CHAPTER III

GENERAL STABILITY CRITERIA

Finite difference approximations to the diffusion equation can give anomalous results, as just mentioned. For example, depending upon the values of r and γ selected in equation II-42, undamped oscillations, damped oscillations, or smooth curves with time, can result for cases when the exact continuous solution is known to be an infinite sum of decaying exponentials. Although this problem has been the subject of many studies, almost all of them consider only cases of uniform mesh spacing where a partial difference equation exists and can be studied, and usually their results only apply to problems where the boundary temperatures are specified functions of time (infinite heat-transfer coefficient). For the asymmetric network or other non-uniform mesh spacing the partial difference equation does not exist, and this is one of the reasons that the matrix formulation and solutions just shown constitute a more general method of studying these difference approximations. Although the stability definitions and criteria based on the matrix solutions are almost trivial, in order to put them in proper perspective, a brief consideration of earlier important work is presented. The definitions used in this study are then presented and the criteria derived. The practical use of these criteria is discussed,

followed by a numerical example using an asymmetric network to check the criteria and also to show that the asymmetric distribution of temperature points leads to reasonable and fairly accurate results.

(A) STABILITY--REVIEW AND DEFINITIONS

In describing the numerical approximation of partial differential equations, the words stability and convergence are often used. Neither of these terms has a generally accepted precise definition, although certain concepts usually are associated with each. An unstable solution is usually taken to mean one where the numerical results go to infinity with time or the number of time increments, i.e.,

$$\lim_{n\to\infty} \| t_n \| \to \infty \qquad \text{(III-1)}$$

and is attributed by some authors to round-off error, or the computer's inability to carry an infinite number of decimals. A convergent solution is usually defined as one where the difference solution goes to the continuous solution as both the time and space increments are refined to zero; and thus convergence is often associated with the discretization error. Most of the recent studies have shown that round-off error is usually not significant and that the solution would oscillate with increasing amplitude even if an infinite number of decimals were carried along. If a q_j is greater than one in absolute value, the solution becomes infinite, even when infinite decimals are carried.

One of the most general discussions of stability and convergence is that of Richtmyer (3). He has precisely defined stability, convergence, and consistency on the basis of making a series of approximate

calculations. For each calculation in the series, the approximation is carried from zero to a fixed time. And, for each successive approximate calculation in the series, the time and space increments are reduced so that a sequence of approximate calculations is defined; the last limiting approximate calculation uses a zero time increment and zero space increment. A consistent approximation* is one where, in the limit of this sequence, the difference between the continuous operator and the approximate operator operating on the continuous function goes to zero. A stable approximation requires, for all approximate calculations in the sequence including the limiting final calculation at zero time increment, that an upper limit exists which is never exceeded in the calculations to the constant time. Note that, in the limit when the time increment is zero, n must be infinite. A convergent approximation is one which, in the sequence of approximate calculations, equals the continuous solution exactly in the limit of the sequence. The basic result is that if the initial-value problem is linear and fulfills certain other very general conditions, and further, if the approximation is consistent and stable, it is then convergent, and arbitrarily accurate approximations can be obtained by using very fine space-time increments. This consequence applies not only to linear partial differential equations of an initial-value nature but also to linear integral equations.

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* These definitions are stated in a very precise mathematical form in terms of norms of Banach spaces.

The above result can be applied directly to approximations of the linear diffusion equation based on the regular location of temperature points. In these cases space and time increments can be reduced simultaneously and a more accurate approximation is obtained if the solution remains bounded. For a rectangular network, this could be done by halving the space increment for each new calculation. However, for the more general irregular or asymmetric network, the refinement of the space network has an ambiguous meaning. It is theoretically possible to have a sequence of geometrically similar nodes; i.e., each successive node in the sequence has the distance to its neighbors reduced in proportion while the angles remain constant; but it is usually not possible to define a sequence of networks based on refining a general asymmetric node in that way. Further, in practice, we are not usually concerned with such a sequence of calculations, each using smaller increments, but points have been located based on the above rules to fit the geometry and thermal properties of the solid, taking into account the purpose of the calculation and the capacity of the computing equipment or the cost of solution. It is desired then to select a time increment $\Delta\tau$ and a weighting factor γ that give useful results. Two definitions that are found to be convenient from this point of view, and which are based on the form of the solution of the difference system (equation II-43) are given below for a stable solution and a non-oscillatory solution. The definitions are used throughout the remainder of this study.

Stable Approximation: A stable approximation is defined as one where, in the numerical stepping-out of the solution for a fixed network of points, no calculated temperature goes to infinity with time for all problems where the forcing functions represent a finite amount of energy into or out from the solid from zero to infinite time. Examples of such forcing functions included in this category are step functions and equal sinusoids. This behavior must be independent of the initial temperature distribution. The above definition is equivalent to the usual stability definition used for linear feedback control systems (13, 14).

Non-Oscillatory Approximation: A non-oscillatory approximation gives a stable solution that contains no oscillations except those caused by the oscillations of the boundary forcing functions.

Both these definitions are based on the behavior of the approximate solution and do not insure accuracy. However, the continuous solution of the diffusion equation with no sources (equation II-4) gives both a stable solution and a non-oscillatory solution, according to the above definition, so that in order that an approximation be accurate, it must be stable, if not both stable and non-oscillatory. Further, the stability definition is like Richtmyer's definition in that they both are concerned with the boundedness of the calculated temperatures. In Chapters IV and V the accuracy is studied in much more detail, from a practical standpoint, for cases where the time and space increments are not zero.

(B) NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY AND NON-OSCILLATORY BEHAVIOR

These definitions together with the solution of the system of difference and ordinary differential equations II-39 and II-31 facilitate the derivation of the necessary and sufficient conditions for stable or non-oscillatory solutions. First, since the Y/A matrix is a negative definite matrix, its eigenvalues are all real and negative and the analog system of differential equations is, therefore, always stable and non-oscillatory. Also, if modifications are made in the rules used to compute Y/A, it is necessary and sufficient to insure stability of the analog that the negative definite property be retained.

For the complete difference approximation, the necessary and sufficient condition for stability is that all the difference eigenvalues q_j's be less than one in absolute value, or, more precisely,

$$-1 < q_j \leq 1 \qquad j = 1, 2, \ldots, S \qquad \text{(III-2)}$$

The reason that a q_j of -1 is not allowed is that, if a q_j is -1 and if the fluid temperature has equal oscillations with a frequency of $1/2\Delta\tau$ or a multiple of $1/2\Delta\tau$, a term of the form $n(-1)^n$ occurs in the particular solution; when a term of this form occurs in the solution the q_j of -1 is said to be "excited." Thus the solution would give undamped oscillations for a forcing function representing a finite amount of energy. A q_j of +1 must be allowed for cases where the boundary is adiabatic

(Y_B is zero), as a q_j of 1 is necessary to represent a constant non-zero steady-state solution. Also, if a q_j falls outside these limits, carrying an infinite number of decimals cannot make the solution stable. However, since it can be shown that any error introduced into the calculation is propagated in such a manner that it satisfies a system of difference equations similar to equation II-39, any round-off error introduced into a calculation where a q_j less than -1 exists "increases" or "aggravates" the instability, or, if the solution were such that the g_j which corresponds to this q_j were zero, this round-off error would "activate" this eigenvalue (make g_j non-zero) and the solution would become unbounded. Therefore, if the matrix $[I-\gamma\Delta\tau(Y/A)]^{-1}[I+(1-\gamma)\Delta\tau(Y/A)]$ contains an eigenvalue outside the range shown in equation III-2, the solution is unstable, even if in a particular calculation the coefficient g_j that corresponds to this eigenvalue is zero.

