CHAPTER V

ACCURACY OF TRANSIENT SOLUTIONS USING AN ASYMMETRIC NETWORK OF NODES

(A) INTRODUCTION

As mentioned in the introductory Chapter I, and as pointed out by MacNeal (8), locating the temperature points on the corners of rectangles or other regular polygons such as equilateral triangles, either of which defines a regular grid, is not a satisfactory arrangement for a solid with a curved or irregular boundary. For this type of solid a regular arrangement of nodes usually requires an extremely find grid with a very large number of points to follow the curved boundary closely. A moderate-size regular mesh that does not follow the boundary has been suggested by others (32, 10), by assuming that the nodes closest to the boundary surface actually fall on the boundary. For problems where the temperature on the boundary surface is specified (infinite heat-transfer coefficient) a somewhat better approximation is to interpolate linearly for this temperature (32). However, when the boundary equations have a finite non-zero heat-transfer coefficient, many techniques are available (6, 10, 17, 32), although none of them appears to be completely satisfactory (8), or to be widely used.

MacNeal suggested that by not restricting the location of the points to the corners of regular polygons, but by locating the points within and on the solid boundary conforming to a much less restrictive set of rules, an irregular boundary could be followed exactly, with a fairly coarse network. The equations for each node can then be found by an energy balance, and in general the equation for each node is different. These points are discussed in more detail in Chapter II, section C. A boundary condition involving a heat-transfer coefficient is easily included in the energy balance for a surface node by adding a term representing the energy in from the fluid of the form

$$(t_{f_i,n} - t_{i,n})h\,A_{f_i} = \text{Heat into surface node from fluid at time } n \qquad (V\text{-}1)$$

where i is allowed to take on values of the nodes on the boundary surface; t_{f_i} is the average fluid temperature opposite the i^{th} surface node ; and A_{f_i} is the surface area through which the energy flows. For the two-dimensional problem this area is the length of surface (assuming unit thickness) terminated by the perpendicular bisectors of the i^{th} node (see Figure V-1). The dimensionless conductance between the i^{th} surface node and the i^{th} fluid node is

$$y_{i,f_i} = \frac{h\,A_{f_i}}{k_0} \qquad (V\text{-}2)$$

The volume for heat capacity associated with the surface node for the two-dimensional case is bounded by the perpendicular bisectors of the i^{th} node and the boundary of the solid. Therefore, an approximate solution for an irregular solid, using a well designed fairly coarse network of points, based on MacNeal's rules for locating the points and calculating the conductances and capacitances of the nodes, would be expected to be more accurate than a regular mesh using the same number of points but not following the boundary exactly.

However, further study of the equations for the space truncation error associated with asymmetric nodes, as given in MacNeal, indicates that an approximate solution to the diffusion equation based upon an asymmetric network of nodes does not necessarily follow Richtmyer's (3) definition of consistency (see Chapter III, section A). This is true because Richtmyer has shown that, for an approximation to be consistent, the truncation error must go to zero as the network is refined; however, as shown by MacNeal's equations (but not pointed out or discussed), the series expansion for the contribution to this truncation error of a general asymmetric node shows that terms occur in this difference that do not go to zero as the neighboring nodes are moved closer. This also can be shown to be caused by the asymmetry in the node location. Since, according to the Richtmyer theory, the approximation must be both consistent and stable to be convergent to the continuous solution as the mesh is refined, the accuracy of an

approximate solution found using asymmetric nodes is in doubt. Consequently, the question is for what types of problems is it appropriate to use an asymmetric network, or to include asymmetric nodes in regular networks, and how should these nodes be located.

It should be pointed out that one of the distinguishing features of an asymmetric network of points is that the equation for each of the nodes is in general different from those of the other nodes because the thermal capacities of each of the points are different and because the thermal conductances between points within and on the boundary of the solid (excluding conductances to fluid temperature points) are not the same. Further, these capacities and conductances cannot be predicted by a recurrence relationship. However, for almost every network used in practice nodes with different heat capacities and/or different conductances occur, and these nodes can be considered in this sense asymmetric or irregular. Several examples are: (1) For the simple one-dimensional diffusion problem in Cartesian coordinates in a solid with uniform thermal properties, only for methods based on mesh $\Delta\xi/2$ are all the conductances (excluding those to fluid temperatures) and capacities equal; for method G based on mesh $\Delta\xi$ the two surface nodes at the boundaries have heat capacities that are different from the interior nodes, and further the heat capacities of these nodes cannot be found from a simple recurrence relationship. Hence, in the sense of heat capacities, method G is based on a regular

network with two irregular nodes at the surface. (2) For a one-dimensional problem in radial coordinates, for either spherical or cylindrical coordinates, approximated with a network of points spaced $\Delta\rho$ apart, the capacities and conductances are different for each node and, therefore, each node has a different equation, which is also the case of a general asymmetric network. However, for the regular radial network the capacities and conductances (and thus each equation) can be found from a recurrence relationship. (3) For an approximation based on a regular rectangular mesh spacing in a two-dimensional uniform solid with a curved boundary at a known temperature, the technique of linear interpolation mentioned previously can be shown to be equivalent to assuming that the points which have conductances to the surface nodes are asymmetric nodes and follow MacNeal's rules. These nodes then contribute to the truncation error quantities of the order of $\Delta\xi$ and $\Delta\eta$; while the other nodes contribute a truncation of the order of $(\Delta\xi)^2$ and $(\Delta\eta)^2$.

Thus, these examples show that the use of networks which contain irregular nodes, or for which the equations for each node are different, are common. Each of the irregular nodes in the above three examples can be shown to be consistent, however.

In order to study the accuracy of a general asymmetric network or the effect on accuracy of several irregular nodes in a regular network the classical method is used to study the truncation error and

propagation throughout the calculation. This is based on a method devised by Von Neumann and reported in most works on numerical solution of partial differential equations (2, 3, 4, 10, 17, 18, 21). The following study first defines the truncation error and then shows how this error affects the difference between the continuous and approximate solution. A detailed discussion of the space and time discretization errors is made. This includes a practical discussion of the meaning of consistency and indicates a possible method of making asymmetric nodes consistent. From these discussions some general rules can be formulated about when asymmetric nodes can be used and how they should be located. Some conclusions can also be made about the selection of the time differencing parameters. This investigation uses both analytic and numerical results. Since these are closely related, they are usually discussed together. The emphasis is primarily on the two-dimensional (space) problem, but many of the relationships also apply to one-dimensional problems, and some one-dimensional references are made. Unfortunately, the analytic and numerical studies of the truncation error and its propagation do not usually give as precise information about accuracy as does the comparison of the analytic expressions for the continuous and approximate solutions. However, it is hoped that eventually the results from the two types of error analysis may be related so that by examining the truncation error precise statements can be made

about the errors in eigenvectors, damping factors and initial vector. Consequently, a precise analysis, such as was made for a regular network for the one-dimensional problem, can be made from a knowledge of only the truncation errors.

In the following discussion the various shapes are associated with both networks and nodes, for example, a triangular network or hexagonal nodes. The shape associated with a network is the polygon formed by the branch lengths ℓ_{ij} when the points are interconnected. The shape associated with a node is the polygon formed by the perpendicular bisectors r_{ij} of the branch lengths and is the shape of the area used to calculate the heat capacity.

(B) DEFINITION OF DISCRETIZATION ERROR

A useful definition of the discretization or truncation error that is equivalent to the usual definition of this error can be developed by considering equation II-39 which defines the stepping out of the calculation. This vector equation, rearranged, is:

$$\frac{1}{\Delta\tau}(t_{n+1}-t_n) - \frac{Y}{A}[\gamma t_{n+1}+(1-\gamma)t_n] - \frac{Y_B}{A}[\gamma t_{B,n+1}+(1-\gamma)t_{B,n}] = 0 \qquad (V-3)$$

If the continuous temperature vector T_n is substituted for the approximate solution vector t_n, the right side of the equation is no longer equal to zero:

$$\frac{1}{\Delta\tau}(T_{n+1}-T_n) - \frac{Y}{A}[\gamma T_{n+1}+(1-\gamma)T_n] - \frac{Y_B}{A}[\gamma T_{B,n+1}+(1-\gamma)T_{B,n}] \equiv d_n \qquad (V-4)$$

This equation V-4 defines the discretization error vector d_n; the elements of this vector $d_{i,n}$ constitute the discretization error associated with the i^{th} node. These components are

$$d_{i,n} \equiv \frac{T_{i,n+1}-T_i}{\Delta\tau} - \frac{1}{A_i}\sum_{j=1}^{R} y_{ij}[\gamma(T_{j,n+1}-T_{i,n+1})+(1-\gamma)(T_{j,n}-T_{i,n})] \qquad (V-5)$$

$$i = 1, 2, \ldots, S$$

As this definition of the discretization error vector is based on either the vector equation, or, equivalently, the component equation, it defines the discretization error for both the interior nodes and nodes which have conductances containing a heat-transfer coefficient to fluid temperature points outside the boundaries.

The reason for using this definition for the discretization error rather than the usual definition is that the effect of this discretization error vector upon the error in the approximate solution is easily derived. Further, for the interior points with no conductances to a fluid temperature the discretization error component $d_{i,n}$ is equivalent to the usual definition. This can be easily seen by writing the diffusion equation for the i^{th} node at the n^{th} and $(n+1)$ time increments in the following form,

$$\frac{\partial T_{i,n}}{\partial \tau} - \frac{1}{K_0(C_p\sigma)_i} \nabla' k_i \nabla' T_{i,n} = 0 \qquad \text{(V-6)}$$

$$\frac{\partial T_{i,n+1}}{\partial \tau} - \frac{1}{K_0(C_p\sigma)_i} \nabla' k \nabla' T_{i,n+1} = 0 \qquad \text{(V-7)}$$

and then subtracting an appropriate linear combination of these equations from equation V-5. The discretization error for the points with conductances across an interface to a fluid cannot be found by subtracting the diffusion equation, as those equations are actually approximations to both the diffusion equation and the boundary equations.

Since other works have not usually defined, discussed, or used the discretization error for nodes with conductances involving a heat-transfer coefficient, the above definition is not in conflict. In section D of this chapter the discretization error for both the interior points and points adjacent to fluid temperatures is discussed in more detail.

(C) PROPAGATION OF DISCRETIZATION ERROR

A difference equation for the error in the approximate solution can be derived from the above definition of the discretization error. The solution of this equation indicates how the discretization error affects the error in the approximate solution. The knowledge of how the discretization error affects the accuracy of the approximate solution should then allow a better practical understanding of section D, on how the selection of the differencing parameters affects the size of the discretization error.

