(F) CONCLUSIONS AND SUMMARY--CHAPTER IV

The z-transforms have been successfully used to find the analytic solutions of the one-dimensional partial difference equation of diffusion for several different methods of approximating the continuous boundary equations. These solutions have been compared with the analytic solutions of the approximate partial differential equations for the same problems. The methods for approximating the boundary equations are based on a mesh with adjacent points on the boundary (mesh $\Delta\xi$), methods G and A, and on a mesh with the adjacent points located a distance $\Delta\xi/2$ from the boundary (mesh $\Delta\xi/2$), method C. Use is made of both the usual differencing for the approximations at the boundaries (methods G and C) and a backward difference for the known fluid temperature for method A. Consideration has been given to all selections of differencing parameters from graphical solutions to implicit calculations. The z-transform solutions have proved very useful for studying and understanding quantitatively the errors and oscillatory behavior of these approximate solutions. Not only can the effect of the differencing parameters be studied quantitatively, but the consistent superiority of the accuracy of generalized method C and graphical method A of the graphical methods has been shown analytically. However, these complete analytic solutions, as found by the z-transforms, are not the most practical way to find the sufficient conditions on the differencing parameters to obtain a solution with satisfactory oscillatory behavior, and are not the best way to study the bound for $|q_{min}|$ which is Υ. Simply calculated matrix norms and the techniques of Chapter III should be used for this purpose.

In addition, the following observations and conclusions are made:

(1) When the particular solution is a true steady-state solution, the steady-state solution for the approximate solution is equal to that for the continuous solution for all methods. For problems where a quasi-steady-state solution occurs, the same comments made about the eigenvectors and damping factors for the transient solution probably also apply to the particular solution.

(2) The error in the intercept $g_j c_{mj}$ of a slowly decaying term in the approximate transient solution, on a semi-logarithmic plot versus time, is approximately proportional to $1/S^2$. The error in the slope $[(1/\Delta\tau)\ln q_j]$, of a slowly decaying term in an approximate transient solution on such a plot is approximately proportional to $1/S^2$ and to $\Delta\tau$; the proportionality constant for $\Delta\tau$ is a function of the weighting γ. The errors, both in the intercept and in the slope, are of importance in obtaining accurate transient solutions. These two quantitative statements above are of much more practical importance in selection of the differencing parameters, or in estimating the error,

than the statements that the truncation error for the difference used to approximate the differentials is of the order of $1/S^2$ and of the order of $\Delta\tau$. Further, the error propagation analysis (2) based on the truncation error does not usually lead to such useful or precise quantitative statements.

(3) The trigonometric characteristic roots for the approximate solution are not equal to those of the continuous solutions for problems with a finite non-zero heat-transfer coefficient for any of the methods; the difference between the squares of these roots goes to zero with $1/S^2$. Errors in these roots affect the accuracy of each of the parts of the approximate solution. For methods G, A, and C, for problems with only zero and/or infinite H's, these roots are equal.

(4) The important effects on the accuracy of the slope of a semi-logarithmic graph of a transient term for the slower decaying positive damping factors in the approximate solution caused by changes in the damping factors are: first, for methods and problems where the approximate trigonometric roots are equal to those of the continuous solution, increasing S does not improve the accuracy of the slope as much as making appropriate changes in the time increment $\Delta\tau$ and weighting γ, providing S is five or larger; the weighting γ which gives the most accurate slopes or damping factors is:

$$\gamma_o = \frac{1}{2} - \frac{1}{12S^2\Delta\tau} = \frac{6r-1}{12r} \qquad \text{(IV-308)}$$

Second, for methods and problems where the approximate trigonometric roots are not equal to the continuous roots the best selection of γ is not given by γ_o but additional research must be carried out to determine quantitatively how much different γ must be from γ_o for a given H and S and a given approximate method. Third, for any methods and problems selecting γ significantly larger than γ_o causes the transient solution to decay too rapidly; selecting γ significantly smaller than γ_o causes the transient solution to decay too slowly. Since γ_o is always between zero and one-half, inclusive, the slopes of the transient terms for the backward difference implicit calculation with a γ of 1 always are the least accurate.

(5) The error contributed to the intercepts by the elements of the eigenvector matrix for the approximate method is proportional to the difference between the trigonometric roots for the approximate solution and those for the continuous solution. Thus, for methods and problems where these roots are equal, the error in the intercept is caused only by the error in the initial vector components.

(6) Method C, with no points located on the boundary, has the most accurate trigonometric roots, eigenvectors, and initial vectors of any of the approximations discussed here. Consequently, in general, for a given selection of differencing parameters, the approximate

solution for method C is expected to be the most accurate solution of any discussed here, and should be used when accuracy is of primary importance. Three other important observations about method C are:

a) The accuracy of method C is a direct result of using mesh $\Delta\xi/2$ which does not locate points on the boundary but locates the points a distance $\Delta\xi/2$ away from the boundaries. This leads to a difference formulation that is closer both physically and mathematically to the continuous problem than for any other method. This is because only for methods based on mesh $\Delta\xi/2$ do the following three properties occur simultaneously, (i) the boundary equations for this mesh are a direct discretization of the continuous boundary equations; (ii) no heat capacity is associated with the point on the boundary which is consistent with the continuous boundary equation; (iii) no heat capacity of the solid is neglected. Further, of the possible methods based on mesh $\Delta\xi/2$ or mesh $\Delta\xi$, method C is the only method that has a symmetric Y/A matrix which must have orthogonal eigenvectors; consequently, method C has an orthogonal relationship that is closer mathematically to the orthogonal relationship of the continuous solution, and method C is expected to have the most accurate initial vectors for any initial temperature distribution.

b) The bound for $|q_{min}|$ for method C is independent of the type of linear boundary equations or size of the heat-transfer coefficients at both surfaces; thus, only one relationship between the bound and the differencing parameters r and γ is necessary. However, for problem I with a large heat-transfer coefficient or problem II, method C gives a large weighting to the damping factors which can be negative. This large weighting causes the very oscillatory behavior in Figure IV-1 for a graphical method C for problem II, and it is the reason method C is not suitable, in general, for graphical methods. Indeed, to obtain the high potential accuracy of method C, the time differencing parameters, $\Delta\tau$ and γ, must be selected so that any oscillatory transient terms are negligible.

c) Although method C is expected to give the most accurate approximations in general, for a specific problem with a specific set of differencing parameters, another method might give a more accurate solution because of compensating errors in the intercept and in the slope in the transient terms for the other method. Unless the direction and an estimate of the size of the error in the intercepts is known (the direction and size of the error in the slope can be found from a prior conclusion) this compensation cannot be predicted.

(7) The most accurate approximate solutions for problems in one, two, or three dimensions in any coordinate system are probably obtained using regular meshes with the points located a distance of $\Delta\xi/2$ away from the boundary. This conclusion is based on the more

accurate solutions obtained for method C, and on the fact that to obtain the physical and mathematical advantages mentioned in (6a) above, the adjacent points must be located away from the boundary. For irregular geometries and/or irregular meshes, locating the points away from the surfaces is probably also advantageous.

(8) Of the graphical methods, A is usually the most accurate. Using this method if no more than 8 points are required, an approximate solution of sufficient accuracy can be obtained by a graphical construction for times when no more than two of the terms of the transient solution are significant. If more than 8 points are required, the solution should be calculated numerically, but, because of the simplicity of the difference equation, this can be done readily on a desk calculator; thus, graphical method A is suitable for use in problems where a moderately accurate solution at intermediate and long times is desired, and where the solution is desired quickly and cheaply. Several additional comments should be made about graphical methods:

a) Graphical method A, along with the other graphical methods, requires about twice the number of points to give as accurate an intercept as does method C for problem II because of the direct or indirect effect of the large negative root. If this is a characteristic of graphical methods for other problems, they are not competitive in the number of calculations required to obtain a given accuracy at a constant time, and a generalized method C should be used for accurate solutions.

b) The reason that graphical method A is the most accurate of the graphical methods is that several errors tend to compensate. As the trigonometric roots α_j are symmetric about $\pi/2$ for all the problems considered here, the damping factors always occur in pairs with equal magnitude but opposite sign. Thus, the slopes for the sum of the two paired damping factor terms give straight lines on semi-logarithmic paper only if the points at alternate time intervals are used. Further, this method appears to have the tendency that the amplitude for the negative of the pair of damping factors is small and about the size of the error in the initial vector component-eigenvector product for the positive damping factor, and the intercept is more accurate for (n+m) odd in these problems. Additional compensation is obtained when the intercept error is positive as it is in these problems using graphical method A, because the slope is too steep for graphical solutions and the transient terms in the approximate solution cross the corresponding continuous terms on a semi-logarithmic graph. Also for problem I, the error in the approximate slope for graphical method A is smaller than usual for γ of zero and r of $\frac{1}{2}$, because the difference in trigonometric roots compensates part of the usual error.

c) The forward average improves the accuracy of graphical method A by reducing the size of the oscillatory component and by reducing the error in the initial vector components. However, this latter improvement is dependent upon the fact that the error in the initial vector for graphical method A in all problems studied here is always positive, and under these circumstances, the forward average decreases this error.

(9) The technique of averaging the approximate solution at the beginning and end of a time interval, and applying the average at either end or some time during the time interval, was shown to be equivalent under certain conditions to interpolating as defined by Longwell (9) for graphical solutions, and also equivalent to replacing a step change in fluid temperature with a one-half step initially, and the full step change occurring at the end of the first increment. Averaging reduces the amplitude of any large negative damping factors, and if the average is applied at the center of the time increment, the initial vector component of the slower-decaying positive damping factors is not changed. If the average is applied at the beginning or end of the time increment, the error in their initial vectors can either be reduced or increased, depending upon the direction of that error before averaging; therefore, averaging other than the central average probably should not be used unless the direction of the error in the initial vector is known. The main application of averaging appears to be to graphical methods.

(10) The quantitative relationships derived for the errors as a function of both the problem and the differencing parameters, have

given a procedure that allows the selection of the differencing parameters to obtain a certain accuracy, and at the same time, tends to minimize the amount of calculations. Although this procedure requires estimates of certain quantities in the continuous solution, and is fairly complicated, the following observations and conclusions can be made about the selection of differencing parameters.

a) The number of points S or S_G to be used is fixed by the allowable error in the intercept.

b) Graphical solutions, if they are to be calculated numerically, must not require many more points than the general explicit or implicit method.

c) The only generalized explicit calculations that need to be considered are those where $r_{Ex} > 2(S/S_G)^3$, and in most cases the oscillatory behavior will limit the size of r_{Ex}. The explicit calculation of r of 1/6 is the most accurate calculation possible for a fixed value of S.

d) The oscillatory behavior need never limit the r_{Im} for an implicit calculation, as increasing γ can always remove this restriction. However, a limiting combination of r_{Im} and γ exists for a fixed S that satisfies the restrictions governing both the oscillatory behavior and the accuracy of the damping factors. For an implicit calculation to be considered, it must have an r_{Im} that is greater than 0.58, or the explicit calculation with r of 1/6 is as accurate and requires the same number of multiplications. In general, the r_{Im} must be greater than $3.5\ r_{Ex}$ or $3.5\ (S/S_G)^3$.

(G) SUGGESTIONS FOR FUTURE WORK

In order to extend, generalize, and check the conclusions reached in this discussion of the approximations to the one-dimensional diffusion equation, several further numerical and analytic studies are suggested.

First, for problem I, a study of the trigonometric roots for method C and graphical method A should be made to develop a relationship either in the form of a series or a graphical correlation for the difference $(\nu_j^2 - \mu_j^2)$ as a function of S, H, and j for at least the first several j's. This type of relationship is necessary to allow use of equation IV-287 to estimate the error in the damping factors. Also the characteristic roots for method C and problem I should be further studied to develop a modified relationship corresponding to equation IV-308 for γ_o which allows one to find the optimum γ_o for problems with a finite heat-transfer coefficient. Again, this relationship could be a graphical correlation of H, r, and γ_o. Along these lines, the numerical solution of generalized method A with r of 1/6 could be stepped out to find if significant improvement is possible when the μ_j and ν_j are brought closer together.

Second, a problem where the heat-transfer coefficients at both boundaries are between zero and infinity should be studied both analytically and numerically to be sure that the methods which are accurate for problem I are also more accurate for this more general problem. The analytic study could follow the work here and the numerical study should follow that suggested for problem I.

Third, the approximate solutions for problems where the boundary forcing functions change with time should be studied. This should indicate if any advantage in accuracy accrues to any of the methods in the particular solution. Also, the effect of the particular solution on the initial vector would be shown. Probably the easiest functions and problems to study first would be a sinusoid surface temperature for problem II and a sinusoid flux for problem III. These results might be generalized to problem I without actually deriving its solution. These several solutions might be sufficient to indicate that the comments made about the eigenvectors and damping factors for the several methods also apply to the particular solution as indicated by equation IV-213.

Fourth, the solutions for the one-dimensional problem found here can be extended to two-dimensional problems (two Cartesian space coordinates). This can be done because by subtracting the continuous solutions from 1 for problems I, II, and V, the solutions to the problem with a zero fluid temperature and initial temperature of 1 are obtained; for these problems, the product solution theorem as described in Carslaw and Jeager (1), section 1.15, can be used for the two-dimensional problem to obtain the complete solution of initial vector, eigenvector matrix, and eigenvalue exponentials directly. For an approximate solution, the eigenvalues and the eigenvectors for the Y/A matrices for square meshes can be found by using matrix theorems proved by Rutherford (23). These theorems are analogous to the product solution theorem for the continuous solution and, by multiplying the two solutions together, the complete solution including the initial vector for the two-dimensional problem results.

Fifth, the approximate solutions for problem V, the semi-infinite solution, should be studied in more detail with the possibility of understanding the errors in the approximate solutions for short times and infinite solids. This could be approached by seeing if the approximate solutions can be expressed in terms of a "binomial error function" and then comparing this to the normal error function term in the continuous solution, as mentioned previously.

Sixth, as the initial vector has not been studied previous to this work several useful additional studies can be made. Of these, probably the most important is further investigation, both analytic and numerical, of the conclusion that the most accurate initial vectors are obtained when the A_i in the orthogonality relationship for g_j correspond to an integration of the weighting in the orthogonality relationship for a_j. This could be done by deriving the approximate solutions for meshes corresponding to mesh $\Delta\xi/2$ for cylindrical and spherical coordinates. Several useful studies involving method C based on mesh $\Delta\xi/2$ also can be made. The z-transform solutions for other problems using this method should be found to try to improve the relationship used to estimate the number of points S for an approximation of a certain accuracy. In particular, the number 1/12 appears as the coefficient of $1/S^2$ in the expansions for the errors in method C; and possibly a study of the orthogonality relationship and the discretization error near the surface might give a relationship that would allow us to use this 1/12 as a proportionality constant in the intercept error bound in equation IV-286.

In this study, the finite sum in the orthogonality relationship for the approximate solution might be treated as an estimate of the integral for the continuous solution. Also, in studying the space discretization at the surface, care should be taken since a discontinuity occurs at the boundary. These additional studies of the initial vectors are also warranted, because if a simple way could be found of determining the sign alone of (g_j-a_j) considerable improvement in accuracy could be obtained by cancelling errors. For example, if (g_j-a_j) is known to be positive, then the time differencing parameters can be selected so that the damping factors are small, $\gamma < \gamma_o$, and the approximate transient term crosses a continuous transient term on a semi-logarithmic graph. Also, a forward average could then be used, which would reduce the error in the initial vector. If (g_j-a_j) is negative, then the γ and $\Delta\tau$ are selected so that the damping factors are too large $(\gamma > \gamma_o)$ or the backward average is used. If the coefficient of

$1/S^2$ in the expansion for $(g_j - a_j)$ were also known, spectacular improvement in accuracy could be made by selecting r and the averaging parameter k so that this term would be zero and $(g_j - a_j)$ would be proportional to $1/S^4$. These same studies should also be carried out for graphical method A. For this graphical method, the effective initial vectors should be derived and studied. These vectors would then show not only the size and sign of $(g_j - a_j)$ for the slower decaying damping factors, but would also indicate the relative size of the g_j for the oscillatory components. Hopefully, these expansions could show that $(g_j - a_j)$ is always positive, so that the forward average could always be associated with graphical method A, and that the g_j for the oscillatory components are always about the same size as $(g_j - a_j)$ so that for either odd or even values of n, compensation occurs. Further, a simple way should be developed to find at which values of n odd or even this compensation occurs; again, hopefully, this would be at $(n+m)$ odd as for the problems already studied.

Table IV-1

GENERALIZED METHOD G

Mesh $\Delta\xi$, Equation IV-10, $\beta = \phi = \gamma$

Boundary Equation $\quad 0 \leq h < \infty$

$$t_{0,n+1}\left[1+2r\gamma(1+H_0/S)\right] - 2r\gamma t_{1,n+1} = \left[1-2r(1-\gamma)(1+H_0/S)\right]t_{0,n}$$
$$+ 2r(1-\gamma)t_{1,n} + (2rH_0/S)\left[\gamma t_{f_0,n+1} + (1-\gamma)t_{f_0,n}\right]$$

$$n \geq 0$$

Interior Points

$$-r\gamma t_{m-1,n+1} + (1+2r\gamma)t_{m,n+1} - r\gamma t_{m+1,n+1} = r(1-\gamma)t_{m+1,n}$$
$$+ \left[1+2r(1-\gamma)\right] t_{m,n} + r(1-\gamma)t_{m-1,n}$$

$$m = 1, 2, \ldots, S-1$$

$$n \geq 0$$

Matrices

$$(\Delta\xi)^2 \frac{Y}{A} = \begin{bmatrix} -2(1+H_0/S) & 2 & 0 \ldots 0 & 0 & 0 \\ 1 & -2 & 1 \ldots 0 & 0 & 0 \\ 0 & 1 & -2 \ldots 0 & 0 & 0 \\ \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots 0 & 2 & -2(1+H_S/S) \end{bmatrix}$$

$$(\Delta\xi)^2 \frac{Y_B}{A} = \begin{bmatrix} 2H_0/S & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 2H_S/S \end{bmatrix}$$

Table IV-1 (Cont.)

