NEW RESULTS ON PARAUNITARY FILTER BANKS: ENERGY COMPACTION PROPERTIES, LINEAR PHASE FACTORIZATIONS AND RELATION TO WAVELETS

Thesis by

Anand Soman

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1993

(Defended April 19, 1993)

ACKNOWLEDGEMENT

It is a great pleasure to acknowledge all those individuals without whose support I would not be able to complete this thesis. First and foremost, I would like to thank my advisor, Dr. P. P. Vaidyanathan, for his valuable guidance. He has always been available to offer advice, encouragement and support. I have always admired his intuition and vast knowledge, and I consider myself fortunate to have had an opportunity to work with him. I also appreciate his generous funding over the years. I am indebt to Profs. McEliece, Posner, Simon, Abu-Mostafa, and Beck not only for the courses they taught, but also for sitting on my thesis/candidacy committees. My interaction with my collegues, particularly Tsu-han, Igor, See-May, Ian, Vinay and Truong has been very enriching. I would also like to thank Debbie McGoughan and Danielle Sanine in the EE Dept for their help.

Finally, I must mention that I owe a lot to the sacrifice, support and prayers of my family.

ABSTRACT

Subband coding schemes have been widely used to encode signals from speech, high quality audio, and image sources. The theory of perfect reconstruction filter banks has also been studied extensively. The purpose of this thesis is to study the properties of the so-called paraunitary systems, and issues pertaining to their applications and implementations.

We will begin by proving several properties of paraunitary filter banks. For example, we will prove that all orthonormal discrete-time wavelets can be generated using paraunitary binary trees. We will also extend this result to arbitrary tree-structures and wavelet packets. Next, we will address the two issues involved in the design of a paraunitary subband coding system. 1) the problem of optimal bit allocation among various channels given a fixed bit-rate, and 2) the problem of finding the optimal filter bank (by optimization) to encode a given signal. We will prove several interesting results in this regard. We will then show how generalized polyphase representations can be used to enhance the coding gain of transform coding systems.

In practical applications, one often imposes several other conditions on the individual filters in a filter bank. For example, the linear phase property is found to be important for encoding image signals, whereas the 'pairwise mirror-image' property generally yields filters with better responses and, therefore, better frequency selectivity. The final part of the thesis deals with the implementions of paraunitary systems having such additional properties. We will obtain factorizations for such systems which will be proved to be minimal as well as complete. These factorizations yield structures which are robust, i.e., all the desired properties are retained in spite of coefficient quantization.

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

Transform coding and subband coding are well-known techniques for efficiently encoding data [1]-[5]. They are used in compressing data from speech, high quality audio, image, and other sources. Consider the subband coding scheme shown in Fig. 1.1(a). In this scheme, the input signal $x(n)$ is split into M subbands in the frequency domain by a bank of filters called the analysis filters. The filters typically have responses as shown in Fig. 1.1(b). The outputs of these filters are therefore bandlimited, and hence we can sub-sample them. This is indicated by boxes with $\downarrow M$. The signals in each of the subbands are then processed according to the particular application at hand. At the receiving end, the sampling rates in each of the subbands are increased to their original value by the expanders (indicated by $\uparrow M$), which introduce $M-1$ zero valued samples between each non-zero sample at their input. The signals are then passed through the synthesis filters, whose magnitude responses typically resemble those of the analysis filters. The outputs of the synthesis filters are combined to give the reconstructed signal $y(n)$.

The problem of perfect reconstruction is to choose the analysis and synthesis filters in such a way that (in the absence of subband processing,) $y(n) = x(n) \quad \textit{for all } n.$ The problem of perfect reconstruction is considerably simplified by the polyphase matrix representation of the system which is shown in Fig. 1.2. In this figure, $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the polyphase matrices [5] corresponding to the analysis and synthesis filters respectively. This scheme works as follows: The sequence $x(n)$ is divided into

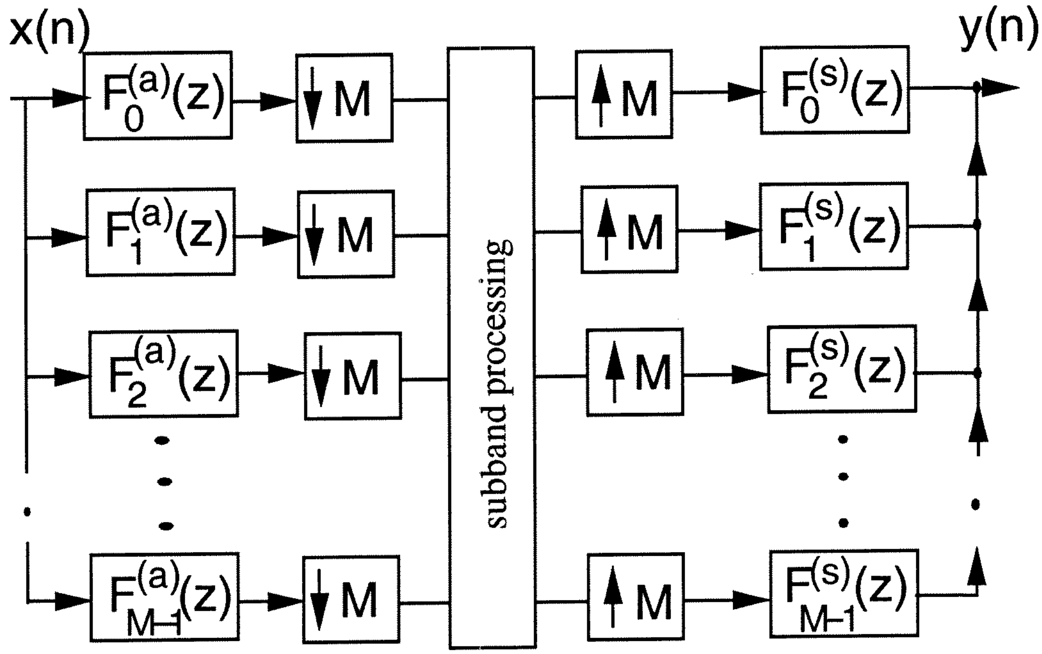


Fig. 1.1(a). A typical subband coding scheme

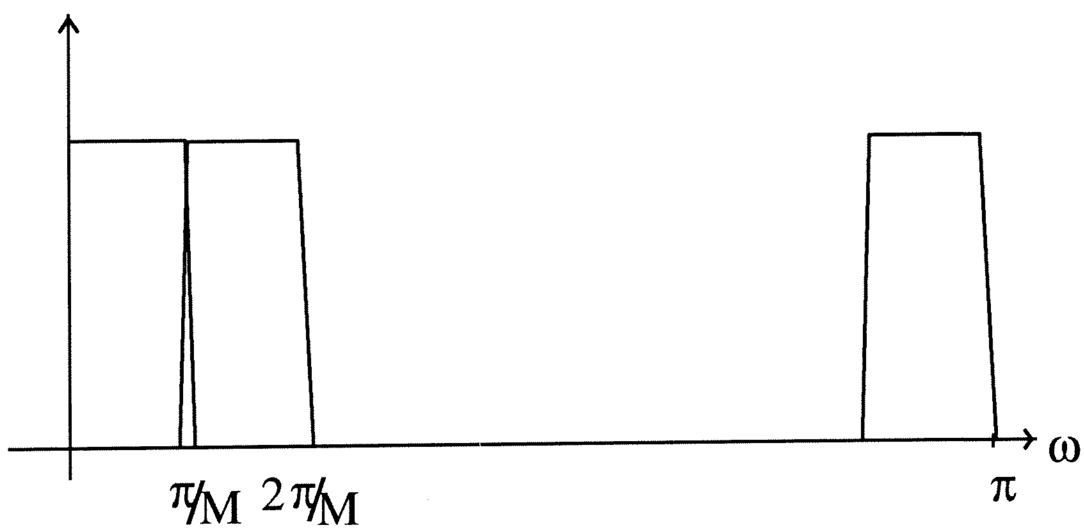


Fig. 1.1(b). Typical appearances of magnitude responses of the above filter bank

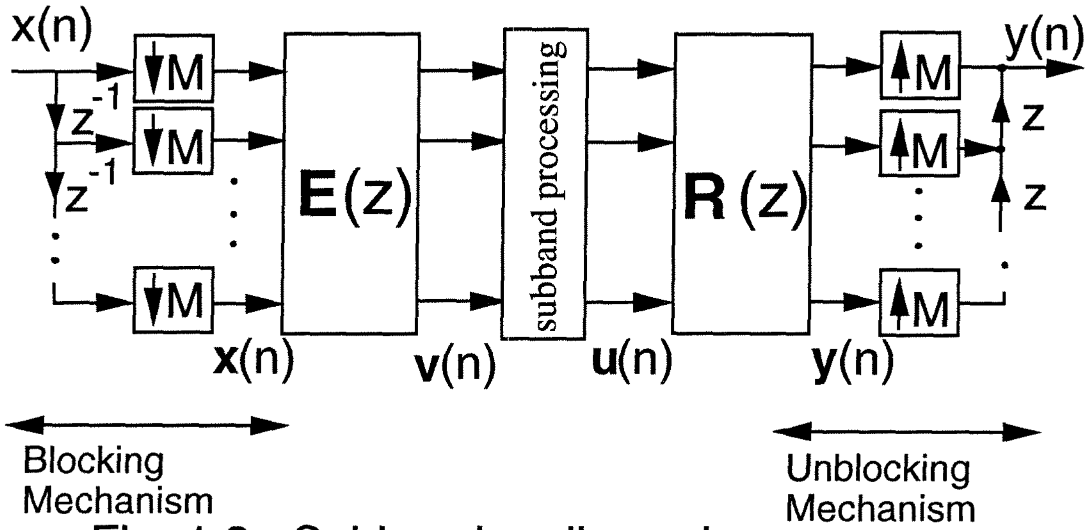


Fig. 1.2. Subband coding scheme showing polyphase matrices

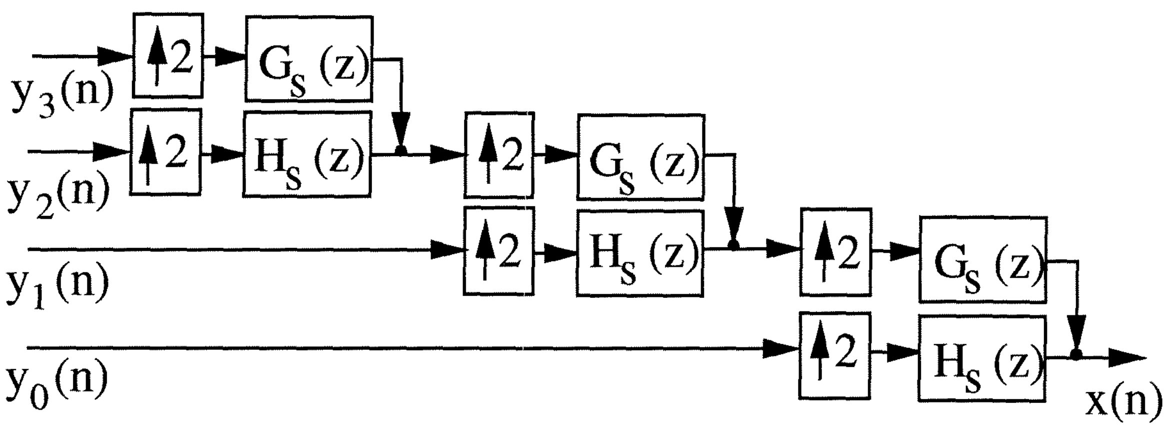


Fig. 1.3. A tree-structured filter bank

non-overlapping blocks of data by grouping together M successive samples from the input stream. These samples form the components of the vector $\mathbf{x}(n)$ which is termed the M-fold blocked version of $x(n)$. Each of these blocks or *vectors* is encoded by the linear transformation $\mathbf{E}(z)$. The M outputs, i.e., the components of the vector $\mathbf{v}(n)$ are the subband signals which are processed. At the receiver, the received vector $\mathbf{u}(n)$ is passed through the transformation $\mathbf{R}(z)$. The output $\mathbf{y}(n)$ is 'unblocked' to give the reconstructed sequence $y(n)$. Any filter bank in Fig. 1.1(a) can be redrawn as in Fig. 1.2, and vice-versa. It is now possible to ensure perfect reconstruction by choosing $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$, and then choosing the matrix $\mathbf{E}(z)$ so that its inverse exists.

While the above approach solves the problem of perfect reconstruction per se, there are some drawbacks to the method. Firstly, $\mathbf{R}(z)$ could be IIR even if $\mathbf{E}(z)$ is FIR. In fact, $\mathbf{R}(z)$ could be unstable. Another problem is that the analysis and synthesis banks could have different complexities as far as implementation is concerned. These problems are overcome by choosing $\mathbf{E}(z)$ to be a paraunitary matrix. A matrix is said to be paraunitary if it satisfies [5]

$$\widetilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I},$$

where $\widetilde{\mathbf{E}}(z) = \mathbf{E}^{\dagger}(1/z^*)$. The system can be guaranteed to have the perfect reconstruction property by having $\mathbf{R}(z) = \widetilde{\mathbf{E}}(z)$. This paraunitary property can be traced back to classical network synthesis [6]. Notice that if $\mathbf{E}(z)$ is chosen to be a FIR paraunitary matrix (such matrices have been shown to exist), we get perfect reconstruction with FIR analysis and synthesis filters.

Organization of the thesis

This thesis is a collection of five papers which address different aspects of the Paraunitary Filter Bank system described above. Four of them (chapters 2,3,4,5) have been accepted for Journal Publication, and one (chapter 6) is under review. Hence each of the chapters is self contained, though they do refer to each other.

Therefore, there is also a slight overlap between the introduction section of these papers.

Consider Fig. 1.3, which shows a tree-structured synthesis filter bank. (There is a corresponding tree-structured analysis filter bank. Also, notice that the uniform subband coding system described in the previous section is in fact a 'One-level' tree.) One can express the sequence $x(n)$ in terms of the sequences $y_k(n)$ as

$$x(n) = \sum_{k=0}^{L} \sum_{m} y_k(m) \eta_{km}(n).$$

The $\eta_{km}(n)$ are called the basis functions, whereas the $y_k(m)$ are the wavelet coefficients. Clearly, the basis functions are related to the synthesis filters in the tree-structure. The relation between paraunitariness of filters and orthonormality of wavelets is known to some extent [7]. In chapter 2, we will further explore this relationship. We will prove several properties of paraunitary systems. The first theorem we will prove states that a binary tree structure with paraunitary matrices on all levels (possibily different) generates orthonormal discrete time wavelets. Furthermore, all orthonormal discrete-time wavelets can be generated using paraunitary binary trees. The other results in this chapter extend these theorems to arbitrary (non-binary) trees, and the resulting basis functions. The bases resulting from such arbitrary tree structures have been referred to as 'wavelet packets' in literature [8].

In chapter 3, we consider one particular application of paraunitary filter banks, namely, subband coding of signals. In this application, we quantize and transmit (or store) each of the subbands. Real life signals often have an unequal distribution of energy across different subbands. In particular, some subbands have most of the energy. This in fact makes data-compression possible. Non-paraunitary subband coding systems have been studied in literature [2],[3],[4], and they have also been used to compress data. The first problem we consider is that of allocating bits to the different subbands from a fixed budget so as to minimize the mean-square error. This problem has been solved by other authors [1], [4] for the two special cases of transform coding and ideal brick-wall filtering. In chapter 3, have obtained bit-allocation results for the general case of non-uniform paraunitary filter banks. Our results also give us bounds on the overall reconstruction error in terms of the quantization errors in each subband, no matter what the frequency responses of the filters are. Furthermore, our results do not assume that the quantization errors are white or uncorrelated. The next problem we address is that of choosing the optimal filter bank to encode a given signal. For the case of transform coding, it is well-known that the optimal transform to encode a signal is the Karhunen-Loeve Transform (KLT) [4]. We will show how one can choose (by optimization) a filter bank to minimize the mean-square error i.e., maximize the "coding gain" given the statistics of the input signal. We will also present simulation results based on the statistics of low-pass and band-pass speech.

In chapter 4, we will discuss the 'generalized polyphase representation' (GPP). The traditional M-fold polyphase representation of a transfer function $H(z)$ is given by

$$H(z) = \sum_{i=0}^{M-1} h_i(z^M) z^{-i},$$

where the $h_i(z)$ are referred to as the M polyphase components of $H(z)$. The right hand side of the above equation is a linear combination of functions $z^{-i}, \quad i = 0, \ldots, M-1$, with the weighting factors being functions of z^M. A natural question which arises is whether an arbitrary transfer function $H(z)$ may be written as a linear combination of functions other than z^{-i} and, furthermore, are there any advantages to be gained by using a different set of functions? In this chapter, we first provide a complete characterization of valid polyphase representations. We will then study an application of the GPP, namely in enhancing the coding gain of transform coding systems. We will prove several interesting properties in this regard.

In chapter 5, we will discuss the linear phase filter bank. In applications to image processing, linear phase property of the filters in the filter bank is found to be important. We begin by answering several theoretical questions pertaining to linear phase paraunitary systems. Next, we develop a *minimal and comlete factorization* for such systems. Further, we *structurally* impose the additional condition that the filters satisfy pairwise mirror-image symmetry in the frequency domain. Imposing this condition significantly reduces the number of parameters to be optimized in the design process. Our factorizations give robust structures for implementing filter banks, i.e., all the desired properties are retained in spite of coefficient quantization. We then demonstrate the use of these filter banks in the generation of M-band orthonormal wavelets. Finally, we use the linear phase paraunitary system to encode signals and provide a comparison with traditional techniques. Several design examples are also given to validate the theory.

In chapter 6, we will discuss minimal and complete factorizations of paraunitary filter banks with filters whose responses have pairwise mirror-image symmetry about $\pi/2$ in the frequency domain. Imposition of this condition has been found to result in filters with better responses, and hence better frequency selectivity.

REFERENCES

[1] J. J. Huang and P. M. Schultheiss, "Block Quantization of Correlated Gaussian Random Variables," IEEE Trans. on Comm. Syst., pp. 289-296, Sept. 1963.

[2] Crochiere, *Multirate Digital Signal Processing,* Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1983.

[3] C. Galand and D. Esteban, "16 Kbps real-time QMF sub-band coding implementation," Proc. Int. Conf. on ASSP, Denver, CO, pp. 332-335, Apr. 1980.

[4] N. S. Jayant and P. Noll, *Digital Coding of Waveforms- Principles and Applications to Speech and Video,* Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1984.

[5] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1993.

[6] V. Belevitch, *Classical Network Theory*, Holden Day, San Fransisco, 1968.

[7] I. Daubechies, "Orthonormal Bases of Compactly Supported Wavelets," Comm. on Pure and App. Math., pp. 909-996, vol. 4, Nov. 1988.

[8] R. Coifman, Y. Meyer, S. Quake and V. Wickerhauser, "Signal Processing with Wavelet Packets," Numerical Algorithms Research Group, Yale University, 1990.

Chapter 2

ON ORTHONORMAL WAVELETS AND PARAUNITARY FILTER BANKS

Abstract[1]

Binary tree-structured filter banks have been employed in the past to generate wavelet bases. It is known that a binary tree-structured filter bank with the same paraunitary polyphase matrix on all levels generates an orthonormal basis. First, we generalize the result to binary trees having different paraunitary matrices on each level. Next, we prove a converse result; that every orthonormal wavelet basis can be generated by a tree-structured filter bank having paraunitary polyphase matrices.

We then extend the concept of orthonormal bases to generalized (i.e., non-binary) tree-structures, and see that a close relationship exists between orthonormality and paraunitariness in this case too. We prove that a generalized tree-structure with paraunitary polyphase matrices produces a orthonormal basis. Since not all bases can be generated by tree-structured filter banks, we prove that if an orthonormal basis can be generated using a tree structure, it can be generated specifically by a paraunitary tree.

I INTRODUCTION

Recently, wavelet transforms have evoked considerable interest in the signal-processing community. They have found applications in several areas such as speech coding, edge-detection, data-compression, extraction of parameters for recognition and di-

[1]Appeared, IEEE Trans. on Signal Processing, March, 1993.

agonostics, etc. [1]-[3]. Since wavelets provide a way to represent a signal on various degrees of resolution, they are a convenient tool for analysis and manipulation of data. In [4], Mallat describes a mutiresolution algorithm for decomposing and reconstructing images. In [2], Mallat and Hwang have shown that the local maxima of the wavelet transform detect the location of irregular structures. They have also shown that it is possible to reconstruct one and two dimensional signals from the local maxima of their wavelet transform. Applications of wavelets to sub-band speech and image coding techniques can be found in [5]-[7]. Wavelets can also be used in the detection of transient signals [6]. Orthonormality is a very desirable property in several of these applications and, indeed, the problem of generating orthonormal wavelets is of considerable interest.

The theory of wavelets was originally developed in the context of continuous time functions [8], [9]. It has since been related to the familiar idea of Quadrature Mirror Filter (QMF) banks. Continuous time wavelets can be obtained from infinite-level binary tree-structured QMF banks, with the *same* filters on each level [10]. This infinite recursion gives rise to two continuous time functions $\psi(t)$ and $\phi(t)$ which are termed as the wavelet function and the scaling function respectively. The wavelet basis is then obtained by dyadic scaling and shifting of the wavelet functon $\psi(t)$. It has been shown [10] that if this basis of continuous time functions is orthonormal, then the *single* QMF pair used to generate them is paraunitary [11].

Subsequently, the notion of wavelets has been extended to discrete time. This is more suitable in a number of signal processing applications. However, there appears to be no universal definition of wavelets in discrete time. Some authors have referred to a one-level paraunitary filter bank as wavelet transforms. This definition is, however, too restrictive. Probably the definition which best captures the notion of wavelets in discrete-time is the idea of having a binary tree with a finite number of levels, simultaneously allowing *different filters on each level*. This definition is fairly general and is also useful from a practical viewpoint. We shall subscribe to this definition in this paper. The idea of wavelets in discrete time therefore reduces to that of a filter bank with dyadically increasing decimation ratios. This idea of a filter bank with dyadic decimation ratios can be generalized to filter banks with non-uniform, non-dyadic decimation ratios. The basis functions corresponding to such non-uniform filter banks have been referred to as wavelet packets [1], [12]. One of the ways to realize such non-uniform filter banks is by using general tree structures.

Given the importance of orthonormal wavelets and wavelet packets in several applications, it becomes natural to seek necessary and sufficient conditions under which these discrete time basis functions are orthonormal. While the relation between orthonormal bases and paraunitary filter banks is known to some extent, there are some extensions which are either not known or not published so far. There also appears to be no published work which can serve as a comprehensive reference for the generalized orthonormal wavelet bases (wavelet packets) and paraunitary filter banks. The aim here is to present a complete study of this relation. The following are the main points of this paper:

1. *Paraunitariness implies wavelet orthonormality:* It is known that if a binary tree is constructed using the same paraunitary block on each level, the resulting discrete time basis is orthonormal. A straightforward extension is that the discrete time basis continues to be orthonormal even if different paraunitary blocks are used on each level (Theorem 1).

2. *Orthonormality implies paraunitariness:* We prove that every orthonormal wavelet basis can be generated using binary tree structured filter banks with paraunitary building blocks (Theorem 2). The proof also shows how we can synthesize the tree, i.e., we can identify the filter pair on each level of the tree, starting from the given orthonormal basis. Furthermore, if a orthonormal wavelet basis is generated using a binary tree, the filters on each level have to be generalised

paraunitary (i.e., parauitary, except for constant scaling) (Corollary 1). These results allow us to generate *all* orthonormal wavelet bases simply by manipulating the coefficients of a set of lattice structures.

3. *Orthonormality of wavelet packets:* We develop the concept of orthonormality for non-uniform filter banks. In particular, we show that if $f_k(n)$ and $f_l(n)$ are two of the basis functions, then the orthonormality condition can be written as

$$\sum_{n=-\infty}^{\infty} f_k(n) f_l^*(n - g_{lk} i) = \delta(k-l)\delta(i).$$

Here, g_{lk} is the *gcd* of (I_k, I_l), the decimation factors corresponding to the two filters. The fact that the *gcd* is involved in the definition has not been brought to attention before. We also prove that if a set of wavelet packets is realized using a general tree structure with paraunitary matrices on each level, the resulting basis is orthonormal (Theorem 3). Since not all bases can be generated using tree structures, the exact converse of this result is not true, unlike the binary case. However, if an orthonormal basis can be generated using a tree structure, we show that it can be generated specifically by a tree having paraunitary filters on each level (Theorem 4). This establishes the relation between paraunitariness and orthonormality in the case of wavelet packets.

Notations Bold-faced quantities denote matrices and vectors, as in $\mathbf{A}$ and $\mathbf{x}$. $\mathbf{A}^T$ denotes the transpose of the matrix $\mathbf{A}$. A superscript asterisk as in $f^*(n)$ denotes conjugation. The tilde-notation as in $\widetilde{\mathbf{H}}(z)$ stands for conjugation of coefficients followed by transposition followed by replacing z by z^{-1}. Consider a transfer function $A(z)$. It can be written in terms of its M polyphase components [11] as follows:

$$A(z) = a_0(z^M) + z^{-1}a_1(z^M) + \ldots + (z^{-M+1})a_{M-1}(z^M). \tag{1.1}$$

This is known as Type I polyphase. Let $H_i(z),\ \ i = 0, \ldots, M-1$, be a set of analysis filters. They can be written as

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{kl}(z^M) \qquad k = 0, \ldots, M-1.$$

The matrix $\mathbf{E}(z) = [E_{k,l}]$ is called the polyphase matrix of the analysis filters. There is also a Type II polyphase representation which is as follows:

$$A(z) = z^{-M+1} a'_0(z^M) + z^{-M+2} a'_1(z^M) + \ldots + a'_{M-1}(z^M). \tag{1.2}$$

Let $F_i(z), \quad i = 0, \ldots, M-1$, be a set of synthesis filters. They can be written as

$$F_k(z) = \sum_{l=0}^{M-1} z^{-M+1+l} R_{lk}(z^M) \qquad k = 0, \ldots, M-1.$$

The matrix $\mathbf{R}(z) = [R_{l,k}]$ is called the polyphase matrix of the synthesis filters.

If $a(n)$ is the inverse transform of $A(z)$, then $a(Mn)$ is called the M-fold decimated version of $a(n)$. In the z-domain, we use a downward arrow to denote the decimation operator; for example, $(A(z))\downarrow_M$ is the z-transform of the M-fold decimated version of $a(n)$. A matrix $\mathbf{E}(z)$ is said to be paraunitary if it satisfies [11]

$$\widetilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I} \quad \textit{for all } z. \tag{1.3}$$

Given a set of M filters $H_k(z), \quad k = 0, \ldots, M-1$, we can define an M by M polyphase matrix for these filters as in [11]. We say that the set of filters forms a paraunitary set (abbreviated PU-set) if their polyphase matrix is paraunitary. In particular, two filters with a paraunitary polyphase matrix are said to form a PU-pair.

The abbreviation '*gcd*' stands for the greatest common divisor. The abbreviation FIR stands for finite impulse response.

II PRELIMINARIES

In this section we shall develop the background useful for dealing with the remainder of the paper. Most of what is presented in this section can be inferred from the work of Daubechies [10]. The filter-bank approach to wavelets has also been presented by Vetterli [12], [15] and by Vaidyanathan [16]. Our notation in this paper will be similar to that in [16].

The wavelet transform provides a time-scale representation of a signal which makes it possible to analyze signals on various degrees of resolution. It is a representation of a signal in terms of a peculiar set of orthonormal functions. The peculiarity of this orthonormal family is that it is obtained by shifting and dilating a single function, often termed as the 'mother wavelet.' Let $x(t)$ be the signal under consideration. Mathematically, we can write its wavelet transform as

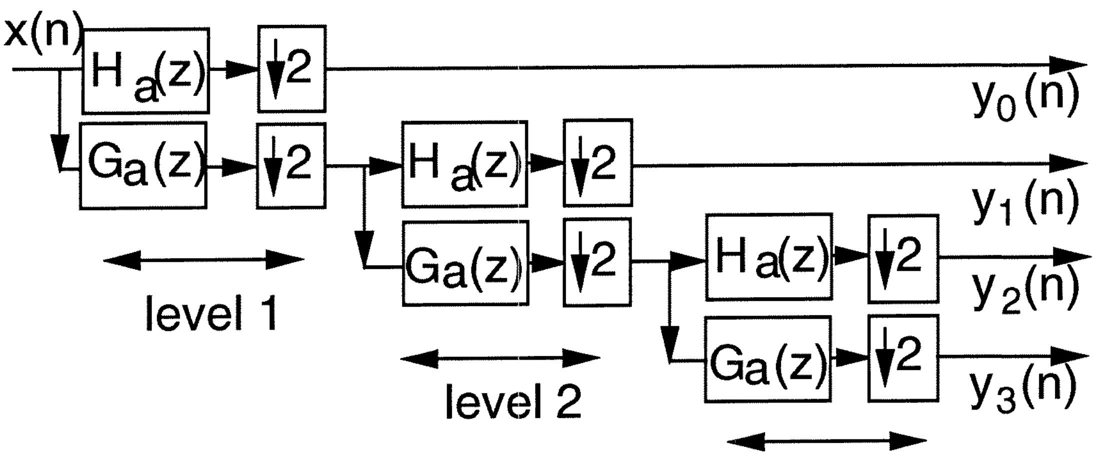
$$X_{CWT}(p,q) = \frac{1}{\sqrt{|p|}}\int_{-\infty}^{\infty} x(t) f(\frac{t-q}{p})dt. \tag{2.1}$$

$X_{CWT}(p,q)$ is referred to as the '*Continuous Wavelet Transform*' (CWT) of the signal $x(t)$. It is called thus because the variables p and q are continuous variables. Note that the family of functions $f(\frac{t-q}{p})$ is generated from a single function $f(t)$ by translations and dilations. 'p' is the dilation parameter, whereas 'q' is the translation parameter. This is a mapping from a one dimensional continuous variable t, to a two dimensional continuous variable (p,q). If we restrict p and q to take discrete values, we obtain a mapping from a one dimensional continuous variable to a two dimensional discrete variable. This is called the *Discrete Wavelet Transform* (DWT).