However, as mentioned previously, the exponentials of the continuous transient solution go to zero with no oscillation, and for the difference solution to have no oscillations, the necessary and sufficient conditions are more restrictive in that all the eigenvalues must also be non-negative.

$$0 \leq q_j \leq 1 \qquad j = 1, 2, \ldots, S \qquad \text{(III-3)}$$

In order to relate the necessary and sufficient conditions in equations III-2 and III-3 to the location of the nodes, Y/A matrix, and the time differencing parameters, equation II-57 is used. First,

remembering that as the λ_j's are negative they can be ranked in the following order:

$$\lambda_{min} < \ldots\ldots < \lambda_{MAX} \leq 0 \qquad \text{(III-4)}$$

where λ_{min} is the smallest λ_j, as they are negative, but has the largest absolute value,

$$|\lambda_{min}| > \ldots\ldots > |\lambda_{MAX}| \geq 0 \qquad \text{(III-5)}$$

The λ_{min} determines the minimum q_j, q_{min}, or the q that would be negative; λ_{MAX} determines q_{MAX}, and λ_{MAX} is equal to or less than zero. Combining these considerations with equation II-57, the inequalities, equations III-2 and III-3, and the fact that $\Delta\tau$, $(1-\gamma)$, and γ are positive, only the lower bound on q can limit the selection of $\Delta\tau$ and γ. The necessary and sufficient conditions are

For Stable Solution:

$$|\lambda_{min}|(1-2\gamma)\Delta\tau < 2 \qquad \text{(III-6)}$$

For Non-Oscillatory Solution:

$$|\lambda_{min}|(1-\gamma)\Delta\tau \leq 1 \qquad \text{(III-7)}$$

These then are the rigorous, necessary, and sufficient conditions to obtain the desired exponential behavior, since when they are fulfilled all q_j's must fall in the required range.

Directly from the criteria, stable solutions are obtained for all values of $\Delta\tau$ if

$$\gamma \geq \tfrac{1}{2} \tag{III-8}$$

and further the solution is never oscillatory if

$$\gamma = 1 \tag{III-9}$$

This agrees with the stability and oscillatory behavior for the one-dimensional problem with equally spaced nodes (15).

Although the eigenvalues for a given Y/A matrix can be found numerically by iteration or Jacobi's method (16), this procedure is probably as time consuming as carrying out several trial combinations of $\Delta\tau$ and γ. However, two easily determined matrix norms are known to be greater than or equal to the absolute value of the eigenvalue which is greatest in absolute value, $|\lambda_{min}|$ (Faddeva (16), pages 55-59). These norms are found by computing the sum of absolute values of the elements in each row and column. The norms are then the maximum of the individual row sums and the maximum of the individual column sums. Defining the element in the i^{th} row and j^{th} column of the Y/A matrix by μ_{ij}, and of the W matrix by w_{ij}, the norms are

Row Norm:

$$\left\|\frac{Y}{A}\right\|_I = \underset{i}{MAX} \sum_{j=1}^{S} |\mu_{ij}| = \underset{i}{MAX} \left[\frac{1}{A_i} \left\{ \sum_{\substack{j=1 \\ j\neq i}}^{R} y_{ij} + \sum_{\substack{j=1 \\ j\neq i}}^{S} y_{ij} \right\} \right] \tag{III-10}$$

Column Norm:

$$\left\|\frac{Y}{A}\right\|_{II} = \underset{j}{MAX} \sum_{i=1}^{S} |\mu_{ij}| = \underset{j}{MAX} \left[\frac{1}{A_j} \sum_{\substack{i=1 \\ i\neq j}}^{R} y_{ij} + \sum_{\substack{i=1 \\ i\neq j}}^{S} \frac{y_{ij}}{A_i} \right] \tag{III-11}$$

Row and Column Norm:

$$\|W\|_I = \|W\|_{II} = \underset{i}{\text{MAX}} \sum_{j=1}^{S} |w_{ij}| = \underset{i}{\text{MAX}} \frac{1}{A_i}\left[\sum_{\substack{j=1 \\ i\neq j}}^{R} y_{ij} + \sum_{\substack{j=1 \\ j\neq i}}^{S} \frac{y_{ij}}{\sqrt{A_i A_j}}\right] \quad \text{(III-12)}$$

The first equality in these three equations is the definition of the norms and represents computing the sum of the absolute values for each row and column of the Y/A and W matrices, followed by selecting the maximum of the row sums for the row norm I and maximum column sum for the column norm II. The second equality shows these sums in terms of the individual elements of the A, Y, and Y_B matrices. The first summation over all the nodes, R, represents the diagonal element of the Y and W matrices; the second summation over the unknown temperature nodes, S, is the sum of the off-diagonal elements in the row or column. The second equality is useful in understanding the effect of the location of the temperature points on the nodes, and also since the W matrix is not usually calculated explicitly, its norm can be found directly from the second equality in equation III-12.

For a lower bound on $|\lambda_{min}|$ we use the fact that for a symmetric matrix such as W $|\lambda_{min}|$ is greater than or equal to the absolute value of the maximum diagonal element. Since W has the same eigenvalues and the same diagonal elements as Y/A, this is a satisfactory lower bound for $|\lambda_{min}|$.

Now defining M as the minimum of the three norms defined by equations III-10, III-11, and III-12, the largest eigenvalue must fall in

the range

$$\underset{i}{\text{MAX}}\ \frac{1}{A_i}\ \sum_{\substack{j=1 \\ i\neq j}}^{R} y_{ij} = \underset{i}{\text{MAX}}\ |\mu_{ii}| = |\mu_{\rho\rho}| \leq |\lambda_{min}| \leq M = \text{min of} \quad \begin{matrix} \left\|\frac{Y}{A}\right\|_I \\ \left\|\frac{Y}{A}\right\|_{II} \\ \|W\|_{II \text{ and } I} \end{matrix} \qquad \text{(III-13)}$$

Therefore, useful and simply found sufficient conditions for stable and non-oscillatory solutions are:

Stable Solution: $M(1-2\gamma)\Delta\tau < 2$ (III-14)

Non-Oscillatory Solution: $M(1-\gamma)\Delta\tau \leq 1$ (III-15)

Again, in the stability criterion, if the value of $M(1-2\gamma)\Delta\tau$ is allowed to equal 2, possibly a q of -1 may occur.

As an example of the usefulness and simplicity of these stability criteria, we will apply the criteria to the problem of one-dimensional heat flow in a solid of constant thermal properties with a regular mesh spacing $\Delta\xi$ starting from both boundaries, Figure IV-2, mesh $\Delta\xi$. The boundary conditions are a specified temperature on one boundary and a zero gradient at the other. Under these conditions the Y/A matrix is

$$\frac{Y}{A} = \frac{-1}{(\Delta\xi)^2} \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{bmatrix} \qquad \text{(III-16)}$$

The three norms in this case are equal and M becomes by inspection

$$M = 4/(\Delta\xi)^2 \qquad \text{(III-17)}$$

Defining the modulus r in the usual manner,

$$r = \frac{\Delta\tau}{(\Delta\xi)^2} = \frac{K\Delta\theta}{(\Delta x)^2} \qquad \text{(III-18)}$$

We obtain directly the familiar sufficient conditions

Stable Solution: $r(1-2\gamma) < \frac{1}{2}$ (III-19)

Non-Oscillatory Solution: $r(1-\gamma) \leq \frac{1}{4}$ (III-20)

Normally these results are derived by a complicated procedure involving separation of variables and Fourier analysis (2, 10, 15, 17) which actually determines the q_j's. However, as shown in Chapter IV, section E-4, the matrix procedure above is more satisfactory, even when the q_j's can in principle be found. If, on the sufficient condition for stability, the value of $r(1-2\gamma)$ is allowed to equal $\frac{1}{2}$, a -1 root can occur. Although this lack of equality would indicate graphical methods

where γ is zero and r is $\frac{1}{2}$ are unstable, as the above condition is merely sufficient, most graphical solutions are stable for many problems; for problems with two adiabatic boundaries or specified flux conditions a -1 root does occur in some graphical methods. It is also known that with a large finite heat-transfer coefficient, some graphical and other approximate solutions become unstable. This result is easily predicted from the matrix norms, as a diagonal element increases linearly with the heat-transfer coefficient, and further, the norm allows an r to be selected that eliminates the undesired behavior. These examples show the inherent simplicity and usefulness of the rigorous criteria in equations III-14 and III-15.

(C) SELECTION OF $\Delta\tau$ AND γ BASED ON STABILITY

At this time, it is convenient to discuss the selection of the differencing parameters $\Delta\tau$ and γ based on the stability criteria and matrix relationships. Obviously, because of the wide variety of problems possible, the different purposes for which the results are to be used, and the capacity of available computing equipment, this discussion must be general so that when one is confronted with a problem a reasonable selection can be made.