GENERALIZED METHOD G

<u>Initial Condition</u> $0 \leq h < \infty$

$t_{m,0} = T(m\Delta\xi,\ 0+)$ $m = 0, 1, \ldots, S$

<u>Fluid Temperature</u>

$t_{f,n} = T_f(n\Delta\tau)$ $n \geq 0$

<u>Stability</u> $0 \leq h < \infty$

$r(1-2\gamma)M \leq 2$ $M = \min\left(\left\|\frac{Y}{A}\right\|_I \text{ or } \left\|\frac{Y}{A}\right\|_{II}\right)$

$$\left\|\frac{Y}{A}\right\|_I = 2(2 + H_{MAX}/S)$$

$$\left\|\frac{Y}{A}\right\|_{II} = \mathrm{MAX}\left\{5 \text{ or } (3 + 2H_{MAX}/S)\right\}$$

<u>Infinite h</u> $H = \infty$

$t_{f,n} = t_{0,n}$ $n \geq 0$

Delete row and column containing infinite H in $(\Delta\xi)^2 Y/A$ and row containing infinite H in $(\Delta\xi)^2 Y_B/A$; in $(\Delta\xi)^2 Y_B/A$ put a one in place of element located at the intersection of column containing infinite H and row adjacent to deleted row.

Stability: $r(1-2\gamma) \leq \frac{1}{2}$

Initial Condition: Fits boundary condition at $n = 0$ for boundaries where $H = \infty$

Table IV-2

GENERALIZED METHOD A

Mesh $\Delta\xi$, Equation IV-10, $\beta = 1$, $\phi = 0$

<u>Boundary Equation</u> $0 \leq h < \infty$ Explicit only

$$t_{0,n+1} = \frac{(1-2r)t_{0,n} + 2rt_{1,n} + (2rH_0/S)t_{f_0,n+1}}{1 + 2rH_0/S}$$

$$n \geq 0$$

<u>Interior Points</u>

$$t_{m,n+1} = rt_{m-1,n} + (1-2r)t_{m,n} + rt_{m+1,0}$$

$$m = 1, 2, \ldots, S-1$$

$$n \geq 0$$

<u>Matrices</u> (Note Y/A matrix is a function of r or $\Delta\tau$.)

$$(\Delta\xi)^2 \frac{Y}{A} = \begin{bmatrix} -\frac{2(1+H_0/S)}{1+2rH_0/S} & \frac{2}{1+2rH_0/S} & 0 \ldots & 0 & 0 \\ 1 & -2 & 1 \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots & \frac{2}{1+2rH_S/S} & -\frac{2(1+H_S/S)}{1+2rH_S/S} \end{bmatrix}$$

$$(\Delta\xi)^2 \frac{Y_B}{A} = \begin{bmatrix} \frac{2H_0/S}{1+2rH_0/S} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & \frac{2H_S/S}{1+2rH_S/S} \end{bmatrix}$$

Table IV-2 (Cont.)

GENERALIZED METHOD A

Initial Distribution $0 \leq h < \infty$

$t_{m,0} = T(m\Delta\xi,\ 0\ -)$ $m = 0, 1, 2, \ldots, S$

Fluid Temperature $0 \leq h < \infty$

$t_{f,n} = T_f(n\Delta\tau)$ $n \geq 0$

Stability

$r \leq \frac{1}{2}$ all h $0 \leq h \leq \infty$

Infinite h $H = \infty$

$t_{0,n+1} = t_{f_0,n+1}$ $n \geq 0$

In $(\Delta\xi)^2 Y/A$ in row containing infinite H set diagonal element equal to $1/r$; other elements in row to zero. In $(\Delta\xi)^2 Y_B/A$ put $1/r$ in position of infinite H.

Initial Condition:

$t_{m,0} = T(m\Delta\xi,\ 0\ -)$ $m = 0, 1, 2, \ldots, S$

$T(m\Delta\xi,\ 0\ -)$ = Initial condition before fluid temperature changes

Note: As $t_{0,0}$ is not required to equal $t_{f_0,0}$ initially, the initial distribution need not necessarily give the surface temperature at 0 or S initially.

Special Case:

If one H is infinite other H zero and $t_{f,n}$ constant, then define implicit method A as implicit method G delayed one $\Delta\tau$:

$$t_{m,n\ A} = t_{m,n-1\ G}$$

Table IV-3

GENERALIZED METHOD C

Mesh $\Delta\xi/2$, Equation IV-29, $\beta = \phi = \gamma$

Boundary Equation $\quad 0 \leq h \leq \infty$

$$t_{\frac{1}{2},n+1}\left[1+ r\left(\frac{1+3H_0/2S}{1+H_0/2S}\right)\right] - r\gamma t_{\frac{3}{2},n+1} = t_{\frac{1}{2},n}\left[1-r(1-\gamma)\left(\frac{1+3H_0/2S}{1+H_0/2S}\right)\right]$$

$$+ r(1-\gamma)t_{\frac{3}{2},n} + \left[\frac{rH_0/S}{1+H_0/2S}\right]\left[(1-\gamma)t_{f_0,n} + t_{f_0,n+1}\right]$$

$$n \geq 0$$

Interior Points

$$-r\gamma t_{m-1,n+1} + (1-2r\gamma)t_{m,n+1} - r\gamma t_{m+1,n+1} = r(1-\gamma)t_{m-1,n}$$

$$+ \left[1-2r(1-\gamma)t_{m,n}\right] + r(1-\gamma)t_{m+1,n}$$

$$m = \frac{3}{2}, \frac{5}{2}, \ldots, S-\frac{3}{2}$$

$$n \geq 0$$

Matrices

$$(\Delta\xi)^2 \frac{Y}{A} = \begin{bmatrix} -\frac{1+3H_0/2S}{1+H_0/2S} & 1 & 0 \ldots 0 & 0 \\ 1 & -2 & 1 \ldots 0 & 0 \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0 & 0 \ldots 1 & -\frac{1+3H_S/2S}{1+H_S/2S} \end{bmatrix}$$

$$(\Delta\xi)^2 \frac{Y_B}{A} = \begin{bmatrix} \frac{H_0/S}{1+H_0/2S} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & \frac{H_S/S}{1+H_S/2S} \end{bmatrix}$$

Table IV-3 (Cont.)

GENERALIZED METHOD C

Initial Condition

$t_{m,0} = T(m\Delta\xi, 0)$ $\qquad m = \frac{1}{2}, \frac{3}{2}, \ldots, S-\frac{1}{2}$

Fluid Temperature

$t_{f,n}\ T_f(n\Delta\tau)$

Stability

$r(1-2\gamma) \leq \frac{1}{2}$ $\qquad 0 \leq h \leq \infty$

Infinite h

Boundary Equation:

$$t_{\frac{1}{2},n+1}(1+3r\gamma) - r\gamma t_{\frac{3}{2},n+1} = t_{\frac{1}{2},n}\left[1-3r(1-\gamma)\right] + r(1-\gamma)t_{\frac{3}{2},n} + 2r\left[(1-\gamma)t_{f_0,n} + t_{f_0,n+1}\right]$$

Table IV-4

GENERALIZED METHOD F

Mesh $\Delta\xi/2$, Equation IV-29, $\beta = \frac{1}{2}$, $\phi = 0$, Explicit $\gamma = 0$

Boundary Equation $0 \leq h \leq \infty$

$$t_{\frac{1}{2},n+1} = \frac{t_{\frac{1}{2},n}\left[1-r\left(\frac{1+H_0/S}{1+H_0/2S}\right)\right] + rt_{\frac{3}{2},n} + \left[\frac{rH_0/2S}{1+H_0/2S}\right]\left[t_{f_0,n} + t_{f_0,n+1}\right]}{1 + r\left(\frac{H_0/2S}{1+H_0/2S}\right)}$$

$$n \geq 0$$

(If H = 0, same equation as explicit method C)

Interior Points

$$t_{m,n+1} = rt_{m-1,n} + (1-2r)t_{m,n} + rt_{m+1,n}$$

$$m = \frac{3}{2}, \frac{5}{2}, \ldots, S-\frac{3}{2}$$

$$n \geq 0$$

Matrices (Note: Equivalent Y/A is a function of r)

$$(\Delta\xi)^2 \frac{Y}{A} = \begin{bmatrix} -\left\{\frac{1+3H_0/2S}{1+(1+r)H_0/2S}\right\} & \left\{\frac{1+H_0/2S}{1+(1+r)H_0/2S}\right\} & 0 \ldots & & \\ 1 & -2 & 1 \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots & \left\{\frac{1+H_S/2S}{1+(1+r)H_S/2S}\right\} & -\left\{\frac{1+3H_S/2S}{1+(1+r)H_S/2S}\right\} \end{bmatrix}$$

$$(\Delta\xi)^2 \frac{Y_B}{A} = \begin{bmatrix} \frac{H_0/2S}{1+(1+r)H_0/2S} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & \frac{H_S/2S}{1+(1+r)H_S/2S} \end{bmatrix}$$

Table IV-4 (Cont.)

GENERALIZED METHOD F

Initial Condition

$t_{m,0} = T(m\Delta\xi,\ 0)$ $\qquad m = \frac{1}{2}, \frac{3}{2}, \ldots, S-\frac{1}{2}$

Fluid Temperature

$t_{f,n} = T_f(n\Delta\tau)$ $\qquad n \geq 0$

Stability

$r \leq \frac{1}{2}$

Infinite h $\qquad H = \infty$

$$t_{\frac{1}{2}} = \frac{(1-2r)t_{\frac{1}{2},n} + rt_{\frac{3}{2},n} + r(t_{f_0,n} + t_{f_0,n+1})}{1 + r}$$

Table IV-5

GRAPHICAL METHODS

General Definition

Generalized methods with

$$\gamma = 0$$

$$r = \frac{1}{2}$$

Graphical Method G

Only defined for:

H's infinite or zero

(Unstable other H's and simple construction not available)

If both H's are zero:

$q_{min} = -1$

Averaged graphical method G, $q_{min} = -1$ does not appear

Graphical Method A

Defined all H

If both H's are zero, $q_{min} = -1$

Averaged graphical method A, $q_{min} = -1$ does not appear

Graphical Method C

Not recommended for $H/S > 2$ (stable all H)

Graphical Method F

Usually used only for both H's = ∞, or one 0, the other ∞

Construction may be difficult or impossible other H's

Table IV-6

AVERAGED METHODS

General Definition

$t_{n+k\ Ave} = \frac{1}{2}(t_n + t_{n+1})$ Applies at time $(n+k)\Delta\tau$ $n \geq 0$

Forward Average, k=0:

$t_{n\ Ave} = \frac{1}{2}(t_n + t_{n+1})$ at time $n\Delta\tau$ $n \geq 0$

(Changes initial condition)

Central Average, k=1/2:

$t_{n+\frac{1}{2}} = \frac{1}{2}(t_n + t_{n+1})$ at time $(n+\frac{1}{2})\Delta\tau$ $n \geq 0$

Backward Average, k=1:

either

$t_{n+1} = \frac{1}{2}(t_n + t_{n+1})$ at time $(n+1)\Delta\tau$ $n \geq 0$

or

$t_n = \frac{1}{2}(t_{n-1} + t_n)$ at time $n\Delta\tau$ $n \geq 1$

In definitions t_n's on right side are found by stepping out the solution for a method using unaveraged temperature vector to advance to (n+1); i.e. $t_{n+k\ Ave}$ is not used to advance. However, if calculation matrices are constant (not a function of n or t_n) using average $t_{n+k\ Ave}$ and an average boundary temperature vector to step to the next time increment gives a solution identical to an averaged solution (see equation IV-44).

Definition Averaged Methods

Averaged Method G -- Backward Average

Averaged Method A -- Forward Average

Averaged Method C -- Backward Average

Averaged Method F -- Forward Average

Table IV-7

z-TRANSFORM OPERATION PAIRS

Difference Function f_n	z-Transform $\bar{f}(z)$
1. f_{n+1}	$z\bar{f}(z) - zf_0$
2. f_{n+2}	$z^2 \bar{f}(z) - z^2 f_0 - zf_1$
3. f_{n+3}	$z^3 \bar{f}(z) - z^3 f_0 - z^2 f_1 - zf_2$
4. f_{n+a} $\quad a \geq 0$	$z^a \bar{f}(z) - z^a f_0 - \ldots - zf_{a-1}$
5. f_{n-b} $\quad b \geq 0$ where $f_n = 0 \quad n < 0$	$z^{-b} \bar{f}(z)$
6. $\Delta f_n = f_{n+1} - f_n$	$(z - 1)\bar{f}(z) - zf_0$
7. nf_n	$- z \frac{d}{dz} \bar{f}(z)$
8. $\frac{1}{n} f_n$	$-\int \frac{\bar{f}(z)}{z} dz$
9. $\frac{1}{n + a} f_n$	$- z^a \int \frac{\bar{f}(z)}{z^{a+1}} dz$
10. $a^n f_n$	$\bar{f}\left(\frac{z}{a}\right)$
11. $\sum_{p=0}^{n} f_p$	$\frac{z}{z - 1} \bar{f}(z)$
12. $\sum_{p=0}^{n} f_p u_{n-p}$	$\bar{f}(z) \bar{u}(z)$

Table IV-7 (Cont.)

z-TRANSFORM OPERATION PAIRS

Difference Function f_n	z-Transform $\bar{f}(z)$
13. $\sum_{j=1}^{\infty} \rho_{j,n}$ where $\rho_{j,n}$ = residues of $z^{n-1}\bar{f}(z)$ For simple poles: $\rho_{j,n} = \lim_{z\to q_j} (z-q_j)f(z)z^{n-1}$ $= \lim_{z\to q_j} \left[\frac{N(z)}{\frac{d}{dz}D(z)}\right]$	$\frac{N(z)}{D(z)z^{n-1}}$ (No Branch Cuts)
14. Initial Value Theorem	$f_0 = \lim_{z\to\infty} \bar{f}(z)$
15. Final Value Theorem	$f_{\infty} = \lim_{z\to 1} (z-1)\,\bar{f}(z)$ (Assumes True Steady State Exists)

Table Modified After Aseltine (25), p. 260

Table IV-8

z-TRANSFORM FUNCTION PAIRS

	Difference Function f_n	z-Transform $\bar{f}(z)$
1.	A = constant	$\frac{z}{z-1} A$
2.	n	$\frac{z}{(z-1)^2}$
3.	n^2	$\frac{z(z+1)}{(z-1)^3}$
4.	n^3	$\frac{z(z^2+4z+1)}{(z-1)^4}$
5.	q^n	$\frac{z}{z-q}$
6.	nq^n	$\frac{qz}{(z-q)^2}$
7.	$\sin \omega_0 n$	$\frac{z \sin \omega_0}{z^2 - 2z\cos \omega_0 + 1}$
8.	$\cos \omega_0 n$	$\frac{z(z - \cos \omega_0)}{z^2 - 2z\cos \omega_0 + 1}$
9.	$\sinh \omega_0 n$	$\frac{z \sinh \omega_0}{z^2 - 2z\cosh \omega_0 + 1}$
10.	$\cosh \omega_0 n$	$\frac{z(z - \cosh \omega_0)}{z^2 - 2z\cosh \omega_0 + 1}$

Table IV-8 (Cont.)

z-TRANSFORM FUNCTION PAIRS

Difference Function f_n	z-Transform $\bar{f}(z)$
11. $e^{-\alpha n} \sin \omega_0 n$	$\dfrac{ze^{-\alpha} \sin \omega_0}{z^2 - 2ze^{-\alpha} \cos \omega_0 + e^{-2\alpha}}$
12. $e^{-\alpha n} \cos \omega_0 n$	$\dfrac{z(z - e^{-\alpha} \cos \omega_0)}{z^2 - 2ze^{-\alpha} \cos \omega_0 + e^{-2\alpha}}$
13. $\dfrac{1}{n!}$	$e^{0.5}$
14. $\dfrac{1}{(2n)!}$	$\cosh(z^{-0.5})$
15. $\delta_{np} = \begin{matrix} 0 & n \neq p \\ 1 & n = p \end{matrix}$	z^{-p}

Table Modified After Aseltine (25), p. 259.

Table IV-9

PROBLEMS TO BE APPROXIMATED

Partial Differential Equation

$$\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{(\partial \xi)^2} \qquad 0 \leq \xi \leq 1$$

Problem I - Finite h and Zero h

Initial	$T(\xi,0) = 0$	$0 \leq \xi \leq 1$
Left Boundary	$-q\left[1 - T(0,\tau)\right] = \frac{\partial T}{\partial \xi}(0,\tau)$	$\tau \geq 0$
Right Boundary	$\frac{\partial T}{\partial \xi}(1,\tau) = 0$	$\tau \geq 0$

Problem II - Infinite h and Zero h

Initial	$T(\xi,0) = 0$	$0 < \xi \leq 1$
Left Boundary	$T(0,\tau) = 1$	$\tau \geq 0$
Right Boundary	$\frac{\partial T(1,\tau)}{\partial \xi} = 0$	$\tau \geq 0$

Problem III - Both h's Zero

Initial	$T(\xi,0) =$	$0 \leq \xi \leq 1$
Left Boundary	$\frac{\partial T}{\partial \xi}(0,\tau) = 0$	$\tau \geq 0$
Right Boundary	$\frac{\partial T}{\partial \xi}(1,\tau) = 0$	$\tau \geq 0$

Problem IV - Both h's Infinite

Initial	$T(\xi,0) = 0$	$0 \leq \xi < 1$
Left Boundary	$T(0,\tau) = 0$	$\tau \geq 0$
Right Boundary	$T(1,\tau) = 1$	$\tau \geq 0$

Problem V - Semi-Infinite Solid

Initial	$T(\xi,0) = 0$	$0 \leq \xi \leq \infty$
Left Boundary	$T(0,\tau) = 1$	$\tau \geq 0$
Right Boundary	$\frac{\partial T}{\partial \xi}(\infty,\tau) = 0$	$\tau \geq 0$

Table IV-10

CONTINUOUS, ANALOG, AND DIFFERENCE SOLUTIONS IN DIFFERENCE FORM

One-Dimensional Diffusion Equation in Cartesian Co-ordinates

Continuous

$$T(m\Delta\xi,\ n\Delta\mathcal{T}) = T_P(m\Delta\xi,\ n\Delta\mathcal{T}) - \sum_{j=1}^{\infty} a_j\ b_j(m\Delta\xi)\left[e^{-\psi_j^2 r}\right]^n$$

where ψ_j defined by $F_C(\psi_j) = 0$ Characteristic Equation

Approximate

$$t_{m,n} = t_{P\ m,n} - \sum_{j=1}^{S} g_j\ c_{mj}\ q_j^{\ n}$$

for difference solution $q_j^{\ n} = \left[\dfrac{1-2r(1-\gamma)(1-\cos\alpha_j)}{1+2r(1-\gamma)(1-\cos\alpha_j)}\right]^n$

for analog solution (only for methods G and C) replace

$q_j^{\ n}$ with $e^{-2S^2(1-\cos\alpha_j)n\mathcal{T}}$

where α_j defined by $F_D(\alpha_j) = 0$ Characteristic Equation

Notes

All Equations and Methods	$n = 0, 1, \ldots, \infty$	
Continuous Solution either	$m = 0, 1, 2, \ldots, S$	Mesh $\Delta\xi$
or	$m = \frac{1}{2}, \frac{3}{2}, \ldots, S-\frac{1}{2}$	Mesh $\Delta\xi/2$
Approximate Method G or A	$m = 0, 1, 2, \ldots, S$	
Approximate Method C or F	$m = \frac{1}{2}, \frac{3}{2}, \ldots, S-\frac{1}{2}$	

For approximate methods summation on j is to number of variable temperature points

Table IV-10 (Cont.)