In order to make the following derivation as general as possible equation V-3 is rewritten to include the effect of round-off error. This can be done by noting that if the approximate solution vectors t_{n+1} and t_n, as found by stepping out the solution, are substituted in equation V-3 the sum of vector-matrix products is not the zero vector, but is a vector e_n made up of small components $\epsilon_{i,n}$ caused by using only a finite number of decimals in the stepping calculation. This equation is, in practice, then

$$\frac{1}{\Delta\tau}(t_{n+1}-t_n)-\frac{Y}{A}\left[\gamma t_{n+1}+(1-\gamma)t_n\right]-\frac{Y_B}{A}\left[\gamma t_{B,n+1}+(1-\gamma)t_{B,n}\right]=\epsilon_n \qquad (V-8)$$

Now subtracting equation V-8 from equation V-4 and substituting v_n for the difference between solution vectors (T_n-t_n), the difference equation for the error in the approximate solution is

$$[I-\gamma\Delta\tau\frac{Y}{A}]v_{n+1}=[I+(1-\gamma)\Delta\tau\frac{Y}{A}]v_n+\Delta\tau\frac{Y_B}{A}[(v_{B,n+1})\gamma+(1-\gamma)(v_B)_n]$$

$$+\Delta\tau(d_n+e_n) \qquad \text{(V-9)}$$

or solving for v_{n+1}:

$$v_{n+1}=[I-\gamma\Delta\tau\frac{Y}{A}]^{-1}[(I+\{1-\gamma\}\Delta\tau\frac{Y}{A})v_n+\Delta\tau\frac{Y_B}{A}(\gamma v_{B,n+1}+\{1-\gamma\}v_{B,n})$$

$$+\Delta\tau(d_n+e_n)] \qquad \text{(V-10)}$$

where $v_{B,n}=T_{B,n}-t_{B,n}$.

The quantity $\Delta\tau(Y_B/A)(\gamma v_{B,n+1}+\{1-\gamma\}v_{B,n})$ is usually zero, as the approximate boundary temperature vector is almost always taken equal to the continuous temperature vector, $T_{B,n}$. However, if an element of $t_{B,n}$ is not equal to the corresponding element in $T_{B,n}$ the effect, as seen from equation V-10, is the same as changing the corresponding element in the discretization error vector. Assuming that no error is made in the boundary vector we have

$$v_{n+1}=[I-\gamma\Delta\tau\frac{Y}{A}]^{-1}[(I+\{1-\gamma\}\Delta\tau\frac{Y}{A})v_n+\Delta\tau(d_n+e_n)] \qquad \text{(V-11)}$$

The similarity between this difference equation and the difference equation II-39 that defines the calculation procedure should be noted. The major difference is that the forcing function vector $[\Delta\tau(Y_B/A)][\gamma t_{B,n+1}+(1-\gamma)t_{B,n}]$ which has non-zero elements only

for nodes with conductances to the fluid temperature is now replaced with the vector $\Delta\tau(d_n + e_n)$ which, in general, has all non-zero elements. Thus, equation V-11 can be considered to be a difference approximation to the partial differential equation of diffusion with a source term:

$$\frac{\partial V}{\partial \tau} = \frac{1}{K_0 C_p \sigma} \nabla' k \nabla' V + D(\xi, \eta, \tau) \qquad (V-12)$$

where $D(\xi, \eta, \tau)$ = known distributed source term expressed in degrees per dimensionless time corresponding to elements in $(d_n + e_n)$

$V(\xi, \eta, \tau)$ = continuous temperature function corresponding to elements in v_n.

This shows that the effect of the discretization and round-off error is that it acts equivalently to an energy source (or sink) distributed throughout the solid.

The solution of the difference equation V-11 for the error, as found by classical means, is:

$$v_n = CQ^n C^{-1} v_0 + \sum_{p=1}^{n} CQ^{n-p} C^{-1} (I - \gamma \Delta\tau \frac{Y}{A})^{-1} (d_{p-1} + e_{p-1}) \Delta\tau \qquad (V-13)$$

where $v_0 = T_0 - t_0$

and where the C and Q matrices are the eigenvector and eigenvalue matrices as defined in equations II-43, II-46, and II-47.

The solution for the error v_n can also be expressed in terms of the calculation matrix $[I-\gamma\Delta\tau(Y/A)]^{-1}[I+(1-\gamma)\Delta\tau(Y/A)]$ by using the eigenvector-eigenvalue matrix identity:

$$C\,Q^{n-p}\,C^{-1} = \left[I-\gamma\Delta\tau\,\frac{Y}{A}\right]^{p-n}\left[I+(1-\gamma)\Delta\tau\,\frac{Y}{A}\right]^{n-p} \qquad \text{(V-14)}$$

The solution equation V-13 has been checked in equation V-11 using the identity in equation V-14, and it is analogous to the solution for a system of ordinary differential equations with constant coefficients as given in Bellman (33), p. 169.

From the above equation, one sees that round-off error is not the cause of instability, but can only aggravate the basic instability caused by the eigenvalues q_j of the calculation matrix. At this point, for simplicity, the round-off error is assumed to be much smaller than the discretization error and it is neglected in the remainder of the discussion:

$$|e_{i,n}| << |d_{i,n}| \qquad i = 1, \ldots, S \qquad n \geq 0 \qquad \text{(V-15)}$$

For a thorough quantitative discussion of the effect of round-off error the references can be consulted (3, 10, 17). The main conclusion is, as given in Richtmyer (3), p. 25, that if the rounding is unbiased and if the quantities are kept in scale, round-off errors accumulate roughly in proportion to the square root of the number of increments n.

In most approximate solutions the stepping-out calculation is started with the continuous temperature vector and v_0 is zero. In this case the error in the approximate solution can be written as a weighted cumulative sum:

$$v_n = \sum_{p=1}^{n} C\,Q^{n-p}\,C^{-1}\left[I-\gamma\Delta\tau\,\frac{Y}{A}\right]^{-1} d_{p-1}\Delta\tau$$

$$= C\,Q^{n-1}\,C^{-1}\left[I-\gamma\Delta\tau\,\frac{Y}{A}\right]^{-1} d_0\Delta\tau + C\,Q^{n-2}\,C^{-1}\left(I-\gamma\Delta\tau\,\frac{Y}{A}\right) d_1\Delta\tau + \ldots$$

$$+ C\,Q^{n-p}\,C^{-1}\left[I-\gamma\Delta\tau\,\frac{Y}{A}\right]^{-1} d_{p-1}\Delta\tau + C\,Q^{n-p-1}\,C^{-1}\left(I-\gamma\Delta\tau\,\frac{Y}{A}\right)^{-1} d_p\Delta\tau + \ldots$$

$$+ C\,Q\,C^{-1}\left(I-\gamma\Delta\tau\,\frac{Y}{A}\right)^{-1} d_{n-2}\Delta\tau + \left(I-\gamma\Delta\tau\,\frac{Y}{A}\right)^{-1} d_{n-1}\Delta\tau \qquad \text{(V-16)}$$

where d_p is the discretization error in stepping from increment p to p+1.

As only stable solutions are considered here, equation V-16 shows that at time n only the discretization errors that occur for several time increments immediately previous to the n^{th} increment are significant.

The above solution equation is a rigorous solution based on no estimates or series expansions. Unfortunately, unless the discretization error d_n is known as a function of time, the summation cannot be expressed in a simpler form; even if it were known as a function of time, in all probability it would be too complicated a function to obtain a simple expression for the summation. Thus the solution is

of limited usefulness. However, an averaged discretization error vector $\bar{d}_n$ can be defined such that the error vector v_n is given by the following equation.

$$v_n = \sum_{p=1}^{n} CQ^{n-p}C^{-1}(I-\gamma\Delta\tau\frac{Y}{A})^{-1} d_{p-1}\Delta\tau \equiv \left[\sum_{p=1}^{n} CQ^{n-p}C^{-1}\right](I-\gamma\Delta\tau\frac{Y}{A})^{-1}\Delta\tau\,\bar{d}_n \tag{V-17}$$

where $\bar{d}_n$ is a vector constant with time that gives the same error at time n, and where the summation $\left[\sum_{p=1}^{n} CQ^{n-p}C^{-1}\right]$ now is a matrix. This vector contains time-averaged components $\bar{d}_{i,n}$. By using the matrix identity in equation V-14, the matrix equations in Fadeeva (16), p. 62, for matrix series, together with other matrix manipulations, the product of the summation matrix and the other matrix is exactly:

$$\left[\sum_{p=1}^{n} C\,Q^{n-p}\,C^{-1}\right]\left[I-\gamma\Delta\tau\frac{Y}{A}\right]^{-1}\Delta\tau = -\left[I - C\,Q^{n}C^{-1}\right]Y^{-1}A \tag{V-18}$$

The solution becomes

$$v_n = -\left[I - C\,Q^{n}\,C^{-1}\right]Y^{-1}A\,\bar{d}_n$$

$$= \left[(I-\gamma\Delta\tau\frac{Y}{A})^{-n}\left(I+\left\{1-\gamma\right\}\Delta\tau\frac{Y}{A}\right)^{n} - I\right]Y^{-1}A\,\bar{d}_n \tag{V-19}$$

If the boundary forcing functions are such that a true steady state exists, the steady-state error is given by

$$v_{SS} = -Y^{-1}\,A\,d_{SS} \tag{V-20}$$

where d_{SS} is given by

$$Y\, T_{SS} + Y_B\, T_B = \Lambda\, d_{SS} \qquad \text{(V-21)}$$

In this case the steady-state discretization error vector d_{SS} is only a function of the number and location of the points, i.e., the Y and Y_B matrices, and obviously it cannot be a function of the time differencing parameters $\Delta\tau$ and γ.

A study of equation V-19 and its special steady-state case, equation V-20, leads to several important practical conclusions about the effect of the discretization error on the accuracy of the solution or smallness of the elements in v_n, even without a specific knowledge of the averaged discretization error vector $\bar{d}_n$. However, in order to relate the information in the next section about the actual discretization error d_n to the following discussion, several general properties of the averaged vector $\bar{d}_n$ should be mentioned. The averaged discretization error vector should for most problems be approximately equal to the discretization error for the several increments just previous to n and the n^{th} increment. This can be seen from writing out the summations in equation V-17, which defines the average $\bar{d}_n$ and by remembering the damping or decaying nature of the Q^n matrix for stable solutions. Moreover, the elements of the actual discretization error vector d_n involve the second- and higher-order derivatives of the continuous temperature solution with space variables, and since

this temperature function is a smooth function for most problems, for a short period of time spanning only a few time increments, the discretization error vectors should not change much. Consequently, the averaged discretization error vector $\bar{d}_n$ can be considered to be about the same as the discretization error vectors that occur just previous to and during the n^{th} increment, and comments made about this discretization error vector d_n in the next section should apply equally to the averaged vector $\bar{d}_n$. The averaged vector $\bar{d}_n$ is a function of time or of n. Further, in general, the elements of $\bar{d}_n$ or d_n are not the same size; however, in the following study the assumption is made that the elements in $\bar{d}_n$ are the same size. Conclusions derived on the basis of this assumption can be easily modified.