In applying the sufficient criteria to the general Y/A matrix for an asymmetric location of nodes, one might ask how much smaller a time increment would be found by using one of these inequalities than the actual minimum necessary time increment. This would be particularly important when the calculations are to be carried to a fixed time where an explicit method or low γ method is desirable, possibly for ease of calculation in the first case or accuracy in the second. From the bounds on λ_{MAX}, equation III-13, and the structure of the specific Y/A matrix, as defined by MacNeal's rules, the largest range for λ_{MAX} is such that the upper bound is at most twice the lower bound. This would occur only when

$$\left\| \frac{Y}{A} \right\|_1 = M = 2\left|\mu_{\rho\rho}\right| = \frac{2}{A_\rho} \sum_{j=1}^{S} y_{\rho i} \qquad \text{(III-21)}$$

Thus the time increment found from the sufficient criteria, equation

III-14 and equation III-15, would never be less than half that necessary. Although in the worst case this could mean doubling the amount of calculations to a constant time τ_1, the criteria do give the correct order of magnitude for $\Delta\tau$. If they show the calculation to be infeasible for the desired γ, consideration should be given to a different arrangement of nodes, or to other approximations that may be made, such as neglecting the heat capacity of nodes in high conductance regions (9). However, for any node arrangement, $\Delta\tau$ can be made as large as desired, and a stable or even a non-oscillatory solution can be obtained, by selecting γ large enough.

In some cases, the time increment can be increased over that shown in the sufficient conditions, equations III-14 and III-15, and useful solutions are obtained. Short of actually numerically determining $|\lambda_{min}|$, one way to do this would be to reduce the range for $|\lambda_{min}|$ and, in particular, to lower M. This can be done by performing a similarity transformation on either the Y/A or W matrices, forming similar matrices with the same eigenvalues, but lower norms. These transformations are

$$U_1 = F_1 \frac{Y}{A} F_1^{-1} \qquad \text{(III-22)}$$

or

$$U_2 = F_2 W F_2^{-1} \qquad \text{(III-23)}$$

and since either of the U's would have the same eigenvalues, $|\lambda_{min}|$

would have to be more than the absolute value of the largest diagonal element and smaller than the smallest norm. A particularly simple and useful form for F is a diagonal matrix with the elements in the i^{th} row designated by φ_i and the j^{th} column by φ_j ; its inverse is then known explicitly, and the new U is also known explicitly. Its diagonal elements are the same as those of Y/A; the off-diagonal element of the i^{th} row and j^{th} column is the corresponding element in the old matrix times φ_i/φ_j. That is, for a transformation on Y/A the elements of U_1 are given by

$$u_{ii} = \mu_{ii} \tag{III-24}$$

$$u_{ij} = \frac{\mu_{ij}\,\varphi_i}{\varphi_j} \tag{III-25}$$

By properly selecting the φ_j's the maximum row (or column) sum can be reduced, while the other row or column sums are increased. Although the lowest sum would occur when all row (or column) sums are equal, this is not practically possible. Also, it is probable that because of the many different matrices possible no one method of selecting the φ's is satisfactory. However, by remembering that φ_j multiplies each off-diagonal element in the j^{th} column and $1/\varphi_j$ multiplies each off-diagonal row element of the j^{th} row, it is feasible to increase the lower row (or column) sums in such a manner that the larger row (or column) sums are significantly reduced. Often it is useful to assume

an approximate sum that falls about midway in the range given by equation III-13 and try to make all row (or column) sums lower than this number.

Another more complicated transformation is the one used in the Jacobi method for eigenvalues of a real symmetric matrix. This method finds the eigenvalues by successive transformations that eliminate the off-diagonal elements. By applying this type of transformation to the largest off-diagonal elements in the W matrix, the range within which the maximum eigenvalue falls (equation III-13) is reduced by increasing the maximum diagonal element and/or reducing the norms. This transformation changes the elements in the two rows and two columns for each element eliminated and equations for the new matrix are given in the references (18). If only a few very large off-diagonal elements were in the matrix this method might prove very useful in reducing the range for $|\lambda_{min}|$.

Since either of the above transformations attacks the rows (or columns) whose sums are much larger than the other sums, neither could be applied to a matrix such as in equation III-16 for a one-dimensional regular mesh spacing. Further, since we can show (Chapter IV, section E-3) that the maximum eigenvalue of the type of matrix in equation III-16 goes to the norms as the dimension of the matrix is increased to infinity, it is concluded that the minimum norm obtained for a Y/A matrix, whose row and column sums are nearly equal, is very probably not much larger than its maximum eigenvalue.

Since the effect of a q_{min} that is negative but small in absolute value is negligible as n becomes large, under some circumstances an oscillatory solution might be tolerated. This would allow a larger $\Delta\tau$ to be used. Consequently we could specify that q_{min} be less than Y in absolute value so that at the lowest time of interest τ_0, or after $\tau_0/\Delta\tau$ time increments, the oscillatory component would be less than Y^{n_0} or $Y^{\tau_0/\Delta\tau}$ and would no longer be significant. The disregarding of the approximate solution for the first few n's must usually be done in any case because the approximate solution has replaced the infinite sum of exponentials of the continuous solution with a finite sum of exponential forms. The sufficient condition to obtain q_{min} less in absolute value than Y is

$$(\Delta\tau)M[1-\gamma(Y+1)] \leq (1+Y) \qquad \text{(III-26)}$$

In general, the value of Y which makes $Y^{\tau/\Delta\tau}$ insignificant at the time of interest is satisfactory for use in equation III-26 to select γ and $\Delta\tau$. For many purposes Y can be as high as 0.6. However, since increasing the time increment reduces all the q_j's, care should be taken that q_{MAX} is significantly larger than Y or the solution will probably show excessive oscillations. Although for most reasonable selections of the time differencing parameters this will not occur, it is possible with a γ of $\frac{1}{2}$ and a very large time increment $\Delta\tau$ to have a stable solution with all the q_j's including q_{MAX} negative. In cases where it is necessary q_{MAX} can be estimated in one of the following

three ways. First a lower bound for q_{MAX} can be derived from the fact that

$$\lambda_{MAX} \geq \mu_{00} \tag{III-27}$$

where μ_{00} is the largest diagonal element of Y/A (smallest in absolute value and negative). This gives for q_{MAX}:

$$q_{MAX} \geq \frac{1 + \mu_{00}(1-\gamma)\Delta\tau}{1 - \mu_{00}\gamma\Delta\tau} \tag{III-28}$$

This lower bound is usually not close to q_{MAX} and consequently it might not be useful. The second way of estimating q_{MAX} for a given time increment is to find the smallest υ_j root for a problem like the one to be approximated; $e^{-\upsilon_1^2\Delta\tau}$ is then an estimate of q_{MAX}. Third, a calculation using a small number of points and time increment $(\Delta\tau)_1$ can be stepped out until the logarithm of the approximate temperature at a point is linear with time. From this the q_{MAX} for the specific calculation can be obtained. Then $(q_{MAX})^{(\Delta\tau)_2/(\Delta\tau)_1}$ is the required approximation to the q_{MAX} for time increment $(\Delta\tau)_2$.

It should be noted that limiting q_{min} makes no allowances for the weighting or coefficients that multiply q_{min}^n in the solution. These coefficients are the associated eigenvector and the coefficient in the initial vector corresponding to the q_{min}. Indeed, the initial motivation for the study in Chapter IV was to find why, under the same conditions, one of two graphical methods with the same difference eigenvalues had

an oscillatory component so large as to make the solution useless, but the second method gave useful approximate solutions.

To eliminate a large oscillatory component, without having to use either a very small time increment or a large weighting factor for all the calculations to very long times, the following scheme can be used. First, the time increment $(\Delta\tau)_1$ and weighting factor γ are selected so that q_{min} is not negative and the contribution after n_1 time increments from $q_{min}^{n_1}$ is small. The estimate of $(\Delta\tau)_1$ and γ_1 can be based on the upper bound M. Then at step n_1 the temperature distribution should be smooth and a larger $(\Delta\tau)_2$ and smaller or zero γ_2 can be used for the rest of the calculation with a reasonable assurance that any oscillatory component is probably insignificant; obviously the selection of $(\Delta\tau)_2$ and γ_2 is still limited as no q can be smaller than -1. To see that this procedure reduces the weighting on the oscillatory component, consider the calculated approximate temperature distribution at increment n_1 as the initial condition for the second choice of $(\Delta\tau)_2$ and γ_2. The solution for this difference calculation becomes

$$t_n = t_{P\,n} - C\,Q_1^{\,n} g \qquad 0 \le n \le n_1 \qquad \tau = n(\Delta\tau)_1 \qquad \text{(III-29)}$$

$$t_n = t_{P\,n} - C\,Q_2^{\,n-n_1}\,Q_1^{\,n_1} g \qquad n \ge n_1 \qquad \text{(III-30)}$$

$$\tau = n_1(\Delta\tau)_1 + (n-n_1)(\Delta\tau)_2$$

where Q_1 is the diagonal matrix of q_{j1}, the difference eigenvalues based on $(\Delta\tau)_1$ and γ_1, and Q_2 is the matrix of q_{j2} based on $(\Delta\tau)_2$ and

γ_2. Now the weighting of $q_{min\ 2}$, which is negative for the second $(\Delta\tau)_2$ and γ_2, is $g_S\, q_{min\ 1}^{n_1}$, which is small, so the oscillations should be negligible for the remaining calculations. In the above equations, III-29 and III-30, no change in the particular solution was shown when the γ and $\Delta\tau$ were changed; thus, the conclusion applies rigorously only for problems where $t_{P\ n}$ is not a function of n but is a true steady-state solution. Although problems where the particular solution does change with time and is a quasi-steady-state solution have not been studied, it is very probable that the above scheme would eliminate spurious oscillations for all but a few types of forcing functions.