CONTINUOUS, ANALOG, AND DIFFERENCE SOLUTIONS IN DIFFERENCE FORM

Corresponding Quantities

	Continuous	Approximate
Particular Solution - Steady State or Quasi-Steady State		
	$T_P(m\Delta\xi, n\Delta\tau)$	$t_{P\ m,n}$
Fourier Coefficient - Component of Initial Vector Weighting Given to j^{th} Eigenvector-Eigenvalue Product		
	$a_j = a_j(\psi_j)$	$g_j = g_j(\alpha_j)$
Component in Row for m^{th} Temperature Point and j^{th} Column of Eigenvector Matrix		
	$b_j(m\Delta\xi) = b_j(m\Delta\xi, \psi_j)$	$c_{mj} = c_{mj}(\alpha_j)$
Damping Factors		
	$e^{-\psi_j^2 r}$	Difference: $\dfrac{1-2r(1-\gamma)(1-\cos\alpha_j)}{1+2r\gamma(1-\cos\alpha_j)}$
	or $e^{-\nu_j^2 \Delta\tau}$	Analog: $e^{-2r(1-\cos\alpha_j)}$
Characteristic Equation		
	$F_C(\psi_j) = 0$	$F_D(\alpha_j) = 0$
Eigenvalue Parameter or Trigonometric Root		
	$\psi_j = \dfrac{\nu_j}{S}$	$\alpha_j = \dfrac{\mu_j}{S}$

Table IV-11

PARTICULAR OR STEADY-STATE SOLUTIONS

Problem I

$$T_P(m\Delta\xi) = t_{P\ m,n} = 1$$

Approximate Methods G, A, and C

Problem II

$$T_P(m\Delta\xi) = t_{P\ m,n} = 1$$

Approximate Methods G, A, and C, and Graphical Method F

Problem III

$$T_P(m\Delta\xi,\ n\Delta\tau) = t_{P\ m,n} = \frac{1}{2}$$

Approximate Methods G, A, and C

Problem IV

$$T_P(m\Delta\xi,\ n\Delta\tau) = t_{P\ m,n} = m\Delta\xi = \frac{m}{S}$$

Approximate Method G

Problem V

$$T_P(m\Delta\xi,\ n\Delta\tau) = t_{P\ m,n} = 1$$

Approximate Method G

($t_{P\ m,n}$ and $T_{P\ m,n}$ steady state for $m < \infty$; steady state for $m = \infty$ is zero as found by adding integral, Table IV-29, to particular solution.)

Table IV-12

CHARACTERISTIC EQUATIONS - PROBLEM I

One h Finite; One h Zero

(Applies in Limits $h \rightarrow \infty$, Problem II; and $h \rightarrow 0$, Problem III)

	Characteristic Equations	Number of Roots	Notes on Roots
Continuous *			
	$(S\psi_j)(\tan S\psi_j) = H$	∞	ψ_j real and positive $0 \leq \psi_j \leq \infty$, $j = 1, 2, \ldots, \infty$
Approximate Methods			All real α_j's fall within $0 \leq \alpha_j \leq \pi$
G**	$S(\sin\alpha_j)(\tan S\alpha_j) = H$	S+1	$\alpha_1, \alpha_2, \ldots, \alpha_S$ real; α_{S+1} complex; $\alpha_{S+1} = \pi + \sqrt{-1}\,\mathcal{g}(\alpha_{S+1})$, $0 \leq \mathcal{g}(\alpha_{S+1}) < \infty$
A	$\frac{S(\sin\alpha_j)(\tan S\alpha_j)}{1 - 2r(1-\cos\alpha_j)} = H$	S+1	If $r < \frac{1}{4}$, α_j's as in method G; If $r \geq \frac{1}{4}$, S+1 roots real
Graphical A, r=1/2			
	$S(\tan\alpha_j)(\tan S\alpha_j) = H$	S+1	All α_j's real
C**	$2S(\tan\frac{\alpha_j}{2})(\tan S\alpha_j) = H$	S	All α_j's real

Notes: 1. Only for $h=\infty$ (Problem II) or $h=0$ (Problem III) are the S or (S+1) α_j's equal to first S or (S+1) ψ_j's for these approximate methods.

2. Characteristic equations depend only on H, not on how fluid temperature changes with time or the initial distribution.

*Continuous characteristic equation in difference form **Characteristic equation independent of γ and r

Table IV-13

CHARACTERISTIC EQUATIONS - PROBLEMS II, III, AND IV

Problem II $h_0 = \infty$ and $h_S = 0$ (Limiting Case Problem I, $H \rightarrow \infty$)

Continuous $\cos S\psi_j = 0$ $\psi_j = \frac{(2j-1)\pi}{2S}$ $j = 1, \ldots, \infty$

Approximate

Methods G, A, and C

$\cos S\,\alpha_j = 0$ $\alpha_j = \frac{(2j-1)\pi}{2S}$ $j = 1, \ldots, S$

$\alpha_j = \psi_j$ for j's $1 \leq j \leq S$

Graphical Method F

$\left[\tan(S-\frac{1}{2})\alpha_j\right] \tan\alpha_j = 2$ $j = 1, \ldots, S$

Problem III Both h's Zero (Limiting Case Problem II, $H \rightarrow 0$)

Continuous $\sin S\psi_j = 0$ $\psi_j = \frac{(j-1)\pi}{S}$ $j = 1, \ldots, \infty$

Approximate

Methods G, A, and C

$\sin S\alpha_j = 0$ $\alpha_j = \frac{(j-1)\pi}{S}$ $j = 1, \ldots, S, S+1$

Method C $j \neq S+1$

$\alpha_j = \psi_j$ for j less than S or S+1

(Note that for j=1, q_1=1, which, although an eigenvalue of the matrices, gives the steady state solution and is not included in transient part of solution.)

The α_{S+1} which equals π corresponds to q_{min} for methods G and A

Problem IV Both h's Infinite

Continuous $\sin S\psi_j = 0$ $\psi_j = \frac{j\pi}{S}$ $j = 1, 2, \ldots, \infty$

Approximate

Method G $\sin S\alpha_j = 0$ $\alpha_j = \frac{j\pi}{S}$ $j = 1, 2, \ldots, S-1$

Problem V No Characteristic Equation

(In limit of characteristic equation of problem II the fact that as $S \rightarrow \infty$ $\Delta\xi \not\rightarrow 0$ shows that ψ and α become continuous because any values of ψ and α satisfy the equation.)

Table IV-14

DAMPING FACTORS

DAMPING FACTORS - EIGENVALUES

Continuous $\quad j = 1, 2, \ldots, \infty$

$$e^{-\nu_j^2 \Delta\tau} = 1 - \nu_j^2 \Delta\tau + \frac{\nu_j^4 (\Delta\tau)^2}{2} - \frac{\nu_j^6 (\Delta\tau)^3}{6} \quad \ldots$$

Approximate All Methods $\quad j = 1, 2, \ldots, S \text{ or } S+1$

$$q_j = \frac{1-2S^2\Delta\tau(1-\gamma)(1-\cos\frac{\mu_j}{S})}{1+2S^2\Delta\tau(\gamma)(1-\cos\frac{\mu_j}{S})} = 1 - \mu_j^2\Delta\tau + \frac{\mu_j^4(\Delta\tau)^2}{2}\, 2\gamma + \left[\frac{1}{6S^2\Delta\tau}\right]$$

$$- \frac{\mu_j^6(\Delta\tau)^3}{6}\left[6\gamma^2 + \frac{\gamma}{S^2\Delta\tau} + \frac{1}{60S^4(\Delta\tau)^2}\right] \quad \ldots$$

DAMPING FACTORS TO n^{th} POWER AS A FUNCTION OF TIME

Continuous

$$e^{-\nu_j^2 n\Delta\tau} = 1 - \nu_j^2 n(\Delta\tau) + \frac{\nu_j^4(n\Delta\tau)^2}{2} - \frac{\nu_j^6(n\Delta\tau)^3}{6} \quad \ldots$$

Approximate All Methods

Difference (Note $\tau = n\Delta\tau$):

$$q_j^n = 1 - \mu_j^2(n\Delta\tau) + \frac{\mu_j^4(n\Delta\tau)^2}{2}\left[1 + \frac{1}{6S^2\tau} - \frac{(1-2\gamma)}{n}\right]$$

$$- \frac{\mu_j^6(n\Delta\tau)^3}{6}\left[1 + \frac{1}{2S^2\tau} - \frac{3(1-2\gamma)}{n} + \frac{1}{60S^4\tau^2} + \frac{2(1-3\gamma+3\gamma^2)}{n^2} - \frac{(1-2\gamma)}{2S^2\tau n}\right] \ldots$$

Analog: $\quad \Delta\tau \longrightarrow 0 \qquad n\Delta\tau \longrightarrow \tau \qquad n \longrightarrow \infty$

$$e^{-2S^2(1-\cos\frac{\mu_j}{S})n\Delta\tau} = 1 - \mu_j^2(n\Delta\tau) + \frac{\mu_j^4(n\Delta\tau)^2}{2}\left[1 + \frac{1}{6S^2\tau}\right]$$

$$- \frac{\mu_j^6(n\Delta\tau)^3}{6}\left[1 + \frac{1}{2S^2\tau} + \frac{1}{60S^4\tau^2}\right] \quad \ldots$$

Table IV-15

DIFFERENCE IN LOGARITHMS OF DAMPING FACTORS

$$\frac{1}{\Delta\tau} \ln q_j = -\mu_j^2 + \mu_j^4 \left[\frac{1}{12S^2} - (\frac{1}{2} - \gamma)\Delta\tau\right]$$

$$- \mu_j^6 \left[(\frac{1}{3} - \gamma + \gamma^2)(\Delta\tau)^2 - \frac{(\frac{1}{2} - \gamma)\Delta\tau}{6S^2} + \frac{1}{360S^4}\right] \quad \cdots$$

$$\ln \frac{e^{-\nu_j^2}}{q_j^{1/\Delta\tau}} = \mu_j^2 - \nu_j^2 - \mu_j^4 \left[\frac{1}{12S^2} - (\frac{1}{2} - \gamma)\Delta\tau\right]$$

$$+ \mu_j^6 \left[(\frac{1}{3} - \gamma + \gamma^2)(\Delta\tau)^2 - \frac{(\frac{1}{2} - \gamma)\Delta\tau}{6S^2} + \frac{1}{360S^4}\right] \quad \cdots$$

<u>Special Cases for γ</u>

Explicit, $\gamma = 0$

$$\ln \frac{e^{-\nu_j^2}}{q_j^{1/\Delta\tau}} = \mu_j^2 - \nu_j^2 - \mu_j^4 \left[\frac{1}{12S^2} - \frac{\Delta\tau}{2}\right]$$

$$+ \mu_j^6 \left[\frac{1}{360S^4} - \frac{\Delta\tau}{12S^2} + \frac{(\Delta\tau)^2}{3}\right] \quad \cdots$$

Implicit, $\gamma = \frac{1}{2} - \frac{1}{12S^2\Delta\tau}$

$$\ln \frac{e^{-\nu_j^2}}{q_j^{1/\Delta\tau}} = \mu_j^2 - \nu_j^2 + \mu_j^6 \left[\frac{(\Delta\tau)^2}{12} - \frac{1}{240S^2}\right] \quad \cdots$$

Implicit, $\gamma = \frac{1}{2}$

$$\ln \frac{e^{-\nu_j^2}}{q_j^{1/\Delta\tau}} = \mu_j^2 - \nu_j^2 - \mu_j^4 \left[\frac{1}{12S^2}\right] + \mu_j^6 \left[\frac{1}{360S^4} + \frac{(\Delta\tau)^2}{2}\right] \quad \cdots$$

Table IV-15 (Cont.)

DIFFERENCE IN LOGARITHMS OF DAMPING FACTORS

Special Cases for γ

Implicit, $\gamma = 1$

$$\ln \frac{e^{-\nu_j^2}}{q_j^{1/\Delta\tau}} = \mu_j^2 - \nu_j^2 - \mu_j^4\left[\frac{1}{12S^2} + \frac{\Delta\tau}{2}\right] + \mu_j^6\left[\frac{1}{360S^4} + \frac{\Delta\tau}{12S^2} + \frac{(\Delta\tau)^2}{3}\right] \cdots$$

Special Cases for γ and $\Delta\tau$ or r

Graphical Explicit, $\gamma = 0$, $r = \frac{1}{2}$

$$\ln \frac{e^{-\nu_j^2}}{q_j^{1/\Delta\tau}} = \mu_j^2 - \nu_j^2 + \mu_j^4\left[\frac{1}{6S^2}\right] + \mu_j^6\left[\frac{2}{45S^4}\right] \cdots$$

Analog, $\Delta\tau \longrightarrow 0$

$$\ln \frac{e^{-\nu^2}}{e^{+\lambda}} = -\nu^2 - \lambda = -\nu^2 + 2S^2(1 - \cos \frac{\mu_j}{S})$$

$$= \mu_j^2 - \nu_j^2 - \mu_j^4\left[\frac{1}{12S^2}\right] + \mu_j^6\left[\frac{1}{360S^2}\right] \cdots$$

Table IV-16

EIGENVECTOR MATRICES

Element in Row Corresponding to m^{th} Point,* j^{th} Column

Continuous Eigenvector Applies Either Mesh $\Delta\xi$ or $\Delta\xi/2$	
For Mesh $\Delta\xi$	$m = 0, 1, 2, \ldots, S$
For Mesh $\Delta\xi/2$	$m = \frac{1}{2}, \frac{3}{2}, \ldots, S-\frac{1}{2}$
Approximate Eigenvector	
Methods G and A, Mesh $\Delta\xi$	$m = 0, 1, 2, \ldots, S$
Method C, Mesh $\Delta\xi/2$	$m = \frac{1}{2}, \frac{3}{2}, \ldots, S-\frac{1}{2}$

<u>Problem I</u> Finite h and zero h	$\alpha_j \neq \psi_j$	
Continuous**		
$b_j(m\Delta\xi) = \cos(S-m)\psi_j$		$j = 1, 2, \ldots, \infty$
Approximate Methods G, A, and C		
$c_{mj} = \cos(S-m)\,\alpha_j$	Methods G and A	$j = 1, 2, \ldots, S+1$
	Method C	$j = 1, \ldots, S$
<u>Problem II</u> Infinite h and zero h		$j = 1, \ldots, \infty$
Continuous**		
$b_j(m\Delta\xi) = \sin m\psi_j$	$\psi_j = \frac{(2j-1)\pi}{2S}$	$j = 1, \ldots, \infty$
Methods G and A		
$c_{mj} = \sin m\psi_j$	$\alpha_j = \psi_j$	$j = 1, \ldots, S$
Method C		
$c_{mj} = \sin m\psi_j$	$\alpha_j = \psi_j$	$j = 1, \ldots, S$
Graphical Method F		
$c_{mj} = \cos(S-m)\,\alpha_j$	$\alpha_j \neq \psi_j$	$j = 1, \ldots, S$

Table IV-16 (Cont.)

EIGENVECTOR MATRICES

Problem III *** Both h's zero

Continuous**

$b_j(m\Delta\xi) = \cos m\psi_j$	$\psi_j = \frac{(j-1)\pi}{S}$	$j = 1, \ldots, \infty$

Methods G and A

$c_{mj} = \cos m\psi_j$	$\alpha_j = \psi_j$	$j = 1, \ldots, S+1$

Method C

$c_{mj} = \cos m\psi_j$	$\alpha_j = \psi_j$	$j = 1, \ldots, S$

Problem IV Both h's infinite

Continuous **

$b_j(m\Delta\xi) = \cos m\psi_j$	$\psi_j = \frac{j\pi}{S}$	$j = 1, 2, \ldots, \infty$

Method G

$c_{mj} = \cos m\psi_j$	$\alpha_j = \frac{j\pi}{S}$	$j = 1, 2, \ldots, S-1$

* m refers to equation for m^{th} point, which is (m+1) row in conventional matrix notation for methods G and A based on mesh $\Delta\xi$ or $(m + \frac{1}{2})$ row in conventional matrix notation for methods C and F.

** Continuous "eigenvector" is actually an eigenfunction if m is allowed to vary continuously from 0 to S.

*** Vector corresponding to $\psi_1 = 0$ is considered as steady state solution, although its corresponding eigenvector is eigenvector of Y/A matrix.