The matrix $[I-C\Omega^nC^{-1}]$ which multiplies the vector $(Y^{-1}A\,\bar{d}_n)$ to give v_n in equation V-19 can be taken as the identity matrix in this analysis. This is permitted because (1) it becomes essentially the identity matrix for sufficiently large n, because of the decaying nature of $(C\Omega^nC^{-1})$; and (2) only a very rough estimate of the relative elements in v_n is to be made. Thus, for values of n, where $[I-C\Omega^nC^{-1}]$ is only very approximately the identity matrix, it can still be assumed to be the identity matrix, as we are usually interested in estimating the position of the decimal point and possibly the first digit for the elements in v_n. For all but very short times it should be possible to obtain this rough estimate by assuming that

$$v_n \simeq - Y^{-1} A \bar{d}_n \qquad (V\text{-}22)$$

Note that v_n is still a function of time as $\bar{d}_n$ changes with time. To quantitatively analyze the accuracy for very short times when $[I\text{-}CQ^nC^{-1}]$ differs considerably from the identity matrix, the effect of this matrix operator would have to be found by numerical calculation. Some insight into the relative size of the elements of $[I\text{-}CQ^nC^{-1}]$ can be gained by noting that it appears in the approximate transient temperature solution for a solid with a zero initial condition.

$$t_n = [I\text{-}CQ^n C^{-1}] t_{P,n} \qquad (V\text{-}23)$$

Thus the conclusions about the manner in which the number and location of points affects the vector $(Y^{-1} A \bar{d}_n)$ can be applied to the error vector v_n for most problems and for all but small values of n.

The first and most obvious of these conclusions is that the vector $A \bar{d}_n$ is formed by the diagonal thermal capacity matrix A and $\bar{d}_n$ as components of $A_i \bar{d}_{i,n}$; that is, the time-averaged discretization error for the i^{th} node is weighted by the thermal capacity of that node. Thus, for a fixed network of nodes, the discretization error of the relatively large nodes is magnified and that for the relatively small nodes has a smaller effect.

However, the above conclusion should not be interpreted to mean that merely by reducing the area of a few or all of the nodes can the approximate solution be made arbitrarily accurate; i.e., the

elements of the error vector v_n do not necessarily go to zero as the area goes to zero. This is because, to reduce the area of the nodes, the number of nodes S, which is the dimension of the square Y^{-1} and A matrices, must be increased proportionately; consequently, the elements in the vector $(Y^{-1}A\,\bar{d}_n)$ are now summations over more elements and the elements in this Y^{-1} matrix remain about the same size. This argument can be put on a more firm basis by considering the norm of the $Y^{-1}A$ matrix for one- and two-dimensional problems based on regular rectangular meshes. For these networks the heat capacities of the nodes are equal and assuming all elements of the time-averaged $\bar{d}_n$ vector are the same, an element of $(Y^{-1}A\,\bar{d}_n)$ is

$$(Y^{-1}A\,\bar{d}_n)_i = (A_i\,\bar{d}_{i,n}) \sum_{j=1}^{S} v_{ij} \qquad (V\text{-}24)$$

$$i = 1, \ldots, S$$

where $(Y^{-1}A\,\bar{d}_n)_i = i^{th}$ element of the vector $(Y^{-1}A\,\bar{d}_n)$

v_{ij} = the element in i^{th} row, j^{th} column, of Y^{-1} matrix.

For both the one-and two-dimensional networks above the elements v_{ij} in the inverse conductance matrix Y^{-1} are shown to be negative using results in the references (6, 29), i.e., Y^{-1} is a non-positive matrix. Consequently, $v_{ij} \leq 0$ for all i and j, and

$$-\sum_{j=1}^{S} v_{ij} = +\sum_{j=1}^{S} |v_{ij}| \qquad (V\text{-}25)$$

and the row norm defined in equation III-10, which is the maximum of the absolute values of the row sums, is a direct measure of the summation in equation V-24, and the product of the norm of Y^{-1} and A_i is a measure of the coefficient of $\bar{d}_{n,i}$.

$$(Y^{-1}A\,\bar{d}_n)_i \leq A_i \|Y^{-1}\|\,\|\bar{d}_{n,i}\| \qquad \text{(V-26)}$$

For the one-dimensional problem with both boundaries specified (problem IV, Table IV-9), approximated with method G, the inverse of the conductance matrix Y is known (29) and the norm-heat capacity product can be shown to be:

$$A_i \|Y^{-1}\| = (\Delta\xi)\|Y^{-1}\| = \frac{1}{8} \qquad \text{S-odd} \qquad \text{(V-27)}$$

$$A_i \|Y^{-1}\| = \frac{1}{8}\left[1 - (\Delta\xi)^2\right] \qquad \text{S-even} \qquad \text{(V-28)}$$

where S is the number of points for which the temperature is to be approximated as a function of time and where $\Delta\xi = \frac{1}{S+1}$. (Note that this S is not the same as would usually be used for method G in Chapter IV, but is S'-1 where S' is equivalent to S, as used in Chapter IV.) In Figure V-2 the product $\|Y^{-1}\|A_i$ and the number of variable temperature points are related to the area A_i based on equations V-27 and V-28. The extrapolation of $A_i\|Y^{-1}\|$ to an infinite number of points or zero nodal area A_i for a uniform unit square solid gives a finite, non-zero result of 1/8. The two-dimensional problem with a known boundary temperature has been studied numerically by calculating

the inverse conductance matrix Y^{-1} as the mesh is refined. In this calculation the network is made up of regular rectangles with constant spacing $\Delta\eta$ and $\Delta\xi$ with points located on the boundary surface. The area of each variable node is then

$$A_i = (\Delta\xi)(\Delta\eta) = \frac{1}{S + S_1 + S_2 + 1} \tag{V-29}$$

where $\Delta\xi = 1/(S_1+1)$

and $\Delta\eta = 1/(S_2+1)$ and S_1 and S_2 are the number of variable points in the ξ and η directions, respectively, and the conductance between nodes connected in the η direction is

$$y_{ij} = \frac{\Delta\xi}{\Delta\eta} \tag{V-30}$$

and in the ξ direction is

$$y_{ij} = \frac{\Delta\eta}{\Delta\xi} \tag{V-31}$$

The first mesh used contained nine variable points (S=9) with $\Delta\xi$ and $\Delta\eta$ equal to one-fourth (S_1 and S_2 equal to 3). The network was refined by first adding a row of points and then a column of points so that the second calculation was based on an S of 12 with an S_2 of 4 and an S_1 of 3, and the third on an S of 16 with both S_1 and S_2 equal to 4. The last network studied contained 64 points, 8 in each row and 8 in each column of the network. The numerical results for $A_i\|Y^{-1}\|$ are related to the area A_i and to the number of variable points S in Figure V-3. These results are somewhat like those for the one-dimensional

problem in that for selection of S resulting in a perfect square ($\Delta\xi$ and $\Delta\eta$ equal) two different lines can be drawn for S odd and S even. These lines are approximately linear and can be extrapolated to a constant value, at zero A_i or infinite S, of about 0.074. The products $A_i \| Y^{-1} \|$ for meshes with unequal $\Delta\xi$ and $\Delta\eta$ lie within the two lines.

From these two studies of the approximations to one- and two-dimensional problems for regular networks, we conclude that the components of the averaged discretization error must be arbitrarily small for the components of the error vector v_n to be made arbitrarily small for a stable solution. This is true because, as the regular network is refined, the coefficients, as measured by $A_i \| Y^{-1} \|$, of the time-averaged discretization error vector $\bar{d}_n$ do not go to zero as the nodal area A_i is reduced to zero. The above conclusion can also be seen to be equivalent to Richtmyer's consistency requirement, as, in order to obtain an arbitrarily accurate approximate solution, the solution must be both stable, $|q| < 1$, and the norm $\|\bar{d}_n\|$ of the discretization error vector must go to zero as the mesh is refined to make the norm $\|v_n\|$ of the error vector go to zero.

The above conclusion has been shown here only for regular networks, where the inverse conductance matrix Y^{-1} has been shown to be a non-positive matrix, for certain methods of refining the network, and for a problem with a specified boundary temperature. However,

since that conclusion is equivalent to the stability-consistency results of Richtmyer, a like conclusion can probably also be made for the asymmetric network. This conclusion is that the norm of the inverse conductance-capacitance matrix, $\|Y^{-1}A\|$, does not go to zero as all nodal areas go to zero. There is no reason to assume that for the irregular location any difference in behavior should occur; indeed the inverse conductance matrices Y^{-1} for four asymmetric networks shown in Tables V-1, V-2, V-3, and V-4 show that this matrix is a non-positive matrix, as the inverse conductance matrices must be for all problems with regular rectangular networks. Although no proof has been attempted that the inverse conductance matrix Y^{-1} for an asymmetric network is a non-positive matrix, the Y matrix has properties like those used in the proof for the matrices for rectangular networks (6).

Of more interest is how an asymmetric network is refined. Although this topic is discussed in more detail in the next section, one might consider adding more nodes in a certain vicinity of an asymmetric network leaving the rest of the network unchanged. These points would be located according to MacNeal's rules. The effect on the accuracy of the approximate solution for the nodes in the unchanged network probably would be insignificant unless the discretization error was also reduced. This is true, even though the area of some of the nodes is reduced, because there are now more nodes and each of the

new nodes probably has a weighting in the inverse conductance matrix Y^{-1} of about the same magnitude as the large nodes had before. Thus the sum of the weighting of the discretization error for this vicinity is not reduced. The error in the approximate solution for the new smaller nodes also probably is not improved unless the components of $\bar{d}_n$ are smaller.