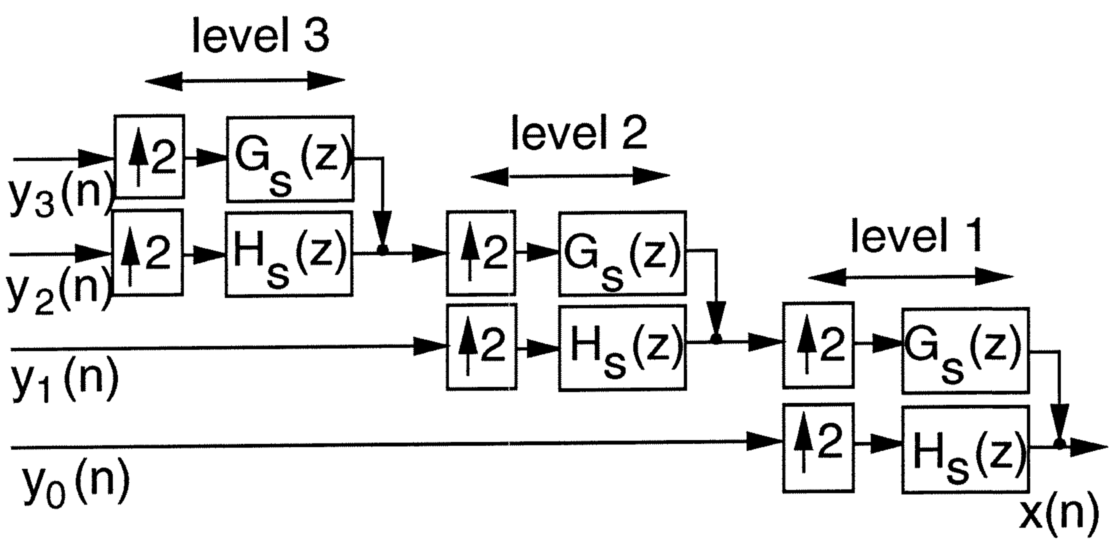
In signal processing literature [15],[16], there has been defined a similar operation as the above for discrete time signals. We say,

$$y_k(n) = \sum_m h_k(m) x(I_k n - m), \quad 0 \leq k \leq L \tag{2.2}$$

is the '*Discrete Time Wavelet Transform*' (DTWT) of the signal $x(n)$. It is common to choose $I_k = 2^{k+1}$, $k = 0, \ldots, L-1$, and $I_L = 2^L$. This is the binary DTWT, often referred to simply as the DTWT [15],[16]. The quantities $y_k(n)$, $k = 0, \ldots, L$ are called the '*Wavelet Coefficients*' of the signal $x(n)$. Eq. (2.2) is a convolution followed by decimation by a factor I_k. The wavelet coefficients can hence be visualized as being obtained by passing the signal through a bank of $L+1$ filters $h_k(n)$, and decimating



(a)



(b)

Fig. 2.1 A tree-structured filter bank used for generating wavelets
(a) The analysis bank

(b) The synthesis bank

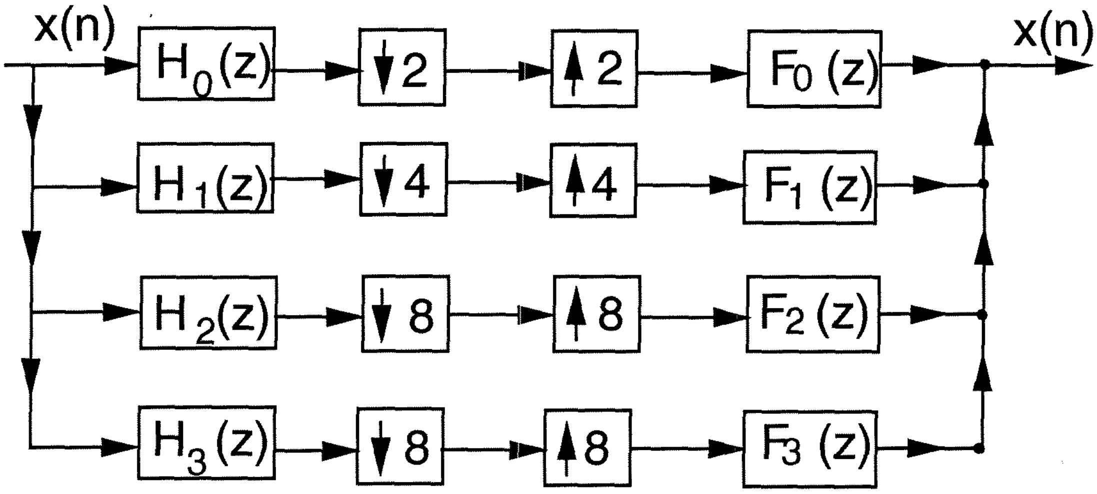


Fig. 2.2(a). A four-channel wavelet filter bank

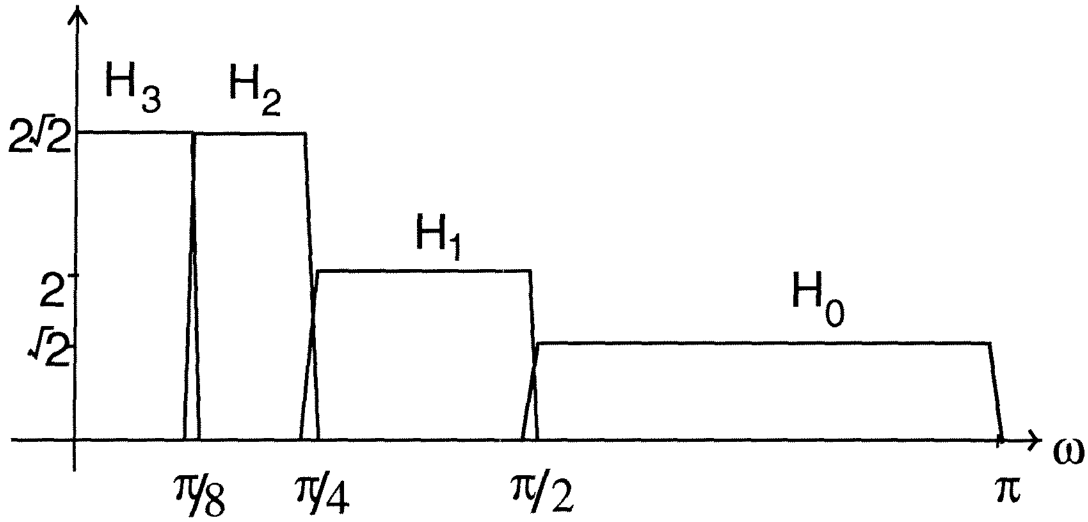


Fig. 2.2(b). Typical appearances of magnitude responses of analysis filters in a three-level tree

the filter outputs by factor I_k. The condition that the filter bank be a maximally decimated system implies that $\sum(1/I_k) = 1$. This viewpoint has been explained in [16]. Specifically, the binary DTWT (which we shall henceforth refer to as DTWT) can be obtained by passing the signal $x(n)$ through a binary tree-structured analysis bank (as shown in Fig. 2.1(a) for the case $L = 3$). Consider the corresponding binary tree-structured synthesis bank shown in Fig. 2.1(b). It is well-known [13] that it is possible to design such perfect-reconstruction tree-structured filter-banks. For the perfect reconstruction system, the signal $x(n)$ can be recovered from its wavelet coefficients as

$$x(n) = \sum_{k=0}^{L}\sum_{m} y_k(m)\eta_{km}(n). \tag{2.3}$$

This is the '*Inverse*' DTWT operation. The $\eta_{km}(n)$ are termed the '*Wavelet Basis Functions.*' The perfect reconstruction binary tree-structured analysis-synthesis system can be redrawn as a traditional filter bank as in Fig. 2.2 (a). Fig. 2.2(b) shows the typical frequency responses of the analysis filters. Note that the amplitudes of the filters increase as the bandwidth decreases, keeping the energy in each of them equal. With reference to Fig. 2.2(a), it can be shown [15], [16] that the synthesis filters are related to the functions $\eta_{km}(n)$ as

$$\eta_{km}(n) = f_k(n - 2^{k+1}m), \quad k = 0, \ldots, L-1 \tag{2.4}$$

$$\eta_{Lm}(n) = f_L(n - 2^{L}m). \tag{2.5}$$

For perfect reconstruction systems, therefore, every signal can be represented in terms of a wavelet transform, and every signal can be recovered from its wavelet coefficients. Note that this is not the case with ordinary Fourier transform. In fact, not all sequences have a Fourier transform!

Orthonormality

The wavelet basis functions in eq. (2.3) are said to be orthonormal if they satisfy

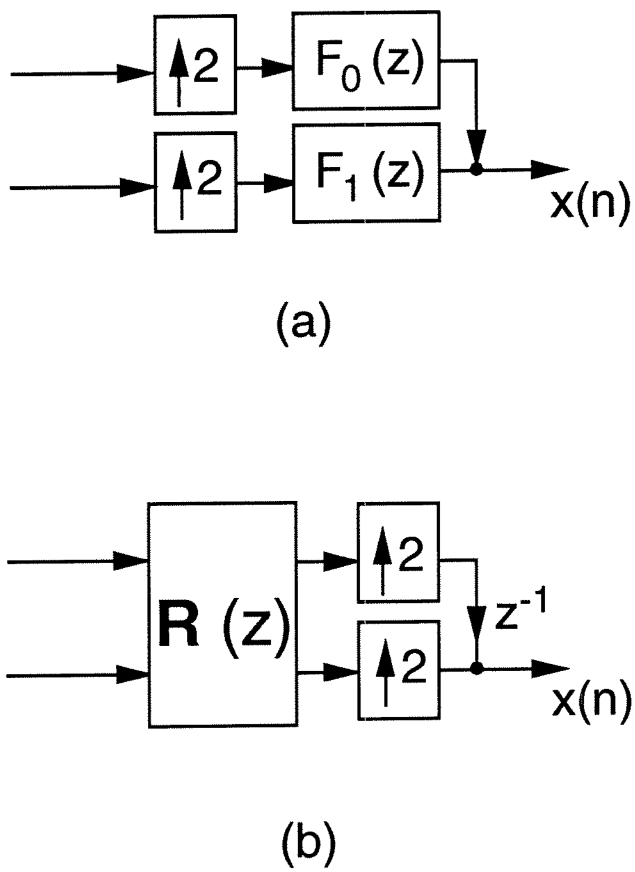


Fig. 2.3 (a) A two channel filter bank
(b) A two channel bank drawn in terms of the polyphase matrix

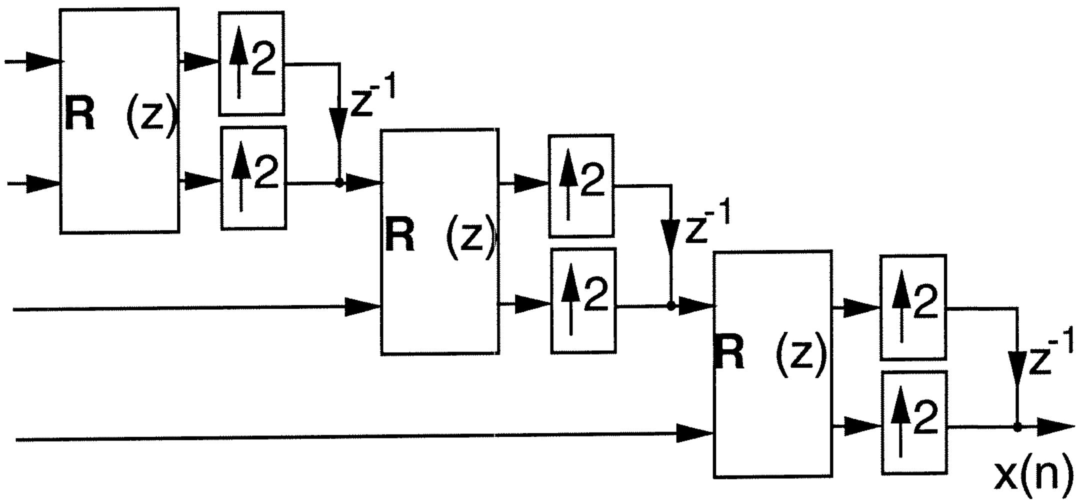


Fig. 2.4. A tree-structured filter bank drawn in terms of the polyphase matrices

the relation [10]

$$\sum_n \eta_{km}(n)\eta_{li}^*(n) = \delta(k-l)\delta(m-i). \tag{2.6}$$

In terms of the filter responses in a binary tree-structured filter bank, this is equivalent to the condition

$$\begin{aligned}
\sum_n f_k(n-2^{k+1}m)f_l^*(n-2^{l+1}i) &= \delta(k-l)\delta(m-i), \quad k,l=0,\ldots,L-1, & (2.7)\\
\sum_n f_L(n-2^L m)f_l^*(n-2^{l+1}i) &= 0, \quad l=0,\ldots,L-1, & (2.8)\\
\sum_n f_L(n-2^L m)f_L^*(n-2^L i) &= \delta(m-i). & (2.9)
\end{aligned}$$

With a change of variables, this becomes

$$\begin{aligned}
\sum_n f_k(n)f_l^*(n-2^{l+1}i) &= \delta(k-l)\delta(i), \quad l\le k, \quad l,k=0,\ldots,L-1, & (2.10)\\
\sum_n f_L(n)f_l^*(n-2^{l+1}i) &= 0, \quad l=0,\ldots,L-1, & (2.11)\\
\sum_n f_L(n)f_L^*(n-2^L i) &= \delta(i). & (2.12)
\end{aligned}$$

In the transform domain this is equivalent to

$$\begin{aligned}
(F_k(z)\widetilde{F}_l(z))\downarrow_{2^{l+1}} &= \delta(k-l), \quad l\le k, \quad k,l=0,\ldots,L-1, & (2.13)\\
(F_L(z)\widetilde{F}_l(z))\downarrow_{2^{l+1}} &= 0, \quad l=0,\ldots,L-1, & (2.14)\\
(F_L(z)\widetilde{F}_L(z))\downarrow_{2^L} &= 1. & (2.15)
\end{aligned}$$

First, consider the case of a one-level 'tree' shown in Fig. 2.3(a), redrawn as in Fig. 2.3(b), where $\mathbf{R}(z)$ is the polyphase matrix of the filters $F_0(z)$ and $F_1(z)$. It has been shown [16] that if the matrix $\mathbf{R}(z)$ is paraunitary, then the filters satisfy the orthonormality condition

$$\sum_n f_k(n-2m)f_l^*(n-2i) = \delta(k-l)\delta(m-i), \quad k,l=0,1. \tag{2.16}$$

In the z-domain, this becomes

$$(F_k(z)\widetilde{F}_l(z))\downarrow_2 = \delta(k-l), \quad k,l=0,1, \tag{2.17}$$

which can be rewritten as,

$$F_k(z)\widetilde{F}_l(z) + F_k(-z)\widetilde{F}_l(-z) = 2\delta(k-l). \tag{2.18}$$

This can be shown [11] to be exactly equivalent to the condition that the filters $F_0(z)$ and $F_1(z)$ form a PU-pair. When $k = l$, we refer to eq. (2.13) as the 'unit-energy condition.' Now consider a general L-level tree, drawn in terms of the polyphase matrix of the filters on all levels. Fig. 2.4 shows the case $L = 3$. Let the polyphase matrix be paraunitary, i.e., it satisfies eq. (1.3). Then, it has been shown in [16] that the wavelet basis generated by that tree is orthonormal.

Orthonormality of basis functions is often a very desirable property in several applications.

III SOME RESULTS ON PARAUNITARY SYSTEMS

In this section we present a few basic results pertaining to paraunitary systems. Some of them are straightforward, but many are fundamental. All are included here for the sake of completeness.

Lemma 1: Let $A(z)$ be a FIR transfer function such that $(A(z))\downarrow_M = 1$, and let $A(z)$ have a factor of the form $c(z^M)$. Then $c(z) = kz^l$, for some constant k and integer l. In other words, the M polyphase components of $A(z)$ cannot have a common factor other than of the form kz^l.

Proof: Let

$$A(z) = c(z^M)a(z). \tag{3.1}$$

Hence, $(c(z^M)a(z))\downarrow_M = 1$. Using the noble identity [11], we get,

$$c(z)\left[(a(z))\downarrow_M\right] = 1. \tag{3.2}$$

Since this is a product of two FIR functions, matching zeros on both sides of the above equation, we get $c(z) = kz^l$.

Now, $A(z)$ can be written in terms of its polyphase components as in eq. (1.1). If the $a_i(z)$ $i = 0, \ldots, M-1$ had a common factor, $A(z)$ could be written as in eq. (3.1), which we have shown is not possible unless the common factor is of the form kz^l. Hence the M polyphase components of a function $A(z)$ satisfying $(A(z))\downarrow_M = 1$ cannot have a common factor other than of the restricted form.

Lemma 2: Given a FIR transfer-function $A(z)$ satisfying

$$(A(z)\widetilde{A}(z))\downarrow_2 = 1, \tag{3.3}$$

we can always find a function $B(z)$ such that $A(z)$ and $B(z)$ form a PU-pair.

Proof: Since $A(z)$ is FIR, it can be multiplied by z^m (for some m positive or negative), so that $C(z) = z^m A(z)$ is causal with $c(0) \neq 0$. Since $A(z)$ satisfies eq. (3.3), $C(z)$ also satisfies eq. (3.3). Hence from [11], we know that the degree N of $C(z)$ is constrained to be odd. Choose

$$D(z) = z^{-N}\widetilde{C}(-z), \tag{3.4}$$

and let $B(z) = z^{-m}D(z)$. If $A(z) = a_0(z^2) + z^{-1}a_1(z^2)$ and $B(z) = b_0(z^2) + z^{-1}b_1(z^2)$, it can be verified that the above choice of $B(z)$ implies the relations

$$b_0(z) = z^{-(N-1)/2}\tilde{a}_1(z) \tag{3.5}$$

$$b_1(z) = -z^{-(N-1)/2}\tilde{a}_0(z). \tag{3.6}$$

It can be shown by direct substitution that with this choice, the filters $A(z)$ and $B(z)$ form a PU-pair.

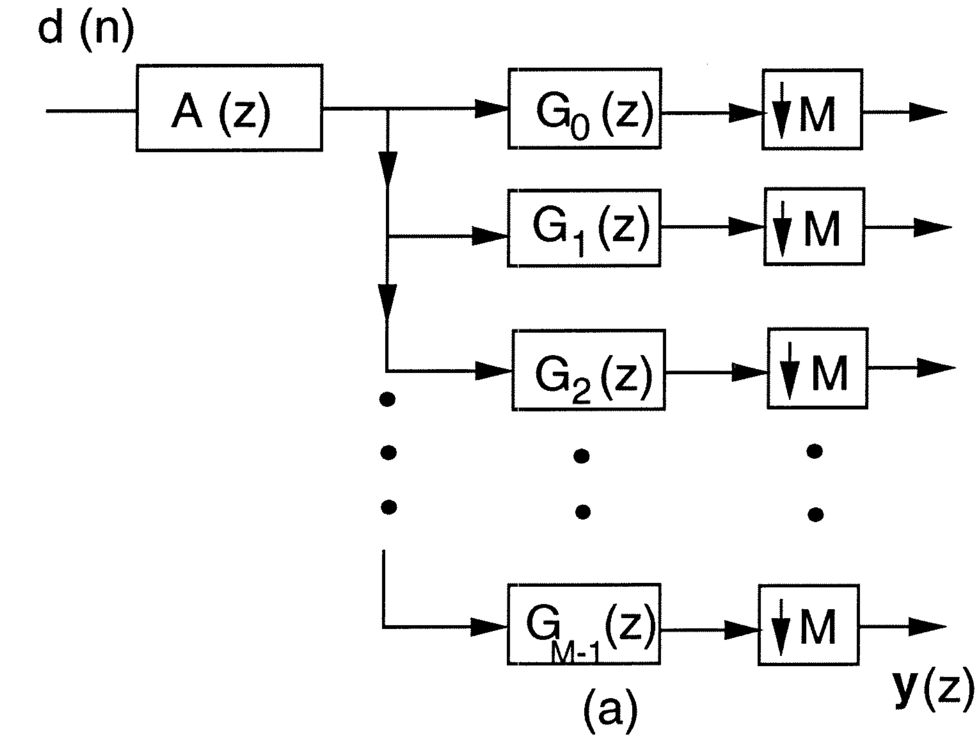
NOTE: Condition (3.3) means that $(A(z)\widetilde{A}(z))$ is a 'halfband' filter [11]. Equivalently, $A(z)$ is a spectral factor of a halfband filter.

Lemma 3: Given a FIR transfer-function $A(z)$ satisfying eq. (3.3), and a function $P(z)$ satisfying

$$(A(z)\widetilde{P}(z))\downarrow_2 = 0, \tag{3.7}$$

$P(z)$ can always be written as

$$P(z) = p(z^2)B(z), \tag{3.8}$$



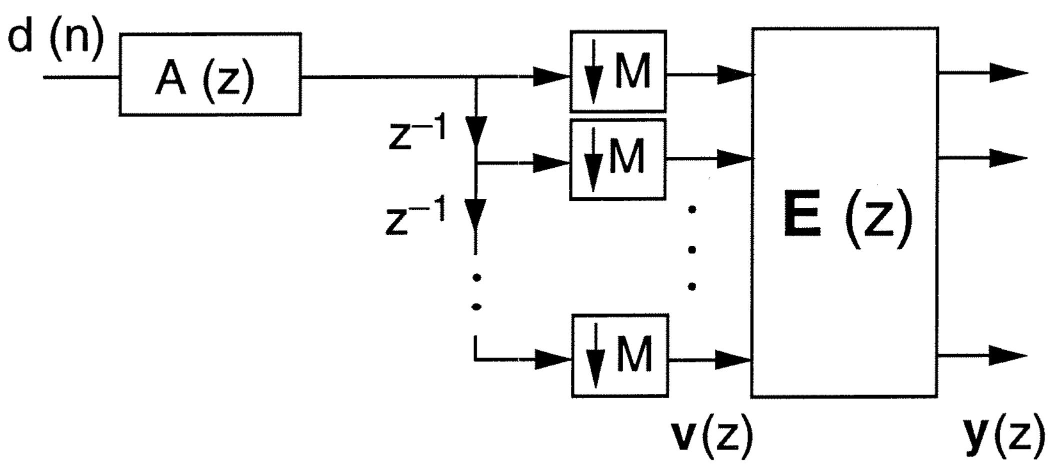


Fig. 2.5. Demonstration for Lemma 5
(a) The M-channel filter bank

(b) The filter bank in terms of the polyphase matrix

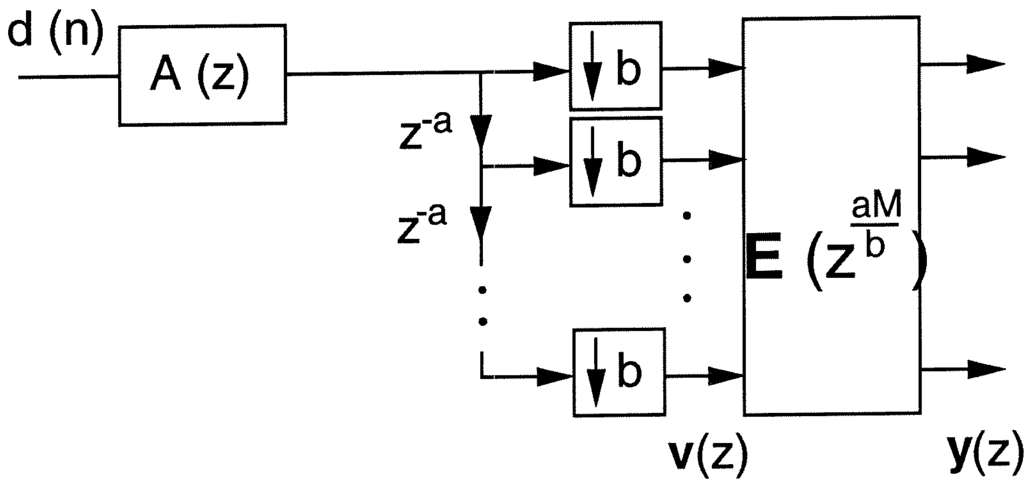


Fig. 2.6. Demonstration for Lemma 6

where $B(z)$ is as defined in Lemma 2.

Proof: The functions can be written in terms of their polyphase components as

$$A(z) = a_0(z^2) + z^{-1}a_1(z^2) \tag{3.9}$$

$$P(z) = p_0(z^2) + z^{-1}p_1(z^2) \tag{3.10}$$

Eq. (3.7) can be written in terms of the polyphase components as,

$$a_0(z)\tilde{p}_0(z) + a_1(z)\tilde{p}_1(z) = 0. \tag{3.11}$$

Note that by Lemma 1, if $a_0(z)$ and $a_1(z)$ have a common factor, it is of the form kz^l. Thus in order that eq. (3.11) be satisfied, we need

$$p_0(z) = p(z)\tilde{a}_1(z)z^{-l} \tag{3.12}$$

$$p_1(z) = -p(z)\tilde{a}_0(z)z^{-l} \tag{3.13}$$

which means $P(z)$ can be written as in eq. (3.8).

Lemma 4: If $A_i(z),\quad i = 0, \ldots, M-1$ form a PU-set, then they cannot have a common factor except of the form kz^l.

Proof: Since $A_i(z),\quad i = 0, \ldots, M-1$ form a PU-set, they satisfy [11]

$$\sum_{i=0}^{M-1} A_i(z)\widetilde{A}_i(z) = M \quad for\ all\ z. \tag{3.14}$$

If $A_i(z),\quad i = 0, \ldots, M-1$ have a common factor $(z - z_0)$ with $z_0 \neq 0$, the left-hand side would vanish at $z = z_0$, violating eq. (3.14).

Lemma 5: Let the filters $G_k(z),\quad k = 0, 1, 2, \ldots, M-1$ form a PU-set. Then,

$$(A(z)G_k(z))\downarrow_M = 0, \quad for\ k = 0, 1, 2, \ldots, M-1 \tag{3.15}$$

implies $A(z) \equiv 0$.

Proof: Consider Fig. 2.5(a) redrawn as Fig. 2.5(b). $\mathbf{E}(z)$ is the Type I polyphase matrix of the filters $G_k(z),\quad k = 0, 1, 2, \ldots, M-1$. Apply an impulse $\delta(n)$ as an input to the system. Eq. (3.15) means that the output of the system $\mathbf{y}(z) \equiv \mathbf{0}$. Since the matrix $\mathbf{E}(z)$ is paraunitary, this means that $\mathbf{v}(z) \equiv \mathbf{0}$. Hence the output of the filter $A(z)$ in response to the impulse $\delta(n)$ is zero, i.e., $A(z) \equiv \mathbf{0}$.

Lemma 6: Let the filters $G_k(z), \quad k = 0, 1, 2, \ldots, M-1$ form a PU-set. Let a and b be relatively prime integers, and let b be a factor of M. If

$$(A(z)G_k(z^a))\downarrow_b = 0, \quad k = 0, 1, 2, \ldots, M-1, \tag{3.16}$$

then $A(z) \equiv 0$.

Proof: In Fig. 2.5(a), imagine that each $G_k(z)$ is replaced with $G_k(z^a)$, and the decimation factor is made b. This can be redrawn in terms of the polyphase matrix as in Fig. 2.6. Again, apply an impulse $\delta(n)$ as an input to the system. Eq. (3.16) means that the output of the system $\mathbf{y}(z) \equiv \mathbf{0}$. Since the matrix $\mathbf{E}(z)$, which is the polyphase matrix of the filters $G_k(z)$ is paraunitary, the matrix $\mathbf{E}(z^{aM/b})$ is also paraunitary. This means that $\mathbf{v}(z) \equiv \mathbf{0}$. Now, since a and b are relatively prime, it can be shown that the output of the filter $A(z)$ is zero, hence proving the Lemma.

Lemma 7: Let $A(z)$ be some rational transfer function, and let L be any integer. Then, there exists a $C(z)$ such that

$$(A(z)\tilde{A}(z))\downarrow_L = C(z)\tilde{C}(z). \tag{3.17}$$

Furthermore, if $A(z)$ is FIR, $C(z)$ is also FIR.

Proof: Observe that $A(e^{j\omega})A^*(e^{j\omega}) \geq 0$, and so is its any L-fold decimated version. Hence we can rewrite it as $C(e^{j\omega})C^*(e^{j\omega})$. By analytic continuation, we have eq. (3.17).

