \quad b_{2n} = 0, \tag{2.33}$$

$$c_{2n-1} = \frac{B_n}{(2n)!}\left[2^{2n} - 2\right], \quad c_{2n} = 0, \tag{2.34}$$

where $\quad B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}$, etc.

Equation 2.31 looks very similar to the equation we began with. To complete this section we show that it is equivalent to a set of conditions similar to conditions A-F for $Q_\alpha(\omega)$. Let

$$\bar{Q}_\alpha(\omega) = \bar{P}_\alpha(\omega) + i\bar{F}_\alpha(\omega) \qquad (\omega > 0). \tag{2.35}$$

In perturbation theory (we are still thinking of all variables as power series in g^2) $\bar{Q}_\alpha(\omega)$ is simply a polynomial in $\ln\omega$, and therefore has an analytic extension into the upper half plane. We now

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* See the Appendix to this chapter.

define $\overline{Q}_\alpha(\omega)$ to be this analytic extension, with a branch line along the negative real axis because of the branch point in $\ln\omega$ at $\omega = 0$. The series $\overline{Q}_\alpha(\omega)$ has the following properties:

A.) $\lim_{\omega \to \infty} \frac{\overline{Q}_\alpha(\omega)}{\omega} = 0$.

D.) $\overline{Q}_\alpha(-\omega^*)^* = -\sum_\beta A_{\alpha\beta}\, \overline{Q}_\beta(\omega) \qquad (\operatorname{Im}\omega > 0)$.

E.) $\operatorname{Im} \overline{Q}_\alpha(\omega) = |\overline{Q}_\alpha(\omega)|^2 \qquad (\omega > 0)$.

F.) $\overline{Q}_\alpha(\omega)$ is an analytic function of ω for $\operatorname{Im}\omega > 0$.

These properties allow us to obtain equation 2.31 except for the constant term. We write Cauchy's theorem for $\frac{\overline{Q}_\alpha(\omega)}{\omega}$, for ω in the upper half plane and using a path in the upper half plane; now let the path approach the real axis except for a large semicircle which we can neglect because of A. The only point on the real axis which gives trouble is $\omega_1 = 0$, so the path must avoid this. Then let ω approach the positive real axis. We now have

$$\frac{\overline{Q}_\alpha(\omega)}{\omega} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\overline{Q}_\alpha(\omega_1)\, d\omega_1}{\omega_1(\omega_1 - \omega - i\epsilon)} , \tag{2.36}$$

with the path $-\infty$ to ∞ detouring the point $\omega_1 = 0$. For the part of the path $-1 < \omega_1 < 1$, divide the integrand into two parts

$$\frac{1}{\omega_1(\omega_1 - \omega - i\epsilon)} = \frac{1}{\omega(\omega_1 - \omega - i\epsilon)} - \frac{1}{\omega\omega_1} . \tag{2.37}$$

The integral over the first term does not become singular as the path approaches the point $\omega_1 = 0$. For the second term we can let the path be the upper half of the unit circle, which we shall denote by C:

$$\begin{aligned} \overline{Q}_\alpha(\omega) = {} & \frac{\omega}{2\pi i} \int_{-\infty}^{-1} \frac{\overline{Q}_\alpha(\omega_1)\, d\omega_1}{\omega_1(\omega_1 - \omega)} + \frac{1}{2\pi i} \int_{-1}^{1} \frac{\overline{Q}_\alpha(\omega_1)}{\omega_1 - \omega - i\epsilon}\, d\omega_1 \\ & + \frac{\omega}{2\pi i} \int_{1}^{\infty} \frac{\overline{Q}_\alpha(\omega_1)}{\omega_1(\omega_1 - \omega - i\epsilon)}\, d\omega_1 - \frac{1}{2\pi i} \int_C \overline{Q}_\alpha(\omega_1) \frac{d\omega_1}{\omega_1} . \end{aligned} \tag{2.38}$$

Now take the real part of this expression; we use the relation

$$\frac{1}{\omega_1 - \omega - i\epsilon} = P.V.\ \frac{1}{\omega_1 - \omega} + i\pi\,\delta(\omega_1 - \omega) \quad , \tag{2.39}$$

and the fact that on the unit circle $\frac{d\omega_1}{\omega_1}$ is imaginary; thus

$$\bar{P}_\alpha(\omega) = \frac{\omega}{\pi}\int_{-\infty}^{-1} \frac{\mathrm{Im}\,\bar{Q}_\alpha(\omega_1)\,d\omega_1}{\omega_1(\omega_1-\omega)} + P.V.\ \frac{1}{\pi}\int_{-1}^{1} \frac{\mathrm{Im}\,\bar{Q}_\alpha(\omega_1)\,d\omega_1}{\omega_1-\omega}$$

$$+ \frac{\omega}{\pi}\,P.V.\int_{1}^{\infty} \frac{\mathrm{Im}\,\bar{Q}_\alpha(\omega_1)\,d\omega_1}{\omega_1(\omega_1-\omega)} - \frac{1}{\pi i}\int_C \mathrm{Re}\,\bar{Q}_\alpha(\omega_1)\,\frac{d\omega_1}{\omega_1} \quad . \tag{2.40}$$

The path C is invariant except for a change in direction, under the transformation $\omega_1 \to -\omega_1^*$; thus since

$$\frac{d(-\omega_1^*)}{-\omega_1^*} = -\frac{d\omega_1}{\omega_1}$$

we can take half of the last integral and rewrite it using crossing symmetry:

$$-\frac{1}{2\pi i}\int_C \mathrm{Re}\,\bar{Q}_\alpha(\omega_1)\,\frac{d\omega_1}{\omega_1} = \frac{1}{2\pi i}\int_C \mathrm{Re}\,\bar{Q}_\alpha(-\omega_1^*)\,\frac{d(-\omega_1^*)}{-\omega_1^*}$$

$$= \frac{1}{2\pi i}\sum_\beta A_{\alpha\beta}\int_C \mathrm{Re}\,\bar{Q}_\beta(\omega_1)\,\frac{d\omega_1}{\omega_1} \quad . \tag{2.41}$$

Now we can apply crossing symmetry to the remaining integrals over negative ω_1, and in this last integral substitute $A_{\alpha\beta} = \delta_{\alpha\beta} + a_\alpha K_\beta$; we obtain

$$\bar{P}_\alpha(\omega) = a_\alpha \sum_\beta \frac{K_\beta}{2\pi}\int_C \mathrm{Re}\,\bar{Q}_\beta(\omega_1)\,\frac{d\omega_1}{\omega_1}$$

$$+ \frac{1}{\pi}\,P.V.\int_0^1 \frac{\bar{F}_\alpha(\omega_1)}{\omega_1-\omega}\,d\omega_1 + \frac{\omega}{\pi}\,P.V.\int_1^\infty \frac{\bar{F}_\alpha(\omega_1)}{\omega_1(\omega_1-\omega)}\,d\omega_1$$

$$+ \frac{1}{\pi}\sum_\beta A_{\alpha\beta}\left\{-\int_0^1 \frac{\bar{F}_\beta(\omega_1)}{\omega_1+\omega}\,d\omega_1 + \omega\int_1^\infty \frac{\bar{F}_\beta(\omega_1)\,d\omega_1}{\omega_1(\omega_1+\omega)}\right\}, \tag{2.42}$$

which agrees with equation 2.31 if the first term is $g_0 a_\alpha$.

Thus we have shown that the asymptotic form $\bar{Q}_\alpha(\omega)$ of the perturbation series is uniquely defined by the following requirements:

A) $\lim_{\omega \to \infty} \frac{\bar{Q}_\alpha(\omega)}{\omega} = 0$.

C) $\sum_\beta \frac{K_\beta}{2\pi} \int_C \frac{Re\, \bar{Q}_\alpha(\omega_1)}{\omega_1} d\omega_1 = g_0$.

D) $\bar{Q}_\alpha(-\omega^*)^* = -\sum_\beta A_{\alpha\beta} \bar{Q}_\beta(\omega) \qquad (Im\, \omega > 0)$.

E) $Im\, \bar{Q}_\alpha(\omega) = |\bar{Q}_\alpha(\omega)|^2 \qquad (\omega > 0)$.

F) $\bar{Q}_\alpha(\omega)$ is analytic in ω for $Im\, \omega > 0$,

where the path C is the upper half of the unit circle, from $\omega_1 = -1$ to $\omega_1 = 1$. There is no requirement corresponding to B of the complete theory.

C.

In this section we develop a method for solving the equations of section B.

To solve equations 2.30 and 2.32 let us assume the expansions*

$$\bar{P}_\alpha(z) = \sum_{n=0}^{\infty} \bar{P}_{\alpha n}(z)\, g_0^{2n+1} \ ,$$

$$\bar{F}_\alpha(z) = \sum_{n=0}^{\infty} \bar{F}_{\alpha n}(z)\, g_0^{2n+2} \tag{2.43}$$

(which means we have redefined $\bar{P}_{\alpha n}$ and $\bar{F}_{\alpha n}$). We can substitute these expressions in equations 2.30 and 2.32; if we differentiate equation 2.30 with respect to z we shall obtain differential equations for $\bar{P}_{\alpha n}$ and $\bar{F}_{\alpha n}$. For example, for n=0 and n=1 we obtain the equations

$$\bar{P}_{\alpha 0}(z) = a_\alpha \left(1 + \sum_\beta \kappa_\beta \int_0^z \bar{F}_{\beta 0}(z_1)\, dz_1 \right) , \tag{2.44}$$

$$\bar{F}_{\alpha 0}(z) = \bar{P}_{\alpha 0}^{2}(z) \ , \tag{2.45}$$

$$\bar{P}_{\alpha 1}(z) = b_1 \frac{d\bar{F}_{\alpha 0}(z)}{dz} + a_\alpha \sum_\beta \kappa_\beta \left\{ \int_0^z \bar{F}_{\beta 1}(z_1)\, dz_1 + c_1 \frac{d\bar{F}_{\beta 0}}{dz} \right\} , \tag{2.46}$$

$$\bar{F}_{\alpha 1}(z) = 2\bar{P}_{\alpha 0}(z)\,\bar{P}_{\alpha 1}(z) + \bar{F}_{\alpha 0}^{2}(z) \ . \tag{2.47}$$

Equations 2.44 and 2.46 may be differentiated once with respect to z yielding differential equations for $\bar{P}_{\alpha 0}$ and $\bar{P}_{\alpha 1}$; they also give the value of $\bar{P}_{\alpha 0}$ and $\bar{P}_{\alpha 1}$ at z=0 so that the solution of the equations is unique. Thus for n=0 we may write

$$\bar{P}_{\alpha 0}(z) = a_\alpha\, u(z) \ , \tag{2.48}$$

* We may assume expansions in g_0^2 rather than g_0 because the b_n and c_n vanish for even n.

and obtain for u(z) the equations

$$u(0) = 1 ,$$

$$\frac{du}{dz} = \theta u^2 , \tag{2.49}$$

where

$$\theta = \sum_\alpha K_\alpha a_\alpha^2 . \tag{2.50}$$

Solving these equations, we have

$$\frac{du}{u^2} = \theta dz , \qquad -\frac{1}{u} = \theta z - 1 ,$$

$$\text{and} \qquad \bar{P}_{\alpha 0}(z) = \frac{a_\alpha}{1 - \theta z} . \tag{2.51}$$

We could go on to solve the equations for $\bar{P}_{\alpha 1}$, $\bar{P}_{\alpha 2}$, etc. in the same manner, but it is simpler to assume the form of the solution and then to show that it is correct. Let g_1 be a power series in g_0:

$$g_1 = g_0 + \sum_{n=1}^{\infty} \delta_n g_0^{2n+1}$$

so that conversely g_0 is a power series in g_1:

$$g_0 = g_1 + \sum_{n=1}^{\infty} \gamma_n g_1^{2n+1} \tag{2.52}$$

(define also $\gamma_0 = 1$). We shall leave the coefficients γ_n to be determined later. Then assume

$$\bar{P}_\alpha = \sum_{n=0}^{\infty} p_{\alpha n} \frac{g_1^{2n+1}}{(1-\theta z)^{2n+1}} , \tag{2.53}$$

$$\bar{F}_\alpha = \sum_{n=0}^{\infty} f_{\alpha n} \frac{g_1^{2n+2}}{(1-\theta z)^{2n+2}} , \tag{2.54}$$

where the $p_{\alpha n}$ and $f_{\alpha n}$ are constants, and now

$$z = \frac{g_1 \ln \omega}{\pi} . \tag{2.55}$$

Because of the change in the definition of z, equation 2.30 is modified by replacing g_o by g_1 everywhere except in $g_o a_\alpha$. Substituting these assumptions in equation 2.30 gives

$$\sum_{n=0}^{\infty} p_{\alpha n}\left[\frac{g_1}{1-\theta z}\right]^{2n+1} = \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} b_m g_1^m \frac{(2\ell+1+m)!}{(2\ell+1)!}\,\theta^m \frac{g_1^{2\ell+2} f_{\alpha\ell}}{(1-\theta z)^{2\ell+2+m}}$$
$$+ a_\alpha\left\{\sum_{n=0}^{\infty}\left[\delta_n g_1^{2n+1} + \sum_\beta \frac{K_\beta g_1^{2n+1} f_{\beta n}}{(2n+1)\theta}\left(\frac{1}{(1-\theta z)^{2n+1}} - 1\right)\right]\right.$$
$$\left. + \sum_\beta K_\beta \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} c_m \frac{(2\ell+1+m)!}{(2\ell+1)!}\,\theta^m f_{\beta\ell}\left(\frac{g_1}{1-\theta z}\right)^{2\ell+m+2}\right\}. \qquad (2.56)$$

Because b_{2m} and c_{2m} are zero only odd powers of $g_1/1-\theta z$ appear on the right hand side (except for the constant terms) which justifies our assuming only odd powers in the expansion of $\overline{P}_\alpha$. From this equation we may extract equations for the δ_n and $p_{\alpha n}$:

$$\delta_n = \frac{1}{(2n+1)\theta}\sum_\beta K_\beta f_{\beta n}\;, \qquad (2.57)$$

$$p_{\alpha n} = \sum_{r=1}^{n} B_{nr}\,\theta^{2r-1} f_{\alpha,n-r} + a_\alpha \sum_\beta K_\beta \sum_{r=0}^{n} \theta^{2r-1} C_{nr} f_{\beta,n-r}\;, \qquad (2.58)$$

where

$$B_{nr} = \frac{B_r}{(2r)!}\,\frac{(2n)!}{(2n-2r+1)!}\left[2^{2r+1}-2\right], \qquad (2.59)$$

$$C_{nr} = \frac{B_r}{(2r)!}\,\frac{(2n)!}{(2n-2r+1)!}\left[2^{2r}-2\right], \qquad (2.60)$$

$$C_{n0} = \frac{1}{(2n+1)}\;. \qquad (2.61)$$

The unitarity condition becomes

$$f_{\alpha n} = \sum_{r=0}^{n} p_{\alpha r}\,p_{\alpha,n-r} + \sum_{r=0}^{n-1} f_{\alpha r}\,f_{\alpha,n-r-1}\;. \qquad (2.62)$$

Equations 2.58 and 2.62 give for $n > 0$ a set of linear inhomogeneous equations for $p_{\alpha n}$ and $f_{\alpha n}$, assuming the coefficients in lower orders to be known. Substituting equation 2.62 into equation 2.58

gives an equation for $p_{\alpha n}$ in terms of a constant $v_{\alpha n}$ which includes all lower-order terms:

$$p_{\alpha n} = \frac{2 a_\alpha}{(2n+1)\theta} \sum_\beta K_\beta p_{\beta o} p_{\beta n} + v_{\alpha n} \quad . \tag{2.63}$$

To show that we can solve this equation it is sufficient to find $u_n = \sum_\beta K_\beta p_{\beta o} p_{\beta n}$; since $p_{\beta o} = a_\beta$ we find

$$u_n = \frac{2}{2n+1} u_n + \sum_\alpha K_\alpha a_\alpha v_{\alpha n} \quad , \tag{2.64}$$

and we see that there is no difficulty. It is evident that our method would not have worked if $\bar{P}_\alpha$ had had to be expanded in g_1 instead of g_1^2 for then the factor which is here $(2n+1)$ would have taken on the value 2; thus the importance of the b_m and c_m vanishing for even m.

Using equations 2.58 and 2.62 I have computed the first five coefficients $p_{\alpha n}$ and $f_{\alpha n}$ for a number of theories, i.e. the symmetric scalar and pseudoscalar and the symmetric scalar Σ-π interactions. The resulting numbers sometimes show no pattern, so I have also computed the expansion of the reciprocal of $\bar{Q}_\alpha$. Define

$$\bar{L}_\alpha(z) = Re \frac{1}{\bar{Q}_\alpha(z)} = \frac{\bar{P}_\alpha}{\bar{P}_\alpha^2 + \bar{F}_\alpha^2} = \frac{\bar{P}_\alpha(z)}{\bar{F}_\alpha(z)} \tag{2.65}$$

(we note that $Im \frac{1}{\bar{Q}_\alpha(z)} = -1$).

We may write the expansion of $L_\alpha(z)$:

$$\bar{L}_\alpha(z) = \sum_{n=0}^{\infty} \ell_{\alpha n} \left[\frac{g_1}{1-\theta z}\right]^{2n-1} \quad . \tag{2.66}$$

For the theories I have looked at the ℓ's are sufficiently simple for one to obtain the remainder of the series by inspection. For example we give in Table I the coefficients $p_{\alpha n}$, $f_{\alpha n}$, and $l_{\alpha n}$ for the symmetric scalar theory.

	p_{1n}	f_{1n}	$2l_{1n}$	p_{2n}	f_{2n}	l_{2n}
n = 0	-2	4	-1	1	1	1
1	14	-40	-3	-1	-1	0
2	-122	364	0	1	1	0
3	1094	-3280	0	-1	-1	0
4	-9842	29524	0	1	1	0

Table I: coefficients $p_{\alpha n}$, $f_{\alpha n}$, and $l_{\alpha n}$ for the symmetric scalar theory.

D.

In this section we report the asymptotic form of the perturbation expansion of Q_α for various theories, and prove that they are correct.

For each theory define

$$y = \frac{g_1}{1-\theta z} \tag{2.67}$$

(we recall that $\theta = \sum_\alpha K_\alpha a_\alpha^2$ and $z = \frac{g_1 \ln \omega}{\pi}$ and that g_1 is a power series in g defined through equations 2.24, 2.52, and 2.57). By the method outlined in section C, I have obtained the asymptotic forms which follow.

1.) "One Dimensional" Theory

This is the simplest form of the Low equation but does not derive from any physical theory. We consider a single amplitude Q, with a = -1 and A = -1, giving $K = 2$ and $\theta = 2$. I obtain

$$\bar{Q} = \frac{-y}{1+iy} \quad . \tag{2.68}$$

2.) Symmetric Scalar Theory

I obtain $\theta = 2$ and

$$\begin{aligned} \bar{Q}_1 &= \frac{-2y}{(1-iy)(1+3iy)} \quad , \\ \bar{Q}_2 &= \frac{y}{(1-iy)} \quad . \end{aligned} \tag{2.69}$$

3.) Symmetric Pseudoscalar Theory

$\theta = 2$ and

$$\begin{aligned} \bar{Q}_1 &= \frac{-4y(1+3y^2)}{(1-iy)^2(1+3iy)^2} \quad , \\ \bar{Q}_2 &= \frac{-y(1-3y^2)}{(1-iy)^2(1+3iy)} \quad , \\ \bar{Q}_3 &= \frac{2y}{(1-iy)^2} \quad . \end{aligned} \tag{2.70}$$

4.) Symmetric Scalar Σ-π Interaction.

This theory is the same as the symmetric scalar theory except that in the coupling Hamiltonian the τ matrices are those of isotopic spin one rather than isotopic spin 1/2. In this theory

$$a_\alpha = \begin{vmatrix} -2 \\ -1 \\ 1 \end{vmatrix} , \quad A_{\alpha\beta} = \frac{1}{6}\begin{vmatrix} 2 & -6 & 10 \\ -2 & 3 & 5 \\ 2 & 3 & 1 \end{vmatrix} , \quad K_\beta = \frac{1}{6}\begin{vmatrix} 2 \\ 3 \\ -5 \end{vmatrix} , \tag{2.71}$$

where α = 1,2,3 are the isotopic spin 0,1,2 amplitudes respectively. I obtain $\theta = 1$ and

$$\begin{aligned} \bar{Q}_1 &= \frac{-2y}{1+2iy} , \\ \bar{Q}_2 &= \frac{-y}{(1+2iy)(1-iy)} , \\ \bar{Q}_3 &= \frac{y}{(1-iy)} . \end{aligned} \tag{2.72}$$

Note that because $\theta = 1$, y is different here than in the other examples.

To prove these results we must show that the amplitudes $\bar{Q}_\alpha$ satisfy the conditions A, C, D, E and F given on page 31. It is evident that our expressions for $\bar{Q}_\alpha$ can be expanded in powers of g, and that the coefficients are polynomials in $\ln\omega$. This is sufficient to satisfy conditions A and F. We can satisfy C by using it to redefine g_1 in terms of g_o [we expect this definition to be equivalent to equations 2.52 and 2.57 but it is unnecessary for us to prove this]. We are left with showing conditions D and E. It is easy to see that they satisfy unitarity, for if we denote the denominator of our expression for $\bar{Q}_\alpha$ by A_α our expression has the form

$$\bar{Q}_\alpha = -\frac{\operatorname{Im} A_\alpha}{A_\alpha} . \tag{2.73}$$

Hence

$$\mathrm{Im}\ \frac{1}{\overline{Q}_\alpha} = \frac{-\overline{F}_\alpha}{\overline{P}_\alpha^2 + \overline{F}_\alpha^2} = -1 \quad . \tag{2.74}$$

We have left to demonstrate crossing symmetry. We shall prove crossing symmetry for the symmetric scalar theory; the proof for the other examples is analogous.

The crossing relation is given by equation 2.7. The transformation $\omega \rightarrow -\omega^*$ becomes for y,

$$\frac{1}{y} = \frac{1}{g_1} - \frac{2}{\pi} \ln\omega \rightarrow \frac{1}{g_1} - \frac{2}{\pi} \ln\omega^* - 2i = \frac{1}{y^*} - 2i \tag{2.75}$$

(for ω in the upper half plane), and hence

$$y \rightarrow \frac{y^*}{1-2iy^*} \quad . \tag{2.76}$$

Thus crossing symmetry requires

$$\left\{ \overline{Q}_\alpha \left(\frac{y^*}{1-2iy^*} \right) \right\}^* = -\sum_\beta A_{\alpha\beta}\, \overline{Q}_\beta\,(y) \quad , \tag{2.77}$$

Now

$$\begin{aligned} \left\{ \overline{Q}_1 \left(\frac{y^*}{1-2iy^*} \right) \right\}^* &= \frac{-2y\,(1+2iy)}{(1+3iy)\,(1-iy)} \\ &= \frac{1}{3\,(1+3iy)(1-iy)} \Big[(-2y) - 4y\,(1+3iy) \Big] \;, \end{aligned} \tag{2.78}$$

$$\begin{aligned} \left\{ \overline{Q}_2 \left(\frac{y^*}{1-2iy^*} \right) \right\}^* &= \frac{y}{1+3iy} \\ &= \frac{1}{3(1+3iy)(1-iy)} \Big[-2\,(-2y) - y\,(1+3iy) \Big] \;. \end{aligned} \tag{2.79}$$

Since $A_{\alpha\beta} = \frac{1}{3} \begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix}$ we have demonstrated crossing symmetry.

E.

In this section we discuss the results reported in section D.

The expressions $\bar{Q}_\alpha$ have been shown to be the asymptotic forms of the complete solutions Q_α only when both are expanded in perturbation theory. Thus we cannot say whether $\bar{Q}_\alpha$ approximates Q_α when ω is large but g is also large enough that the perturbation series diverges. However, for the one example for which the solution Q_α is known $\bar{Q}_\alpha$ is the asymptotic form of Q_α for large ω ; this is the "one-dimensional" theory. In the one-dimensional theory the solution may be found by the method of Dyson et al.[(6)]: it is

$$Q = \frac{-g}{1 - \frac{g}{\pi}\left\{ 2\frac{\sqrt{\omega^2-1}}{\omega} \ln\left[\omega + \sqrt{\omega^2-1}\right] - 2 - i\pi\frac{\sqrt{\omega^2-1}}{\omega}\right\}} \,. \tag{2.80}$$

For large ω we obtain

$$\begin{aligned} Q &\simeq \frac{-g}{1 - \frac{g}{\pi}\left\{2\ln\omega + 2\ln 2 - 2 - i\pi\right\}} \\ &= \frac{-g_1}{1 - \frac{g_1}{\pi}\left\{2\ln\omega - i\pi\right\}} \,, \end{aligned} \tag{2.81}$$

if

$$g_1 = \frac{g}{1 - \left[2\ln 2 - 2\right]\frac{g}{\pi}} \,. \tag{2.82}$$

From equations 2.67 and 2.68 we find that $\bar{Q}$ is equal to the right hand expression in equation 2.81, and one can verify that equation 2.82 also holds.

There are two properties of the sum of the asymptotic power series which are interesting to note. The first is its behavior for very large ω (keeping g^2 fixed).

We find for very large ω that y $\simeq \frac{-\pi}{\theta \ln \omega}$ which is small, and therefore

$$\bar{P}_\alpha \simeq -\frac{a_\alpha \pi}{\theta \ln \omega} , \tag{2.83}$$

$$\bar{F}_\alpha \simeq \frac{a_\alpha^2 \pi^2}{\theta^2 \ln^2 \omega} . \tag{2.84}$$

If $\bar{Q}_\alpha$ is the asymptotic form of Q_α, it follows from equation 2.84 that a dispersion integral over F_α without subtractions, i.e.

$$\frac{1}{\pi}\int_1^\infty \frac{F_\alpha(\omega_1)}{\omega_1 \pm \omega} d\omega_1$$

converges, and from equation 2.83 we find that P_α goes to zero for large ω, hence Q_α satisfies an unsubtracted dispersion relation

$$Q_\alpha(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Im} Q_\alpha(\omega_1)}{\omega_1 - \omega - i\epsilon} d\omega_1 . \tag{2.85}$$

Although it is interesting, I do not believe this result is of any use.

The other question is whether $\bar{Q}_\alpha$ has any singularities in the upper half ω-plane, in which case the dispersion relation for Q_α would be incorrect (again assuming Q_α is asymptotic to $\bar{Q}_\alpha$). We find that a singularity occurs only in the one-dimensional theory, where $\bar{Q}$ has a pole at

$$1 + iy = 0 , \tag{2.86}$$

or

$$1 - \frac{2g_1}{\pi} \ln \omega + ig_1 = 0 \text{, i.e. } \omega = i e^{\frac{\pi}{2g_1}} , \tag{2.87}$$

and hence $\bar{Q}$ has a singularity for ω large and on the positive imaginary axis if g_1 is small and positive. For the symmetric scalar and pseudoscalar theories $\bar{Q}_\alpha$ has poles for

$$1 - iy = 0 \text{, or } 1 + 3iy = 0 , \tag{2.88}$$

i.e. $$1 - \frac{2g_1}{\pi} \ln \omega \, {}^{+3}_{-1} (ig_1) = 0 \quad , \tag{2.89}$$

and neither equation can be satisfied for ω in the upper half plane. The same holds true for the symmetric scalar Σ-π interaction.

Finally, we note that the method of transforming the dispersion equation 2.9 into a differential equation 2.30 is generally applicable to dispersion relations when it is known that the amplitudes in question behave logarithmically for large ω , although it is unlikely that in more complicated situations one will be able to get as much information as we have obtained for the Low equation. Even in the Low equation for the symmetric pseudoscalar Σ-π interaction, for example, the differential equations for the terms $\overline{P}_{\alpha o}(z)$ and $\overline{F}_{\alpha o}(z)$ are coupled, non-linear first order equations which I have been unable to solve analytically.

F.

In this section we apply the ideas of Gell-Mann and Low[16] to the one-meson approximation. We shall show that they give an alternate proof that when the crossing matrix has only one negative eigenvalue the asymptotic form of the perturbation expansion of the one-meson approximation depends only on the variable y.

Ordinarily one defines the renormalized electric charge e in terms of the low-energy behavior of electrons and photons. As a result the only dimensional constant which enters quantum electrodynamics is the electron mass m, and therefore quantities such as the photon propagator depend in a complicated way on m. Gell-Mann and Low discovered that if one renormalized quantum electrodynamics in terms of the behaviour of photons at a non-zero four-momentum squared (say λ^2), thereby introducing an alternative unit of length, one could then let the mass of the electron go to zero (keeping λ^2 fixed) and obtain a definite limit for such quantities as the photon propagator. The existence of this limit limits the form of the asymptotic form for large momentum squared (k^2) of the photon propagator. In fact, if D_{FC} is the renormalized propagator (now renormalized in the usual fashion) Gell-Mann and Low showed that in perturbation theory, for $k^2 \gg m^2$

$$D_{FC} \cong \frac{1}{e^2 k^2} f\left\{ \ln\frac{k^2}{m^2} + \frac{\psi(e^2)}{e^2} \right\} , \tag{2.90}$$

where f and ψ are power series in a single argument, which must be determined from ordinary perturbation theory. To see how similar this is to our results for the one-meson approximation, substitute ω for k^2/m^2 and $-\frac{\pi}{\theta g_1}$ for $\frac{\psi(e^2)}{e^2}$, and the argument of f becomes

$$\ln\omega - \frac{\pi}{6g_1} = -\frac{\pi}{6}\frac{1}{y}\,, \tag{2.91}$$

i.e. Gell-Mann and Low's result is analogous to our result that the asymptotic form of perturbation theory for the one-meson approximation depends only on y.