Table IV-17

COMPONENTS OF INITIAL VECTOR - PROBLEM I

One h Finite; Other h Zero; Fluid Temperature Zero; Initial Condition Zero

$$T(m\Delta\xi,\ 0-) = T(m\Delta\xi,\ 0+) = 0 \qquad 0 \leq m\Delta\xi \leq 1$$

Continuous $\qquad j = 1, 2, \ldots, \infty$

$$a_j = \frac{2H}{(\nu_j)\left[(H+1)(\sin \nu_j) + (\nu_j)(\cos \nu_j)\right]}$$

Method G $\qquad j = 1, 2, \ldots, S+1$

$$g_j = \frac{2H}{\left[(2S)(\tan \frac{\mu_j}{2S})\right]\left[(H+\cos \frac{\mu_j}{S})(\sin \mu_j) + (S)(\sin \frac{\mu_j}{S})(\cos \mu_j)\right]}$$

Generalized Method A, Explicit $\qquad j = 1, 2, \ldots, S+1$

$$g_j = \frac{2H}{\left[2S \tan\frac{\mu_j}{2S}\right]\left[\left\{H\left[1-2r(1-\cos\frac{\mu_j}{S})\right] + \cos\frac{\mu_j}{S}\right\}(\sin \mu_j) + (S+\frac{2rH}{S})(\sin\frac{\mu_j}{S})(\cos \mu_j)\right]}$$

Graphical Method A, r = 1/2 $\qquad j = 1, 2, \ldots, S+1$

$$g_j = \frac{2H}{\left[2S \tan \frac{\mu_j}{2S}\right]\left[(H+1)(\cos \frac{\mu_j}{S})(\sin \mu_j) + (S+\frac{H}{S})(\sin \frac{\mu_j}{S})(\cos \mu_j)\right]}$$

Method C $\qquad j = 1, 2, \ldots, S$

$$g_j = \frac{2H}{\left[2S \sin \frac{\mu_j}{2S}\right]\left[(H+1)(\sin \mu_j) + (2S+\frac{H}{2S})(\tan \frac{\mu_j}{2S})(\cos \mu_j)\right]}$$

Averaged Methods

G and C multiply by $\qquad \dfrac{q_j + 1}{2q_j} \qquad$ Backward

A multiply by $\qquad \dfrac{q_j + 1}{2} \qquad$ Forward

Table IV-18

COMPONENTS OF INITIAL VECTOR - PROBLEM II

$h = \infty$

Initial Condition $T(m\Delta\xi,\ 0-) = 0 \qquad 0 \le m\Delta\xi \le 1$

$$T(m\Delta\xi,\ 0+) = \begin{pmatrix} 0 & 0 < m\Delta\xi \le 1 \\ 1 & 0 = m\Delta\xi \end{pmatrix}$$

Continuous $\quad \nu_j = \frac{(2j-1)\pi}{2} \qquad j = 1, 2, \ldots, \infty$

$a_j = \frac{2}{\nu_j}$

Approximate $\qquad j = 1, 2, \ldots, S$

g_j	Expansion for $(g_j - a_j)$ Small $\frac{\nu_j}{S}$, $0 \le \frac{\nu_j}{S} < \pi$
Method G $\quad \frac{1}{S \tan \frac{\nu_j}{2S}}$	$-\frac{\nu_j}{6S^2} - \frac{\nu_j^3}{360S^4} \ldots$
Generalized Method A, Explicit or Implicit $\quad \frac{1}{S(\tan \frac{\nu_j}{2S})q_j}$	$\frac{\nu_j(12r-1)}{6S^2} - \frac{\nu_j^3}{360}\left[1+120r\{1-6r(1-\gamma)\}\right] \ldots$
Graphical Method A $\quad \frac{1}{S(\tan \frac{\nu_j}{2S})(\cos \frac{\nu_j}{S})}$	$\frac{5\nu_j}{6S^2} - \frac{\nu_j^3}{360}(119) \ldots$
Method C $\quad \frac{1}{S \sin \frac{\nu_j}{2S}}$	$\frac{\nu_j}{12S^2} + \frac{7\nu_j^3}{2880S^4} \ldots$
Graphical Method F $\quad \frac{4}{\tan\frac{\mu_j}{2S}\left[3(S+\frac{1}{2})(\sin\frac{\{2S+1\}\mu_j}{2S})+(S-\frac{3}{2})(\sin\frac{\{2S-3\}\mu_j}{2S})\right]}$ where μ_j are the roots of the characteristic equation, Table IV-13.	

Table IV-19

COMPONENTS OF INITIAL VECTOR - PROBLEM III

Both h's Zero

Initial Condition $T(m\Delta\xi, 0+) = T(m\Delta\xi, 0-) = m\Delta\xi$

Continuous

$$a_j = \frac{2\left[1-(-1)^j\right]}{\nu_j^2} \quad \text{where } \nu_j = j\pi \qquad j = 1, 2, \ldots, \infty$$

Approximate $\qquad j = 1, 2, \ldots, S-1$ or S

g_j	Expansion for $(g_j - a_j)$ Small $\frac{\nu_j}{S}$
Methods G and A Equivalent $$\frac{\left[1-(-1)^j\right]}{S^2\left(1-\cos\frac{\nu_j}{S}\right)}$$	$$\left[1-(-1)^j\right]\left[\frac{1}{6S^2}+\frac{\nu_j^2}{120S^4}\cdots\right.$$
Method C $$\frac{\left[1-(-1)^j\right]}{2S^2\left(\tan\frac{\nu_j}{2S}\right)\left(\sin\frac{\nu_j}{2S}\right)}$$	$$-\left[1-(-1)^j\right]\left[\frac{1}{12S^2}+\frac{7\nu_j^2}{960S^2}\cdots\right.$$

Table IV-20

COMPONENTS OF INITIAL VECTOR - PROBLEM IV

Both h's Infinite

Initial Condition $T(m\Delta\xi,\ 0-) = 0 \qquad 0 \leq m\Delta\xi \leq 1$

$$T(m\Delta\xi,\ 0+) = \begin{pmatrix} 0 & m\Delta\xi = 0 \\ 0 & 0 < m\Delta\xi < 1 \\ 1 & m\Delta\xi = 1 \end{pmatrix}$$

<u>Continuous</u>

$$a_j = \frac{-2(-1)^j}{\nu_j} \qquad \text{where } \nu_j = j\pi$$

<u>Approximate</u> $\qquad j = 1, 2, \ldots, S-1$

g_j	Expansion for $(g_j - a_j)$ Small ν_j
Method G	
$\dfrac{-(-1)^j \sin \frac{\nu_j}{S}}{2S(\sin \frac{\nu_j}{2S})^2}$	$(-1)^j\left[\dfrac{\nu_j}{6S^2} + \dfrac{\nu_j^3}{360S^4} \cdots\right.$
Averaged Method G	$(g_{j\ Ave} - a_j)$
$\dfrac{-(-1)^j (\sin \frac{\nu_j}{S})(q_j + 1)}{4S(\sin \frac{\nu_j}{2S})^2\ q_j}$	$(-1)^j\left[\dfrac{\nu_j}{6S^2}(1-6r) + \dfrac{\nu_j^3}{360S^4}\left\{1-360r^2(1-\gamma)\right\}\right.$ $\cdots$

Table IV-21

EFFECTIVE INITIAL VECTOR COMPONENTS

Graphical Solutions - Problem II

Continuous

$$a_j = \frac{2}{\nu_j} \qquad \nu_j = \frac{(2j-1)\pi}{2} \qquad j = 1,\ 2,\ \ldots,\ \infty$$

Approximate Graphical

The following expressions result from pairs of q_j's of equal magnitude but opposite in sign, combined in a single q, with the corresponding initial vector components combined into effective weightings.

$g_{j\,E} = g_j + (-1)^n\, g_{S+1-j}\, \dfrac{c_{m\ S+1-j}}{c_{mj}}$ $1 \leq j < \frac{S}{2}$	Expansion for $(g_{j\,E} - a_j)$ Small ν_j $\left\|\frac{\nu_j}{S}\right\| < \frac{\pi}{2}$
Method G $\dfrac{\text{ctn}\,\frac{\nu_j}{2S} - (-1)^{m+n}(\tan\frac{\nu_j}{2S})}{2S}$	$-\nu_j\left[\dfrac{1+3(-1)^{m+n}}{6S^2}\right] - \nu_j^3\left[\dfrac{1+15(-1)^{m+n}}{350S^4}\right]\ldots$
Method A $\dfrac{\text{ctn}\,\frac{\nu_j}{2S} + (-1)^{m+n}(\tan\frac{\nu_j}{2S})}{S\cos\frac{\nu_j}{S}}$	$\nu_j\left[\dfrac{5+3(-1)^{m+n}}{6S^2}\right] + \nu_j^3\left[\dfrac{119+105(-1)^{m+n}}{360S^4}\right]\ldots$
Method C $\dfrac{1}{S\sin\frac{\nu_j}{2S}} + \dfrac{(-1)^{n+m-\frac{1}{2}}(\text{ctn}\,\frac{m\nu_j}{S})}{\cos\frac{\nu_j}{2S}}$	$\dfrac{a_j(-1)^{n+m-\frac{1}{2}}}{2m} + \nu_j\left[\dfrac{2m+(-1)^{n+m-\frac{1}{2}}(3-8m^2)}{24mS^2}\right]\ldots$

Table IV-22

COMPARISON OF GRAPHICAL METHOD F

Problem II, S=5, r=1/2, γ=0

$\Delta\tau$ = 0.02

Trigonometric Roots

Continuous, Methods G, A, and C	18°	54°	90°	126°	162°
Radians, ψ_j	0.3142	0.9426	1.5710	2.1994	2.8278
Graphical Method F	17.955°	52.62°	83.05°	111.5°	144.4°
α_j	0.3133	0.9183	1.4495	1.946	2.5199

Continuous Solution (First Five Damping Factors, Mesh $\Delta\xi/2$)

$$\begin{pmatrix} T_{\frac{1}{2}} \\ T_{\frac{3}{2}} \\ T_{\frac{5}{2}} \\ T_{\frac{7}{2}} \\ T_{\frac{9}{2}} \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.1992 & 0.1927 & 0.1800 & 0.1621 & 0.1397 \\ 0.5780 & 0.4192 & 0.1800 & -0.0285 & -0.1261 \\ 0.9003 & 0.3001 & -0.1800 & -0.1286 & 0.1000 \\ 1.1345 & -0.0664 & -0.1800 & 0.1797 & -0.0642 \\ 1.2576 & -0.3782 & 0.1800 & 0.0826 & 0.0221 \end{bmatrix} \begin{pmatrix} 0.9518^n \\ 0.6414^n \\ 0.2912^n \\ 0.0891^n \\ 0.0184^n \end{pmatrix}$$

Graphical Method F (Mesh $\Delta\xi/2$)

$$\begin{pmatrix} t_{\frac{1}{2}} \\ t_{\frac{3}{2}} \\ t_{\frac{5}{2}} \\ t_{\frac{7}{2}} \\ t_{\frac{9}{2}} \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.2084 & 0.2800 & 0.3346 & 0.1523 & 0.0243 \\ 0.5951 & 0.5100 & 0.1216 & -0.1676 & -0.0592 \\ 0.9236 & 0.3391 & -0.3051 & -0.0295 & 0.0720 \\ 1.1623 & -0.0982 & -0.1955 & 0.1892 & -0.0578 \\ 1.2876 & -0.4583 & 0.2578 & -0.1092 & 0.0220 \end{bmatrix} \begin{pmatrix} 0.9513^n \\ 0.6071^n \\ 0.1210^n \\ -0.3665^n \\ -0.8129^n \end{pmatrix}$$

Averaged Graphical Method F (Mesh $\Delta\xi/2$)

$$\begin{pmatrix} t_{\frac{1}{2}} \\ t_{\frac{3}{2}} \\ t_{\frac{5}{2}} \\ t_{\frac{7}{2}} \\ t_{\frac{9}{2}} \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.2033 & 0.2250 & 0.1875 & 0.0482 & 0.0023 \\ 0.5806 & 0.4098 & 0.6816 & -0.0531 & -0.0055 \\ 0.9011 & 0.2725 & -0.1710 & -0.0093 & -0.0067 \\ 1.1339 & -0.0789 & -0.1096 & 0.0599 & 0.0054 \\ 1.2556 & -0.3682 & 0.1450 & -0.0346 & -0.0021 \end{bmatrix} \begin{pmatrix} 0.9513^n \\ 0.6071^n \\ 0.1210^n \\ -0.3665^n \\ -0.8129^n \end{pmatrix}$$

Table IV-23

COMPARISON OF SOLUTIONS, GRAPHICAL METHOD A

Problem II, $S=5$, $r=1/2$, $\gamma=0$, $\Delta\tau=0.02$

<u>Continuous Solution</u> (First Five Damping Factors, Mesh $\Delta\xi$)

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.3934 & 0.3433 & 0.2546 & 0.1472 & 0.0437 \\ 0.7484 & 0.4036 & 0 & -0.1730 & -0.0832 \\ 1.0300 & 0.1311 & -0.2546 & 0.0562 & 0.1145 \\ 1.2109 & -0.2495 & 0 & 0.1069 & -0.1346 \\ 1.2732 & -0.4244 & 0.2546 & -0.1819 & 0.1415 \end{bmatrix} \begin{bmatrix} 0.9518^n \\ 0.6414^n \\ 0.2912^n \\ 0.0891^n \\ 0.0184^n \end{bmatrix}$$

<u>Graphical Method A</u> (Mesh $\Delta\xi$)

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.4103 & 0.5403 & -^{*} & -0.1402 & -0.0103 \\ 0.7804 & 0.6352 & - & 0.1649 & 0.0196 \\ 1.0741 & 0.2064 & - & -0.0536 & -0.0269 \\ 1.2627 & -0.3925 & - & -0.1019 & 0.0317 \\ 1.3277 & -0.6679 & - & 0.1734 & -0.0333 \end{bmatrix} \begin{bmatrix} 0.9511^n \\ 0.5878^n \\ -^{*} \\ -0.5878^n \\ -0.9851^n \end{bmatrix}$$

(* Zero roots not shown)

<u>Effective Solution Graphical Method A</u> (Mesh $\Delta\xi$)

n even

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.4000 & 0.4000 \\ 0.8000 & 0.8000 \\ 1.4472 & 0.1528 \\ 1.2944 & -0.4944 \\ 1.2944 & -0.4945 \end{bmatrix} \begin{bmatrix} 0.9511^n \\ 0.5878^n \end{bmatrix}$$

n odd

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.4206 & 0.6805 \\ 0.7608 & 0.4703 \\ 1.1010 & 0.2600 \\ 1.2310 & -0.2906 \\ 1.3610 & 0.8413 \end{bmatrix} \begin{bmatrix} 0.9511^n \\ 0.5878^n \end{bmatrix}$$

<u>Averaged Graphical Method A</u> (Mesh $\Delta\xi$)

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.4002 & 0.4289 & -^{*} & -0.0289 & -0.0002 \\ 0.7614 & 0.5043 & - & 0.0340 & 0.0005 \\ 1.0478 & 0.1638 & - & -0.0110 & -0.0006 \\ 1.2016 & -0.3116 & - & -0.0210 & 0.0008 \\ 1.2953 & -0.5303 & - & 0.0358 & -0.0009 \end{bmatrix} \begin{bmatrix} 0.9511^n \\ 0.5878^n \\ -0.5878^n \\ -0.9511^n \end{bmatrix}$$

(* Zero roots not shown)

<u>Effective Solution Averaged Graphical Method A</u>

n even

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.4000 & 0.4000 \\ 0.7619 & 0.5383 \\ 1.0472 & 0.1528 \\ 1.2024 & -0.3346 \\ 1.2944 & -0.4945 \end{bmatrix} \begin{bmatrix} 0.9511^n \\ 0.5878^n \end{bmatrix}$$

n odd

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 0.4004 & 0.4578 \\ 0.7608 & 0.4703 \\ 1.0484 & 0.1748 \\ 1.2310 & -0.2906 \\ 1.2962 & -0.5661 \end{bmatrix} \begin{bmatrix} 0.9511^n \\ 0.5878^n \end{bmatrix}$$

Table IV-24

SERIES TO MULTIPLY BY "UNAVERAGED" INITIAL VECTOR COMPONENTS TO OBTAIN SERIES REPRESENTATION OF INITIAL VECTOR COMPONENTS FOR AVERAGED METHODS

(Only for Use on Non-Oscillatory Part of Effective Initial Vectors Components)

General Average Applied at Time $(n+k)\Delta\tau$ $\qquad t_{n+k\ \text{Ave}} = \frac{1}{2}(t_n + t_{n+1})$

$$\frac{q_j^{+1}}{2q_j^{k}} = 1 + \frac{(k-\frac{1}{2})r\mu_j^2}{S^2} + \frac{r^2\mu_j^4}{S^4}\left[\left(\frac{1}{12r}+\gamma\right)\left(\frac{1}{2}-k\right) + \frac{k^2}{2}\right]$$
$$- \frac{r^3\mu_j^6}{S^6}\left[\left(\gamma^2 + \frac{\gamma}{6r} + \frac{1}{360r^2}\right)\left(\frac{1}{2}-k\right) + \frac{k^2}{12}\left(\frac{1}{r}+12\gamma\right) - \frac{(2k+1)(k+1)k}{12}\right] \cdots$$

Forward Average $\quad k=0 \quad$ Time $n(\Delta\tau)$

$$\frac{q_j^{+1}}{2} = 1 - \frac{r\mu_j^2}{2S^2} + \frac{r\mu_j^4}{24S^4}(1+12r\gamma) - \frac{r\mu_j^6}{720S^6}\left[60r\gamma(6r\gamma+1)+1\right] \cdots$$

Central Average $\quad k=1/2 \quad$ Time $(n+1/2)\Delta\tau$

$$\frac{q_j^{+1}}{2\sqrt{q_j}} = 1 + \frac{r^2\mu_j^4}{8S^4} - \frac{r^2\mu_j^6}{48S^6}\left[1+6r(2\gamma-1)\right] \cdots$$

Backward Average $\quad k=1 \quad$ Time $(n+1)\Delta\tau$

$$\frac{q_j^{+1}}{2q_j} = 1 + \frac{r\mu_j^2}{2S^2} + \frac{r\mu_j^4}{24S^4}\left[12r(1-\gamma)-1\right] + \frac{r\mu_j^6}{720S^6}\left[1+60r(1-\gamma)(6r\{1-\gamma\}-1)\right] \cdots$$