If $A(z)$ is FIR, its L-fold decimated version is FIR, and so $C(z)$ is also FIR.

IV ORTHONORMAL WAVELETS AND BINARY TREE-STRUCTURED FILTER BANKS

In this section, we study further the connection between orthonormality of wavelet bases and paraunitariness of matrices in a binary tree-structured filter bank. All

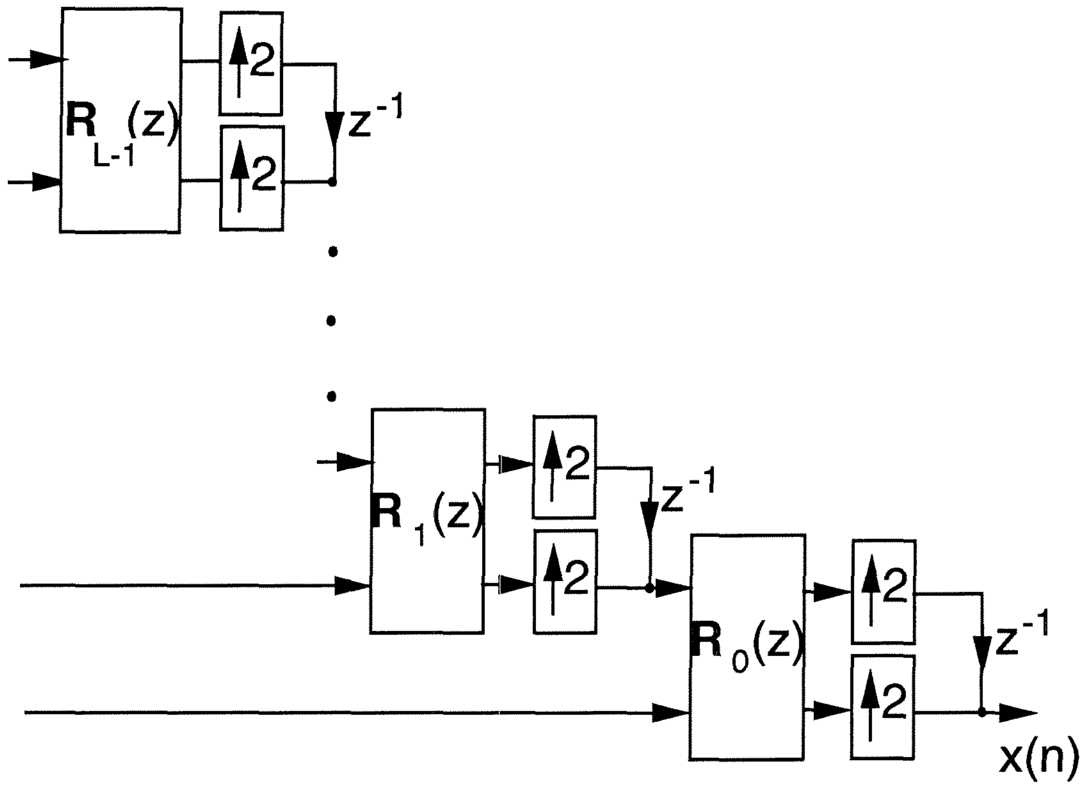


Fig. 2.7. A binary tree-structured synthesis bank in terms of polyphase matrices

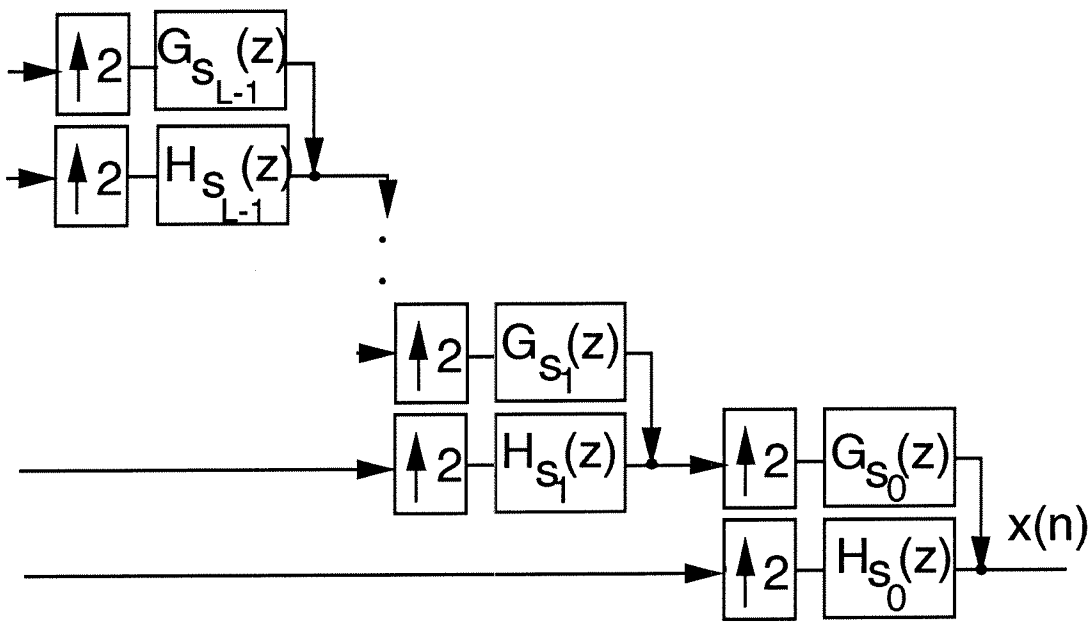


Fig. 2.8(a). A binary tree-structured synthesis bank

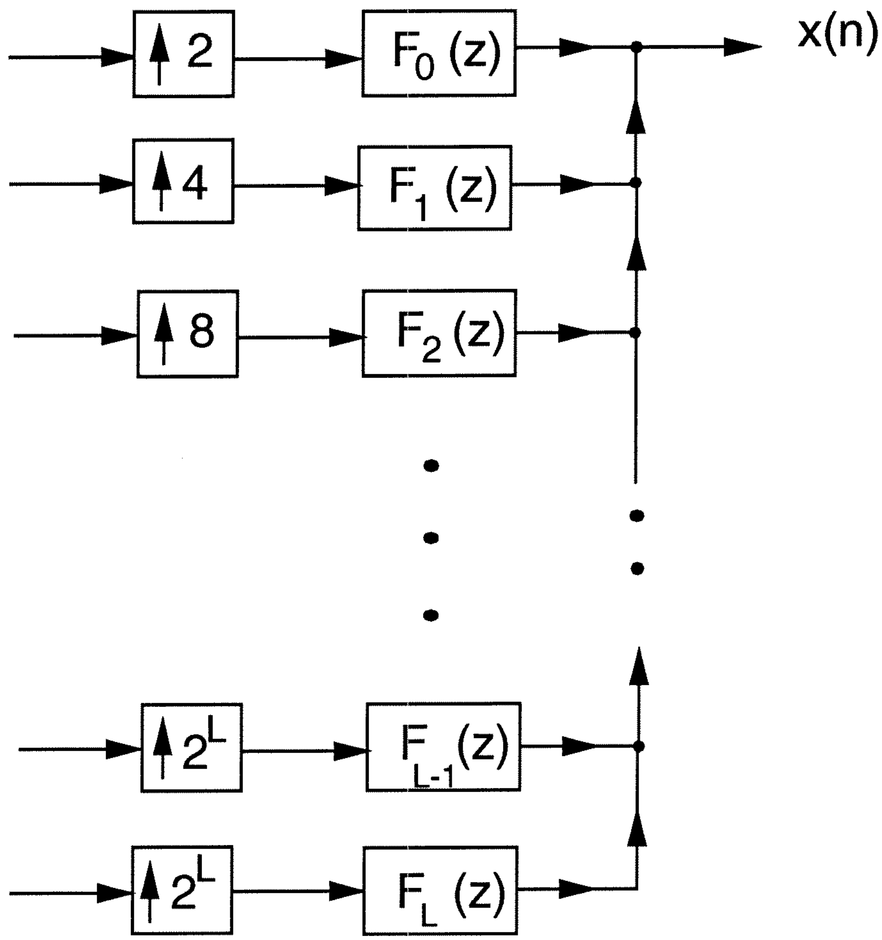


Fig. 2.8(b). A binary tree-structure redrawn as a traditional filter bank

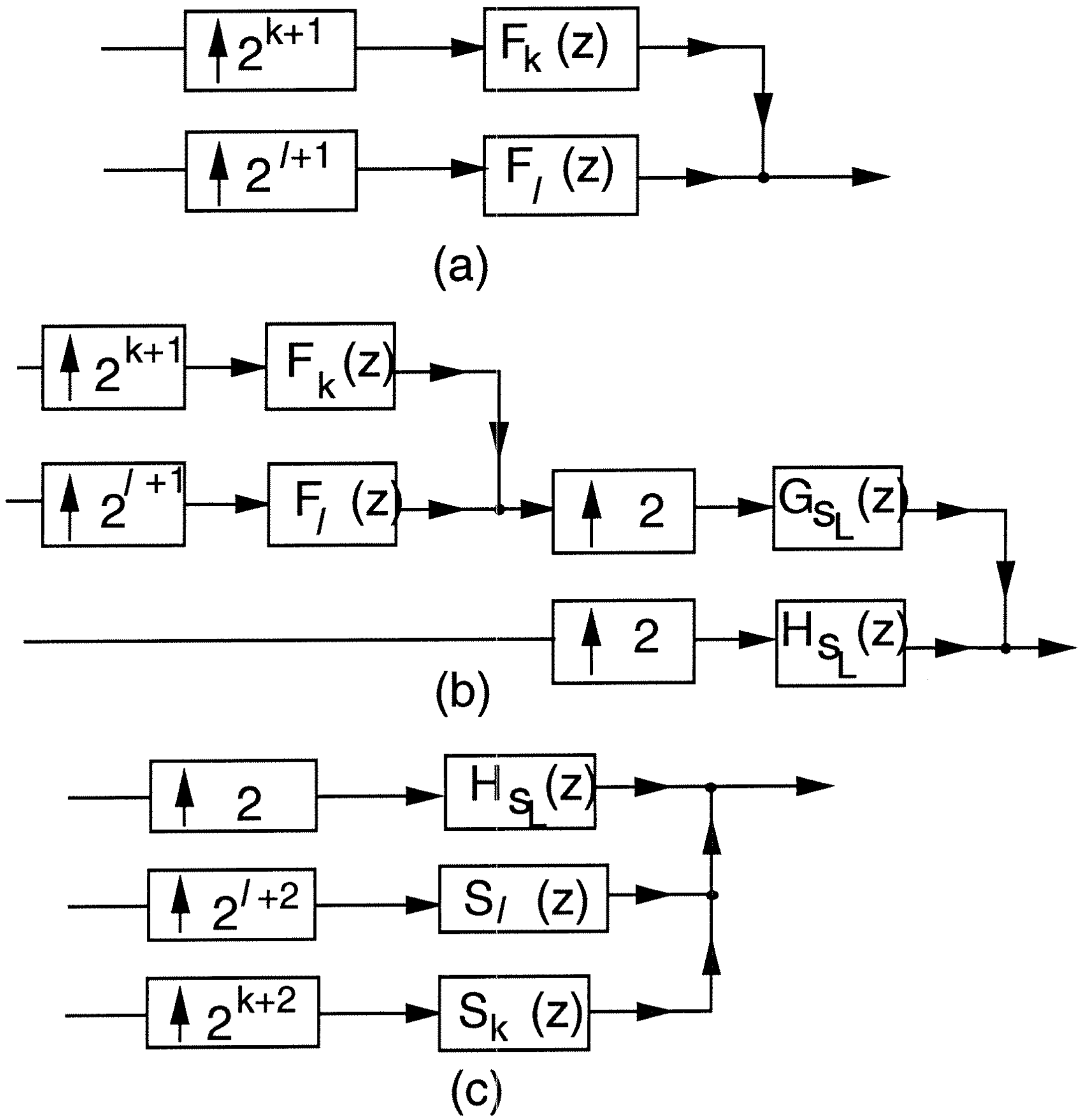


Fig. 2.9. (a) Two branches of an L-level tree
(b) Filters on the (L+1)-level are added
(c) Tree resulting from the addition of one more level

wavelet bases we consider are of finite duration, or FIR, unless stated otherwise. Finite duration wavelets have been referred to as 'compactly supported wavelets' in [10]. Consider an L-level binary tree-structured synthesis filter bank used traditionally for generating a wavelet basis, drawn in terms of the polyphase matrices of the filters. Fig. 2.4 is an example for $L = 3$. It is known [16] that if the matrix $\mathbf{R}(z)$ is paraunitary, i.e., it satisfies eq. (1.3), the wavelet basis generated by this tree-structure is orthonormal.

First, we shall consider a simple generalization of Fig. 2.4. Consider Fig. 2.7. This is also a binary tree-structured synthesis filter bank, but the filters (and hence their polyphase matrix) on each level are different. We now prove:

Theorem 1: Consider an L-level binary tree-structured filter bank. Let the polyphase matrices on each level, $\mathbf{R}_i(z)$ $i = 0, \ldots, L-1$ be paraunitary. Then, the wavelet basis generated by this tree is orthonormal.

Proof: The proof of this result is a straightforward generalization of the one given in [16], for tree structures having the same paraunitary matrix on each level. We present it here for the sake of completeness.

We prove this result by induction. Consider an L-level tree (Fig. 2.8(a)) drawn as a traditional synthesis filter bank (Fig. 2.8(b)). The filters $F_k(z)$ are given by the relations

$$F_0(z) = H_{s_0}(z) \tag{4.1}$$

$$F_k(z) = H_{s_k}(z^{2^k})\prod_{i=0}^{k-1} G_{s_i}(z^{2^i}) \quad k = 1, \ldots, L-1 \tag{4.2}$$

$$F_L(z) = G_{s_{L-1}}(z^{2^{L-1}})\prod_{i=0}^{L-2} G_{s_i}(z^{2^i}) \tag{4.3}$$

The tree has $L + 1$ branches. Fig. 2.9(a) shows two of these branches, with $k \geq l$. Suppose we add another level to the tree. This adds a new branch and modifies the existing branches as shown in Fig. 2.9(b). Assuming that

a) the wavelet basis is orthonormal for the L-level tree, and that

b) the new set of filters $(G_{s_L}(z), H_{s_L}(z))$ has a polyphase matrix which is paraunitary,

we prove that the wavelet-basis generated by the $(L+1)$-level tree is also orthonormal. From the paraunitariness of their polyphase matrix, we know that

$$(G_{s_L}(z)\widetilde{G}_{s_L}(z))\downarrow_2 = 1 \quad (H_{s_L}(z)\widetilde{H}_{s_L}(z))\downarrow_2 = 1 \quad (G_{s_L}(z)\widetilde{H}_{s_L}(z))\downarrow_2 = 0. \tag{4.4}$$

Orthonormality of the L-level tree implies

$$(F_k(z)\widetilde{F}_l(z))\downarrow_{2^{l+1}} = \delta(k-l), \quad 0 \le l \le k \le L-1. \tag{4.5}$$

The three branches of the $L+1$-level tree shown in Fig. 2.9(b) can be redrawn as in Fig. 2.9(c), where

$$S_k(z) \equiv F_k(z^2)G_{s_L}(z), \quad S_l(z) \equiv F_l(z^2)G_{s_L}(z). \tag{4.6}$$

Thus,

$$(S_k(z)\widetilde{S}_l(z))\downarrow_{2^{l+2}} = (F_k(z^2)\widetilde{F}_l(z^2)G_{s_L}(z)\widetilde{G}_{s_L}(z))\downarrow_{2^{l+2}} \tag{4.7}$$

$$= (F_k(z)\widetilde{F}_l(z)(G_{s_L}(z)\widetilde{G}_{s_L}(z))\downarrow_2)\downarrow_{2^{l+1}} = \delta(k-l). \tag{4.8}$$

We have used eqs. (4.4) and (4.5) to arrive at the final answer. Also,

$$(S_k(z)\widetilde{H}_{s_L}(z))\downarrow_2 = F_k(z)(G_{s_L}(z)\widetilde{H}_{s_L}(z))\downarrow_2 \tag{4.9}$$

$$= 0, \tag{4.10}$$

using eq. (4.4). This is sufficient to prove that the wavelet basis generated by the $L+1$-level tree is orthonormal.

♠

We shall now consider the converse of the above question, namely, is it possible to generate all orthonormal wavelet bases using binary tree-structured filter banks? It turns out that this is true in the case of finite duration discrete-time orthonormal wavelets.

Theorem 2: Every finite duration discrete-time orthonormal wavelet basis can be generated by a binary tree-structured filter bank having paraunitary matrices on all levels.

Proof: Let $y_k(m)$ denote the wavelet coefficients at resolution k, and let $\eta_{km}(n)$ denote the wavelet basis functions. We recall that the original signal $x(n)$ can be reconstructed from its wavelet coefficients as in eq. (2.3). We also recall that in a tree-structured filter bank, the synthesis filters are related to the wavelet basis functions as in eqs. (2.4) and (2.5). Consider a binary tree-structured synthesis bank, drawn as a conventional filter bank in Fig. 2.8(b). Notice the increasing interpolation ratios. Since we are dealing with compactly supported bases, the filters $F_k(z)$ are FIR. We shall also assume that they are causal with $f_k(0) \neq 0$. This assumption is not restrictive, since we are dealing with FIR functions, and any FIR function can be brought into this form by a suitable advance/delay operation. Orthonormality of the wavelet basis implies eqs. (2.7) to (2.15). For the sake of convenience, we reproduce below the orthonormality condition in the z-domain,

$$\begin{aligned}
(F_k(z)\widetilde{F}_l(z))\downarrow_{2^{l+1}} &= \delta(k-l), \quad 0 \leq k \leq l \leq L-1. \\
(F_L(z)\widetilde{F}_l(z))\downarrow_{2^{l+1}} &= 0 \quad l = 0, \ldots, L-1, \\
(F_L(z)\widetilde{F}_L(z))\downarrow_{2^L} &= 1.
\end{aligned}$$

Our task is to show that a set of functions $F_i(z)$ satisfying the above condition can always be generated using a tree-structured filter-bank having paraunitary matrices on all levels. In other words, given the filters $F_k(z)$ in Fig. 2.8(b), we want to obtain filters $(G_{s_i}(z), H_{s_i}(z))$ in Fig. 2.8(a) such that they form PU-pairs for all i. We now give a constructive proof showing that this is always possible.

Choose

$$H_{s_0}(z) = F_0(z). \tag{4.11}$$

Hence from eq. (2.13),

$$(H_{s_0}(z)\widetilde{H}_{s_0}(z))\downarrow_2 = 1. \tag{4.12}$$

Choose

$$G_{s_0}(z) = z^{-N_0}\widetilde{H}_{s_0}(-z), \tag{4.13}$$

where N_0 is the degree of $H_{s_0}(z)$. Note that $G_{s_0}(z)$ and $H_{s_0}(z)$ are both FIR, and $g_{s_0}(0) \neq 0$, $h_{s_0}(0) \neq 0$. Lemma 2 assures us that $G_{s_0}(z)$ and $H_{s_0}(z)$ form a PU-pair. Now from eqs. (2.13) and (2.14),

$$(F_j(z)\widetilde{F}_0(z))\downarrow_2 = 0 \quad \mathit{for}\ j = 1, 2, ..., L. \tag{4.14}$$

Hence by Lemma 3, the $F_j(z)$ are expressible as

$$F_j(z) = f_j'(z^2)G_{s_0}(z), \tag{4.15}$$

where the $f_j'(z)$ are FIR and do not have a zero at infinity.

Now in particular, $F_1(z)$ is expressible as

$$F_1(z) = f_1'(z^2)G_{s_0}(z). \tag{4.16}$$

Choose

$$H_{s_1}(z) = f_1'(z). \tag{4.17}$$

From eq. (2.13) we know that,

$$(F_1(z)\widetilde{F}_1(z))\downarrow_4 = 1, \tag{4.18}$$

therefore,

$$(H_{s_1}(z^2)G_{s_0}(z)\widetilde{H}_{s_1}(z^2)\widetilde{G}_{s_0}(z))\downarrow_4 = 1. \tag{4.19}$$

Using noble identity, this becomes

$$\left[H_{s_1}(z)\widetilde{H}_{s_1}(z)(G_{s_0}(z)\widetilde{G}_{s_0}(z))\downarrow_2\right]\downarrow_2 = 1, \tag{4.20}$$

and hence, from the fact that $G_{s_0}(z)$ belongs to a PU-pair,

$$(H_{s_1}(z)\widetilde{H}_{s_1}(z))\downarrow_2 = 1. \tag{4.21}$$

Choose

$$G_{s_1}(z) = z^{-N_1}\widetilde{H}_{s_1}(-z), \tag{4.22}$$

where N_1 is the degree of $H_{s_1}(z)$. Hence $G_{s_1}(z)$ and $H_{s_1}(z)$ form a PU-pair, and both are FIR with $g_{s_1}(0) \neq 0$ and $h_{s_1}(0) \neq 0$.

Now from eq. (2.13),

$$(F_j(z)\widetilde{F}_1(z))\downarrow_4 = 0 \quad for\ j = 2, \ldots, L. \tag{4.23}$$

Using eqs. (4.15) and (4.17) this becomes

$$\left[f_j'(z^2)G_{s_0}(z)\widetilde{H}_{s_1}(z^2)\widetilde{G}_{s_0}(z)\right]\downarrow_4 = 0, \tag{4.24}$$

which simplifies to

$$(f_j'(z)\widetilde{H}_{s_1}(z))\downarrow_2 = 0 \quad j = 2, 3, \ldots, L. \tag{4.25}$$

Hence by Lemma 3, the $f_j'(z)$ are expressible as

$$f_j'(z) = f_j''(z^2)G_{s_1}(z) \quad j = 2, 3, \ldots, L, \tag{4.26}$$

where the filters $f_j''(z)$ are FIR and do not have a zero at infinity. Hence using eq. (4.15), we get,

$$F_j(z) = f_j''(z^4)G_{s_1}(z^2)G_0(z) \quad j = 2, \ldots, L. \tag{4.27}$$

Now in particular, $F_2(z)$ is expressible as

$$F_2(z) = f_2''(z^4)G_{s_1}(z^2)G_{s_0}(z), \tag{4.28}$$

with $f_2''(z)$ being FIR. Hence choose

$$H_{s_2}(z) = f_2''(z). \tag{4.29}$$

Therefore,

$$F_2(z) = H_{s_2}(z^4)G_{s_1}(z^2)G_{s_0}(z). \tag{4.30}$$

In general, since $F_k(z)$ is orthogonal to $F_i(z)$, $for\ i = 0, 1, 2, \ldots, k-1$, it can be expressed as

$$F_k(z) = H_{s_k}(z^{2^k})G_{s_{k-1}}(z^{2^{k-1}}) \ldots G_{s_0}(z), \tag{4.31}$$

and from the unit-energy property (eq. (2.13)) we have

$$(H_{s_k}(z)\widetilde{H}_{s_k}(z))\downarrow_2 = 1. \tag{4.32}$$

At every stage, the filters $G_{s_k}(z)$ are chosen such that

$$G_{s_k}(z) = z^{-N_k}\widetilde{H}_{s_k}(-z), \tag{4.33}$$

where N_k is the degree of $H_{s_k}(z)$. Hence the filters $(G_{s_i}(z), H_{s_i}(z))$ form PU-sets on all levels. They are all FIR, causal, and do not have a zero at infinity.

In particular, on the final level we have,

$$F_{L-1}(z) = H_{s_{L-1}}(z^{2^{L-1}})G_{s_{L-2}}(z^{2^{L-2}}) \ldots G_{s_0}(z) \tag{4.34}$$

with

$$(H_{s_{L-1}}(z)\widetilde{H}_{s_{L-1}}(z))\downarrow_2 = 1. \tag{4.35}$$

Since the function $F_L(z)$ is orthogonal to $F_i(z)$, $for\ i = 0, 1, 2, \ldots, L-2$, following the general procedure, it can be expressed as

$$F_L(z) = g(z^{2^{L-1}})G_{s_{L-1}}(z^{2^{L-2}}) \ldots G_{s_0}(z), \tag{4.36}$$

where $g(z)$ is FIR. Let $G_{s_{L-1}}(z) = z^{-N_{L-1}}\widetilde{H}_{s_{L-1}}(-z)$. Since $F_L(z)$ is also orthonormal to $F_{L-1}(z)$, by Lemma 3, $g(z) = g'(z^2)G_{s_{L-1}}(z)$. It can be verified that the unit-energy condition (eq. (2.16)) applied to $F_L(z)$ implies $g'(z) = 1$. Thus,

$$F_L(z) = G_{s_{L-1}}(z^{2^{L-1}})G_{s_{L-1}}(z^{2^{L-2}}) \ldots G_{s_0}(z), \tag{4.37}$$

with $G_{s_{L-1}}(z)$ and $H_{s_{L-1}}(z)$ forming a PU-pair.

Thus, we have shown that given any finite duration discrete-time orthonormal basis $F_i(z)$, *for* $i = 0, 1, 2, \ldots, L$, it is always possible to generate it using a tree-structured filter bank having paraunitary matrices on all levels; i.e., Fig. 2.8(b) can always be redrawn as Fig. 2.8(a), with the filters $(G_{s_i}(z), H_{s_i}(z))$ forming PU-pairs.

♠

Now consider Fig. 2.8(a). We know that $G_{s_i}(z)$ belongs to a PU-pair, is causal, and $g_{s_i}(0) \neq 0$. Hence it cannot have a factor of the form $c(z^2)$, other than a constant. Thus no factor (except a constant) of $G_{s_i}(z)$ can be moved left across the interpolators. Also, since the filters $(G_{s_i}(z), H_{s_i}(z))$ on each level form a PU-pair, are causal, FIR, and without a zero at infinity, by Lemma 4, they cannot have a common factor other than a constant. Hence no factor (except a constant) comon to these two can be moved right across the interpolators. This gives the following corollary to the above Theorem.

Corollary 1: If a finite duration orthonormal wavelet basis is generated using a tree-structured filter bank, the polyphase matrices of the filters on each level *have to* satisfy the condition

$$\widetilde{\mathbf{E}}_i(z)\mathbf{E}_i(z) = c_i\mathbf{I} \quad \textit{for all } z, \quad i = 0. \ldots, L-1. \tag{4.38}$$

From the above theorem, we have another corollary regarding the linearity of phase of the wavelet basis.

Corollary 2: If an orthonormal wavelet basis has filters with linear phase, the filters $G_{s_i}(z)$ and $H_{s_i}(z)$ on each level have the form $c_1 z^{-l_1} + c_2 z^{-l_2}$ for some constants c_1 and c_2 and integers l_1 and l_2.

To see this, note that orthonormality of the wavelet basis implies that the polyphase matrices on *each* level have to satisfy eq. (4.38). In particular, since $F_0(z)$ has linear phase, we know from [14] that $H_{s_0}(z) = c_1 z^{-l_1} + c_2 z^{-l_2}$. Since $G_{s_0}(z)$ and $H_{s_0}(z)$ form a PU-set, $G_{s_0}(z)$ can also be written in such a form. Now, since $F_1(z)$ and $G_{s_0}(z)$ both have linear phase, $H_{s_1}(z)$ also has linear phase, and it is therefore restricted to the form stated above. Therefore, $G_{s_0}(z)$ also has this form. Continuing such an argument down the tree, we see that each of the filters $G_{s_i}(z)$ and $H_{s_i}(z)$ have the form $c_1 z^{-l_1} + c_2 z^{-l_2}$.