We shall now analyze the equations for the asymptotic form derived earlier, in the spirit of Gell-Mann and Low. We have measured the energy ω in units of the meson mass, and defined the coupling constant g_o in terms of the low energy behaviour of the asymptotic form. Thus the meson mass is our only unit of mass, and we cannot expect to obtain a reasonable limit if we let this mass go to zero, unless we introduce a new unit first. To see exactly how this should be done, consider the equations for the asymptotic form (equation 2.31):

$$\begin{aligned}\bar{P}_\alpha(\omega) &= g_o a_\alpha \\ &+ \frac{1}{\pi}\,\mathrm{P.V.}\int_0^1 \frac{\bar{F}_\alpha(\omega_1)\,d\omega_1}{\omega_1-\omega} + \frac{\omega}{\pi}\,\mathrm{P.V.}\int_1^\infty \frac{\bar{F}_\alpha(\omega_1)\,d\omega_1}{\omega_1(\omega_1-\omega)} \\ &+ \sum_\beta A_{\alpha\beta}\left\{-\frac{1}{\pi}\int_0^1 \frac{\bar{F}_\beta(\omega_1)\,d\omega_1}{\omega_1+\omega} + \frac{\omega}{\pi}\int_1^\infty \frac{\bar{F}_\beta(\omega_1)\,d\omega_1}{\omega_1(\omega_1+\omega)}\right.\,,\end{aligned} \tag{2.31}$$

$$\bar{F}_\alpha(\omega) = \bar{P}_\alpha^2(\omega) + \bar{F}_\alpha^2(\omega) \qquad (\omega > o) \tag{2.32}$$

If we did not take the meson mass to be 1, but rather let it have a value m, the only change would be to replace 1 by m as a limit in the integrals. The stumbling block, mathematically, to letting m $\to$ o is the factor $\frac{1}{\omega_1}$ in the second and fourth integrals, which makes them divergent if we replace 1 by o.

To eliminate the factors $\frac{1}{\omega_1}$ requires our introducing a new coupling constant h, expressed in terms of the properties of $\bar{Q}_\alpha$ in the neighborhood of a value ω_0 of ω ; in this way we introduce a new measure of energy. To achieve this and avoid defining separate constants for each value of α requires a little care but can be done as follows: write

$$\frac{1}{\omega_1 - \omega} = \frac{\omega + \omega_0}{(\omega_1 + \omega_0)(\omega_1 - \omega)} + \frac{1}{\omega_1 + \omega_0} \quad ; \tag{2.92}$$

$$\frac{\omega}{\omega_1(\omega_1 - \omega)} = \frac{\omega + \omega_0}{(\omega_1 + \omega_0)(\omega_1 - \omega)} + \frac{-\omega_0}{\omega_1(\omega_1 + \omega_0)} \quad , \tag{2.93}$$

$$\frac{-1}{\omega_1 + \omega} = \frac{\omega - \omega_0}{(\omega_1 + \omega_0)(\omega_1 + \omega)} - \frac{1}{\omega_1 + \omega_0} \quad , \tag{2.94}$$

$$\frac{\omega}{\omega_1(\omega_1 + \omega)} = \frac{\omega - \omega_0}{(\omega_1 + \omega_0)(\omega_1 + \omega)} + \frac{\omega_0}{\omega_1(\omega_1 + \omega_0)} \quad . \tag{2.95}$$

Now we introduce the assumption that A has only one negative eigenvalue, so that (equation 2.4)

$$A_{\alpha\beta} = \delta_{\alpha\beta} + a_\alpha \kappa_\beta \quad . \tag{2.4}$$

This allows us to lump the effect of the second terms in our expressions above into a new constant h:

$$\begin{aligned} h(\omega_0, g_0) = {} & g_0 \\ & + \sum_\beta \kappa_\beta \left\{ -\frac{1}{\pi} \int_0^1 \frac{\bar{F}_\beta(\omega_1)\, d\omega_1}{\omega_1 + \omega_0} + \frac{\omega_0}{\pi} \int_1^\infty \frac{\bar{F}_\beta(\omega_1)\, d\omega_1}{\omega_1(\omega_1 + \omega_0)} \right\} ; \end{aligned} \tag{2.96}$$

and obtain

$$\begin{aligned} \bar{P}_\alpha(\omega) = {} & h\, a_\alpha + \frac{\omega + \omega_0}{\pi}\, \mathrm{P.V.} \int_0^\infty \frac{\bar{F}_\alpha(\omega_1)\, d\omega_1}{(\omega_1 + \omega_0)(\omega_1 - \omega)} \\ & + \sum_\beta A_{\alpha\beta} \frac{(\omega - \omega_0)}{\pi} \int_0^\infty \frac{F_\beta(\omega_1)}{(\omega_1 + \omega_0)(\omega_1 + \omega)}\, d\omega_1 \quad . \end{aligned} \tag{2.97}$$

Let

$$\bar{Q}_\alpha(\omega, g_o) = \bar{P}_\alpha(\omega) + i\,\bar{F}_\alpha(\omega) \qquad (2.98)$$

give the functional dependence (more precisely, the power series expansion) of the asymptotic form <u>in terms of ω and g_o.</u> Now from equation 2.97 above and the unitarity condition it is evident that in terms of h instead of g_o, $\bar{P}_\alpha$ and $\bar{F}_\alpha$ depend only on the ratio $\frac{\omega}{\omega_o}$. Thus let R_α give the power series expansion $\bar{P}_\alpha + i\,\bar{F}_\alpha$ <u>in terms of ω/ω_o and h</u>. We now have the functional equation

$$R_\alpha\left(\frac{\omega}{\omega_o},\; h(\omega_o, g_o)\right) = \bar{Q}_\alpha(\omega, g_o) \quad , \qquad (2.99)$$

which holds for all values of ω, ω_o and g_o. This equation contains the information we need about $\bar{Q}_\alpha$; the reason it contains any is because R_α would normally depend on <u>three</u> variables-- ω, ω_o and h, and only for special forms of $\bar{Q}_\alpha$ can R_α depend on only ω/ω_o and h. To obtain the information we want from this equation we shall use the method of Bogoliubov and Shirkov(17), for the most part. It must also be kept in mind that R_α is a power series in h, h a power series in g_o and $\bar{Q}_\alpha$ in a power series in g_o, with leading terms

$$R_\alpha = h\, a_\alpha + \ldots \quad ,$$
$$\bar{Q}_\alpha = g_o\, a_\alpha + \ldots \quad ,$$
$$h = g_o + \ldots \quad .$$

First we observe that the power series for h can be expressed in terms of the power series for R_α and for $\bar{Q}_\alpha$, say by the equation

$$R_\alpha(1, h(\omega_o, g_o)) = \bar{Q}_\alpha(\omega_o, g_o) \quad . \qquad (2.100)$$

Given the power series for R_α in terms of h, this allows us to obtain h as a power series in $\bar{Q}_\alpha$:

$$h(\omega_0, g_0) = H\left[\bar{Q}_\alpha(\omega_0, g_0)\right] \quad . \tag{2.101}$$

Thus we can change variables from h to $\bar{Q}_\alpha$ in R_α, giving a new power series T_α say:

$$R_\alpha\left(\frac{\omega}{\omega_0}, H(\bar{Q}_\alpha)\right) = T_\alpha\left(\frac{\omega}{\omega_0}, \bar{Q}_\alpha\right) \quad , \tag{2.102}$$

and now

$$T_\alpha\left(\frac{\omega}{\omega_0}, \bar{Q}_\alpha(\omega_0, g_0)\right) = \bar{Q}_\alpha(\omega, g_0) \quad . \tag{2.103}$$

Differentiate this expression with respect to ω, and afterwards set $\omega_0 = \omega$: we obtain

$$\frac{1}{\omega} T_{\alpha,1}\left(1, \bar{Q}_\alpha(\omega, g_0)\right) = \frac{\partial \bar{Q}_\alpha}{\partial \omega}(\omega, g_0) \quad , \tag{2.104}$$

where

$$T_{\alpha,1}(x, y) = \frac{\partial T_\alpha}{\partial x}(x, y) \quad . \tag{2.105}$$

Let

$$\psi(\bar{Q}_\alpha) = T_{\alpha,1}(1, \bar{Q}_\alpha) \quad . \tag{2.106}$$

Treating g_0 as a constant the partial derivative becomes total and

$$\frac{d\bar{Q}_\alpha}{\psi(\bar{Q}_\alpha)} = \frac{d\omega}{\omega} \quad . \tag{2.107}$$

Let

$$\int^{\bar{Q}_\alpha} \frac{dx}{\psi(x)} = F(\bar{Q}_\alpha) \quad , \tag{2.108}$$

then

$$F\left[\bar{Q}_\alpha(\omega, g_0)\right] = \ln\omega - \ln\omega_0 + F(\bar{Q}_\alpha(\omega_0, g_0)) \,. \tag{2.109}$$

Let ω_o be a number, say 1; then if we write

$$\phi(g_0) = F\left[\bar{Q}_\alpha(1, g_0)\right] \,, \tag{2.110}$$

we have

$$\bar{Q}_\alpha(\omega, g_0) = F^{-1}\left[\ln\omega + \phi(g_0)\right] \,. \tag{2.111}$$

It is not difficult to see that these operations are legitimate with power series of the type we are considering; we assume no a_α is o, and $T_{\alpha,1}(x,y)$ has a power series whose first term is of order y^2 since the coefficient of y vanishes. Detailed calculation then shows that for small g_o,

$$\phi(g_0) \simeq -\frac{\pi}{e g_0} \,, \tag{2.112}$$

so that we can now define

$$g_1 = -\frac{e}{\pi}\,\frac{1}{\phi(g_0)} \,. \tag{2.113}$$

This completes our demonstration that the method of Gell-Mann and Low can be applied to the one-meson approximation.

APPENDIX

We must evaluate the integrals for b_n and c_n (equations 2.26 - 2.28). First we make a change of variable to $t_1 = \frac{1}{t}$ in the second integral in both equations 2.26 and 2.27. It is then evident that e_n and c_n, and therefore b_n, vanish for even n. We obtain also

$$e_{2n-1} = \frac{-2}{\pi^{2n}(2n-1)!}\int_0^1 \frac{\ln^{2n-1} t}{1-t}\, dt \quad , \tag{2A.1}$$

$$c_{2n-1} = \frac{-2}{\pi^{2n}(2n-1)!}\int_0^1 \frac{\ln^{2n-1} t}{1+t}\, dt \quad . \tag{2A.2}$$

The first integral is known[19]; when our definition of the Bernoulli numbers is used (our B_n is the B_{2n} of reference 19) we obtain

$$e_{2n-1} = 2^{2n}\frac{B_n}{(2n)!} \quad . \tag{2A.3}$$

To calculate c_{2n-1} we observe that

$$e_{2n-1} - c_{2n-1} = \frac{-4}{\pi^{2n}(2n-1)!}\int_0^1 \frac{(\ln t)^{2n-1}}{1-t^2}\, t\, dt \quad , \tag{2A.4}$$

and making a change of variable to $u = t^2$, we obtain

$$e_{2n-1} - c_{2n-1} = 2\,\frac{e_{2n-1}}{2^{2n}} \quad , \tag{2A.5}$$

and therefore

$$c_{2n-1} = [2^{2n} - 2]\frac{B_n}{(2n)!} \quad , \tag{2A.6}$$

$$b_{2n-1} = [2^{2n+1} - 2]\frac{B_n}{(2n)!} \quad . \tag{2A.7}$$

Equations 2A.6 and 2A.7 are precisely equations 2.33 and 2.34 of the text.

III. ASYMPTOTIC FORM OF THE LOW EQUATION (CONTINUED)

In chapter II we derived equations for the asymptotic form of the perturbation expansion of the one-meson approximation. We also gave the solution of these equations for various theories (symmetric scalar, etc.).

For the purpose of understanding the Low equation in the one-meson approximation, as a mathematical problem, it is interesting to study it for crossing matrices $A_{\alpha\beta}$ and Born approximations a_α picked at random (i.e. not derived from any physical interaction). We still require that (equations 2.2 and 2.3 of chapter II)

$$\sum_\beta A_{\alpha\beta} A_{\beta\gamma} = \delta_{\alpha\gamma} \quad , \tag{3.1}$$

$$\sum_\beta A_{\alpha\beta} a_\beta = -a_\alpha \quad , \tag{3.2}$$

so that crossing symmetry is still a symmetry property of the solution. To use the methods of chapter II we must require that $A_{\alpha\beta}$ have the form (equation 2.4 of chapter II)

$$A_{\alpha\beta} = \delta_{\alpha\beta} + a_\alpha K_\beta \quad , \tag{3.3}$$

with (equation 2.5 of chapter II)

$$\sum_\beta a_\beta K_\beta = -2 \quad . \tag{3.4}$$

The crossing matrices of the theories considered in the previous chapter all have one other property, namely

$$\sum_\beta A_{\alpha\beta} = 1 \quad , \tag{3.5}$$

i.e. that $\begin{vmatrix} 1 \\ 1 \end{vmatrix}$ or $\begin{vmatrix} 1 \\ 1 \end{vmatrix}$ is an eigenvector of A with eigenvalue + 1.

In this chapter we present the asymptotic form of the perturbation series for all two-dimensional crossing matrices $A_{\alpha\beta}$ and Born approximations a_α which satisfy equations 3.1 - 3.5. These include the symmetric scalar theory, but as far as the author knows, no other physical interaction.

The important result is that, except for exceptional cases, the power series in y giving the asymptotic form for these crossing matrices is divergent for any non-zero value of y. One of the exceptional cases is the symmetric scalar theory, for we saw in chapter II that its power series in y converges for small y. Specifically, we find that the perturbation series in y can be rewritten as a continued fraction, and that for special choices of the crossing matrix the continued fraction is finite; but when the continued fraction is infinite the corresponding power series diverges for any value of y.

It is a remarkable fact that we have had to consider <u>non-physical</u> crossing matrices in order to discover this property of the one-meson approximation, namely that its perturbation expansion can be divergent for any value of g^2. We have no explanation for this phenomenon.

In this chapter we present the continued fraction mentioned above and investigate some of the properties of the function it represents; we have not however made a complete study of this problem.

Condition 3.5 means

$$\sum_\beta K_\beta = 0 \quad ; \qquad (3.6)$$

if equations 3.4 and 3.6 are satisfied, equations 3.1, 3.2, 3.3, and

3.5 will also hold. Thus we may choose the a_α arbitrarily, the K_β then being determined. However multiplying all the a_α by a scale factor does not change one's theory but is instead equivalent to a change in the coupling constant keeping the a_α fixed. We find it convenient to choose the scale factor in advance by requiring that

$$\sum_\beta K_\beta a_\beta^2 = \theta = 4 \quad . \tag{3.7}$$

From equations 3.4 and 3.6 we obtain

$$K_1 = \frac{2}{a_2 - a_1} \quad , \tag{3.8}$$

$$K_2 = \frac{-2}{a_2 - a_1} \quad , \tag{3.9}$$

and hence equation 3.7 becomes

$$a_1 + a_2 = -2 \quad . \tag{3.10}$$

The condition that $\begin{vmatrix} 1 \\ 1 \end{vmatrix}$ be an eigenvector of A with eigenvalue +1 means that the S matrix elements satisfy crossing symmetry. Define

$$\bar{S}_\alpha = 1 + 2i\,\bar{Q}_\alpha \quad , \tag{3.11}$$

then (from condition D for the asymptotic form, in chapter II)

$$\left[\bar{S}_\alpha (-\omega^*) \right]^* = \sum_\beta A_{\alpha\beta}\, \bar{S}_\beta (\omega) \quad . \tag{3.12}$$

The solution of the equations for the asymptotic form was found by the author by trial and error from a knowledge of the first few coefficients in the expansion of $\bar{Q}_\alpha$ in powers of y.* The expansion

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* See page 37 of chapter II for the definition of y.

was found to be quite simple if expressed as a continued fraction. We obtain

$$\tan \bar{\delta}_\alpha = \frac{\bar{Q}_\alpha}{1+i\bar{Q}_\alpha} = \cfrac{a_\alpha y}{1 - \cfrac{(4-a_\alpha^2)y^2}{1 - \cfrac{(16-a_\alpha^2)y^2}{1 - \cfrac{(36-a_\alpha^2)}{1 - \cdots}}}} \, . \tag{3.13}$$

Unfortunately, this fraction diverges for real y and the same is true of the power series it represents. However for complex y the fraction converges to the expression[20]

$$\cfrac{a_\alpha y}{1 - \cfrac{(4-a_\alpha^2)y^2}{1-\cdots}} = i \tanh \left\{ \frac{1}{2} \int_0^\infty \frac{\sinh a_\alpha u}{u \cosh u} e^{-\frac{iu}{2y}} du \right\} . \tag{3.14}$$

Define

$$I(q) = \frac{1}{4} \int_0^1 \left\{ \frac{e^{-qu}}{\cosh u} - 1 \right\} \frac{du}{u} + \frac{1}{4} \int_1^\infty \frac{e^{-qu}}{\cosh u} \frac{du}{u} , \tag{3.15}$$

and let

$$z = \frac{1}{2y} = \frac{1}{2y_1} - \frac{2}{\pi} \ln \omega \tag{3.16}$$

(since $\theta = 4$). According to equation 3.14 we would have

$$\bar{\delta}_\alpha = i\, I(-a_\alpha + iz) - i\, I(a_\alpha + iz) \, . \tag{3.17}$$

Since $\frac{\sinh a_\alpha u}{u \cosh u}$ expanded in powers of u has a finite radius of convergence (unless a_α is an even integer), it is a simple calculation to see that the expansion of $\bar{\delta}_\alpha$ in powers of y is totally divergent (unless a_α is an even integer).

We should like the phase shift $\bar{\delta}_\alpha$ to be real when z is real.

This means we should like I(q) to be symmetric about the <u>imaginary</u> axis of q. Unfortunately this is not the case: to determine the nature of I(q) when q is say positive imaginary we rotate the path of integration in the u plane to the negative imaginary axis. On this new path e^{-qu} and cosh u and $\frac{du}{u}$ are real but cosh u has zeros which must be avoided giving I(q) an imaginary part.

We investigate the function I(q) in the Appendix to this chapter. In particular, it satisfies (equation 3A.10 of the Appendix)

$$I(q) = I(-q) + \frac{\pi i}{4} + \frac{1}{2} \ln \left\{ \frac{1 - i e^{-iq\frac{\pi}{2}}}{1 + i e^{-iq\frac{\pi}{2}}} \right\} , \tag{3.18}$$

and it has branch lines for $q < -1$ as shown in figure 3. Thus let us redefine the phase shift so that if $a_\alpha > -1$

$$\bar{\delta}_\alpha = i \left[I(a_\alpha - iz) + \frac{\pi i}{4} - I(a_\alpha + iz) \right] , \tag{3.19}$$

while if $a_\alpha < -1$

$$\bar{\delta}_\alpha = i \left[I(-a_\alpha + iz) - I(-a_\alpha - iz) - \frac{\pi i}{4} \right] . \tag{3.20}$$

Since the change in the definition of $\bar{\delta}_\alpha$ is just a logarithmic factor which vanishes more rapidly than $\frac{1}{z^n}$ for any n when z is large and approximately real, this change will not affect the perturbation expansion of $\bar{\delta}_\alpha$. Now however since I is real for real q, and therefore

$$I(q^*) = I^*(q) , \tag{3.21}$$

our expressions for $\bar{\delta}_\alpha$ are real.

We must now investigate whether these expressions for satisfy crossing symmetry, and whether they lead to singularities in

FIGURE 3

The region of analyticity of the function $I(q)$: it has branch lines on the negative real q axis as shown.

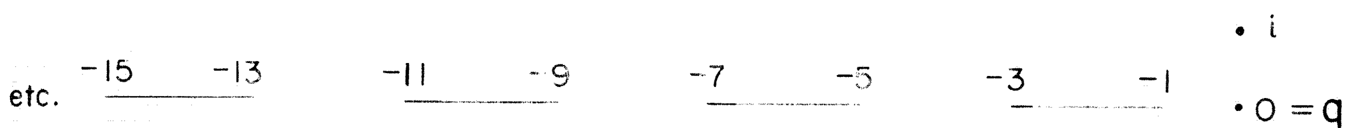

FIGURE 3

$\bar{Q}_\alpha$ in the upper half ω -plane.

To test crossing symmetry we use equation 3A.9 of the Appendix:

$$I(q+2) + I(q) = c - \frac{1}{2}\ln(q+1) \tag{3.22}$$

for $q \nless -1$: c is a constant, and we must take the principal branch of the logarithm. Let us suppose that $a_1 > -1$, $a_2 < -1$ which we can do since $a_1 + a_2 = -2$. Then

$$\begin{aligned}\bar{\delta}_2 &= i\left[I(2+a_1+iz) - I(2+a_1-iz) - \frac{\pi i}{4}\right] \\ &= i\left[-I(a_1+iz) + I(a_1-iz) - \frac{\pi i}{4}\right] - \frac{i}{2}\ln\left\{\frac{1+a_1+iz}{1+a_1-iz}\right\} \\ &= \bar{\delta}_1 + \frac{\pi}{2} - \frac{i}{2}\ln\left\{\frac{1+a_1+iz}{1+a_1-iz}\right\} .\end{aligned} \tag{3.23}$$

The transformation $\omega \to -\omega^*$ gives

$$z = \frac{1}{2g_1} - \frac{2}{\pi}\ln\omega \to z^* - 2i \quad . \tag{3.24}$$

Thus

$$\begin{aligned}[\bar{S}_1(-\omega^*)]^* &= \exp\left\{-2\left[I(a_1-iz^*-2) - I(a_1+iz^*+2) + \frac{\pi i}{4}\right]\right\}^* \\ &= \exp\left\{-2\left[-I(a_1+iz) + I(a_1-iz) - \frac{\pi i}{4}\right] + \ln\left\{\frac{a_1+iz-1}{a_1-iz+1}\right\}\right\} \\ &= \exp\left\{2i\bar{\delta}_1 + \pi i + \ln\left(\frac{a_1+iz-1}{a_1-iz+1}\right)\right\} ,\end{aligned} \tag{3.25}$$

$$\begin{aligned}[\bar{S}_2(-\omega^*)]^* &= \exp\left\{-2\left[I(-a_2+iz^*+2) - I(-a_2-iz^*-2) - \frac{\pi i}{4}\right]\right\}^* \\ &= \exp\left\{2i\bar{\delta}_2 - \pi i + \ln\left(\frac{-a_2-iz+1}{-a_2+iz-1}\right)\right\} .\end{aligned} \tag{3.26}$$

Thus we have for α = 1 or 2 and α' = 2 or 1 respectively

$$\bar{S}_{\alpha'}(\omega) = -\bar{S}_{\alpha}(\omega)\left(\frac{1+a_\alpha+iz}{1+a_\alpha-iz}\right), \tag{3.27}$$

$$[\bar{S}_\alpha(-\omega^*)]^* = -\bar{S}_\alpha(\omega)\left(\frac{-a_\alpha+1-iz}{-a_\alpha-1+iz}\right), \tag{3.28}$$

and from equations 3.8, 3.9, and 3.10,

$$K_\alpha = \frac{-2}{a_\alpha-(-2-a_\alpha)} = \frac{-1}{a_\alpha+1},$$

$$K_{\alpha'} = -K_\alpha ,$$

so that

$$\sum_{\beta=1}^{2} A_{\alpha\beta}\bar{S}_\beta(\omega) = \left(1-\frac{a_\alpha}{a_\alpha+1}\right)\bar{S}_\alpha + \frac{a_\alpha}{a_\alpha+1}\left(\frac{1+a_\alpha+iz}{1+a_\alpha-iz}\right)\left(-\bar{S}_\alpha(\omega)\right)$$

$$= \left(\frac{1-a_\alpha-iz}{1+a_\alpha-iz}\right)\bar{S}_\alpha(\omega)$$

$$= [\bar{S}_\alpha(-\omega^*)]^* . \tag{3.29}$$

Hence crossing symmetry (equation 3.12) is satisfied.

Now we must see whether $\bar{Q}_\alpha$ is analytic in ω in the upper half plane. The upper half ω plane corresponds to the region $-2 < \mathrm{Im} z < o$ or $o < \mathrm{Re}\, iz < 2$. Since $I(q)$ has singularities only for $q \lesssim -1$ we need to consider only the case $\mathrm{Im}\, iz \simeq o$. Considering equations 3.19 and 3.20 we see that there is no trouble if $a_1 > 1$ (and hence $a_2 < -3$) for then if $o < \mathrm{Re}\, iz < 2$ we never intersect the branch lines of I. Also since $a_2 < -1$ there is no trouble with $\bar{\delta}_2$. If $a_1 < 1$ we may use equation 3.22 to see that even though $\bar{\delta}_1$ has singularities in this region, S_1 has the form

$$\bar{S}_1 = \frac{a_1-iz+1}{a_1+iz+1}e^{2i\delta_1'}$$

(where δ_1' is analytic) and $\bar{S}_1$ has a zero but no poles in the region $0 < \mathrm{Re}\, iz < 2$. Since

$$\bar{S}_\alpha = 1 + 2i\bar{Q}_\alpha \ ,$$

we see that $\bar{Q}_\alpha$ has no singularities in the upper half ω -plane.

It is interesting to note that the expression we have chosen for $\bar{\delta}_\alpha$ does not agree with our previous expression for the symmetric scalar theory. This is because the expression we derived in the previous chapter is just a special case of equation 3.13. Let us see what the expression of this chapter gives for the symmetric scalar theory. In the units of this chapter we have

$$a_1 = -4 \ , \quad a_2 = 2 \ ,$$

We can evaluate equations 3.19 and 3.20 by relating these expressions back to the continued fraction of equation 3.13.

We obtain (note the y used here differs from that of chapter II)

$$\bar{S}_1 = \frac{(1+2iy)}{(1-2iy)} \frac{(1-6iy)}{(1+6iy)} \frac{(\omega + ie^{\frac{\pi}{4g_1}})}{(\omega - ie^{\frac{\pi}{4g_1}})} \ ; \tag{3.30}$$

$$\bar{S}_2 = \frac{(1+2iy)}{(1-2iy)} \frac{(\omega + ie^{\frac{\pi}{4g_1}})}{(\omega - ie^{\frac{\pi}{4g_1}})} \ , \tag{3.31}$$

where

$$y = \frac{g_1}{1 - \frac{4}{\pi} g_1 \ln \omega} \ . \tag{3.32}$$

It would appear that these expressions have a pole at $\omega = ie^{\frac{\pi}{4g_1}}$, but this is cancelled by a zero in $1+2iy$. If g_0 is positive we must multiply $\bar{S}_1$ and $\bar{S}_2$ by -1 in order that they will approach 1 as $g \to 0$.

APPENDIX

In this appendix we investigate the properties of the function

$$I(q) = \frac{1}{4}\int_0^1 \left\{\frac{e^{-qu}}{\cosh u} - 1\right\} \frac{du}{u} + \frac{1}{4}\int_1^\infty \frac{e^{-qu}}{\cosh u}\,\frac{du}{u} \;. \tag{3A.1}$$

Since $\cosh u \simeq e^u$ when u is large, this integral converges for Re $q > -1$. Thus $I(q)$ is analytic in q for Re $q > -1$. To obtain the analytic continuation of $I(q)$ to other values of q, we observe that if $-1 <$ Re $q < o$ and Im $q < o$, the path of integration of the second integral may be rotated through the upper half of the complex u-plane so that it ends at $u = -\infty$, provided we take into account the poles of the integrand that are passed. If we then make a change of variable we obtain

$$\begin{aligned} I(q) = {} & \frac{1}{4}\int_0^{-1} \left\{\frac{e^{qu}}{\cosh u} - 1\right\} \frac{du}{u} + \frac{1}{4}\int_{-1}^{\infty} \frac{e^{qu}}{\cosh u}\,\frac{du}{u+i\epsilon} \\ & + \frac{\pi i}{2}\sum_{n=1}^{\infty} \frac{e^{-iq(2n-1)\pi/2}}{(2n-1)\frac{i\pi}{2}\sinh\left[i(2n-1)\frac{\pi}{2}\right]} \;. \end{aligned} \tag{3A.2}$$

In the first integral we change the path so that it runs from $u = o$ to $u = 1$ and thence in the lower half plane to $u = -1$ and make the obvious cancellation. Also, since $\sinh i(2n-1)\frac{\pi}{2} = -i(-1)^n$ the infinite sum becomes a series for the arc tangent, which may be written as a logarithm. Thus we have

$$\begin{aligned} I(q) = {} & \int_0^1 \left\{\frac{e^{qu}}{\cosh u} - 1\right\} \frac{du}{u} + \frac{\pi i}{4} + \frac{1}{4}\int_1^\infty \frac{e^{qu}}{\cosh u}\,\frac{du}{u} \\ & + \frac{1}{2}\ln\left\{\frac{1 - i e^{-iq\pi/2}}{1 + i e^{-iq\pi/2}}\right\} \;. \end{aligned} \tag{3A.3}$$

By assumption Im $q < o$; hence $|e^{-iq\pi/2}| < 1$ and we define the logarithm by requiring that

$$-\frac{\pi}{2} < \operatorname{Im} \ln \left\{ \frac{1 - ie^{-iq\pi/2}}{1 + ie^{-iq\pi/2}} \right\} < \frac{\pi}{2} \quad ,$$

so that it will agree with the infinite sum.

Thus if $-1 < \operatorname{Re} q < o$, and $\operatorname{Im} q < o$,

$$I(q) = I(-q) + \frac{\pi i}{4} + \frac{1}{2} \ln \left\{ \frac{1 - ie^{-iq\pi/2}}{1 + ie^{-iq\pi/2}} \right\} \quad , \qquad (3A.4)$$

and this equation allows us to extend I(q) to the whole third quadrant since neither I(-q) nor the logarithm have any singularities there.

Since I(q) is real for $q > -1$ the Schwartz reflection principle shows that I(q) is also analytic inside the second quadrant. Thus I(q) is analytic in q everywhere except possibly on the negative real axis for q -1.

The mapping

$$J(q) = \frac{\pi i}{4} + \frac{1}{2} \ln \left\{ \frac{1 - ie^{-iq\pi/2}}{1 + ie^{-iq\pi/2}} \right\}$$

is shown in figure 4. We see that J(q) is real for real q except on the branch lines [$4n + 1 \leq q \leq 4n + 3$ for integral n], on which $\operatorname{Im} J(q) = \frac{\pi}{2}$ if approached from the lower half plane. Thus I(q) is real for negative q except on the branch lines. By the reflection principle then I(q) is analytic except on the branch lines $4n + 1 \leq q \leq 4n + 3$ for negative integers n. On the branch lines we have $\operatorname{Im} I(q \pm i\ ) = \mp \frac{\pi}{2}$. The branch lines of I(q) are shown in figure 3.