A second way of reducing oscillations is by averaging the calculation for two successive times n and $(n+1)$ and then applying the result at n, $(n+1)$, or an intermediate value of n. By examining the form of the solution equation it is seen that this has the effect of multiplying each weighting g_j of the j^{th} eigenvector-eigenvalue product by $(q_j+1)/2$:

$$g_{j\ \text{averaged}} = \left(\frac{q_j+1}{2}\right) g_{j\ \text{unaveraged}} \qquad \text{(III-31)}$$

If q_{min} is close to -1, and q_{MAX} is close to +1, averaging significantly reduces the weighting q_{min} of the negative eigenvalue-eigenvector product without changing the weighting g_{MAX} of the q_{MAX} eigenvalue-eigenvector product. However, the weighting for the moderate sized

q's is radically changed. The effect of averaging is discussed more precisely for one-dimensional problems in Chapter IV, sections B-7, D-2 and E-6. It is shown to be mostly useful in graphical methods in which the q_{min} has the same absolute value as q_{MAX}.

The criteria shown in equations III-14 and III-15 should be compared and related to another criterion often suggested (9) for the explicit method. This other criterion is that none of the coefficients of $[I + \Delta\tau(Y/A)]$ be negative to avoid wild oscillations. For cases of uniform mesh spacing simple arguments show that this is a sufficient requirement (3). For the asymmetric network and the explicit method this amounts to the condition that

$$\Delta\tau \leq \frac{1}{|\mu_{\rho\rho}|} \qquad \text{(III-32)}$$

Referring to equations III-13 and III-15, this is seen to be equivalent to assuming that the absolute value of the minimum eigenvalue, $|\lambda_{min}|$, is less than, or equal to, $2|\mu_{\rho\rho}|$ and equivalent to substituting $2|\mu_{\rho\rho}|$ into the stability ctieria. Although this condition is sufficient for stability, if the time increment is selected by the equality, the solution probably contains an oscillatory component, and also it does not eliminate the possibility of a -1 root. In graphical methods for one dimension the time increment used conforms to the equality in equation III-32 and, as mentioned previously, oscillations that make the solution useless are sometimes obtained. The criteria derived in this section

have the advantage of being almost as simple and yet rigorous and, with proper modifications, they allow the selection of γ and $\Delta\tau$ to obtain the type of solution required with almost the largest time increment possible.

In conclusion, to obtain a satisfactory approximate solution, the time increment and weighting γ must be selected to give a stable solution. The precise selection of the time increment and γ must then be based on the specific purpose of the approximation. In general, the combination of time increment and γ should be selected which gives an accurate representation of how the continuous exponential varies with time, and simultaneously minimizes the calculations necessary, and keeps any oscillations from destroying the usefulness and accuracy of these results. Although it is probably not possible to derive a general set of rules that would apply to all asymmetric distributions, this question is considered in detail in Chapter IV for one-dimensional regular networks in Cartesian coordinates and some generalizations are possible.

For some problems the selection of the time increment may be fixed or limited by considerations other than stability or oscillatory effects. For example, if the boundary temperature contains oscillatory components, the time increment must be smaller than one-half the smallest period of oscillation of the oscillatory components. This can be demonstrated rigorously from Shannon's sampling theorem (12) or

can be seen by considering that, if the value of the oscillating function is not sampled at least this often, a wholly inadequate description of the oscillations results.

(D) NUMERICAL EXAMPLE

In order to numerically check the necessary and sufficient criteria for stable and non-oscillatory solutions, a transient heat transfer problem was solved using an asymmetric distribution of temperature points for several combinations of time increments and weighting factors. In addition, the norms of Y/A and W were studied to show the usefulness of the sufficient conditions and the closeness of the bounds for $|\lambda_{min}|$. The presentation of an example problem also affords an opportunity to mention some of the practical points concerning the calculation of the Y/A matrix, and procedures for solving the equations for the implicit methods. The problem selected is one where the solution for the partial differential equation is known, so that it also can be used to study the accuracy.

The problem is to find the transient temperature distribution in a solid with a unit square cross-section of uniform thermal properties, and a uniform initial temperature distribution of unity, which has its surfaces maintained at zero for all times greater than zero. Taking the center of the solid as the origin, the solution of the two-dimensional partial differential equation can be found as the product of solutions of one-dimensional problems:

$$T(\xi,\eta,\tau) = \Omega(\xi,\tau)\Omega(\eta,\tau) \qquad -\tfrac{1}{2} \le \xi \le \tfrac{1}{2} \qquad \text{(III-33)}$$

$$-\tfrac{1}{2} \le \eta \le \tfrac{1}{2}$$

$$\tau \ge 0$$

where

$$\Omega(\xi,\tau) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)} e^{-\pi^2(2j+1)^2\tau} \cos(2j+1)\pi\xi \qquad \text{(III-34)}$$

This problem is important because its solution has the same eigenvalues and eigenfunctions as the transient solution for problems where the temperatures of the surfaces of the unit square are specified as any function of space and/or time. In these other cases, only the particular solution $T_P(\xi,\eta,\tau)$, which is zero above, and the Fourier coefficients a_{ij}, which are $16(-1)^{i+j}/\pi^2(2j+1)(2i+1)$ above, would be changed.

For the approximate solution, points need be located only on the minimum area necessary, considering the symmetry of the problem. This area is the 1/8 of the cross-section as shown by the triangle in Figure III-1. Nine points where the temperature is an unknown function of time are located within the solid, and five points where the temperature is known as a function of time are located along the surface. The variable temperature points are located away from the axes of symmetry, which can be considered as adiabatic boundaries. By locating the points away from the boundary, the number of nodes required to be calculated for the small cross-section is not raised, but the equivalent of the total number of nodes in the whole cross-section is

significantly raised. Moreover, since the whole area of the surface nodes A, B, C, D, and E is assumed to be at the boundary temperature of zero, the heat capacity of the solid corresponding to the total heat capacity of these nodes is neglected. The network of points was made irregular so that the effects of asymmetric location on the accuracy could be studied. However, this could be considered as a practical arrangement for a problem where only the corner temperatures are important.

The Y/A matrix was then calculated using exact geometric relationships for all the lengths, and thus errors are probably less than 0.01 per cent in the elements of Y/A. In a practical case, sufficient accuracy probably is obtained by making a large-scale drawing and measuring the lengths. A simple digital computer program could be devised to determine the Y/A matrix from arrays of the r_{ij} and ℓ_{ij} in a matrix form, together with the effective thermal conductivities for each connector, and heat capacity data for each node. A program to find Y/A from just the location of the temperature points, the specification of the connected points, and the effective thermal properties, probably could be devised although it would be very complicated.

The Y/A matrix for the arrangements of nodes in Figure III-1, as computed using a desk calculator, is in Table III-1. The first row in this matrix contains the coefficients in the ordinary differential equation for node 1, the second row for node 2, etc. Since a numerical

check was to be made of the necessary and sufficient criteria, the eigenvalues of the matrix were determined using a method based on the general numerical method of iterative multiplications of the Y/A matrix (16), and these are in Table III-2. From the $|\lambda_{min}|$ of 1442, the limiting time increments for stable and non-oscillatory solutions have been calculated for γ's of 0, $\frac{1}{2}$, and 1 from equations III-14 and III-15, and are summarized in the table. In order to check these criteria, the approximate solution was stepped out using equation II-39 for these γ's and for time increments on both sides of the critical values. The results are shown as graphs relating the calculated approximate temperature at node 1 and dimensionless time in Figure III-2 for the explicit case of γ of zero, and in Figure III-3 for the implicit γ's of $\frac{1}{2}$ and 1. Node 1 was selected as it is the most sensitive to instability and oscillatory behavior. Also, for comparison, the continuous solution of the partial differential equation for that point is shown.