Graphical Methods $\quad r=1/2 \quad \gamma=0$

Forward: $$\frac{q_j^{+1}}{2} = 1 - \frac{\mu_j^2}{4S^2} + \frac{\mu_j^4}{48S^4} - \frac{\mu_j^6}{1440S^6} \cdots$$

Central: $$\frac{q_j^{+1}}{2\sqrt{q_j}} = 1 + \frac{\mu_j^4}{32S^4} + \frac{\mu_j^6}{96S^6} \cdots$$

Backward: $$\frac{q_j^{+1}}{2q_j} = 1 + \frac{\mu_j^2}{4S^2} + \frac{5\mu_j^4}{48S^4} + \frac{61\mu_j^6}{1440S^6} \cdots$$

Table IV-25

AVERAGED INITIAL VECTOR COMPONENTS - PROBLEM II

Continuous

$$a_j = \frac{\nu_j}{2} \qquad \nu_j = \frac{(2j-1)\pi}{2} \qquad j = 1, 2, \ldots, \infty$$

Approximate*

$g_{j\,Ave}$	Expansion ($g_{j\,Ave} - a_j$) Small ν_j
Averaged Methods A and G (Equivalent) $\dfrac{q_j + 1}{2q_j\, S \tan \frac{\nu_j}{2S}}$	$\dfrac{\nu_j(6r-1)}{6S^2} + \dfrac{\nu_j^3\left[60r\{6r(1-\gamma)-1\}-1\right]}{360S^4} \ldots$
Averaged Method C $\dfrac{q_j + 1}{2q_j\, S \sin \frac{\nu_j}{2S}}$	$\dfrac{\nu_j(12r+1)}{12S^2} + \dfrac{\nu_j^3\left[7+120r\{24r(1-\gamma)-1\}\right]}{2880S^4} \ldots$

* Approximate initial vectors for averaged solutions found by multiplying unaveraged initial vector by

$$\frac{q_j + 1}{2q_j} \quad \text{(Backward) for methods G and C}$$

or by

$$\frac{q_j + 1}{2} \quad \text{(Forward) for method A}$$

Table IV-26

AVERAGED INITIAL VECTOR COMPONENTS - PROBLEM III

Continuous

$$a_j = \frac{2\left[1-(-1)^j\right]}{\nu_j^2} \qquad \nu_j = j\pi \qquad j = 1, 2, \ldots, \infty$$

Approximate

$$j = 1, 2, \ldots, S \text{ or } S+1$$

$g_{j\,Ave}$	Expansion for $(g_{j\,Ave} - a_j)$ Small ν_j
Averaged Method G $\dfrac{\left[1-(-1)^j\right](q_j+1)}{2q_jS^2(1-\cos\frac{\nu_j}{S})}$	$\left[1-(-1)^j\right]\left[\dfrac{(6r+1)}{6S^2} + \dfrac{\nu_j^2}{120S^4}\left\{1+120r(1-\gamma)\right\} \ldots\right.$
Averaged Method A (Explicit) $\dfrac{\left[1-(-1)^j\right](q_j+1)}{2S^2(1-\cos\frac{\nu_j}{S})}$	$-\left[1-(-1)^j\right]\left[\dfrac{6r-1}{6S^2} + \dfrac{\nu_j^2}{120S^4} \ldots\right.$
Averaged Method C $\dfrac{\left[1-(-1)^j\right](q_j-1)}{2q_j(\tan\frac{\nu_j}{2S})(\sin\frac{\nu_j}{2S})}$	$\left[1-(-1)^j\right]\left[\dfrac{12r-1}{12S^2} + \dfrac{\nu_j^2}{960S^4}\left\{7+120r\left[4r(1-\gamma)+1\right]\right\} \ldots\right.$

Table IV-27

SERIES TO MULTIPLY BY OSCILLATORY SERIES IN EFFECTIVE INITIAL VECTOR TO OBTAIN OSCILLATORY SERIES IN AVERAGED SOLUTION

(Graphical Solution, $\gamma=0$, $r=1/2$ Only)

General Average at Time $(n+k)\Delta\tau$

$$\frac{1-q_j}{2q_j^{\,k}} = \frac{\mu_j^2}{4S^2} + \frac{\mu_j^4}{48S^4}(4k-1) \quad \ldots$$

Forward Average k=0 Time $n\Delta\tau$

$$\frac{1-q_j}{2} = \frac{\mu_j^2}{4S^2} - \frac{\mu_j^4}{48S^4} \quad \ldots$$

Central Average k=1/2 Time $n\Delta\tau$

$$\frac{1-q_j}{2\sqrt{q_j}} = \frac{\mu_j^2}{4S^2} + \frac{\mu_j^4}{48S^4} \quad \ldots$$

Backward Average k=1 Time $(n+1)\Delta\tau$

$$\frac{1-q_j}{2q_j} = \frac{\mu_j^2}{4S^2} + \frac{\mu_j^4}{16S^4} \quad \ldots$$

To find expansions for effective initial vector components for averaged graphical methods: write series for effective initial vector component into the sum of a series of non-oscillatory terms and a series of oscillatory terms containing $(-1)^n$ as a factor; multiply non-oscillatory series by appropriate series on Table IV-22 for $\gamma=0$ and $r=1/2$; multiply oscillatory series by appropriate series on this table and recombine series to obtain series for the effective initial vector components for an averaged initial vector.

Table IV-28

AVERAGED EFFECTIVE INITIAL VECTOR COMPONENTS - PROBLEM II

Continuous

$$a_j = \frac{2}{\nu_j} \qquad \nu_j = \frac{(2j-1)\pi}{2} \qquad j = 1, 2, \ldots, \infty$$

Approximate Effective Initial Vectors

$1 \le j \le \frac{S}{2}$ g_j Ave E	Expansion for $(g_j \text{ Ave E} - a_j)$ $0 \le \frac{\nu_j}{S} \le \frac{\pi}{2}$
Averaged Methods A and G (Equivalent)	
$\frac{1}{2S}\left[\text{ctn}\frac{\nu_j}{2S} + \frac{\text{ctn}\frac{\nu_j}{2S}}{\cos\frac{\nu_j}{S}} + \frac{(-1)^{m+n}(\tan\frac{\nu_j}{2S})(1-\cos\frac{\nu_j}{S})}{\cos\frac{\nu_j}{S}}\right]$	$\frac{\nu_j}{3S^3} + \frac{\nu_j^3\left[59+45(-1)^{n+m}\right]}{360S^4}\ldots$
Averaged Method C	
$\frac{1}{2S\cos\frac{\nu_j}{S}}\left[\frac{1+\cos\frac{\nu_j}{S}}{\sin\frac{\nu_j}{2S}} + \frac{(-1)^{n+m-\frac{1}{2}}(\cos\frac{\nu_j}{S}-1)(\text{ctn}\frac{m\nu_j}{S})}{\cos\frac{\nu_j}{2S}}\right]$	$\frac{\nu_j\left[7m-3(-1)^{n+m-\frac{1}{2}}\right]}{12mS^2}\ \ldots$

Table IV-29

COMPLETE SOLUTION - PROBLEM V

Semi-Infinite Solid

Infinite h at $m\Delta\xi = 0$

Initial $T(m\Delta\xi) = 0 \qquad 1 \leq (m\Delta\xi) \leq \infty$

Continuous

$$T(m\Delta\xi, n\Delta\tau) = 1 - \text{erf}\,\frac{m}{2\sqrt{rn}} = 1 - \frac{2}{\pi}\int_0^\infty \frac{(\sin m\psi)e^{-\psi^2 rn}\,d\psi}{\psi}$$

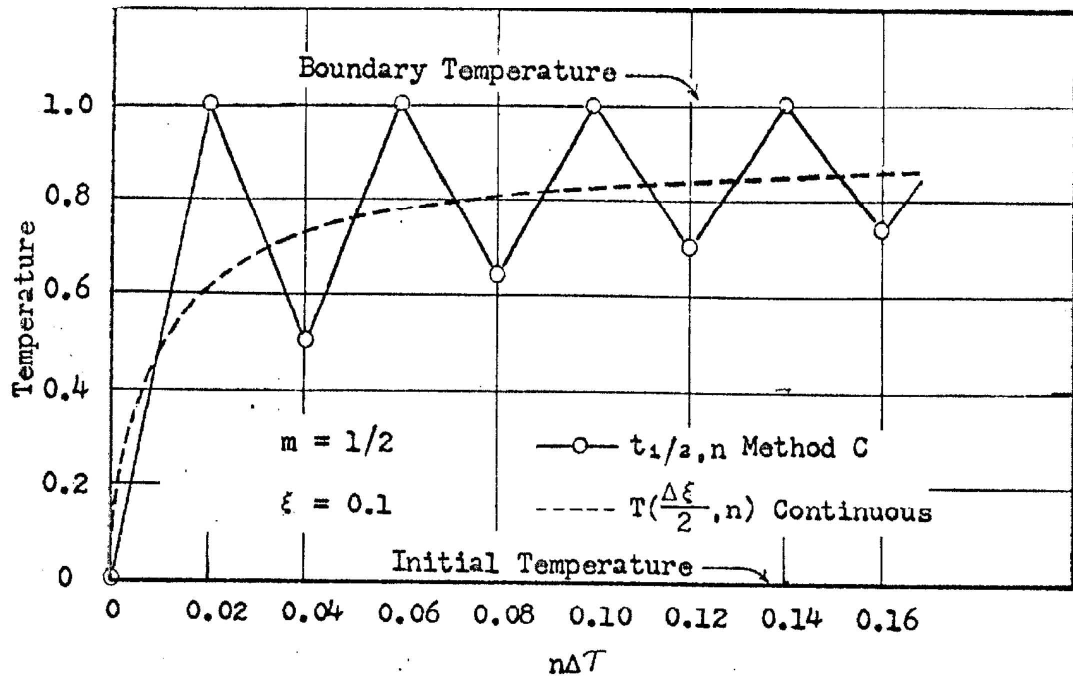
$$= 1 - \text{erf}\,\frac{m\Delta\xi}{2\sqrt{n\Delta\tau}} = 1 - \frac{2}{\pi}\int_0^\infty \frac{\left[\sin(m\Delta\xi)\nu\right]e^{-\nu^2\Delta\tau n}\,d\nu}{\nu}$$

Approximate

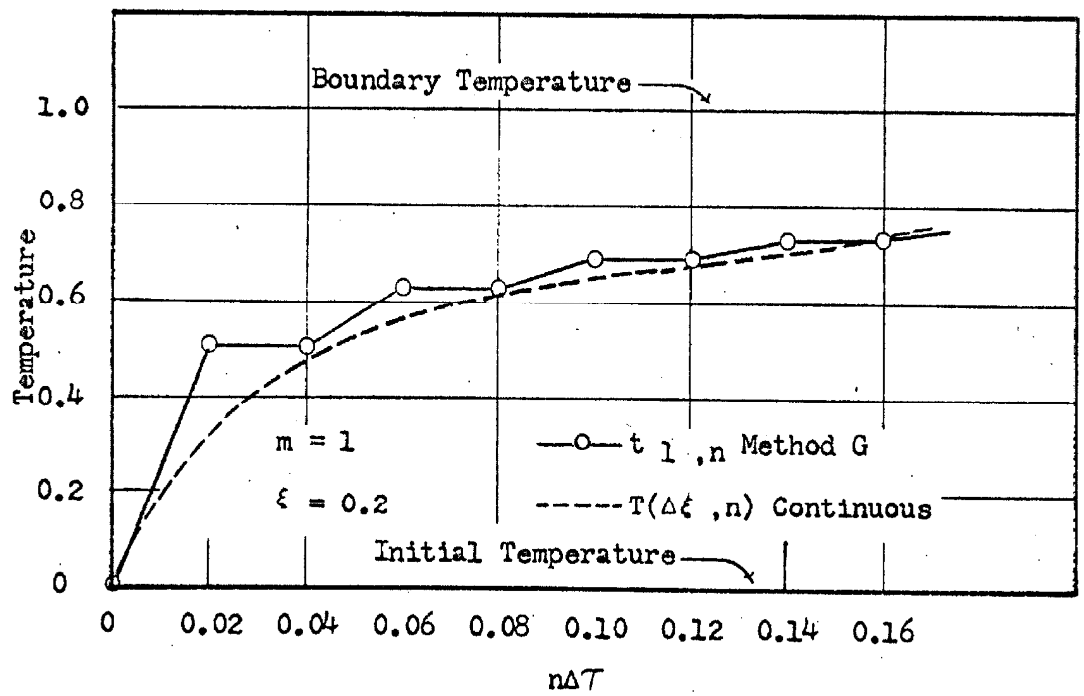
Method G:

$$t_{m,n} = 1 - \frac{1}{\pi}\int_0^\pi \frac{\left[\sin m\alpha\right]\left[1-2r(1-\cos\alpha)\right]^n d\alpha}{\tan\frac{\alpha}{2}}$$

$$= 1 - \frac{1}{\pi}\int_0^{\frac{\pi}{\Delta\xi}} \frac{\Delta\xi\left[\sin m(\Delta\xi)\mu\right]\left[1-2r(1-\cos\mu\,\Delta\xi)\right]^n d\mu}{\tan\frac{(\Delta\xi)\mu}{2}}$$

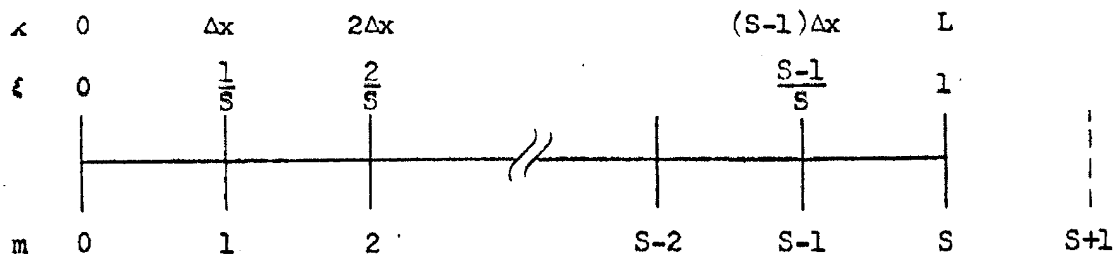


a. Method C, Mesh Δξ /2

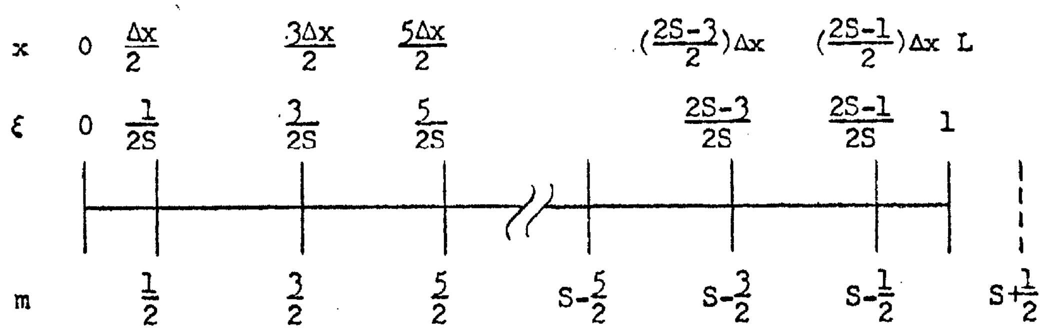


b. Method G, Mesh Δξ .

Figure IV-1. Comparison of graphical methods G and C.



a. MESH Δξ



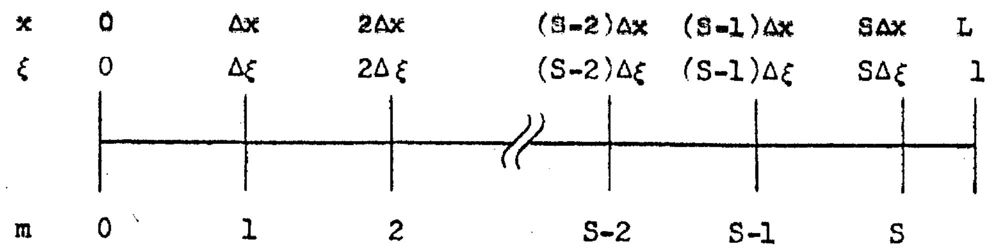
b. MESH Δξ/2

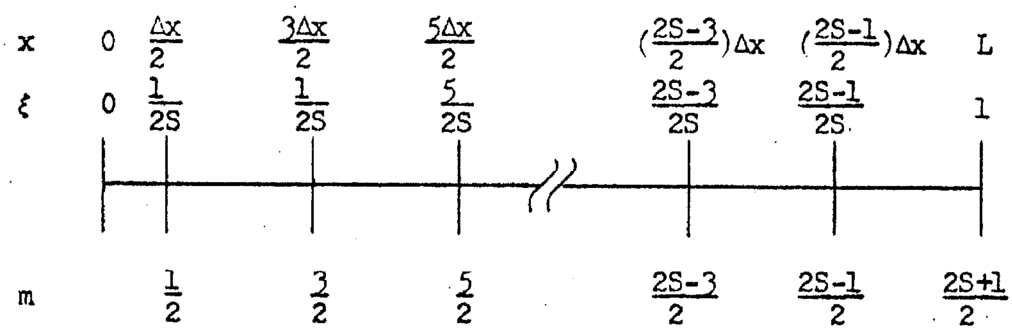
Relationships $\xi = \frac{x}{L} \qquad \Delta\xi = \frac{\Delta x}{L} = \frac{1}{S}$

$$\xi = m\Delta\xi = \frac{m}{S}$$

$$x = m\Delta x = \frac{m}{S}L$$

Figure IV-2. Meshes $\Delta\xi$ and $\Delta\xi$ /2 .





Relationships $\quad \xi = \frac{x}{L} \qquad \Delta\xi = \frac{\Delta x}{L} = \frac{1}{S+\frac{1}{2}}$

$$\xi = m\Delta\xi = \frac{m}{S+\frac{1}{2}}$$

$$x = m\Delta x = \left(\frac{m}{S+\frac{1}{2}}\right)L$$

Figure IV-3. Other meshes.