We need one formula involving I. We observe that (if $\operatorname{Re} q > -1$)

FIGURE 4

The mapping

$$J(q) = \frac{\pi i}{4} + \frac{1}{2} \ln \left\{ \frac{1 - ie^{-iq\pi/2}}{1 + ie^{-iq\pi/2}} \right\} .$$

The region outside the branch lines BCDEB in the q plane is mapped onto the region between the lines BCD and BED in the J plane. The lines DAB in the q plane are mapped onto the real axis of the J plane.

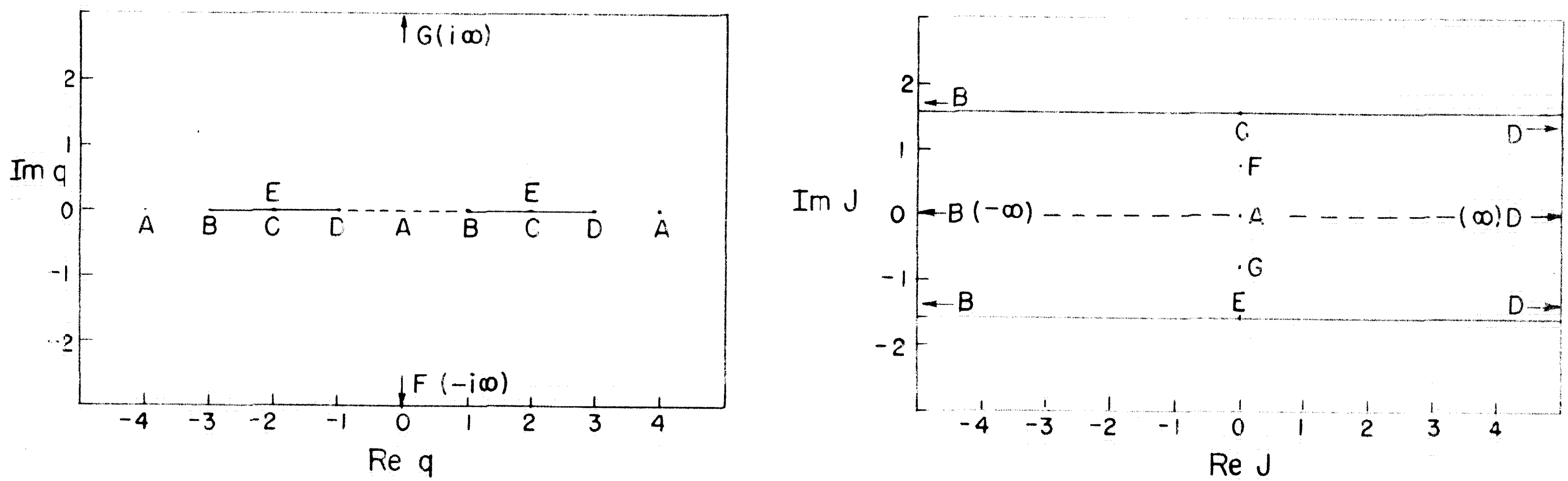

FIGURE 4

$$I(q+2) + I(q) = \frac{1}{4}\int_0^1 \left\{ \frac{e^{-qu}}{\cosh u}(1+e^{-2u}) - 2 \right\} \frac{du}{u} + \frac{1}{4}\int_1^\infty \frac{e^{-qu}}{\cosh u}(1+e^{-2u}) \frac{du}{u}$$

$$= \frac{1}{2}\int_0^1 \left\{ e^{-qu-u} - 1 \right\} \frac{du}{u} + \frac{1}{2}\int_1^\infty e^{-(q+1)u} \frac{du}{u}$$

$$= \frac{1}{2}\int_0^1 \left\{ e^{-x} - 1 \right\} \frac{dx}{x} + \frac{1}{2}\int_1^\infty e^{-x} \frac{dx}{x} - \frac{1}{2}\int_1^{q+1} \frac{dx}{x}$$

$$= c - \frac{1}{2} \ln (q+1) \quad , \tag{3A.5}$$

where c is a constant independent of q. By analytic continuation then

$$I(q+2) + I(q) = c - \frac{1}{2} \ln (q+1) \tag{3A.6}$$

for all q except $q < -1$, and we define the logarithm to be on its principal branch.

In summary, I(q) is an analytic function of q except for a series of branch lines on the negative real axis, shown in figure 3. Except on the branch line, I(q) is real when q is real. It satisfies the following identities:

$$I(q^*) = I^*(q) \qquad \text{(q not on the branch lines)} , \tag{3A.7}$$

$$\operatorname{Im} I(q \pm i\epsilon) = \mp \frac{\pi}{2} \qquad \text{(q on a branch line)} , \tag{3A.8}$$

$$I(q+2) + I(q) = c - \frac{1}{2} \ln (q+1) \qquad (q \nless -1) , \tag{3A.9}$$

$$I(q) = I(-q) + \frac{\pi i}{4} + \frac{1}{2} \ln \left\{ \frac{1 - ie^{-iq\pi/2}}{1 + ie^{-iq\pi/2}} \right\} \tag{3A.10}$$

($q \nless -1$ and $q \ngtr 1$ in equation 3A.10). In equation 3A.9

$$-\pi < \operatorname{Im} \ln < \pi .$$

In equation 3A.10

$$-\frac{3\pi}{2} < \operatorname{Im} \ln < \frac{\pi}{2} .$$

IV. SOLUTION OF THE SYMMETRIC SCALAR THEORY

In this chapter we present the power series solution of the one-meson approximation to the symmetric scalar theory; for comparison, we shall also discuss the solution of the charged scalar theory of Lee and Serber[(6)]. A method for obtaining the solution of the symmetric scalar theory is outlined in the appendix. Unlike the method reported in chapter II for obtaining the solution for the asymptotic form, the methods for obtaining the exact solution are either very cumbersome or very specialized; they show no promise for problems other than the symmetric scalar theory; thus we do not treat them in great detail. In the text we shall show that the solution we report satisfies the conditions (given in chapter I) that define the solution. We give the solution only for no cutoff i.e.

$$v^2(k) - 1 \quad . \tag{4.1}$$

We use the notation of chapter I. We observe that we may use the relation (equation 1.9 of chapter I)

$$S_\alpha(\omega) = 1 + 2i\,\frac{\sqrt{\omega^2-1}}{\omega}\,Q_\alpha(\omega) \tag{4.2}$$

to extend the definition of $S_\alpha(\omega)$ to the complex ω plane excluding the negative branch cut $\omega < -1$ (remember that $Q_\alpha(\omega)$ is defined on the positive branch cut $\omega > 1$ as the limit from the upper half complex plane; we define $S_\alpha(\omega)$ for $\omega > 1$ likewise). It is convenient for the purpose of this chapter to have the properties B to E of Q_α given in chapter I restated as properties of S_α. Since $\frac{\sqrt{\omega^2-1}}{\omega}$ is odd under the transformation $\omega \to -\omega$ we see that if

$$\sum_\beta A_{\alpha\beta} = 1 \tag{4.3}$$

then crossing symmetry (property D) gives

$$S_\alpha(-\omega) = \sum_\beta A_{\alpha\beta} S_\beta(\omega) . \tag{4.4}$$

Equation 4.3 is satisfied by the crossing matrices of both the charged and symmetric scalar theory. Thus properties B, C, D and E' (the unitarity condition used in the one-meson approximation) become

B.) $S_\alpha(\omega)$ is real for $-1 < \omega < 1$.

C.) $S_\alpha(\omega) = -\dfrac{2g^2 a_\alpha}{\omega}$ for very small ω .

D.) $S_\alpha(-\omega) = \sum_\beta A_{\alpha\beta} S_\beta(\omega)$ (ω not on a branch cut) ,

E'.) $|S_\alpha(\omega)| = 1$ $(\omega > 1)$.

The solution of Lee and Serber(6) for the charged scalar theory is as follows:

$$S_1(\omega) = \frac{1 - g^2\left[\dfrac{1 - i\sqrt{\omega^2-1}}{\omega}\right]}{1 - g^2\left[\dfrac{1 + i\sqrt{\omega^2-1}}{\omega}\right]} , \tag{4.5}$$

$$S_2(\omega) = \frac{1 + g^2\left[\dfrac{1 - i\sqrt{\omega^2-1}}{\omega}\right]}{1 + g^2\left[\dfrac{1 + i\sqrt{\omega^2-1}}{\omega}\right]} . \tag{4.6}$$

It is easy to verify that these expressions satisfy B, C, D, and E', and that scattering amplitudes obtained by means of equation 4.2 satisfy condition A and are analytic in the cut ω plane, when expanded in perturbation theory.

The important properties of the solution given above are as follows:

1.) They do not depend on $\ln \omega$ when ω is large; instead for large

ω, $S_\alpha(\omega)$ approaches a limit

$$S_\alpha(\infty) = \frac{1 + (g^2 a_\alpha) i}{1 - (g^2 a_\alpha) i} \quad . \tag{4.7}$$

2.) The perturbation expansion has a finite radius of convergence and is uniformly convergent for all ω when g^2 is small.

3.) The solution analytically extended beyond the radius of convergence of the perturbation expansion has no singularities other than poles; for real g^2 these poles occur only for $-1 < \omega < 1$ and can therefore be associated with bound states.

The presence of bound states changes the dispersion equation (equation 1.15 of chapter I) satisfied by the amplitudes, but this is of no particular importance since physically bound states are expected to occur when the coupling constant is large enough.

Now we present the solution of the symmetric scalar theory.

Define α as a power series in g^2 by the relation

$$g^2 = \alpha \left\{ (1-\alpha^2)\left(1 + \frac{2}{\pi} \tan^{-1}\alpha\right) - \frac{2}{\pi}\alpha \right\} , \tag{4.8}$$

and let

$$\phi = \tan^{-1}\alpha \quad . \tag{4.9}$$

Define γ and x and u by the equations

$$\gamma = \frac{1-\alpha^2}{\alpha}\left(1 + \frac{2}{\pi}\tan^{-1}\alpha\right) , \tag{4.10}$$

$$x = 1 + \frac{i\sqrt{\omega^2 - 1}}{\omega} \qquad \text{(}\omega\text{ in the cut plane or on the real axis approached from above),} \tag{4.11}$$

$$u = \left(\frac{2x}{1-x^2}\right)\gamma + 1 - \frac{4}{\pi}\tan^{-1}x \quad . \tag{4.12}$$

Equation 4.11 gives a one to one mapping of the cut ω -plane onto the interior of the unit circle in the x plane, and the first quarter of the unit circle in the x plane corresponds to the branch cut $\omega > 1$ approached from above. See figure 5. The function $\tan^{-1}x$ is defined by writing

$$\tan^{-1}x = \frac{1}{2i} \ln\left(\frac{1+ix}{1-ix}\right) \tag{4.13}$$

and requiring

$$-\frac{\pi}{2} < \operatorname{Im} \ln\left(\frac{1+ix}{1-ix}\right) < \frac{3\pi}{2} ,$$

for values of x inside the unit circle or on the first quarter of the unit circle (this corresponds to the usual definition of $\tan^{-1}x$, for $-1 < x < 1$). In terms of x

$$\omega = \frac{2x}{1+x^2} , \tag{4.14}$$

$$\sqrt{1-\omega^2} = \frac{1-x^2}{1+x^2} . \tag{4.15}$$

The two S matrix elements of the symmetric scalar theory are

$$S_1(\omega) = \left(\frac{u-1}{u+1}\right)\left(\frac{u+3}{u-3}\right)\left(\frac{1-\alpha^2/x^2}{1-\alpha^2x^2}\right) , \tag{4.16}$$

$$S_2(\omega) = \left(\frac{u-1}{u+1}\right)\left(\frac{1-\alpha^2/x^2}{1-\alpha^2x^2}\right) . \tag{4.17}$$

We now prove that expressions 4.16 and 4.17 satisfy the conditions of chapter I. First consider unitarity. For $\omega > 1$ i.e. x on the first quarter of the unit circle, we obtain

$$u = i\left\{\left(\frac{\omega}{\sqrt{\omega^2-1}}\right)\gamma - \frac{2}{\pi}\ln\left[\omega + \sqrt{\omega^2-1}\right]\right\} \tag{4.18}$$

FIGURE 5

Mapping of the cut ω plane into the unit circle of the x plane by the relation

$$\omega = \frac{2x}{1+x^2} .$$

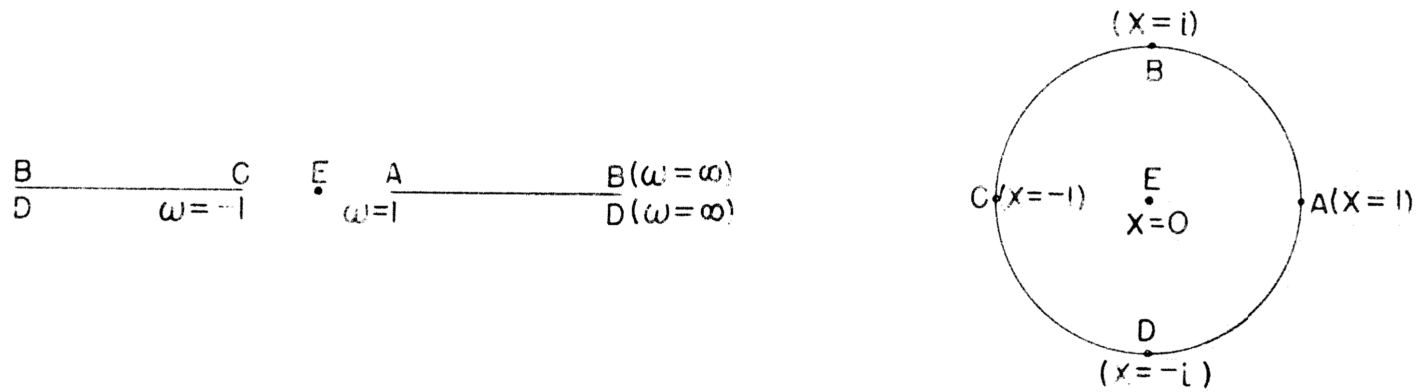

FIGURE 5

(to obtain this specific expression requires some algebra, but the fact that u is imaginary, which is all we need, follows from the fact that $\frac{1 + ix}{1 - ix}$ is positive imaginary when x is on the first quarter of the unit circle; hence Re $\tan^{-1} x = \frac{\pi}{4}$ and u is imaginary). Since $\frac{1}{x} = x^*$ when x is on the unit circle, we see that each of the factors in S_1 and S_2 is unitary, and thus condition E' is satisfied. Now look at crossing symmetry. The transformation $\omega \to -\omega$ becomes $x \to -x$; and since $\tan^{-1} x$ is odd we have

$$u(-x) = -u(x) + 2 \quad . \tag{4.19}$$

Thus

$$\begin{aligned} S_1(-\omega) &= \left(\frac{-u(x)+1}{-u+3}\right)\left(\frac{-u+5}{-u-1}\right)\left(\frac{1-\alpha^2/x^2}{1-\alpha^2 x^2}\right) \\ &= \left(\frac{u-1}{u+1}\right)\left(\frac{1-\alpha^2/x^2}{1-\alpha^2 x^2}\right)\left\{-\frac{1}{3}\,\frac{u+3}{u-3} + \frac{4}{3}\,\frac{u-3}{u-3}\right\} \\ &= A_{11} S_1(\omega) + A_{12} S_2(\omega) \quad , \end{aligned} \tag{4.20}$$

$$\begin{aligned} S_2(-\omega) &= \left(\frac{-u(x)+1}{-u+3}\right)\left(\frac{1-\alpha^2/x^2}{1-\alpha^2 x^2}\right) \\ &= \left(\frac{u-1}{u+1}\right)\left(\frac{1-\alpha^2/x^2}{1-\alpha^2 x^2}\right)\left\{\frac{2}{3}\,\frac{u+3}{u-3} + \frac{1}{3}\,\frac{u-3}{u-3}\right\} \\ &= A_{21} S_1(\omega) + A_{22} S_2(\omega) \end{aligned} \tag{4.21}$$

(the matrix $A_{\alpha\beta}$ for the symmetric scalar theory is given on page 8 of chapter I).

Near zero energy ($\omega \simeq 0$ or $x \simeq 0$) we have

$$u(x) \simeq 1 + 2\left(\gamma - \frac{2}{\pi}\right)x \quad , \tag{4.22}$$

$$\omega \simeq 2x \quad , \tag{4.23}$$

$$S_1(\omega) \simeq \frac{4}{\omega}\left(\gamma - \frac{2}{\pi}\right)\alpha^2 \quad , \tag{4.24}$$

$$S_2(\omega) \simeq -\frac{2}{\omega}\left(\gamma - \frac{2}{\pi}\right)\alpha^2 \quad . \tag{4.25}$$

By equations 4.8 and 4.10 (the numbers a_α are given in chapter I) we see that C is satisfied. Since u and x are real for $-1 < \omega < 1$, B is satisfied.

Since γ is a power series in g whose first term is $\frac{1}{g^2}$, and α a power series beginning with g^2, and since for large ω

$$u \simeq i\left\{\gamma - \frac{2}{\pi}\ln 2\omega\right\}, \tag{4.26}$$

it is easily verified that Q_α has a power series expansion in g whose individual terms satisfy condition A (see chapter I).

Finally we must show that the individual terms in the expansion are analytic in the cut ω plane. To do this it is sufficient to show that the Q_α are themselves analytic in the cut ω plane for small (real or complex) values of g, and that their power series expansion in g is uniformly convergent in finite regions of the cut ω plane, when g is small.

First we observe that for small g, α is small, and x is an analytic function of ω in the cut ω plane; thus the factor

$$\frac{1 - \alpha^2/x^2}{1 - \alpha^2 x^2}$$

is analytic. Secondly, u(x) is analytic for ω in the cut plane (which excludes $\omega = 1$ i.e. $x = 1$). Thus $S_\alpha(\omega)$ can fail to be analytic only because of poles at $u = -1$ or $u = 3$. It is shown in the appendix that for small (real or complex) values of g i.e. large values of γ, and values of x inside the unit circle not too near $x = \pm i$, the equations $u(x) = -1$ and $u(x) = 3$ each have exactly one root. The points $x = \pm i$ correspond to $\omega = \infty$ and hence do not concern us. However we observe that for $x = \alpha$

$$u(\alpha) = 2 + \frac{4}{\pi}\tan^{-1}\alpha + 1 - \frac{4}{\pi}\tan^{-1}\alpha - 3 , \tag{4.27}$$

$$u(-\alpha) = -2 - \frac{4}{\pi}\tan^{-1}\alpha + 1 + \frac{4}{\pi}\tan^{-1}\alpha = -1 . \tag{4.28}$$

Thus the roots we seek are $x = \alpha$ and $x = -\alpha$ for $u = 3$ and $u = -1$ respectively. However, at these points

$$1 - \alpha^2/x^2 = 0 . \tag{4.29}$$

Hence the apparent singularities in S_α are cancelled by a vanishing numerator. By the statement that $u(x) = -1$ or $u(x) = 3$ have only simple roots, the simple root of $1 - \alpha^2/x^2$ will completely kill the root of the denominator (i.e. it is impossible for the ratio $\frac{0}{0}$ to be ∞). Thus for small g^2 (real or complex) in finite regions of the cut ω plane, S_α and hence Q_α are analytic in ω . Furthermore, γ and α are analytic functions of g when g is small, since the equation relating g^2 to α is one to one when g^2 and α are small. We have already shown that S_α has no poles for small g, thus it is analytic in g and hence has a convergent power series expansion in g for small g. This completes our argument.

Thus we have indeed reported the power series solution of the symmetric scalar theory in the one-meson approximation.

Now we must consider its properties, considering in particular the questions proposed in chapter I.

First we observe that the S matrix elements S_α are meromorphic functions of the parameter ϕ, for any value of ω [i.e. they have poles, but no essential singularities for any value of ϕ]. This is not true, as a function of g^2. The value of g^2 is plotted versus the value of ϕ in figure 6. Evidently ϕ is a multiple valued function of g^2,

FIGURE 6

Plot of g^2 versus $\emptyset$, where g^2 is the renormalized coupling constant for the symmetric scalar theory, and $\emptyset$ a convenient parameter (see text).

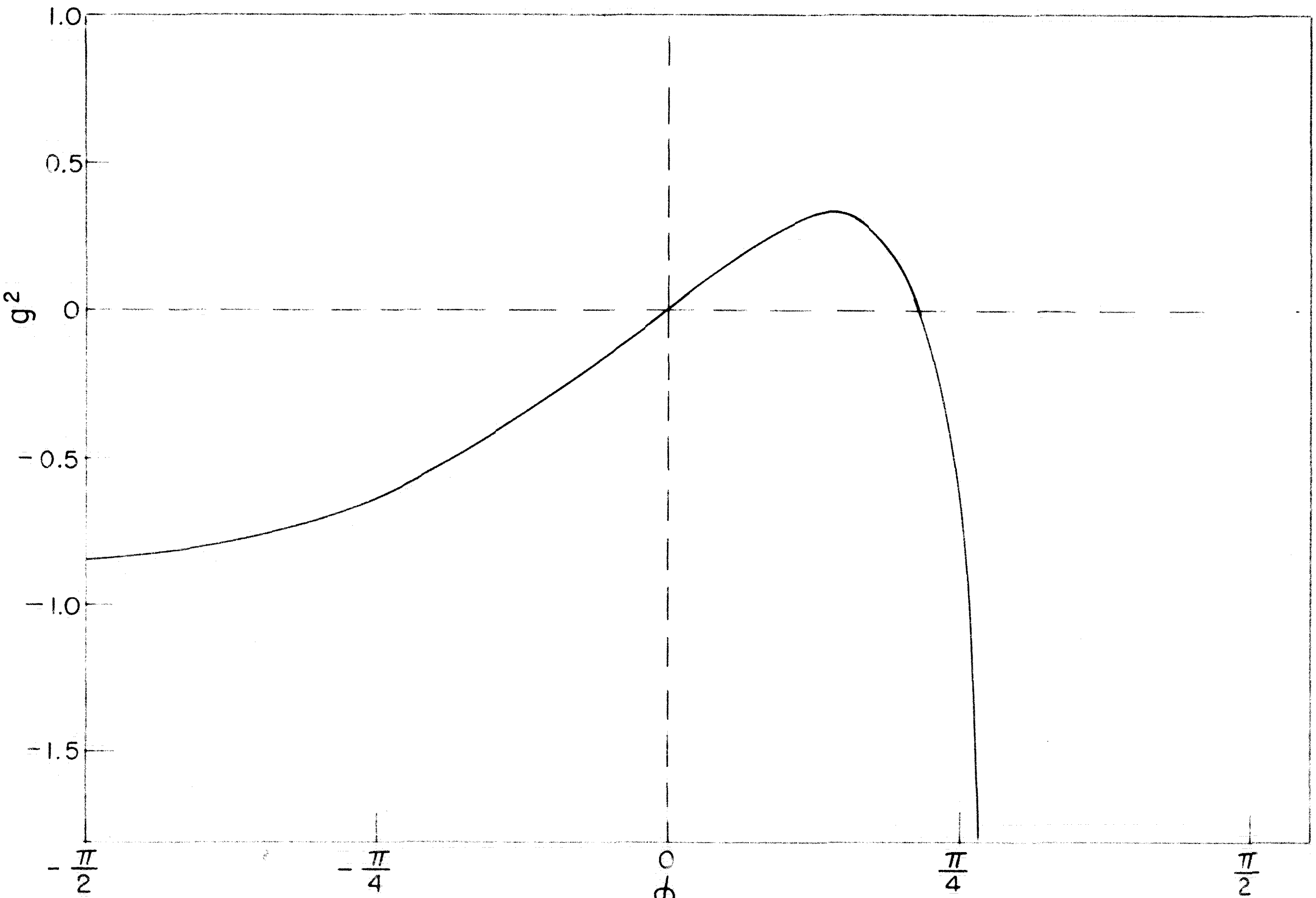

FIGURE 6

with a branch point at $g^2 \simeq .33$ and $\phi \simeq \frac{3}{5}\frac{\pi}{4}$. We shall see below that there are bound state poles except for

$$-\frac{\pi}{4} < \phi < \frac{\pi}{4} ,$$

but even in this region of no bound state poles, there are two solutions for each value of g^2 between $g^2 = -.64$ and $g^2 = .33$. It must be noted here that since g^2 and not g appears in the Low equation we can legitimately consider negative values of g^2, for which our solution still satisfies unitarity and crossing symmetry (unitarity is not satisfied if g^2 is complex). Because of the branch point at $g^2 = .33$, perturbation theory diverges for $|g^2| > .33$ (it may diverge for $|g^2| < .33$ depending on the value of ω).

Secondly, we inquire whether our solution $S_\alpha(\omega)$ has any singularities other than bound state poles (i.e. other than poles for $-1 < \omega < 1$). The possible causes for poles in $S_\alpha(\omega)$ are evidently

1.) $u(x) = 3$.

2.) $u(x) = -1$.

3.) $x = \pm 1/\alpha$.

We show in the appendix that for real γ , the equations $u(x) = 3$ and $u(x) = -1$ have exactly one root each for values of x inside or on the unit circle, and that this root occurs on the real axis. Since this corresponds to a root for $-1 < \omega < 1$ it simply means a bound state. Similarly $x = \frac{\pm 1}{\alpha}$ can occur only for real x, and again is just a bound state. We conclude that the sum of the power series extended beyond its radius of convergence remains a solution of the equations of the one-meson approximation except for the effect of bound states.

We saw that the roots of $u(x) = 3$ and $u(x) = -1$ do not cause poles in S_α when g^2 is small; this result evidently holds so long as $\tan^{-1}x$ has the same definition as $\tan^{-1}\alpha$ when $x = \alpha$, and providing $|\alpha| < 1$. This means that the roots of $u(x) = 3$ or -1 cause bound states unless $-\frac{\pi}{4} < \phi < \frac{\pi}{4}$. There is also a bound state for $x = \frac{1}{\alpha}$ if $|\alpha| > 1$, except as noted below.

A peculiarity of the solution is that it has another branch for complex ϕ, for which g^2 and δ and α^2 (but not α) are real, and hence unitarity, etc. are satisfied since the solution actually involves only δ and α^2. This other branch is for values of ϕ having the form

$$\phi = -\frac{\pi}{2} + i\psi \quad . \tag{4.30}$$

On this branch

$$\alpha = \tan\left(-\frac{\pi}{2} + i\psi\right) = \frac{-1}{\tan(i\psi)} = \frac{i}{\tanh\psi} \quad . \tag{4.31}$$

Since

$$1 + \frac{2}{\pi}\tan^{-1}\alpha = \frac{2}{\pi} i\psi$$

is imaginary, both g^2 and δ are real. It would appear that since $\alpha = \frac{i}{\tanh\psi}$ is larger than one in absolute value we now have a complex pole in $S_\alpha(\omega)$, at

$$x = \frac{1}{\alpha} = -i\tanh\psi \quad . \tag{4.32}$$

However this pole is cancelled by a zero in the denominator of S_α. The factor $u - 1$ appears in the denominator of both the S_α. It vanishes when

$$\delta - \frac{4}{\pi}\frac{(1-x^2)}{x}\tan^{-1}x = 0 \quad , \tag{4.33}$$

that is,

$$\frac{1-\alpha^2}{\alpha}\left(1+\frac{2}{\pi}\tan^{-1}\alpha\right) = \frac{1-x^2}{x}\left(\frac{2}{\pi}\tan^{-1}x\right) . \qquad (4.34)$$

Let $\alpha = -\frac{1}{\beta}$; this becomes

$$\frac{1-\beta^2}{\beta}\left(\frac{2}{\pi}\tan^{-1}\beta\right) = \frac{1-x^2}{x}\left(\frac{2}{\pi}\tan^{-1}x\right) , \qquad (4.35)$$

which holds for $x = \pm\beta$ if $-\frac{\pi}{4} < \tan^{-1}\beta < \frac{\pi}{4}$ i.e. $-\frac{3\pi}{4} < \emptyset < -\frac{\pi}{4}$. By extension it also holds for $\emptyset$ of the form $\emptyset = -\frac{\pi}{2} + i\psi$; but $x = \pm\beta$ means $x = \mp\frac{1}{\alpha}$ and hence for $-\frac{3\pi}{4} < \emptyset < -\frac{\pi}{4}$ or $\emptyset = -\frac{\pi}{2} + i\psi$, the points $x = \pm\frac{1}{\alpha}$ are not singular points of S_α.

This completes our discussion of the symmetric scalar theory. It has the property in common with the charged scalar theory of having no singularities other than bound states, but contrary to the charged scalar theory it is not a single-valued function of the coupling constant g^2. It is easy to see that its asymptotic form is the expression obtained in chapter II.

APPENDIX

I

In this appendix we prove four related statements. Let

$$u(x) = \gamma\left(\frac{2x}{1-x^2}\right) - \frac{4}{\pi}\tan^{-1}x + 1 , \tag{4A.1}$$

Then we show

A. For real γ and for x inside the unit circle, the equation $u(x) = 3$ has exactly one root x, which lies on the real axis.

B. For large, complex γ and for x inside the unit circle and not too close to $x = \pm i$, the equation $u(x) = 3$ has exactly one root.

C. and D. The same results hold for the equation $u(x) = -1$.

First we observe that

$$u(-x) = -u(x) + 2 , \tag{4A.2}$$

from which it follows that $u(x) = -1$ if and only if $u(-x) = 3$. Hence statements C and D follow from A and B.

To show that $u(x) = 3$ has a root x on the real axis, we observe that $u(x) = 3$ may be written (defining $\sigma(x)$ at the same time)

$$\gamma = \sigma(x) = \frac{1-x^2}{2x}\left\{2 + \frac{4}{\pi}\tan^{-1}x\right\} . \tag{4A.3}$$

Since $\sigma(x) = 0$ for $x = \pm 1$, and since $\sigma(x) \rightarrow \pm\infty$ as x goes to 0 from positive or negative values, respectively, it must take on all values in between. Thus $\sigma(x) = \gamma$ has a root for $-1 < x < 1$ if x is real.

