TRANSIENT HEAT TRANSFER IN SOLIDS

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ABSTRACT

The accuracy of approximate calculations of transient temperature distributions in solids is studied, emphasizing easily applied methods for both irregular and regular solids with a heat-transfer coefficient boundary condition. General stability criteria are shown which are easily found and used, and which apply for regular or asymmetric networks.

Analytic solutions of the one-dimensional partial difference equation are found by z-transforms. A method with points located half an increment inside the surface gives the most accurate solutions. A graphical method is shown analytically to give approximations of engineering accuracy. A procedure is also presented for finding the differencing parameters which give a solution of a specified accuracy with the minimum calculations.

Transient temperature distributions of satisfactory accuracy can be calculated for irregular solids using a relatively coarse network, by locating points that describe the boundary a short distance within, rather than on, the boundary, and by making the interior network as regular as possible.

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NOMENCLATURE

A	S x S diagonal matrix with dimensionless thermal capacity along diagonal
A_i	$\frac{1}{4L^2 \sigma_0 C_{p0}} \sum_{\substack{j=1 \\ j \neq i}}^{S} r_{ij} \ell_{ij} \sigma_i C_{pi}$, dimensionless thermal capacity of i^{th} node; also i^{th} diagonal element of A matrix
A_{f_i}	surface area between fluid and node i through which energy flows
a	vector of continuous Fourier coefficients determined to fit continuous initial condition $T(\xi, 0)$, $0 \leq \xi \leq 1$
a_j	j^{th} continuous Fourier coefficient in vector a
B	continuous eigenvector arranged in matrix
$b_j(x)$	j^{th} orthogonal function to fit continuous boundary conditions
C	matrix whose columns are eigenvectors of Y/A matrix
C_p	heat capacity, Btu/°F-lb
c_i	i^{th} column eigenvector of difference system $\lambda_i c_i = \frac{Y}{A} c_i$
D	S x S diagonal matrix with $1/\sqrt{A_i}$ in diagonal position
$D(z)$	part of denominator of z-transform that goes to zero as z goes to a pole
D or D_j	replacement factor for $S\psi$ or ν in approximate characteristic equation

d	differential operator
d_n	discretization error vector at time n, defined by equation V-4
$d_{i,n}$	discretization error vector associated with i^{th} node at time n
E	finite difference operator, $E\,t_n = t_{n+1}$
e	base of napierian or natural logarithm
e_n	round-off error vector occurring at time n, defined by equation V-8
F	constant in undetermined parameter solution
F	diagonal transformation matrix
F_C	continuous characteristic equation
F_D	characteristic equation for approximate method
$f(\tau)$	function of time
$\bar{f}(z)$	z-transform of difference function, f_n
G	constant in solution of boundary value difference equation
g	$C^{-1}(t_0 - t_{P,0}) = C'A(t_0 - t_{P,0})$ difference initial vector that is vector of coefficients required to fit the initial condition
g_j	j^{th} component of initial vector
$\bar{g}(z)$	z-transform of difference function g_n, used only in z-transform table

H	hL/k = dimensionless heat transfer coefficient
h	heat transfer coefficient, Btu/hr, ft^2, °F
I	identity matrix which is a diagonal matrix with 1 in the diagonal positions
J	limit of the integrals as defined in equations IV-194 and IV-196
J	number of terms in transient solution
K	$k/\sigma C_p$ = thermometric conductivity, ft^2/hr
k	thermal conductivity, Btu/hr, ft, °F
L	reference length, thickness of slab for one-dimensional problem
ℓ_{ij}	distance between i^{th} and j^{th} node
ℓ_{i,f_i}	distance along normal to boundary from surface to i^{th} node
M	minimum of row norm of Y/A, column norm of Y/A, and norm of W
N	number of non-zero multiplications to calculate to τ_1
N	number of nodes with non-zero conductances to the i^{th} node
N(z)	part of z-transform that remains analytic as z goes to a pole, i.e., as D(z) goes to zero
n	number of time increments; as a subscript occurring at end of n^{th} increment

n	direction normal to boundary of solid directed toward interior
P	constant in solution of boundary value difference equation
p	index of summation, an integer
p	proportionality constant
Q	S x S diagonal matrix of difference eigenvalues dia $q_1\ q_2 \cdots q_S$
$\vec{\dot{q}}$	heat flux per unit area
$\underline{\vec{\dot{q}}}$	total heat flux
q_j	j^{th} difference eigenvalue
R	total number of temperature nodes in problem including fluid and boundary nodes
r	$K\Delta\theta/(\Delta x)^2 = \Delta\tau/(\Delta\xi)^2$ = difference modulus
r_{ij}	length of perpendicular bisector between i^{th} and j^{th} nodes
S	total number of temperature points whose temperatures are to be found as functions of time; dimensions of Y matrix and number of elements in temperature vector. Note that in Chapter IV, for methods based on mesh $\Delta\xi$, S is number of subdivisions, which does not always conform to number of active temperature points.
S_1	variable points in ξ direction, regular network, two-dimensional problem