The elements of the inverse conductance matrix Y^{-1} are the weightings for the discretization error-area products. That is, the element in the i^{th} row and j^{th} column, v_{ij}, of the Y^{-1} matrix is the weighting given to the $A_j \bar{d}_{jn}$ product in the equation for the error for the i^{th} node $v_{i,n}$. Because of the symmetry of the Y^{-1} matrix this coefficient v_{ij} is also v_{ji} and is the weighting of the product $A_i d_{i,n}$ in the equation for the error of the j^{th} node. Consequently, conclusions made about the effect of the $A_j \bar{d}_{jn}$ on the i^{th} node also apply to the effect of the $A_i \bar{d}_{i,n}$ on the j^{th} node. These effects have been studied by numerically calculating the inverse conductance matrices, and correlating the size of the elements with the position of the nodes in the networks.

The inverse conductance matrix Y^{-1} is shown on Tables V-1, V-2, V-3, and V-4, for four asymmetric networks together with the Y matrix and a sketch of the network of points. Also shown are the matrices for two regular rectangular networks for 9 points. For the asymmetric networks the 9 x 9 matrix is for the numerical example

used in Chapter III and has a specified temperature along the horizontal boundary. The 8 x 8 matrices are for the same geometry but the hypotenuse boundary now also has a specified temperature. The shapes I and II are from solid-propellant curing problem used in the original Longwell (9) reference. The regular rectangular networks are for a square and a rectangular problem with specified boundary conditions.

A study of the inverse conductance matrix Y^{-1}, the conductance matrix Y, and the sketch allows the following conclusions to be made about the effect on the i^{th} error $v_{i,n}$ of the components of the discretization error heat capacity vector $(A_i \bar{d}_{i,n})$ based on equation V-22, for both regular and asymmetric networks.

(1) The diagonal element v_{ii} of the inverse conductance matrix has a larger absolute value than the off-diagonal elements in the i^{th} row or i^{th} column. Consequently the product $(A_i \bar{d}_{i,n})$ or the area-weighted discretization error associated with the i^{th} node is weighted more in the error for the i^{th} node, $v_{i,n}$, than any other area-weighted discretization error product. This property has been demonstrated to hold for rectangular networks, and inverse conductance matrices obtained for the asymmetric network indicate that it holds there also.

(2) The area products $(A_j \bar{d}_{j,n})$ for those nodes which have conductances to the i^{th} node (non-zero elements in Y matrix, $y_{ij} \neq 0$) are usually weighted the next in the equation for the error at node i. The weighting is larger for those with the larger conductances to the i^{th} node.

(3) The weighting for nodes that can be considered to have a high equivalent thermal resistance between them (usually nodes distant from each other) are small and the discretization error products for these nodes influence the error at i negligibly.

The conclusions about the relative effect of the area-weighted discretization error vector have been checked numerically using the asymmetric network shown in Figure III-1 to solve for the steady-state condition where the horizontal surface is at unity, the vertical boundary is adiabatic, and the hypotenuse is at a temperature of $\frac{1}{2}$. For the steady-state problem equation V-22 gives the error exactly and is not an approximation. This is equivalent to a steady-state problem for the unit square with its horizontal boundaries at one and its vertical boundaries at zero. The continuous solution as derived by suitable coordinate transformations and combinations from a solution in Carslaw and Jaeger (1) is:

$$T_{SS}(\xi,\eta) = \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{[\sin(2j+1)\pi(\xi+0.5)]\,[\cosh(2j+1)\pi\eta]}{(2j+1)\,[\cosh(\frac{2j+1}{2})\pi]} \qquad (V-32)$$

The approximate solution was computed from

$$t_{SS} = -Y^{-1}Y_B t_B \qquad (V-33)$$

where the Y matrix is modified from that used in the transient problem in Chapter III, by assuming anti-symmetry of the nodes about the hypotenuse. This is equivalent to taking nodes on the hypotenuse with

a temperature of one half or to assuming that the temperature difference between the temperature of anti-symmetric nodes opposite 1, 3, 5, 7, and 8, and $\frac{1}{2}$ is equal to the temperature difference between the nodes on the hypotenuse ($\frac{1}{2}$) and nodes 1 3, 5, 7, and 8. Suitable modifications were made in the Y_B matrix. Note that the temperature at node 9 is now specified. Three approximate solutions were calculated using temperatures at node A of $\frac{1}{2}$, $\frac{3}{4}$, and 1. An argument for using each of these temperatures at A is that in the continuous problem the temperature at node A can be considered indeterminate. However, the continuous solution gives one half at this point; therefore, use of the temperatures of $\frac{3}{4}$ and 1 can be considered to be the equivalent of changing the area-weighted discretization error at point 1 (see equation V-10).

A summary of these calculations is in Table V-7. These show that increasing this boundary temperature from $\frac{1}{2}$ to 1 changes the area-weighted discretization error for node 1 from -0.0468 to 0.0740, while the other discretization errors remain constant. This change in discretization error changes the error at node 1 from 0.0067 to -0.0107, and also changes the error significantly at its neighbor nodes 2 and 3. However, no significant change is observed in nodes located at a distance, for example, nodes 6, 7, and 8. Thus the conclusions about the relative influence of the discretization error are confirmed. The accuracy of the approximate solution for the steady-state solution

is discussed in more detail later; however, it can be seen that the asymmetric network does give satisfactory results for this problem.

1. Conclusions

The important conclusions based on equation V-19 for the propagation of the discretization error which should be remembered in the following discussion on the discretization error and in locating points are:

(1) The error of an approximate solution cannot be made small merely by adding more points and reducing the area unless the discretization errors are decreased by the addition of the points. An important corollary to this conclusion is that the size of the discretization errors directly affects the error in the approximate solution, and one wishes to locate the points in such a manner that the discretization errors are small, particularly in a region of interest for the specific geometry, for the specific boundary conditions and forcing functions.

(2) For a fixed network of points the discretization error at a node is weighted by the heat capacity of that node; thus, the discretization error for a relatively large node is weighted more than that for a small node.

(3) The discretization error weighted with the nodal heat capacity ($A_i \bar{d}_{i,n}$) associated with the i^{th} node affects the accuracy of the solution at the i^{th} node more than it affects the accuracy of the

adjoining nodes. It affects the error at these adjoining nodes more than those located some distance away which have a high equivalent thermal resistance to the i^{th} node. However, the sum of the discretization errors for many nodes on the i^{th} node, even though each of these nodes does have a high equivalent resistance to the i^{th} node, can give a significant contribution to the error at i.

(D) DISCRETIZATION ERROR

In this section series expansions are developed for the discretization error associated with the selection of the time discretization parameters of γ and $\Delta\tau$, and the space discretization error associated with the number and location of points. On the basis of these expansions several conclusions can be made about the accuracy of asymmetric networks, ways of improving the accuracy, and about the concept of consistency. Some practical conclusions also can be made about the selection of the differencing parameters.

1. Division of Discretization Error with Time and Space Differencing Parameters

In most discussions of the discretization error the total discretization error, as defined by equations V-4 or V-5, is divided into two parts. One part is associated with the time differencing parameters γ and $\Delta\tau$, and the other part with the space differencing parameters. This can be seen to be a logical procedure as the approximate formulation in Chapter II, section C, treated the two discretizations separately. However, the precise way in which this division is made affects both the size of the discretization associated with time and space and the series expansion for both discretizations. However, despite the fact that the size of the discretization errors associated

with time and space are affected by the precise division, the total discretization error which is the sum of the time and space errors is the same as defined by equation V-3.

Indeed, for the one-dimensional problem with a regular mesh, Richtmyer shows one series expansion for the total discretization error that can be derived by not dividing the discretization error. That expression is

$$d_{m,n} = \frac{\partial^4 T_{m,n}}{\partial \xi^4} \left[\Delta\tau \left(\frac{1}{2} - \gamma \right) - \frac{1}{12} (\Delta\xi)^2 \right] + \cdots \qquad \text{(V-34)}$$

$$\tfrac{1}{2} < m < S - \tfrac{1}{2}$$

where m is for an interior point.

Note that if the coefficient of $\partial^4 T_{m,n}/\partial \xi^4$ is set equal to zero equation IV-262 for γ_o results. Equation V-34 can be derived by writing equation V-5 in terms of m and $\Delta\xi$ for the one-dimensional problem and by subtracting the diffusion equation at point m at time n, which is

$$\frac{\partial T_{m,n}}{\partial \tau} - \frac{\partial^2 T_{m,n}}{\partial \xi^2} = 0 \qquad \text{(V-35)}$$

This gives

$$\frac{T_{m,n+1} - T_{m,n}}{\Delta\tau} - \frac{\partial T_{m,n}}{\partial \tau}$$

$$- \frac{1}{(\Delta\xi)^2} \left[(T_{m-1,n+1} - 2T_{m,n+1} + T_{m+1,n+1})\gamma + (T_{m-1,n} - 2T_{m,n} + T_{m+1,n})(1-\gamma) \right]$$

$$+ \frac{\partial^2 T_{m,n}}{\partial \xi^2} = d_{m,n} \qquad \text{(V-36)}$$

Then the quantities $(T_{m,n+1})$, $(T_{m+1,n+1})$, $(T_{m-1,n+1})$, $(T_{m+1,n})$, and $(T_{m-1,n})$ are written in terms of $T_{m,n}$ using a two-variable Taylor series expansion in $\Delta\xi$ and $\Delta\tau$ which, together with derivative identities based on the diffusion equation, gives equation V-34.