Now to show that $\sigma(x) = \gamma$ has only one root we use the theorem that the number of roots minus the number of poles of a function in a region of the complex plane is given by the change of the argument of the function on one counterclockwise circuit around the boundary of the region. Thus we study the function $\sigma(x) - \gamma$ for x on the unit circle. Let us map the unit circle onto the upper half-plane by the mapping

$$z = i\left\{\frac{1-ix}{1+ix}\right\} \quad , \tag{4A.4}$$

$$x = -i\left\{\frac{1+iz}{1-iz}\right\} \quad . \tag{4A.5}$$

In terms of z, σ becomes

$$\begin{aligned} \sigma &= \frac{1}{2}\left\{ i\left(\frac{1-iz}{1+iz}\right) + i\left(\frac{1+iz}{1-iz}\right)\right\}\left\{ 2 - \frac{2}{\pi i}\ln(-iz)\right\} \\ &= \frac{1-z^2}{1+z^2}\left\{ 3i - \frac{2}{\pi}\ln z\right\} \quad , \end{aligned} \tag{4A.6}$$

where $0 \leq \text{Im} \ln z \leq \pi$. One revolution counterclockwise around the unit circle in the x-plane becomes one pass from $z = -\infty$ to $z = +\infty$ on the real axis and back on a large simicircle in the upper half plane.

Consider now $[\sigma(z) - \gamma]$, where γ is real. On the real axis of the z-plane, Im $\sigma(z)$ vanishes only for $z = \pm 1$ where Re $\sigma(z)$ vanishes also. On a large semicircle in the upper half z plane

$$\sigma(z) - \gamma \simeq -3i + \frac{2}{\pi}\ln z - \gamma \quad , \tag{4A.7}$$

and Im $[\sigma(z) - \gamma]$ does not vanish on this semicircle. Thus the only points on this path (real axis and semicircle) for which Im $[\sigma(z) - \gamma]$ vanishes are $z = \pm 1$, at which Re $[\sigma(z) - \gamma] = -\gamma$. On this path, therefore, $[\sigma(z) - \gamma]$ does not take on both positive and negative real values, and its change of argument on this path is zero. However, since $[\sigma(z) - \gamma]$ has only one pole (at $z = i$) inside this path it must have only one root inside the path. Since the location of the semicircle is arbitrary so long as $|z|$ is large on it, we have proved statement A.

Now suppose γ is large and complex. The points $x = i$ and $x = -i$ become $z = \infty$ and $z = o$ respectively; thus we should follow the argument of $[\sigma(z) - \gamma]$ on a path which excludes $z = o$. However provided the large semicircle is not too large and provided in going around $z = o$ the path does not approach $z = i$, $\sigma(z)$ will be bounded on the path and if $|\gamma|$ is large enough the change in argument of $[\sigma(z) - \gamma]$ will be zero. Thus statement B holds.

APPENDIX

II

In this appendix we outline a procedure for obtaining the solution of the symmetric scalar theory. It is not precisely the method used by the author; for his method moved mountains to reach molehills and is not worth repeating.

We begin by observing that the asymptotic form reported in chapter II for the symmetric scalar theory when expressed in terms of $\bar{S}_\alpha$ is

$$\bar{S}_1 = \left(\frac{1+iy}{1-iy}\right)\left(\frac{1-3iy}{1+3iy}\right), \tag{4A.8}$$

$$\bar{S}_2 = \left(\frac{1+iy}{1-iy}\right). \tag{4A.9}$$

Let $z = \frac{1}{y}$, then

$$\bar{S}_1 = \left(\frac{z+i}{z-i}\right)\left(\frac{z-3i}{z+3i}\right), \tag{4A.10}$$

$$\bar{S}_2 = \left(\frac{z+i}{z-i}\right). \tag{4A.11}$$

Let us forget the definition of y given in chapter II, and consider equations 4A.10 and 4A.11 without specifying what z is. We observe that by the results of the text if z is real for $\omega > 1$ and satisfies

$$z(-\omega) = -z(\omega) - 2i \tag{4A.12}$$

then the S_α satisfy crossing symmetry and unitarity. (Let $u = iz$ and refer to equations 4.20 and 4.21). However equations 4A.10 and 4A.11 are too restrictive; they relate two variables S_1 and S_2 to one variable z, and we cannot satisfy the other requirements on S_α for low energies with such a representation. If we could generalize these expressions to involve an independent parameter and preserve crossing symmetry and unitarity, we would be in a position to make progress. This is easy to do; let t be a function of ω even under crossing:

$$t(-\omega) = t(\omega) \tag{4A.13}$$

and let $t(\omega)$ be real for $\omega > 1$. Then the expressions

$$S_1 = \left(\frac{z+i}{z-i}\right)\left(\frac{z-3i}{z+3i}\right)\left(\frac{1+it}{1-it}\right) , \tag{4A.14}$$

$$S_2 = \left(\frac{z+i}{z-i}\right)\left(\frac{1+it}{1-it}\right) \tag{4A.15}$$

satisfy unitarity and crossing symmetry. This representation is evidently possible; the advantage of it is that we have related S_1 and S_2 to two functions z and t whose imaginary parts we know for both positive and negative real axis.

Specifically, we now examine the functions

$$\frac{\sqrt{\omega^2-1}}{\omega}\, z(\omega) \qquad \text{and} \qquad \frac{\omega}{\sqrt{\omega^2-1}}\, t(\omega) .$$

Call these functions z and t instead of our previous definition. We can find expressions for z and t in terms of Q_1 and Q_2; these turn out to be

$$z(\omega) = 3\left\{ \frac{1 + i\frac{\sqrt{\omega^2-1}}{\omega}\left[Q_1(\omega) + Q_2(\omega)\right]}{Q_2(\omega) - Q_1(\omega)} \right\} , \tag{4A.16}$$

$$t(\omega) = \frac{Q_1 + 2Q_2 + 2i\frac{\sqrt{\omega^2-1}}{\omega}Q_2\left[2Q_1 + Q_2\right]}{3 + 3i\frac{\sqrt{\omega^2-1}}{\omega}\left[Q_1 + 2Q_2\right] - 2\left(\frac{\omega^2-1}{\omega^2}\right)Q_2\left[2Q_1 + Q_2\right]} . \tag{4A.17}$$

From the unitarity and crossing conditions we see that

$$z(-\omega) = z(\omega) + 2i\frac{\sqrt{\omega^2-1}}{\omega} , \quad (4A.18)$$

$$t(-\omega) = -t(\omega) , \quad (4A.19)$$

for ω in the cut plane, besides which z and t are real for $-1 < \omega < 1$ and for $1 < \omega$ approached from above. Since Q_1 and Q_2 are supposed to be analytic in ω, the functions $z(\omega)$ and $t(\omega)$ will be analytic also except for poles where their denominators vanish. We can therefore write dispersion relations for z and t; these will involve some parameters for their poles but otherwise can be evaluated since we know the imaginary parts of z and t on the entire real axis (z has one for $\omega < -1$ and t has none). Now for small g^2 we assume Q_1 and Q_2 are approximated by their Born approximation $g^2 a_\alpha$ except for $\ln\omega \gtrsim \frac{1}{g^2}$ i.e. except for ω as large or larger than $e^{\frac{1}{g^2}}$. This allows us to obtain the number and approximate location of poles of $z(\omega)$ and $t(\omega)$. We neglect possible poles for $\omega \sim e^{\frac{1}{g^2}}$ (in practice they do not contribute to the power series, but may cause terms which do not have power series expansions). One then finds that z has a simple pole at $\omega = 0$, while t has two poles symmetrically located about $\omega = 0$ (by crossing symmetry). When the fact that $Q_\alpha(0) = g^2 a_\alpha$ exactly is included, and the dispersion relations calculated, one has

$$z(\omega) = \gamma - \frac{2}{\pi}\frac{\sqrt{\omega^2-1}}{\omega}\ln\left[\omega + \sqrt{\omega^2-1}\right] , \quad (4A.20)$$

$$t(\omega) = \frac{2\beta^2\omega}{\omega^2 - 2\beta^2} , \quad (4A.21)$$

where β and γ are constants depending on g: β is small and γ large when g is small. The constants β and γ can be determined by two requirements

1.) $Q_\alpha(o) = g^2 a_\alpha$.

2.) That $S_\alpha(\omega)$ have no singularities for small ω and g^2 except at $\omega = o$. In terms of the present definition of z, and for small ω

$$S_1(\omega) = \left\{ \frac{z(\omega) - \frac{1}{\omega}}{z(\omega) + \frac{1}{\omega}} \right\} \left\{ \frac{z + \frac{3}{\omega}}{z - \frac{3}{\omega}} \right\} \left\{ \frac{1 - \frac{t}{\omega}}{1 + \frac{t}{\omega}} \right\} , \tag{4A.22}$$

$$S_2(\omega) = \left\{ \frac{z - \frac{1}{\omega}}{z + \frac{1}{\omega}} \right\} \left\{ \frac{1 - \frac{t}{\omega}}{1 + \frac{t}{\omega}} \right\} , \tag{4A.23}$$

Evidently $z(\omega) + \frac{1}{\omega}$ and $z - \frac{3}{\omega}$ each vanish for some small value of ω no matter how large γ is. Thus we must choose t so that $t = \omega$ when $z = -\frac{1}{\omega}$ or $\frac{3}{\omega}$; this condition results in a relation between β and γ . The result is the solution given in the text.

V. REVIEW OF METHODS FOR IMPROVING PERTURBATION THEORY

In this chapter we discuss the concept of "improving perturbation theory", and review the determinantal method[8] and the Cini-Fubini method[10] for accomplishing this, for a particle scattering in a potential. We use the scattering length for the square well potential to illustrate.

We shall first define the determinantal function and then using it as an example consider the meaning of "improving perturbation theory".

The Schrödinger equation for the radial wave function u(r) of a particle scattering in an s state in a potential $\lambda V(r)$ is

$$\left\{ -\frac{d^2}{dr^2} + \lambda V(r) \right\} u(r) = E\,u(r) \quad . \tag{5.1}$$

We have set the appropriate constants equal to one and removed a factor of r from u(r) to simplify the equation. The factor λ allows us to vary the strength of the potential without changing its shape. E is the energy of the particle.

The S matrix element for a particle of energy E scattering in the potential is obtained by solving equation 5.1 for a function u(r) satisfying the boundary conditions

1. $u(o) = o$.
2. $u(r) = e^{-ikr} + Se^{ikr}$ for large r ,

where $k = \sqrt{E}$ and S is the S matrix element. The phase shift δ is given by $S = e^{2i\delta}$.

One may also solve equation 5.1 for a function v(r) satisfying the boundary conditions

1. $v(o) = 0$.

2. $\left.\frac{dv}{dr}\right|_{r=o} = 1$.

The function v(r), unlike u(r), is an analytic function of λ for all real and complex values of λ [this follows from the theory of differential equations; see reference (21)]. For large r the function v(r) has the form

$$v(r) \simeq D \frac{e^{-ikr}}{-2ik} + D^* \frac{e^{ikr}}{2ik} \quad . \tag{5.2}$$

The function D is the determinantal function, and is an analytic function of λ . Because the functions u(r) and v(r) can differ only by a constant factor we obtain

$$S = D^* / D \quad , \tag{5.3}$$

and hence the power series for S will diverge for values of λ greater than any value λ_o for which

$$D(\lambda_o) = 0 . \tag{5.4}$$

Now write the power series expansion of D and S:

$$D(\lambda) = \sum_{n=o}^{\infty} D_n \lambda^n \quad , \tag{5.5}$$

$$S(\lambda) - \sum_{n=o}^{\infty} S_n \lambda^n \quad . \tag{5.6}$$

Since the series for D converges for large λ while that for S does not we must have for sufficiently large n

$$\left| \frac{S_n}{S_o} \right| >> \left| \frac{D_n}{D_o} \right| \tag{5.7}$$

(actually this need not hold for all sufficiently large n; a more precise statement is that given any n_o there exists an $n > n_o$ for which inequality 5.7 holds). Equation 5.7 is the basis of the determinantal method for improving perturbation theory, which we must now explain.

For very small λ the scattering of the particle is given by the second term in the expansion of S (the first term is one):

$$S(\lambda) \simeq 1 + \lambda S_1 \quad . \tag{5.8}$$

For very accurate calculations one would include the second term $\lambda^2 S_2$, but this term will not change the qualitative features of the scattering.

For very large λ we cannot use the power series for S since it diverges; the series for D converges but many terms in the series must be calculated before a reasonable estimate of the sum can be obtained; in practice this is out of the question if say 1000 terms of the series must be calculated.

We are interested in the intermediate region, where λ is neither very large nor very small, and the second and third terms in the series for S and D are about the same size as the first term; we want to find a function which can be estimated roughly by the first few terms in its power series, preferably only three or four. Now if the inequality 5.7 extends to small values of n, i.e. n = 4 or 5, the determinantal function D will evidently be better for this purpose than the function S, which is the function usually expanded in perturbation theory. This is to say that the approximation

$$S \simeq \frac{D_0^* + \lambda D_1^* + \lambda^2 D_2^*}{D_0 + \lambda D_1 + \lambda^2 D_2} \tag{5.9}$$

is probably more accurate for intermediate values of λ than the approximation

$$S \simeq 1 + \lambda S_1 + \lambda^2 S_2 \quad . \tag{5.10}$$

Whether inequality 5.7 holds for small n must of course be determined separately for each scattering problem that one considers; we shall illustrate with the scattering length for the square well potential. For very small k the S matrix element has the form

$$S = 1 - 2ika \tag{5.11}$$

where a is the scattering length, and we neglect terms of order k^2. Let the potential V(r) be

$$V(r) = 1 \qquad (0 \le r \le 1) \; ; \; V(r) = 0 \qquad (1 \le r) \quad . \tag{5.12}$$

It is easy to obtain

$$a = 1 - \frac{\tanh\sqrt{\lambda}}{\sqrt{\lambda}} \quad , \tag{5.13}$$

which has the power series expansion[22]

$$a = \frac{\lambda}{3} - \frac{2}{15}\lambda^2 + \frac{17}{315}\lambda^3 - \frac{62}{2835}\lambda^4 + \cdots \quad . \tag{5.14}$$

The D function has the form

$$D = E + ikF \tag{5.15}$$

where the functions E and F are analytic in λ ; the scattering length a is

$$a = \frac{F}{E} \quad , \tag{5.16}$$

For the square-well potential (equation 5.12) E and F are

$$E = \cosh\sqrt{\lambda} \quad , \tag{5.17}$$

$$F \quad \cosh\sqrt{\lambda} - \frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}} \quad , \tag{5.18}$$

Expanded in power series

$$E = 1 + \frac{\lambda}{2} + \frac{\lambda^2}{24} + \frac{\lambda^3}{720} \quad , \tag{5.19}$$

$$F = \frac{\lambda}{3} + \frac{\lambda^2}{30} + \frac{\lambda^3}{840} + \frac{\lambda^4}{45360} + \cdots \qquad (5.20)$$

Omitting the fourth term in the series for E causes an error of 10% when $|\lambda| \simeq 4$, and at this value the fourth term in the series for F is negligible. However the fourth term in the series for a is 10% of the first for $|\lambda| = 2.5$. Thus we see that in this example the use of the first three terms in the series for the determinantal functions E and F is useful up to $|\lambda| \simeq 4$, as compared with the limit $|\lambda| \simeq 2.5$ if we use the first three terms in the series for a. Observe that for $|\lambda| = 4$, the second term in the series for E is twice the first term; thus using three terms of the series is essential to obtain qualitative as well as quantitative understanding of the scattering length.

Now we look at the other method for obtaining a good perturbation approximation. This is an approximation to the tangent of the phase shift obtained by Cini and Fubini(10) as the result of a variational principle of Lippman and Schwinger(9). Specifically Lippman and Schwinger (among other things) wrote down an integral for the tangent of the phase shift which is stationary with respect to the solution of the Schrodinger equation. Cini and Fubini used perturbation theory, with variable coefficients replacing powers of λ , as the trial function inside the integral, and found the values of the coefficients which make the integral stationary. One then gets a sequence of approximations to tan δ, depending on the number of terms from perturbation theory used in the trial function. It is easy to show that this sequence of approximations is the successive approximants to a continued fraction of the form

$$\tan\delta = \cfrac{a_1\lambda}{1 + a_2\lambda + \cfrac{a_3\lambda^2}{1 + a_4\lambda + \cfrac{a_5\lambda^2}{1 + \cdots}}} \qquad (5.21)$$

The numbers a_1, a_2, etc., are calculated by expanding the continued fraction in powers of λ and requiring this series to be the expansion of tan δ in powers of λ ; thus if the expansion of tan δ is

$$\tan\delta = b_1\lambda + b_2\lambda^2 + b_3\lambda^3 + \cdots , \tag{5.22}$$

we obtain

$$a_1 = b_1 , \tag{5.23}$$

$$-a_2 a_1 = b_2 , \tag{5.24}$$

$$a_2^2 a_1 - a_3 a_1 = b_3 , \tag{5.25}$$

etc.

The Cini-Fubini approximations to tan δ are obtained by keeping only an even number of the coefficients a_n; thus the first two approximations are

$$\tan\delta \simeq \frac{a_1\lambda}{1 + a_2\lambda} \tag{5.26}$$

and

$$\tan\delta \simeq \cfrac{a_1\lambda}{1 + a_2\lambda + \cfrac{a_3\lambda^2}{1 + a_4\lambda}} \tag{5.27}$$

To see the implications of these results look at the corresponding expressions for cot δ :

$$\cot\delta \simeq \frac{1}{a_1\lambda} + \frac{a_2}{a_1} , \tag{5.28}$$

$$\cot\delta \simeq \frac{1}{a_1\lambda} + \frac{a_2}{a_1} + \left(\frac{a_3}{a_1}\right)\frac{\lambda}{(1 + a_4\lambda)} . \tag{5.29}$$

As an example we examine the square well potential again (equation 5.12). At low energies k cot δ is well behaved; we obtain (neglecting terms of order k)

$$-\frac{1}{a} = k\cot\delta = -3\left\{ \frac{1}{\lambda} + \frac{2}{5} - \left(\frac{1}{525}\right)\cfrac{\lambda}{1 + \frac{2}{45}\lambda - \left(\frac{1}{6237}\right)\cfrac{\lambda^2}{1 + \frac{2}{117}\lambda - \cdots}} \right\} . \tag{5.30}$$

Here the third term $\frac{1}{525}\lambda$ contributes 10% of the first for $|\lambda| \simeq 6$. If we neglect the $\frac{1}{525}\lambda$ term it is just as if we were looking at the first two terms in the power series expansion of k cot δ .

We see that in this example the best simple but reasonable approximation for obtaining the phase shift for intermediate values of λ $(1 \leq |\lambda| \leq 6)$ is to approximate k cot δ (i.e. $\frac{k}{S-1}$) by keeping the first two terms in its power series expansion.

If we go to the next variational approximation i.e. neglect

$$\frac{1}{6237}\,\frac{\lambda^2}{1+\cdots}$$

in equation 5.30, we have an approximation good for $|\lambda| \leq 20$. For comparison, the series for S diverges at $|\lambda| = 2.5$. Evidently in this simple example (where perturbation approximations are unnecessary in practice since we know the exact solution) the determinantal or Cini-Fubini methods give considerably better perturbation approximations than the series for S directly, if we go just one or two terms beyond the Born approximation.

I am indebted to Jon Shirley who pointed out to me the excellent behaviour of the series for cot δ for the purpose of obtaining a perturbation approximation to S for the Schrödinger equation.

VI. TEST OF THE DETERMINANTAL AND CINI-FUBINI METHODS USING THE SYMMETRIC PSEUDOSCALAR THEORY

In this chapter we examine the determinantal method and the Cini-Fubini variational method for improving perturbation theory, as applied to the symmetric pseudoscalar theory in the one-meson approximation.

In the previous chapter we explained what is meant by "improving perturbation theory" and illustrated by means of the square well potential how the determinantal method and the Cini-Fubini method do just that. For field theory, Marshall Baker[(8)] has shown how to write the S matrix elements S_α as a ratio of two functions $\frac{D_\alpha}{D_\alpha}$, by using analogy with the Schrodinger theory (we shall give the formulas below), but he cannot claim that the functions D_α are analytic functions of the coupling constant g^2 when g^2 is large (the exact solution for the symmetric scalar theory reported in chapter IV requires that D_α be a multiple-valued function of g^2 in that example). Similarly the Cini-Fubini method can be applied without change since to calculate one needs to know only the power series for tan δ, but there is no variational principle which says that the approximants of the continued fraction should give a good approximation to tan δ. Thus from a strictly objective viewpoint we might just as well be looking at functions like

$$\mathrm{gd}\left\{ J_0(\delta) \right\}$$

(gd is the gudermannian function) to see if their power series show any better convergence properties than ordinary perturbation series. However, since the determinantal method and the Cini-Fubini method existed as definite proposals at the time this thesis was prepared, and since the tests reported in this chapter turn out favorably, I have not bothered to look for other methods. For simplicity, I have looked at the power series for the reciprocal of the scattering amplitude instead of the continued fraction which results from the Cini-Fubini method; since we shall see that the power series for the reciprocal is quite useful there seemed no point in looking at the continued fraction.

The example we use for testing these methods is the symmetric pseudoscalar theory in the one-meson approximation, using the cutoff proposed by Salzman and Salzman:

$$v^2(k) = e^{-k^2/49} \tag{6.1}$$

(we use the notation of chapter I). This example has the double advantage that we can compare perturbation approximations with the numerical results of Salzman and Salzman, and that the Salzmans' results describe the qualitative features of low energy pion-nucleon p-wave scattering (i.e. small phase shifts in the $\alpha = 1$ and $\alpha = 2$ states but a resonance in the $\alpha = 3$ state), whereas the Born approximation does not.

First we must establish some matters of notation and definition. Baker defines the determinantal function by the equations

$$\frac{D_\alpha^*(\omega)}{D_\alpha(\omega)} = S_\alpha(\omega) \qquad (\omega > 1), \tag{6.2}$$

$$D_\alpha(\omega) = 1 + \frac{\omega}{\pi}\int_1^\infty \frac{\operatorname{Im} D_\alpha(\omega_1)}{\omega_1(\omega_1 - \omega - i\epsilon)}\, d\omega_1 . \tag{6.3}$$

We assume expansions of the form

$$S_\alpha(\omega) = \sum_{n=0}^\infty g^{2n} S_{\alpha,n}(\omega) , \tag{6.4}$$

$$D_\alpha(\omega) = \sum_{n=0}^{\infty} g^{2n} D_{\alpha n}(\omega) \quad . \tag{6.5}$$

It is not difficult to see that if the terms $S_{\alpha n}$ are known for all values of ω , then the functions $D_{\alpha n}(\omega)$ can be calculated by means of equations 6.2 and 6.3 (of course the calculation of $D_{\alpha n}$ assumes that one has already computed $D_{\alpha,n-1}$, $D_{\alpha,n-2}$ etc.). For this chapter only, we redefine the function Q_α:

$$S_\alpha(\omega) = 1 + 2i\frac{k^3 v^2(k)}{6\pi\omega} Q_\alpha(\omega) \quad . \tag{6.6}$$

This changes the unitarity condition to

$$\operatorname{Im} Q_\alpha(\omega) = \frac{k^3 v^2(k)}{6\pi\omega} |Q_\alpha(\omega)|^2 \qquad (\omega > 1) \quad . \tag{6.7}$$

With this definition of Q_α, but with the requirements A, B, C and D of chapter I unchanged, the coupling constant g^2 becomes the rationalized renormalized coupling constant. Salzman and Salzman's calculations assume $g^2 = 1$. Define also

$$L_\alpha(\omega) = \operatorname{Re} \frac{1}{Q_\alpha(\omega)} \quad , \tag{6.8}$$

and let

$$L_\alpha(\omega) = \sum_{n=0}^{\infty} g^{2n-2} L_{\alpha n}(\omega) \quad . \tag{6.9}$$

It has been shown previously that the determinantal method shows some agreement with Salzman and Salzman's numerical results[8], and that the Cini-Fubini method is in rough agreement with experiment[23]. The purpose of this investigation is

1.) To compare these methods quantitatively with Salzman and Salzman.

2.) To discover whether the terms in the series for D and L become small after the first few, so that there is a genuine basis for using these functions in perturbation approximations.

To accomplish this the expansions of D_α and L_α have been calculated through n = 8. Preliminary computations were performed on the Burroughs 205 at Cal Tech, and more precise calculations were done on the IBM 709 at the Western Data Processing Center and on the IBM 704 at the M.I.T. Computation Center. Details on how the calculation was done are given in the appendix.

The principal results of this calculation are presented in the three tables and three figures which follow.

Tables II and III

In tables II and III we present the power series expansions through n = 8 of $L_\alpha(\omega)$ and $D_\alpha(\omega)$, for all three states α and for $\omega = 0$ and $\omega = 7.43$ (ω is given in units of the meson mass). For presenting these power series we adopt a standard format. First we give the first term of the series (under the heading "B.A." for "Born Approximation"). Then for each n we give the n^{th} term of the series divided by the first term. For example, for $\alpha = 1$ and $\omega = 0$ the series for $L_1(\omega)$ is

$$L_1(0) = \frac{-1}{g^2}\left\{.25 + .135\,g^2 - .0081\,g^4 - \cdots\right\} . \tag{6.10}$$

Our presentation is equivalent to rewriting this series as

$$L_1(0) = \frac{-.25}{g^2}\left\{1 + .540\,g^2 - .032\,g^4 - \cdots\right\} . \tag{6.11}$$

In tables II and III the functions $\mathrm{Re}D_\alpha(\omega)$ and $\mathrm{Im}D_\alpha(\omega)$ are treated as separate functions. Also, as $\omega \to 1$ each term in the series for $\mathrm{Im}D_\alpha(\omega)$ vanishes as $\sqrt{\omega^2-1}$, but evidently in our method of presenting the series this affects only the term labeled "B.A.", for when divided by the leading term the series for $\mathrm{Im}D_\alpha(\omega)$ is finite at $\omega = 1$. Thus from Table II we see that for $\omega \simeq 1$,

$$\frac{\mathrm{Im}\,D_{1n}(\omega)}{\mathrm{Im}\,D_{11}(\omega)} = 1.000 \qquad (n=1)$$

$$= -.116 \qquad (n=2) \qquad \text{etc.} \tag{6.12}$$

The numbers given in tables II and II are accurate to $\pm$ 2 in the last digit except for n = 6, 7, or 8 where the numbers give an indication only and are not guaranteed to be accurate.

Table II

Power Series Expansion of $L_\alpha(\omega)$ and $D_\alpha(\omega)$ for $\omega = 0$.

See page 98 .

$\alpha = 1$

n	L	ReD	ImD
B.A.	-.2500	1.000	0
0	1.000	1.000	0
1	.540	.424	1.000
2	-.032	-.128	-.116
3	.016	-.024	-.033
4	.030	.001	-.026
5	.001	.000	-.014
6	-.018	.007	.010
7	-.010	.010	.021
8	.000	.002	.008

$\alpha = 2$

n	L	ReD	ImD
B.A.	-1.000	1.000	0
0	1.000	1.000	0
1	.048	.106	1.000
2	.260	.016	.058
3	.062	-.042	-.247
4	-.031	-.030	-.107
5	-.037	.007	.067
6	-.019	.023	.086
7	.003	.012	.022
8	.021	-.005	-.030

$\alpha = 3$

n	L	ReD	ImD
B.A.	+.500	1.000	0
0	1.000	1.000	0
1	-.444	-.212	1.000
2	-.032	-.128	.232
3	-.032	-.052	.007
4	.006	-.006	-.009
5	.025	.015	-.008
6	.012	.019	-.015
7	-.008	.008	-.006
8	-.015	-.008	.010

Table III

Power Series Expansion of $L_\alpha(\omega)$ and $D_\alpha(\omega)$, for $\omega = 7.43$.

See page 98 .

$\alpha = 1$	n	L	ReD	ImD
	B.A.	-.2500	1.000	3.771
	0	1.000	1.000	0
	1	-1.064	-1.477	1.000
	2	-.062	.316	-.413
	3	-.031	.093	-.062
	4	.043	.043	.033
	5	.050	.029	.019
	6	.008	-.020	.017
	7	-.026	-.052	.015
	8	-.040	-.033	-.003

$\alpha = 2$	n	L	ReD	ImD
	B.A.	-1.000	1.000	.943
	0	1.000	1.000	0
	1	-.576	-.369	1.000
	2	.495	-.039	.207
	3	.196	.178	-.415
	4	.039	.112	-.358
	5	-.032	-.050	.032
	6	-.060	-.103	.250
	7	-.034	-.040	.178
	8	.022	.037	-.021

$\alpha = 3$	n	L	ReD	ImD
	B.A.	.5000	1.000	-1.885
	0	1.000	1.000	0
	1	-.088	.738	1.000
	2	-.062	.316	.827
	3	-.018	-.007	.450
	4	.051	-.110	.102
	5	.041	-.081	-.109
	6	-.018	-.021	-.159
	7	-.047	.029	-.079
	8	-.022	.048	.049

Table IV

In this table we present the expansions of $L_\alpha(\omega)$, $Q_\alpha(\omega)$ and $L_\alpha(-\omega + i\epsilon)$, for $\alpha = 1$ and $\omega = 7.43$. For the purpose of this table only, we define

$$L_\alpha(\omega) = \frac{1}{Q_\alpha(\omega)} \quad . \tag{6.13}$$

In this table the real and imaginary parts of L_α and Q_α are treated as part of the same function; thus for $\alpha = 1$ and $\omega = 7.43$

$$L_\alpha(\omega) = -.\frac{2500}{g^2}\left\{1 + [-1.064 + 3.771\,i]\,g^2 - .062\,g^4 - \cdots\right\}. \tag{6.14}$$

The numbers for $Q_\alpha(\omega)$ and $L_\alpha(-\omega + i\epsilon)$ have been rounded off to the nearest integer or (in the higher orders) to four significant figures.