S_2	variable points in η direction, regular network, two-dimensional problems
s	Laplace transform complex variable
T	continuous temperature
$T(\xi, \eta, \tau)$	exact continuous solution temperature, two-dimensional problem
$T(\xi, \tau)$	$T(m\Delta\xi,\ n\Delta\tau)$ = continuous solution, one-dimensional problem
$T(\tau)$	S-dimensional exact continuous solution vector for certain temperature points
$T_f(\tau)$	continuous fluid temperature
t_n	vector of difference temperatures at end of n^{th} time increment
$t_{i,n}$ or $t_{m,n}$	difference temperature of i^{th} or m^{th} node
$t_{f,n}$	temperature of fluid as used in difference calculation
U	S x S diagonal matrix used to reduce bound on $\lvert\lambda_{min}\rvert$
U	total number of off-diagonal elements above diagonal in Y/A matrix, $= \sum_{i=1}^{S} u_i$
$\bar{u}(z)$	$z^{p-1}\ \bar{f}(z)$ = z-transform modified so that z^n occurs in numerator
u_i	number of off-diagonal elements above diagonal in i^{th} row of Y/A matrix

$\bar{u}$ — average number of off-diagonal elements above diagonal in Y/A matrix, = U/S

V — volume or dimensionless volume

V — bound on elements in error vector

$v_{m,n}$ — $T_{m,n} - t_{m,n}$ = element in error vector v_n

v_n — $T_n - t_n$ = error vector, difference between continuous solution vector and approximate solution vector

W — $D \frac{Y}{A} D$ = S x S symmetric matrix with same norms, diagonal elements, and eigenvalues as Y/A; off-diagonal element is $(y_{ij}/\sqrt{A_i A_j})$; i^{th} diagonal element is $\frac{-1}{A_i} \sum_{\substack{j=1 \\ j \neq i}}^{R} y_{ij}$

x — x-direction in Cartesian coordinates

Y — S x S matrix of dimensionless conductances; off-diagonal element is y_{ij}; i^{th} diagonal element is $- \sum_{\substack{j=1 \\ j \neq i}}^{R} y_{ij}$

$\frac{Y}{A}$ or Y/A — $A^{-1} Y$ = S x S matrix used to approximate Laplacian operator; off-diagonal element in i^{th} row, j^{th} column (y_{ij}/A_i); i^{th} diagonal element is $- \frac{1}{A_1} \sum_{\substack{j=1 \\ j \neq i}}^{R} y_{ij}$

$\frac{Y_B}{A}$ or Y_B/A — $A^{-1} Y_B$ = matrix of thermal conductances to nodes of known temperature as a function of time; element in i^{th} row, j^{th} column (y_{ij}/A_i) $\quad i = 1, \ldots, S$

$j = S+1 \ldots R$

y — y-direction in Cartesian coordinates

y_{ij} — dimensionless conductance between nodes i and j

y_{i,f_i} — dimensionless conductance between i^{th} adjacent node and i^{th} fluid node

z — complex transform variable for difference equations

Subscripts

Ave — referring to average

B — referring to boundary

E — effective initial vector or initial vector component for graphical solution

Ex — explicit calculation, $\gamma = 0$

f — referring to surrounding fluid, f_0 at boundary 0, f_S at boundary S

f_i — refers to fluid opposite adjacent node i

G — graphical solution, $\gamma = 0$, $r = \frac{1}{2}$

H — homogeneous solution

I — referring to interpolation

Im — implicit calculation, $\gamma \neq 0$

i — referring to i^{th} row in matrix; i^{th} temperature node or i^{th} diagonal position in diagonal matrix; index of summation; i^{th} component of vector

i — refers to interface in equations V-88 and V-89

J — last significant term in transient solution

Subscripts

j	refers to j^{th} column in matrix; j^{th} element in vector; j^{th} diagonal position in diagonal matrix; index of summation; and j^{th} temperature node adjacent to i^{th}
k	refers to averaged solution, $t_{n+k\ Ave}$, time at which applied is $(n+k)\Delta\tau;0 \leq k \leq 1$
m	refers to temperature point in one dimension in a regular mesh where m has relationship to spatial position, i.e. in Figure IV-2 $\xi = m\Delta\xi$, $m = 0, \frac{1}{2}, 1, \ldots, S-\frac{1}{2}, S$ depending on case
n	subscripted quantity applies at end of n^{th} time increment
P	particular solution
p	index of summation, summation variable
R	the R^{th} point
RS	refers to ramp-step modification
S	the S^{th} point
SS	steady state
0, 1, 2	zero, 1st, 2nd, etc. point, or time increment
I	refers to solid I with thermal properties k_I and $(\sigma C_p)_I$
II	refers to solid II with thermal properties k_{II} and $(\sigma C_p)_{II}$