For the explicit calculation an unstable solution is expected for a time increment $\Delta\tau$ greater than 0.001387, equal undamped oscillations for a $\Delta\tau$ of 0.001387, damped oscillations for a $\Delta\tau$ between 0.001387 and 0.000694, and smooth damped exponentials for a positive $\Delta\tau$ less than 0.000694. These expected types of solutions were obtained as shown in Figure III-2. Referring to Figure III-2, the midpoint of equal oscillations shown for the limiting time increment for stability, 0.001387, (b) decreases and approaches zero for large times.

The oscillatory solution for a $\Delta\tau$ of 0.001, although useless for the first four increments, does give a fair approximation for long times. The solution found by using the maximum time increment which gives no negative coefficients in the calculation matrix, $[I + \Delta\tau(Y/A)]$ (Longwell's sufficient criteria, equation III-32), although oscillatory, gives a better approximation to the continuous solution shown in (f). Approximate solutions using this criterion for similar problems also show oscillations, but they are so rapidly damped (19) that they are not significant. The smallest increment shown, $\Delta\tau = 0.0001$ (e), is seen to give an accurate approximation to the continuous solution. Not shown are good approximations with time increments of 0.0005 and 0.000694.

The implicit calculations were calculated by finding the inverse matrix and then stepping out the solution explicitly using equation II-39. For a γ of $\frac{1}{2}$ the solution is always expected to be stable, and should be non-oscillatory for positive time increments less than 0.001387; for a γ of 1, neither instability nor oscillatory behavior is expected. The graphs, together with other numerical results in Figure III-3, show that the expectations are fulfilled. However, for the large $\Delta\tau$ of 0.01 and a γ of $\frac{1}{2}$ the solution oscillates (damped) so badly that it is useless as an approximation. The corresponding calculation for a γ of 1 at a $\Delta\tau$ of 0.01, although a smooth damped exponential, is not an accurate approximation as discussed later. Comparison of the

non-oscillatory solutions for a small time increment of 0.0005 with the continuous solution, Figure III-3 (b), (d), and (e), shows that the approximation with γ of $\frac{1}{2}$ appears to be more accurate than with a γ of 1 or zero.

Although a detailed discussion of accuracy is deferred until the last chapter, several brief comments can be made. First, the approximate solutions that are stable and which have only small oscillations are reasonably accurate. More accurate results are obtained for the smaller nodes, 1, 2, and 3; less accurate results for the large nodes 6 and 7, as would be expected. For the large time increments the most accurate results are with a γ of $\frac{1}{2}$. As the time increments are reduced, the errors appear to approach a constant for each node indicating that the approximate solution approaches the analog as the time increment goes to zero.

To show that, when $\Delta\tau$ and γ are selected so that a q of -1 occurs, the calculated temperature can go to infinity even though the surface temperatures are bounded, the explicit calculation for the limiting time increment for stability was repeated but with the surface temperature of the nodes A, B, C, D, E oscillating between +1 and -1.

$$t_f = (-1)^n \tag{III-35}$$

The temperatures of the surface nodes and of node 1 are shown in Figure III-4. The temperature of node 1 oscillates with a linearly increasing amplitude indicating that a term of the form $n(-1)^n$ must be

in its solution. Consequently, the solution is unbounded and the limiting $\Delta\tau$ and γ combination, although bounded for some boundary conditions, is unstable according to the definition of stability used here. Also, this shows that with an oscillatory forcing function the time increment should not be a multiple of the period of oscillation or the oscillation might excite an oscillatory difference eigenvalue which does not occur in the continuous solution.

Thus, the necessary and sufficient criteria for stable and non-oscillatory behavior have been checked. However, practically, the eigenvalue $|\lambda_{min}|$ probably would not be obtained, but the sufficient conditions given by equations III-14 and III-15 in terms of the matrix norms would have been used. The calculated matrix norms for the Y/A matrix are in Table III-2. The row norm $\| \frac{Y}{A} \|_I$ is the smallest at 1618.8 and the largest (in absolute value) diagonal element is 1266.7, which defines the range for $|\lambda_{min}|$. If M had been used as 1618.8 the sufficient time increment computed for a constant γ would be about 12 per cent lower than the necessary and sufficient increment. Also, from the diagonal element of Y/A that is largest in absolute value, maximum of $|\mu_{ii}|$ of 1266.7, a time increment based on the minimum norm of Y/A would not be more than 22 per cent lower than necessary. The symmetrized Y/A matrix, or W matrix, can be calculated directly or from the elements of Y/A, and is shown in

Table III-3. Its norm is lower than those of Y/A and is 1583.3, which, if used in the sufficient condition, would give a $\Delta\tau$ ten per cent smaller than necessary. A simple diagonal similarity transformation on the W matrix as suggested in equation III-23 gives a U matrix which is shown together with the transformation matrix in Table III-4. The U matrix has a minimum norm (column) of 1501.8 which is only four per cent greater than the $|\lambda_{min}|$. The trial and error calculation of the F matrix and the simultaneous calculation of U required about an hour's time using a slide rule and adding machine; the transformation F was selected so that it reduced the off-diagonal elements in the fourth and fifth rows and second and third columns without increasing the other column sums.

A numerical study was also made of the norms of Y/A, maximum diagonal element $\underset{i}{MAX}\,|\mu_{ii}|$, and $|\lambda_{min}|$ for the two Y/A matrices used by Longwell in his rocket motor study (9). For his shape 1, fourteen equations were used. The minimum norm of Y/A itself was only 9 per cent greater than $|\lambda_{min}|$ and about 13 per cent greater than the maximum $|\mu_{ii}|$. Sixteen equations were needed for his shape 2, and the minimum norm was 8 per cent larger than $|\lambda_{min}|$ and 35 per cent larger than the maximum $|\mu_{ii}|$.

From the limited experience of these three problems, the norm of Y/A appears to be a good estimate of the absolute value of the eigenvalue which determines the stability and oscillatory behavior of

the difference solution. Replacing $|\lambda_{min}|$ with the norm to compute the limiting time increment gives, on the average, a time increment only about 10 per cent too small. However, in these problems the minimum norm of Y/A does not approach being 100 per cent greater than the maximum diagonal element, as is theoretically possible. The only case when this occurs is when a node with the largest diagonal element, $\sum_{j=1}^{R} y_{ij}/A_i$, is connected only to interior temperature nodes where the temperature is to be calculated or when it is bounded by an adiabatic boundary. However, it is probable that in the most frequently encountered problems, the maximum diagonal element occurs for a point which has connectors to a known fluid temperature and thus one of the conductances appears in the Y_B/A matrix. Consequently, the row sum is less than twice the diagonal element which is the sum over all conductances. In these cases, the norm of the Y/A matrix probably can be used satisfactorily without further calculation; for the problems where the norm and diagonal element give too large a range for $|\lambda_{min}|$, the calculation of the norm for W and/or a diagonal transformation may be used.

The number and location of the points also affects the limiting time increment. Increasing the number of points reduces the dimensionless heat capacity A_i's of some or all of the points which in turn raises the norm, diagonal elements, and $|\lambda_{min}|$ requiring a lower time increment or an implicit method with a relatively high γ. From the

previous consideration of the Y/A matrix, the elements associated with the node or nodes having the smallest dimensionless heat capacities A_i's are the ones that usually determine the norm of the matrix and the maximum diagonal element. Consequently, these nodes can greatly reduce the allowable time increment. Thus, to prevent one or a few nodes from restricting the time increment too severely, the points should be located, in so far as practical, so that the elements in Y/A are of the same order of magnitude.

Locating S points in a regular fashion for a solid of uniform thermal properties gives a Y/A matrix with smaller norms and smaller maximum diagonal element than the Y/A matrix for locating the S points so that some nodes have smaller heat capacities. The Y/A for the asymmetric case also very probably has a large $|\lambda_{min}|$, and thus a smaller time increment must be used.

(E) ALTERNATE CALCULATION FOR IMPLICIT METHODS

In calculating the approximate solution for the implicit methods, the inverse matrix $[I-\gamma\Delta\tau\,(Y/A)]^{-1}$ was first computed with an elimination method; then the calculation matrices of $[I-\gamma\Delta\tau(Y/A)]^{-1}[I+(1-\gamma)\Delta\tau(Y/A)]$ and, if necessary, $[I-\gamma\Delta\tau(Y/A)]^{-1}[\Delta\tau(Y_B/A)]$ were found. The approximate solution was then stepped out using equation II-39 explicitly as a matrix-vector product, using the known temperature vector at the beginning of the increment and known boundary temperature vector. Although this method is simple, it suffers from two disadvantages. First, as the elements of the inverse matrix $[I-\gamma\Delta\tau(Y/A)]^{-1}$ contain round-off error, significant additional errors might be introduced into the approximate solution by the successive multiplications by that matrix. Of more importance is that under certain circumstances methods of solution of the system of equations II-39 are available that require fewer of the time-consuming multiplication operations than does a vector-matrix product. Thus, for these cases, the system of equations II-39 can be solved for each time increment for t_{n+1} more rapidly than the vector-matrix product can be calculated.