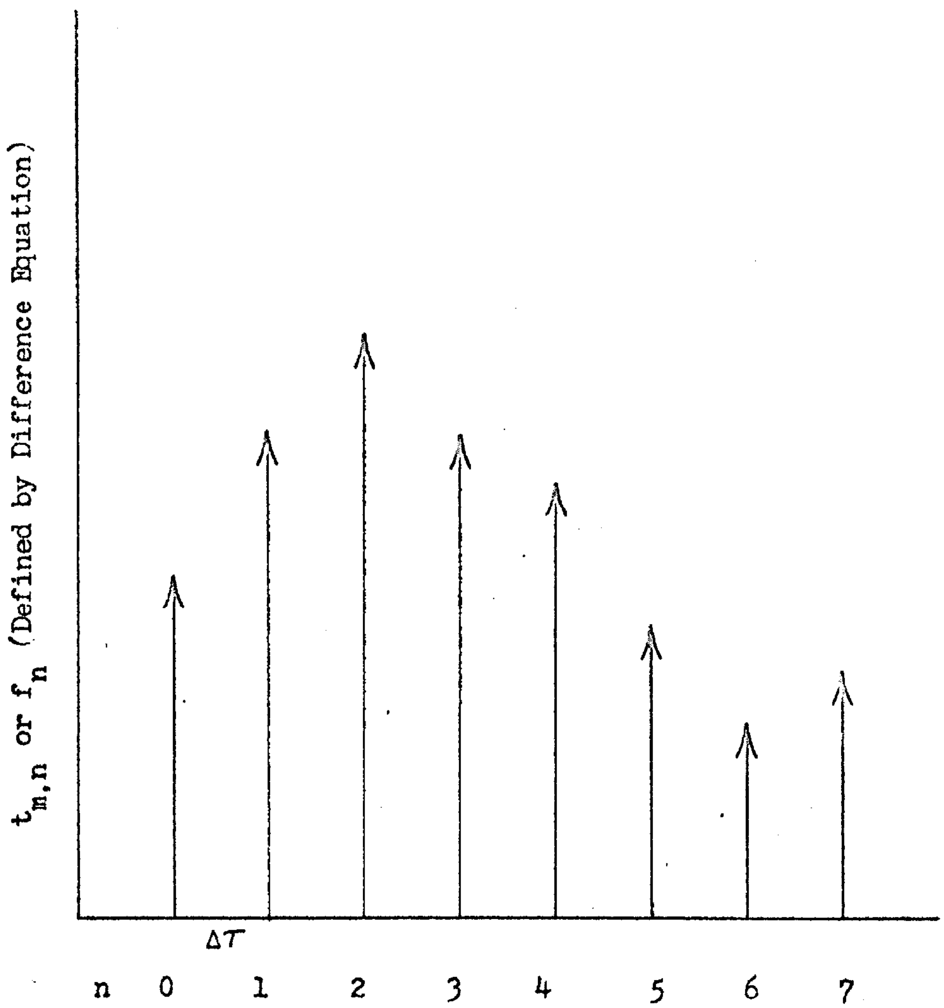
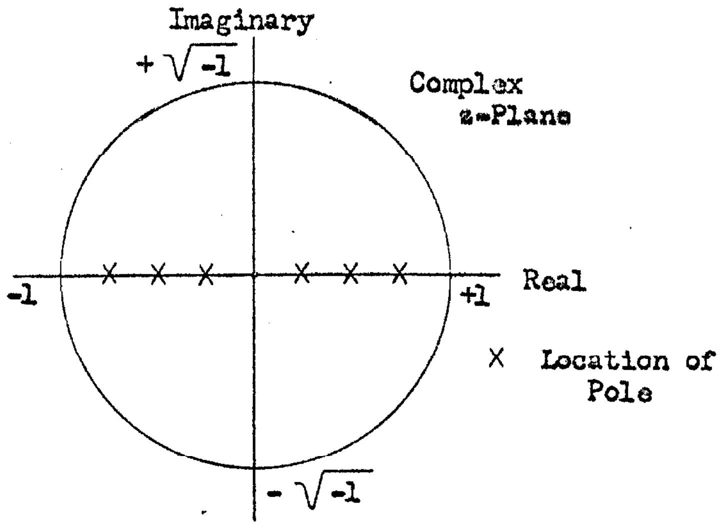
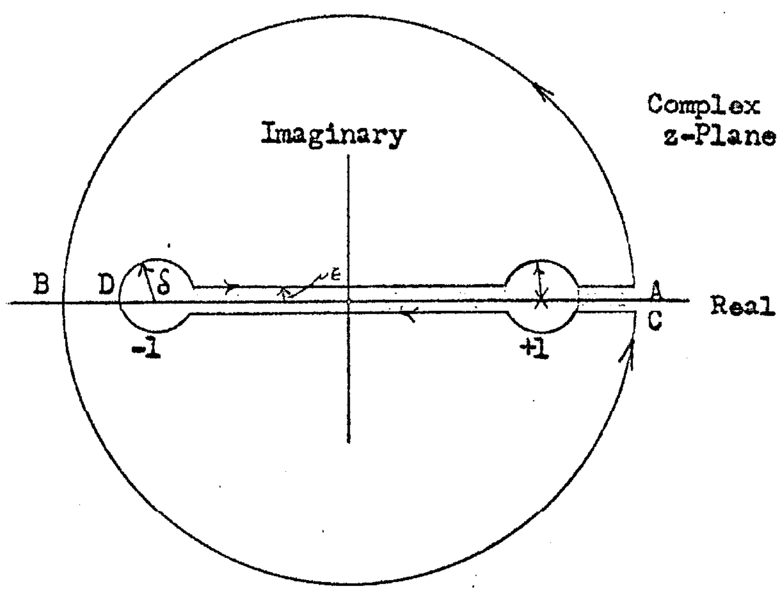


Figure IV-4. Example of a sampled or difference signal.



a. Integration Contour for Inverting Transform with Poles Inside Unit Circle (Stable Solution).



b. Integration Contour for Inverting Transform with Pole at +1 and Branch Cut from -1 to +1.

$$\bar{t}_m = \left[\frac{z}{z-1}\right]\left[\frac{1}{z+\sqrt{z^2-1}}\right]^m$$

Figure IV-5. Contour integrations for inverting z-transforms.

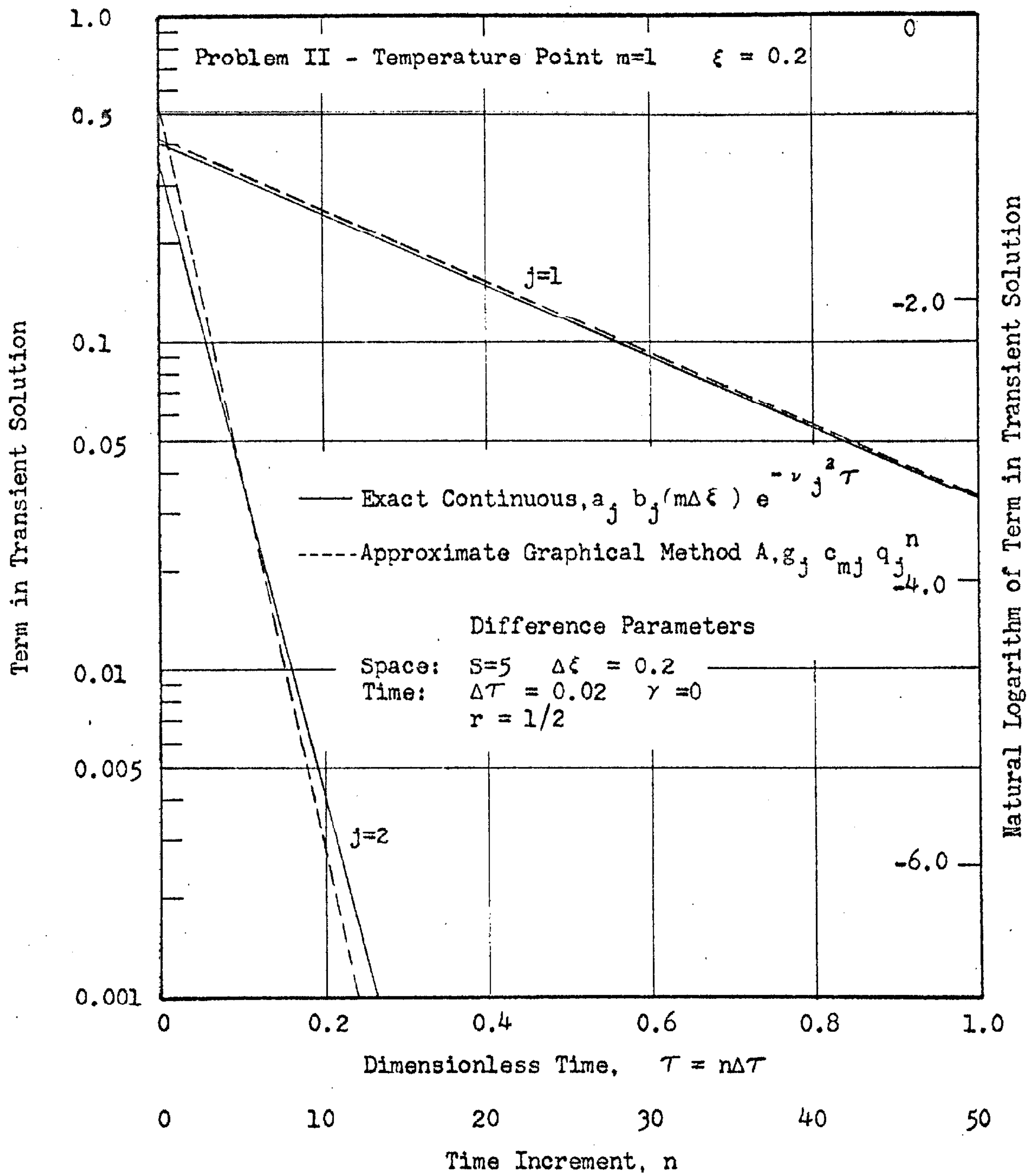


Figure IV-6. Semi-logarithmic graph of term in transient solution versus time.

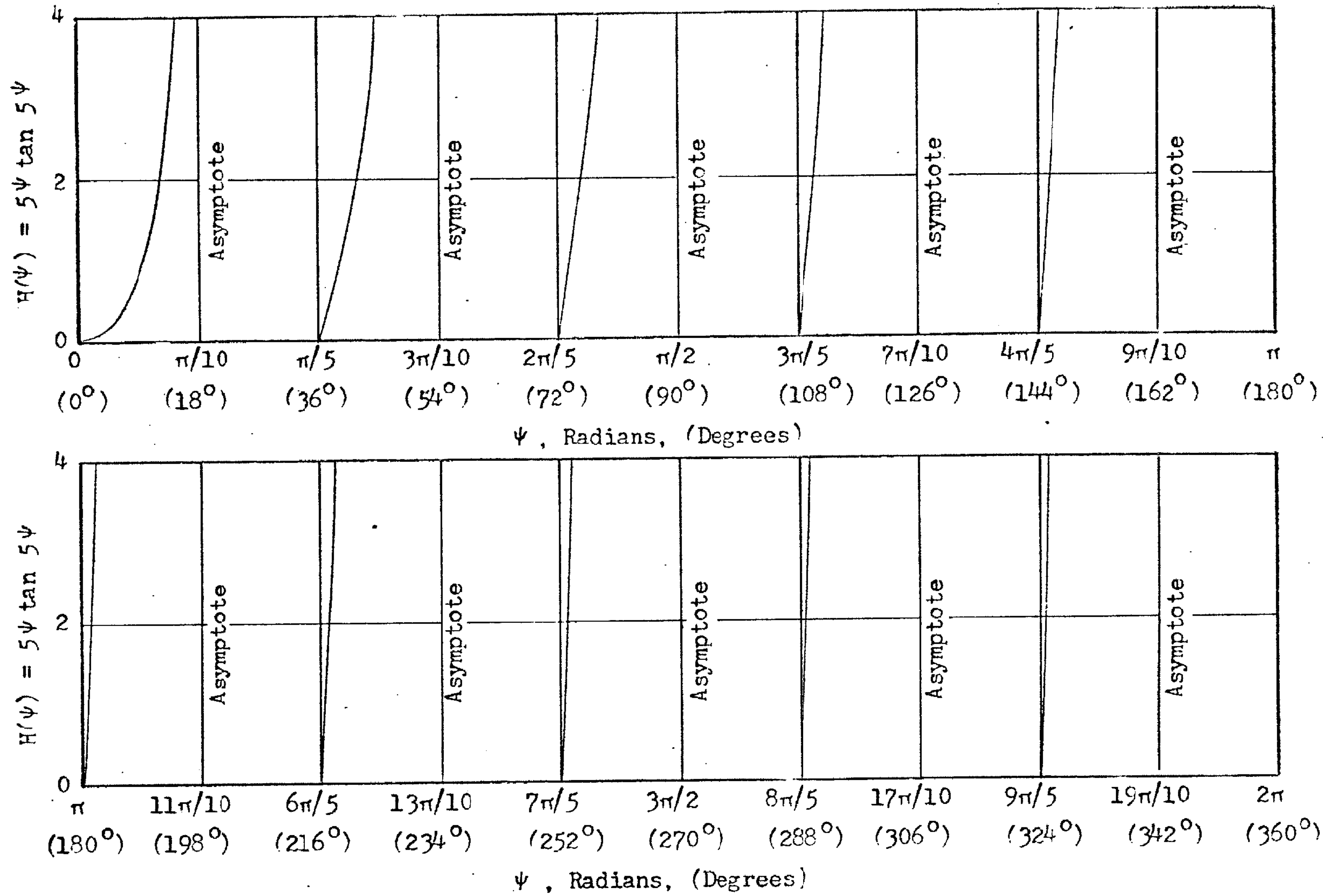


Figure IV- 7. Characteristic equation, continuous solution, first ten roots.
Note: $\nu = 5\psi$

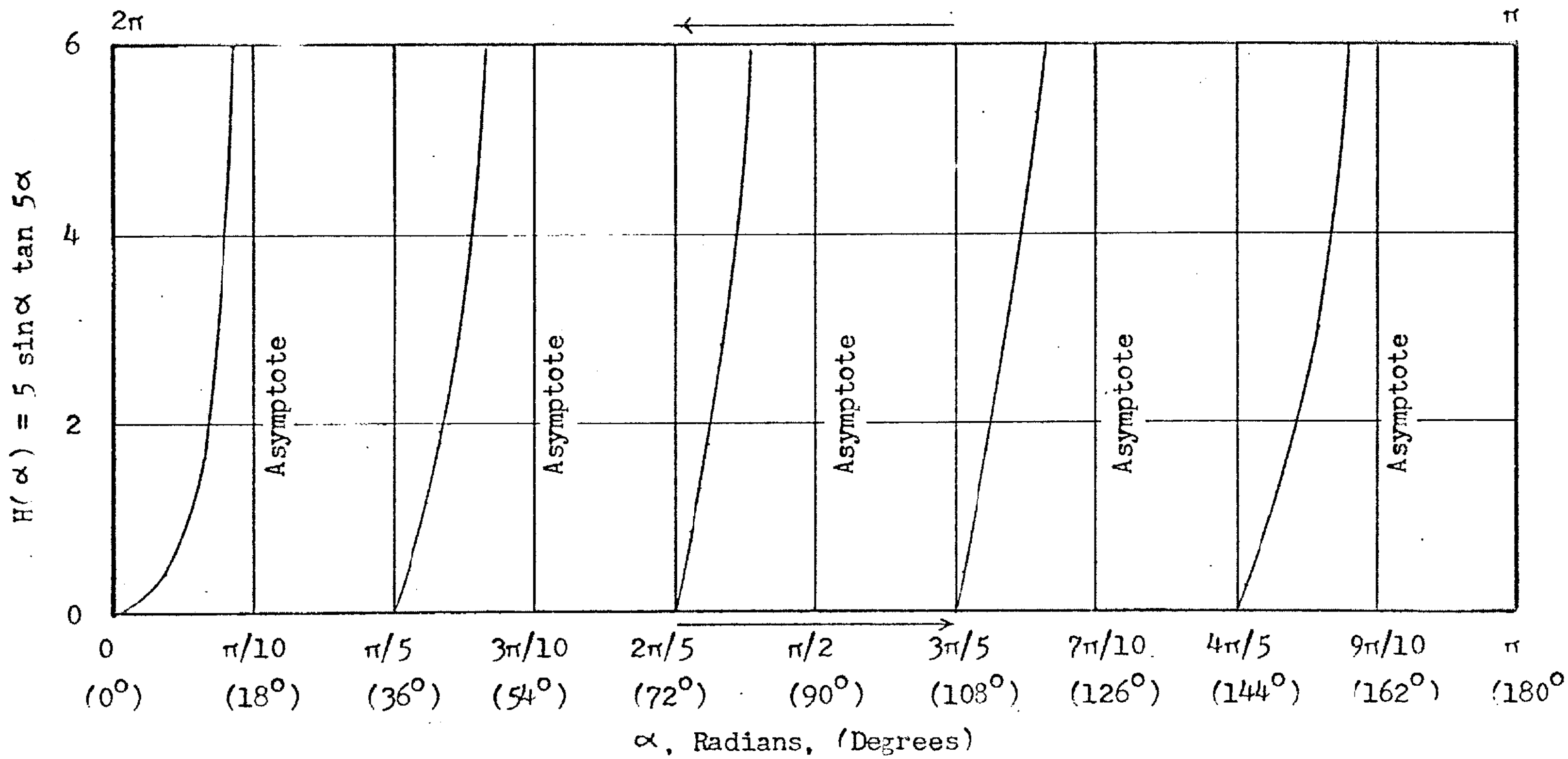


Figure IV-8 . Characteristic equation, method G, real roots for S=5, first five roots.
Note: $\mu = 5\alpha$

