

TRANSIENT HEAT TRANSFER IN SOLIDS

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## ABSTRACT

The accuracy of approximate calculations of transient temperature distributions in solids is studied, emphasizing easily applied methods for both irregular and regular solids with a heat-transfer coefficient boundary condition. General stability criteria are shown which are easily found and used, and which apply for regular or asymmetric networks.

Analytic solutions of the one-dimensional partial difference equation are found by z-transforms. A method with points located half an increment inside the surface gives the most accurate solutions. A graphical method is shown analytically to give approximations of engineering accuracy. A procedure is also presented for finding the differencing parameters which give a solution of a specified accuracy with the minimum calculations.

Transient temperature distributions of satisfactory accuracy can be calculated for irregular solids using a relatively coarse network, by locating points that describe the boundary a short distance within, rather than on, the boundary, and by making the interior network as regular as possible.

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## NOMENCLATURE

A	S x S diagonal matrix with dimensionless thermal capacity along diagonal
$A_i$	$\frac{1}{4L^2 \sigma_0 C_{p0}} \sum_{\substack{j=1 \\ j \neq i}}^S r_{ij} \ell_{ij} \sigma_i C_{pi}$ , dimensionless thermal capacity of $i^{\text{th}}$ node; also $i^{\text{th}}$ diagonal element of A matrix
$A_{f,i}$	surface area between fluid and node i through which energy flows
a	vector of continuous Fourier coefficients determined to fit continuous initial condition $T(\xi, 0)$ , $0 \leq \xi \leq 1$
$a_j$	$j^{\text{th}}$ continuous Fourier coefficient in vector a
B	continuous eigenvector arranged in matrix
$b_j(x)$	$j^{\text{th}}$ orthogonal function to fit continuous boundary conditions
C	matrix whose columns are eigenvectors of Y/A matrix
$C_p$	heat capacity, Btu/°F-lb
$c_i$	$i^{\text{th}}$ column eigenvector of difference system $\lambda_i c_i = \frac{Y}{A} c_i$
D	S x S diagonal matrix with $1/A_i$ in diagonal position
D(z)	part of denominator of z-transform that goes to zero as z goes to a pole
D or $D_j$	replacement factor for $S\psi$ or $v$ in approximate characteristic equation

d	differential operator
$d_n$	discretization error vector at time n, defined by equation V-4
$d_{i,n}$	discretization error vector associated with $i^{\text{th}}$ node at time n
E	finite difference operator, $E t_n = t_{n+1}$
e	base of napierian or natural logarithm
$e_n$	round-off error vector occurring at time n, defined by equation V-8
F	constant in undetermined parameter solution
F	diagonal transformation matrix
$F_C$	continuous characteristic equation
$F_D$	characteristic equation for approximate method
$f(\tau)$	function of time
$\bar{f}(z)$	z-transform of difference function, $f_n$
G	constant in solution of boundary value difference equation
g	$C^{-1}(t_0 - t_P, 0) = C^1 A(t_0 - t_P, 0)$ difference initial vector that is vector of coefficients required to fit the initial condition
$g_j$	$j^{\text{th}}$ component of initial vector
$\bar{g}(z)$	z-transform of difference function $g_n$ , used only in z-transform table



H	$hL/k =$ dimensionless heat transfer coefficient
h	heat transfer coefficient, $\text{Btu/hr, ft}^2, ^\circ\text{F}$
I	identity matrix which is a diagonal matrix with 1 in the diagonal positions
J	limit of the integrals as defined in equations IV-194 and IV-196
J	number of terms in transient solution
K	$k/\sigma C_p =$ thermometric conductivity, $\text{ft}^2/\text{hr}$
k	thermal conductivity, $\text{Btu/hr, ft, } ^\circ\text{F}$
L	reference length, thickness of slab for one-dimensional problem
$l_{ij}$	distance between $i^{\text{th}}$ and $j^{\text{th}}$ node
$l_{i, f_i}$	distance along normal to boundary from surface to $i^{\text{th}}$ node
M	minimum of row norm of $Y/A$ , column norm of $Y/A$ , and norm of $W$
N	number of non-zero multiplications to calculate to $\tau_1$
N	number of nodes with non-zero conductances to the $i^{\text{th}}$ node
$N(z)$	part of $z$ -transform that remains analytic as $z$ goes to a pole, i. e., as $D(z)$ goes to zero
n	number of time increments; as a subscript occurring at end of $n^{\text{th}}$ increment