The equation to be solved is

$$[I-\gamma\Delta\tau\,\frac{Y}{A}]t_{n+1} = [I+(1-\gamma)\Delta\tau\,\frac{Y}{A}]t_n + \frac{\Delta\tau\,Y_B}{A}(\gamma\,t_{B\,n+1} + \left\{1-\gamma\right\}t_{B\,n}) \qquad \text{(III-36)}$$

where the quantity on the right side of the equality represents a known

vector and t_{n+1} is to be found. The method for solution of the one-dimensional case is based on simplifications in the elimination method which are possible because only the three major diagonals of $[I-\gamma\Delta\tau(Y/A)]$ and $[I+(1-\gamma)\Delta\tau(Y/A)]$ contain non-zero elements. For two- or three-dimensional problems, where the temperature points are located on the corners of a rectangle or rectangular solids, line iterations and alternating direction methods (6) are useful. However, they do not give the exact solution of the system of equations III-36, but actually represent a slightly different method, and further it appears that they cannot be modified for an asymmetric distribution of points. A modification of the elimination solution for the one-dimensional case can be devised which can solve the system of equations for the irregular network more rapidly than the vector-matrix product can be formed, providing the multiplication by zero is avoided, or providing it is significantly faster than multiplication by non-zero elements. This method for the general asymmetric network is done by using an elimination technique to convert the system of equations III-36 to an upper triangular system, which can then be solved explicitly. After determining the constants on the right side of the equation (note that because of the many zeros in both matrices on the right side the vector-matrix products here are formed much more rapidly than would be the case with the inverse matrix which probably has no zeros) the conversion to an upper triangular system requires the following steps:

(1) Solve the first equation for the first variable, $t_{1,n+1}$. As $[I+\gamma \Delta\tau(Y/A)]$ is sparse, it should only require three or four divisions by the diagonal element μ_{11}.

(2) Eliminate $t_{1,n+1}$ from all the following equations by substituting the expression found in step (1). Since $[I+\gamma \Delta\tau(Y/A)]$ is a sparse matrix, this should require substitution in only two or three equations.

(3) Repeat steps 1 and 2 for the next equation and variable, eliminating the next temperature point from all succeeding equations. Continue until all the coefficients below the diagonal in $[I+\gamma \Delta\tau(Y/A)]$ are zero. Now the last variable, $t_{S,n+1}$, is determined explicitly by the last equation. Knowing $t_{S,n+1}$, $t_{S-1,n+1}$ is calculated explicitly from the (S-1) equation. Following this procedure back through the equations each component of t_{n+1} is found. This procedure, including the calculation of the known vector on the right side of the equation, requires a number of multiplicative operations involving non-zero elements directly proportional to S; on the other hand, the inverse matrix-vector product has S^2 multiplicative operations. Consequently, the advantage of using the modified elimination method depends upon the proportionality constant and the number of points used.

The proportionality constant depends upon exactly where the non-zero elements are located in the $[I-\gamma \Delta\tau(Y/A)]$ matrix. For the one-dimensional case where each node has two neighbors, 8S

multiplicative operations are needed to solve the system, including the determinations of the constants on the right side of the equation. For a two-dimensional problem where each node is surrounded by an average of four nodes, the solution requires about 16S multiplicative operations. Thus, if S is greater than 8 or 16 for the one- or two-dimensional system, the system of equations III-36 can be solved at each time increment more rapidly than the inverse matrix-vector product can be constructed.

(F) CONCLUSIONS

In Chapter III the criteria for stability and non-oscillatory approximations were derived as simple relationships involving the time increment $\Delta\tau$, the weighting factor γ, and the eigenvalue of the Y/A matrix with the largest absolute value for a fixed network of points. The necessary and sufficient criteria for both stable approximations and non-oscillatory approximations were then checked numerically by approximating a problem using an asymmetric network of points.

Under any practical conditions, it is too time consuming to determine this eigenvalue. Therefore, sufficient conditions were derived in terms of easily calculated norms of the Y/A matrices. These matrix norms were then shown to give the same stable and non-oscillatory criteria usually found by Fourier analysis for one-dimensional problems with a regular distribution of temperature points. It was also shown that these sufficient conditions can never give a time increment less than $\frac{1}{2}$ of the time increment actually required. Applications of these sufficient conditions to the example problem and to two other two-dimensional problems using an asymmetric distribution of points showed that these norms usually are only 4 to 15 per cent larger than the absolute value of the critical eigenvalue $|\lambda_{min}|$, which indicates that the time increment determined by the sufficient conditions is expected

to be just slightly smaller than its limiting value to obtain the required behavior.

In addition to these points, a weighting γ of $\frac{1}{2}$ was shown always to give a stable solution and a weighting γ of 1 always to give a non-oscillatory solution. Also a negative q_{min} can be tolerated providing q_{min}^{n} is negligible at the times of interest. Further, a proof was given that, after starting a solution using a small time increment for several time increments, a larger time increment could be used for the remaining calculations without excessive oscillations. It was also noted that by averaging the calculated temperatures for successive time increments, the weighting of the oscillatory components was reduced.

However, a solution which is stable or non-oscillatory is not necessarily an accurate approximation. A stable solution is bounded as is the continuous solution. A non-oscillatory solution only oscillates as does the continuous solution. But although the general behavior of the approximation might be like that of the continuous solution, nevertheless the eigenvectors, the large damping factors q_j, and the initial vector components might not be good approximations to the corresponding quantities in the continuous solution as would be necessary for an accurate solution. The question of accuracy for one- and two-dimensional problems is the subject of Chapters IV and V.

Table III-1

CONDUCTANCE/CAPACITANCE MATRIX, Y/A

1/8 - Square Cross-Section

$\frac{Y}{A} =$

$j \rightarrow$ / $i \downarrow$	1	2	3	4	5	6	7	8	9	Row Sum $\sum_{j=1}^{9} \lvert \mu_{ij} \rvert$
1	-1266.7	49.80	219.30	0	0	0	0	0	0	1535.80
2	29.35	-974.15	430.81	57.44	86.16	0	0	0	0	1577.91
3	160.30	541.10	-812.59	0	109.19	0	0	0	0	1623.18
4	0	19.26	0	-302.22	50.55	23.11	7.04	0	0	402.18
5	0	34.46	34.78	60.30	-170.94	0	41.36	0	0	341.84
6	0	0	0	16.51	0	-99.01	46.37	15.48	0	177.37
7	0	0	0	63.92	31.16	58.30	-121.75	25.96	0	301.09
8	0	0	0	0	0	25.35	33.80	-87.28	28.13	174.56
9	0	0	0	0	0	0	0	320.00	-320.00	640.00
$\sum_{i=1}^{9} \lvert \mu_{ij} \rvert$	1456.35	1618.77	1497.48	500.39	448.00	205.77	250.32	448.72	348.13	

Column Sum

Table III-2

EIGENVALUES, NORMS, AND STABILITY CRITERIA

1/8 - Square Cross-Section

a. Negative of Eigenvalues

	$-\lambda$
$\lvert\lambda_{min}\rvert$	1442
	1241
	401
	354
	323
	175
	121
	80
	19

b. Norms

$$\left\|\frac{Y}{A}\right\|_I = \max_i \sum_{j=1}^{9} \left|\mu_{ij}\right| = 1618.8 \qquad i=3$$

$$\left\|\frac{Y}{A}\right\|_{II} = \max_j \sum_{i=1}^{9} \left|\mu_{ij}\right| = 1623.18 \qquad j=2$$

$$\max_i \left|\mu_{ii}\right| = 1266.7$$

c. Stability and Oscillatory Criteria

γ =	0	1/2	1
Limiting $\Delta\tau$			
$\lvert\lambda_{min}\rvert(1-2\gamma)\Delta\tau \leq 2^*$	0.001387	∞	∞
$\lvert\lambda_{min}\rvert(1-\gamma)\Delta\tau \leq 1^{**}$	0.0006937	0.001387	∞