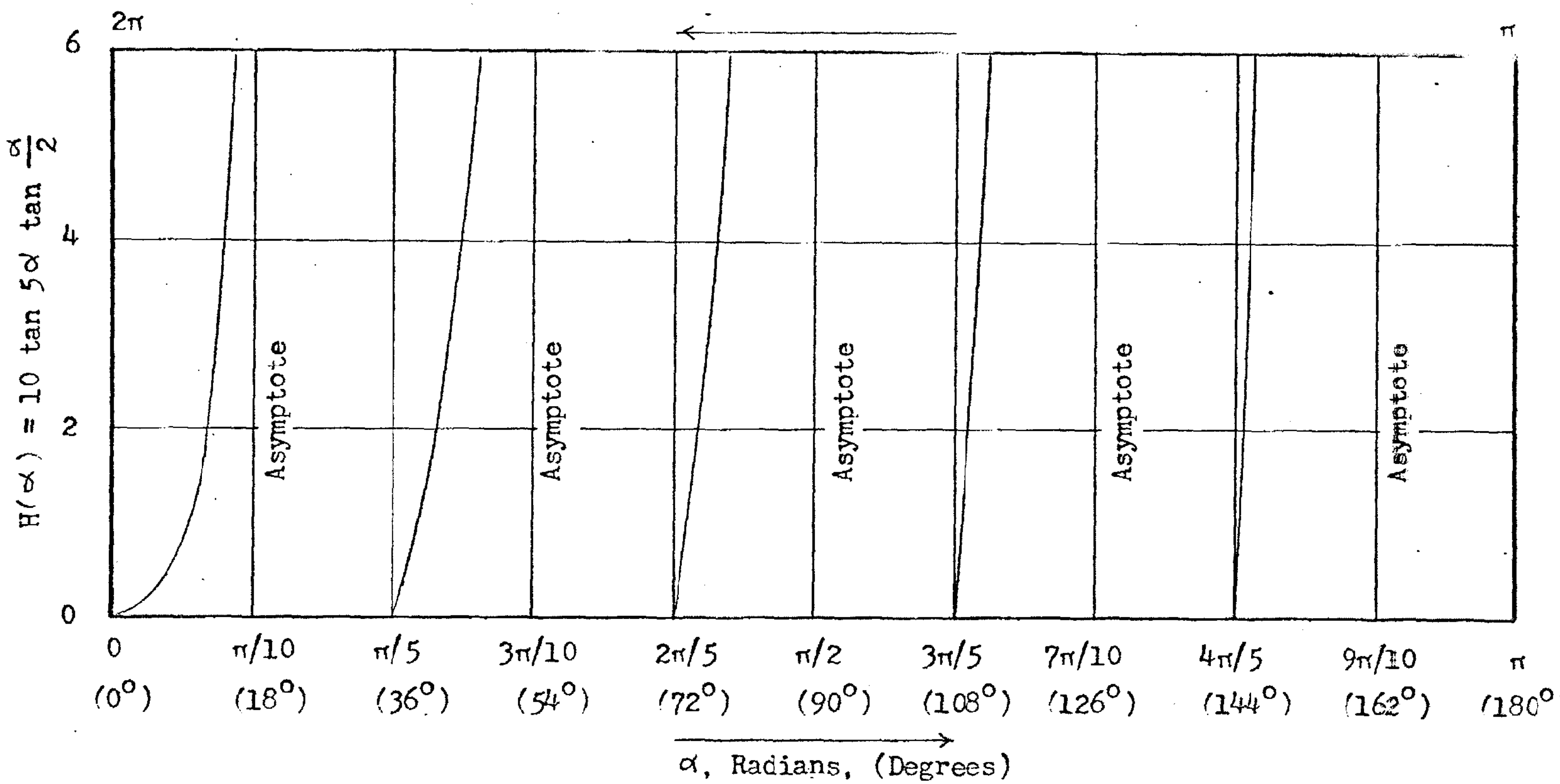


Figure IV-9 . Characteristic equation, method C, S=5, five roots.
Note: $\mu = 5\alpha$

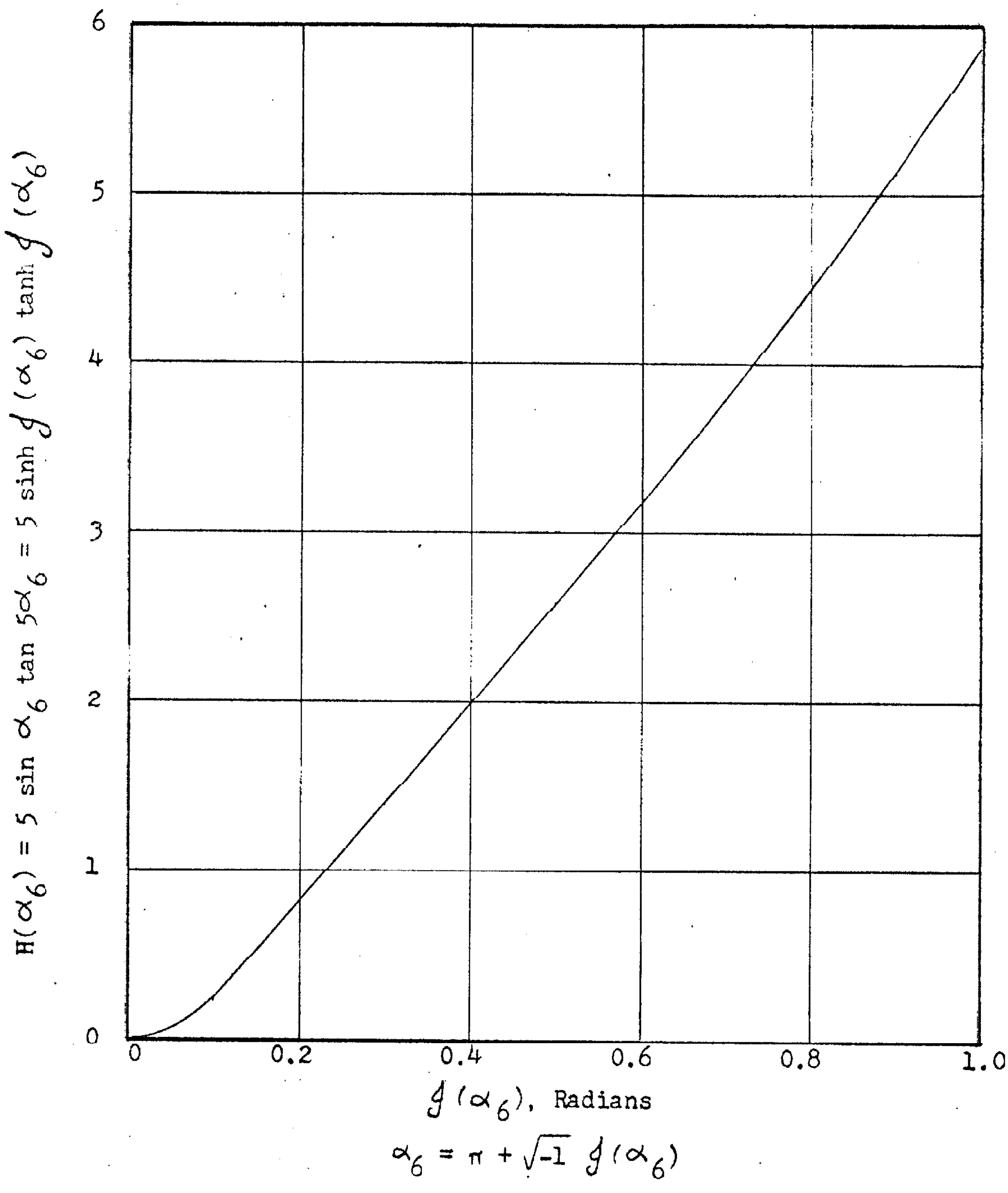


Figure IV-10. Characteristic equation, method G, imaginary root (sixth root) for S=5.
Note: $\mu = 5\alpha$

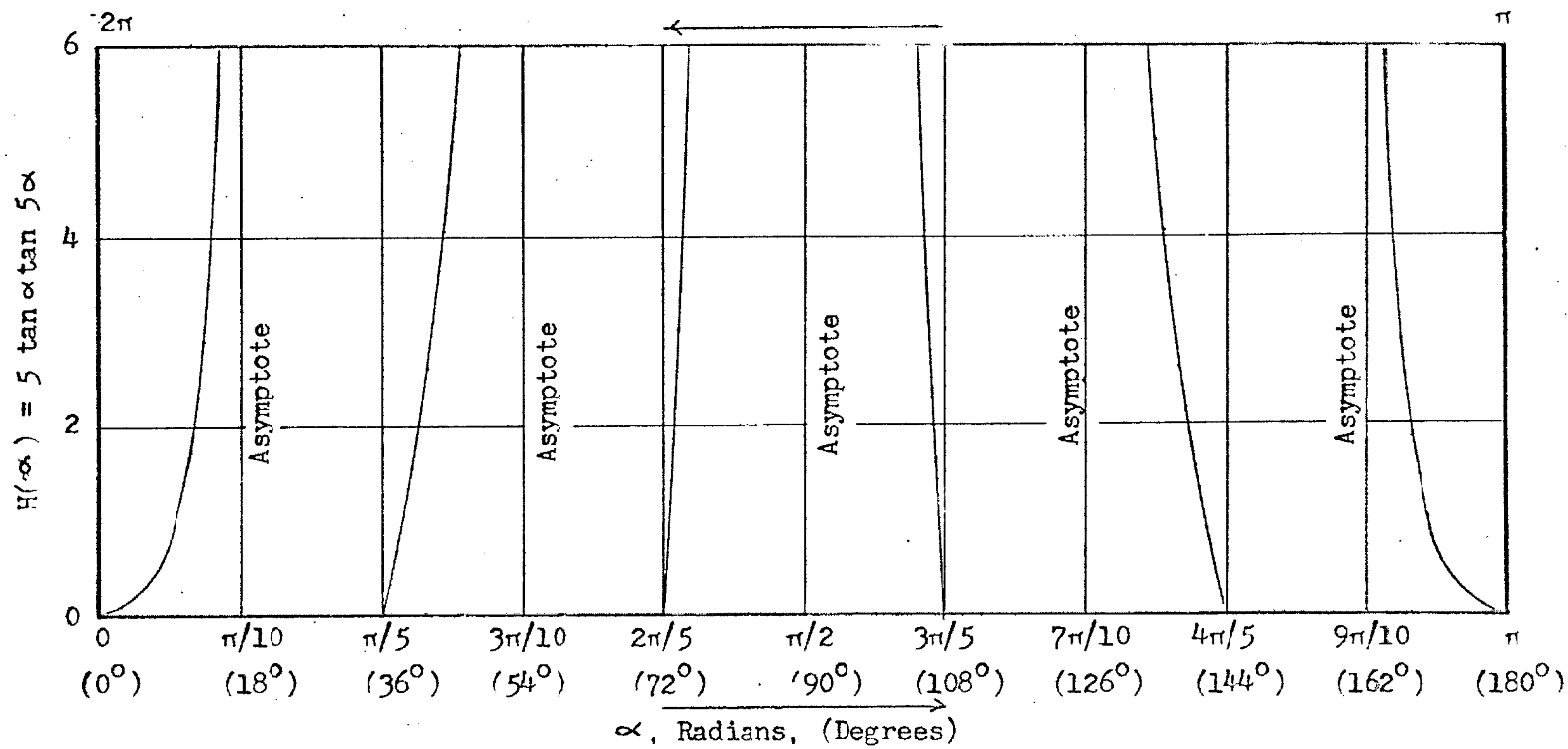
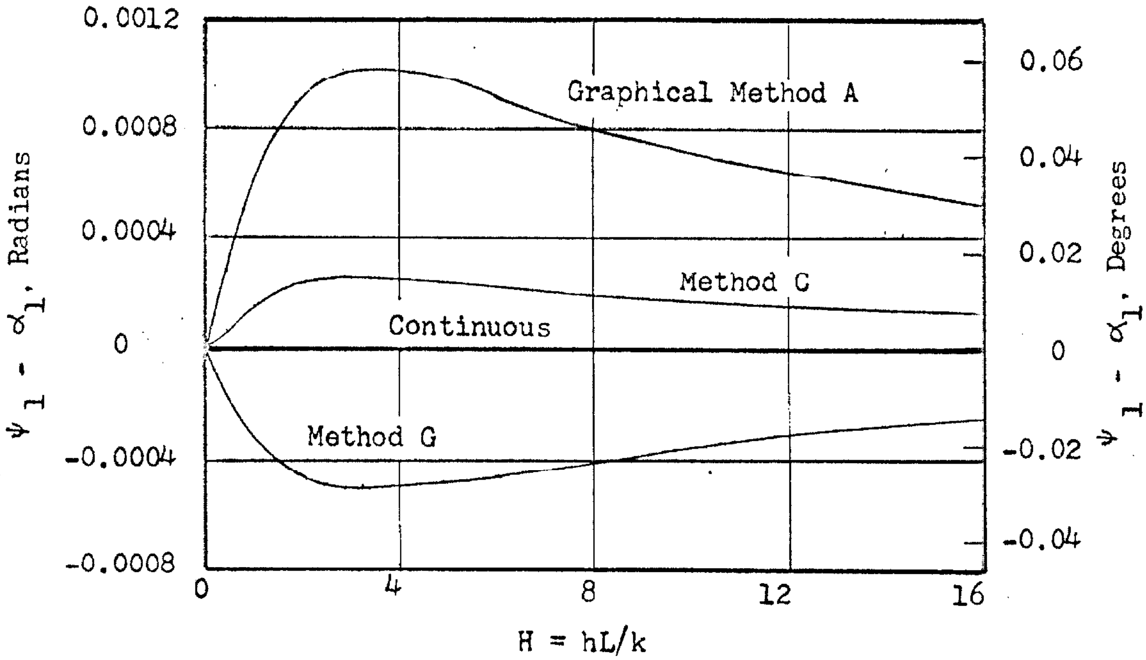


Figure IV-11. Characteristic equation, graphical method A, S=5, all six roots
Note: $\mu = 5\alpha$



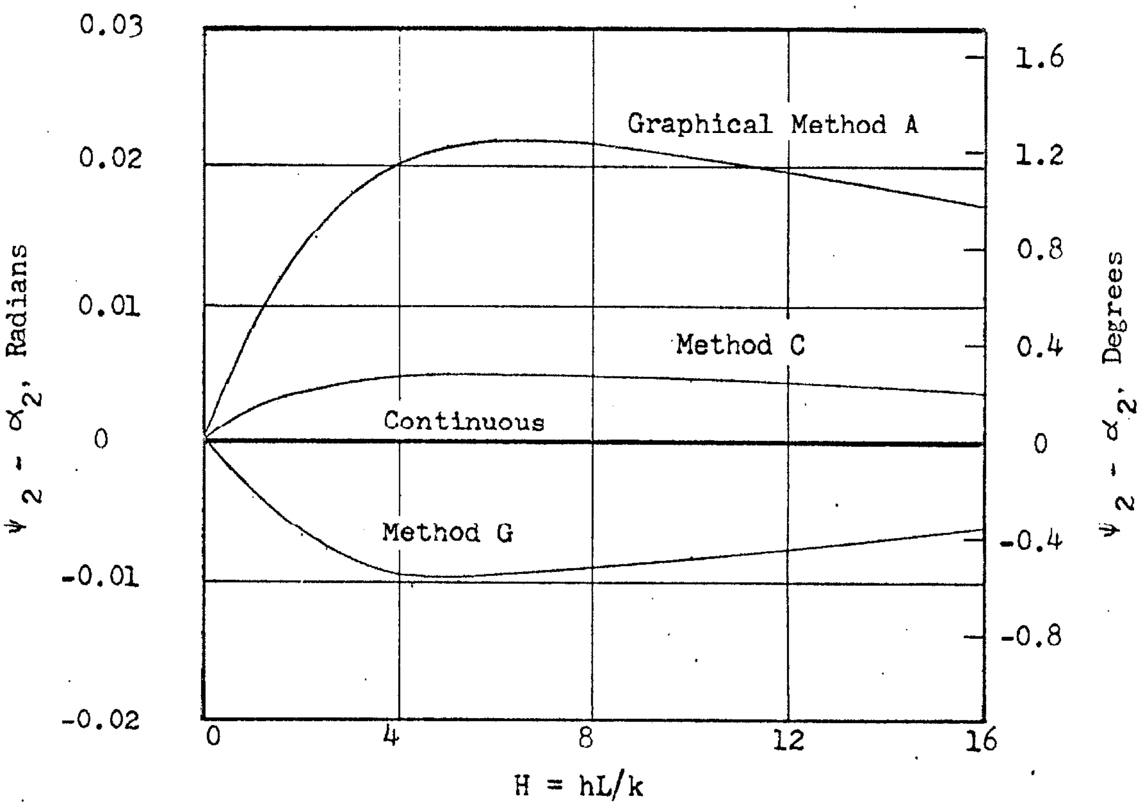


Figure IV-12. Error in first two roots, method G, graphical method A, and method C, S=5.