Table IV

Power Series for $L_\alpha(\omega)$, $Q_\alpha(\omega)$, and $L_\alpha(-\omega)$, for $\omega = 7.43$ and $\alpha = 1$.

Here we define

$$L_\alpha(\omega) = \frac{1}{Q_\alpha(\omega)} \quad .$$

n	ReL	ImL	ReQ	ImQ	ReL(-ω)	ImL(-ω)
B.A.	-.2500		-4.000		+.2500	
0	1.000	0	1	0	1	0
1	-1.064	3.771	1	-4	0	-2
2	-.062	0	-13	-8	-2	-1
3	-.031	0	-44	40	-7	10
4	.043	0	104	208	36	42
5	.050	0	893	-170	225	-103
6	.008	0	314	-3534	-180	-1086
7	-.026	0	-12930	-4947	-4859	-490
8	-.040	0	-32350	43250	-7733	20220

Figure 7

To show the agreement between partial sums of the series for L_α and the numerical results of Salzman and Salzman, we have computed the partial sums*

$$\sum_{n=0}^{n_0} \frac{L_{\alpha n}(\omega)}{L_{\alpha 0}(\omega)}$$

for $1 \leq n_0 \leq 8$, for $\alpha = 1$ and $\alpha = 3$, and for $3 \leq n_0 \leq 8$ for $\alpha = 2$. We define

$$g_\alpha(\omega) = \frac{g^2 a_\alpha}{Q_\alpha(\omega)} \tag{6.15}$$

so that these partial sums are partial sums for the function Re $g_\alpha(\omega)$. We then plot the Salzmans' results for Re $g_\alpha(\omega)$ as a continuous curve, and the range of values of the partial sums are shown for discrete points. For example, at $\omega = 4$ it was found that

$$1.67 \leq \sum_{n=0}^{n_0} \frac{L_{2n}(\omega)}{L_{20}(\omega)} \leq 1.79 , \tag{6.16}$$

for $3 \leq n_0 \leq 8$.

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* To compare with Salzman and Salzman we have put $g^2 = 1$.

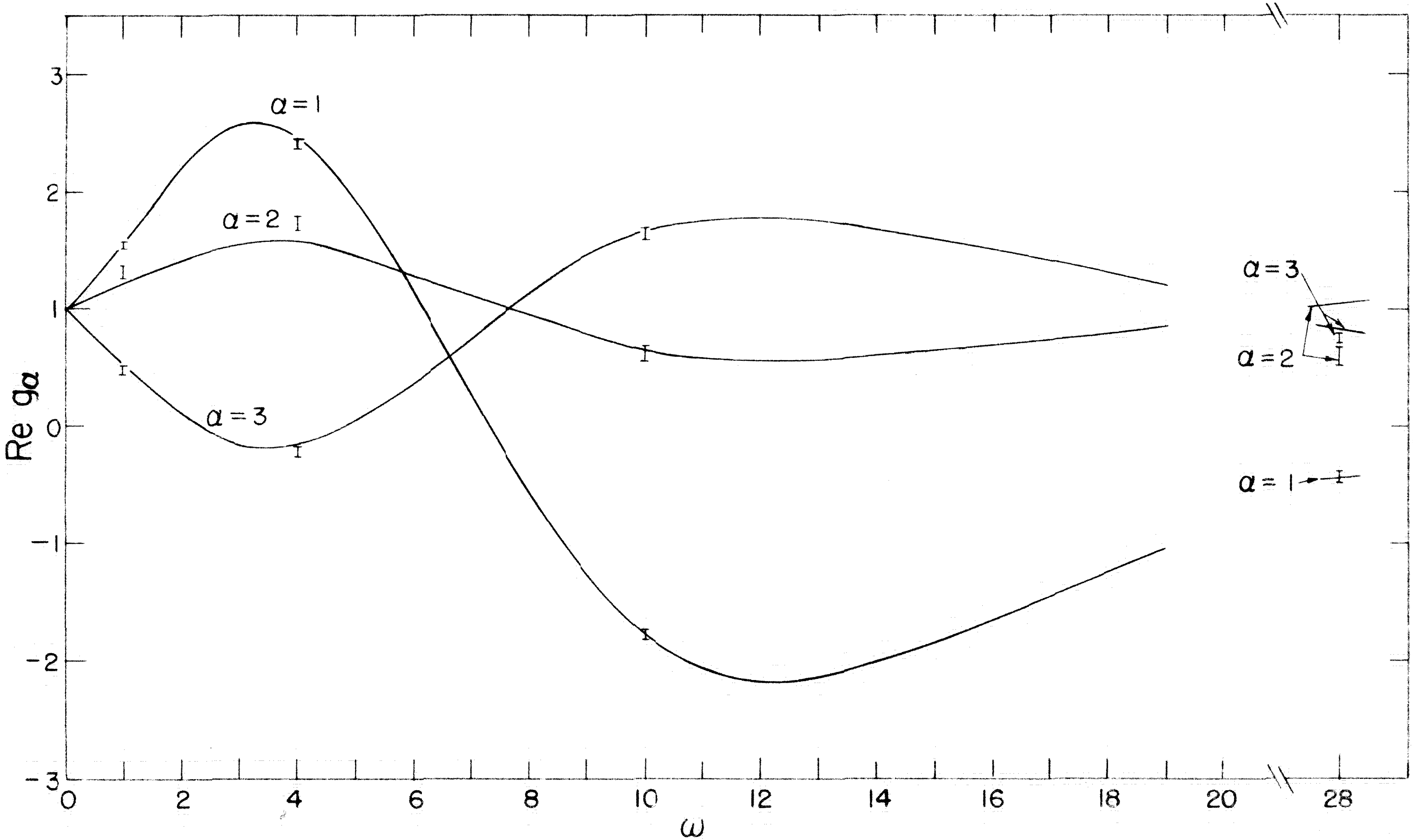

Re g_α
3
2
1
0
-1
-2
-3
0
2
4
6
8
10
12
14
16
18
20
28
ω
$\alpha=1$
$\alpha=2$
$\alpha=3$
$\alpha=3$
$\alpha=2$
$\alpha=1$

FIGURE 7

Figure 8

To show the agreement between partial sums of the series for D_α and the results of Salzman and Salzman, we have computed the partial sums

$$E_{\alpha n_0}(\omega) = \sum_{n=0}^{n_0} D_{\alpha n}(\omega) \quad , \tag{6.17}$$

and computed g_α from the formula

$$\mathrm{Re}\ g_\alpha(\omega) = \left\{ \frac{\mathrm{Re}\ E_{\alpha n_0}(\omega)}{\mathrm{Im}\ E_{\alpha n_0}(\omega)} \right\} \mathrm{Im}\ D_{\alpha 1}(\omega) \tag{6.18}$$

Once again we plot Salzman's results for Re $g_\alpha(\omega)$ as a continuous curve and show the range of values of Re $g_\alpha(\omega)$ obtained from equation 6.18, for discrete points. For $\alpha = 1$ we show the results for $4 \leq n_0 \leq 8$; for $\alpha = 2$ we take $6 \leq n_0 \leq 8$ and for $\alpha = 3$ we take $5 \leq n_0 \leq 8$. Note that we are limiting n_0 more than we did for L_α.

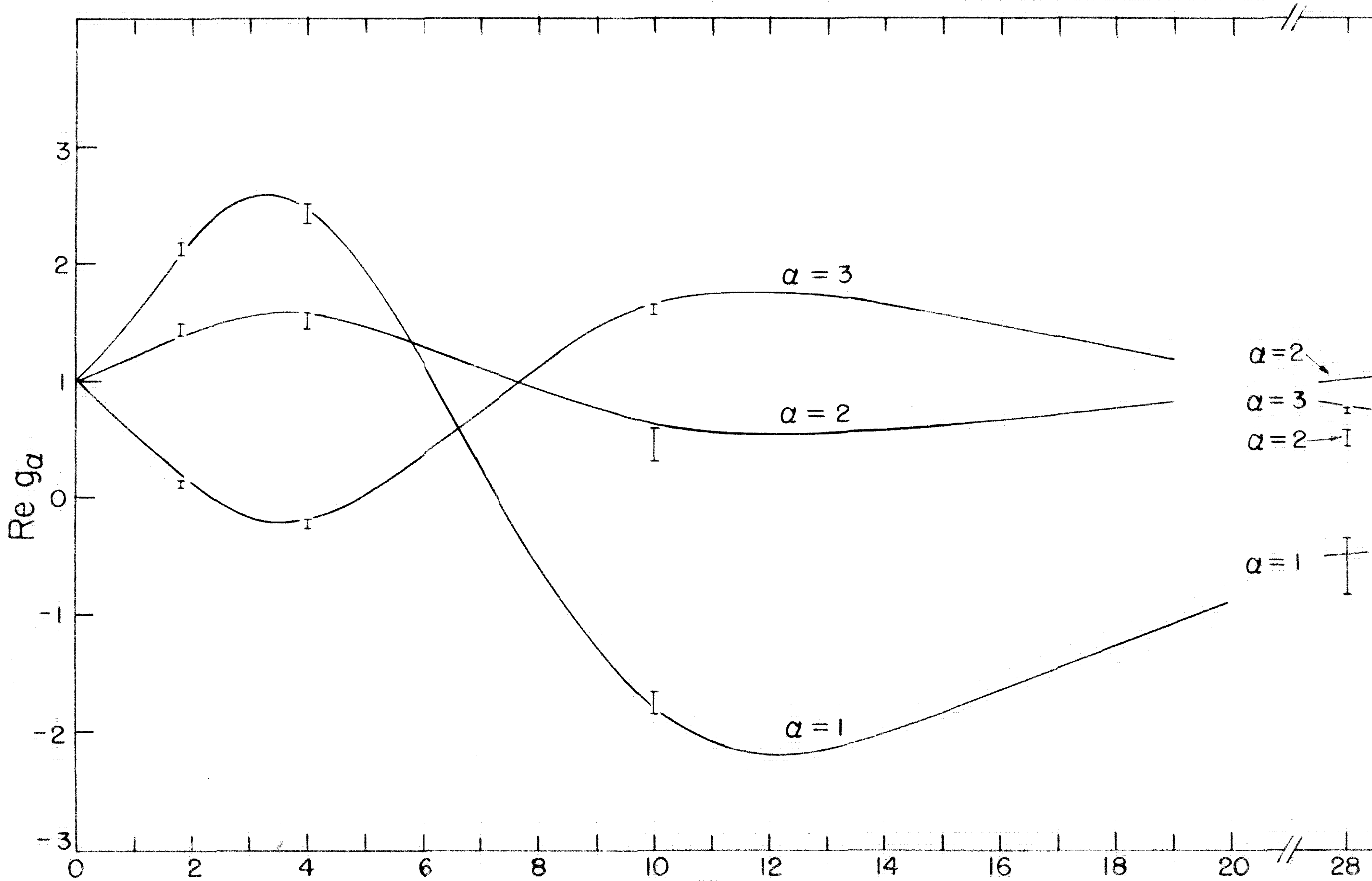
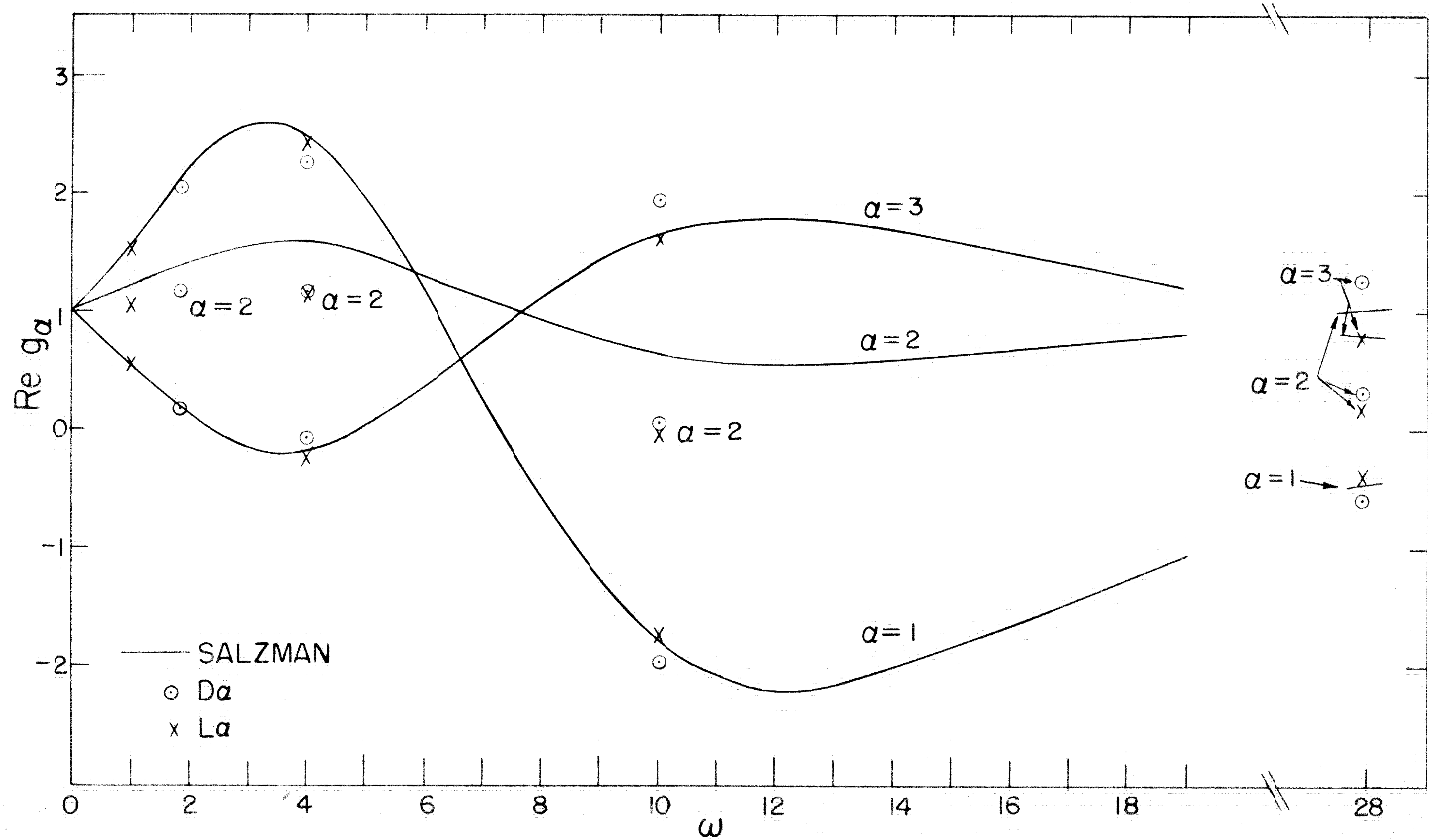

Re g_α
3
2
1
0
-1
-2
-3
0
2
4
6
8
10
12
14
16
18
20
28
ω
$\alpha = 3$
$\alpha = 2$
$\alpha = 1$
$\alpha = 2$
$\alpha = 3$
$\alpha = 2$
$\alpha = 1$

FIGURE 8

Figure 9

In figure 9 we show the agreement of simple determinantal or Cini-Fubini approximations with Salzman's results. Precisely we compare the sum of the first two terms in the series for L_α, and the sum of the first three terms in the series for D_α.



Re g_α
3
2
1
0
-1
-2
$\alpha=3$
$\alpha=2$
$\alpha=2$
$\alpha=2$
$\alpha=2$
$\alpha=1$
$\alpha=3$
$\alpha=2$
$\alpha=1$
SALZMAN
$D\alpha$
$L\alpha$
0
2
4
6
8
10
12
14
16
18
28
ω

FIGURE 9

It should be noted that Salzman's numbers for the $\alpha = 2$ state were not in very good agreement with the Low equation, and in particular at high energy since the limit

$$\lim_{\omega \to \infty} \mathrm{Re}\, g_\alpha(\omega)$$

should be independent of α. In Salzman's results this limit is the same for the $\alpha = 1$ and $\alpha = 3$ states but is quite different in the $\alpha = 2$ state. Thus the power series results for the $\alpha = 2$ state may be more accurate than Salzman's, but since the determinantal numbers disagree with the numbers calculated from the series for L_α, we cannot say which is correct until all the results can be tested by substitution into the Low equation, and this I have not done.

The curves I have shown for the Salzmans' results have been copied from their figure 2 [reference (5)]. I have defined $g_\alpha(\omega)$ to agree with their definition, and their coupling constant f^2 is related to our g^2 by the equation

$$f^2 = g^2/4\pi \quad . \tag{6.19}$$

This completes the presentation of the results of the calculations.

There are a number of noteworthy features in the results. First consider the power series for L_α. We see that in the $\alpha = 1$ and $\alpha = 3$ states, the third and higher terms are generally a few percent of the first term, and in the $\alpha=2$ state the fifth and higher terms are a few percent. We are unable to say whether these series actually converge or not, but since positive and negative terms are about equally frequent, we see that the partial sums of the series do not change very much (figure 7). Since the partial sums in the $\alpha = 1$ and $\alpha = 3$ states are dominated by the first two terms, we can get a rough estimate for the radius of convergence of the series for

$$Q_\alpha(\omega) = \frac{1}{L_\alpha(\omega) - i\frac{k^3 v^2(k)}{6\pi\omega}} \tag{6.20}$$

by looking at Salzman's results in figure 7. Whenever

$$|\,\mathrm{Re}\; g_1(\omega) - 1\,| > 1 \;,$$

or

$$|\,\mathrm{Re}\; g_3(\omega) - 1\,| > 1 \;,$$

the series for the corresponding Q_α will diverge for $g^2 = 1$. The minimum radius of convergence encountered for Q_α is $g^2 \simeq .25$ for $\alpha = 1$ and $\omega = 7.5$ (see table IV); here the divergence is due to $\frac{k^3 v^2(k)}{6\pi\omega}$ being four times the leading term of L_α, when $g^2 = 1$. On the other hand, near $\omega = 1$ the series for Q_α seems to be moderately convergent.

The difference between the behaviour of the series for Q_α and for L_α is best seen from the fact that for any value of ω the higher order terms for L_α range from a few percent of the leading term to zero, while for Q_α the higher order terms are sometimes 1000 times the leading term and sometimes .1 times the leading term, depending on the value of ω.

There is less distinction between the series for L_α and the series for D_α; in fact, they are remarkably similar.

We observe that the higher order terms in the expansion of $\frac{L_\alpha(\omega)}{L_{\alpha 0}(\omega)}$ are small for all positive values of ω, yet can become very large for negative values of ω: see table IV.

Now look at the agreement of the determinantal method and the method of using the power series for L_α, with the results of Salzman. They both work quite well, but to get a small spread in the results for the determinantal method I had to include a minimum of four terms whereas in the $\alpha = 1$ and $\alpha = 3$ state I could consider a minimum of two terms from the series for L_α, and this is an advantage if the same is true in more complicated problems, where calculating five terms in the series is not so easy. In figure 9 we compare the sum of the first two terms of the series for L_α with the sum of the first three terms of the series for D_α (note that if two terms in L_α are known the first three terms of D_α can be calculated), and we see that in the $\alpha = 1$ and $\alpha = 3$ states, the L_α approximation is somewhat better, but both are poor in the $\alpha = 2$ state. Furthermore, the result that we get good agreement in the $\alpha = 1$ and $\alpha = 3$ states but not the $\alpha = 2$ state is easily predicted, by looking at the third term in the series for L_α.

I feel that the conclusion to be drawn from this is that the approximation of Cini and Fubini namely

$$L_\alpha(\omega) \simeq \frac{1}{g^2} L_{\alpha 0}(\omega) + L_{\alpha 1}(\omega) \tag{6.21}$$

can be very good in field theory but that one must calculate $L_{\alpha 2}(\omega)$ and determine whether it is small, before using this approximation seriously.

APPENDIX TO CHAPTER VI

The purpose of this appendix is to explain the details of the calculation of the power series expansion of L_α and D_α.

First let us outline the simplest method for performing this calculation, and then discuss the difficulties with it. The simplest approach is to use the following equations, each expanded in powers of g^2:*

$$\mathrm{Im}\, Q_\alpha(\omega) = \frac{k^3 v^2(k)}{6\pi\omega} \left| Q_\alpha(\omega) \right|^2 \qquad (\omega > 1), \qquad (6A.1)$$

where $k = \sqrt{\omega^2 - 1}$,

$$\mathrm{Re}\, Q_\alpha(\omega) = g^2 a_\alpha + \frac{\omega}{\pi} \mathrm{P.V.} \int_1^\infty \frac{\mathrm{Im}\, Q_\alpha(\omega_1)}{\omega_1(\omega_1 - \omega)} d\omega_1 \qquad (6A.2)$$

$$+ \sum_\beta A_{\alpha\beta} \frac{\omega}{\pi} \int_1^\infty \frac{\mathrm{Im}\, Q_\beta(\omega_1)}{\omega_1(\omega_1 + \omega)} d\omega_1 \qquad (\omega > 1),$$

$$\mathrm{Im}\, D_\alpha(\omega) = -\frac{k^3 v^2(k)}{6\pi\omega} Q_\alpha(\omega) D_\alpha(\omega) \qquad (\omega > 1), \qquad (6A.3)$$

$$D_\alpha(\omega) = 1 + \frac{\omega}{\pi} \mathrm{P.V.} \int_1^\infty \frac{\mathrm{Im}\, D_\alpha(\omega_1)}{\omega_1(\omega_1 - \omega)} d\omega_1 \qquad (\omega > 1), \qquad (6A.4)$$

$$L_\alpha(\omega) = \frac{1}{Q_\alpha(\omega)} , \qquad (6A.5)$$

These five equations allow one to calculate the perturbation series for L_α and D_α to any order one might require. To see how they look we write these equations as they appear when Q_α, D_α and L_α are expanded in powers of g^2. Let

---

* In this appendix, we define L_α differently than in the text, but the change is trivial.

$$Q_\alpha(\omega) = \sum_{n=0}^{\infty} g^{2+2n} \, Q_{\alpha n}(\omega) \quad , \tag{6A.6}$$

$$D_\alpha(\omega) = \sum_{n=0}^{\infty} D_{\alpha n}(\omega) \, g^{2n} \quad , \tag{6A.7}$$

$$L_\alpha(\omega) = \sum_{n=0}^{\infty} g^{2n-2} \, L_{\alpha n}(\omega) \quad . \tag{6A.8}$$

Equations 6A.1 to 6A.5 become

$$\mathrm{Im}\, Q_{\alpha 0}(\omega) = 0 \quad , \tag{6A.9}$$

$$(n>0): \quad \mathrm{Im}\, Q_{\alpha n}(\omega) = \frac{k^3 v^2(k)}{6\pi\omega} \left\{ \sum_{m=0}^{n-1} Q_{\alpha m}(\omega)\, Q^*_{\alpha, n-1-m}(\omega) \right\} , \tag{6A.10}$$

$$\mathrm{Re}\, Q_{\alpha 0}(\omega) = a_\alpha \, , \tag{6A.11}$$

$$(n>0): \quad \mathrm{Re}\, Q_{\alpha n}(\omega) = \frac{\omega}{\pi} \, \mathrm{P.V.} \int_1^\infty \frac{\mathrm{Im}\, Q_{\alpha n}(\omega_1)}{\omega_1(\omega_1 - \omega)} \, d\omega_1 + \sum_\beta A_{\alpha\beta} \frac{\omega}{\pi} \int_1^\infty \frac{\mathrm{Im}\, Q_{\beta n}(\omega_1)}{\omega_1(\omega_1 + \omega)} \, d\omega_1 \, , \tag{6A.12}$$

and similarly for D_α; equation 6A.5 may be written

$$Q_{\alpha 0}(\omega)\, L_{\alpha n}(\omega) = - \sum_{m=0}^{n-1} L_{\alpha m}(\omega)\, Q_{\alpha, n-m}(\omega) \, . \tag{6A.13}$$

Our equations are now almost exactly as they would be written in FORTRAN, if the program were to be run on a 704, except for the method of doing the integrals.

In a machine calculation we calculate the functions $Q_{\alpha n}(\omega)$ and $D_{\alpha n}(\omega)$ at a finite number of points; call these points ω_m, where $1 \leq m \leq m_0$. To do the integrals we can use Simpson's rule (the principal value singularity requires special treatment: see below); the simplest way is to choose the points ω_m to be equally spaced between $\omega = 1$ and about $\omega = 16$ at which point the cutoff has become very small. If we let

$$\Delta\omega = \omega_{m+1} - \omega_m \tag{6A.14}$$

be the spacing of the points, we can immediately write down the approximation for the crossed integral in 6A.12:

$$\frac{\omega}{\pi}\int_1^\infty \frac{\mathrm{Im}\, Q_{\beta n}(\omega_1)}{\omega_1(\omega_1+\omega)}\, d\omega_1 \simeq \sum_{m=1}^{m_0} \theta_m \frac{\mathrm{Im}\, Q_{\beta n}(\omega_m)\Delta\omega}{\omega_m(\omega_m+\omega)} , \tag{6A.15}$$

where

$$\begin{aligned} \theta_1 &= \theta_{m_0} = \tfrac{1}{3} , \\ \theta_2 &= \theta_4 = \cdots = \theta_{m_0-1} = 4/3 , \\ \theta_3 &= \theta_5 = \cdots = \theta_{m_0-2} = 2/3 , \end{aligned} \tag{6A.16}$$

and m_0 must be odd.

To take care of the principal value integral, we observe that we need to calculate these integrals only when ω has one of the values ω_m, for we must know Re $Q_{\alpha n}(\omega_m)$ in order to calculate Im $Q_{\alpha,n+1}(\omega_m)$, which we need to calculate the integrals in the next order. Now we can subtract from the principal value integral another integral if the latter is zero. In particular suppose we are computing the principal value integral for $\omega = \omega_m$ where m is even. Then

$$\begin{aligned} \frac{\omega}{\pi}\, \mathrm{P.V.}\int_1^\infty \frac{\mathrm{Im}\, Q_\alpha(\omega_1)}{\omega_1(\omega_1-\omega)}\, d\omega_1 &= \frac{\omega}{\pi}\int_1^{\omega_{m-1}} \frac{\mathrm{Im}\, Q_\alpha(\omega_1)}{\omega_1(\omega_1-\omega)}\, d\omega_1 \\ &+ \frac{\omega}{\pi}\int_{\omega_{m+1}}^\infty \frac{\mathrm{Im}\, Q_\alpha(\omega_1)}{\omega_1(\omega_1-\omega)}\, d\omega_1 \\ &+ \frac{\omega}{\pi}\int_{\omega_{m-1}}^{\omega_{m+1}} \left\{ \frac{\mathrm{Im}\, Q_\alpha(\omega_1)}{\omega_1} - \frac{\mathrm{Im}\, Q_\alpha(\omega)}{\omega} \right\} \frac{d\omega_1}{\omega_1-\omega} . \end{aligned} \tag{6A.17}$$

The first two integrals are done by Simpson's Rule; the last one no longer has a singular integrand since the denominator vanishes when the numerator does, so we can again use Simpson's Rule. Since we cannot calculate the integrand at $\omega_1 = \omega$ we approximate it as the average of the values of the integrand at the two endpoints. Thus

$$\frac{\omega}{\pi} \int_{\omega_{m-1}}^{\omega_{m+1}} \left\{ \frac{\operatorname{Im} Q_\alpha(\omega_1)}{\omega_1} - \frac{\operatorname{Im} Q_\alpha(\omega)}{\omega} \right\} \frac{d\omega_1}{\omega_1 - \omega}$$

$$\simeq \frac{\omega}{\pi} \Delta\omega \left\{ \frac{\operatorname{Im} Q_\alpha(\omega_{m-1})}{\omega_{m-1}} \frac{1}{\omega_{m-1} - \omega_m} + \frac{\operatorname{Im} Q_\alpha(\omega_{m+1})}{\omega_{m+1}} \frac{1}{\omega_{m+1} - \omega_m} \right\} \tag{6A.18}$$

(the term we subtracted cancels out since $\omega_{m-1} - \omega_m = -[\omega_{m+1} - \omega_m]$) All this amounts to is revising the coefficients of Simpson's Rule. A similar result can be obtained if m is odd; the relevant changes in equations A.16 are

1.) m even

$$\theta_{m-1} = \frac{4}{3}, \quad \theta_m = 0, \quad \theta_{m+1} = \frac{4}{3}.$$

2.) m odd

$$\theta_{m-1} = \frac{5}{3}, \quad \theta_m = 0, \quad \theta_{m+1} = 5/3.$$

If $\omega = 1$ the integrand becomes singular at $\omega_1 = 1$, the endpoint of the integration. Because $\operatorname{Im}Q_{\alpha n}$ vanishes as $\sqrt{\omega^2-1}$ at $\omega = 1$ the integrand is integrable, but our methods do not give a good approximation to this integral. However it is not necessary to do any calculations for $\omega = 1$ since we know $\operatorname{Im} Q_{\alpha n}$ vanishes for $\omega = 1$. There is no difficulty at the upper endpoint since we can simply take our last value ω_{m_0} large enough so that the integrand is negligibly small at ω_{m_0} even if there is a principal value singularity there (of course if $\omega = \omega_{m_o}$ we put $\theta_{m_o} = 0$, and we can set $\theta_{m_o-1} = \frac{5}{3}$ as usual, but since the integrand is small we may neglect the terms which come from $\omega_1 - \omega_{m_o+1}$ etc.).

The difficulty with the above scheme is the following. In table IV we show the power series expansion of $L_\alpha(\omega)$, $Q_\alpha(\omega)$ and $L_\alpha(-\omega)$ for $\alpha = 1$ and $\omega = 7.43$. According to the above scheme we compute the expansion of Q_α, and then by means of equation 6A.13 calculate the expansion of L_α. From Table IV, it is evident that this requires extracting very small numbers from very large numbers; for example, for $\alpha = 1$, $\omega = 7.43$, and $n = 8$ the individual terms on the right hand side of equation 6A.13 are each about 50,000, whereas the answer is about .05; to get this number to 10% accuracy requires knowing the individual terms on the right hand side to one part in 10^7 ! This requires doing the integrals to enormous accuracy. Unfortunately the computer program if efficiently written spends all its time doing sums such as equation 6A.15, and therefore the total time the program takes varies as the square of the number of values m_o for ω that we take. Thus we have to see how to calculate the integrals to high accuracy without using a large number of points.