V ORTHONORMALITY OF WAVELET PACKETS

The basis functions of a filter bank with non-uniform decimation ratios are called wavelet packets [1], [17]. Fig. 2.10 shows a schematic of such a filter bank. If $y_k(n)$ are the inputs to the synthesis filters, then (assuming that this is perfect reconstruction system) we can write

$$x(n) = \sum_{k=0}^{M} \sum_{m} y_k(m) f_k(n - I_k m) \tag{5.1}$$

where I_k is the interpolator preceeding $F_k(z)$, and $M+1$ is the total number of filters. From the similarity of the above equation with eq. (2.3), we refer to the quantities $y_k(n)$ as the generalized wavelet coefficients. The set of functions $f_k(n - I_k m)$ is the generalized wavelet basis, or wavelet packets. Analogous to eq. (2.6), we say the basis functions are orthonormal if they satisfy the relation

$$\sum_{n=-\infty}^{\infty} f_k(n - I_k m) f_l^*(n - I_l i) = \delta(k-l)\delta(m-i). \tag{5.2}$$

With a change of variables this can be rewritten as (see Appendix A)

$$\sum_{n=-\infty}^{\infty} f_k(n) f_l^*(n - g_{lk} i) = \delta(k-l)\delta(i), \tag{5.3}$$

where g_{lk} is the *gcd* of (I_k, I_l). In the z-domain, this becomes,

$$(F_k(z)\widetilde{F}_l(z))\downarrow_{g_{lk}} = \delta(k-l). \tag{5.4}$$

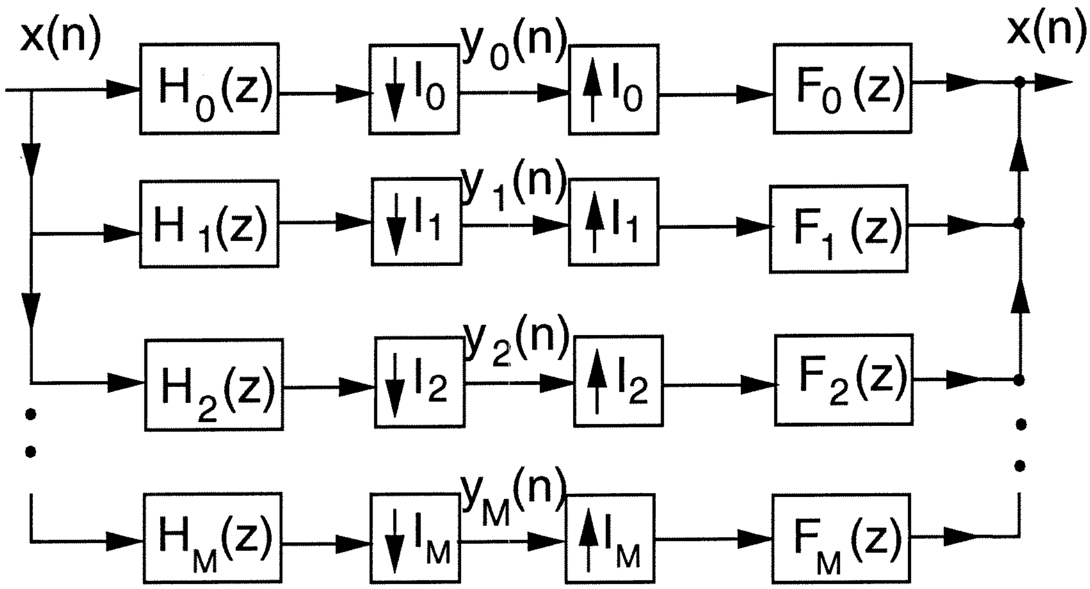


Fig. 2.10. A non-uniform filter bank

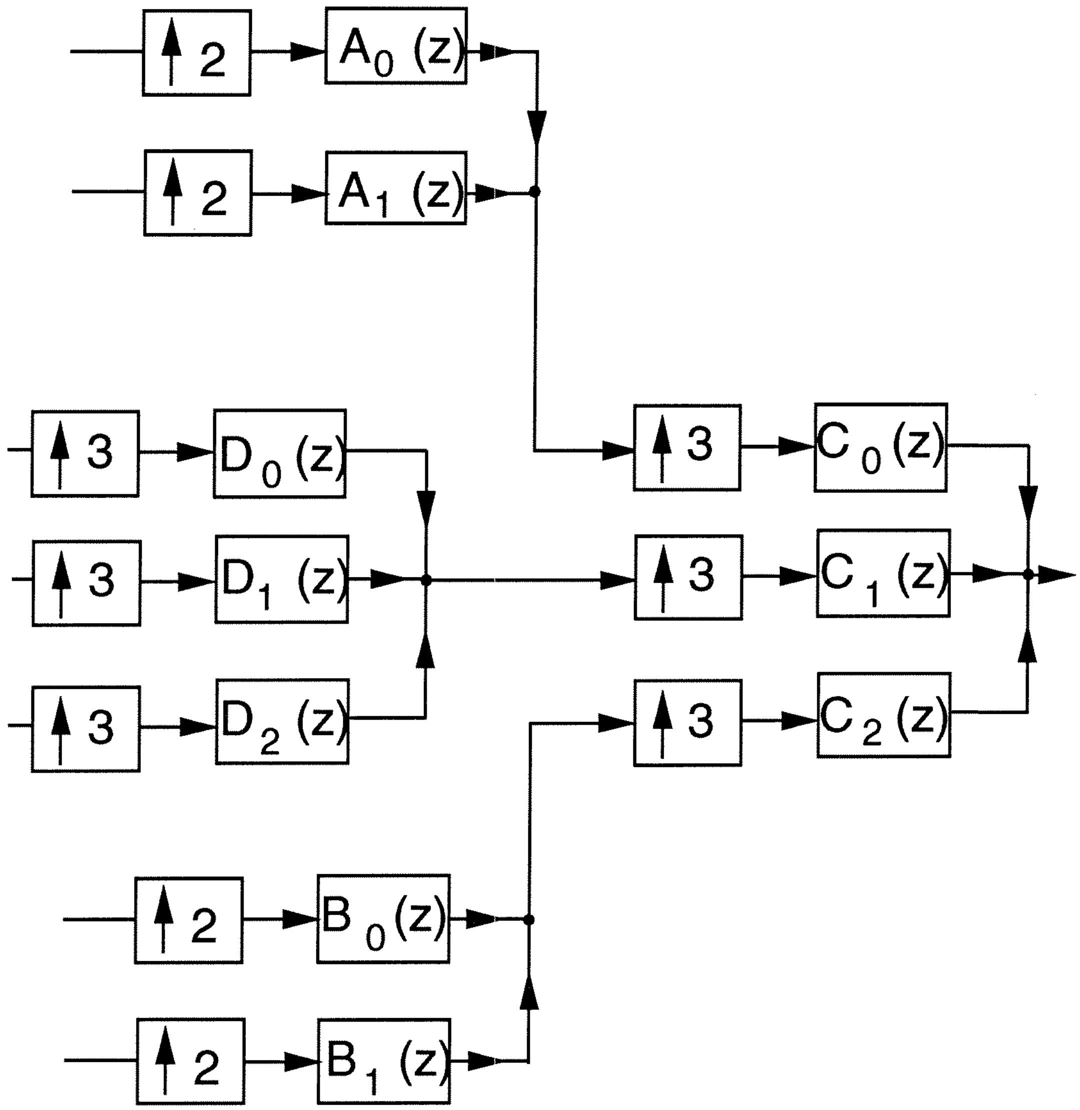


Fig. 2.11 (a). A generalized tree-structured synthesis bank

VI GENERALIZED TREE-STRUCTURES AND ORTHONORMAL BASES

In section III we saw that wavelet bases could be generated using a binary tree-structured QMF filter bank. The wavelet basis functions were seen to be orthonormal if the filters on each level of the tree had a polyphase matrix which was paraunitary, and conversely. Fig. 2.1(b) shows a binary tree-structured filter bank traditionally used for generating wavelet bases. Now consider a general tree-structured filter bank. Fig. 2.11(a) shows one such example of the synthesis bank. This is associated with a corresponding analysis bank not shown in the figure.

Consider the binary tree-structured synthesis bank (Fig. 2.1 (b)), and a general tree-structured synthesis bank (Fig. 2.11(a)). Comparing the two, we note two important differences. Firstly, (going right to left) we see that in a general tree, any branch on a certain level can divide further, whereas in a binary tree, only one of the two branches on any level branches out further. Secondly, for a generalized tree, each level may have different number of filters, in contrast to a binary tree in which each level has exactly two filters. Thus, a maximally decimated filter bank in which one or more branches on any level split further into branches is called an "arbitrary tree structured" filter bank. In the context of generalized tree-structures, we need to rigorously define what we mean by a 'level.' In a tree-structured filter bank, filters whose outputs go into a single adder are said to be on the same level. Consider for example, Fig. 2.11(a). This tree has *four* levels, namely i) $(D_0(z), D_1(z), D_2(z))$ ii) $(A_0(z), A_1(z))$ iii) $(B_0(z), B_1(z))$ iv) $(C_0(z), C_1(z), C_2(z))$. To see this, note that, for example, the outputs of the filters $D_0(z)$, $D_1(z)$ and $D_2(z)$ go into a single adder (denoted by a heavy dot), and hence they are on the same level. On the other hand, the ouputs of the filters $D_0(z)$ and $B_0(z)$ do not go into the same adder, and thus they are said to be on different levels. The word 'level' used in the case of arbitrary trees does not have the strict connotation of 'depth' as it does in the English language, or as in the case of binary trees (Fig. 2.1). Consequently, for generalized trees the levels are not numbered as they are in the case of binary trees. We do define something called the 'input level,' however. If none of the branches in a certain level further divide into branches (while going right to left in a synthesis core tree), such a level is called an 'input level.' Note that there can exist more than one input level for a general tree, whereas a binary tree has a distinct input level. For example, Fig. 2.11(a) has three input levels, namely i) $(D_0(z), D_1(z), D_2(z))$ ii) $(A_0(z), A_1(z))$ iii) $(B_0(z), B_1(z))$. Note that $(C_0(z), C_1(z), C_2(z))$ is not an input level.

We can guarantee perfect reconstruction property for such filter banks by appropriately choosing filters on each level. The tree-structure therefore gives rise to a set of wavelet packets. The generalized tree-structure can be redrawn as a traditional filter bank as in Fig. 2.11(b). Fig. 2.11(c) shows typical appearances of frequency responses of such a tree-structure. Taking a cue from traditional wavelet theory, we now ask the question: Is there a relationship between the paraunitariness of filters on each level of the generalized tree and orthonormality of the resulting basis? The answer to this is provided by the two theorems in this section.

In this section, too, we are dealing with finite duration discrete functions.

Theorem 3: If an arbitrary tree-structured FIR filter bank, such as one in Fig. 2.11(a), has filters on each level forming PU-sets, then the functions $f_k(n - I_k m)$ generated by that tree form an orthonormal basis.

Proof: We prove this result by induction. We know from [16] that the result is true for a 1-level tree, i.e., we know that if a set of filters has a polyphase matrix which is paraunitary, the filters form an orthonormal basis. We now assume that the result holds for an L-level tree, and adding levels to the tree, we show that the functions generated by the new tree also form an orthonormal basis. Consider Fig. 2.12(a). The functions in both filter banks are assumed to be orthonormal, i.e., they satisfy

$$(P_k(z)\widetilde{P}_l(z))\downarrow_{g_1} = \delta(k-l), \quad g_1 = gcd(I_k, I_l) \tag{6.1}$$

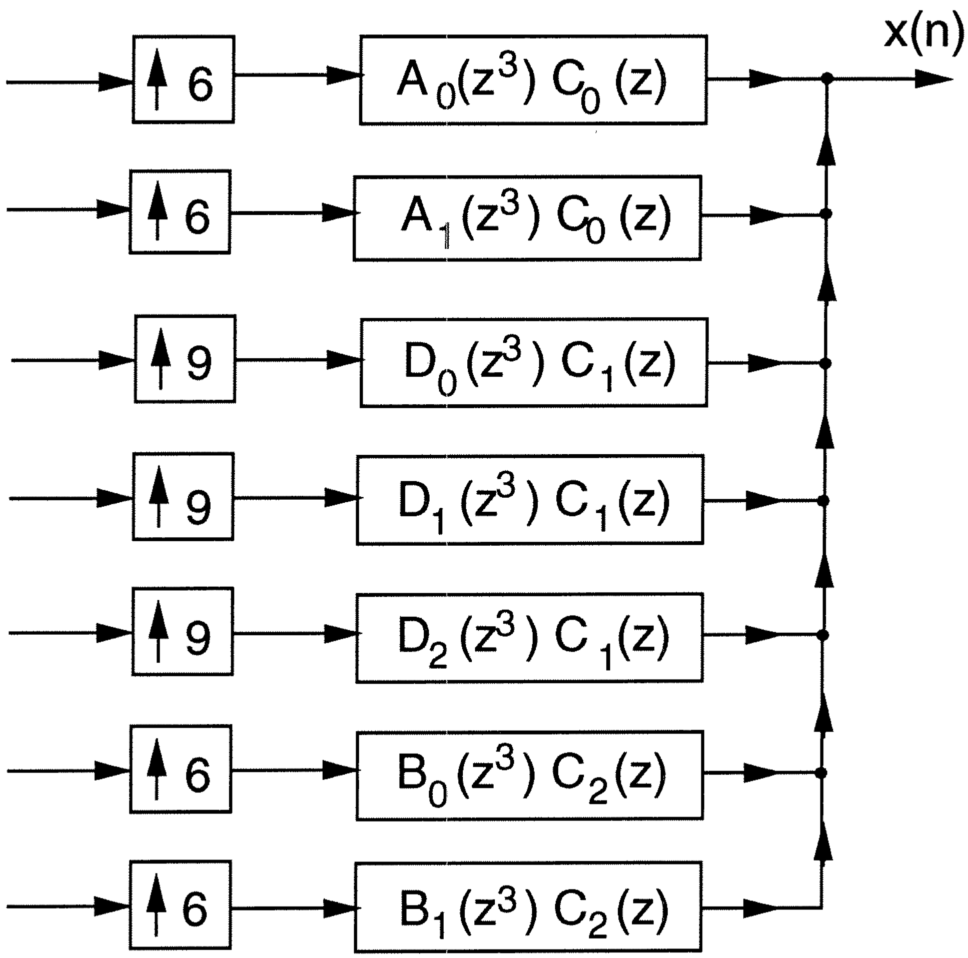


Fig. 2.11 (b). A generalized tree drawn as a traditional synthesis filter bank

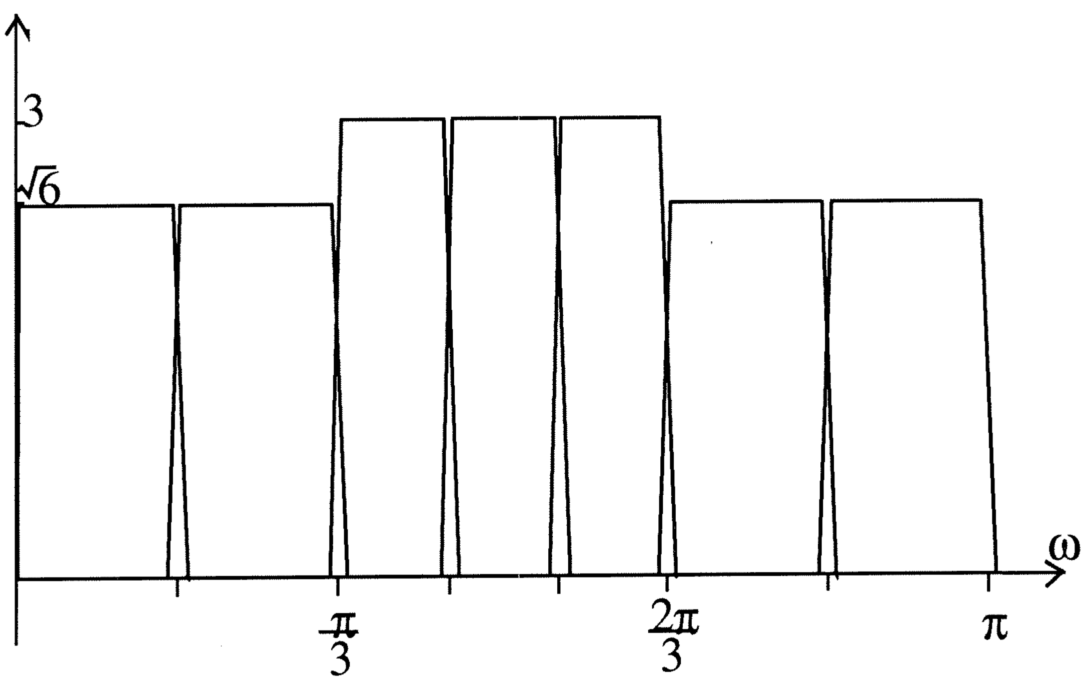


Fig. 2.11(c). Typical responses of filters produced by a generalized tree

$$(Q_m(z)\widetilde{Q}_n(z))\downarrow_{g_2} = \delta(m-n), \quad g_2 = gcd(J_m, J_n). \tag{6.2}$$

We now combine these two by a common level to obtain the tree structure shown in Fig. 2.12(b). The M filters on the new level added are paraunitary, i.e., they satisfy,

$$(T_i(z)\widetilde{T}_j(z))\downarrow_M = \delta(i-j) \quad i,j = 0,\ldots,M-1. \tag{6.3}$$

This new tree can be redrawn again as in Fig. 2.12(c), where the filters are given as,

$$S_k(z) = P_k(z^M)T_i(z) \tag{6.4}$$

$$S_l(z) = P_l(z^M)T_i(z) \tag{6.5}$$

$$S_m(z) = Q_m(z^M)T_j(z) \tag{6.6}$$

$$S_n(z) = Q_n(z^M)T_j(z). \tag{6.7}$$

To prove orthogonality of the new basis, it is sufficient to show that

$$(S_k(z)\widetilde{S}_k(z))\downarrow_{I_kM} = 1, \tag{6.8}$$

and that

$$(S_k(z)\widetilde{S}_l(z))\downarrow_{g_3} = 0, \quad g_3 = gcd(I_kM, I_lM), \tag{6.9}$$

$$(S_k(z)\widetilde{S}_m(z))\downarrow_{g_4} = 0, \quad g_4 = gcd(I_kM, J_mM). \tag{6.10}$$

Consider,

$$(S_k(z)\widetilde{S}_k(z))\downarrow_{I_kM} = (P_k(z^M)T_i(z)\widetilde{P}_k(z^M)\widetilde{T}_i(z))\downarrow_{I_kM}. \tag{6.11}$$

Using the noble identity, this becomes

$$(P_k(z)\widetilde{P}_k(z)(T_i(z)\widetilde{T}_i(z))\downarrow_M)\downarrow_{I_k} = (P_k(z)\widetilde{P}_k(z))\downarrow_{I_k} \tag{6.12}$$

$$= 1. \tag{6.13}$$

This proves eq. (6.8). The unit-energy property for the other transfer functions can be verified likewise.

Now, since $g_3 = gcd(I_kM, I_lM)$ and $g_1 = gcd(I_k, I_l)$, we have $g_3 = Mg_1$. Hence,

$$(S_k(z)\widetilde{S}_l(z))\downarrow_{g_3} = (P_k(z^M)T_i(z)\widetilde{P}_l(z^M)\widetilde{T}_i(z))\downarrow_{Mg_1} \tag{6.14}$$

$$= (P_k(z)\widetilde{P}_l(z)(T_i(z)\widetilde{T}_i(z))\downarrow_M)\downarrow_{g_1} \tag{6.15}$$

$$= (P_k(z)\widetilde{P}_l(z))\downarrow_{g_1} \tag{6.16}$$

$$= 0, \tag{6.17}$$

using eq. (6.1). This proves eq. (6.9).

Now, $g_4 = gcd(J_mM, I_kM)$, so it is a multiple of M; let $g_4 = aM$. Hence,

$$(S_k(z)\widetilde{S}_m(z))\downarrow_{g_4} = (P_k(z^M)T_i(z)\widetilde{Q}_m(z^M)\widetilde{T}_j(z))\downarrow_{a.M} \tag{6.18}$$

$$= (P_k(z)\widetilde{Q}_m(z)(T_i(z)\widetilde{T}_j(z))\downarrow_M)\downarrow_a \tag{6.19}$$

$$= (P_k(z)\widetilde{Q}_m(z)(0))\downarrow_a \tag{6.20}$$

$$= 0, \tag{6.21}$$

which proves eq. (6.10). Orthogonality of other pairs can similarly be verified.

Hence we have shown that the functions generated by the new tree also form an orthonormal basis. Since any tree-structured filter bank can be synthesized by this process of adding new levels, it proves our theorem.

We now turn our attention to the converse of this result. Unfortunately, the exact converse of the result in the previous theorem is not true. To see this, one only needs to consider a simple example of a tree-structured filter bank drawn in Fig. 2.11(a). This can be redrawn as in Fig. 2.11(b). Let the polyphase matrices of the filters in Fig. 2.11(a) be paraunitary. By the previous theorem, we know that the wavelet basis generated by this tree is orthonormal. Now consider the filter bank shown in

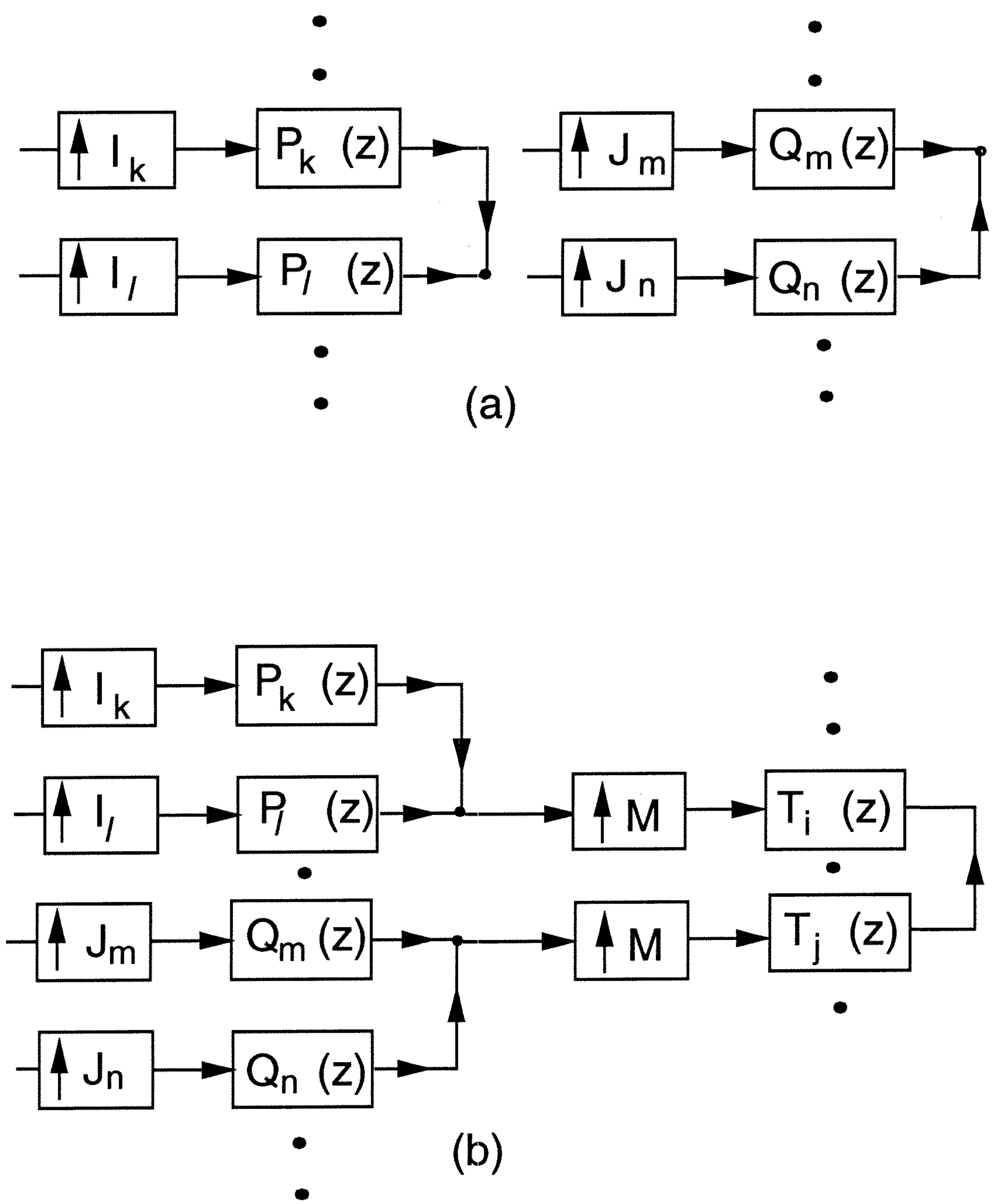


Fig. 2.12. (a). Two different filter banks
(b). The two filter banks in (a) combined by adding one more level of filters

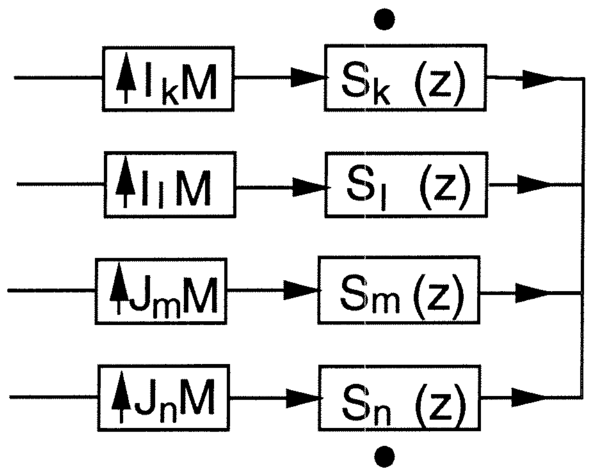
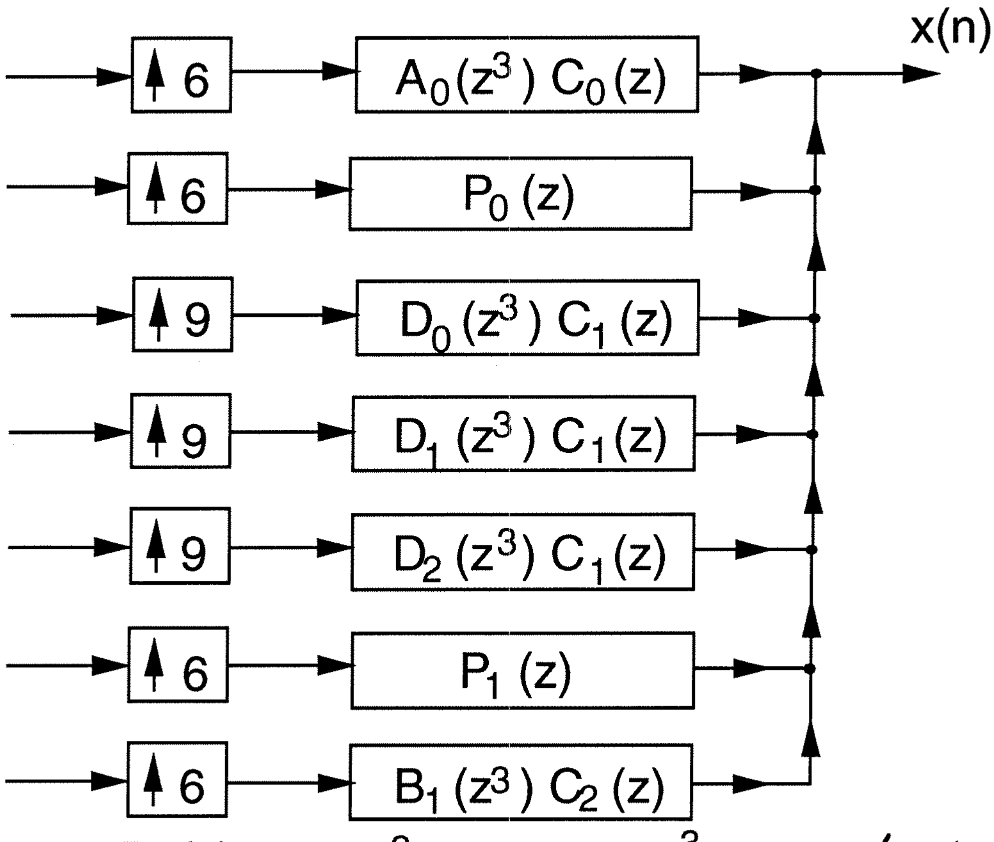


Fig. 2.12 (c). Tree structure of Fig. 11(b) redrawn



$$P_0(z) = (A_1(z^3)C_0(z) + B_0(z^3)C_2(z))/\sqrt{2}$$

$$P_1(z) = (A_1(z^3)C_0(z) - B_0(z^3)C_2(z))/\sqrt{2}$$

Fig. 2.13. A filter bank which cannot be generated using a tree structure

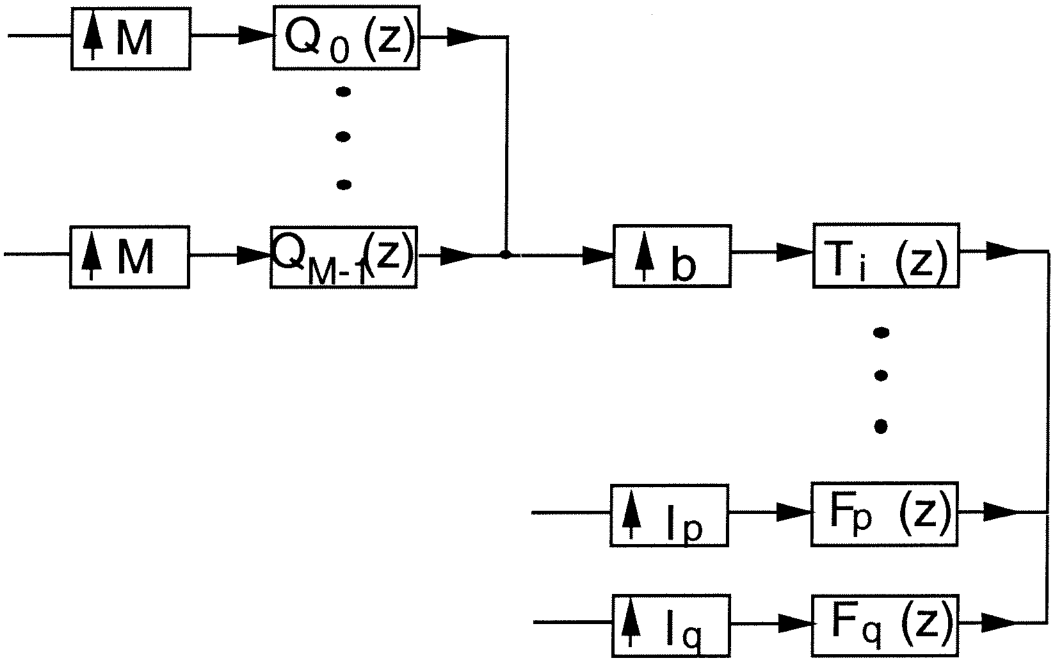


Fig. 2.14 (a). A filter bank showing only one final level

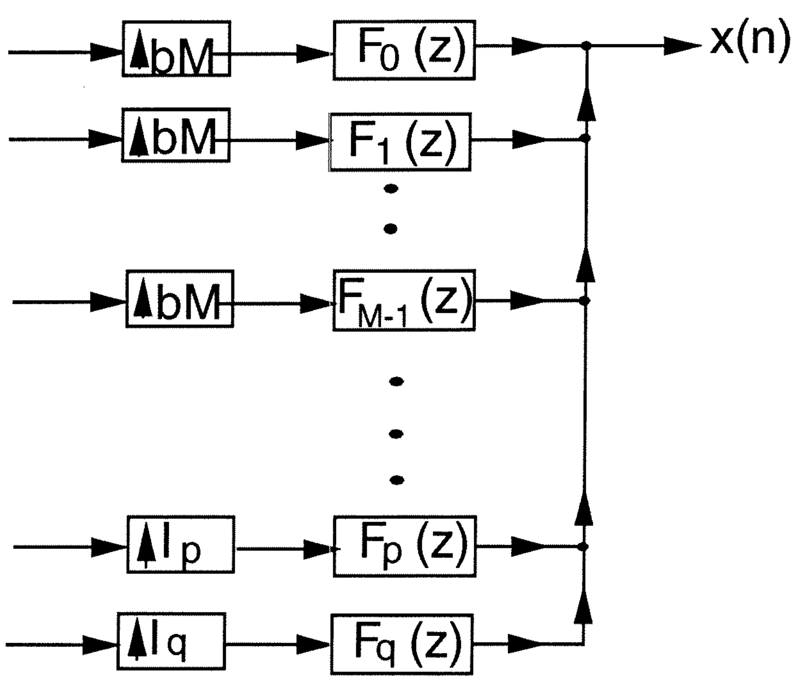


Fig. 2.14 (b). The filter bank in Fig. 2.13(a) redrawn as a traditional filter bank