* Stable

** Non-Oscillatory

Table III-3

SYMMETRIZED Y/A MATRIX, W = D Y D

1/8 - Square Cross-Section

$W =$

j → / i ↓	1	2	3	4	5	6	7	8	9
1	-1266.7	38.2	187.4	0	0	0	0	0	0
2	38.2	-974.2	482.8	33.3	54.6	0	0	0	0
3	187.4	482.8	-812.6	0	61.6	0	0	0	0
4	0	33.3	0	-302.2	55.2	19.5	21.2	0	0
5	0	54.6	61.6	55.2	-170.9	0	35.9	0	0
6	0	0	0	19.5	0	-99.0	52.0	19.8	0
7	0	0	0	21.2	35.9	52.0	-121.8	29.6	0
8	0	0	0	0	0	19.8	29.6	-87.3	94.9
9	0	0	0	0	0	0	0	94.9	-320.0
Column Sum $\sum_{i=1}^{9} \lvert w_{ij} \rvert$	1492.3	1583.1	1544.4	431.4	378.2	190.3	260.5	231.6	414.9

Table III-4

U MATRIX, $U = F\,W\,F^{-1}$, SIMILAR TO W AND Y/A MATRICES

1/8 - Square Cross-Section

$U =$

j → / i ↓	1	2	3	4	5	6	7	8	9
1	-1266.7	38.2	187.4	0	0	0	0	0	0
2	38.2	-974.2	482.8	1000	545	0	0	0	0
3	187.4	482.8	-812.6	0	616	0	0	0	0
4	0	1.1	0	-302.2	18.4	130	21.2	0	0
5	0	5.5	6.2	165.6	-170.9	0	107.7	0	0
6	0	0	0	2.9	0	-99 .0	7.8	1	0
7	0	0	0	21.2	12.0	348	-121.8	9.9	0
8	0	0	0	0	0	396	88.8	-87.3	9.5
9	0	0	0	0	0	0	0	94.9	-320
Column Sum $\sum_{i=1}^{9} \lvert u_{ij} \rvert$	1492.3	1501.8	1489.0	1491.9	1362	973	348	1047	330

F^{-1} = dia 1 1 1 30 10 200 30 10 1

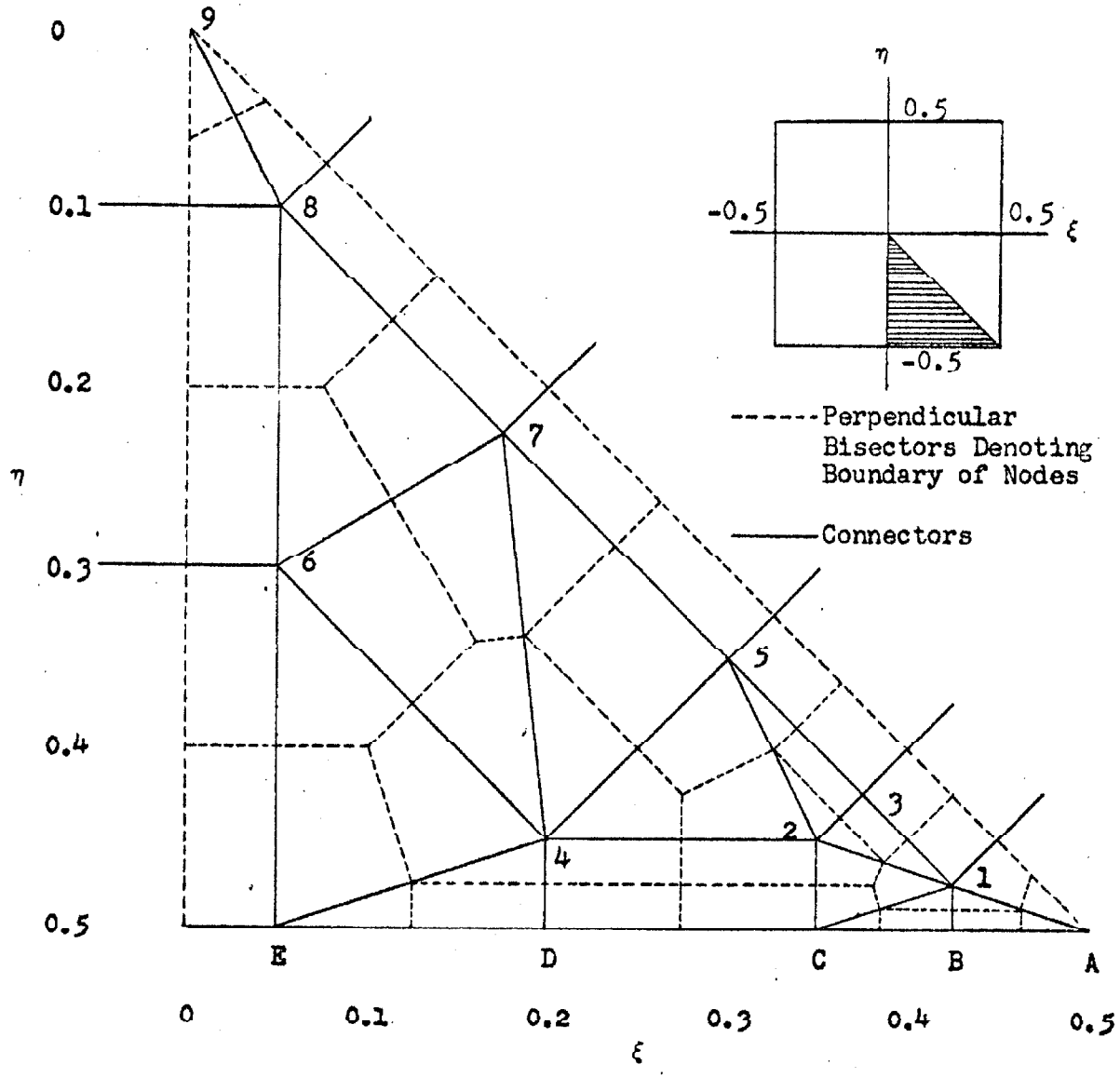


Figure III-1. Layout of nodes on square cross-section. Nodes 1-9, interior points where temperature is a known function of time. Nodes A-E, temperature nodes in boundary vector.