$n$	direction normal to boundary of solid directed toward interior
$P$	constant in solution of boundary value difference equation
$p$	index of summation, an integer
$P$	proportionality constant
$Q$	$S \times S$ diagonal matrix of difference eigenvalues dia $q_1, q_2, \dots, q_S$
$\vec{q}$	heat flux per unit area
$\vec{Q}$	total heat flux
$q_j$	$j^{\text{th}}$ difference eigenvalue
$R$	total number of temperature nodes in problem including fluid and boundary nodes
$r$	$K\Delta\theta/(\Delta x)^2 = \Delta\tau/(\Delta\xi)^2 =$ difference modulus
$r_{ij}$	length of perpendicular bisector between $i^{\text{th}}$ and $j^{\text{th}}$ nodes
$S$	total number of temperature points whose temperatures are to be found as functions of time; dimensions of $Y$ matrix and number of elements in temperature vector. Note that in Chapter IV, for methods based on mesh $\Delta\xi$ , $S$ is number of subdivisions, which does not always conform to number of active temperature points.
$S_1$	variable points in $\xi$ direction, regular network, two-dimensional problem

$S_2$	variable points in $\eta$ direction, regular network, two-dimensional problems
$s$	Laplace transform complex variable
$T$	continuous temperature
$T(\xi, \eta, \tau)$	exact continuous solution temperature, two-dimensional problem
$T(\xi, \tau)$	$T(m\Delta\xi, n\Delta\tau)$ = continuous solution, one-dimensional problem
$T(\tau)$	$S$ -dimensional exact continuous solution vector for certain temperature points
$T_f(\tau)$	continuous fluid temperature
$t_n$	vector of difference temperatures at end of $n^{\text{th}}$ time increment
$t_{i,n}$ or $t_{m,n}$	difference temperature of $i^{\text{th}}$ or $m^{\text{th}}$ node
$t_{f,n}$	temperature of fluid as used in difference calculation
$U$	$S \times S$ diagonal matrix used to reduce bound on $ \lambda_{\min} $
$U$	total number of off-diagonal elements above diagonal in $Y/A$ matrix, $= \sum_{i=1}^S u_i$
$\bar{u}(z)$	$z^{p-1} \bar{f}(z) = z$ -transform modified so that $z^0$ occurs in numerator
$u_i$	number of off-diagonal elements above diagonal in $i^{\text{th}}$ row of $Y/A$ matrix

$\bar{u}$	average number of off-diagonal elements above diagonal in Y/A matrix, = U/S
V	volume or dimensionless volume
V	bound on elements in error vector
$v_{m,n}$	$T_{m,n} - t_{m,n}$ = element in error vector $v_n$
$v_n$	$T_n - t_n$ = error vector, difference between continuous solution vector and approximate solution vector
W	$D \frac{Y}{A} D = S \times S$ symmetric matrix with same norms, diagonal elements, and eigenvalues as Y/A; off-diagonal element is $(y_{ij}/\sqrt{A_i A_j})$ ; $i^{\text{th}}$ diagonal element is $\frac{-1}{A_i} \sum_{\substack{j=1 \\ j \neq i}}^R y_{ij}$
x	x-direction in Cartesian coordinates
Y	$S \times S$ matrix of dimensionless conductances; off-diagonal element is $y_{ij}$ ; $i^{\text{th}}$ diagonal element is $-\sum_{\substack{j=1 \\ j \neq i}}^R y_{ij}$
$\frac{Y}{A}$ or Y/A	$A^{-1} Y = S \times S$ matrix used to approximate Laplacian operator; off-diagonal element in $i^{\text{th}}$ row, $j^{\text{th}}$ column $(y_{ij}/A_i)$ ; $i^{\text{th}}$ diagonal element is $-\frac{1}{A_i} \sum_{\substack{j=1 \\ j \neq i}}^R y_{ij}$
$\frac{Y_B}{A}$ or $Y_B/A$	$A^{-1} Y_B =$ matrix of thermal conductances to nodes of known temperature as a function of time; element in $i^{\text{th}}$ row, $j^{\text{th}}$ column $(y_{ij}/A_i)$ $i = 1, \dots, S$ $j = S+1 \dots R$