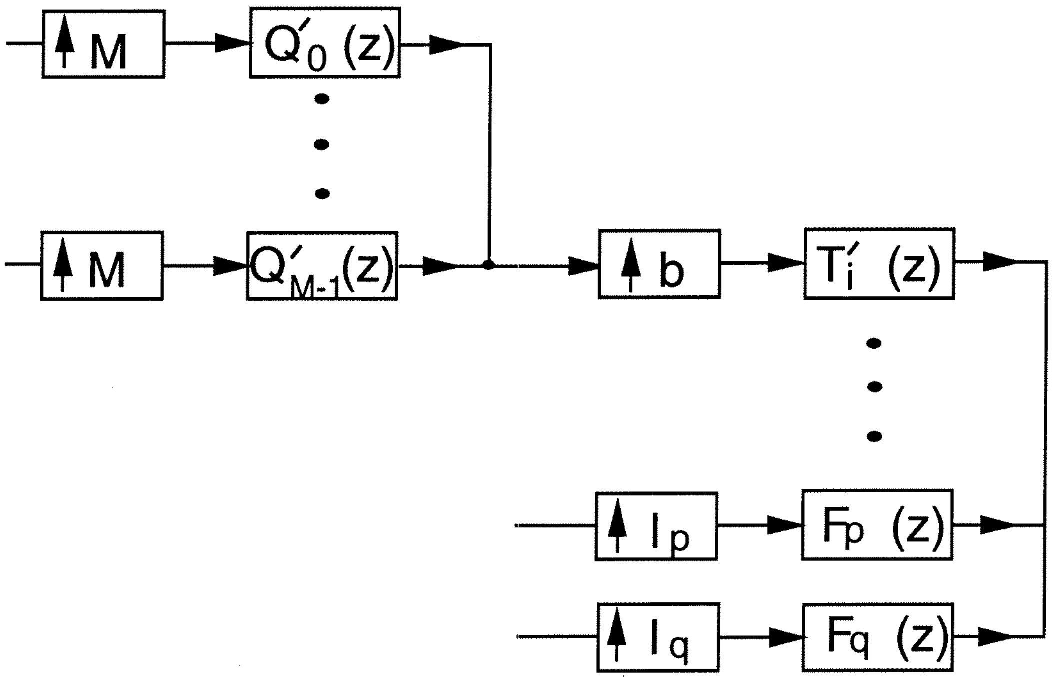


Fig. 2.14 (c). A filter bank showing only one final level with modified filters

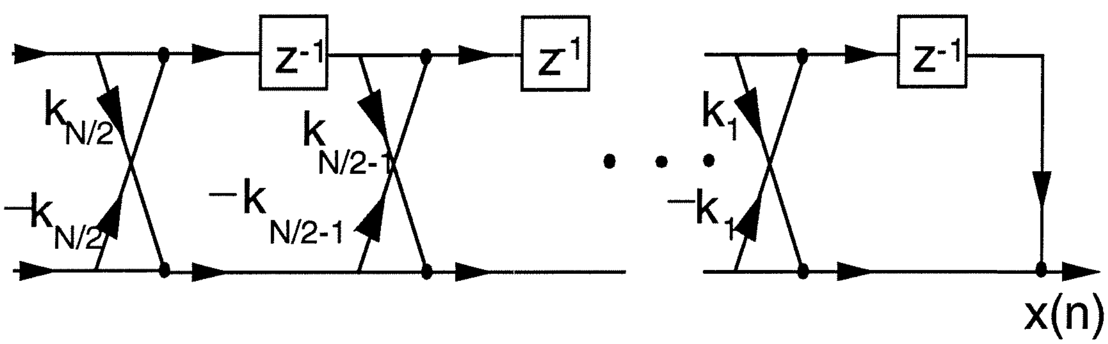


Fig. 2.15. Lattice structure for implementing a two channel synthesis bank with paraunitary polyphase matrix

Fig. 2.13. $P_0(z)$ and $P_1(z)$ are the sum and difference of two filters in Fig. 2.11(b), as defined in the Fig. 2.13. Note that these filters form an orthonormal basis, too however, these cannot be generated using a tree-structure. The reason is as follows. For this filter bank to be represented as a tree, we need that $P_0(z)$ be expressible as $A_1(z^3)C_0(z)$ or as $B_0(z^3)C_2(z)$ (compare with Fig. 2.11(b)). Neither is possible if $C_0(z) \neq C_2(z)$. But since $C_0(z)$ and $C_2(z)$ came from a PU-set to start with, the condition $C_0(z) \neq C_2(z)$ is guaranteed (by orthonormality). Thus, this filter bank cannot be generated using a tree structure.

We can, however, prove the following weaker result.

Theorem 4: Let $F_k(z)$ be a set of FIR transfer functions satisfying eq. (5.4). If they can be generated using a tree-structured filter bank, they can be generated *specifically* by a tree having PU-sets on all levels.

Before proving this theorem we will prove the following two lemmas.

Lemma 8 : Let $F_k(z)$ be a set of FIR basis functions which can be generated using a tree-structured filter bank, and let them satisfy the orthonormality condition (eq. (5.4)). Then, the filters on an input level of the tree can be made to form a PU-set.

Proof: Given the FIR nature of the transfer functions involved, we shall assume without loss of generality that they are all causal. Consider the filter bank shown in Fig. 2.14(a). The filters $Q_i(z), \quad i = 0, \ldots, M-1$ are the filters on an input level of the tree. This can be redrawn as in Fig. 2.14(b). We are told that the functions $F_m(z)$ form an orthonormal basis, i.e., they satisfy eq. (5.4). We are to show that it is possible to choose a set of filters $Q'_i(z)$ in Fig. 2.14(c) for the input level such they form a PU-set.

Now, by orthonormality,

$$(F_k(z)\widetilde{F}_l(z))\downarrow_{g_6} = \delta(k-l), \quad g_6 = bM \quad k,l = 0, \ldots, M-1, \tag{6.22}$$

where,

$$F_k(z) = Q_k(z^b)T_i(z), \quad k = 0, \ldots, M-1. \tag{6.23}$$

Hence eq. (6.22) means,

$$(Q_k(z^b)T_i(z)\widetilde{Q}_l(z^b)\widetilde{T}_i(z))\downarrow_{b.M} = \delta(k-l), \quad k,l = 0,\ldots,M-1, \tag{6.24}$$

i.e.,

$$(Q_k(z)\widetilde{Q}_l(z)(T_i(z)\widetilde{T}_i(z))\downarrow_b)\downarrow_M = \delta(k-l). \tag{6.25}$$

But by Lemma 7,

$$(T_i(z)\widetilde{T}_i(z))\downarrow_b = B(z)\widetilde{B}(z), \tag{6.26}$$

for some FIR $B(z)$. Substituting in eq. (6.25) we have,

$$(Q_k(z)\widetilde{Q}_l(z)(B(z)\widetilde{B}(z)))\downarrow_M = \delta(k-l). \tag{6.27}$$

Define a new set of transfer functions

$$Q_i'(z) = Q_i(z)B(z) \quad i = 0,\ldots,M-1. \tag{6.28}$$

Hence,

$$(Q_k'(z)\widetilde{Q}_l'(z)\downarrow_M = \delta(k-l) \quad k,l = 0,\ldots,M-1. \tag{6.29}$$

This means that the functions $Q_k'(z)$ $k = 0,\ldots,M-1$ form a PU-set. By Lemma 4, they cannot have a common factor, except a constant. Eq. (6.25) now becomes

$$(Q_k'(z)\widetilde{Q}_l'(z)(T_i'(z)\widetilde{T}_i'(z))\downarrow_b)\downarrow_M = \delta(k-l) \tag{6.30}$$

with

$$(T_i'(z)\widetilde{T}_i'(z))\downarrow_b = 1. \tag{6.31}$$

Note that now

$$F_k(z) = Q_k'(z^b)T_i'(z), \quad k = 0,\ldots,M-1. \tag{6.32}$$

This proves that the filters on an input level can be made to form a PU-set. In other words, Fig 2.14(a) can be redrawn as Fig. 2.14(c) where the $Q_k'(z)$, $k = 0,\ldots,M-1$ form a PU-set.

Lemma 9: Consider Fig. 2.14(c), drawn alternatively as in Fig. 2.14(b). Let $F_m(z)$ be a set of orthonormal FIR functions satisfying eq. (5.4). Remove an input level of the tree on which the filters formed a PU-set, i.e., remove the filters $Q'_k(z), \quad k = 0, \ldots, M-1$ and the interpolators ($\downarrow M$) in Fig. 2.14(c). Then the remaining part of the tree also gives an orthonormal basis.

Proof: We showed in the previous Lemma that the filters $Q'_i(z) \quad i = 0, \ldots, M-1$ constitute a PU-set on one input level of the tree. Removing these filters gives rise to a modified filter bank. We have to prove that the filters in this modified bank give an orthonormal basis, i.e., we have to show that

$$(T'_i(z)\widetilde{T}'_i(z))\downarrow_b = 1, \tag{6.33}$$

and that

$$(T'_i(z)\widetilde{F}_p(z))\downarrow_{g_8} = 0, \qquad g_8 = gcd(b, I_p). \tag{6.34}$$

We have proved eq. (6.33) while proving Lemma 8. Hence we only need to prove eq. (6.34). From the orthonormality of the original basis (Fig. 2.14(b)), we have,

$$(F_k(z)\widetilde{F}_p(z))\downarrow_{g_7} = 0 \quad k = 0, \ldots, M-1. \tag{6.35}$$

where $g_7 = gcd(Mb, I_p)$. Using eq. (6.32), this becomes,

$$(Q'_k(z^b)T'_i(z)\widetilde{F}_p(z))\downarrow_{g_7} = 0 \quad k = 0, \ldots, M-1. \tag{6.36}$$

Now, g_8 is a factor of g_7; let $g_7 = c.g_8$, where c has to be a factor of M. Hence from the above equation we have,

$$(Q'_k(z^b)T'_i(z)\widetilde{F}_p(z))\downarrow_{g_8.c} = 0 \quad k = 0, \ldots, M-1. \tag{6.37}$$

By using the noble identity, this becomes

$$(Q'_k(z^{b/g_8})(T'_i(z)\widetilde{F}_p(z))\downarrow_{g_8})\downarrow_c = 0 \quad k = 0, \ldots, M-1. \tag{6.38}$$

It can be verified that b/g_8 is indeed an integer, enabling us to write eq. (6.38). Let $b/g_8 = d$. Then it can also be verified that d and c are relatively prime. Hence using Lemma 6, we get,

$$(T_i'(z)\widetilde{F}_p(z))\downarrow_{g_8} = 0 \tag{6.39}$$

which completes the proof.

Using the above two Lemmas, Theorem 4 is easy to prove.

Proof of Theorem 4: Consider the given filter bank which is known to have been generated by using a tree-structure. The functions generated by this tree are given to form an orthonormal basis. Every tree has at least one input level. Using Lemma 8 we know that the filters on this input level can be made to form a paraunitary set. Remove these filters. By Lemma 9, the remaining tree also gives an orthonormal basis. Hence we can repeatedly apply Lemma 8 and Lemma 9 to finally reduce the given tree to a one-level tree. But we know from [16], that for a one-level tree, if the functions form an orthonormal basis, the filters have a paraunitary polyphase matrix. This proves Theorem 4.

♠

NOTE: A corollary similar to Corollary 1 can be proved in this case too. Namely, if an orthonormal basis is generated by a generalized tree-structured filter bank, then the polyphase matrices on all levels *have to* satisfy eq. (4.38). The proof involves mainly bookkeeping of constants while going over Lemma 8 and Lemma 9, and we do not reproduce it here.

VII IMPLEMENTATION OF PARAUNITARY FILTER BANKS

Perfect-reconstruction QMF filter banks have been studied before [11], [13]. The problem of design and implementation of such filter banks has been addressed by Vaidyanathan and Hoang in [18]. In this paper the authors have described a lattice structure for realizing QMF banks. The resulting filters have a paraunitary polyphase matrix. Fig. 2.15 shows this lattice structure. This lattice is robust in the sense that the paraunitariness of polyphase matrices is preserved in spite of coefficient quantization. Moreover, the lattice has a hierarchical property, i.e., higher order PU-pairs can be obtained from lower order PU-pairs simply by adding more lattice sections. Another important property of the lattice is that by changing the lattice coefficients we can generate all PU-sets. This property makes the lattice particularly important with reference to orthonormal wavelets. We showed in section IV that all possible orthonormal wavelet bases could be generated using a tree-structured filter bank which had paraunitary matrices on all levels. Thus, if we constructed the tree-structure using the above mentioned lattice, we could generate all orthonormal wavelet bases simply by manipulating the lattice coefficients. Moreover, orthonormality would be preserved under coefficient quantization.

Extensions of this structure to M-channel filter banks can be found in [11]. Results of section VI indicate that the M-channel lattice could be used to realize the 'generalized wavelet bases' mentioned therein.

VIII CONCLUSIONS

In this paper we have investigated the relationship between orthonormality of wavelet basis and paraunitariness of matrices in a tree-structured filter bank. We started by proving a few interesting results on multirate paraunitary systems in section III. Using these, in section IV, we showed that a binary tree with paraunitary matrices on all levels (possibly different) generates an orthonormal wavelet basis. More importantly, we proved that all orthonormal bases could be generated by a tree-structured filter bank having paraunitary polyphase matrices on all levels and that the polyphase matrices in fact have to be generalized paraunitary. Knowing the connection between paraunitariness and a special lattice structure, we concluded that all orthonormal wavelet bases could be generated by manipulating the coefficients of the lattice. Hence paraunitariness of polyphase matrices is a necessary and sufficient condition for wavelet orthonormality.

In section V, we have developed the equations governing orthonormality for general discrete time bases. The relation derived in this section showed that the *gcd* of the two decimation factors plays a role in the orthonormality equation for two functions.

Using these relations, in section VII, we studied the concept of orthonormality with respect to arbitrary tree structured filter banks. We showed that a tree with paraunitary polyphase matrices gives an orthonormal basis; conversely, a set of orthonormal functions which can be generated using a tree can be generated specifically by a paraunitary tree. This proves the equivalence of paraunitariness and orthonormality in the context of arbitrary tree structures. The generalized tree structure would be convenient in the analysis of waveforms in which the frequency characteristics are not monotonic, as Fig. 2.11(c) suggests.

APPENDIX A

Consider the orthonormality relation for genereralized wavelets (eq.(5.2)) reproduced below for the sake of convenience.

$$\sum_{n=-\infty}^{\infty} f_k(n - I_k m) f_l^*(n - I_l i) = \delta(k-l)\delta(m-i) \qquad \textit{for all integers } m, i.$$

Put $(n - I_k m) = p$. Hence $n = I_k m + p$. Therefore, the above equation becomes

$$\sum_{p=-\infty}^{\infty} f_k(p) f_l^*(p - (I_l i - I_k m)) = \delta(k-l)\delta(m-i). \tag{A.1}$$

Let $g = gcd(I_k, I_l)$. Hence there exists a j such that $(I_l i - I_k m) = gj \quad \textit{for all } m, i.$ Also, by Euclid's identity, there exist integers $\hat{I}_l$ and $\hat{I}_k$ such that $(I_l \hat{I}_l - I_k \hat{I}_k) = g$. Using these two facts, it can be shown that the condition (A.1) is identical to the condition

$$\sum_{p=-\infty}^{\infty} f_k(p) f_l^*(p - gi) = \delta(k-l)\delta(i), \tag{A.2}$$

where g is the *gcd* of (I_k, I_l). A change of dummy variables results in eq. (5.3).

REFERENCES

[1] R. Coifman, Y. Meyer, S. Quake and V. Wickerhauser, "Signal Processing with Wavelet Packets," Numerical Algorithms Research Group, Yale University, 1990.

[2] S. Mallat and W. L. Hwang, "Singularity Detection and Processing with Wavelets," preprint.

[3] R. Kronland-Martinet, "The wavelet transform for analysis, synthesis, and processing of speech and music sounds," Computer Music Journal, 12-4 (1988) 11-20.

[4] S. Mallat, "A theory for multiresolution signal decomposition: the wavelet representation," IEEE Trans. on Pattern Anal. Machine Intell., pp. 674-693, vol. 11, July 1989.

[5] S. Mallat, "Multifrequency Channel Decomposition of Images and Wavelet Models," IEEE Trans. on ASSP, vol. 37, pp. 2091-2110, 1989.

[6] F. Tuteur, "Wavelet Transformations in Signal Detection," Proc. of Intern. Conf. on ASSP, pp. 1435-1438, New York, April 1988.

[7] M. Antonin, M. Barlaud, P. Mathieu and I. Daubechies, "Image Coding using Vector Quantization in Wavelet Transform Domain," Proc. of Intern. Conf. on ASSP, pp. 2297-2300, Albuquerque, NM,1990.

[8] A. Grossman and J. Morlet, "Decomposition of Hardy Functions into Square Integrable Wavelets of Constant Shape," SIAM J. Math. Anal. pp. 723-736, 1984.

[9] I. Daubechies, A. Grossman, Y. Meyer, "Painless Non-Orthogonal Expansions," J. Math. Phys. pp. 1271-1283, 1986.

[10] I. Daubechies, "Orthonormal Bases of Compactly Supported Wavelets," Comm.

on Pure and App. Math., pp. 909-996, vol. 4, Nov. 1988.

[11] P. P. Vaidyanathan, "Multirate Digital Filters, Filter Banks, Polyphase Networks, and Applications: A Tutorial," Proc. of IEEE, pp. 56-93, Vol. 78, No. 1, Jan 1990.

[12] O. Rioul and M. Vetterli, "Wavelets and Signal Processing," IEEE Signal Proc. Magazine, Oct. 1991.

[13] M. J. T. Smith and T. P. Barnwell, III, "Exact Reconstruction Techniques for Tree-Structured Sub-band Coders," IEEE Trans. on ASSP, pp. 434-441, vol. 34, June 1986.

[14] P. P. Vaidyanathan, "On Power-Complementary FIR filters," IEEE Trans. on Circuits and Systems, pp. 1308-1310, vol. 32, Dec. 1985.

[15] M. Vetterli and C. Herley, "Wavelets and Filter Banks: Theory and Design," Internal Report, Columbia University, 1990.

[16] P. P. Vaidyanathan, "Lossless systems in Wavelet Transforms," Proc. IEEE Int. Symp. Circuits and Systems, Singapore, 1991.

[17] H. Zou and A. H. Tewfik, "Design and Parametrization of Compactly Supported Wavelets and Wavelet Packets," Internal Report, Univ. of Minnesota, May, 1990.

[18] P. P. Vaidyanathan and P. Q. Hoang, "Lattice Structures for Optimal design and Robust Implementation of Two-Channel Perfect Reconstruction QMF Banks," IEEE Trans. on ASSP, pp. 81-94, vol. 36, Jan. 1988.

Chapter 3

CODING GAIN IN PARAUNITARY ANALYSIS/SYNTHESIS SYSTEMS

Abstract[1]

Subband coders have been used in the past to decompose a signal into subbands. The signals in each subband are quantized before transmission. The problem of optimal bit allocation involves allocating bits to the individual quantizers from a fixed budget so as to minimize the overall reconstruction error variance. The problem has been addressed in the past for two cases, namely orthogonal transform coding, and ideal brick-wall filtering. Both of these are special cases of the so-called 'paraunitary' filter banks. The results which were proved for these special cases have been used without proof for other non-paraunitary subband coding schemes. We present here a formal proof that these bit-allocation results hold for the entire class of paraunitary subband coders. Next, we address the problem of finding an optimal paraunitary subband coder, so as to maximize the coding gain of the system.

We then analyze the bit-allocation problem for the case of paraunitary tree-structured filter banks, such as those used for generating orthonormal wavelets. The even more general case of non-uniform filter banks is next considered. In all cases we show that under optimal bit allocation, the variance of the errors introduced by each of the quantizers have to be equal. Expressions for coding gains for these systems have also been derived.

[1]To appear, IEEE Trans. on Signal Processing, June 1993.

I INTRODUCTION

Transform coding and subband coding are well-known techniques for efficiently encoding data [1]-[5]. They are used in data compression of speech, image, and other random signals. Consider the subband coding scheme shown in Fig. 3.1. In this scheme, the input signal $x(n)$ is split into M subbands in the frequency domain by a bank of filters called the analysis filters. The outputs of these filters are bandlimited, and hence we can sub-sample them. This is indicated by boxes with $\downarrow M$. The signals in each of the subbands are then independently quantized and transmitted. At the receiving end, the sampling rates in each of the subbands are increased once again to their original value by the interpolaters (indicated by $\uparrow M$). They are then passed through the synthesis filters. The outputs of the synthesis filters are combined to give the reconstructed signal $y(n)$.

Another way of representing the same scheme is shown in Fig. 3.2. In this figure, $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the polyphase matrices [6] corresponding to the analysis and synthesis filters respectively. The sequence $x(n)$ is divided into non-overlapping blocks of data by grouping together M successive samples. These samples form the components of the vector $\mathbf{x}(n)$ which is termed the M-fold blocked version of $x(n)$. Formally, we write

$$\mathbf{x}^T(n) = [x(nM)\ x(nM-1)\ \ldots\ x(nM-M+1)]. \tag{1.1}$$

Each of these blocks or *vectors* is encoded by the linear transformation $\mathbf{E}(z)$. The M outputs, i.e., the components of the vector $\mathbf{v}(n)$ are the subband signals which are quantized and transmitted. At the receiver, the received vector $\mathbf{u}(n)$ is passed through the transformation $\mathbf{R}(z)$. The output $\mathbf{y}(n)$ is 'unblocked' to give the reconstructed sequence $y(n)$. A special case of this scheme is the transform coding scheme [1], [2] which we shall now review.

Transform coding

In a traditional transform coding scheme, the polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ mentioned earlier are chosen to be constant matrices. The matrix at the transmitting end is often chosen to be an orthogonal matrix ($\mathbf{A}$ say), so that if the matrix $\mathbf{A}^T$ is used at the receiver, the system becomes a perfect reconstruction system (i.e., $y(n) = x(n)$) in the absence of quantizers. In the presence of quantizers, a natural objective in such a coding system is to minimize the reconstruction error between the input and the output. The variance of the reconstruction error is chosen as a suitable criterion for minimization [2]. Define the error vector to be the difference between the input vector and the output vector, i.e.,

$$\mathbf{r}(n) = \mathbf{y}(n) - \mathbf{x}(n). \tag{1.2}$$

Assuming this error to be a zero-mean, Wide Sense Stationary (abbreviated *WSS*) vector-process, the reconstruction error variance is,

$$\sigma_r^2 = (\frac{1}{M})E[\mathbf{r}^T(n)\mathbf{r}(n)]. \tag{1.3}$$

In a conventional Pulse Code Modulation (PCM) scheme, the input samples are independently quantized and transmitted over the channel. This is equivalent to making the transform matrix in Fig. 3.2 an identity matrix. The coding gain [2] of the transform coding system is defined as the ratio of the error variance in a PCM system to the error variance in the transform coding system, i.e.,

$$G_{TC} = \frac{\sigma^2_{r,PCM}}{\sigma^2_{r,TC}}. \tag{1.4}$$

Now let us turn to the individual quantizers. Let R_i be the number of bits allocated to the quantizer Q_i and let $\sigma^2_{v_i}$ be the variance of the input to that quantizer. The range of values that the input to the quantizer can take is divided into 2^{R_i} intervals. The weight of the most significant bit is taken to be proportional to σ_{v_i}, so that the probability of overflow is the same for each i. The variance of the error introduced by the ith quantizer is then given as [2],

$$\sigma_i^2 = \epsilon^2 2^{-2R_i} \sigma_{v_i}^2 \quad , \tag{1.5}$$

where ϵ is a constant. Now, suppose the total number of bits available for quantizing all the M subband signals is fixed, i.e.,

$$R = (R_0 + R_1 + R_2 + \ldots + R_{M-1})/M \; = constant. \tag{1.6}$$

The design issues in the transform coding scheme then are as follows:

1) How does one allocate bits to the individual quantizers under the constraint imposed by eq. (1.6), and

2) How does one choose the orthogonal transformation $\mathbf{A}$ so as to maximize the coding gain of the system?

In the case of the transform coding schemes, it has been shown [1], [2] that the optimal allocation of bits is that which makes all the individual quantizer error variances equal. Under optimal bit allocation, it has also been shown [2] that the coding gain of the system becomes

$$G_{TC} = \frac{(1/M)(\sum_{i=0}^{M-1} \sigma_{v_k}^2)}{(\prod_{i=0}^{M-1} \sigma_{v_k}^2)^{1/M}}, \tag{1.7}$$

which is the ratio of the arithmetic mean to the geometric mean of the $\sigma_{v_k}^2$. This is maximized if the transform matrix $\mathbf{A}$ is the Karhunen-Loeve Transform (KLT).