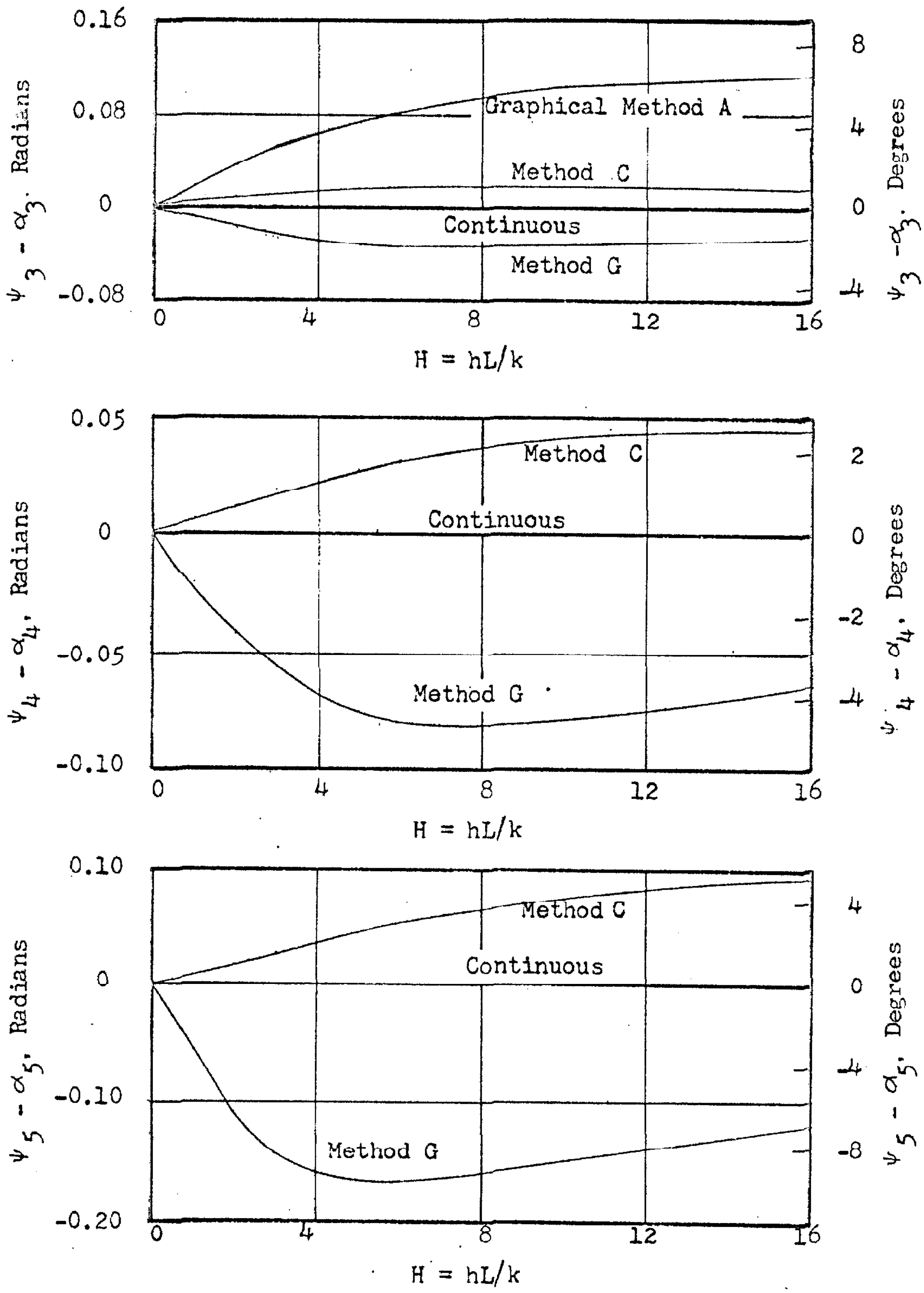


Figure IV-13. Error in roots three, four, and five, S=5.

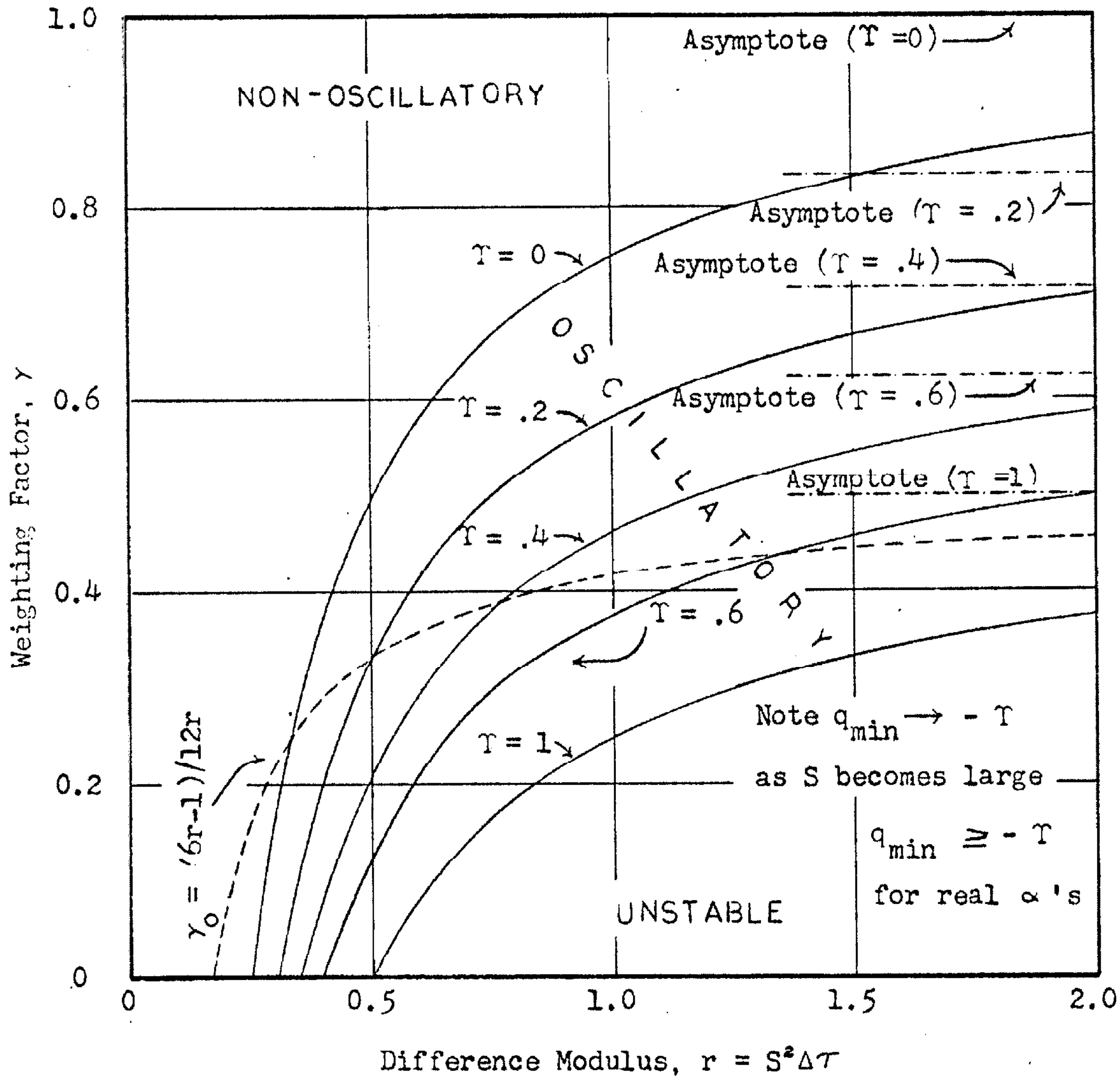


Figure IV-14. Sufficient conditions such that $q_{min} \geq -T$ for methods where minimum norm of Y/A matrix is equal to or less than $4S^2$, or all trigonometric roots α_j are real.

$$\gamma = \frac{4r-(1+T)}{4r(1+T)}$$

Selection of r and γ above and to the left of a line for constant T assures $q_{min} \geq -T$.

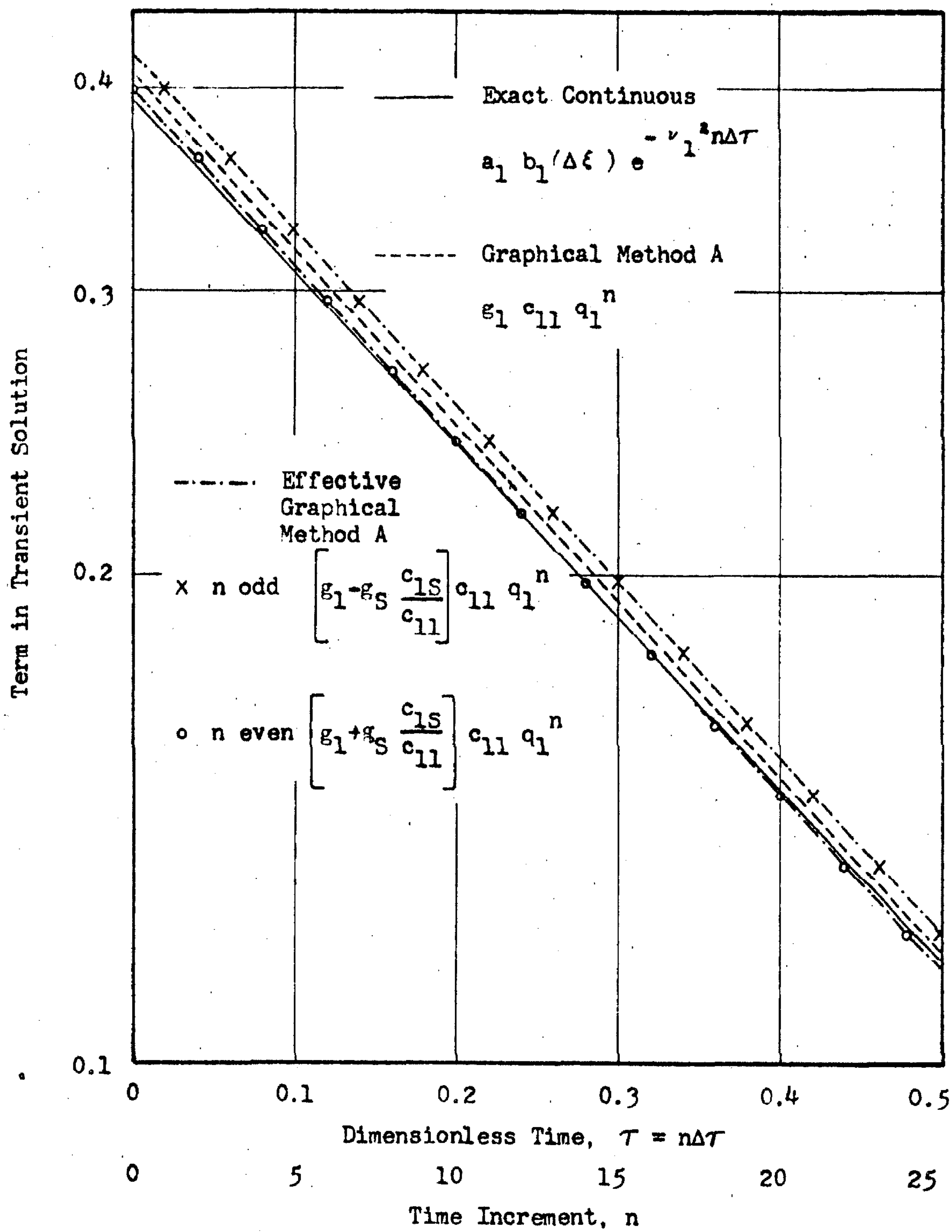


Figure IV-15. Semi-logarithmic graph of term in transient solution versus time showing effective initial vector component for graphical method A.

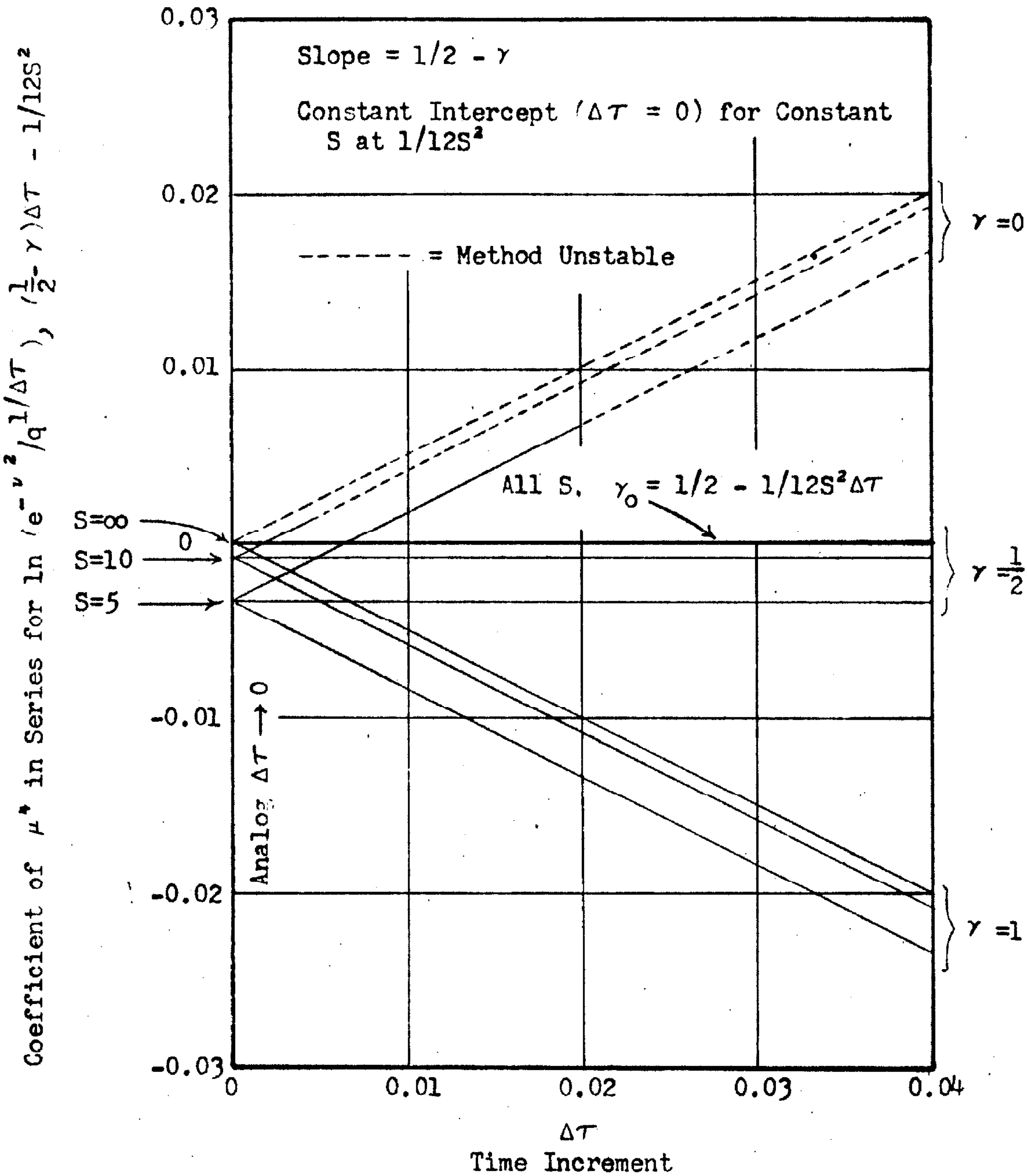


Figure IV-16. Coefficient of μ^4 term in series for the logarithm of $e^{-\nu^2}/q^{1/\Delta\tau}$ as a function of the differencing parameters.

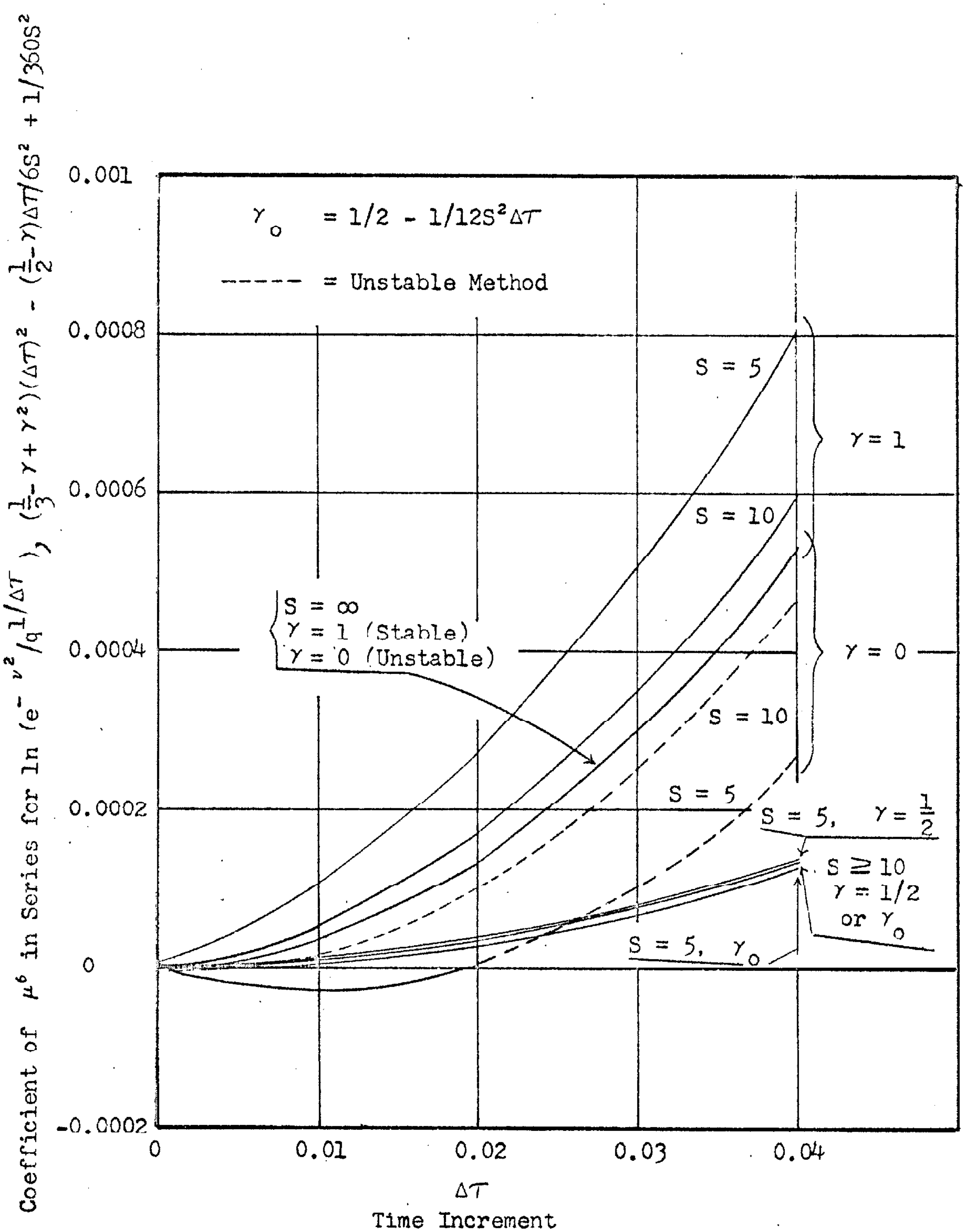


Figure IV-17. Coefficient of μ^6 term in series for the logarithm of $e^{-\nu^2}/q^{1/\Delta\tau}$ as a function of the differencing parameters.