y	y-direction in Cartesian coordinates
$y_{ij}$	dimensionless conductance between nodes i and j
$y_{i, f_i}$	dimensionless conductance between $i^{\text{th}}$ adjacent node and $i^{\text{th}}$ fluid node
z	complex transform variable for difference equations

### Subscripts

Ave	referring to average
B	referring to boundary
E	effective initial vector or initial vector component for graphical solution
Ex	explicit calculation, $\gamma = 0$
f	referring to surrounding fluid, $f_0$ at boundary 0, $f_S$ at boundary S
$f_i$	refers to fluid opposite adjacent node i
G	graphical solution, $\gamma = 0$ , $r = \frac{1}{2}$
H	homogeneous solution
I	referring to interpolation
Im	implicit calculation, $\gamma \neq 0$
i	referring to $i^{\text{th}}$ row in matrix; $i^{\text{th}}$ temperature node or $i^{\text{th}}$ diagonal position in diagonal matrix; index of summation; $i^{\text{th}}$ component of vector
i	refers to interface in equations V-88 and V-89
J	last significant term in transient solution

Subscripts

j	refers to $j^{\text{th}}$ column in matrix; $j^{\text{th}}$ element in vector; $j^{\text{th}}$ diagonal position in diagonal matrix; index of summation; and $j^{\text{th}}$ temperature node adjacent to $i^{\text{th}}$
k	refers to averaged solution, $t_{n+k \text{ Ave}}$ , time at which applied is $(n+k)\Delta\tau$ ; $0 \leq k \leq 1$
m	refers to temperature point in one dimension in a regular mesh where m has relationship to spatial position, i.e. in Figure IV-2 $\xi = m\Delta\xi$ , $m = 0, \frac{1}{2}, 1, \dots, S-\frac{1}{2}, S$ depending on case
n	subscripted quantity applies at end of $n^{\text{th}}$ time increment
P	particular solution
p	index of summation, summation variable
R	the $R^{\text{th}}$ point
RS	refers to ramp-step modification
S	the $S^{\text{th}}$ point
SS	steady state
0, 1, 2	zero, 1st, 2nd, etc. point, or time increment
I	refers to solid I with thermal properties $k_I$ and $(\sigma C_p)_I$
II	refers to solid II with thermal properties $k_{II}$ and $(\sigma C_p)_{II}$

Greek Letters

$\alpha$	trigonometric parameter in difference characteristic equation
$\alpha$	geometric parameter for trapezoidal nodes
$\beta$	weighting given to energy in from fluid at time (n+1)
$\beta_{ij}$	angle between positive $\xi$ -axis and the connector $l_{ij}$
$\Gamma$	closed curve for integration in complex plane
$\gamma$	weighting given to temperature vector at $t_{n+1}$ in implicit formulation, $0 \leq \gamma \leq 1$
$\gamma_o$	$\frac{(6r-1)}{12r}$ or $\frac{1}{2} - \frac{1}{12S^2 \Delta \tau}$
$\Delta$	difference operator
$\delta$	radius of circles about branch points in complex plane, Figure IV-5
$\delta_{j\rho}$	Kronecker delta function = $\begin{matrix} 1 & j = \rho \\ 0 & j \neq \rho \end{matrix}$
$\epsilon$	small number
$\epsilon$	distance from branch cut in complex plane, Figure IV-5
$\epsilon_{i,n}$	component of round-off error vector at $i^{\text{th}}$ node at time n
$\zeta$	dimensionless z-coordinate = $z/L$
$\eta$	dimensionless y-coordinate = $y/L$
$\theta$	time, hr
$\theta$	angle coordinate for cylindrical coordinates
$\Lambda$	diagonal matrix of eigenvalues of Y/A matrix