Paraunitary Filter Banks

A linear transformation $\mathbf{E}(z)$ is said to be paraunitary if it satisfies [6]

$$\widetilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I}, \tag{1.8}$$

where $\widetilde{\mathbf{E}}(z)$ is obtained from $\mathbf{E}(z)$ by conjugating the coefficients, transposing, and replacing z by z^{-1}. The paraunitary property is essentially an extension of the unitary property to linear, time-invariant systems with memory. Paraunitary transformations

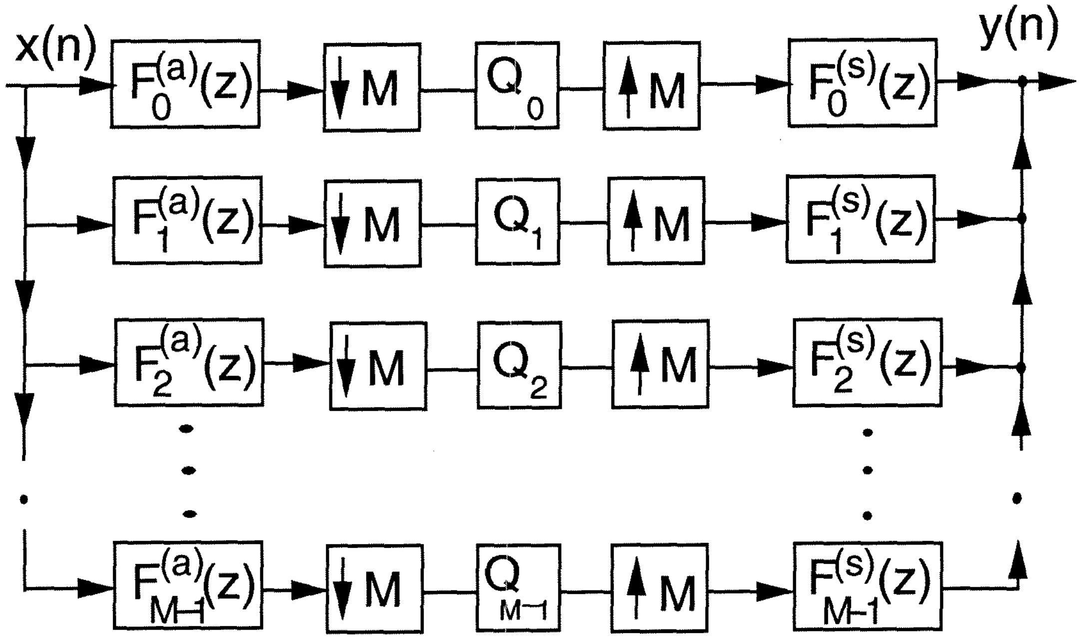


Fig. 3.1. A typical subband coding scheme

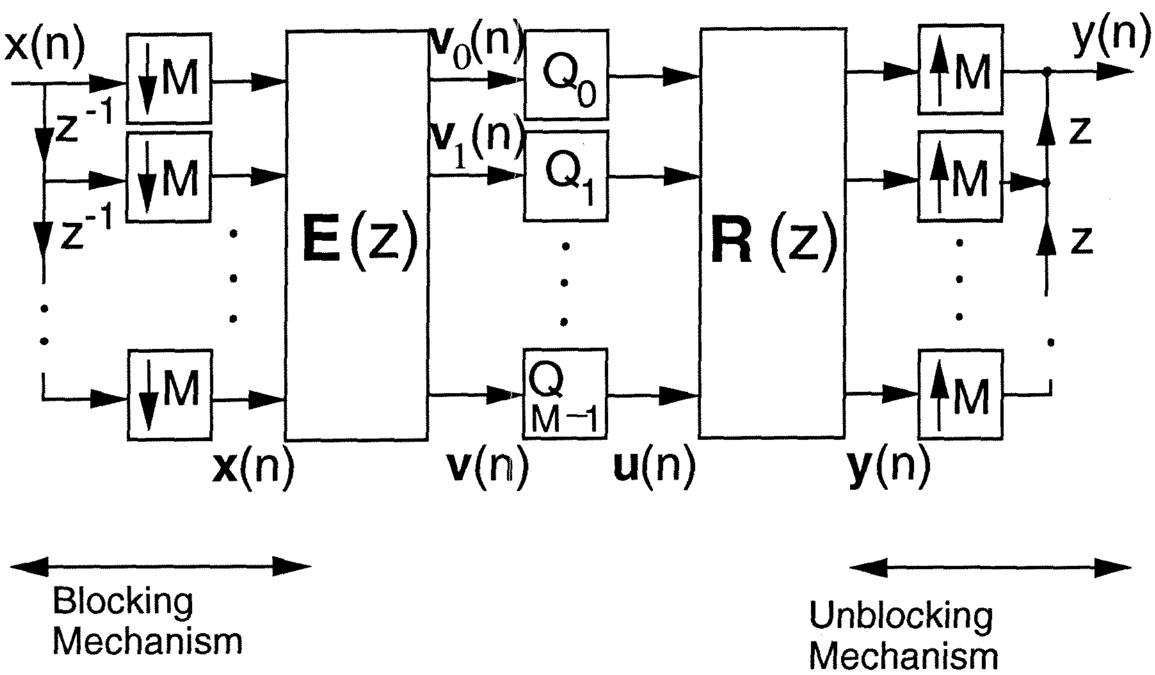


Fig. 3.2. Subband coding scheme showing polyphase matrices

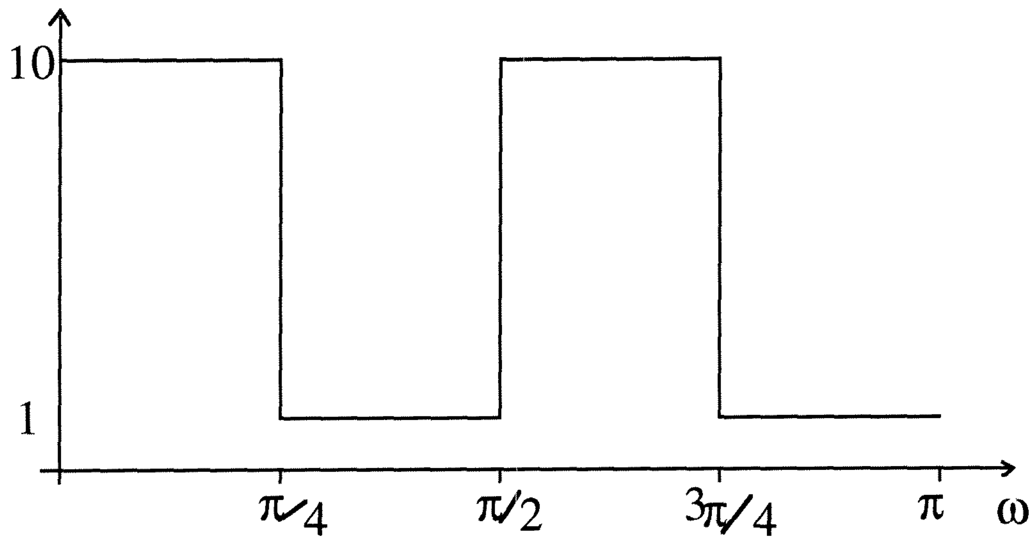


Fig. 3.3. Example of power spectrum for which brick-wall filters give zero coding gain

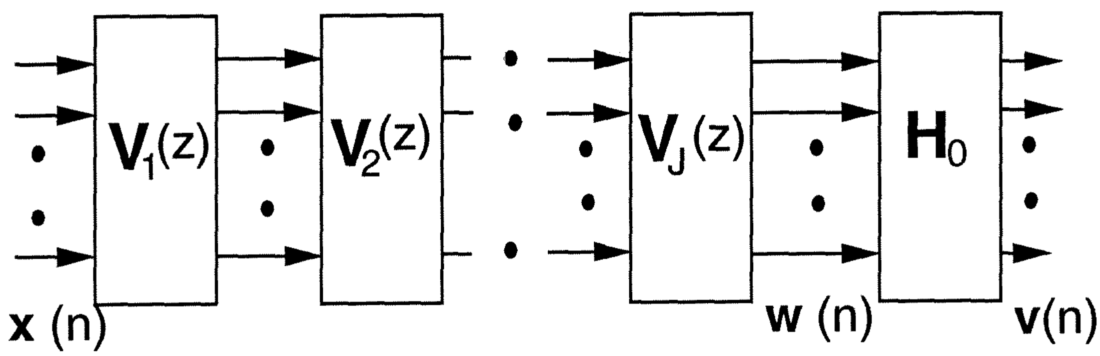


Fig. 3.4. Cascade implementation of a Paraunitary matrix

are important in subband coding because the subband coding system shown in Fig. 3.2 can be made to have perfect reconstruction property (in the absence of quantizers) by choosing the matrix $\mathbf{E}(z)$ to be paraunitary, and choosing $\mathbf{R}(z)$ to be $\widetilde{\mathbf{E}}(z)$ [7]. The analysis and synthesis filters are then related as $F_k^{(s)}(z) = \widetilde{F}_k^{(a)}(z)$. Secondly, it has been shown [6] that paraunitary transformations can be realized using lattice/cascade structures. In the case of the paraunitary subband coding system, we define the coding gain in a likewise manner, i.e.,

$$G_{PU} = \frac{\sigma^2_{r,PCM}}{\sigma^2_{r,PU}}. \tag{1.9}$$

One can ask questions similar to those asked before, namely,

1) How does one allocate bits to the quantizers, and

2) How does one choose the paraunitary transformation $\mathbf{E}(z)$ so as to maximize the coding gain?

The transform coding scheme discussed previously is one special case of paraunitary subband coders. Now consider ideal subband coders (using brickwall filters as shown in Fig. 7.46 in [4]). This can also be shown (by invoking eq. (36) in [7]) to be a special case of paraunitary subband coders. In this case, too, under optimal bit allocation, eq. (1.7) holds [2]. Eq. (1.7) has been used without proof in the context of Lapped Orthogonal Transform (LOT) in [8] also, which are a special case of paraunitary systems.

The problem of optimal bit allocation itself has been mentioned in [3], [9]-[11] for non-paraunitary subband coders. In this paper, we derive conditions for bit-allocation optimality of general paraunitary subband coders and formally prove that a result similar to eq. (1.7) holds for this entire class. We also address the problem of finding the optimal paraunitary transformation so as to maximize the coding gain. Next we consider the bit-allocation problem in the context of paraunitary tree-structured filter banks such as those used for generating wavelets [12], [13]. Our final extension of this analysis is to the case of general non-uniform filter banks.

Notations and Definitions

The notations used in this paper are as follows: Bold-faced quantities denote matrices and vectors, as in $\mathbf{E}$ or $\mathbf{x}$. $\mathbf{A}^T$ denotes the transpose of the matrix $\mathbf{A}$. $Tr(\mathbf{A})$ denotes the trace of the matrix $\mathbf{A}$, i.e., it is the sum of the diagonal terms of the matrix. $det(\mathbf{A})$ denotes the determinant of the matrix $\mathbf{A}$. The tilde-notation, as in $\widetilde{\mathbf{E}}(z)$, stands for conjugation of coefficients followed by transposition, followed by replacing z by z^{-1}. As defined earlier, a matrix $\mathbf{E}(z)$ is said to be paraunitary if it satifies eq. (1.8). Note that if $\mathbf{E}(z)$ is a square paraunitary matrix, it also satisfies

$$\mathbf{E}(z)\widetilde{\mathbf{E}}(z) = \mathbf{I}. \tag{1.10}$$

We shall deal with FIR matrices with real coefficients. The FIR nature is required if we constrain both the analysis and synthesis filters to be stable [6]. Let N be the order of $\mathbf{E}(z)$;

$$\mathbf{E}(z) = \mathbf{e}(0) + \mathbf{e}(1)z^{-1} + \mathbf{e}(2)z^{-2} + \ldots + \mathbf{e}(N)z^{-N}. \tag{1.11}$$

Therefore,

$$\widetilde{\mathbf{E}}(z) = \mathbf{e}^T(0) + \mathbf{e}^T(1)z + \mathbf{e}^T(2)z^2 + \ldots + \mathbf{e}^T(N)z^N. \tag{1.12}$$

Expressed in time-domain the square paraunitary relations (1.8) and (1.10) imply

$$\sum_m \mathbf{E}^T(m-l)\mathbf{E}(m) = \delta(l)\mathbf{I}, \tag{1.13}$$

$$\sum_m \mathbf{E}(m-l)\mathbf{E}^T(m) = \delta(l)\mathbf{I}. \tag{1.14}$$

All signals considered are real. A vector random process $\mathbf{x}(n)$ is said to be WSS if $E[\mathbf{x}(n)] = \mathbf{m_x}$, independent of n, and

$$E[\mathbf{x}(n)\mathbf{x}^T(n-k)] = \mathbf{R}(k) \quad \textit{for all } n, k. \tag{1.15}$$

A random process $x(n)$ is Cyclo Wide Sense Stationary with a period M, abbreviated as $(CWSS)_M$, if its M-fold blocked version $\mathbf{x}(n)$ as defined in eq. (1.1) is a WSS

vector process. Conversely, if $\mathbf{x}(n)$ is a WSS vector process, then the sequence $x(n)$ obtained by unblocking it is $(CWSS)_M$. A vector process $\mathbf{p}(n)$ of size ML is said to be an M-fold blocked version of another vector process $\mathbf{s}(n)$ of size L if they are related as

$$\mathbf{p}^T(n) = [\mathbf{s}^T(nM)\ \mathbf{s}^T(nM-1)\ldots\mathbf{s}^T(nM-M+1)]. \tag{1.16}$$

We say a vector process $\mathbf{s}(n)$ is $(CWSS)_M$ if its M-fold blocked version $\mathbf{p}(n)$ as defined above is a WSS vector process. Let $\mathbf{R}(z)$ be a multi-input multi-output system. The matrix $\mathbf{B}(z)$ is called the M-fold blocked version of the matrix $\mathbf{R}(z)$ if

$$\mathbf{B}(z) = \begin{bmatrix} \mathbf{R}_0(z) & \mathbf{R}_1(z) & \mathbf{R}_2(z) & \ldots & \mathbf{R}_{M-1}(z) \\ z^{-1}\mathbf{R}_{M-1}(z) & \mathbf{R}_0(z) & \mathbf{R}_1(z) & \ldots & \mathbf{R}_{M-2}(z) \\ z^{-1}\mathbf{R}_{M-2}(z) & z^{-1}\mathbf{R}_{M-1}(z) & \mathbf{R}_0(z) & \ldots & \mathbf{R}_{M-3}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{-1}\mathbf{R}_1(z) & \ldots & \ldots & \ldots & \mathbf{R}_0(z) \end{bmatrix} \tag{1.17}$$

where $\mathbf{R}_i(z)$ are the polyphase components [6] of the original matrix $\mathbf{R}(z)$, given by

$$\mathbf{R}(z) = \mathbf{R}_0(z^M) + z^{-1}\mathbf{R}_1(z^M) + z^{-2}\mathbf{R}_2(z^M) + \ldots + z^{-(M-1)}\mathbf{R}_{M-1}(z^M). \tag{1.18}$$

The reason for calling $\mathbf{B}(z)$ the blocked version of $\mathbf{R}(z)$ is as follows. Let $\mathbf{s}(n)$ be an input to the system $\mathbf{R}(z)$, and let $\mathbf{y}(n)$ be the corresponding output. Let $\mathbf{p}(n)$ be the M-fold blocked version of $\mathbf{s}(n)$ as in eq. (1.16), and let $\mathbf{y}_B(n)$ be the M-fold blocked version of $\mathbf{y}(n)$. Then it can be verified that the input $\mathbf{p}(n)$ to the system $\mathbf{B}(z)$ produces the output $\mathbf{y}_B(n)$. The proof of this when $\mathbf{R}(z)$ is a scalar can be found in [14], and the case where $\mathbf{R}(z)$ is a matrix is a straightforward generalization of this proof.

In the figures, the acronym $MIMO$ stands for multi-input, multi-output. LTI stands for linear, time-invariant. Boxes with $\downarrow M$ and $\uparrow L$ stand for decimation by a factor M and interpolation by a factor L respectively, as defined in [6].

II PARAUNITARY SUBBAND CODERS

The two design issues can be considered seperately. First, we present a strategy for optimal bit-allocation so as to minimize the reconstruction error variance. This bit-allocation will hold irrespective of the paraunitary transformation used. We will then deal with the problem of finding an optimal FIR paraunitary transformation $\mathbf{E}(z)$.

Bit allocation result

First we prove the following lemma:

Lemma 1 : Consider a paraunitary multi-input, multi-output system $\mathbf{E}(z)$. Let $\mathbf{x}(n)$, the input to this system, be a zero-mean vector WSS process, and let $\mathbf{y}(n)$ be the output vector. Then,

$$E[\mathbf{x}^T(n)\mathbf{x}(n)] = E[\mathbf{y}^T(n)\mathbf{y}(n)]. \tag{2.1}$$

Proof: Let $\mathbf{S_{xx}}(e^{j\omega})$ and $\mathbf{S_{yy}}(e^{j\omega})$ be the power spectra of the vector sequences $\mathbf{x}(n)$ and $\mathbf{y}(n)$. Then,

$$\mathbf{S_{yy}}(e^{j\omega}) = \mathbf{E}(e^{j\omega})\mathbf{S_{xx}}(e^{j\omega})\mathbf{E}^\dagger(e^{j\omega}). \tag{2.2}$$

Since $Tr\left[\mathbf{E}(e^{j\omega})\mathbf{S_{xx}}(e^{j\omega})\mathbf{E}^\dagger(e^{j\omega})\right] = Tr\left[\mathbf{E}^\dagger(e^{j\omega})\mathbf{E}(e^{j\omega})\mathbf{S_{xx}}(e^{j\omega})\right]$, we have,

$$Tr\left[\mathbf{S_{yy}}(e^{j\omega})\right] = Tr\left[\mathbf{S_{xx}}(e^{j\omega})\right]. \tag{2.3}$$

Integrating the diagonal terms on both sides of the above equation gives us the required result.

♠

Now consider the system shown in Fig. 3.2. $\mathbf{x}(n)$ and $\mathbf{y}(n)$, both vectors of size M, are the blocked versions of $x(n)$ and $y(n)$ respectively. Let the quantizers be modeled as zero-mean, WSS noise sources, uncorrelated with the input. Not that we do *not* assume the noise sources to be white or mutually uncorrelated. Let the total number of bits allocated to all the quantizers be fixed (eq. 1.6).

Theorem 1: Let $\mathbf{E}(z)$ be a FIR paraunitary matrix and let $\mathbf{R}(z) = \widetilde{\mathbf{E}}(z)$. Then the reconstruction error variance of the system in Fig. 3.2 is minimized when the variances σ_i^2 of the errors introduced by each of the quantizers are equal, i.e., $\sigma_i^2 = \sigma^2$ for all i.

Proof : Let $\mathbf{q}(n) = \mathbf{u}(n) - \mathbf{v}(n)$ and $\mathbf{r}(n) = \mathbf{y}(n) - \mathbf{x}(n)$. Here, $\mathbf{q}(n)$ is the vector whose individual components are the quantizer errors, and $\mathbf{r}(n)$ is the reconstruction error vector. So $\mathbf{r}(n)$ is the output of $\widetilde{\mathbf{E}}(z)$ in response to $\mathbf{q}(n)$, or equivalently, $\mathbf{q}(n)$ is the output of the system $\mathbf{E}(z)$ in response to the input $\mathbf{r}(n)$. Hence using eq. (1.11) we have

$$\mathbf{q}(n) = \sum_{m=0}^{N} \mathbf{e}(m)\mathbf{r}(n-m). \tag{2.4}$$

Therefore, applying Lemma 1,

$$E[\mathbf{q}^T(n)\mathbf{q}(n)] = E[\mathbf{r}^T(n)\mathbf{r}(n)]. \tag{2.5}$$

From the definition of $\mathbf{q}(n)$, we have

$$E[\mathbf{q}^T(n)\mathbf{q}(n)] = \sum_{i=0}^{M-1} \sigma_i^2. \tag{2.6}$$

so that $E[\mathbf{r}^T(n)\mathbf{r}(n)] = \sum_{i=0}^{M-1}\sigma_i^2$. Our problem is therefore to minimize $\sum_{i=0}^{M-1}\sigma_i^2$. Since the $\sigma_{v_i}^2$ depend only on the input statistics and the paraunitary transformation $\mathbf{E}(z)$, we can show (using eq. 1.5) that eq. (1.6) is equivalent to the condition $\left(\prod_{i=0}^{M-1}\sigma_i^2\right)^{1/M} = \textit{constant}$. We know [15] that the arithmetic mean of a set of non-negative numbers is always greater than or equal to their geometric mean, i.e., $\sum_{i=0}^{M-1}\sigma_i^2 \geq M\left(\prod_{i=0}^{M-1}\sigma_i^2\right)^{1/M} = Mc^{1/M}$, with equality if and only if the σ_i^2 are the same for all i. Thus the above $\sum \sigma_i^2$ in eq. (2.6) is minimized if and only if all the σ_i^2 are equal, i.e.,

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = ... = \sigma_{M-1}^2. \tag{2.7}$$

We can therefore show that the following is the optimal bit allocation:

$$R_i = R + (1/2)log_2\left[\frac{\sigma_{v_i}^2}{(\prod_{i=0}^{M-1}\sigma_{v_i}^2)^{1/M}}\right]. \tag{2.8}$$

In [2], [3], it was shown that the above equation holds for two special cases of paraunitary transformations. One must remember that it assumes high bit rates.

Optimal Paraunitary Transforms

This problem has been mentioned in literature for a few special types of paraunitary transformations. In [8], the author has dealt with the problem in context of the LOT, which are degree one paraunitary transformations with a particular form. A more recent work [16] deals with the Extended Lapped Transform (ELT) which are paraunitary transformations of higher degrees, but again constrained to take a special form. By optimizing the filter responses, the author demonstrates coding gains approaching those of filter banks with ideal (brick-wall) filters. However, optimizing the filter responses is not necessarily the appropriate strategy, because ideal filters need not necessarily maximize the coding gain. To see this consider by way of an example a power spectral density which is as shown in Fig. 3.3. A two-channel filter bank with brick-wall filters gives a coding gain of unity, whereas a filter bank with filters $F_1^{(a)}(z) = 1 + z^{-1}$ and $F_2^{(a)}(z) = 1 - z^{-1}$ gives a coding gain of 1.0238.

In [8], the problem of finding the optimal basis functions for the LOT so as to maximize the coding gain has been formulated as a constrained optimization problem, to be solved by the method of Lagrange multipliers. The optimum basis functions are found in a sequential manner. However, it is not clear that such a sequential optimization would yield a global minimum. Another open problem therein is whether or not the optimal basis functions are the eigenvectors of the so-called 'extended autocorrelation matrix,' as they are in the case of the KLT. The extended autocorrelation matrix is the autocorrelation matrix of size ML, corresponding to the input sequence, where $L - 1$ is the order of the paraunitary transformation. We have the following proposition:

Proposition 2.1: The optimal basis functions are not necessarily the eigenvectors of the extended autocorrelation matrix.

Proof: Consider a two-channel paraunitary subband coding system. We know that the extended autocorrelation matrix is positive-definite and Toeplitz. It is easy to construct a positive-definite Toeplitz matrix with distinct eigenvalues. Hence it is a valid autocorrelation matrix of some process. Its eigenvectors are either symmetric or antisymmetric, i.e, the resulting filters have to be linear phase. However, we know that [6] a two-channel linear phase paraunitary subband coding system can only have trivial filters.

We suggest the following scheme for directly finding an optimal paraunitary transform so as to maximize the coding gain.

We have $\sigma^2_{r,PU} = (1/M)\sum_{i=0}^{M-1}\sigma_i^2$ which under optimal bit-allocation becomes $(\prod_{i=0}^{M-1}\sigma_i^2)^{1/M}$. Therefore, using eq. (1.5) we get

$$\sigma^2_{r,PU} = \epsilon^2 2^{-2R}(\prod_{i=0}^{M-1}\sigma^2_{v_k})^{1/M}. \tag{2.9}$$

Now, $\sigma^2_{r,PCM} = \epsilon^2(1/M)2^{-2R}\sum_{i=0}^{M-1}\sigma^2_{v_i}$, and hence the coding gain of the paraunitary system becomes

$$G_{PU} = \frac{(1/M)(\sum_{i=0}^{M-1}\sigma^2_{v_k})}{(\prod_{i=0}^{M-1}\sigma^2_{v_k})^{1/M}}, \tag{2.10}$$

where $\sigma^2_{v_k}$ is the variance of the input to the kth quantizer. Note that this is true only for paraunitary transform matrices, and not for arbitrary subband coders. Both the brickwall subband coder and the orthogonal transform coder satisfy this because both are special paraunitary subband coders.

Using Lemma 1, we have that $\sum_{i=0}^{M-1}\sigma^2_{v_k} = \sum_{i=0}^{M-1}\sigma^2_{x_k}$, where $\sigma^2_{x_k}$ is the variance of kth element if $\mathbf{x}(n)$. Hence the numerator in eq. (2.10) is completely determined by the input statistics. The problem therefore is to minimize $(\prod_{i=0}^{M-1}\sigma^2_{v_k})^{1/M}$ in the denominator of eq. (2.10). For a two-channel system, this is equivalent to minimizing $\sigma^2_{v_1}$. For the general case, let $\mathbf{R}_{\mathbf{xx}}(i)$ and $\mathbf{R}_{\mathbf{vv}}(i)$ denote the autocorrelation matrices of the vector random sequences $\mathbf{x}(n)$ and $\mathbf{v}(n)$ respectively. From matrix theory we know that $det(\mathbf{R_{vv}}(0)) \leq (\prod_{i=0}^{M-1} \sigma_{v_k}^2)$, with equality if and only if the matrix $\mathbf{R_{vv}}(0)$ is diagonal. In transform coding case, $det(\mathbf{R_{vv}}(0)) = det(\mathbf{R_{xx}}(0))$, and is hence determined by the input statistics. The coding gain G_{TC} is maximized by making $det(\mathbf{R_{vv}}(0)) = (\prod_{i=0}^{M-1} \sigma_{v_k}^2)$, i.e., by chosing the transform to diagonalize $\mathbf{R_{xx}}(0)$. This is done by the KLT. In paraunitary subband coding however, $det(\mathbf{R_{vv}}(0))$ is not invariant and can in fact be made less than $det(\mathbf{R_{xx}}(0))$. Thus the problem is to choose the paraunitary transformation $\mathbf{E}(z)$ (of a fixed degree) so as to minimize $det(\mathbf{R_{vv}}(0))$.

From [17], we know that every FIR paraunitary matrix $\mathbf{E}(z)$ of degree J can be written as

$$\mathbf{E}(z) = \mathbf{H}_0 \mathbf{V}_J(z) \mathbf{V}_{J-1}(z) \ldots \mathbf{V}_1(z). \tag{2.11}$$

Here, $\mathbf{H}_0$ is a constant unitary matrix, and the $\mathbf{V}_i(z)$ are degree-one paraunitary systems of the form

$$\mathbf{V}_i(z) = \mathbf{I} - \mathbf{v}_i \mathbf{v}_i^\dagger + \mathbf{v}_i \mathbf{v}_i^\dagger z^{-1}, \tag{2.12}$$

where $\mathbf{v}_i$ are unit norm vectors (Fig. 3.4). Thus the unit norm vectors $\mathbf{v}_i$ and the constant orthogonal matrix $\mathbf{H}_0$ completely specify the paraunitary system.

The proposed optimization of the coding gain proceeds as follows. With reference to Fig. 3.4, for given input statistics, it is possible to evaluate the $det(\mathbf{R_{ww}}(0))$ in terms of the system parameters (vectors $\mathbf{v}_i$). Minimization of this determinant can then be carried out using an iterative minimization scheme such as the one based on the quasi-Newton techniques. We used a standard subroutine EO4JAF from the NAG FORTRAN library [18].

After having carried out the minimization of the said determinant, the final block in Fig. 3.4, which is the constant unitary matrix $\mathbf{H}_0$, is chosen to be the KLT matrix whose columns are the eigenvectors of the matrix $\mathbf{R_{ww}}(0)$. $\mathbf{H}_0$ cannot alter the value of $det(\mathbf{R_{ww}}(0))$, and hence the choice of $\mathbf{H}_0$ does not enter into the optimization process directly.

Experimental results: Figs. 3.5 and 3.6 show the maximum possible coding gain of paraunitary subband coders for different number of channels M. The abscissa indicates the degree of the paraunitary transformation, J. Hence the length of the filters in each case is less than or equal to $M(J+1)$. A paraunitary transformation of order zero implies the usual KLT. The input in Fig. 3.5 was bandpass speech, whereas in Fig. 3.6 it was lowpass speech. Fig. 3.7 shows the responses of the resulting filters after optimizing for the coding gain. They correspond to the case in which the input was lowpass speech.

From the plots, we find that it is possible to achieve significant improvements in the coding gain by using paraunitary transformations. Moreover, the optimal coding gain seems to saturate quickly with increasing degree of the paraunitary transformation, so that it is sufficient to use transformations of small degrees.

III THE DISCRETE-TIME WAVELET TRANSFORM

Now consider a further extension of the subband coding idea described above, namely, tree-structured filter banks [19], [20]. Fig. 3.8 shows a special case of a 3-level, binary tree-structured filter bank, drawn in terms of the polyphase matrices of the filters on each level; but the following discussion holds for a general L-level binary tree-structured filter bank.

The input signal $x(n)$ is coded (or 'transformed') by passing it through the analysis bank (Fig. 3.8(a)). The synthesis bank (Fig. 3.8(b)) performs the inverse transformation on the quantized versions of $y_k(n)$. There are several ways to ensure that this system has the perfect-reconstruction property. One of these is to choose the analysis filters $G_{a_i}(z)$, $H_{a_i}(z)$ such that their polyphase matrix $\mathbf{E}_i(z)$ is paraunitary and then choose the synthesis filters as

$$G_{s_i}(z) = \widetilde{G}_{a_i}(z), \qquad H_{s_i}(z) = \widetilde{H}_{a_i}(z) \quad i = 0, \ldots, L-1. \tag{3.1}$$

Choosing the synthesis filters in this manner means that the polyphase matrices

$\mathbf{R}_i(z)$ corresponding to the synthesis filters are also paraunitary. It also means that the synthesis filters are non-causal, but since they are FIR, non-causality does not matter.

The relation between the above filter bank system and *wavelet transforms* has been known for quite some time [12], [21]-[24]. With reference to Fig. 3.8(a), the quantities $y_k(n)$ are called the wavelet transform coefficients of the signal $x(n)$. Assuming that perfect-reconstruction property holds, (in the absence of quantizers) we have $y(n) = x(n)$, and we can express

$$x(n) = \sum_{k=0}^{L-1} \sum_{m=-\infty}^{\infty} y_k(m) f_k^{(s)}(n - 2^{k+1}m) + \sum_{m=-\infty}^{\infty} y_L(m) f_L^{(s)}(n - 2^L m), \qquad (3.2)$$

with $f_k^{(s)}(n)$ being the impulse response of the filter $F_k^{(s)}(z)$. In other words, we have obtained an expansion for the signal $x(n)$ in terms of the wavelet coefficients $y_k(n)$ and the wavelet basis functions $f_k^{(s)}(n - 2^{k+1}m)$.

In the above perfect-reconstruction system, if the polyphase matrices at each level of the analysis bank are paraunitary, then the wavelet basis can be shown to be orthonormal [23], which is often a desirable property. The basis functions then satisfy the relations [23]

$$\sum_{n=-\infty}^{\infty} f_k^{(s)}(n - 2^{k+1}m) f_l^{(s)*}(n - 2^{l+1}i) = \delta(k-l)\delta(m-i), \quad k, l = 0, \ldots, L-1$$

$$\sum_{n=-\infty}^{\infty} f_k^{(s)}(n - 2^{k+1}m) f_L^{(s)*}(n - 2^L i) = 0, \quad k = 0, \ldots, L-1$$

$$\sum_{n=-\infty}^{\infty} f_L^{(s)}(n - 2^L m) f_L^{(s)*}(n - 2^L i) = \delta(m-i). \qquad (3.3)$$

Here $f_l^{(s)*}(n)$ is obtained from $f_l^{(s)}(n)$ by conjugation of coefficients.