Although the development of a single series such as equation V-34 for the two-dimensional problem with a regular mesh would be desirable, unfortunately this would require a Taylor series expansion in three variables, $\Delta\tau$, $\Delta\xi$, and $\Delta\eta$. Even for a regular mesh, such an expansion would be very cumbersome, both to derive and to reduce to understandable expressions. Thus, the discretization error for the interior points is compared to the diffusion equation at times n and (n+1) weighted by γ and $(1-\gamma)$, respectively. Subtracting $(1-\gamma)$ times equation V-6 and γ times equation V-7 from equation V-5, there is obtained:

$$d_{i,n} = \left[\frac{T_{i,n+1} - T_{i,n}}{\Delta\tau} - \left\{ \gamma \frac{\partial T_{i,n+1}}{\partial\tau} + (1-\gamma) \frac{\partial T_{i,n}}{\partial\tau} \right\} \right]$$

$$- \gamma \left[\frac{1}{A_i} \sum_{j=1}^{l} y_{ij} (T_{j,n+1} - T_{i,n+1}) - \frac{1}{K_0 (C_p \sigma)_i} \nabla' k \nabla' T_{i,n+1} \right]$$

$$- (1-\gamma) \left[\frac{1}{A_i} \sum_{j=1}^{S} y_{ij} (T_{j,n} - T_{i,n}) - \frac{1}{K_0 (C_p \sigma)_i} \nabla' k \nabla' T_{i,n} \right] \qquad \text{(V-37)}$$

Now the discretization error associated with the time parameters can be defined as:

$$\varphi_{i,n} \equiv \frac{T_{i,n+1}-T_{i,n}}{\Delta\tau} - \left\{\gamma \frac{\partial T_{i,n+1}}{\partial \tau} + (1-\gamma)\frac{\partial T_{i,n}}{\partial \tau}\right\} \qquad \text{(V-38)}$$

and the space discretization error as:

$$\sigma_{i,n} = \frac{1}{A_i}\sum_{j=1}^{l} y_{ij}(T_{j,n}-T_{i,n}) - \frac{1}{K_0(C_p\sigma)_i}\nabla' k_i \nabla' T_{i,n} \qquad \text{(V-39)}$$

and

$$\sigma_{i,n+1} = \frac{1}{A_i}\sum_{j=1}^{R} y_{ij}(T_{j,n+1}-T_{i,n+1}) - \frac{1}{K_0(C_p\sigma)_i}\nabla' k_i \nabla' T_{i,n+1} \qquad \text{(V-40)}$$

Thus the total discretization error is:

$$d_{i,n+1} = \varphi_{i,n} - [\gamma\sigma_{i,n+1} + (1-\gamma)\sigma_{i,n}] \qquad \text{(V-41)}$$

This division of the discretization error has the disadvantage that the total contribution from the space error, $[(1-\gamma)\sigma_{i,n} + \gamma\sigma_{i,n+1}]$, is a function of the time differencing parameters γ and $\Delta\tau$, but it has the advantage that an expansion for $\sigma_{i,n}$ or $\sigma_{i,n+1}$ is only a two-variable series, and thus is significantly simpler than the three-variable expansion. This definition of the space discretization error also gives the usual definition for the steady-state problem.

Equations V-39 and V-40 apply for points or nodes which have conductances only to other nodes within or on the solid boundary, that

is, not to a fluid node through a finite non-zero heat-transfer coefficient. The difference equations for nodes, whether located on the boundary, as in method G, or away from the boundary, as in method C, that have a heat-transfer coefficient conductance to a fluid node, are an approximation to both the diffusion equation and to the boundary equation:

$$H[T_f(\tau) - T(\xi_B, \eta_B, \tau)] = - \frac{k \partial T}{\partial \eta} (\xi_B, \eta_B, \tau) \qquad (V\text{-}42)$$

Therefore, the discretization error of a node adjacent to a fluid node should be compared to the above equation in addition to the diffusion equation. However, no published discussion of this type of comparison has been found, and even for method C and the one-dimensional problem no useful expression or expansion has been derived for the discretization error. Very probably the difficulty in developing a meaningful expression for the discretization error at such an adjacent node is that, when h is finite, the series expansions in effect cross the solid boundary or a discontinuity.

For the special cases of zero or infinite heat-transfer coefficient, the adjacent nodes can be treated as interior nodes. For the zero heat-transfer coefficient condition the temperature distribution can be considered as continuous and symmetric about the boundary; for the infinite heat-transfer coefficient condition, where the surface is at a constant specific temperature, the temperature distribution can be considered as continuous and anti-symmetric about the boundary. To

consider the adjacent node as an interior node in these circumstances, additional points are located across the boundary such that if the boundary has no flux across it, the temperature at each of the additional points would be the same as at a corresponding point in the interior of the solid; for the infinite heat-transfer coefficient, the temperature difference between the additional point and the constant surface temperature would be equal to the temperature difference between the surface and the corresponding point within the solid. The adjacent node then can be treated as an interior node with neighbors corresponding to the original points within the solid and the necessary initial points as neighbors.

The above procedure, although valid in principle for curved boundaries, is useful only for straight-line boundaries. This is true because only for straight-line boundaries can the location of the additional temperature points be found easily, by reflecting the network about the boundary. Further, this technique for adjacent nodes under these conditions should only be considered as a temporary method of studying their discretization errors because eventually an expansion should be developed which would contain the heat-transfer coefficient as a parameter, and which would give results for zero and infinite coefficients as special cases. Moreover, it should also be noted that the series expansions for the errors as developed above would be identical for infinite and zero h, and that the difference in discretization

error for the two different conditions would be caused in these expansions by widely different values of the derivatives under the two conditions.

The adjacent nodes for methods G and C for the one-dimensional problem can be studied using equations V-39 and V-40 for constant surface temperature or adiabatic conditions. For both methods, under these conditions, the discretization error for all nodes including adjacent nodes is given by equation V-34 and the relationship for γ_o, as found from the damping expansions or the discretization error, is an optimum value; however, for the problem with a finite heat-transfer coefficient equation V-34 still applies for the interior points, but apparently the discretization error for the node adjacent to the fluid is of a different form so that γ_o is no longer an optimum value. However, from the complete solution we know that the actual error $v_{m,n}$ for all points is of the order of $1/S^2$ and of $\Delta\tau$; this fact together with a consideration of equation V-19 allows us to conclude that the discretization error for the adjacent nodes in methods G and C is of the order of $1/S^2$ and $\Delta\tau$. One of the most interesting and important future studies suggested by this work is that of finding an expansion for the discretization error for nodes adjacent to a fluid with finite heat-transfer conductances.

2. Time Discretization Error

A series expansion for the time discretization error can be derived by developing a series expansion for $(T_{i,n+1}-T_{i,n})/\Delta\tau$ about both n and (n+1). Then, using the weighting factor γ on the expansion about (n+1) and $(1-\gamma)$ on that about n, there results:

$$\varphi_{i,n}=\frac{T_{i,n+1}-T_{i,n}}{\Delta\tau}-\left[\gamma\frac{\partial T_{i,n+1}}{\partial\tau}+(1-\gamma)\frac{\partial T_{i,n}}{\partial\tau}\right]=\frac{\Delta\tau}{2}\left[(1-\gamma)\frac{\partial^2 T_{i,n}}{\partial\tau^2}-\gamma\frac{\partial^2 T_{i,n+1}}{\partial\tau^2}\right]$$

$$+\frac{(\Delta\tau)^2}{6}\left[(1-\gamma)\frac{\partial^3 T_{i,n}}{\partial\tau^3}+\gamma\frac{\partial^3 T_{i,n+1}}{\partial\tau^3}\right]$$

$$+\frac{(\Delta\tau)^3}{24}\left[(1-\gamma)\frac{\partial^4 T_{i,n}}{\partial\tau^4}-\gamma\frac{\partial^4 T_{i,n+1}}{\partial\tau^4}\right]+\cdots \qquad \text{(V-43)}$$

Thus, the contribution from the time discretization is about proportional to $\Delta\tau$. The effect of the weighting can be seen by considering that the second time derivatives evaluated at n and (n+1) are about equal which would mean that the coefficient of the time increment term is:

$$\varphi_{i,n}\simeq\frac{\Delta\tau}{2}(1-2\gamma)\frac{\partial^2 T_{i,n}}{\partial\tau^2}+\cdots \qquad \text{(V-44)}$$

where the coefficients of the succeeding derivatives at (n+1) would be changed slightly. Although this expression indicates that the optimum γ is $\frac{1}{2}$, it must be remembered that $\varphi_{i,n}$ is only part of the

contribution to the total discretization error $d_{i,n}$. When this is considered for regular networks the best weighting γ is usually less than $\frac{1}{2}$ and the γ of $\frac{1}{2}$ gives a difference solution very close to that of the analog solution. The comparison of approximate solutions for the transient problem in Chapter III indicates that much the same conclusions about the effect of the time increment and γ on accuracy made for the one-dimensional regular network problems in Chapter IV also apply to the two-dimensional problem with an asymmetric network. This would be expected, as the time discretization error appears in the same form for both the one- and two-dimensional problems. These results are discussed in section F in this chapter.

(E) SPACE DISCRETIZATION ERROR

Since the total discretization error is important, and since the contribution from the time discretization error appears to be satisfactory, the success or failure of an asymmetric network depends on the size of the contribution from the space discretization error, $-[\sigma_{i,n}(1-\gamma) + \sigma_{i,n+1}\gamma]$. The main problem with using an asymmetric location of the points is shown to be in the quantities, σ_i's. The following develops a series expansion for the space discretization error at time n, for points within and on the surface of a uniform solid but which have no conductances to a fluid. This is then followed by a summary of expansions for $\upsilon_{i,n}$ for several types of nodes, and a discussion of consistency and practical implications of the space discretization error is included.

1. Expansion for Space Discretization Error, Interior Node.

Consider a general asymmetric node for a two-dimensional problem within a solid with constant thermal properties having a thermometric conductivity of K_0, with continuous temperature distribution in the vicinity of the node. Such a node is shown in Figure V-4. The neighbors for the i^{th} node are assumed to have subscripts numbered counter-clockwise. Moreover, a dimensionless Cartesian coordinate system is superimposed on the system. In order to formulate the series

expansion in the dimensionless coordinates, the dimensionless lengths λ_{ij} and ρ_{ij} are defined according to:

$$\lambda_{ij} = \frac{\ell_{ij}}{L} \tag{V-45}$$

and

$$\rho_{ij} = \frac{r_{ij}}{L} \tag{V-46}$$

where λ_{ij} = dimensionless distance between nodes i and j

ρ_{ij} = dimensionless length of perpendicular bisector of line connecting nodes i and j

The node is assumed to have a total of N neighbors, although 5 are shown in the figure. The specification of the λ_{ij} and ρ_{ij} serves to completely define the geometry of the nodes. However, for convenience in making the expansion the angle β_{ij} between the positive ξ axis and the leg λ_{ij} is introduced. The λ_{ij}'s and β_{ij}'s are equivalent to radial coordinates of neighbor nodes. The angle parameter β_{ij} is not an independent variable and from the geometry the following relationship between ρ_{ij}, λ_{ij}, and β_{ij} can be derived.