The example shown in Table IV is extreme; usually the series for Q_α is not so divergent, but there is still a problem of accuracy, and one part in 10^6 or 10^7 is highly desirable. The first thing we need is a more accurate integration formula. There are two principal sources of error in our simple Simpson's Rule formula:

1.) Im $Q_{\alpha n}(\omega)$ behaves as $\sqrt{\omega^2-1}$ when $\omega \simeq 1$.

2.) Im $Q_{\alpha n}(\omega)$ decreases rapidly when ω is large .

The behaviour of Im$Q_{\alpha n}$ for $\omega \simeq 1$ causes considerable error because

Simpson's Rule, whose error is normally proportional to $(\Delta\omega)^4$, has a considerably larger error when the integrand or its derivatives become very large within the range of integration. The second difficulty arises because one would like most of the points ω_m to be in the region where Im $Q_{\alpha n}(\omega)$ is large; for large ω the integrand is quite small, and yet if we do not use enough points in the large energy region, the integrand will change by a factor of 5 or 10 between points ω_m and Simpson's Rule will be incorrect by 50% or more.

The way to circumvent these difficulties is as follows. The essence of Simpson's Rule is to fit the integrand by a parabola and integrate the parabola instead. Since the above difficulties arise from the nature of the known function $k^3v^2(k)$, we can look for an integration formula in which only the function

$$\frac{\mathrm{Im}\, Q_{\alpha n}(\omega)}{k^3 v^2(k)}$$

is fitted by a parabola; this is a function which has continuous derivatives of all orders for $1 \leq \omega \leq \infty$ and for large ω does not drop off sharply. One such formula is what we may call a "modified Simpson's Rule with a kernel". In its general form, we consider the integral

$$I = \int_{\omega_A}^{\omega_B} f(\omega)\, g(\omega)\, d\omega \quad , \tag{6A.19}$$

where $f(\omega)$ is the kernel, supposed to be a known function, and $g(\omega)$ is a function unknown except for its values at three points ω_1, ω_2 and ω_3. We can now write down a formula for I in terms of $g(\omega_1)$, $g(\omega_2)$, $g(\omega_3)$, whose coefficients are integrals over $f(\omega)$; we shall require that this formula be <u>exact</u> if $g(\omega)$ is a polynomial of second degree in ω. The required formula is

$$I = \sum_{m=1}^{3} r_m\, g(\omega_m) \quad , \tag{6A.20}$$

where

$$r_m = \int_{\omega_A}^{\omega_B} f(\omega)\; P_m(\omega)\; d\omega \,, \tag{6A.21}$$

and

$$\begin{aligned} P_1(\omega) &= \frac{(\omega-\omega_2)(\omega-\omega_3)}{(\omega_1-\omega_2)(\omega_1-\omega_3)} \,, \\ P_2(\omega) &= \frac{(\omega-\omega_1)(\omega-\omega_3)}{(\omega_2-\omega_1)(\omega_2-\omega_3)} \,, \\ P_3(\omega) &= \frac{(\omega-\omega_1)(\omega-\omega_2)}{(\omega_3-\omega_1)(\omega_3-\omega_2)} \,. \end{aligned} \tag{6A.22}$$

The $Pm(\omega)$ are all second degree polynomials for which we have required 6A.20 to be exact; equating 6A.19 and 6A.20 gives 6A.21. If $f(\omega)$ is identically one we obtain the ordinary form of Simpson's Rule.

The beauty of equation 6A.20 is that we may take the kernel to be

$$f(\omega_1) = \frac{k^3 v^2(k)}{\omega_1(\omega_1-\omega)} \,, \tag{6A.23}$$

that is, to include the principal value singularity. For the integral

$$I = \frac{\omega}{\pi}\, P.V. \int_1^\infty \frac{\operatorname{Im} Q_{\alpha n}(\omega_1)\, d\omega_1}{\omega_1(\omega_1-\omega)} \,, \tag{6A.24}$$

the range of integration will be divided into many intervals, and the result will be of the form

$$I \simeq \sum_{m=1}^{m_0} r_m \frac{Q_{\alpha n}(\omega_m)}{k_m^3\, v^2(k_m)} \,. \tag{6A.25}$$

If $\omega = \omega_m$ where m is even then r_m will be a simple principal value integral. If $\omega = \omega_m$ where m is odd (but not 1) then r_m will be the sum of two terms:

$$r_m = \lim_{\epsilon \to 0} \left\{ \int_{\omega_{m-2}}^{\omega_m - \epsilon} f(\omega) \frac{(\omega - \omega_{m-1})(\omega - \omega_{m-2})}{(\omega_m - \omega_{m-1})(\omega_m - \omega_{m-2})} d\omega \right.$$

$$\left. + \int_{\omega_m + \epsilon}^{\omega_{m+2}} f(\omega) \frac{(\omega - \omega_{m+1})(\omega - \omega_{m+2})}{(\omega_m - \omega_{m+1})(\omega_m - \omega_{m+2})} d\omega \right\} , \qquad (6A.26)$$

and since the polynomials in ω both approach 1 as $\omega \to \omega_m$ this limit exists. Whether m is even or odd the r_ℓ for ℓ near m will be ordinary integrals since even if they appear to have a singularity at $\omega = \omega_m$ this is cancelled by the vanishing of the polynomial at $\omega = \omega_m$.

Thus in a more accurate calculation one starts by computing the r_m; these can be done with many points, and one may change the variable of integration to eliminate the singularity at $\omega = 1$ (e.g. to $t = \sqrt{\omega^2 - 1}$). One is then free to choose the points ω_m to fit one's best guess for the behaviour of $\frac{\operatorname{Im} Q_{\alpha n}(\omega)}{k^3 v^{-2}(k)}$.

Besides improving the accuracy of the numerical approximations to the integrals, it seemed worthwhile to rearrange the calculation so that dispersion integrals were calculated for the $L_{\alpha n}(\omega)$ directly instead of the Re $Q_{\alpha n}(\omega)$; since $Q_{\alpha n}(\omega)$ and $L_{\alpha n}(\omega)$ have the same analytic properties they satisfy the same type of dispersion relation. One might think that because $L_{\alpha n}(\omega)$ is consistently small for positive values of ω its imaginary part for negative values of ω might also be small, but this is not the case (see Table IV). The precise calculations used a more complicated set of formulae which included a dispersion relation for $L_\alpha(\omega)$ instead of $Q_\alpha(\omega)$, but whether this accomplished much I do not know. These formulae are (they must be expanded in perturbation theory before use):

$$\operatorname{Im} Q_\alpha(\omega) = \frac{k^3 v^{-2}(k)}{6\pi\omega} \, | Q_\alpha(\omega) |^2 , \qquad (6A.27)$$

$$Q_\alpha(-\omega-i\epsilon) = -\sum_\beta A_{\alpha\beta}\, Q_\beta(\omega) \qquad (\omega \geq 1)\,, \tag{6A.28}$$

$$G_\alpha(\omega) = |\, Q_\alpha(-\omega-i\epsilon)\,|^2 \qquad (\omega \geq 1)\,, \tag{6A.29}$$

$$H_\alpha(\omega) = \frac{6\pi\omega}{k^3 v^{-2}(k)} \; \frac{\mathrm{Im}\, Q_\alpha(-\omega-i\epsilon)}{G_\alpha(\omega)} \qquad (\omega \geq 1)\,, \tag{6A.30}$$

$$\begin{aligned} L_\alpha(\omega) = {} & \frac{1}{g^2 a_\alpha} + \frac{\omega}{\pi} \int_1^\infty \frac{k_1^3 v^{-2}(k_1)}{6\pi\omega_1^2} \; \frac{H_\alpha(\omega_1)\, d\omega_1}{\omega_1+\omega} \\ & - \frac{\omega}{\pi} \int_1^\infty \frac{k_1^3 v^{-2}(k_1)}{6\pi\omega_1^2} \; \frac{d\omega_1}{\omega_1-\omega-i\epsilon} \qquad (\omega \geq 1)\,, \end{aligned} \tag{6A.31}$$

$$Q_\alpha(\omega) = \frac{1}{L_\alpha(\omega)} \qquad (\omega \geq 1)\,. \tag{6A.32}$$

In the actual calculations these functions were calculated for forty values of ω , and as a check on the calculation the program was rerun with the functions calculated for 29 values of ω ; for the first six orders there was good agreement, but in the last three orders some discrepancies arose due to the numerical approximations to the integrals losing accuracy. The actual calculations we report in Tables II and III are probably accurate except for n = 8 but we can claim accuracy with complete confidence only for $n \leq 5$, where the two calculations agree.

The final calculation took ten minutes on the IBM 704; much more time was required to debug the program.

We complete this appendix with some remarks more relevant to the calculations of pion-pion scattering (see chapter VIII), which presents similar problems. The calculation of the pion-pion scattering amplitudes is computationally equivalent to the above, except with the cutoff taken to be

$$v^2(k) \simeq \frac{1}{k^2} \tag{6A.33}$$

when k is large; this presents a new difficulty in computing the integrals in equation 6A.12. The amplitudes $Q_{\alpha n}$ now behave logarithmically in ω when ω is large, as shown in chapter II. Therefore they have an essential singularity at $\omega = \infty$ and we cannot choose this as one of the points ω_m. We have then a maximum value ω_{m_0} for which we shall compute $Q_{\alpha n}(\omega)$. Consider now the problem of calculating Re $Q_{\alpha n}(\omega_{m_0})$; the difficulty is that the integral for Re $Q_{\alpha n}(\omega_{m_0})$ requires that we know Im $Q_{\alpha n}(\omega)$ for $\omega > \omega_{m_0}$. This problem did not arise in the cutoff case because we could choose ω_{m_0} large enough so that the integrals for $\omega > \omega_{m_0}$ were negligible. It is possible to do the same here but a suitable value of ω_{m_0} is $\omega_{m_0} \simeq 10^{100}$, which is impractical. Thus we must use an integral formula which extrapolates Im $Q_{\alpha n}(\omega)$ beyond $\omega = \omega_{m_0}$. I have adopted the procedure of writing the Simpson's Rule formula for $\omega_{m_0-2} \leq \omega \leq \omega_{m_0}$ so that it is exact if Im $Q_{\alpha n}(\omega)$ is a second order polynomial in $\ln\omega$, and then letting the integrals for the coefficients r_m run to ∞ instead of ω_{m_0}. Looking at the general formula for the modified Simpson's Rule, equations 6A.20 to 6A.22, the change required to make it exact if $g(\omega)$ is a second degree polynomial in $\ln\omega$ is to write

$$P_1(\omega) = \frac{(\ln\omega - \ln\omega_2)(\ln\omega - \ln\omega_3)}{(\ln\omega_1 - \ln\omega_2)(\ln\omega_1 - \ln\omega_3)},$$

etc.

In conclusion I shall list the bugs that appeared in a FORTRAN program prepared to carry out the above ideas--not the routine mistakes, but details that one might not notice.

1.) The function $\frac{\mathrm{Im}\, Q_{\alpha n}(\omega)}{k^3 v^2(k)}$ must be computed for $\omega = 1$; since it has the form $\frac{0}{0}$ care must be exercised in its computation. The IBM 704 is designed so that normally it does not stop when it divides by 0; instead it sets the result equal to 0 and continues.

2.) The IBM 704 treats numbers smaller than 10^{-37} as if they were very large numbers ($\sim 10^{37}$), or by special arrangement they can be replaced by 0. Either way may give difficulty, thus one must be careful when k is so large that $k^3 v^2(k)$ becomes a number $\sim 10^{-16}$ or so (since the program will involve terms proportional to $[k^3 v^2(k)]^2$). (This difficulty of course occurs only for the Gaussian cutoff, and not in the pion-pion problem.)

3.) The integration formulae 6A.20 to 6A.22 leaves the spacing of the points ω_1, ω_2 , and ω_3 arbitrary, and this is a useful feature; the only point one must watch is that sometimes the function $\frac{1}{\omega - \omega_m}$ will have a singularity <u>just</u> <u>beyond</u> the range of integration, and when this happens one must be sure the integral 6A.21 is computed with enough points to be accurate.

4.) When the modified Simpson's Rule is used to calculate the integral from ω_{m_0-2} to ∞ , one must choose the points ω_{m_0-2} , ω_{m_0-1} , and ω_{m_0} sufficiently far apart so that the function that it fits to $\mathrm{Im}\ Q_{\alpha n}(\omega)$ will not be much larger than $\mathrm{Im}\ Q_{\alpha n}(\omega)$, due to small departures of $\mathrm{Im}\ Q_{\alpha n}(\omega)$ from logarithmic behaviour. This point arose in particular because I first used an eight-point integration formula for the last interval instead of the usual three-point Simpson's Rule; but in any case to get a reasonable fit to $\mathrm{Im}\ Q_{\alpha n}(\omega)$ beyond $\omega = \omega_{m_0}$, the three points must be taken reasonably far apart. In the actual calculation of pion-pion scattering the last three points were roughly $\omega_{m_0} = \sqrt{2}\,\omega_{m_0-1}$, $\omega_{m_0-1} = \sqrt{2}\,\omega_{m_0-2}$, and this worked well enough for the purpose involved.

VII. A FORMAL STUDY OF THE CHEW-MANDELSTAM EQUATIONS

In this and the next chapters we make a study of the equations proposed by Chew and Mandelstam for the s and p wave pion-pion scattering amplitudes. These equations differ from the equations of the one-meson approximation to the Low equation in that the crossing relation is more complicated. The techniques we have used to study the Low equation can be applied to the Chew-Mandelstam equations, in general with less success.

Like the Low equation, the Chew-Mandelstam equations involve a Born approximation a_α and a crossing matrix $A_{\alpha\beta}$. Here we assign the numbers α as follows:

$\alpha = 1$: $I = 0$ s state

$\alpha = 2$: $I = 2$ s state

$\alpha = 3$: $I = 1$ p state.

The statistics of pions forbids the existence of even I p-states or odd I s-states, and in the Chew-Mandelstam approximation scattering in higher partial waves and inelastic scattering are neglected. We shall have occasion to discuss the theory of the interaction of neutral pions in the absence of charged pions, in which case there is only one s state and no p-wave states.

For pion-pion scattering the crossing matrix satisfies

$$\sum_\beta A_{\alpha\beta} A_{\beta\gamma} = 4\,\delta_{\alpha\gamma}\,, \tag{7.1}$$

$$\sum_\beta A_{\alpha\beta}\, a_\beta = 2 a_\alpha\,, \tag{7.2}$$

and we shall see that the s-wave part of the crossing matrix has the form

$$A_{\alpha\beta} = -\delta_{\alpha\beta} - a_\alpha K_\beta \quad (\alpha = 1 \text{ or } 2;\ \beta = 1 \text{ or } 2) \,. \tag{7.3}$$

With respect to its role in the Chew-Mandelstam equations the crossing matrix $A_{\alpha\beta}$ is the analogy of $(- A_{\alpha\beta})$ in the Low equation.

In our investigation of the equations of Chew and Mandelstam we shall present a calculation of the perturbation expansion of the solution to sixth order in the coupling constant, and a preliminary analysis of the results. Unlike the case of the Low equation, where we essentially reproduced Salzman's results, the power series we shall obtain disagrees qualitatively with the numerical results of Chew and Mandelstam. Before presenting the calculations we shall present two theorems which are a serious barrier to any attempt to analyze the results of the calculations; in fact with just the analysis carried out so far on the power series it is impossible to say whether its predictions for the phase shifts are significant or not. The most important prediction is that if there is a low energy p-wave resonance, it is extremely narrow. See chapter VIII.

The theorems we shall prove are:

1.) The equations of Chew and Mandelstam have no solution for positive values of the coupling constant.

2.) The perturbation expansion of the solution of the Chew-Mandelstam equations for the interaction of neutral pions diverges for any non-zero value of the coupling constant, at least for large values of the energy.

These theorems will be explained and proved later in this chapter. I wish to point out here that these theorems do not reflect any difference between the Chew-Mandelstam equations and the Low equation, for <u>the same theorems hold for the Low equation if the crossing matrix and Born approximation are the same</u>. For this theorem it is actually the s-wave part of the Born approximation and crossing matrix that is relevant. The reason for theorem 1 will be seen to be that the a_α all are negative and the κ_β all positive, while theorem 2 requires that $a\kappa$ be less than or equal to -3 (for the neutral pion theory there is only one state).

We begin by presenting the equations of Chew and Mandelstam. For their derivation and justification see reference (7). Let ν be the square of the vector momentum of one of the pions in the center of mass system of the two pions, measured in units of the pion mass. Let $S_\alpha(\nu)$ be the S matrix element for the elastic scattering of two pions in the state α, and let $Q_\alpha(\nu)$ be given by the equation

$$S_\alpha(\nu) = 1 + 2i\sqrt{\frac{\nu}{\nu+1}}\; Q_\alpha(\nu) \quad , \quad 0 \le \nu < \infty \; . \qquad (7.4)$$

In the approximation of Chew and Mandelstam, $Q_\alpha(\nu)$ has the following properties:

A.) $Q_\alpha(\nu)$ is the boundary value of an analytic function of ν in the complex ν plane; this analytic function has branch lines for $0 \le \nu \le \infty$ and $-\infty < \nu \le -1$, and $Q_\alpha(\nu)$ is the limit of this analytic function as ν approaches the positive real axis from above. Exactly as for the Low equation, we shall define $Q_\alpha(\nu)$ for complex ν to be this analytic function and retain the original definition (equation 7.4) when ν is real and positive.

B.) $Q_\alpha(\nu)$ is real for $-1 < \nu < 0$.

C.) $\operatorname{Im} Q_\alpha(\nu) = \sqrt{\frac{\nu}{\nu+1}}\, | Q_\alpha(\nu)|^2$, for $\nu > 0$.

D.) $\lim\limits_{\nu \to \infty} \frac{Q_\alpha(\nu)}{\nu} = 0$,

These four properties are except for trivial changes properties of the solution of the one-meson approximation to the Low equation. Besides the above there are two more properties which are more elaborate than their analogy in the static model, the reason being that only s waves (or p waves) were coupled in the static model, whereas in the pion-pion problem it is only an approximation to neglect the higher partial waves. First we have crossing symmetry. In the pion-pion problem crossing symmetry is simple for the amplitudes when considered as a function of energy and momentum transfer; to obtain an approximate crossing relation involving the s- and p-wave partial wave amplitudes, Chew and Mandelstam approximated the two-variable amplitude by the sum of its s- and p-wave parts; after applying crossing symmetry they projected out the s- and p-wave parts. They do this actually only for the imaginary part of the scattering amplitude, whose partial wave expansion should be more convergent than that of the real part. See reference (7). The result of this approximation is the equations

1.) $\alpha = 1$ or 2 , $\nu > 0$:

$$\operatorname{Im} Q_\alpha(-\nu-1+i\epsilon) = \frac{-1}{\nu+1} \int_0^\nu \Big\{ \sum_{\beta=1}^{2} A_{\alpha\beta} \operatorname{Im} Q_\beta(\nu_1) + 3\left(1 - \frac{2\nu}{\nu_1}\right) A_{\alpha 3} \operatorname{Im} Q_3(\nu_1) \Big\}\, d\nu_1 \,, \tag{7.5}$$

2.) $\alpha = 3$, $\nu > 0$:

$$\operatorname{Im} Q_3(-\nu-1+i\epsilon) = \frac{-1}{\nu+1} \int_0^\nu \left[1 - 2\left(\frac{\nu_1+1}{\nu+1}\right)\right] \Big\{ \sum_{\beta=1}^{2} A_{\alpha\beta} \operatorname{Im} Q_\beta(\nu_1) + 3\left(1 - \frac{2\nu}{\nu_1}\right) A_{\alpha 3} \operatorname{Im} Q_3(\nu_1) \Big\}\, d\nu_1 \,, \tag{7.6}$$

This is not the only form which we can consider for crossing symmetry, for if the d waves can be neglected we would expect p waves to make much less of a contribution to these crossing relations than the s waves, and at least for rough or qualitative purposes one should be able to neglect the p waves also. Indeed it is found that in computing the perturbation expansion of Q_α the effect of the p-wave term in equations 7.5 and 7.6 is usually 3 - 6 percent. Thus we may consider an alternate form of the crossing condition in which we set $A_{\alpha 3} = 0$ for all α.

The second property is essentially the definition of the coupling constant λ ; this becomes a property of the Q_α because we must be able to obtain subtraction constants for all three states using only one constant (actually, only two of the three; see below). This is again simple for the amplitudes as a function of energy and momentum transfer, but for the partial wave amplitudes Q_α we get a complicated equation (d waves etc. neglected) (α = 1 or 2):

$$Q_\alpha\left(-\tfrac{2}{3}\right) = \lambda a_\alpha + \frac{1}{\pi}\int_0^\infty \left\{-\frac{3}{2}\ln\left(1-\frac{2}{3}\frac{1}{\nu_1+1}\right) - \frac{1}{\nu_1+2/3}\right\} \left\{\sum_{\beta=1}^{2} A_{\alpha\beta}\,\mathrm{Im}\,Q_\beta(\nu_1) + 3\left(1+\frac{2}{3}\frac{1}{\nu_1}\right) A_{\alpha 3}\,\mathrm{Im}\,Q_3(\nu_1)\right\} d\nu_1 \; . \tag{7.7}$$

For the usual symmetric theory of pions

$$a_\alpha = \begin{vmatrix} -5 \\ -2 \end{vmatrix} \; , \qquad A_{\alpha\beta} = \frac{1}{3}\begin{vmatrix} 2 & 10 & 6 \\ 2 & 1 & -3 \\ 2 & -5 & 3 \end{vmatrix} \; . \tag{7.8}$$

For the neutral theory of pions

$$a = -3 \quad , \qquad A = 2 \; , \tag{7.9}$$

We obtain the equivalent of a subtraction constant for the p-wave amplitude $Q_3(\nu)$ from the requirement that it vanish at $\nu=0$.

Conditions A, B, and D mean that we may obtain a dispersion relation for $Q_\alpha(\nu)$ exactly as we did for the solution of the Low equation. For the s states we write Cauchy's theorem for $\frac{Q_\alpha(\nu)}{\nu+2/3}$ and obtain

$$Q_\alpha(\nu) = Q_\alpha\left(-\tfrac{2}{3}\right) + \frac{(\nu+2/3)}{\pi}\int_{-\infty}^{-1} \frac{\operatorname{Im} Q_\alpha(\nu_1+i\epsilon)}{(\nu_1+2/3)(\nu_1-\nu)}\,d\nu_1$$
$$+ \frac{(\nu+2/3)}{\pi}\int_{0}^{\infty} \frac{\operatorname{Im} Q_\alpha(\nu_1)}{(\nu_1+2/3)(\nu_1-\nu)}\,d\nu_1 \tag{7.10}$$

(ν not on a branch cut). For the p-wave amplitude we write Cauchy's theorem for $\frac{Q_3(\nu)}{\nu}$, and since the residue at $\nu=0$ is zero we obtain

$$Q_3(\nu) = \frac{\nu}{\pi}\int_{-\infty}^{-1} \frac{\operatorname{Im} Q_3(\nu_1+i\epsilon)}{\nu_1(\nu_1-\nu)}\,d\nu_1 + \frac{\nu}{\pi}\int_{0}^{\infty} \frac{\operatorname{Im} Q_3(\nu_1)}{\nu_1(\nu_1-\nu)}\,d\nu_1 , \tag{7.11}$$

again for ν not on a branch cut.

The equations for the s-wave amplitudes can be put in a relatively simple form which emphasizes the close parallel between these equations and the Low equation. The algebra is worked out in the appendix. The result of combining equations 7.5, 7.7 and 7.10 and putting $A_{\alpha 3} = 0$ is (for $\alpha = 1$ or 2):

$$Q_\alpha(\nu) = \lambda a_\alpha + \frac{(\nu+2/3)}{\pi}\int_0^\infty \frac{\operatorname{Im} Q_\alpha(\nu_1)}{(\nu_1+2/3)(\nu_1-\nu)}\,d\nu_1$$
$$- \sum_{\beta=1}^{2} A_{\alpha\beta}\,\frac{1}{\nu}\int_1^{\nu+1}\left\{\frac{(\nu_1-2/3)}{\pi}\int_0^\infty \frac{\operatorname{Im} Q_\beta(\nu_2)}{(\nu_2+2/3)(\nu_2+\nu_1)}\,d\nu_2\right\} d\nu_1 \tag{7.12}$$

(ν not on a branch cut), where the path from $\nu_1 = 1$ to $\nu_1 = \nu + 1$ must not cross the negative real axis. Except for the trivial change in the point the subtraction is made (from 0 to $-\frac{2}{3}$), the only difference between this and the Low equation is that the crossed term here involves taking an average over the type of crossed term present in the Low equation.*

The equations for the p-wave state, or for the s-wave states when the p-wave contribution is included, are somewhat more complicated.

We note that when p-wave effects are included equation 7.12 is modified by a term whose dependence on α is proportional to $A_{\alpha 3}$.

Now consider the first of the two theorems to be proved in this chapter, namely that the Chew-Mandelstam equations have no solution when λ is positive. This theorem is known to Chew and Mandelstam but their proof is unsatisfactory: they show that the determinantal function $D_\alpha(\nu)$ defined by

$$Q_\alpha(\nu) = \frac{N_\alpha(\nu)}{D_\alpha(\nu)} , \qquad (7.13)$$

$$\operatorname{Im} D_\alpha(\nu) = -\sqrt{\frac{\nu}{\nu+1}}\, N_\alpha(\nu) \qquad (\nu \geqslant 0) , \qquad (7.14)$$

$$D_\alpha(\nu) = 1 + \frac{(\nu + 2/3)}{\pi} \int_0^\infty \frac{\operatorname{Im} D_\alpha(\nu_1)}{(\nu_1 + 2/3)(\nu_1 - \nu)}\, d\nu_1 \qquad (\nu \not> 0) , \qquad (7.15)$$

$$D_\alpha(\nu) = \lim_{\epsilon \to 0} D_\alpha(\nu + i\epsilon) \qquad (\nu > 0) \qquad (7.16)$$

has a zero for negative real ν but this means a pole for $Q_\alpha(\nu)$ only if $N_\alpha(\nu)$ does not have a compensating zero. For example, the one

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* It is understood that the corresponding crossing matrix in the Low equation will be $-A_{\alpha\beta}$.

dimensional Low equation (see chapter II) can be treated in the same way, and there Chew and Mandelstam's arguments would predict a pole in $Q(\omega)$ for negative real ω, while in fact $Q(\omega)$ has a pole only for <u>imaginary</u> ω.

The theorem we prove still holds if inelastic scattering is included in the unitarity condition; but since the proof depends on the approximation of neglecting the higher partial waves, the exact theory of pion-pion scattering may still be consistent.

The inconsistency in Chew and Mandelstam's equations is due to their high energy behaviour (see our proof) whereas their validity as an approximation to the exact pion-pion scattering problem is at best only for low energies, and one is thus tempted to try to solve the equations approximately for low energies, hoping that the inconsistency will appear only at large energies. Chew and Mandelstam try to achieve this by solving equations for $N_\alpha(\nu)$ and $D_\alpha(\nu)$, which are consistent equations; they do not report how well their results agree with the original equations[7,24]. In chapter VIII we try to achieve an approximation by means of perturbation theory, with indifferent success.

Now we prove that Mandelstam's equations have no solution when λ is positive.

Assume that we have a solution of Mandelstam's equations for a positive value of λ. Define

$$Q(\nu) = \frac{1}{3} Q_1(\nu) + \frac{2}{3} Q_2(\nu) , \qquad (7.17)$$

and let $Q(\nu) = P(\nu) + i F(\nu)$,

$$Q_\alpha(\nu) = P_\alpha(\nu) + i F_\alpha(\nu) \qquad (\nu \geq 0) . \qquad (7.18)$$

$Q(\nu)$ satisfies a dispersion equation of the form of equation 7.10. However, from the value of the crossing matrix given in equation 7.8 we see that Im $Q(\nu)$ for $\nu < -1$ depends via the crossing relation only on F_1 and F_2, not on F_3, for positive energies, and the same holds for the subtraction constant. In fact Im $Q(\nu)$ at negative energies and the subtraction constant depend on F_1 and F_2 in the form $\frac{1}{3} F_1 + \frac{2}{3} F_2$ i.e. on F only; since no p waves are involved the integral over $F(\nu)$ for negative ν can be transformed into the form of equation 7.12, and we obtain

$$P(\nu) = -3\lambda + \frac{(\nu-\nu_0)}{\pi} \, P.V. \int_0^\infty \frac{F(\nu_1)\, d\nu_1}{(\nu_1-\nu_0)(\nu_1-\nu)} - \frac{2}{\nu} \int_1^{\nu+1} \left\{ \frac{(\nu_1+\nu_0)}{\pi} \int_0^\infty \frac{F(\nu_2)}{(\nu_2-\nu_0)(\nu_2+\nu_1)} \, d\nu_2 \right\} d\nu_1 \,, \tag{7.19}$$

where $\nu_0 = -\frac{2}{3}$.