For an L-level binary tree, let $\mathbf{x}(n)$ be the 2^L-fold blocked version of the input $x(n)$, and $\mathbf{y}(n)$ be the 2^L-fold blocked version of the output $y(n)$. Let $\mathbf{r}(n)$ be the error vector as defined in eq. (1.2). We now repeat our question on optimal bit allocation: assuming that the polyphase matrices on each level of the binary tree

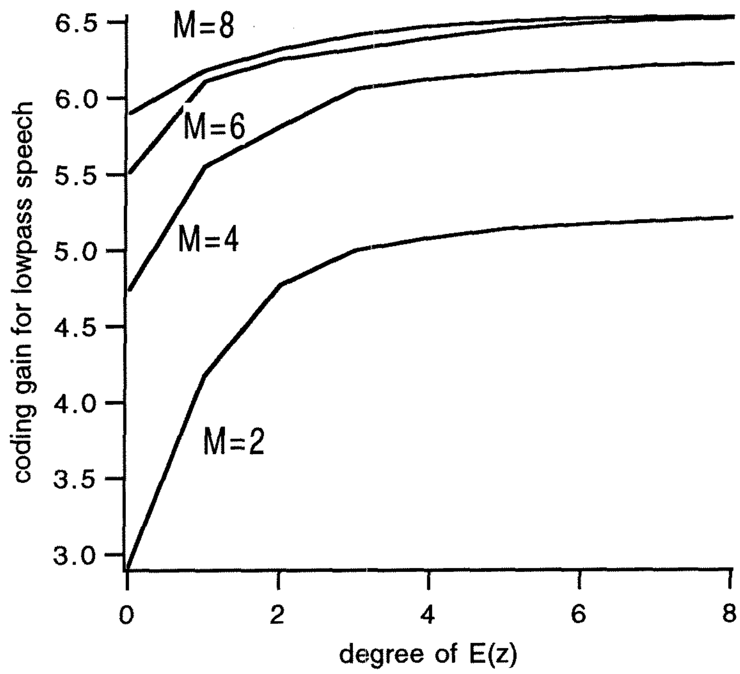

M=8
M=6
M=4
M=2
coding gain for lowpass speech
6.5
6.0
5.5
5.0
4.5
4.0
3.5
3.0
0
2
4
6
8
degree of E(z)

Fig. 3.5.

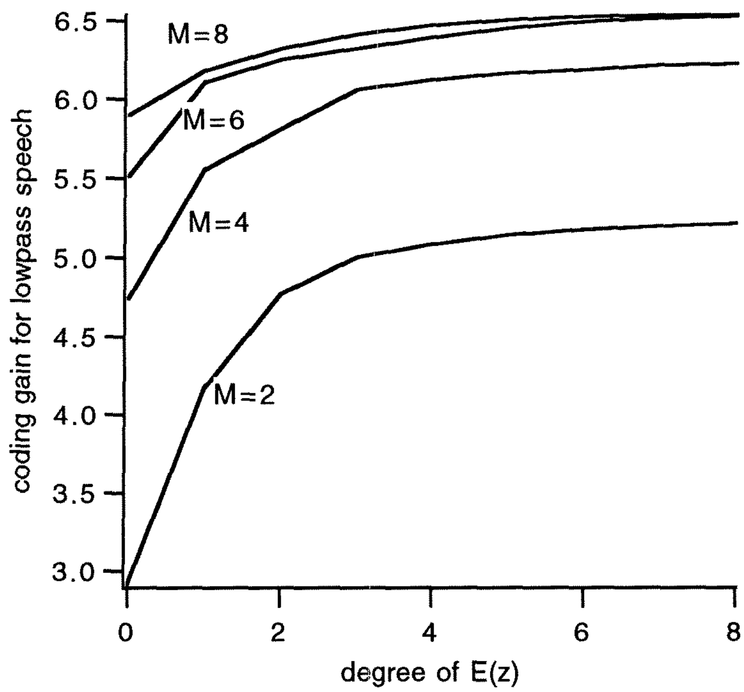

M=8
M=6
M=4
M=2
coding gain for lowpass speech
6.5
6.0
5.5
5.0
4.5
4.0
3.5
3.0
0
2
4
6
8
degree of E(z)

Fig. 3.6.

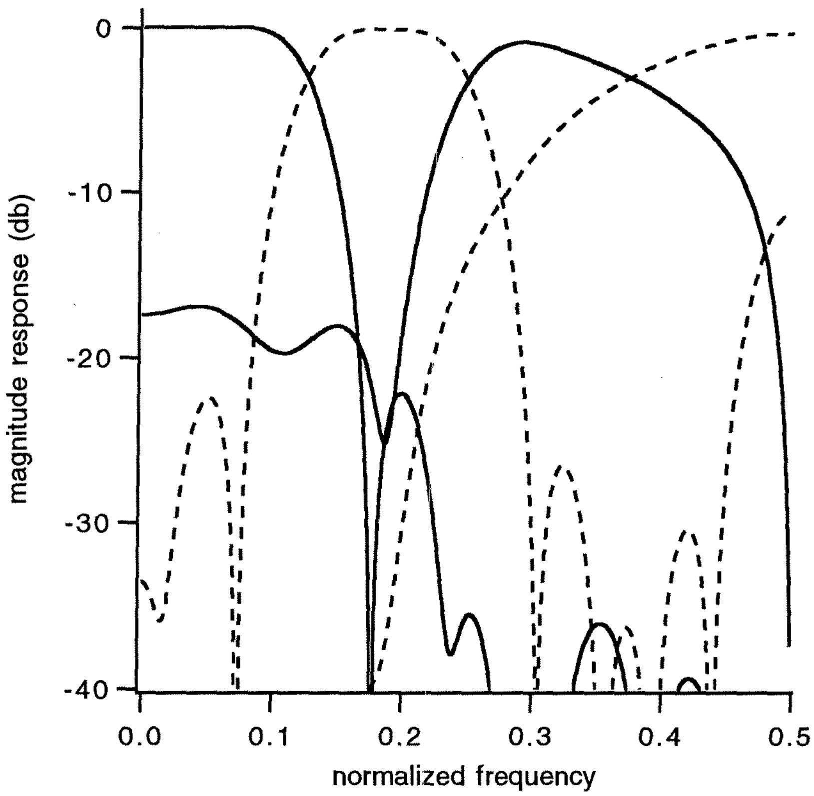


Fig. 3.7. Filter responses after optimizing for coding gain

are paraunitary, and with proper statistical assumptions, what is the optimal way to allocate the bits among the quantizers (with their total number being fixed) so that the reconstruction error variance is minimized? We will answer this question in Theorem 2. Our result can be extended to any perfect-reconstruction sub-band coding system with paraunitary polyphase matrices.

Bit allocation result

We first state two lemmas which shall be used in the proof of our main result of this section.

Lemma 2 : Let $\mathbf{R}(z)$ be an N by N paraunitary matrix. Then its M-fold blocked version $\mathbf{B}(z)$ as defined in eq. (1.17) is also paraunitary.

Proof : Consider the system shown in Fig. 3.9(a). $\mathbf{s}(n)$ and $\mathbf{y}(n)$ are vectors of size N. Since the matrix $\mathbf{R}(z)$ is paraunitary, we have,

$$\sum_n \mathbf{y}^T(n)\mathbf{y}(n) = \sum_n \mathbf{s}^T(n)\mathbf{s}(n). \tag{3.4}$$

Now consider the system in Fig. 3.9(b). $\mathbf{B}(z)$ is the M-fold blocked version of $\mathbf{R}(z)$. Let $\mathbf{p}(n)$ be the M-fold blocked version of $\mathbf{s}(n)$, i.e.,

$$\mathbf{p}^T(n) = [\mathbf{s}^T(nM)\ \mathbf{s}^T(nM-1)\ \ldots\ \mathbf{s}^T(nM-M+1)], \tag{3.5}$$

and let $\mathbf{t}(n)$ be the M-fold blocked form of the corresponding output $\mathbf{y}(n)$, i.e.,

$$\mathbf{t}^T(n) = [\mathbf{y}^T(nM)\ \mathbf{y}^T(nM-1)\ \ldots\ \mathbf{y}^T(nM-M+1)]. \tag{3.6}$$

$\mathbf{p}(n)$ and $\mathbf{t}(n)$ are both vectors of length NM. For an arbitrary vector sequence of length M input to the system $\mathbf{R}(z)$ in Fig. 3.9(a), the total energy at the output equals the total input energy. Thus for the system in Fig. 3.9(b), the output energy equals the input energy for any input $\mathbf{p}(n)$, and so $\mathbf{B}(z)$ is paraunitary [25].

Lemma 3: Consider the system in Fig. 3.10(a) :

(a) Let $\mathbf{s}^T(n) = [s_0(n)\ s_1(n)]$. Then, if $\mathbf{s}(n)$ is a $(CWSS)_{2^M}$ vector process, $y(n)$ is $(CWSS)_{2^{M+1}}$.

(b) Further, let $\mathbf{s}_0(n)$ and $\mathbf{s}_1(n)$ denote vectors of length 2^M formed by blocking $s_0(n)$ and $s_1(n)$ respectively, and $\mathbf{y}_B(n)$ denote the vector of length 2^{M+1} formed by blocking $y(n)$. Then, if $\mathbf{R}(z)$ is paraunitary,

$$E[\mathbf{y}_B^T(n)\mathbf{y}_B(n)] = E[\mathbf{s}_0{}^T(n)\mathbf{s}_0(n)] + E[\mathbf{s}_1{}^T(n)\mathbf{s}_2(n)] \tag{3.7}$$

Proof: Note that $\mathbf{y}_B(n)$ can alternatively be defined as the 2^M-fold blocked version of the vector process $\mathbf{y}(n)$ shown in Fig. 3.10(a). We can redraw the system in Fig. 3.10(a) as in Fig. 3.10(b), where $\mathbf{p}(n)$ and $\mathbf{y}_B(n)$ are vectors of length 2^{M+1}. Also, notice that $\mathbf{p}(n)$, the 2^M-fold blocked version of $\mathbf{s}(n)$ is a WSS vector process.

The matrix $\mathbf{B}(z)$ is as follows:

$$\mathbf{B}(z) = \begin{bmatrix} \mathbf{R}_0(z) & \mathbf{R}_1(z) & \mathbf{R}_2(z) & \dots & \mathbf{R}_{2^M-1}(z) \\ z^{-1}\mathbf{R}_{2^M-1}(z) & \mathbf{R}_0(z) & \mathbf{R}_1(z) & \dots & \mathbf{R}_{2^M-2}(z) \\ z^{-1}\mathbf{R}_{2^M-2}(z) & z^{-1}\mathbf{R}_{2^M-1}(z) & \mathbf{R}_0(z) & \dots & \mathbf{R}_{2^M-3}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{-1}\mathbf{R}_1(z) & \dots & \dots & \dots & \mathbf{R}_0(z) \end{bmatrix}$$

where $\mathbf{R}_i(z)$ are the polyphase components of the original matrix $\mathbf{R}$, given by

$$\mathbf{R}(z) = \mathbf{R}_0(z^{2^M}) + z^{-1}\mathbf{R}_1(z^{2^M}) + z^{-2}\mathbf{R}_2(z^{2^M}) + \ldots + z^{-(2^M-1)}\mathbf{R}_{2^M-1}(z^{2^M}). \tag{3.8}$$

Since $\mathbf{y}_B(n)$ is obtained by passing $\mathbf{p}(n)$ through a linear system, $\mathbf{y}_B(n)$ is also WSS. Hence, $y(n)$, which is formed by unblocking $\mathbf{y}_B(n)$, is $(CWSS)_{2^{M+1}}$, proving part (a).

Note that since $\mathbf{B}(z)$ is a blocked version of a paraunitary matrix, it is also paraunitary (Lemma 2). Therefore, using Lemma 1, we get

$$E[\mathbf{y}_B^T(n)\mathbf{y}_B(n)] = E[\mathbf{p}^T(n)\mathbf{p}(n)]. \tag{3.9}$$

But from the definition of these quantites,

$$E[\mathbf{p}^T(n)\mathbf{p}(n)] = E[\mathbf{s}_0^T(n)\mathbf{s}_0(n)] + E[\mathbf{s}_1{}^T(n)\mathbf{s}_1(n)]. \tag{3.10}$$

Therefore,

$$E[\mathbf{y}_B^T(n)\mathbf{y}_B(n)] = E[\mathbf{s}_0^T(n)\mathbf{s}_0(n)] + E[\mathbf{s}_1{}^T(n)\mathbf{s}_1(n)], \tag{3.11}$$

proving part (b).

♠

We now present the main result of this section. Consider a general L-level tree structured FIR-filter bank such as one used for generating wavelet basis functions (Fig. 3.8 shows a special case of a three-level binary tree). Let the polyphase matrices $\mathbf{E}_i(z)$ be paraunitary, and let $\mathbf{R}_i(z) = \widetilde{\mathbf{E}}_i(z)$, so that $y(n) = x(n)$ in the absence of quantizers. Let the quantizers be modelled as zero-mean, WSS, mutually uncorrelated noise sources. Let the total bit rate be constant.

Theorem 2: The reconstruction error variance of the binary tree-structured filter bank is minimized when the variances σ_i^2 of the errors introduced by each of the quantizers are equal, i.e., $\sigma_i^2 = \sigma^2$ for all i.

Proof: Consider a general tree-structured analysis-synthesis system. The polyphase matrices $\mathbf{E}_i(z)$ and $\mathbf{R}_i(z)$ on each level are assumed paraunitary. The bits allocated to each quantizer R_i are related to the variance of the error introduced by that quantizer σ_i^2 as in eq. (1.5).

Since the system in the absence of quantizers performs perfect-reconstruction, and since we have assumed that noise and signal are uncorrelated, it is sufficient to consider only the synthesis bank to study the effect of noise on the final reconstruction error. Consider the system in Fig. 3.11. The error-signals $e_i(n)$ are all WSS. Hence, in particular, they are $(CWSS)_M$ for any integer M.

By Lemma 3, we know that $s_{L-1}(n)$ is $(CWSS)_2$. Now, $e_{L-2}(n)$ is also $(CWSS)_2$ and $e_{L-2}(n)$ and $s_{L-1}(n)$ are uncorrelated and, hence, they are jointly $(CWSS)_2$. Applying Lemma 3 again, we have that $s_{L-2}(n)$ is $(CWSS)_4$. In general, $e_k(n)$ and $s_{k+1}(n)$ are jointly $(CWSS)_{2^{L-k-1}}$; hence, $s_k(n)$ is $(CWSS)_{2^{L-k}}$.

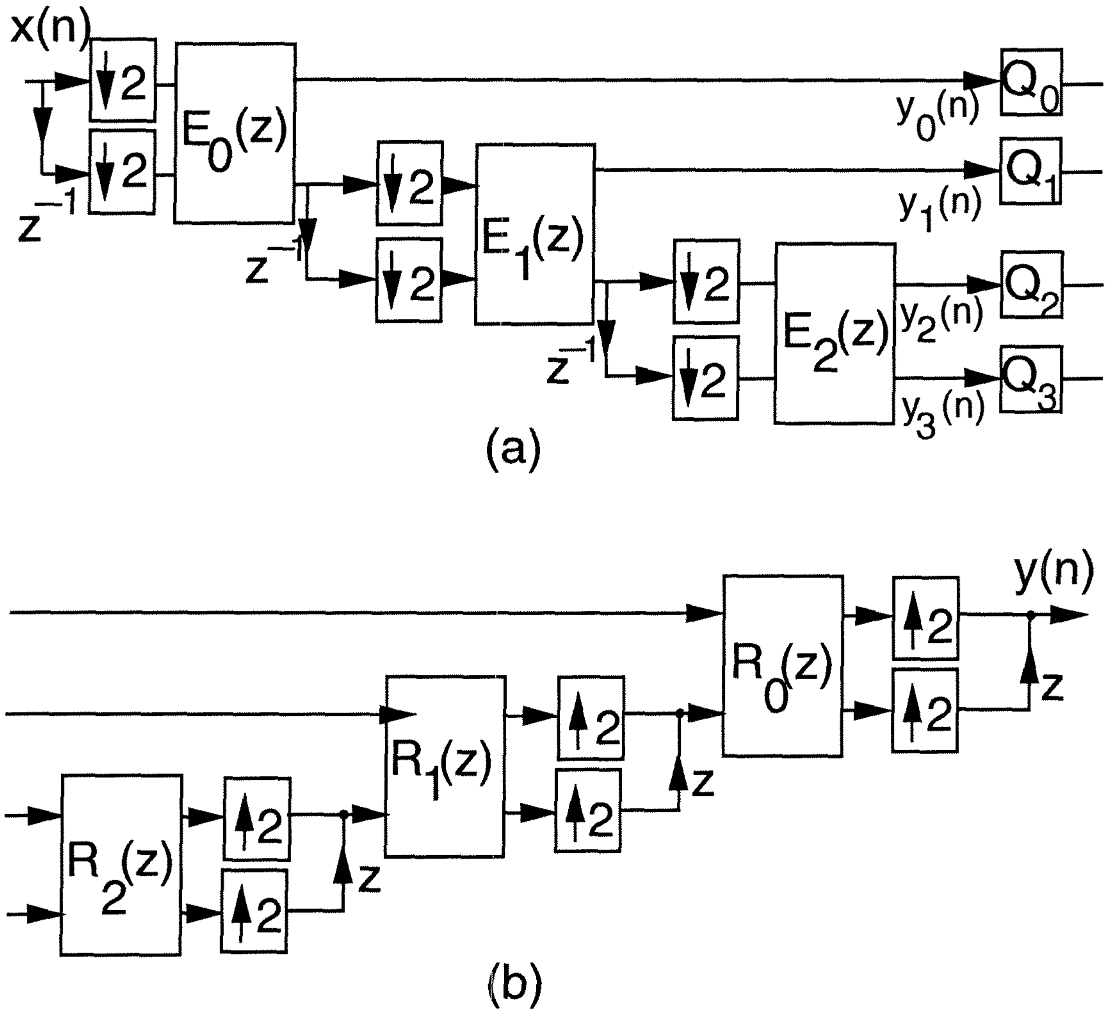
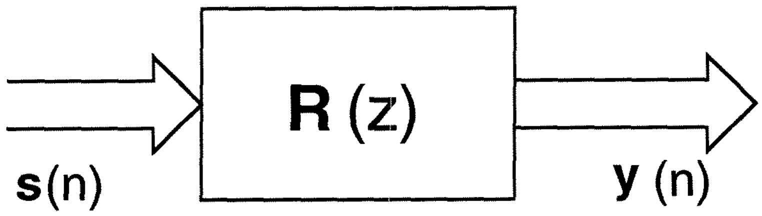
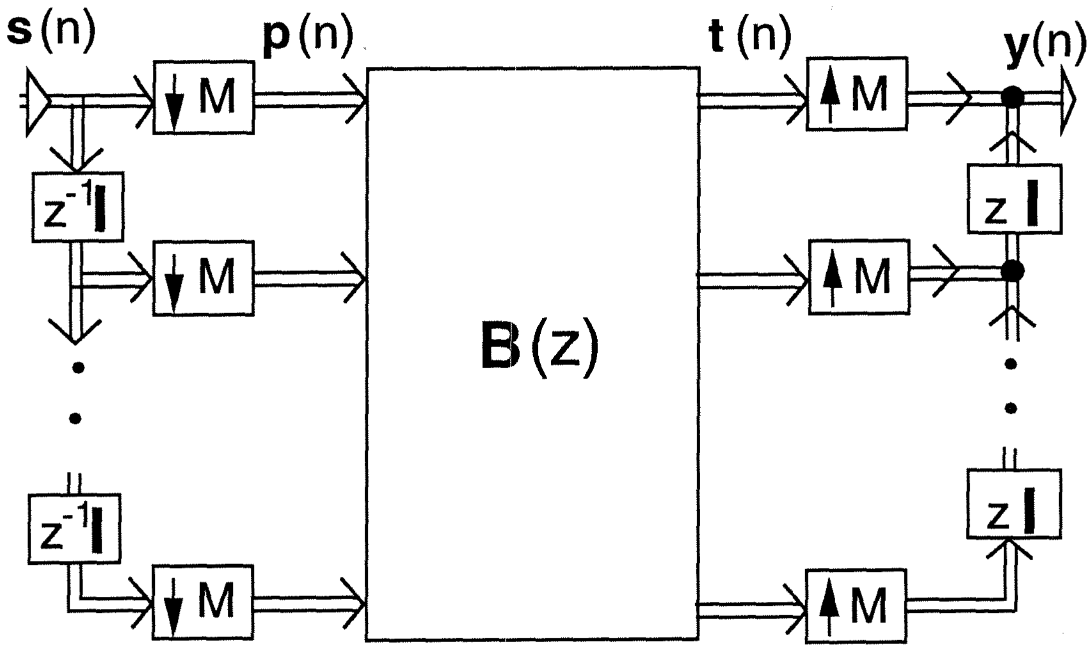


Fig. 3.8. A binary tree-structured QMF bank in terms of the polyphase matrices
(a) the analysis bank
(b) the synthesis bank



(a)



(b)

Fig. 3.9. (a) A MIMO system, and
(b) its M-fold blocked version

Let $\mathbf{E}_k(n)$ and $\mathbf{s}_{k+1}(n)$ denote the 2^{L-k-1}-fold blocked versions of $e_k(n)$ and $s_{k+1}(n)$ respectively for $k = 0, \ldots, L-2$; for example, $\mathbf{s}_0(n)$ is a 2^L-fold blocked version of $s_0(n)$, whereas $\mathbf{E}_0(n)$ and $\mathbf{s}_1(n)$ are 2^{L-1} fold blocked versions of $e_0(n)$ and $s_1(n)$ respectively, and so on. The tree can be redrawn in terms of the polyphase matrices on all levels, similar to Fig. 3.8.

By applying Lemma 3 to each level of the tree, we get the following set of equalities:

$E[\mathbf{y}^T(n)\mathbf{y}(n)] = \left(E[\mathbf{E}_0^T(n)\mathbf{E}_0(n)] + E[\mathbf{s}_1{}^T(n)\mathbf{s}_1(n)]\right),$

$E[\mathbf{s}_1{}^T(n)\mathbf{s}_1(n)] = \left(E[\mathbf{E}_1^T(n)\mathbf{E}_1(n)] + E[\mathbf{s}_2{}^T(n)\mathbf{s}_2(n)]\right),$

$E[\mathbf{s}_2{}^T(n)\mathbf{s}_2(n)] = \left(E[\mathbf{E}_2^T(n)\mathbf{E}_2(n)] + E[\mathbf{s}_3^T(n)\mathbf{s}_3(n)]\right)$, in general,

$$E[\mathbf{s}_k^T(n)\mathbf{s}_k(n)] = \left(E[\mathbf{E}_k^T(n)\mathbf{E}_k(n)] + E[\mathbf{s}_{k+1}^T(n)\mathbf{s}_{k+1}(n)]\right), \quad k = 0, .., L-2, \qquad (3.12)$$

and finally,

$$E[\mathbf{s}_{L-1}^T(n)\mathbf{s}_{L-1}(n)] = \left(E[e_{L-1}^2(n)] + E[e_L^2(n)]\right). \qquad (3.13)$$

Hence from the above equations, one can write,

$$\begin{aligned} E[\mathbf{y}^T(n)\mathbf{y}(n)] &= E[\mathbf{E}_0^T(n)\mathbf{E}_0(n)] + E[\mathbf{E}_1^T(n)\mathbf{E}_1(n)] + E[\mathbf{E}_2^T(n)\mathbf{E}_2(n)] + \\ &\quad \ldots + E[e_{L-1}(n)e_{L-1}(n)] + E[e_L(n)e_L(n)]) \qquad (3.14) \end{aligned}$$

i.e., the variance of the overall reconstruction error $\frac{1}{2^L}E[\mathbf{y}^T(n)\mathbf{y}(n)] = \sigma_r^2$ is given as

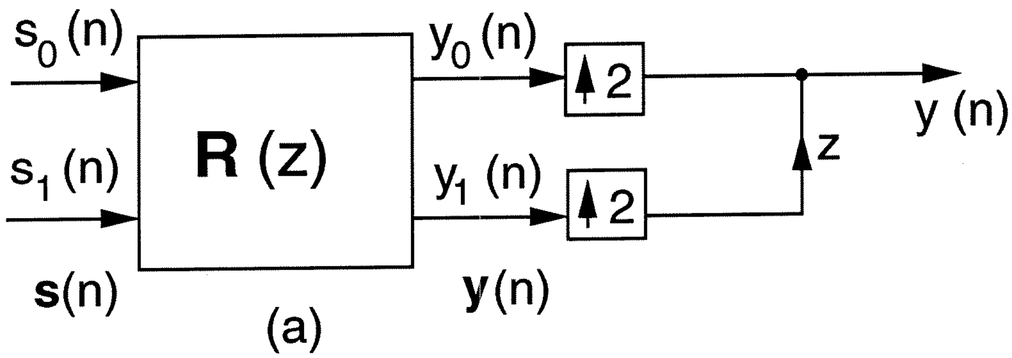
$$\sigma_r^2 = (\frac{1}{2^L})\left(2^{L-1}\sigma_0^2 + 2^{L-2}\sigma_1^2 + 2^{L-3}\sigma_2^2 + \ldots + \sigma_{L-1}^2 + \sigma_L^2\right), \qquad (3.15)$$

where σ_i^2 are the individual quantizer variances. Our constraint is that the total bit-rate is constant. This means that

$$(2^{L-1}R_0 + 2^{L-2}R_1 + 2^{L-3}R_2 + \ldots + R_{L-1} + R_L)/2^L = constant = R. \qquad (3.16)$$

To obtain the optimal bit allocation, we minimize the reconstruction error variance σ_r^2 in eq. (3.15) under the above constraint on the bit-rate. This can be done by the method of Lagrange multipliers, and it gives the following optimal bit allocation:

$$R_i = R + (1/2)log_2\left[\frac{\sigma_{v_i}^2}{D}\right] \qquad i = 0, \ldots, L, \qquad (3.17)$$



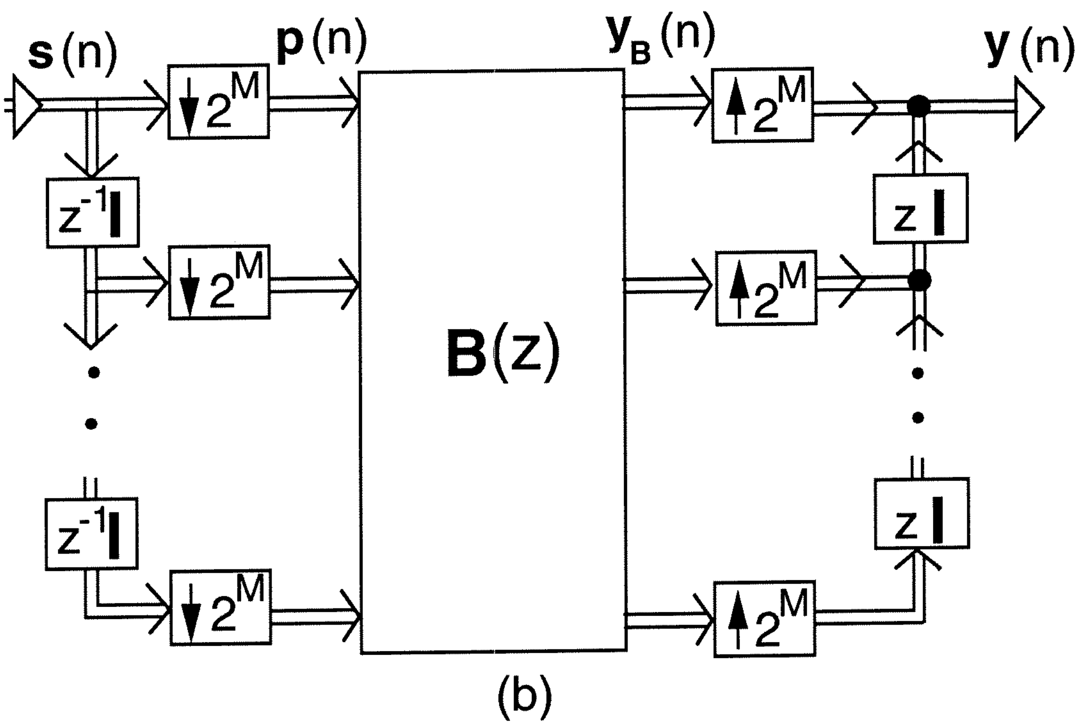


Fig. 3.10. (a) A two-input one-output system
(b) Blocked version of the system in (a)

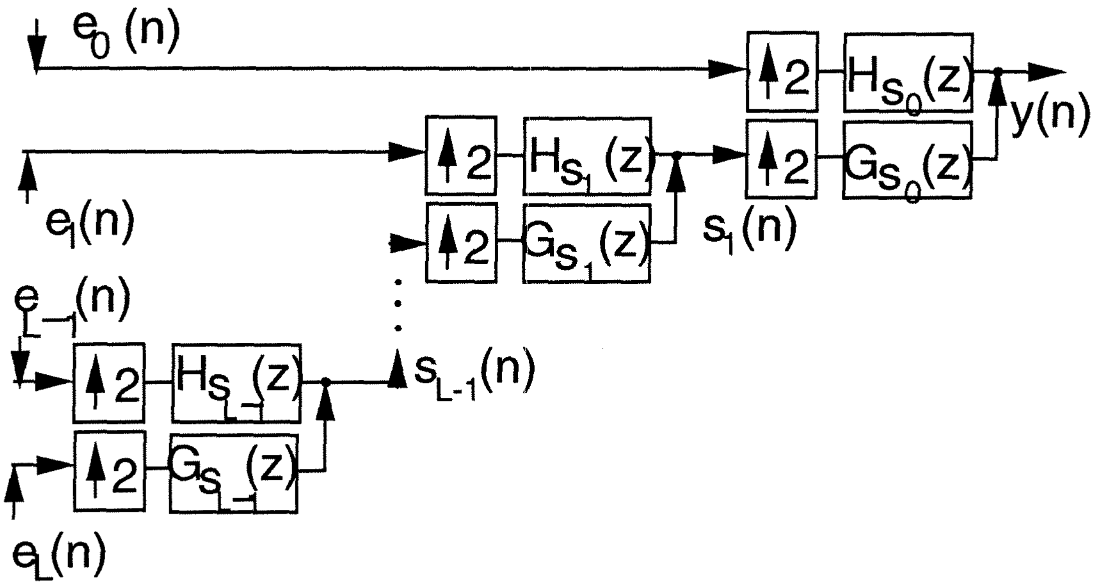


Fig. 3.11. A binary tree-structured synthesis bank

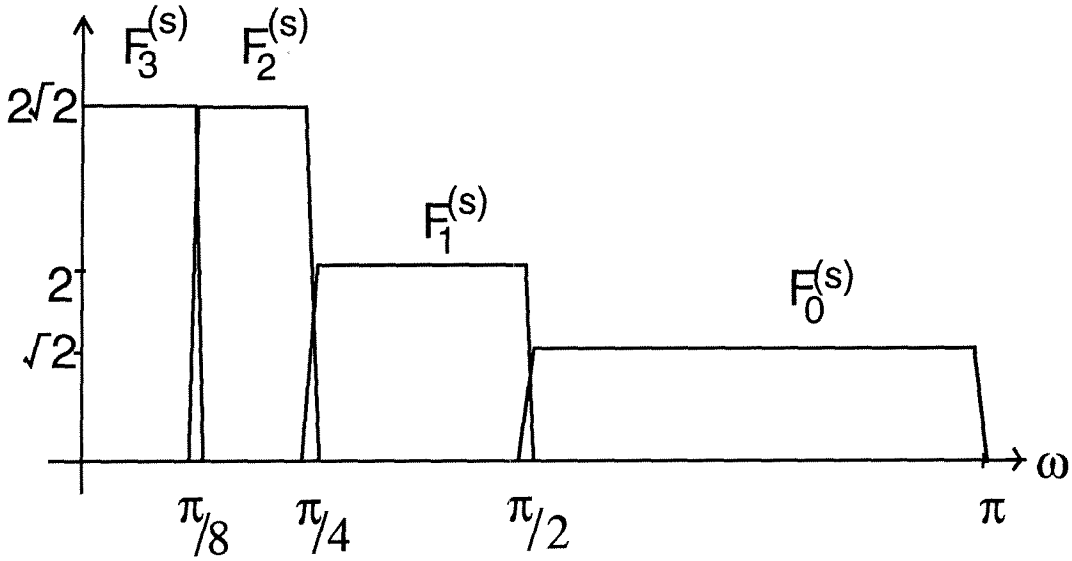


Fig. 3.12. Typical appearances of magnitude responses of synthesis filters in a 3-level tree.

where $D = (\sigma_{v_L}^2)^{1/2^L} \prod_{j=0}^{L-1} (\sigma_{v_j}^2)^{1/2^{j+1}}$ and $\sigma_{v_i}^2$ are the variances of the inputs to the quantizers. It can be verified that under optimal bit allocation the variances of the errors introduced by each of the quantizers are equal.