$$\rho_{ij} = \frac{\lambda_{i,j+1}\sin(\beta_{i,j+1}-\beta_{ij}) - \lambda_{ij}\sin(\beta_{i,j+1}-\beta_{i,j-1}) + \lambda_{i,j-1}\sin(\beta_{ij}-\beta_{i,j-1})}{[\sin(\beta_{i,j+1}-\beta_{ij})][\sin(\beta_{ij}-\beta_{i,j-1})]} \tag{V-47}$$

where j has a geometrical meaning because of the counter-clockwise numbering. When j is 1, the subscript (j-1) means N and similarly when j is N, the subscript (j+1) is 1. With these substitutions equation V-39 for $\sigma_{i,n}$ is:

$$\sigma_{i,n} = \left[\frac{4}{\sum_{j=1}^{N} \rho_{ij}\,\lambda_{ij}}\right]\left[\sum_{j-1}^{N}\left(\frac{\rho_{ij}}{\lambda_{ij}}\right)\,(T_{j,n}-T_{i,n})\right]-\left[\frac{\partial^2 T_{i,n}}{\partial\xi^2}+\frac{\partial^2 T_{i,n}}{\partial\eta^2}\right] \qquad (V\text{-}48)$$

where N = number of neighboring nodes

i = subscripts of nodes with no conductance to fluid temperatures

j = refers to subscripts of neighbor nodes with non-zero conductance to the i^{th} node, in these equations

$$\rho_{ij}/\lambda_{ij} = y_{ij} = r_{ij}/\ell_{ij}$$

$$A_i = \sum_{j=1}^{N} \rho_{ij}\,\lambda_{ij} = \sum_{j=1}^{N} r_{ij}\,\ell_{ij}/L^2$$

The first step in developing an expansion for $\sigma_{i,n}$ is to expand the quantity (T_j-T_i) using the Taylor series. This is, symbolically,

$$T_{j,n}-T_{i,n} = \sum_{p=1}^{\infty} \frac{1}{p!}\left[\xi_j\,\frac{\partial}{\partial\xi} + \eta_j\,\frac{\partial}{\partial\eta}\right]^p T_{i,n} \qquad (V\text{-}49)$$

where ξ_j and η_j are the coordinates of the j^{th} neighbor based on the coordinate system with the origin at node i. These coordinates are, from the geometry,

$$\xi_j = \lambda_{ij} \cos\beta_{ij} \tag{V-50}$$

$$\eta_j = \lambda_{ij} \sin\beta_{ij} \tag{V-51}$$

and the expansion becomes, after being weighted by the conductance,

$$y_{ij}(T_{j,n}-T_{i,n}) = \frac{\rho_{ij}}{\lambda_{ij}} \sum_{p=1}^{\infty} \frac{1}{p!} \left[\lambda_{ij}(\cos\beta_{ij} \frac{\partial}{\partial\xi} + \sin\beta_{ij} \frac{\partial}{\partial\eta})\right]^p T_{i,n} \tag{V-52}$$

The first several terms of this series are in Table V-8.

The summation of these terms over all neighboring points must now be found.

$$\sum_{j=1}^{N} y_{ij}(T_{j,n}-T_{i,n}) = \sum_{j=1}^{N} \frac{\rho_{ij}}{\lambda_{ij}} \sum_{p=1}^{\infty} \frac{\left[\lambda_{ij}(\cos\beta_{ij} \frac{\partial}{\partial\xi} + \sin\beta_{ij} \frac{\partial}{\partial\eta})\right]^p T_{i,n}}{p!} \tag{V-53}$$

This is best studied by interchanging the summation signs and considering the finite summations on j for the first several p's. MacNeal has shown that for p of 1 the summation on j is zero.

$$\sum_{j=1}^{N} \rho_{ij}\left[\cos\beta_{ij}\ \frac{\partial T_{j,n}}{\partial \xi} + \sin\beta_{ij}\ \frac{\partial T_{i,n}}{\partial \eta}\right]$$

$$= \frac{\partial T_{i,n}}{\partial \xi} \sum_{j=1}^{N} \rho_{ij}\cos\beta_{ij} + \frac{\partial T_{i,n}}{\partial \eta} \sum_{j=1}^{N} \rho_{ij}\sin\beta_{ij}$$

$$= 0 \qquad \text{(V-54)}$$

The argument is that the summations which are the coefficient of the first derivatives in equation V-54 are zero because the ρ_{ij} described a closed polygon. For a p of 2, noting that the derivatives can be removed from the summation, and after a simple algebraic manipulation and rearrangement, the summation on j gives:

$$\sum_{j=1\ p=2}^{N} \equiv$$

$$\frac{1}{2}\sum_{j=1}^{N} \rho_{ij}\lambda_{ij}\left[(\cos\beta_{ij})^2\ \frac{\partial^2 T_{i,n}}{\partial \xi^2} + 2\cos\beta_{ij}\sin\beta_{ij}\ \frac{\partial^2 T_{i,n}}{\partial \xi\,\partial \eta} + (\sin\beta_{ij})^2\ \frac{\partial^2 T_{i,n}}{\partial \eta^2}\right]$$

$$= \left[\frac{\partial^2 T_{i,n}}{\partial \xi^2} + \frac{\partial^2 T_{i,n}}{\partial \eta^2}\right]\left[\frac{\sum_{j=1}^{N} \rho_{ij}\lambda_{ij}\left\{(\cos\beta_{ij})^2 + (\sin\beta_{ij})^2\right\}}{4}\right]$$

$$+ \left[\frac{\partial^2 T_{i,n}}{\partial \xi^2} - \frac{\partial^2 T_{i,n}}{\partial \eta^2}\right]\left[\frac{\sum_{j=1}^{N} \rho_{ij}\lambda_{ij}\left\{(\cos\beta_{ij})^2 - (\sin\beta_{ij})^2\right\}}{4}\right]$$

$$+\left[\frac{\partial^2 T_{i,n}}{\partial\xi\,\partial\eta}\right]\left[\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}\cos\beta_{ij}\sin\beta_{ij}\right] \qquad (V\text{-}55)$$

Now using trigonometric identities the summation for p of 2 can be written as

$$\sum_{j=1}^{N}\sum_{p=2} = \left[\frac{\partial^2 T_{i,n}}{\partial\xi^2} + \frac{\partial^2 T_{i,n}}{\partial\eta}\right]\left[\frac{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}}{4}\right]$$

$$+\left[\frac{\partial^2 T_{i,n}}{\partial\xi^2} - \frac{\partial^2 T_{i,n}}{\partial\eta^2}\right]\left[\frac{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}\cos 2\beta_{ij}}{4}\right] \qquad (V\text{-}56)$$

$$+\left[\frac{\partial^2 T_{i,n}}{\partial\xi\,\partial\eta}\right]\left[\frac{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}\sin 2\beta_{ij}}{2}\right]$$

And in the same way the summations for p's of 3 and greater can be obtained. The expansion of the space discretization error $\sigma_{i,n}$ is then found by dividing each of the sums of each of these summations on j by the dimensionless heat capacity A_i or $\sum_{j=1}^{N}(\rho_{ij}\lambda_{ij}/4)$ and subtracting the Laplacian operator evaluated at point i and n, $[(\partial^2 T_{i,n}/\partial\xi^2) + (\partial^2 T_{i,n}/\partial\eta^2)]$. The first terms of the expansion for $\sigma_{i,n}$ are:

$$\sigma_{i,n}=\left[\frac{\partial^2 T_{i,n}}{\partial\xi^2}-\frac{\partial^2 T_{i,n}}{\partial\eta^2}\right]\left[\frac{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}\cos 2\beta_{ij}}{\sum_{j-1}^{N}\rho_{ij}\lambda_{ij}}\right]+$$

$$+\left[\frac{\partial^2 T_{i,n}}{\partial\xi\,\partial\eta}\right]\left[\frac{2\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}\sin 2\beta_{ij}}{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}}\right]+$$

$$+\left[\frac{\partial^3 T_{i,n}}{\partial\xi^3}\right]\left[\frac{2\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}^2(\cos\beta_{ij})^3}{3\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}}\right]+$$

$$+\left[\frac{\partial^3 T_{i,n}}{\partial\xi^2\partial\eta}\right]\left[\frac{2\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}^2(\cos\beta_{ij})^2(\sin\beta_{ij})}{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}}\right]+$$

$$+\left[\frac{\partial^3 T_{i,n}}{\partial\xi\,\partial\eta^2}\right]\left[\frac{2\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}^2(\cos\beta_{ij})(\sin\beta_{ij})^2}{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}}\right]+$$

$$+\left[\frac{\partial^3 T_{i,n}}{\partial\eta^3}\right]\left[\frac{2\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}^2(\sin\beta_{ij})^3}{3\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}}\right] +$$

$$+\left[\frac{\partial^4 T_{i,n}}{\partial\xi^4}\right]\left[\frac{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}^3(\cos\beta_{ij})^4}{6\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}}\right] +$$

$$+\left[\frac{\partial^4 T_{i,n}}{\partial\xi^3\partial\eta}\right]\left[\frac{2\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}^3(\cos\beta_{ij})^3(\sin\beta_{ij})}{3\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}}\right] +$$

$$+\left[\frac{\partial^4 T_{i,n}}{\partial\xi^2\partial\eta^2}\right]\left[\frac{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}^3(\cos\beta_{ij})^2(\sin\beta_{ij})^2}{\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}}\right] +$$

$$+\left[\frac{\partial^4 T_{i,n}}{\partial\xi\partial\eta^3}\right]\left[\frac{2\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}^3(\cos\beta_{ij})(\sin\beta_{ij})^3}{3\sum_{j=1}^{N}\rho_{ij}\lambda_{ij}^3}\right] +$$

$$+\left[\frac{\partial^4 T_{i,n}}{\partial \eta^4}\right]\left[\frac{\sum_{j=1}^{N} \rho_{ij}\lambda_{ij}^3(\sin\beta_{ij})^4}{6\sum_{j=1}^{N} \rho_{ij}\lambda_{ij}}\right] + \cdots \qquad \text{(V-57)}$$

In the above equation no attempt has been made to use trigonometric identities or other manipulations to simplify the terms corresponding to p's of 3 and 4. Geometric relationships such as equation V-47 could be used to eliminate one of the quantities λ, ρ, or β, from the above equation but such a substitution is too cumbersome to be useful. Although the above equation was based upon a coordinate system with its origin at the center of the node, this coordinate system serves mainly to describe the geometric parameters λ_{ij} and β_{ij}, and the expansion above is independent of the origin of the coordinate system. The derivatives and their coefficients in the expansion are a function of the relative direction of the ξ or η axis, although the space discretization error as defined by equation V-39 obviously has no such dependence.

Despite the complexity of the expansion for the space discretization error equation V-57 is useful for making several observations about the nature of the space error and for studying the space discretization error at a node as a function of the location of its neighboring nodes by actually finding the coefficients of the derivatives.