$\lambda_i$	eigenvalue of Y/A matrix
$\lambda_{ij}$	distance between nodes i and j in dimensionless coordinates, $= l_{ij}/L$
$\mu$	multiplicity of a pole
$\mu_{ij}$	element in $i^{\text{th}}$ row $j^{\text{th}}$ column of Y/A matrix
$\mu_{00}$	diagonal element of Y/A matrix with minimum absolute value
$\mu_{\rho\rho}$	diagonal element of Y/A matrix with maximum absolute value
$\mu_j$	$S\alpha_j = \alpha_j/\Delta\xi =$ trigonometric parameter root in characteristic equation for approximate solution in continuous form
$\nu_j$	trigonometric parameter root of characteristic equation for continuous solution
$\nu_{ij}$	element in $i^{\text{th}}$ row and $j^{\text{th}}$ column of inverse conductance matrix $Y^{-1}$
$e^{-\nu_j^2}$	diagonal matrix with $e^{-\nu_j^2}$ as the $j^{\text{th}}$ diagonal element
$\Phi$	scaling term, $\text{MAX}_{m_1, m_2}   t_{m_1, 0} - t_{m_2, 0}  $ where $m_1$ and $m_2$ take on all values of mesh and fluid temperature flux
$\varphi$	weighting given to boundary node heat loss to interior node, Chapter IV
$\varphi_j$	elements in F diagonal matrix
$\varphi_{i,n}$	part of discretization error associated with time differencing



Greek Letters

$\Psi$	angle parameter for solution of steady-state problem, equation V-61
$\psi_j$	$v_j \Delta \xi = v_j / S =$ trigonometric parameter in characteristic equation for continuous solution in difference form
$\xi$	dimensionless x-coordinate = $x/L$
$\pi$	ratio circumference to diameter of circle = 3.1416 ...
$\rho$	dimensionless radial coordinate in cylindrical coordinates
$\rho_j$	residue at $j^{\text{th}}$ pole
$\rho_{ij}$	length of perpendicular bisectors between nodes i and j in dimensionless coordinates, = $r_{ij}/L$
$\rho'_{ij}$	solution to system of equations V-77 to V-80, length to replace $\rho_{ij}$ to eliminate zero-order error terms
$\sigma$	specific weight, lb/ft <sup>3</sup>
$\sigma(\xi)$	weighting factor in orthogonal function relationship
$\tau$	$K\theta/L^2 =$ dimensionless time
$Y$	allowable limit on absolute value of $q_{\min}$ or upper bound on $q_{\min}$
$\Omega(\xi, \tau)$	$\frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)} \cos(2j+1)\pi\xi e^{-(2j+1)^2 \pi^2 \tau}$ , function in continuous solution for specified surface temperatures
$\omega$	angular frequency, radians/unit time
$\omega_0$	frequency in difference notation = $\omega / \Delta \tau$

Miscellaneous Symbols

$\rightarrow$	designates a physical vector
$\vec{\nabla}$	operator del, gradient of scalar
$\vec{\nabla} \cdot$	divergence operator
$L \vec{\nabla} \cdot$	$L \vec{\nabla} \cdot$ = del operator in dimensionless coordinates
$\nabla^2$	Laplacian operator = $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
$L^2 \nabla^2$	$L^2 \nabla^2$ = Laplacian operator in dimensionless coordinates
	$\xi$ and $\eta$
$\infty$	infinity
$'$ $S$	applied to a matrix means its transpose
$\sum_{j=p}$	summation on j from p to S
$\int$	integral of
$\partial$	partial differential operator
$\mathcal{I}$	imaginary part of a complex number
$\mathcal{L}$	Laplace transform of
$\mathcal{Z}$	z-transform of
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\equiv$	is defined as
$\approx$	approximately equal to
$\rightarrow$	approaches
$\nrightarrow$	does not approach
$  \quad  $	absolute value of
$\  \quad \ $	norm of

Miscellaneous Symbols

cos	cosine of
ctn	cotangent of
det	determinant of
dia...	diagonal matrix = matrix whose elements are all zero except those along the diagonal
sin	sine of
tan	tangent of
ln	natural logarithm of
MAX	maximum of
min	minimum of