From the unitarity condition (condition C) we see that $F_1(\nu)$ and $F_2(\nu)$ are positive when ν is positive. Since $\nu_0 = -\frac{2}{3}$ and ν_1 is greater than 1, the last term of equation 7.19 is negative when ν is positive. Thus for $\nu > 0$,

$$P(\nu) < -3\lambda + \frac{(\nu-\nu_0)}{\pi} \, P.V. \int_0^\infty \frac{F(\nu_1)\, d\nu_1}{(\nu_1-\nu_0)(\nu_1-\nu)} \,. \tag{7.20}$$

From unitarity we may obtain bounds for P and F. Suppose $\nu > 1$; then

$$F_\alpha(\nu) > \frac{1}{\sqrt{2}} F_\alpha^2(\nu) \,, \text{ i.e.}$$
$$F(\nu) = \frac{1}{3} F_1(\nu) + \frac{2}{3} F_2(\nu) < \sqrt{2} \,. \tag{7.21}$$

Also (for $\nu > 1$)

$$F_\alpha(\nu) > \frac{1}{\sqrt{2}} P_\alpha^2(\nu) \,, \text{ and hence}$$

$$\left(\frac{1}{3} P_1 + \frac{2}{3} P_2 \right)^2 \leq \frac{P_1^2}{9} + \frac{4}{9} |P_1 P_2| + \frac{4}{9} P_2^2 ,$$

but since $|P_1 P_2| < \frac{1}{2}(P_1^2 + P_2^2)$,

$$P^2(\nu) \leq \frac{1}{3} P_1^2 + \frac{2}{3} P_2^2 < \sqrt{2}\, F(\nu) < 2 . \qquad (7.22)$$

The inconsistency we wish to find can be understood from an examination of equation 7.20. For large ν one would expect the integral to be determined mainly from values of ν_1 less than ν, say values of ν_1 less than $\frac{\nu}{100}$. Then we can set $\nu_1 - \nu \simeq -\nu$ and replace the upper limit by $\frac{\nu}{100}$. Then both terms in equation 7.20 are negative; furthermore either $\lim_{\nu_1 \to \infty} F(\nu_1) = 0$ or the second term in equation 7.20 becomes arbitrarily large as we let ν increase; the second possibility cannot be allowed because $P(\nu)$ is bounded, and the first possibility cannot be allowed because our argument says $|P(\nu)|$ is greater than 0 and therefore by equation 7.22 $F(\nu)$ cannot approach zero.

To translate this argument into a rigorous proof of inconsistency we must deal with the principal value singularity. To do this we consider an integral over $P(\nu)$, and evidently this integral should depend as far as possible on values of $P(\nu)$ for very large values of ν, that is it should be very slowly convergent. A suitable choice is to define

$$J(x) = \int_x^\infty \frac{P(\nu)}{(\nu + 2/3)\ln^3 \nu}\, d\nu , \qquad (7.23)$$

which converges only because of the slowly increasing function $\ln \nu$ in the denominator. By equation 7.22, $J(x)$ is absolutely convergent, and

$$|J(x)| < \sqrt{2} \int_x^\infty \frac{1}{\nu \ln^3 \nu} \, d\nu = \frac{\sqrt{2}}{2} \frac{1}{\ln^2 x} . \tag{7.24}$$

The integral of $\frac{1}{\nu \ln^3 \nu}$ will be used repeatedly in the arguments below. In the following we shall always assume x very large, so that we can make free use of inequalities of the form

$$\frac{1}{\nu + 2/3} > \frac{1}{2\nu}, \quad \ln\left(\frac{\nu}{2}\right) > \frac{1}{2} \ln \nu ,$$

$$\text{and} \quad \ln^2 \nu > N \ln \nu ,$$

where N is any number we wish to use less than 1000 (i.e. any number not arbitrarily large), and so forth.

Using equation 7.20 we obtain

$$\begin{aligned} J(x) < &- 3\lambda \int_x^\infty \frac{1}{(\ln^3 \nu)(\nu + 2/3)} \, d\nu \\ &+ \frac{1}{\pi} \int_x^\infty \frac{1}{\ln^3 \nu} \left\{ P.V. \int_0^\infty \frac{F(\nu_1) \, d\nu_1}{(\nu_1 + 2/3)(\nu_1 - \nu)} \right\} d\nu . \end{aligned} \tag{7.25}$$

It is shown in the appendix that the order of integration in the second term can be interchanged. Only very weak assumptions are made; for details see the appendix. This gives (using an upper bound for the first integral)

$$\begin{aligned} J(x) < &- \frac{3}{4} \lambda \frac{1}{\ln^2 x} \\ &- \frac{1}{\pi} \int_0^\infty \frac{F(\nu_1)}{(\nu_1 + 2/3)} \left\{ P.V. \int_x^\infty \frac{1}{\ln^3 \nu} \frac{1}{(\nu - \nu_1)} \, d\nu \right\} d\nu_1 . \end{aligned} \tag{7.26}$$

Now consider the principal value integral

$$I = P.V. \int_x^\infty \frac{1}{\ln^3 \nu} \frac{1}{(\nu - \nu_1)} \, d\nu . \tag{7.27}$$

If ν_1 is less than x this has a lower bound

$$I > \int_x^\infty \frac{1}{\nu \ln^3 \nu} \, d\nu = \frac{1}{2} \frac{1}{\ln^2 x} \quad . \tag{7.28}$$

If $\nu_1 > x$ it is shown in the appendix that I remains positive. Thus (since $F(\nu_1)$ is positive),

$$J(x) < \frac{1}{4 \ln^2 x} \left\{ -3\lambda - \frac{2}{\pi} \int_1^x \frac{F(\nu_1)}{(\nu_1 + 2/3)} \, d\nu_1 \right\} , \tag{7.29}$$

or

$$| J(x) | > \frac{1}{4 \ln^2 x} \left\{ 3\lambda + \frac{2}{\pi} \int_1^x \frac{F(\nu_1)}{(\nu_1 + 2/3)} \, d\nu_1 \right\} . \tag{7.30}$$

By equation 7.24, $|J(x)| \ln^2 x$ is bounded independently of the value of x, so that

$$\int_1^\infty \frac{F(\nu_1)}{(\nu_1 + 2/3)} \, d\nu_1$$

must converge. Thus if x is large enough, we have

$$\lambda > \frac{2}{\pi} \int_x^\infty \frac{F(\nu_1)}{(\nu_1 + 2/3)} \, d\nu_1 \quad , \tag{7.31}$$

Now by equation 7.24 , $F(\nu_1) > \frac{1}{2} P^2(\nu_1)$.

Thus

$$| J(x) | > \frac{1}{4 \ln^2 x} \left\{ 2\lambda + \frac{1}{\pi} \int_1^\infty \frac{P^2(\nu_1)}{(\nu_1 + 2/3)} \, d\nu_1 \right\} . \tag{7.32}$$

Now from equation 7.32

$$|J(x)| > \frac{\lambda}{2\ln^2 x} , \tag{7.33}$$

which means that $|P(\nu)|$ must be greater than $\frac{\lambda}{2}$ for some values of ν (since $\int_x^\infty \lambda \frac{1}{2\nu \ln^3 \nu} d\nu < \frac{\lambda}{4}\frac{1}{\ln^2 x}$). Let S_x be the set of points $\nu \geq x$ for which $|P(\nu)| > \frac{\lambda}{2}$. Evidently to satisfy equation 7.33 we must have

$$\int_{S_x} |P(\nu)| \frac{1}{(\nu + 2/3)\ln^3 \nu} d\nu > \frac{\lambda}{4}\frac{1}{\ln^2 x} . \tag{7.34}$$

Now we can obtain the inconsistency. With the use of equations 7.24 and 7.34 we obtain

$$\frac{1}{\ln^2 x} > \frac{1}{4\ln^2 x}\left\{ 2\lambda + \frac{1}{\pi}\int_{S_x} \left[|P(\nu_1)| \ln^3 \nu_1 \right]\left[\frac{|P(\nu_1)|}{(\nu_1 + 2/3)\ln^3 \nu_1} \right] d\nu_1 \right\}$$

$$> \frac{1}{4\ln^2 x}\left\{ 2\lambda + \frac{1}{\pi}\frac{\lambda}{2}\ln^3 x \int_{S_x} \frac{|P(\nu_1)|}{(\nu_1 + 2/3)\ln^3 \nu_1} d\nu_1 \right\} \tag{7.35}$$

$$> \frac{1}{\ln^2 x}\left\{ \frac{\lambda}{2} + \frac{\lambda^2}{8\pi}\ln x \right\} .$$

Since this equation has been derived for arbitrarily large x, we have an inconsistency, regardless of the magnitude of λ .

Our second theorem states that the asymptotic form for large ν of the perturbation expansion of the solution of the Chew-Mandelstam equations for the neutral pion interaction diverges for any non-zero value of λ .

In the Chew-Mandelstam equations for the neutral pion interaction there is one s-wave amplitude and no p-wave amplitudes, so that we may start from equation 7.12. Let

$$Q(\nu) = P(\nu) + iF(\nu) \qquad (\nu > 0) , \tag{7.36}$$

and let ν approach the real axis from above, in equation 7.12. Using equation 7.9 we obtain

$$\begin{aligned} P(\nu) = &- 3\lambda + \frac{(\nu + 2/3)}{\pi} \, P.V. \int_0^\infty \frac{F(\nu_1)\, d\nu_1}{(\nu_1 + 2/3)(\nu_1 - \nu)} \\ &- \frac{2}{\nu} \int_1^{\nu+1} \left\{ \frac{(\nu_1 - 2/3)}{\pi} \int_0^\infty \frac{F(\nu_2)}{(\nu_2 + 2/3)(\nu_2 + \nu_1)} \, d\nu_2 \right\} d\nu_1 . \end{aligned} \tag{7.37}$$

We now obtain equations for the asymptotic forms of the perturbation expansions of P and F, for large values of ν . The method is analogous to the method used in chapter II for the Low equation, so we shall be brief. Let

$$F(\nu) = \overline{F}(\nu) + F'(\nu) , \tag{7.38}$$

where $\overline{F}(\nu)$ is the asymptotic form of $F(\nu)$, i.e. a double power series in λ and $\ln \nu$, while $F'(\nu)$ is the remainder of $F(\nu)$. We write equation 7.37, for large ν and neglecting terms of order $\frac{1}{\nu}$, in such a way that each term either contributes to the definition of a new constant λ_0 or is a double power series in λ and $\ln \nu$. The uncrossed integral in equation 7.37 may be written (for large ν)

$$\begin{aligned} &- \frac{1}{\pi} \int_0^\infty \frac{F'(\nu_1)}{(\nu_1 + 2/3)} \, d\nu_1 \quad - \frac{1}{\pi} \int_0^1 \frac{\overline{F}(\nu_1)}{(\nu_1 + 2/3)} \, d\nu_1 \\ &\qquad + \frac{2}{3\pi} \int_1^\infty \frac{\overline{F}(\nu_1)}{\nu_1 (\nu_1 + 2/3)} \, d\nu_1 \\ &+ \frac{1}{\pi} \, P.V. \int_0^1 \frac{\overline{F}(\nu_1)\, d\nu_1}{(\nu_1 - \nu)} \quad + \frac{\nu}{\pi} \, P.V. \int_1^\infty \frac{\overline{F}(\nu_1)}{\nu_1 (\nu_1 - \nu)} \, d\nu_1 \end{aligned}$$

and the crossed integral may be written

$$- \frac{2}{\nu} \int_1^{\nu+1} \left\{ \frac{1}{\pi} \int_0^\infty \frac{F'(\nu_2)}{(\nu_2 + 2/3)} \, d\nu_2 + \frac{1}{\pi} \int_0^1 \frac{\overline{F}(\nu_2)}{(\nu_2 + 2/3)} \, d\nu_2 \right.$$

$$- \frac{2}{3\pi} \int_1^\infty \frac{\overline{F}(\nu_2)}{\nu_2(\nu_2 + 2/3)} \, d\nu_2$$

$$\left. - \frac{1}{\pi} \int_0^1 \frac{\overline{F}(\nu_2)}{\nu_2 + \nu_1} \, d\nu_2 + \frac{\nu_1}{\pi} \int_1^\infty \frac{\overline{F}(\nu_2)}{\nu_2(\nu_2 + \nu_1)} \, d\nu_2 \right\} d\nu_1 \, .$$

Thus we may define

$$\lambda_0 = \lambda + \frac{1}{\pi} \int_0^\infty \frac{F'(\nu_1)}{(\nu_1 + 2/3)} \, d\nu_1 + \frac{1}{\pi} \int_0^1 \frac{\overline{F}(\nu_1)}{(\nu_1 + 2/3)} \, d\nu_1$$
$$- \frac{2}{3\pi} \int_1^\infty \frac{\overline{F}(\nu_1) \, d\nu_1}{\nu_1(\nu_1 + 2/3)} \, , \qquad (7.39)$$

and obtain for large ν

$$\overline{P}(\nu) = -3\lambda_0 + \frac{1}{\pi} \, \mathrm{P.V.} \int_0^1 \frac{\overline{F}(\nu_1)}{(\nu_1 - \nu)} \, d\nu_1 + \frac{\nu}{\pi} \, \mathrm{P.V.} \int_1^\infty \frac{\overline{F}(\nu_1)}{\nu_1(\nu_1 - \nu)} \, d\nu_1$$
$$- \frac{2}{\nu} \int_1^{\nu+1} \left\{ -\frac{1}{\pi} \int_0^1 \frac{\overline{F}(\nu_2)}{\nu_2 + \nu_1} \, d\nu_2 + \frac{\nu_1}{\pi} \int_1^\infty \frac{\overline{F}(\nu_2) \, d\nu_2}{\nu_2(\nu_2 + \nu_1)} \right\} d\nu_1 \, . \qquad (7.40)$$

This equation is very similar to equation 2.31 of chapter II, and we can use the results of chapter II to obtain from equation 7.40 an equation similar to equation 2.30. First we must examine

$$I = \frac{1}{\nu} \int_1^{\nu+1} \ln^m \nu_1 \, d\nu_1$$

for large values of ν . Neglecting terms of order $\frac{1}{\nu}$ this integral is

$$I = \sum_{\ell=0}^{m} (-1)^\ell \frac{m!}{(m-\ell)!} \ln^{m-\ell} \nu \, . \qquad (7.41)$$

Thus in analogy with equation 2.30, we obtain

$$\bar{P}(z) = -3\lambda_o + \sum_{n=0}^{\infty} e_{2n-1}\lambda_o^{2n-1}\frac{d^{2n-1}}{dz^{2n-1}}\bar{F}(z)$$

$$-2\sum_{m=0}^{\infty}(-1)^m\frac{\lambda_o^m}{\pi^m}\frac{d^m}{dz^m}\left\{\sum_{n=0}^{\infty} c_{2n-1}\lambda_o^{2n-1}\frac{d^{2n-1}}{dz^{2n-1}}\bar{F}(z)\right\}, \qquad (7.42)$$

where

$$z = \frac{\lambda_o}{\pi}\ln\nu \quad , \qquad (7.43)$$

$$e_{2n-1} = \frac{2^{2n}}{(2n)!}B_n \quad , \qquad (7.44)$$

$$c_{2n-1} = [2^{2n}-2]\frac{B_n}{(2n)!} \qquad (7.45)$$

(we define $B_o = -1$),

The unitarity condition for the asymptotic forms is

$$\bar{F}(z) = \bar{P}^2(z) + \bar{F}^2(z) \; . \qquad (7.46)$$

Now let us write $\bar{P}(z)$ and $\bar{F}(z)$ explicitly as double power series in λ_o and z:

$$\bar{P}(z) = -\lambda_o\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} p_{rs}(-\lambda_o)^r z^s \quad , \qquad (7.47)$$

$$\bar{F}(z) = \lambda_o^2\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f_{rs}(-\lambda_o)^r z^s \quad . \qquad (7.48)$$

Substituting equations 7.47 and 7.48 into equation 7.46 gives

$$f_{rs} = \sum_{\mu=0}^{r}\sum_{\nu=0}^{s} p_{\mu\nu}p_{r-\mu,s-\nu} + \sum_{\mu=0}^{r-2}\sum_{\nu=0}^{s} f_{\mu\nu}f_{r-2-\mu,s-\nu} \quad , \qquad (7.49)$$

and making the same substitution in equation 7.42 gives

$$p_{rs} = 3\delta_{r0}\delta_{s0} + \sum_{n=0}^{r} D_n f_{r-n,\,s+n-1} \frac{(s+n-1)!}{s!} , \qquad (7.50)$$

where

$$D_n = -e_{n-1} + 2\sum_{m=0}^{n/2} \frac{c_{n-1-2m}}{\pi^{2m}} \quad (n \text{ even}) , \qquad (7.51)$$

$$D_n = 2\sum_{m=0}^{(n-1)/2} \frac{c_{n-2-2m}}{\pi^{2m+1}} \quad (n \text{ odd}) . \qquad (7.52)$$

Since the c_n are all positive, and since

$$2c_{2n-1} \geq |e_{2n-1}| \quad \text{(with equality only for } n = 1) , \qquad (7.53)$$

the D_n also are all positive.

Equations 7.49 and 7.50 can be solved by iteration to obtain p_{rs} and f_{rs} for any values of r and s; hence p_{rs} and f_{rs} are positive. Therefore

$$p_{rs} > D_r f_{0,\,s+r-1} \frac{(s+r-1)!}{s!} . \qquad (7.54)$$

If r is even, write r = 2n; we obtain

$$D_r > 2c_{2n-1} - e_{2n-1} = [2^{2n} - 4] \frac{B_n}{(2n)!} , \qquad (7.55)$$

For large n [see Dwight, reference (25)]

$$B_n > \frac{(2n)!}{\pi^{2n}\, 2^{2n-1}} , \qquad (7.56)$$

and hence for large n

$$D_r > \frac{1}{\pi^{2n}} = \frac{1}{\pi^r} . \qquad (7.57)$$

For odd r write r = 2n + 1; we obtain

$$D_r > \frac{2C_{2n-1}}{\pi} = \frac{2}{\pi}\left[2^{2n}-2\right]\frac{B_n}{(2n)!} > \frac{1}{\pi^{2n+1}} = \frac{1}{\pi^r} \ , \tag{7.58}$$

Hence for large r

$$p_{rs} > \frac{(s+r-1)!}{s!}\ \frac{1}{\pi^r}\ f_{o,s+r-1} \ . \tag{7.59}$$

However it is easy to evaluate $f_{o,n}$, which is found to be

$$f_{o,n} = (n+1)\, q^{n+1} \ . \tag{7.60}$$

Hence

$$p_{r,s} > \frac{(s+r)!}{s!}\ \frac{q^{s+r}}{\pi^r} \ , \tag{7.61}$$

Now look at the effect of this on the perturbation expansion of $\overline{P}(z)$. We have

$$\overline{P} = -\lambda_o \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} p_{rs}\,(-\lambda_o)^r z^s \ . \tag{7.62}$$

But $z = \frac{\lambda_o}{\pi}\ln\nu$; thus if we write n = r + s, m = s , then

$$\overline{P} = -\lambda_o \sum_{n=0}^{\infty} (-\lambda_o)^n \sum_{m=0}^{n} p'_{nm}\,(-\ln\nu)^m , \tag{7.63}$$

where

$$p'_{nm} = \frac{p_{n-m,m}}{\pi^m} , \tag{7.64}$$

and hence

$$p'_{nm} > \frac{q^n}{\pi^n}\ \frac{n!}{m!} \ . \tag{7.65}$$

This inequality guarantees the divergence of the expansion of $\overline{P}_\alpha$ in powers of λ_o for fixed ν if $\ln\nu$ is negative; there is a possibility that the series will converge for one positive value of $\ln \nu$, but at least it must diverge for almost all positive values of $\ln \nu$.

APPENDIX

I

We show that equations 7.5, 7.7 and 7.10 of chapter VII can be rewritten

$$Q_\alpha(\nu) = \lambda a_\alpha + \frac{(\nu+2/3)}{\pi}\int_0^\infty \frac{\operatorname{Im} Q_\alpha(\nu_1)}{(\nu_1+2/3)(\nu_1-\nu)}\,d\nu_1$$

$$-\sum_{\beta=1}^{2} A_{\alpha\beta}\frac{1}{\nu}\int_1^{\nu+1}\left\{\frac{(\nu_1-2/3)}{\pi}\int_0^\infty \frac{\operatorname{Im} Q_\beta(\nu_2)}{(\nu_2+2/3)(\nu_2+\nu_1)}\,d\nu_2\right\}d\nu_1 , \qquad (7A.1)$$

if the p-wave contribution (the term multiplied by $A_{\alpha 3}$) is neglected.

Let

$$Q_\alpha(\nu) = \lambda a_\alpha + \frac{(\nu+2/3)}{\pi}\int_0^\infty \frac{\operatorname{Im} Q_\alpha(\nu_1)\,d\nu_1}{(\nu_1+2/3)(\nu_1-\nu)} + \sum_{\beta=1}^{2} A_{\alpha\beta}\, U_\beta(\nu) \qquad (7A.2)$$

when the p-wave contribution is neglected; using equations 7.5, 7.7, and 7.10 of the text we obtain

$$U_\beta(\nu) = -\frac{(\nu+2/3)}{\pi}\int_1^\infty \frac{1}{(\nu_1+\nu)(\nu_1-2/3)}\left\{\frac{1}{\nu_1}\int_0^{\nu_1-1} \operatorname{Im} Q_\beta(\nu_2)\,d\nu_2\right\}d\nu_1$$

$$+\frac{1}{\pi}\int_0^\infty\left\{-\frac{3}{2}\ln\left(1-\frac{2}{3}\frac{1}{\nu_1+1}\right)-\frac{1}{\nu_1+2/3}\right\}\operatorname{Im} Q_\beta(\nu_1)\,d\nu_1 , \qquad (7A.3)$$

Interchanging the order of integration in the first term makes it

$$-\frac{(\nu+2/3)}{\pi}\int_0^\infty \operatorname{Im} Q_\beta(\nu_2)\left\{\int_{\nu_2+1}^\infty \frac{d\nu_1}{\nu_1(\nu_1+\nu)(\nu_1-2/3)}\right\}d\nu_2 ,$$

The inner integral can be obtained by standard methods, the result being

$$\int_{\nu_2+1}^{\infty} \frac{1}{\nu_1(\nu_1+\nu)(\nu_1-2/3)}\, d\nu_1$$

$$= \frac{-1}{\nu(\nu+2/3)} \ln\left(1+\frac{\nu}{\nu_2+1}\right) - \frac{3}{2}\frac{1}{(\nu+2/3)} \ln\left(1-\frac{2}{3}\frac{1}{\nu_2+1}\right) . \qquad (7A.4)$$

Hence

$$U_\beta(\nu) = \frac{1}{\pi}\int_0^\infty \left\{ \frac{1}{\nu} \ln\left(1+\frac{\nu}{\nu_1+1}\right) - \frac{1}{\nu_1+2/3} \right\} \operatorname{Im} Q_\beta(\nu_1)\, d\nu_1 . \qquad (7A.5)$$

Differentiating under the integral sign

$$\frac{d}{d\nu}\left[\nu\, U_\beta(\nu)\right]$$

$$= \frac{1}{\pi}\int_0^\infty \left\{ \frac{1}{\nu_1+1+\nu} - \frac{1}{\nu_1+2/3} \right\} \operatorname{Im} Q_\beta(\nu_1)\, d\nu_1 \qquad (7A.6)$$

$$= -\frac{(\nu+1/3)}{\pi}\int_0^\infty \frac{\operatorname{Im} Q_\beta(\nu_1)}{(\nu_1+2/3)(\nu_1+\nu+1)}\, d\nu_1 .$$

However

$$\nu\, U_\beta(\nu) \to 0 \qquad (7A.7)$$

as $\nu \to 0$ and hence

$$\nu\, U_\beta(\nu) = -\int_0^\nu \left\{ \frac{(\nu_1+1/3)}{\pi} \int_0^\infty \frac{\operatorname{Im} Q_\beta(\nu_2)\, d\nu_2}{(\nu_2+2/3)(\nu_2+\nu_1+1)} \right\} d\nu_1 , \qquad (7A.8)$$

or

$$U_\beta(\nu) = -\frac{1}{\nu}\int_1^{\nu+1} \left\{ \frac{(\nu_1-2/3)}{\pi} \int_0^\infty \frac{\operatorname{Im} Q_\beta(\nu_2)\, d\nu_2}{(\nu_2+2/3)(\nu_2+\nu_1)} \right\} d\nu_1 , \qquad (7A.9)$$

which gives equation 7A.1.

APPENDIX

II

In this appendix we prove that one may interchange the order of integration in the double infinite integral

$$\int_{x_0}^{\infty} \frac{1}{\ln^3 x} \left\{ \text{P.V.} \int_1^{\infty} \frac{F(y)}{y(y-x)} \, dy \right\} dx \quad .$$

In order to prove this we make the following assumptions:

1.) $|F(y)|$ is bounded for all y i.e. there exists a constant a such that

$$|F(y)| \leq a \ ,$$

2.) The principal value integral is bounded for finite values of x. That is, for all x satisfying $x_0 \leq x \leq x_A$ it has a bound A (depending on x_A):

$$\left| \text{P.V.} \int_1^{\infty} \frac{F(y)}{y(y-x)} \, dy \right| < A(x_A) \quad \text{for} \quad x_0 \leq x \leq x_A \ . \qquad (7A.10)$$

3.) The principal value integral is defined as

$$\text{P.V.} \int_1^{y_1} \frac{F(y)}{y(y-x)} \, dy = \lim_{\epsilon \to 0} \int_1^{y_1} \frac{y-x}{(y-x)^2+\epsilon^2} \, \frac{F(y)}{y} \, dy \ . \qquad (7A.11)$$

We shall assume $x_0 > 2$, and that this principal value integral exists for all $x \geq x_0$ except $x = y_1$. We assume further that this limit is uniform for all closed intervals of the x axis for which $x \geq x_0$ and $x \neq y_1$.

It is not difficult to show that the first two assumptions hold for the integral considered in the text. The third assumption is made to simplify the proof; it is probably not necessary and probably also excludes only highly pathological functions.

Define

$$T(x_1, y_1) = \int_{x_0}^{x_1} \left\{ \text{P.V.} \int_1^{y_1} \frac{F(y)}{y(y-x)} \frac{1}{\ln^3 x} dy \right\} dx \; , \tag{7A.12}$$

$$S(x_1, y_1) = \int_1^{y_1} \left\{ \text{P.V.} \int_{x_0}^{x_1} \frac{F(y)}{y(y-x) \ln^3 x} dx \right\} dy \; . \tag{7A.13}$$

Our proof involves proving two lemmas:

Lemma 1: $T(x_1, y_1) = S(x_1, y_1)$.

Lemma 2: $\lim_{x_1 \to \infty} S(x_1, y_1)$ and $\lim_{y_1 \to \infty} S(x_1, y_1)$ are each uniform in the other variable.

The completion of the proof is trivial.

Lemma 1. We prove that we may interchange the order of integration in the finite double integral

$$T(x_1, y_1) = \int_{x_0}^{x_1} \left\{ \mathrm{P.V.} \int_1^{y_1} \frac{F(y)}{y(y-x)\ln^3 x}\, dy \right\} dx \ .$$

The proof is non-trivial because the principal value integral is undefined for $x = y_1$.

Let

$$R(x, y_1) = \mathrm{P.V.} \int_1^{y_1} \frac{F(y)}{y(y-x)}\, dy \ , \tag{7A.14}$$

and let us obtain an upper bound for R. For $x > y_1$ and $x > 2$ we obtain (using assumption 1)

$$\begin{aligned} |R(x, y_1)| &< a \int_1^{y_1} \frac{1}{x-y}\, dy \\ &< a \ln|x - y_1| + a \ln x \ . \end{aligned} \tag{7A.15}$$

Let $A = A(x_1)$. For $x_0 \leq x \leq y_1$ and $x \leq x_1$ we use assumption 2: we have

$$R(x, y_1) = R(x, y_1 + 1) - \int_{y_1}^{y_1+1} \frac{F(y)}{y(y-x)}\, dy \ , \tag{7A.16}$$

and

$$\begin{aligned} |R(x, y_1+1)| &< |R(x, \infty)| + \left| \int_{y_1+1}^{\infty} \frac{F(y)}{y(y-x)}\, dy \right| \\ &\leq A + a \int_{y_1+1}^{\infty} \frac{1}{y(y-x)}\, dy \\ &\leq A + \frac{a}{x_0} \ln(y_1 + 1) \ , \end{aligned} \tag{7A.17}$$

$$\left| \int_{y_1}^{y_1+1} \frac{F(y)\,dy}{y(y-x)} \right| < a\,|\ln(y_1-x)| + a\ln(y_1+1) \quad . \tag{7A.18}$$

In summary, for $x_0 \le x \le x_1$ and given values of x_0, x_1 and y, $R(x, y_1)$ has a bound of the form

$$|R(x, y_1)| < B + a\,|\ln|y_1 - x|| \quad , \tag{7A.19}$$

where B is a constant independent of x.

Define

$$R_0(x, y_1, \epsilon) = \int_1^{y_1} \frac{y-x}{(y-x)^2+\epsilon^2} \; \frac{F(y)}{y}\,dy \quad . \tag{7A.20}$$

By assumption 3, $|R_0(x, \infty, \epsilon)|$ has a bound A' for $x_0 \le x \le x_1$ if ϵ is small enough; and since

$$\left| \frac{x-y}{(x-y)^2+\epsilon^2} \right| \le \frac{1}{|x-y|} \quad , \tag{7A.21}$$

the argument for $R(x,y_1)$ applies equally well for $R_0(x,y_1,\epsilon)$ if ϵ is small enough.

We now show that $T(x_1, y_1)$ can be approximated by the double integral of a regular function, for which the order of integration can be interchanged. Divide the range of integration $x_0 \le x \le x_1$ into two parts: I is that part if any for which $y_1 - \epsilon_1 \le x \le y_1 + \epsilon_1$, while II is the rest; ϵ_1 will be chosen later. Now for any ϵ, and for any ϵ_2 small enough we can choose ϵ_1 small enough so that

$$\left| \int_I R(x, y_1) \frac{dx}{\ln^3 x} \right| < \frac{\epsilon}{4} \quad , \tag{7A.22}$$

$$\left| \int_{I} R_o(x, y_1, \epsilon_2) \frac{dx}{\ln^3 x} \right| < \frac{\epsilon}{4} , \tag{7A.23}$$

since the length of I can be taken as small as we please and since R and R_o are bounded by integrable functions of x. Furthermore by assumption 3 we may now choose ϵ_2 small enough so that

$$\left| \int_{II} R(x, y_1) \frac{dx}{\ln^3 x} - \int_{II} R_o(x, y_1, \epsilon_2) \frac{dx}{\ln^3 x} \right| < \frac{\epsilon}{4} . \tag{7A.24}$$

Thus for any ϵ we can choose ϵ_2 so that

$$\left| T(x_1, y_1) - \int_{x_o}^{x_1} \left\{ \int_{1}^{y_1} \frac{(y-x)}{(y-x)^2+\epsilon_2^2} \frac{F(y)}{y \ln^3 x} dy \right\} dx \right| < \epsilon . \tag{7A.25}$$

We have thus approximated $T(x_1, y_1)$ by the double integral of a non-singular function in which the order of integration can be interchanged. The same result can be proved for $S(x_1, y_1)$ and thus

$$S(x_1, y_1) = T(x_1, y_1) . \tag{7A.26}$$

Lemma 2. We prove that the limits $\lim_{x_1\to\infty} S(x_1,y_1)$ and $\lim_{y_1\to\infty} S(x_1,y_1)$ are each uniform in the other variable.