A summary of several nodes and the expansions for the corresponding space discretization errors is in Table V-9. In this summary a sketch is used to show the node shape, coordinate system, and usually only the first non-zero terms are shown in the expansion. The geometric parameters are shown in the most convenient form for the particular geometry rather than any one consistent method. The selection of nodes for the summary has been made to contrast regular nodes and asymmetric nodes and is not meant to be a complete catalogue of all nodes. The errors for the nodes are based on equation V-57 and hence they apply when the following assumptions are valid: (1) the capacity of the node and the conductances to the neighboring nodes are based on MacNeal's rules; (2) the location of the nodes follows MacNeal's rules; (3) the i^{th} node is an internal node, that is all of its neighboring nodes are within an area which has constant thermal properties of volumetric heat capacity, C_p, and thermal conductivity, k. This means that none of the conductances cross an interface into a second solid or to a fluid with a finite heat-transfer coefficient.

2. General Comments--Space Discretization Error.

A study of the expansion for the space discretization error (equation V-57) shows that each term is the product of the second- or higher-order derivatives and a coefficient which is a ratio of finite sums. The derivative part of the product depends on the location of the node within the solid, the time, the solid boundary, and how the

forcing functions change with both time and space variables. Thus, these derivatives cannot be controlled, as they are functions of the individual problem. Since these derivatives are of the second- or higher-order, the space discretization error is zero for problems or times where the temperature gradient is independent of position.

The coefficients of the derivatives depend upon the geometry of the node and its neighbors; hence, some control over the size of the discretization error can be exercised by arranging the nodes so that these coefficients are small, or at least so that the coefficients of derivatives that are likely to be large are small. These coefficients determine the consistency of the approximation, in the Richtmyer sense. Richtmyer has stated (page 43) that, "...for problems with constant coefficients consistency can be determined by simply examining the truncation error term of the difference formula." Further, in the discussion of the propagation of the discretization error, consistency is important if more, smaller nodes are to replace large nodes to improve accuracy. Thus, most of the following discussion is on consistency and how these coefficients affect the space discretization error.

3. Consistency.

The consistency of a general asymmetric node can be studied by studying the above expansion (equation V-57) and finding how the space discretization error changes as the node is made smaller. The

time discretization has already been shown to be consistent as the contribution of the time discretization error ($\varphi_{i,n}$) to $d_{i,n}$ goes to zero as the time increment $\Delta\tau$ goes to zero. Before proceeding to the study of the expansion, it should be noted that, in Richtmyer's definition of consistency, actually both the time increments and the mesh increments for the whole network go to zero together; consequently, the consistency at a node does not have meaning in the precise Richtmyer sense. However, in view of Richtmyer's statement quoted above, a consistent node can be defined as one where the space discretization error goes to zero as its area or a characteristic linear dimension goes to zero. A network of such nodes would follow Richtmyer's definition of consistency, providing that the network could be refined in such a way that each succeeding mesh contained only consistent nodes. A further discussion of these concepts is best delayed until after a discussion of the coefficients in the expansion.

A study of the coefficients in the expansion indicates that the coefficients for the second derivatives, the hyperbolic term, $(\partial^2 T_{i,n}/\partial\xi^2 - \partial^2 T_{i,n}/\partial\eta^2)$, and the mixed partial derivative $(\partial^2 T_{i,n}/\partial\xi\,\partial\eta)$, depend only on the geometry of the node and its neighbors, and not upon the distance between nodes, or the area of the i^{th} node. In order to show this the longest distance between the i^{th} node and any of its neighbors is designated λ. For a node where the distances to each of its neighbors remain in proportion, and the angles

remain constant, reducing this length gives a sequence of geometrically similar nodes, but with smaller areas. The coefficient of the hyperbolic term is

$$\frac{\sum_{j-1}^{N} \left(\frac{\rho_{ij}}{\lambda}\right)\left(\frac{\lambda_{ij}}{\lambda}\right)(\cos\beta_{ij})}{\sum_{j=1}^{N} \left(\frac{\rho_{ij}}{\lambda}\right)\left(\frac{\lambda_{ij}}{\lambda}\right)}$$

and that for the mixed derivative is:

$$\frac{\sum_{j=1}^{N} \left(\frac{\rho_{ij}}{\lambda}\right)\left(\frac{\lambda_{ij}}{\lambda}\right)(\sin 2\beta_{ij})}{\sum_{j=1}^{N} \left(\frac{\rho_{ij}}{\lambda}\right)\left(\frac{\lambda_{ij}}{\lambda}\right)}$$

Since λ_{ij}/λ and the angles β_{ij} are constant the ratios ρ_{ij}/λ are also constant, and therefore these coefficients are independent of the λ or area of the nodes. Therefore, the space discretization errors for such nodes, where the coefficients of the hyperbolic term and the mixed partial derivative are identical and not zero, have a zero-order error term in their series expansions. Using the same arguments, we can say that the coefficients of the third derivatives depend upon the geometry, and, for a sequence of geometrically similar nodes, these coefficients are proportional to λ. In general, the p^{th} order derivatives have coefficients, for geometrically similar nodes, that are proportional to a linear distance (λ) in the node raised to the (p-2)

power, or λ^{p-2}. In the following a term in the expansion proportional to λ^{p-2} and the p space derivative is called a (p-2) order term, i.e., when p is 2, the term is a zero-order term.

A study of the series expansions for the asymmetric nodes in Table V-9 shows that many of the asymmetric nodes do indeed have zero-order error terms, and, hence, if they are refined in a sequence of geometrically similar nodes, they are inconsistent nodes. Further, a comparison of the expansions for the discretization error of the trapezoidal nodes, F, G, H, and I among themselves, and with the expansions for the rectangular nodes C and D, shows that the coefficients for the zero-order error terms are proportional to the geometrical parameters that are associated with the irregularity, or difference from the rectangular node. That is, the coefficient of these terms for node I, the most regular of the trapezoids, is proportional to the square of the sine of the angle by which two of the legs differ from being a rectangle. And, as the lengths and angles are further distorted in nodes H, G, and F, terms proportional to the differences in lengths of the legs and to the sines and cosines of the angles are added to the coefficients of these non-zero terms. Consequently, the conclusion must be made that irregularity or asymmetry in the geometry of the nodes produces the zero-order error terms. And further, if such an asymmetric network, containing asymmetric nodes with zero-order error terms, is refined to zero so that each succeeding network contains

only nodes which are geometrically similar to those in the original asymmetric network, the discretization does not go to zero, and Richtmyer's consistency condition is not satisfied.

In order to verify that, in the expansion for $\sigma_{i,n}$, no unnoticed simplification actually makes the zero-order error terms zero, a numerical study of the space-discretization error was made for nodes J and L from Table V-9. In this study, a node of the shape and proportions of node J or L was assumed at a position of $\xi = 0$ and $\eta = 1$. in a solid bounded at $\xi = \pm \pi/2$, $\eta = 0$, and $\eta = \infty$, with a steady-state temperature distribution and the following boundary conditions:

$$T(\xi, 0) = 1 \qquad -\frac{\pi}{2} < \xi < \frac{\pi}{2} \qquad \text{(V-58)}$$

$$T(\eta, \tfrac{\pi}{2}) = 0 \qquad 0 < \eta \leq \infty \qquad \text{(V-59)}$$

$$T(\eta, -\tfrac{\pi}{2}) = 0 \qquad 0 < \eta \leq \infty \qquad \text{(V-60)}$$

The solution to this problem is given in Churchill (28), page 194, and is

$$T = \frac{2}{\pi} \Psi \qquad \text{(V-61)}$$

where Ψ is given by:

$$\tan \Psi = \frac{\cos\xi}{\sinh\eta} \qquad 0 \leq \Psi \leq \frac{\pi}{2}$$

The temperature at the node and its neighbors can be computed from this expression, and the space discretization error can be found directly from its definition. Also, the second- and third-order derivatives can be found, and the constant and linear terms as functions of

the length λ can be calculated for the series expansions. Thus, the space discretization error can be checked in two ways, directly from the definition,

$$\sigma_{i,SS} = \frac{1}{A_i} \sum_{j=1}^{N} y_{ij}(T_j - T_i)_{SS} \qquad (V\text{-}62)$$

and approximately from the constant (second derivatives) and linear (third derivatives) terms.

The results of these calculations for nodes J and L are shown in Figures V-5 and V-6, respectively. The expansion for node J in Table V-9 shows it to be a consistent node, and the calculation of the space discretization error from the exact defining relationship shows this to be true. Further, the discretization, as found from equation V-62, follows fairly closely the behavior that would be expected merely on the basis of the linear term of the expansion. The expansion for node L indicates that a zero-order term should occur. The calculation, based on the exact, defining equation for σ_i, shows a σ_i of about -0.1257 at a λ of zero, or just about the value predicted by the expansion. Further, the discretization error estimated by the expansion follows closely that found by the defining equation. Great care was required in calculating the space discretization error from the defining equation, because, as the node becomes smaller, $(T_j - T_i)$ becomes very small, and a great many significant figures must be calculated in both T_j and T_i to insure that the quantity $(T_j - T_i)$ contains any

significant figures. As a result, these calculations were made using 10-place tables for the cosine (34), hyperbolic sine (34), and the arc tangent (35), and the temperatures were calculated to 10 decimal places. The scattering of the points for both nodes for the smallest λ's is due to round-off error, and more accurate calculations of T_j and T_i would be required to remove this scatter. However, the calculations are sufficiently accurate to show the validity of the expansions, and that the zero-order error term is a reality for node L.

The regular trapezoid, node I, shown in Table V-9, is another asymmetric node which has a zero-order error term. This node is like the regular rectangular node D, except that the two neighboring nodes, which are located on the ξ-axis for the rectangle, lie above the ξ-axis, so that the legs from node I to these nodes make an angle α with the ξ-axis. The area associated with the node is then trapezoidal in shape. The expansion for the space discretization error is:

$$\sigma_{I,n} = -(\sin\alpha)^2 \left[\frac{\partial^2 T_{I,n}}{\partial \xi^2} - \frac{\partial^2 T_{I,n}}{\partial \eta^2} \right] + \lambda_1 \sin\alpha(\cos\alpha)^2 \frac{\partial^3 T_{I,n}}{\partial \xi^2 \partial \eta} \qquad \text{(V-63)}$$

$$+ \left[\frac{\sin\alpha}{3} \right] \left[\lambda_1 (\sin\alpha)^2 - \frac{\lambda_2^2}{\lambda_1} \right] \frac{\partial^3 T_{I,n}}{\partial \eta^3} + \cdots$$

As previously noted, the cause of the zero-order term is the asymmetry introduced by the inclination of two of the legs at an angle of α to the ξ-axis. And this node must give a non-zero space discretization error at zero area, if the refinement is made using a sequence of geometrically similar nodes.

$$\lim_{\lambda \to 0} \sigma_{I,n} = -(\sin\alpha)^2 \left[\frac{\partial^2 T_{I,n}}{\partial \xi^2} - \frac{\partial^2 T_{I,n}}{\partial \eta^2} \right] \qquad \text{(V-64)}$$

$\frac{\lambda_1}{\lambda}$, $\frac{\lambda_2}{\lambda}$, and α are constant

This non-zero error would be expected to be small if the angle α is small.