Greek Letters

α	trigonometric parameter in difference characteristic equation
α	geometric parameter for trapezoidal nodes
β	weighting given to energy in from fluid at time (n+1)
β_{ij}	angle between positive ξ -axis and the connector ℓ_{ij}
Γ	closed curve for integration in complex plane
γ	weighting given to temperature vector at t_{n+1} in implicit formulation, $0 \leq \gamma \leq 1$
γ_o	$\frac{(6r-1)}{12r}$ or $\frac{1}{2} - \frac{1}{12S^2 \Delta \tau}$
Δ	difference operator
δ	radius of circles about branch points in complex plane, Figure IV-5
$\delta_{j\rho}$	Kronecker delta function = $\begin{cases} 1 & j = \rho \\ 0 & j \neq \rho \end{cases}$
ϵ	small number
ϵ	distance from branch cut in complex plane, Figure IV-5
$\epsilon_{i,n}$	component of round-off error vector at i^{th} node at time n
ζ	dimensionless z-coordinate = z/L
η	dimensionless y-coordinate = y/L
θ	time, hr
θ	angle coordinate for cylindrical coordinates
Λ	diagonal matrix of eigenvalues of Y/A matrix

λ_i	eigenvalue of Y/A matrix
λ_{ij}	distance between nodes i and j in dimensionless coordinates, $= \ell_{ij}/L$
μ	multiplicity of a pole
μ_{ij}	element in i^{th} row j^{th} column of Y/A matrix
μ_{00}	diagonal element of Y/A matrix with minimum absolute value
$\mu_{\rho\rho}$	diagonal element of Y/A matrix with maximum absolute value
μ_j	$S\alpha_j = \alpha_j/\Delta\xi$ = trigonometric parameter root in characteristic equation for approximate solution in continuous form
ν_j	trigonometric parameter root of characteristic equation for continuous solution
ν_{ij}	element in i^{th} row and j^{th} column of inverse conductance matrix Y^{-1}
$e^{-\nu^2}$	diagonal matrix with $e^{-\nu_j^2}$ as the j^{th} diagonal element
Φ	scaling term, $\underset{m_1, m_2}{MAX} \lvert t_{m_1,0} - t_{m_2,0} \rvert$ where m_1 and m_2 take on all values of mesh and fluid temperature flux
φ	weighting given to boundary node heat loss to interior node, Chapter IV
φ_j	elements in F diagonal matrix
$\varphi_{i,n}$	part of discretization error associated with time differencing

Greek Letters

Ψ	angle parameter for solution of steady-state problem, equation V-61
ψ_j	$\nu_j \Delta \xi = \nu_j / S$ = trigonometric parameter in characteristic equation for continuous solution in difference form
ξ	dimensionless x-coordinate = x/L
π	ratio circumference to diameter of circle = 3.1416 ...
ρ	dimensionless radial coordinate in cylindrical coordinates
ρ_j	residue at j^{th} pole
ρ_{ij}	length of perpendicular bisectors between nodes i and j in dimensionless coordinates, $= r_{ij}/L$
ρ'_{ij}	solution to system of equations V-77 to V-80, length to replace ρ_{ij} to eliminate zero-order error terms
σ	specific weight, lb/ft^3
$\sigma(\xi)$	weighting factor in orthogonal function relationship
τ	$K\theta/L^2$ = dimensionless time
Υ	allowable limit on absolute value of q_{min} or upper bound on q_{min}
$\Omega(\xi, \tau)$	$\frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)} \cos(2j+1)\pi\xi \; e^{-(2j+1)^2 \pi^2 \tau}$, function in continuous solution for specified surface temperatures
ω	angular frequency, radians/unit time
ω_0	frequency in difference notation = $\omega / \Delta\tau$

Miscellaneous Symbols

$\rightarrow$	designates a physical vector
$\vec{\nabla}$	operator del, gradient of scalar
$\vec{\nabla}\cdot$	divergence operator
$\vec{\nabla}'$	$L\ \vec{\nabla}\cdot$ = del operator in dimensionless coordinates
∇^2	Laplacian operator $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
∇'^2	$L^2\ \nabla^2$ = Laplacian operator in dimensionless coordinates ξ and η
∞	infinity
$'$	applied to a matrix means its transpose
$\sum_{j=p}^{S}$	summation on j from p to S
$\int$	integral of
∂	partial differential operator
$\mathcal{I}$	imaginary part of a complex number
$\mathcal{L}$	Laplace transform of
$\mathcal{Z}$	z-transform of
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\equiv$	is defined as
$\simeq$	approximately equal to
$\rightarrow$	approaches
$\not\rightarrow$	does not approach
$\mid\ \mid$	absolute value of
$\|\ \|$	norm of

Miscellaneous Symbols

cos	cosine of
ctn	cotangent of
det	determinant of
dia...	diagonal matrix = matrix whose elements are all zero except those along the diagonal
sin	sine of
tan	tangent of
ln	natural logarithm of
MAX	maximum of
min	minimum of