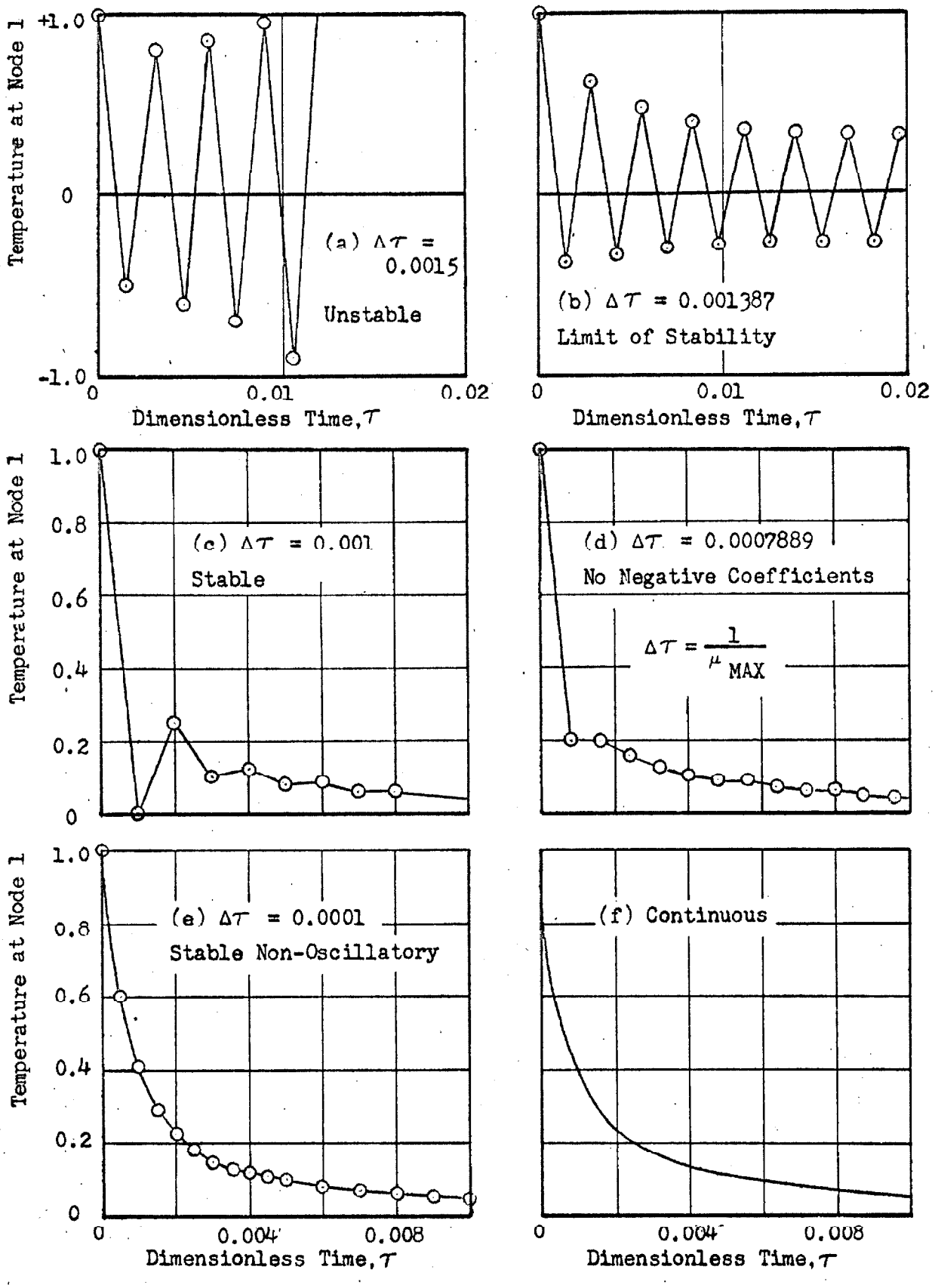


Figure III-2. Behavior of approximate solutions, $\gamma = 0$.

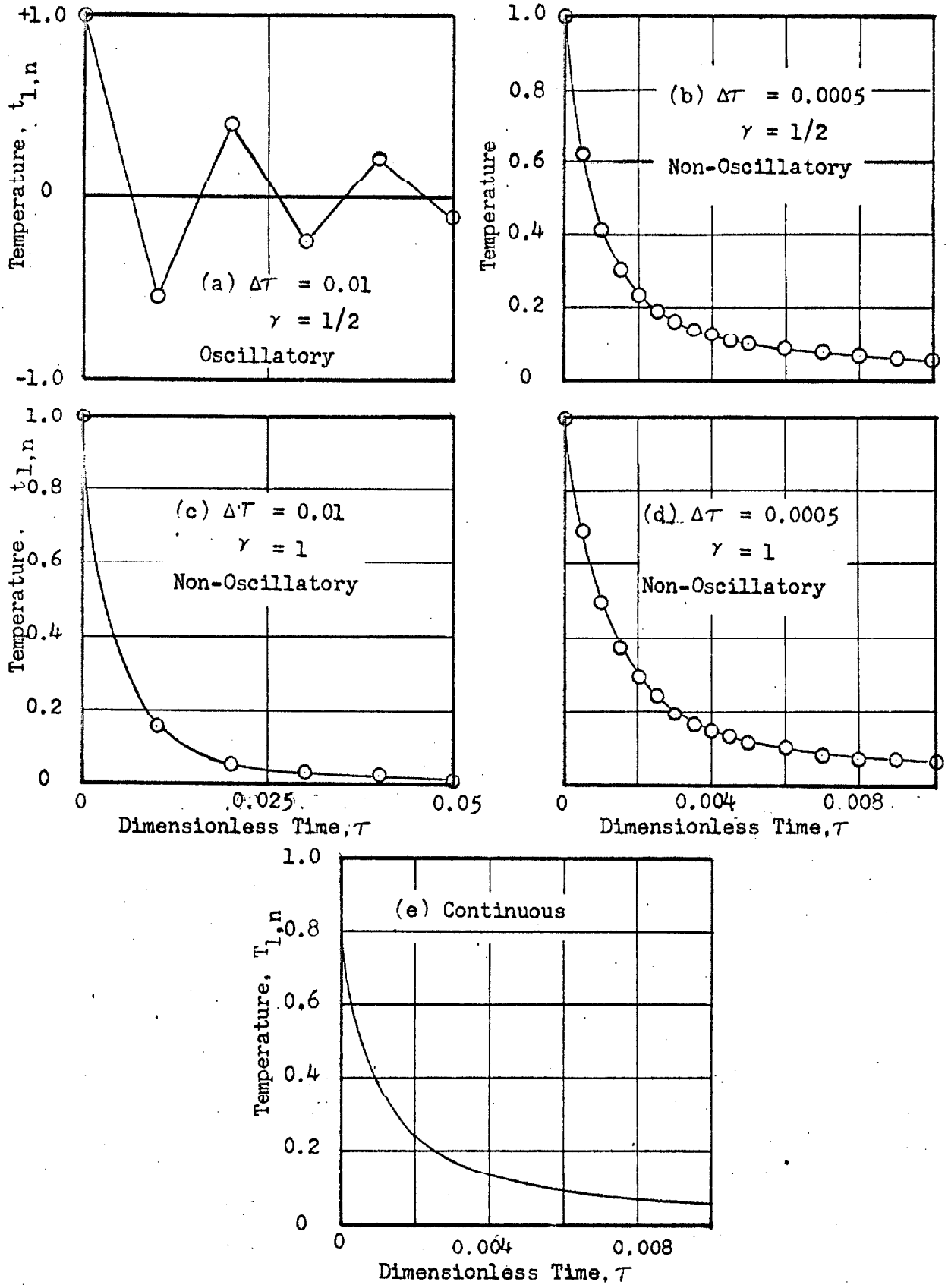
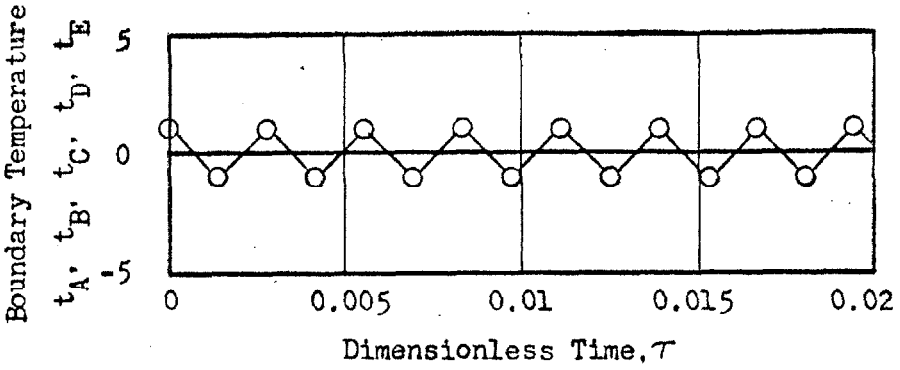


Figure III-3. Behavior of approximate solutions, $\gamma = 1/2$ and $\gamma = 1$.



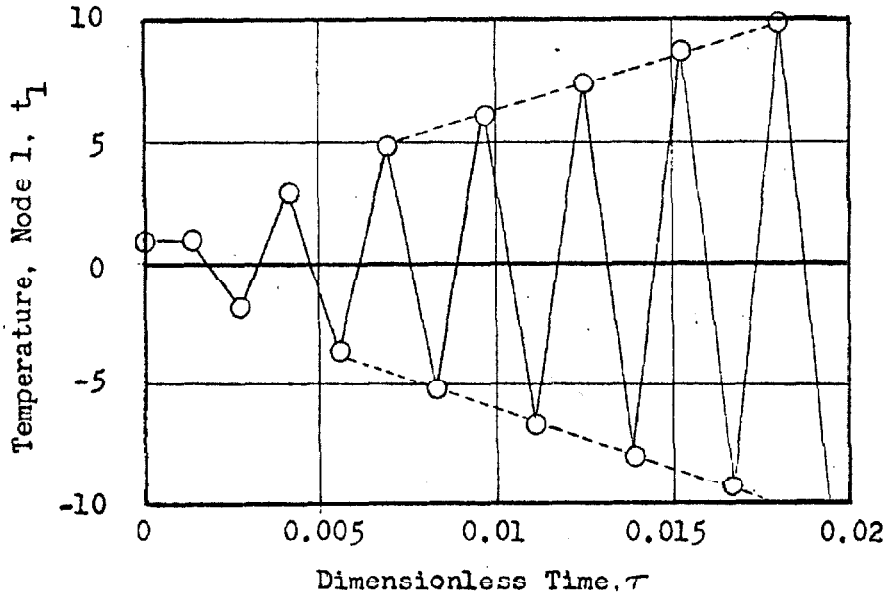


Figure III-4. Calculation at limit of stability, $\gamma = 0$, $\Delta\tau = 0.001387$, with oscillatory boundary conditions.