However, a network of regular trapezoidal nodes can be used as an approximation to a two-dimensional problem in cylindrical coordinates, with gradients in the radial and angular directions, if MacNeal's rules are followed in setting up the network. Such a network is in Figure V-7. The main difference between this approximation and that usually used for cylindrical coordinates is that the perpendicular bisectors of the radial conductors are used for the conductances, and to bound the node, rather than the $\rho \Delta \theta$ arc lengths. Because of the geometry of the network, α is related to the angle increment, and the length λ_2 is equal to the radial increment:

$$\alpha = \frac{\Delta\theta}{2} \tag{V-65}$$

$$\lambda_2 = \Delta\rho \tag{V-66}$$

Making these substitutions in the expansion (equation V-57) gives:

$$\sigma_{I,n} = \tfrac{1}{2}(\cos\Delta\theta - 1)\left[\frac{\partial^2 T_{I,n}}{\partial\xi^2} - \frac{\partial^2 T_{I,n}}{\partial\eta^2}\right] + \lambda_1\left[\sin\frac{\Delta\theta}{2}\left(\cos\frac{\Delta\theta}{2}\right)^2\right]\frac{\partial^3 T_{I,n}}{\partial\xi^2\partial\eta} +$$

$$+\left[\frac{\sin\frac{\Delta\theta}{2}}{3}\right]\left[\lambda_1\left(\sin\frac{\Delta\theta}{2}\right)^2 - \frac{(\Delta\rho)^2}{\lambda_1}\right]\left[\frac{\partial^3 T_{I,n}}{\partial\eta^3}\right] + \cdots \tag{V-67}$$

Now, if the network and the trapezoidal nodes are considered an approximation to the Laplacian in cylindrical coordinates, and the network and nodes are refined such that the angle increment, $\Delta\theta$, and radial increment, $\Delta\rho$, go to zero, the space discretization error goes to zero:

$$\lim_{\substack{\Delta\theta\to 0\\ \Delta\rho\to 0}} \sigma_{I,n} \to 0 \tag{V-68}$$

and the network and the nodes satisfy Richtmyer's condition for consistency.

Thus, the important point can be made that the manner in which a network is refined determines whether or not a network of nodes satisfies Richtmyer's consistency condition. Likewise, the manner in which an area is made smaller determines whether or not the space discretization error at a node is reduced as its area is made smaller.

Practical Aspects of Consistency. The meaning of the consistency condition, for a network containing asymmetric nodes, can now be interpreted as follows: that the shape of the asymmetric node in a network, or the geometry of its neighbors, must be changed as more nodes are added, if the accuracy of the approximation is to be improved. Thus, in general, the additional nodes should be added so that the coefficients of the hyperbolic and mixed second-order derivatives go to zero, which requires that

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} \rho_{ij}\lambda_{ij}\cos 2\beta_{ij} \rightarrow 0 \qquad i = 1, \ldots \qquad \text{(V-69)}$$

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} \rho_{ij}\lambda_{ij}\sin 2\beta_{ij} \rightarrow 0 \qquad i = 1, \ldots \qquad \text{(V-70)}$$

Since, for any practical asymmetric network, one is able to add nodes so that the network becomes at least an irregular rectangular network, containing nodes of type C, Table V-9, the asymmetric network can be considered to be a consistent approximation. In refining the mesh all the node shapes need not go to rectangles, but to any of the consistent node shapes shown in the table, or any which are possible. In particular, it would be more logical to consider that the refinements of an asymmetric triangular network could approach the network of equilateral triangles with hexagonal-shaped nodes, type N, Table V-9, rather than a rectangular network.

In most cases of a general asymmetric network, one is not able to refine the network to zero mesh spacing in a manner that allows all, or even some, of the nodes to be geometrically similar to the nodes in the coarse network. Thus, the expansion for $\sigma_{i,n}$ (equation V-57) does not, when interpreted correctly, show that the network of asymmetric nodes is inconsistent. Further, in most practical situations the network is not going to be refined, and in those situations the expansion allows an estimate of the space discretization error if sufficient information is known, or can be approximated, about the second- and higher-order derivatives.

Because Richtmyer's consistency, stability, and convergence definitions are based on successive refinements to zero of both space and time increments, the main practical consequences apply mainly to space networks containing a relatively large number of points (S probably much greater than thirty). When such a fine space mesh is to be used, both Richtmyer's theory and practical considerations dictate the use of a rectangular network or a network of equilateral triangles for the interior points. This is true for the following reasons: (1) The space discretization error is assured to be small only for small, consistent nodes, such as nodes D and N. Further, it should be noted that even radial nodes such as node I become rectangular as $\alpha \rightarrow 0$, according to the refinement just discussed. (2) If a large number of points is used in a regular network, curved or irregular boundaries

can be followed closely, and such boundaries can be followed exactly by the addition of only a few irregular nodes along the boundary.

(3) Since no partial difference equation or recurrence relationship is available for asymmetric nodes, their conductances and capacities must be calculated specifically for each node. Setting up the difference equations can require an amount of calculation which would be prohibitive if done on a desk calculator, and bothersome to program on a digital computer.

Possible Ways of Eliminating Zero-order Error Terms. Before leaving the topic of consistency, it should be mentioned that consistent approximations can be found for irregular nodes of many shapes by solving a system of equations which make the coefficients of the first and second derivatives in the expansion for $\sigma_{i,n}$ zero (10). This can be done by replacing the elements in the row for such a node in the Y/A matrix, $(1/A_i)(\rho_{ij}/\lambda_{ij})$, with a weighting μ_{ij}. Thus equation V-54 becomes:

$$\sigma_{i,n} = \sum_{j=1}^{N} \mu_{ij}(T_{j,n}-T_{i,n}) - \frac{\partial^2 T_{i,n}}{\partial \xi^2} - \frac{\partial^2 T_{i,n}}{\partial \eta^2} \qquad (V-71)$$

Now, using the expansions of $(T_{j,n}-T_{i,n})$, and setting the coefficients of $\partial T_{i,n}/\partial \xi$, $\partial T_{i,n}/\partial \eta$, $\partial^2 T_{i,n}/\partial \xi \partial \eta$ equal to zero, and of $\partial^2 T_{i,n}/\partial \xi^2$ and $\partial^2 T_{i,n}/\partial \eta^2$ equal to one, the following system of equations is obtained:

$$\sum_{j=1}^{N} \lambda_{ij}(\cos\beta_{ij})\mu_{ij} = 0 \qquad (V-72)$$

$$\sum_{j=1}^{N} \lambda_{ij}(\sin\beta_{ij})\mu_{ij} = 0 \qquad (V-73)$$

$$\tfrac{1}{2}\sum_{j=1}^{N} (\lambda_{ij})^2(\sin\beta_{ij})(\cos\beta_{ij})\mu_{ij} = 0 \qquad (V-74)$$

$$\tfrac{1}{2}\sum_{j=1}^{N} (\lambda_{ij})^2(\cos\beta_{ij})^2\mu_{ij} = 1 \qquad (V-75)$$

$$\tfrac{1}{2}\sum_{j=1}^{N} (\lambda_{ij})^2(\sin\beta_{ij})^2\mu_{ij} = 1 \qquad (V-76)$$

The μ_{ij} that satisfy these equations would then be the off-diagonal elements in the i^{th} row, j^{th} position, in the equation for the i^{th} node; the diagonal element is, for the i^{th} row, $\sum_{j=1}^{N} \mu_{ij}$. The existence or uniqueness of the μ_{ij} that satisfy the above equations is not guaranteed, but depends upon the number of neighbors N and their geometrical relationships. Indeed, Forsythe and Wasow (10) state that there are geometrical situations for which no solution exists.

A second, and essentially equivalent, system of equations which has an unknown that can be given a geometrical interpretation can be derived in the same way. This is done by determining a length ρ'_{ij} to replace the length of the perpendicular bisector ρ_{ij} in both the

conductances and heat capacity of a node. An analysis like that above shows that the ρ'_{ij} must satisfy the following system of homogeneous equations:

$$\sum_{j=1}^{N} \cos\beta_{ij}\rho'_{ij} = 0 \qquad \text{(V-77)}$$

$$\sum_{j=1}^{N} \sin\beta_{ij}\rho'_{ij} = 0 \qquad \text{(V-78)}$$

$$\sum_{j=1}^{N} \lambda_{ij}(\sin 2\beta_{ij})\,\rho'_{ij} = 0 \qquad \text{(V-79)}$$

$$\sum_{j=1}^{N} \lambda_{ij}(\cos 2\beta_{ij})\,\rho'_{ij} = 0 \qquad \text{(V-80)}$$

The first two equations are satisfied if the ρ'_{ij} form a closed polygon; the second two equations have the same form as those for the conditions under which a MacNeal node is consistent (equations V-69 and V-70). In principle, the zero-order error terms can be eliminated from the discretization error of all nodes in some asymmetric networks. This could be done by solving a system of equations such as equations V-72 to V-76 for μ_{ij} or V-77 to V-80 for ρ'_{ij} for each node in the network. From a solution for a node one row in the Y/A matrix would be generated. Because of the time-consuming nature of solving such systems of equations, this is probably not practical. However, such a

technique could be useful if solutions were required only for a few irregular nodes which are used to follow an irregular boundary, the rest of the network containing regular, consistent nodes. Unfortunately, because the above analysis applies only to interior nodes, the system of equations does not apply to adjacent nodes with heat-transfer coefficient conductances to a fluid temperature, thus further restricting their utility.

However, before the solution of either system of equations to eliminate zero-order error terms became a general technique, the following studies would have to be carried out: first, the geometrical conditions for which a solution to the equation exists should be found. Second, if several solutions to the equations exist, a criterion for selecting one of them should be found. Third, a determination should be made of the effect of using the solution of these equations on the coefficients of the third and fourth derivatives in the expansion, to be sure that these coefficients are not greatly increased. Fourth, a study would have to be made of the properties of the Y/A matrix which contains elements that are a solution of the above system, to see if modifications in the stability and non-oscillatory criteria are required.