We defined $S(x_1,y_1)$ to be

$$S(x_1,y_1) = \int_1^{y_1} \frac{F(y)}{y}\, P.V. \int_{x_0}^{x_1} \frac{1}{(y-x)\ln^3 x}\, dx\, dy \quad . \tag{7A.27}$$

Let

$$Q(y,x_2,x_1) = P.V. \int_{x_1}^{x_2} \frac{dx}{(y-x)\ln^3 x} \quad , \tag{7A.28}$$

To prove the uniformity of the two limits it is sufficient to show

1.) $\lim_{x_1\to\infty} \int_1^{\infty} \frac{a}{y} |Q(y,\infty,x_1)|\, dy = 0$,

2.) $\int_{y_1}^{\infty} \frac{a}{y} |Q(y,x_2,x_0)|\, dy$ is small for large y_1

independently of the value of x_2.

The proof involves estimates of Q. First let us estimate the principal value part. Let

$$Q = Q_1 + Q_2 \quad , \tag{7A.29}$$

where Q_1 is the integral over any part of the interval $x_1 \le x \le x_2$ which intersects the interval $\frac{y}{2} \le x \le \frac{3y}{2}$, and Q_2 is the remainder. For Q_1 we may use the mean value theorem on $\frac{1}{\ln^3 x}$:

$$\frac{1}{\ln^3 x} = \frac{1}{\ln^3 y} + \frac{6\,\Theta(x)}{y \ln^4(\frac{y}{2})}(y-x) \quad , \tag{7A.30}$$

where $|\Theta(x)| < 1$ for $\frac{y}{2} \le x \le \frac{3y}{2}$ (we may assume y is large since for 1, y needs to be of order x_1 to intersect the range of integration, and y_1 is large for 2). Now if the interval $\frac{y}{2} \leq x \leq \frac{3y}{2}$ lies completely within the range of integration, only the second term contributes to Q_1 and we obtain

$$|Q_1(y, x_2, x_1)| < \frac{6}{\ln^4(\frac{y}{2})} < \frac{12}{\ln^4 y} \quad \text{(since y is large)} . \tag{7A.31}$$

If $\frac{y}{2} \leq x \leq \frac{3y}{2}$ intersects one of the endpoints, say x_1 then we have to include the effect of the first term:

$$\begin{aligned} |Q_1(y, x_2, x_1)| &< \frac{|\ln|y - x_1|| + \ln y}{\ln^3 y} + \frac{12}{\ln^4 y} \\ &< \frac{|\ln|y - x_1||}{\ln^3 y} + \frac{2}{\ln^2 y} , \end{aligned} \tag{7A.32}$$

that is, this estimate applies for $\frac{2}{3} x_1 \leq y \leq 2x_1$ and a similar one involving x_2 for $\frac{2}{3} x_2 \leq y \leq 2x_2$. Now consider 1 specifically. For $y \leq 2x_1$ we obtain for Q_2

$$|Q_2(y, \infty, x_1)| < \int_{x_1}^{\infty} \frac{3}{x \ln^3 x} dx = \frac{3}{2} \frac{1}{\ln^2 x_1} , \tag{7A.33}$$

since in the actual range of integration of Q_2, $y \leq \frac{2}{3} x$ and hence $|y - x| > \frac{1}{3} x$. For $y > 2x_1$

$$|Q_2(y, \infty, x_1)| < \int_{x_1}^{\frac{y}{2}} \frac{2}{y \ln^3 x} dx + \int_{\frac{3y}{2}}^{\infty} \frac{3}{x \ln^3 x} dx \tag{7A.34}$$

using the same reasoning. Now since x_1 is large we have in the first integral

$$\frac{2}{\ln^3 x} < \frac{4}{\ln^3 x} - \frac{12}{\ln^4 x} = \frac{d}{dx} \frac{4x}{\ln^3 x} . \tag{7A.35}$$

Hence for $y > 2x_1$ for $Q_1 + Q_2$ we have

$$|Q(y, \infty, x_1)| < \frac{4}{\ln^3 y} + \frac{3}{2} \frac{1}{\ln^2 y} + \frac{12}{\ln^4 y} < \frac{2}{\ln^2 y} . \tag{7A.36}$$

Putting this information together

$$\int_1^\infty \frac{a}{y} \left| Q(y,\infty,x_1) \right| dy < \int_1^{2x_1} \frac{3}{2} \frac{a}{y} \frac{1}{\ln^2 x_1} dy$$

$$+ \int_{2x_1}^\infty \frac{2a}{y \ln^2 y} dy + \int_{2/3 x_1}^{2x_1} \frac{a}{y} \left\{ \frac{|\ln|y-x_1|| + 2\ln y}{\ln^3 y} \right\} dy \ .$$

(7A.37)

Since $\int \ln u = u \ln u - u$ it is evident that all these terms are small (of order $\frac{1}{\ln x_1}$) when x_1 is large and 1 is proved. Now look at 2. When $y > \frac{2}{3} x_2$ we see that

$$\left| Q_2(y,x_2,x_0) \right| < \int_{x_0}^{x_2} \frac{2}{y} \frac{dx}{\ln^3 x} < \frac{4x_2}{y \ln^3 x_2} \ . \qquad (7A.38)$$

For $y < \frac{2}{3} x_2$

$$\left| Q_2(y,x_2,x_0) \right| < \int_{x_0}^{y/2} \frac{2}{y} \frac{dx}{\ln^3 x} + \int_{3/2 y}^{x_2} \frac{3}{x} \frac{dx}{\ln^3 x} \ , \qquad (7A.39)$$

and hence

$$\left| Q(y,x_2,x_0) \right| < \frac{12}{\ln^4 y} + \frac{4}{\ln^3 y} + \frac{3}{2} \frac{1}{\ln^2 y} < \frac{2}{\ln^2 y} \ . \qquad (7A.40)$$

Putting this together gives

$$\int_{y_1}^\infty \frac{a}{y} \left| Q(y,x_2,x_0) \right| dy < \int_{\mathrm{Max}(y_1, 3/2 x_2)}^\infty \frac{4a}{y^2} \frac{x_2}{\ln^3 x_2} dy$$

$$+ \int_{y_1}^{2/3 x_2} \frac{2a}{y \ln^2 y} dy + \int_{2/3 x_2}^{2x_2} \frac{a}{y} \left\{ \frac{|\ln|y-x_2|| + 2\ln y}{\ln^3 y} \right\} dy \ ,$$

(7A.41)

where the second and third terms appear only if $2x_2 > y_1$. It is now easy to see each of these terms has a maximum near $x_2 = y_1$ considered as functions of x_2, and this maximum decreases as $\frac{1}{\ln y_1}$ when y_1 is large: thus 2 is proved.

To complete our proof we need to show that

$$\lim_{x_1\to\infty}\lim_{y_1\to\infty} S(x_1, y_1) = \lim_{y_1\to\infty}\lim_{x_1\to\infty} S(x_1, y_1) , \tag{7A.42}$$

Let

$$S_1(x_1) = \lim_{y_1\to\infty} S(x_1, y_1) , \tag{7A.43}$$

$$S_2(y_1) = \lim_{x_1\to\infty} S(x_1, y_1) . \tag{7A.44}$$

Then from lemma 2 we may find an N for any $\in$, such that if $x_1 > N$, $y_1 > N$ then

$$| S(x_1, y_1) - S_1(x_1) | < \frac{\in}{2} , \tag{7A.45}$$

$$| S(x_1, y_1) - S_2(y_1) | < \frac{\in}{2} . \tag{7A.46}$$

and hence

$$| S_1(x_1) - S_2(y_1) | < \in . \tag{7A.47}$$

By Cauchy's theorem we now obtain

$$\lim_{x_1\to\infty} S_1(x_1) = \lim_{y_1\to\infty} S_2(y_1) . \tag{7A.48}$$

Q.E.D.

The reader who finds this appendix long and difficult is referred to reference (26).

APPENDIX

III

We show that

$$I = P.V. \int_{x}^{\infty} \frac{1}{\ln^3 \nu} \frac{1}{\nu - \nu_1} d\nu \qquad (7A.49)$$

is positive for any large values of x and ν_1 with

$$\nu_1 > x \quad . \qquad (7A.50)$$

Consider two cases, $\nu_1 < 2x$ and $\nu_1 > 2x$. In the first case we write

$$I = I_1 + I_2 \, , \qquad (7A.51)$$

where

$$I_1 = P.V. \int_{x}^{2\nu_1 - x} \frac{1}{\ln^3 \nu} \frac{1}{\nu - \nu_1} d\nu \, , \qquad (7A.52)$$

$$I_2 = \int_{2\nu_1 - x}^{\infty} \frac{1}{\ln^3 \nu} \frac{1}{\nu - \nu_1} d\nu \, . \qquad (7A.53)$$

By the mean value theorem

$$\frac{1}{\ln^3 \nu} = \frac{1}{\ln^3 \nu_1} + \frac{6\Theta(\nu)}{\nu_1 \ln^4 (\frac{\nu_1}{2})} (\nu - \nu_1) \, , \qquad (7A.54)$$

where for $x \leq \nu \leq 2\nu_1 - x$, $|\Theta(\nu)| < 1$ since $\frac{\nu_1}{2} < x$. Substituting into the integral for I_1 we get

$$|I_1| < \frac{6(2\nu_1 - 2x)}{\nu_1 \ln^4 (\frac{\nu_1}{2})} < \frac{1}{8 \ln^2 \nu_1} \, , \qquad (7A.55)$$

if x (and hence ν_1) is large enough. Also

$$I_2 > \int_{2\nu_1 - x}^{\infty} \frac{1}{\ln^3 \nu} \frac{d\nu}{\nu} = \frac{1}{2} \frac{1}{\ln^2 (2\nu_1 - x)} > \frac{1}{4 \ln^2 \nu_1} \, , \qquad (7A.56)$$

and hence $I = I_1 + I_2$ is positive. If $\nu_1 > 2x$ we divide the integral into three parts:

$$I = I_1 + I_2 + I_3 \ , \tag{7A.57}$$

where

$$I_1 = \int_x^{\nu_1/2} \frac{1}{\ln^3 \nu} \ \frac{1}{\nu - \nu_1} \ d\nu \ , \tag{7A.58}$$

$$I_2 = \text{P.V.} \int_{\nu_1/2}^{3\nu_1/2} \frac{1}{\ln^3 \nu} \ \frac{1}{\nu - \nu_1} \ d\nu \ , \tag{7A.59}$$

$$I_3 = \int_{3\nu_1/2}^{\infty} \frac{1}{\ln^3 \nu} \ \frac{1}{\nu - \nu_1} \ d\nu \ . \tag{7A.60}$$

Now

$$|I_1| < \frac{2}{\nu_1} \int_x^{\nu_1/2} \frac{1}{\ln^3 \nu} \ d\nu \ . \tag{7A.61}$$

However

$$\frac{1}{\ln^3 \nu} < \frac{2}{\ln^3 \nu} - \frac{6}{\ln^4 \nu} = \frac{d}{d\nu} \left\{ \frac{2\nu}{\ln^3 \nu} \right\} \tag{7A.62}$$

(for large ν) hence

$$|I_1| < \frac{4}{\ln^3 \nu_1} < \frac{1}{8 \ln^2 \nu_1} \tag{7A.63}$$

when ν_1 is large. The treatment of I_2 and I_3 corresponds exactly to the treatment of I_1 and I_2 in the case $\nu_1 < 2x$ hence again $I = I_1 + I_2 + I_3$ is positive.

POWER SERIES EXPANSION OF THE CHEW-MANDELSTAM EQUATIONS

In this chapter we examine the perturbation expansion of the reciprocal of the scattering amplitude for pion-pion scattering, as obtained from the equations of Chew and Mandelstam.

We use the notation of chapter VII. We recall that we defined the states α as follows

$\alpha = 1$: $I = 0$, s state

$\alpha = 2$: $I = 2$, s state

$\alpha = 3$: $I = 1$, p state

and defined $Q_\alpha(\nu)$ by the relation

$$e^{2i\delta_\alpha(\nu)} = 1 + 2i\sqrt{\frac{\nu}{\nu+1}}\, Q_\alpha(\nu) , \tag{8.1}$$

where δ_α is the phase shift for the scattering of two pions in the state α when each pion has momentum $\sqrt{\nu}$ in the center of mass system. We define

$$L_\alpha(\nu) = \frac{1}{Q_\alpha(\nu)} , \tag{8.2}$$

and write

$$L_\alpha(\nu) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \lambda^n L_{\alpha n}(\nu) \tag{8.3}$$

(for $\alpha = 1$ or 2),

and

$$L_3(\nu) = \frac{1}{\lambda^2} \sum_{n=0}^{\infty} \lambda^n L_{3n}(\nu) . \tag{8.4}$$

Since the power series for $Q_3(\nu)$ starts with a λ^2 term instead of a term proportional to λ , the series for $L_3(\nu)$ begins with a λ^{-2} term.

The functions $L_{\alpha n}$ have been calculated through n = 5, for the s states, and through n = 4 for the p state. The calculations were performed on the IBM 704 computer at the M.I.T. Computation Center. The method of calculation was similar to the method used for the Low equation in the one-meson approximation; see the appendix to chapter VI.

Some results of the calculation are shown in table V. The presentation is the same as was used for the results of the calculations for the Low equation (see chapter VI), except that each term is given for λ = .14; thus

$$L_1(0) = -1.428\left(\frac{.14}{\lambda}\right)\left\{1 + .319\left(\frac{\lambda}{.14}\right) + .014\left(\frac{\lambda}{.14}\right)^2 + \cdots\right\}, \qquad (8.5)$$

$$L_3(.08) = 1860\left(\frac{.14}{\lambda}\right)^2\left\{1 - .506\left(\frac{\lambda}{.14}\right) - .120\left(\frac{\lambda}{.14}\right)^2 - \cdots\right\}. \qquad (8.6)$$

The first three terms in the series for L_1 and L_2 , and the first two terms in the series for L_3 should agree with a calculation of the first three terms in perturbation theory for pion-pion scattering using Feynman diagrams, since in perturbation theory the approximation of neglecting higher partial waves and inelastic scattering affects only fourth order and higher (since these are approximations in the calculation of the imaginary part of the scattering amplitude). The numbers we give are in satisfactory agreement with the calculations of Baker and Zachariasen[27] using Feynman diagrams.

From table V it appears that none of the series is genuinely convergent for λ = .14, but the series for L_1 and L_2 should still give

Table V

POWER SERIES EXPANSION OF $L_\alpha(\nu)$ for $\lambda = .14$.

$\alpha = 1$	n	$\nu=0$	$\nu=2.13$	$\nu=6.71$
	B.A.	-1.428	-1.428	-1.428
	0	1.000	1.000	1.000
	1	.319	-.242	-.654
	2	.014	.048	.102
	3	.009	.026	.049
	4	.009	.031	.065
	5	.014	.050	.107

$\alpha = 2$	n	$\nu=0$	$\nu=2.13$	$\nu=6.71$
	B.A.	-3.571	-3.571	-3.571
	0	1.000	1.000	1.000
	1	.051	-.316	-.639
	2	-.014	-.027	-.023
	3	-.010	-.025	-.041
	4	-.010	-.029	-.047
	5	-.007	-.013	.000

$\alpha = 3$	n	$\nu=.08$	$\nu=2.13$	$\nu=6.71$
	B.A.	1860	118	66.2
	0	1.000	1.000	1.000
	1	-.506	-.849	-1.239
	2	-.120	-.079	.044
	3	-.176	-.211	- .240
	4	-.310	-.377	- .423

a rough approximation at least for small ν . The higher order terms in the series for L_3 are uncomfortably large but since the third term in the series for L_3 is moderately small there is a possibility that the first two terms can be used to give a rough approximation to L_3. To test this hypothesis I have chosen to approximate all three states by the sum of the first two terms in their power series expansions: we write

$$L_\alpha(\nu) = \frac{1}{\lambda} L_{\alpha 0}(\nu) + L_{\alpha 1}(\nu) \qquad (\alpha = 1 \text{ or } 2), \tag{8.7}$$

$$L_3(\nu) = \frac{1}{\lambda^2} L_{30}(\nu) + \frac{1}{\lambda} L_{31}(\nu), \tag{8.8}$$

and

$$Q_\alpha(\nu) = \frac{1}{L_\alpha(\nu) - i\sqrt{\frac{\nu}{\nu+1}}} \quad . \tag{8.9}$$

These equations give us values of Re $Q_\alpha(\nu)$ and Im $Q_\alpha(\nu)$ for $\nu \geq 0$. To test these values I have substituted the values for Im $Q_\alpha(\nu)$ into the dispersion and crossing equations (equations 7.5, 7.6, 7.7, 7.10, and 7.11 of chapter VII), thus obtaining an alternate calculation of Re $Q_\alpha(\nu)$; the test of the approximation is how well the two values of Re $Q_\alpha(\nu)$ agree.

The results are shown in table VI. We define

$$P_\alpha(\nu) = \text{Re} \frac{1}{L_\alpha(\nu) - i\sqrt{\frac{\nu}{\nu+1}}} \quad , \tag{8.10}$$

and $\bar{P}_\alpha(\nu)$ = Re $Q_\alpha(\nu)$ as calculated through the dispersion and crossing equations. For this purpose equations 7.5, 7.6, 7.7, 7.10 and 7.11 of chapter VII can be combined to give

$$\bar{P}_\alpha(\nu) = \lambda a_\alpha + \left(\frac{\nu + 2/3}{\pi}\right) \text{P.V.} \int_0^\infty \frac{\text{Im } Q_\alpha(\nu_1)\, d\nu_1}{(\nu_1 + 2/3)(\nu_1 - \nu)} + \sum_{\beta=1}^{3} U_{\alpha\beta}(\nu) \tag{8.11}$$

(for $\alpha = 1$ or 2), and

$$\bar{P}_3(\nu) = \frac{\nu}{\pi} \text{ P.V.} \int_0^\infty \frac{\operatorname{Im} Q_3(\nu_1)}{\nu_1(\nu_1 - \nu)} d\nu_1 + \sum_{\beta=1}^{3} U_{3\beta}(\nu) , \tag{8.12}$$

where the crossed terms $U_{\alpha\beta}$ are given as follows:

$$U_{\alpha\beta}(\nu) = \frac{A_{\alpha\beta}}{\pi} \int_0^\infty \left\{ \frac{1}{\nu} \ln\left(1 + \frac{\nu}{\nu_1 + 1}\right) - \frac{1}{\nu_1 + 2/3} \right\} \operatorname{Im} Q_\beta(\nu_1) \, d\nu_1 \tag{8.13}$$

for $\alpha = 1$ or 2 and $\beta = 1$ or 2,

$$\begin{aligned} U_{\alpha 3}(\nu) = A_{\alpha 3} \int_0^\infty & \frac{3}{\pi} \frac{1}{\nu_1} \\ \times & \left\{ \left(\frac{\nu_1 + 2}{\nu} + 2 \right) \ln\left(1 + \frac{\nu}{\nu_1 + 1}\right) - 1 \right\} \operatorname{Im} Q_3(\nu_1) \, d\nu_1 \end{aligned} \tag{8.14}$$

for $\alpha = 1$ or 2,

$$\begin{aligned} U_{3\beta}(\nu) = A_{3\beta} \int_0^\infty & \frac{1}{\pi} \frac{1}{\nu} \\ \times & \left\{ \left(1 + \frac{2\nu_1 + 2}{\nu} \right) \ln\left(1 + \frac{\nu}{\nu_1 + 1}\right) - 2 \right\} \operatorname{Im} Q_\beta(\nu_1) \, d\nu_1 \end{aligned} \tag{8.15}$$

for $\beta = 1$ or 2, and

$$\begin{aligned} U_{33}(\nu) = A_{33} \int_0^\infty & \frac{3}{\pi} \frac{1}{\nu_1} \\ \times & \left\{ \left(2 + \frac{5\nu_1 + 6}{\nu} + \frac{2(\nu_1 + 1)(\nu_1 + 2)}{\nu^2} \right) \ln\left(1 + \frac{\nu}{\nu_1 + 1}\right) \right. \\ & \left. - \left(4 + \frac{2\nu_1 + 4}{\nu} \right) \right\} \operatorname{Im} Q_3(\nu_1) \, d\nu_1 \quad . \end{aligned} \tag{8.16}$$

In table VI, for various ν and α we show $P_\alpha(\nu)$ and $\bar{P}_\alpha(\nu)$ and then the various terms which sum to give $\bar{P}_\alpha(\nu)$, namely the Born approximation λa_α, the uncrossed integral (principal value integral) and the three crossed integrals $U_{\alpha\beta}(\nu)$.

In table VII, for various values of ν and α we show the

Table VI

COMPARISON OF $P_\alpha(\nu)$ with $\overline{P}_\alpha(\nu)$ AND THE VARIOUS PARTS OF $\overline{P}_\alpha(\nu)$

$\alpha = 1$	ν	$P_\alpha(\nu)$	$\overline{P}_\alpha(\nu)$	B.A.	Principle value Integral	Crossed Integrals $\beta=1$	$\beta=2$	$\beta=3$
	0	-.5309	-.5281	-.7000	.2175	-.0296	-.0592	.0432
	1.00	-.573	-.594	-.700	.203	-.063	-.132	.098
	1.96	-.585	-.652	-.700	.180	-.088	-.188	.144
	4.00	-.554	-.792	-.700	.101	-.128	-.288	.223
	5.96	-.488	-.926	-.700	.008	-.155	-.359	.280

$\alpha = 2$	ν	$P_\alpha(\nu)$	$\overline{P}_\alpha(\nu)$	B.A.	Principle value Integral	Crossed Integrals $\beta=1$	$\beta=2$	$\beta=3$
	0	-.2664	-.2677	-.2800	.0694	-.0296	-.0059	-.0216
	1.00	-.319	-.312	-.280	.093	-.063	-.013	-.049
	1.96	-.361	-.340	-.280	.119	-.088	-.019	-.072
	4.00	-.437	-.377	-.280	.172	-.128	-.029	-.112
	5.96	-.488	-.401	-.280	.210	-.155	-.036	-.140

$\alpha = 3$	ν	$P_\alpha(\nu)$	$\overline{P}_\alpha(\nu)$	B.A.	Principle value Integral	Crossed Integrals $\beta=1$	$\beta=2$	$\beta=3$
	0	.01348*	.01358*	0*	.00951*	.01315*	-.01359*	.00451*
	1.00	.0163	.0172	0	.0133	.0103	-.0115	.0050
	1.96	.0468	.0497	0	.0419	.0170	-.0196	.0104
	3.4596	.565	.572	0	.558	.024	-.030	.019
	3.61	-.451	-.445	0	-.460	.025	-.030	.020
	5.96	-.080	-.065	0	-.089	.032	-.041	.033

* For $\nu = 0$ the p wave terms are divided by ν to make them non-zero.

Table VII

COMPARISON OF $\overline{P}_\alpha(\nu) - P_\alpha(\nu)$, CROSSED INTEGRAL PART OF $\overline{P}_\alpha(\nu)$ AND THE p WAVE CONTRIBUTION TO $\overline{P}_\alpha(\nu)$. THE COLUMNS ARE LABELLED AS FOLLOWS

D for $\overline{P}_\alpha(\nu) - P_\alpha(\nu)$

C for Crossed Integral

p for p wave part of Crossed Integral

$\alpha = 1$	ν	D	C	p
	0	.003	-.046	.043
	1.00	-.021	-.097	.098
	1.96	-.067	-.132	.144
	4.00	-.238	-.193	.223
	5.96	-.438	-.234	.280

$\alpha = 2$	ν	D	C	p
	0	-.001	-.057	-.022
	1.00	.007	-.125	-.049
	1.96	.021	-.178	-.072
	4.00	.060	-.269	-.112
	5.96	.087	-.331	-.140

$\alpha = 3$	ν	D	C	p
	0	.0001*	.0041*	.0045*
	1.00	.0009	.0038	.0050
	1.96	.0029	.0078	.0104
	3.4596	.007	.014	.019
	3.61	.006	.015	.020
	5.96	.015	.024	.033

* The p wave results for $\nu = 0$ have been divided by ν to make them non-zero.

difference $\overline{P}_\alpha(\nu) - P_\alpha(\nu)$, the sum of the crossed integrals $\sum_{\beta=1}^{3} U_{\alpha\beta}(\nu)$ for $\overline{P}_\alpha(\nu)$ and the p wave contribution $U_{\alpha 3}$ to the crossed integral.

These results were obtained for $\lambda = .14$. This value of λ was chosen so that there would be a low energy p-wave resonance as proposed by Fraser and Fulco[28], in our approximation. This resonance is very narrow as can be seen from the formula

$$\cot \delta_3(\nu) \simeq 9(\nu - 3.56) \ , \tag{8.17}$$

which is valid in the neighborhood of the resonance in the approximation of keeping only the first two terms in the expansion of $L_3(\nu)$.

Since it is the crossed terms in the dispersion equation which make the solution of the Chew-Mandelstam equations non-trivial, it seemed most reasonable to compare the difference $\overline{P}_\alpha(\nu) - P_\alpha(\nu)$ with the crossed term in the formula for $\overline{P}_\alpha(\nu)$; the results are shown in table VII.* Considering that the crossed terms are themselves small the agreement of the $\alpha = 2$ and $\alpha = 3$ states seems reasonable, but the agreement of the $\alpha = 1$ state is poor, and thus the solution is unacceptable.

Because of the difficulty that the Chew-Mandelstam equations have no solution when λ is positive, there seems to be no way to make the power series approximation the basis of a numerical solution (i.e. the input of an iteration scheme).

What we have reported is not a complete investigation of the problem; further work includes testing the power series approximation for negative λ , when the equations of Mandelstam and Chew can have a

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* I am indebted to Drs. Chew and Mandelstam for this suggestion.

solution. Also a search must be made for an approximate set of equations which do not neglect higher partial waves in the crossing condition, and see whether the difficulties for positive values of persist when higher partial waves are included. If such an approximation is found, it is unlikely to change appreciably the perturbation expansion of $L_\alpha(\lambda)$ since we noted that the p-wave contribution to the crossed term in low orders of perturbation theory was small. Thus it might be found that our power series approximation to $L_\alpha(\nu)$ satisfied such an approximation. In table VII we list the p-wave contribution to $\overline{P}_\alpha(\nu)$, and observe that it is large, and greatly improves the agreement of $\overline{P}_\alpha(\nu)$ to $P_\alpha(\nu)$ for small ν ; thus it may be that the effect of higher partial waves are also important when testing the <u>sum</u> of the power series.

Chew and Mandelstam[24] have obtained numerical solutions of their equations which have no p-wave resonance and thus disagree qualitatively with the power series prediction for positive λ ; for negative values of λ there is rough agreement between Chew and Mandelstam's results and the power series, but it must still be determined whether the power series gives a solution for negative λ and if so whether it is the solution of Chew and Mandelstam.

REFERENCES

1. F. E. Low, Phys. Rev. 97, 1392 (1955).
2. G. C. Wick, Rev. Mod. Phys. 27, 339 (1955).
3. G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956).
4. M. Gell-Mann and M. L. Goldberger, Phys. Rev. 96, 1433 (1954).
5. G. Salzman and F. Salzman, Phys. Rev. 108, 1619 (1957).
6. Castillejo, Dalitz and Dyson, Phys. Rev. 101, 453 (1956).
7. G. F. Chew and S. Mandelstam, UCRL 8728 (April, 1959), and Phys. Rev., to be published (1960).
8. M. Baker, Ann. Phys. 4, 271 (1958).
9. B. A. Lippman and J. Schwinger, Phys. Rev. 79, 469 (1950).
10. M. Cini and S. Fubini, Nuovo cimento 11, 142 (1954).
11. R. Norton and A. Klein, Phys. Rev. 109, 991 (1958).
12. C. A. Hurst, Proc. Cambridge Phil. Soc. 48, 625 (1952).
13. W. Thirring, Helv. Phys. Acta 26, 33 (1953).
14. A. Petermann, Phys. Rev. 89, 1160 (1953) and Arch. Sci. (Geneva) 6, 5 (1953).
15. R. Utiyama and T. Imamura, Progr. Theoret. Phys. (Kyoto) 9, 431 (1953).
16. M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).
17. N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience Publishers, Inc., New York, N.Y., 1959), pp. 511-552.
18. M. Gell-Mann (Private Communication).
19. Grobner and Hofreiter, Integraltafel (Springer-Verlag, Vienna, Austria, 1950), Vol. II, p. 76, formula 9a.
20. H. S. Wall, Analytic Theory of Continued Fractions (Van Nostrand Co., New York, 1948), p. 371. I am indebted to Dr. Gordon Baym for suggesting this reference.

REFERENCES (continued)

21. E. L. Ince, Ordinary Differential Equations (Longmans, Green and Company, London, 1927), Chapter 3.

22. Mathematical Tables from Handbook of Chemistry and Physics (Chemical Rubber Publishing Company, Cleveland, Ohio, 1949), ninth edition, p. 272.

23. Sartori and Wataghin, Nuovo cimento 12, 144 (1954).

24. G. F. Chew, S. Mandelstam, and H. P. Noyes, UCRL 9001 (November, 1959), and Phys. Rev., to be published (1960).

25. H. B. Dwight, Tables of Integrals and other Mathematical Data (The Macmillan Company, New York, 1957), third edition, p. 11, formula 47.1.

26. F. J. Dyson, Phys. Rev. 117, 1616 (1960).

27. M. Baker and F. Zachariasen, Bull. Am. Phys. Soc. 5, 368 (1959), and Phys. Rev., to be published (1960).

28. W. F. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959), and Phys. Rev. 117, 1609 (1960).