As in the case of Theorem 1, the above result is valid only for high bit rates.

The total reconstruction error variance under optimal bit allocation becomes

$$\sigma_r^2 = \epsilon^2 D 2^{-2R}. \tag{3.18}$$

The coding gain of the system is, therefore,

$$G_{PU} = \sigma_x^2 / D, \tag{3.19}$$

where σ_x^2 is the variance of the input.

To obtain a physical insight into this result, consider the typical appearances of the filter responses in a tree-structured bank (Fig. 3.12, for a 3-level tree). If the polyphase matrices on each level of the tree are paraunitary, these filters have equal energy irrespective of their frequency characteristics. This can be proved as follows. Putting $k = l$ and $m = i = 0$ in eq. (3.3) we get

$$\sum_{n=-\infty}^{\infty} f_k^{(s)}(n) f_k^{(s)*}(n) = 1, \quad k = 0, \ldots, L. \tag{3.20}$$

This means that all filters have unit energy. Hence it is indeed appealing intuitively to equalize the variance of the error in each subband.

Notice that in the proof of the above result, the assumption that the noise sources are mutually uncorrelated is a slightly stronger assumption than we actually need. It would have been enough to assume that the vectors $\mathbf{s}_k(n)$, $k = 0, \ldots, L-1$ are WSS.

IV BIT ALLOCATION IN ORTHONORMAL NON-UNIFORM FILTER BANKS

Consider the filter bank shown in Fig. 3.13. This is an example of a general non-uniform filter bank. The numbers n_i need not necessarily be powers of 2, as they were in the case of the discrete-time wavelet transforms. Non-uniform filter banks with perfect reconstruction property have been shown to exist. The filter bank is said to be orthogonal, if the synthesis filters satisfy the following condition [26]

$$\sum_{n=-\infty}^{\infty} f_k^{(s)}(n) f_l^{(s)*}(n-gi) = \delta(k-l)\delta(i), \tag{4.1}$$

where g is the greatest common divisor of (n_k, n_l). One way to realize perfect reconstruction non-uniform filter banks is to generate them via tree-structures. Orthonormality of the resulting non-uniform system can be ensured by choosing the polyphase matrices on each level to be paraunitary [26]. Hence, the analysis of the previous section can be extended in a straightforward manner to orthogonal non-uniform filter banks which arise from tree-structures. However, it must be noted that not all orthonormal non-uniform filter banks can be generated using tree-structures [26]. In this case, a different approach needs to be taken to arrive at the bit-allocation results. This is the topic of the present section.

The trick is to reduce the orthogonal non-uniform filter bank shown in Fig. 3.13 to a uniform paraunitary filter bank. This idea has found mention in [27]. This will then enable us to directly use the results developed in section II.

Let L be the least common multiplier of the decimation ratios n_i, and let $L = n_i k_i, \quad i = 0, \ldots, M$. It can be verified that $L = \sum_{i=0}^{M} k_i$. Now consider one branch of the non-uniform bank, as shown in Fig. 3.14(a). This can be redrawn as in Fig. 3.14(b). The unblocking mechanism shown only interleaves the samples (i.e., no addition of two non-zero samples takes place). Hence, the quantizer can be moved across the unblocking mechanism into each of the branches. Finally, the individual branch we started with can be redrawn as in Fig. 3.14(c). If we use the preceding technique to represent all branches of the non-uniform filter bank, the resulting system is an L-channel *uniform* filter bank. It can been shown [28] that if the original non-uniform filter bank is orthogonal, the polyphase matrix corresponding to the new L-channel uniform filter bank is paraunitary. Hence, applying Lemma 1 to the uniform

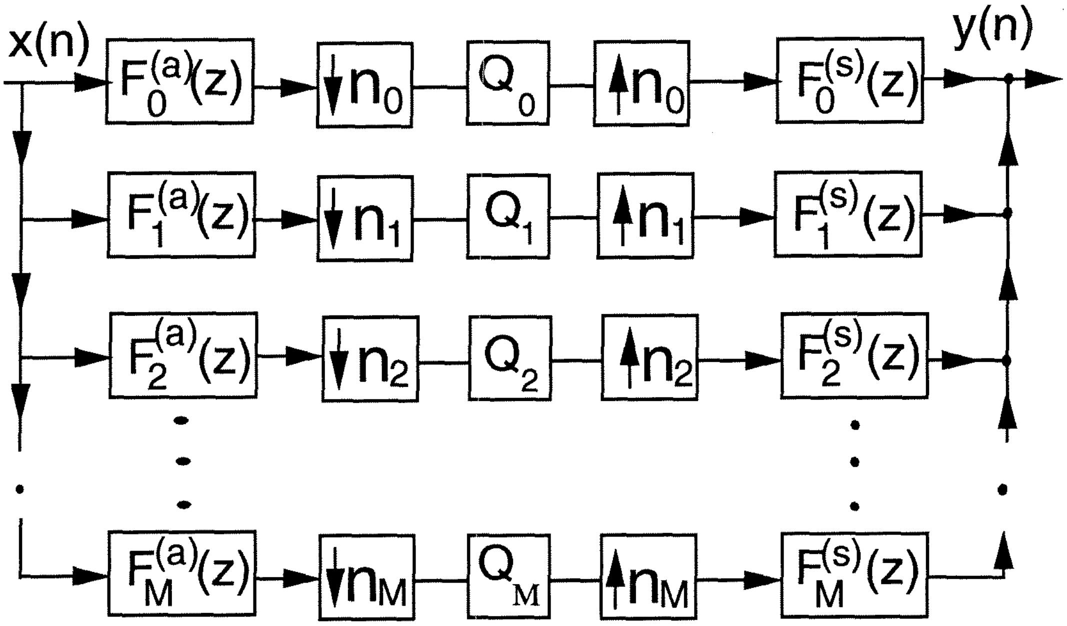


Fig. 3.13. A non-uniform filter bank

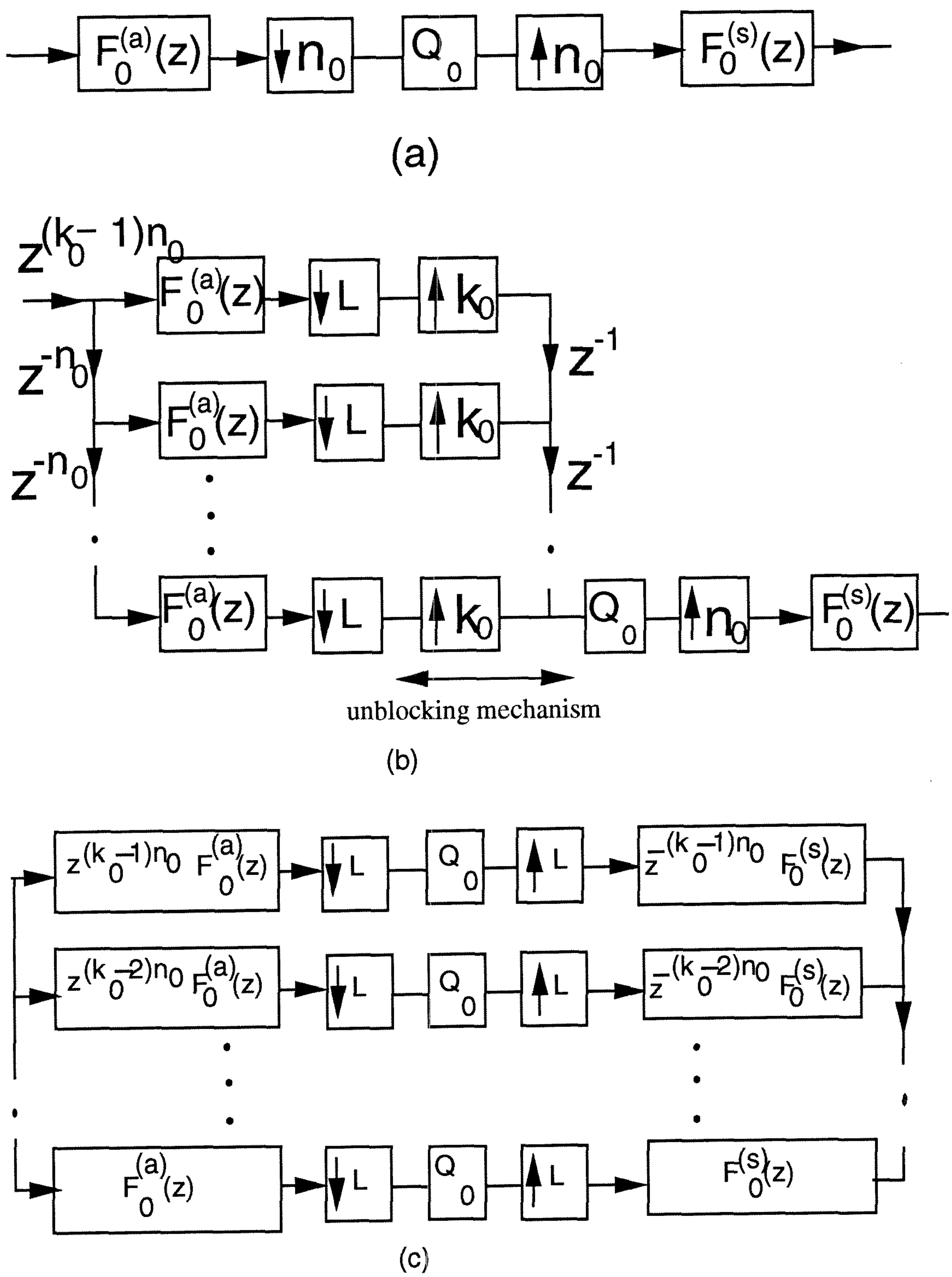


Fig. 3.14. Transforming a non-uniform filter bank to a uniform one

system, we get

$$\sigma_r^2 = (1/L)\sum_{i=0}^{M} k_i \sigma_i^2, \tag{4.2}$$

where σ_i^2 is the variance of the error introduced by the ith quantizer in Fig. 3.13.

The problem is to minimize this reconstruction error variance under the constraint of constant bit rate, namely,

$$(\sum_{i=0}^{M} k_i R_i)/L = constant = R. \tag{4.3}$$

The optimal bit allocation is again found by the Lagrange multiplier method. For optimality, the number of bits allocated to the ith quantizer is given by

$$R_i = R + (1/2)log_2\left[\frac{\sigma_{v_i}^2}{D_n}\right] \qquad i = 0, \ldots, M, \tag{4.4}$$

where $D_n = \prod_{j=0}^{M}(\sigma_{v_j}^2)^{1/n_j}$. The $\sigma_{v_i}^2$ are, as usual, the variances of the inputs to the quantizers. Once again, it can be verified that under this condition the variance of the errors at the location of each of the quantizers are equal.

The total reconstruction error variance under optimal bit allocation is given by

$$\sigma_r^2 = \epsilon^2 D_n 2^{-2R}. \tag{4.5}$$

The coding gain of the system is, therefore,

$$G_{PU} = \sigma_x^2/D_n, \tag{4.6}$$

where σ_x^2 is the variance of the input.

IV CONCLUSIONS

In this paper we have proved results for bit-allocation in sub-band coding schemes using paraunitary matrices. To start with, we proved some basic results for simple paraunitary LTI systems. Using these we have then derived bit-allocation results for a more complex system, namely the tree-structured filter bank. For a binary tree-structured filter bank, we showed that under the constraint that the total bit-rate is fixed, the individual quantizer error variances have to be equal under optimal bit allocation. These theorems can be extended readily to general tree-structures. We finally performed this analysis for non-uniform orthogonal filter banks which cannot be derived from tree-structures. It would also be possible to parametrize wavelet filter banks in terms of the polyphase matrices on each level of the tree-structure and optimize the overall coding gain, though this has not been done in this paper. This procedure would result in an optimized wavelet for the given signal statistics.

In the case of uniform paraunitary subband coders, we presented a scheme to directly optimize the coding gain. Experimental results were presented.

In speech and image coding applications, one might use other criterion for minimizing the reconstruction error, rather than minimizing its variance. For instance, we could attach different (non-negative) weights w_k to each of the subbands, and then try to minimize the weighted sum of the variances, i.e., $\sum_k w_k \sigma_k^2$. The fact that the arithmetic mean of a set of non-negative numbers is always greater than or equal to their geometric mean can still be used. It is easy to see that in order to minimize the weighted sum, each of the $w_k \sigma_k^2$ would have to be equal. In other words, the variances σ_k^2 would have to be inversely proportional to the weights w_k, again an intuitively appealing result.

REFERENCES

[1] J. J. Huang and P. M. Schultheiss, "Block Quantization of Correlated Gaussian Random Variables," IEEE Trans. on Comm. Syst., pp. 289-296, Sept. 1963.

[2] N. S. Jayant and P. Noll, *Digital Coding of Waveforms- Principles and Applications to Speech and Video,* Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1984.

[3] J. W. Woods and S. D. O'Neil, "Subband coding images," IEEE Trans. on ASSP, Vol. 34, pp. 1278-1288, Oct. 1986.

[4] Crochiere, *Multirate Digital Signal Processing*, Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1983.

[5] C. Galand and D. Esteban, "16 Kbps real-time QMF subband coding implementation," Proc. Int. Conf. on ASSP, Denver, CO, pp. 332-335, Apr. 1980.

[6] P. P. Vaidyanathan, "Multirate Digital Filters, Filter Banks, Polyphase Networks, and Applications: A Tutorial," Proc. of IEEE, Vol.78, No.1, Jan. 1990.

[7] P. P. Vaidyanathan, "Theory and Design of M-channel Maximally Decimated Filters with Arbitrary M, having Perfect Reconstruction Property," IEEE Trans. on ASSP, pp. 476-492, Apr. 1987.

[8] H. S. Malvar, "Lapped Transforms for Efficient Transform/Subband coding," IEEE Trans. on ASSP, Vol. 38, No. 6, June 1990.

[9] A. Akansu and Liu, "On Signal Decomposition Techniques," Optical Eng. J., pp. 912-920, Vol. 30, No. 7, July 1991.

[10] W. A. Pearlman, "Performance Bounds for Subband Coding," Ch. 1 of *Subband Image Coding*, edited by J. Woods, Kluwer, 1991.

[11] A. Segall, "Bit allocation and encoding for vector sources," IEEE Trans. on Info. Theory, pp. 162-169, March 1976.

[12] I. Daubechies, "Orthonormal Bases of Compactly Supported Wavelets," Comm. on Pure and Appl. Math. Vol.4, pp. 909-996, Nov.1988.

[13] O. Rioul and M. Vetterli, "Wavelets and Signal Processing," IEEE Signal Processing Magazine, pp. 14-38, Oct. 1991.

[14] P.P. Vaidyanathan and S. K. Mitra, "Polyphase Networks, Block Digital Filtering, and Alias-free QMF Banks: A Unified Approach Based on Pseudo-circulants," IEEE Trans. on ASSP, Vol. 36, March 1988.

[15] E. F. Beckenbach and R. E. Bellman, *Inequalities*, Berlin, Springer, 1961.

[16] H. S. Malvar, "Extended Lapped Transforms: Properties, Applications, and Fast Algorithms," preprint.

[17] P. P. Vaidyanathan, T. Q. Nguyen, Z. Doganata, T. Saramaki, " Improved Technique for design of perfect reconstruction FIR QMF banks with lossless polyphase matrices," IEEE Trans. on ASSP, pp. 1042-1056, July 1989.

[18] NAG Fortran Library, Numerical Algorithms Group Inc., Downers Grove, IL.

[19] M. J. T. Smith and T. P. Barnwell, III, "A Procedure for Designing Exact Reconstruction Filter Banks for Tree-Structured Sub-band Coders," Proc. of IEEE Int. Conf. on ASSP, San Diego, March 1984.

[20] F. Mintzer, "Filters for Distortion-free Two-band Multirate Filter Banks," IEEE Trans. on ASSP, pp. 626-630, June 1985.

[21] M. Vetterli and C. Herley, "Wavelets and Filter Banks: Relationships and New Results," Proc. of ICASSP, pp. 1723-1726, 1990.

[22] S. G. Mallat, "A Theory for Multiresolution Signal Decomposition: The Wavelet Representation," IEEE Trans. on Pattern Anal. and Mach. Intel, pp. 674-693, July 1989.

[23] P. P. Vaidyanathan, "Lossless systems in Wavelet Transforms," IEEE Int. Sym. on Cir. Syst., pp. 116-119, Singapore, June 1991.

[24] G. Evangelista, "Orthogonal Wavelet Transforms and Filter Banks," Proc. of 23rd Asilomar Conf., 1989.

[25] P. P. Vaidyanathan and Z. Doğanata, "The role of lossless systems in modern digital signal processing: A tutorial," Special Issue on Circuits and Systems, IEEE Trans. on Education, Aug. 1989.

[26] A. Soman and P. P. Vaidyanathan, "Paraunitary Filter Banks and Wavelet Packets," Proc. of ICASSP, pp. 397-400, San Fransisco, 1992.

[27] P. Hoang and P. P. Vaidyanathan, "Non-uniform Multirate Filter Banks: Theory and Design," Proc. of ISCAS, Portland, 1989.

[28] I. Djokovic and P. P. Vaidyanathan, "Some Results on Orthonormal Filter Banks," preprint.

Chapter 4

GENERALIZED POLYPHASE REPRESENTATION AND APPLICATION TO CODING GAIN ENHANCEMENT

Abstract[1]

Generalized polyphase representations (GPP) have been mentioned in literature in the context of several applications. In this paper, we provide a characterization for what constitutes a valid GPP. Then, we study an application of GPP, namely in improving the coding gains of transform coding systems. We also prove several properties of the GPP.

I INTRODUCTION

The polyphase representation is a useful tool in multirate applications [1]-[3],[11]. It has been extensively used in the design of digital filter banks. The M-fold polyphase representation of a transfer function $H(z)$ is given by

$$H(z) = \sum_{i=0}^{M-1} h_i(z^M) z^{-i}, \tag{1.1}$$

where the $h_i(z)$ are referred to as the M polyphase components of $H(z)$. The right hand side of eq. (1.1) is a linear combination of functions $z^{-i}, \quad i = 0, \ldots, M-1$, with the weighting factors being functions of z^M. Such a representation holds for both finite and infinite impulse response (FIR and IIR) transfer functions. Moreover, $H(z)$ is FIR if and only if all its polyphase components are FIR. A natural question

[1]To appear, IEEE Trans. on Circuits and Systems.

which arises is whether an arbitrary transfer function $H(z)$ may be written as a linear combination of functions other than z^{-i}, while retaining the desirable properties [1]-[3] of the traditional polyphase representation. Furthermore, are there any advantages to be gained by using a different set of functions?

In [3], the author has mentioned the so-called 'generalized polyphase representation' (GPP). It has been shown that using a GPP, it is possible to efficiently quantize the coefficients of a digital filter. It has also been shown therein that the GPP gives a second derivation of the so-called Interpolated FIR (IFIR) filter technique [8]. In [4], further applications of GPP have been studied. However, neither of these references addresses the issue of what constitutes a valid generalized polyphase representation. In this paper we first provide a complete characterization of valid polyphase representations (section II). In section III, we study another application of the GPP, namely in enhancing the coding gain of transform coding systems. We prove several interesting properties in this regard.

The notation used in this paper closely follows that used in [3]. Bold faced quantities denote vectors and matrices. Let $x(n)$ be a real, wide sense stationary (WSS) random process. The correlation function $\rho(k)$ of this process is defined as $\rho(k) = E[x(n)x(n-k)]$. If $\mathbf{x}(n)$ is a WSS vector random process, its M by M autocorrelation matrix is defined as $\mathbf{R_{xx}}(k) = E[\mathbf{x}(n)\mathbf{x}^T(n-k)]$. AR(N) refers to an autoregressive process of order N [6]. The abbreviation *gcd* stands for 'greatest common divisor.' In the figures, the boxes with $\uparrow M$ and $\downarrow M$ stand for interpolators and decimators respectively, as defined in [2],[3].

II GENERALIZED POLYPHASE REPRESENTATIONS

In this section we first define a 'valid polyphase representation' (VPP) and then provide a characterization of all such representations.

Definition: Let $\mathbf{u}(z) = [u_0(z)\ u_1(z)\ \ldots\ u_{M-1}(z)]^T$. This is said to be a valid polyphase representation (VPP) if

(a) every rational function $B(z)$ can be represented as $B(z) = \sum_{i=0}^{M-1} b_i(z^M)u_i(z)$, where the $b_i(z)$ are rational

(b) $b_i(z)$ are FIR if and only if $B(z)$ is FIR.

It can be shown that with this definition, $\mathbf{u}(z)$ is guaranteed to be FIR.

We now characterize all such VPPs. Let $\mathbf{e}(z) = [1\ z^{-1}\ z^{-2}\ \ldots\ z^{-M+1}]^T$. This is, therefore, the basis for the usual polyphase representation. Let the vector $\mathbf{u}(z)$ defined above be given the usual polyphase representation $\mathbf{u}(z) = \mathbf{V}(z^M)\mathbf{e}(z)$. This means that $\mathbf{V}(z)$ is the conventional polyphase matrix [3] of the elements of the vector $\mathbf{u}(z)$. Note that $\mathbf{V}(z)$ is FIR. We have the following result:

Lemma 2.1: $\mathbf{u}(z)$ is a valid polyphase basis if and only if $\det[\mathbf{V}(z)] = cz^k$ for $c \neq 0$ and integer k .

Proof: First assume that $\mathbf{u}(z)$ is a VPP. Then every transfer function can be represented in terms of the elements of $\mathbf{u}(z)$. In particular, $\mathbf{e}(z)$ can be written in terms of $\mathbf{u}(z)$ as

$$\mathbf{e}(z) = \mathbf{E}(z^M)\mathbf{u}(z). \tag{2.1}$$

But, $\mathbf{u}(z) = \mathbf{V}(z^M)\mathbf{e}(z)$. Hence

$$\mathbf{e}(z) = \mathbf{E}(z^M)\mathbf{V}(z^M)\mathbf{e}(z). \tag{2.2}$$

Now, $\mathbf{E}(z)\mathbf{V}(z)$ is the traditional polyphase matrix of $\mathbf{e}(z)$ with respect to $\mathbf{e}(z)$. Therefore, $\mathbf{E}(z)\mathbf{V}(z) = \mathbf{I}$. Since, $\mathbf{E}(z)$ and $\mathbf{V}(z)$ are both FIR, we have the result that $\det[\mathbf{V}(z)]$ is a power of z.

Conversely, let $\det[\mathbf{V}(z)]$ be a power of z. We know that any transfer function $H(z)$ can be represented as $H(z) = h^T(z^M)\mathbf{e}(z)$. Using eq. (2.1) this becomes $H(z) = h^T(z^M)\mathbf{E}(z^M)\mathbf{u}(z)$. Hence $H(z)$ can be represented in terms of $\mathbf{u}(z)$. Since $\mathbf{V}(z)$ is FIR and $\det[\mathbf{V}(z)]$ is a power of z, $\mathbf{E}(z^M)$ should be FIR (using $\mathbf{E}(z^M)\mathbf{V}(z^M) = \mathbf{I}$). Hence $h^T(z^M)\mathbf{E}(z^M)$ is also FIR for FIR $H(z)$. This proves the converse. $\diamond$

III CODING GAIN ENHANCEMENT USING GPP

In this section, we shall study a specific application of the generalized polyphase representation.

Consider Fig. 4.1 with $J_i = 1, \;\; i = 1, \ldots, M-1$. This is therefore the familiar case of Transform coding. Such schemes are used in data compression of speech, images and other signals. In such a scheme, the input string is divided into non-overlapping blocks $\mathbf{x}(n)$ of length M by grouping together M successive samples. Each block is encoded by multiplying it with a transform matrix $\mathbf{A}$. The transform coefficients $\mathbf{s}(n)$ are independently quantized. At the receiver, the inverse transformation $\mathbf{A}^{-1}$ is applied to the received vector $\mathbf{t}(n)$ to produce the output vector, which is 'unblocked' to obtain the output sequence. The case where the transform matrix is orthogonal ($\mathbf{A}^T = \mathbf{A}^{-1}$) is called Orthogonal Transform Coding [6], and is the one most commonly used in practice. The scheme can also be drawn as a filter bank as in Fig. 4.2.

There are two issues involved in the design of transform coding systems; namely, allocating the bits to the individual quantizers, and choosing the 'optimal' transform matrix $\mathbf{A}$ so as to maximize the coding gain. The optimal bit allocation result [6] says that the distribution of bits which minimizes the reconstruction error variance is the one that makes the individual quantizer error variances equal. Also, it is well known that the transform matrix $\mathbf{A}$ which maximizes the coding gain of the system is the Karhunen-Loeve Transform (KLT), whose rows are the eigenvectors of the input autocorrelation matrix [6]. The coding gain then becomes

$$G_{TC} = \frac{\sigma_x^2}{(\det[\mathbf{R}_{\mathbf{xx}}(0)])^{1/M}}. \tag{3.1}$$

An aspect of the transform coding scheme which has not received attention so far are the variations of the blocking/unblocking mechanisms (Fig. 4.1). Notice that in a traditional transform coding system, this mechanism is responsible for blocking M *successive* samples of the input data. However, it is possible in case of certain inputs to exploit the correlations between non-adjacent samples of the input data so as to enhance the coding gain. This would be particularly important when data from several sources is multiplexed into one bit-stream. Specifically, it is possible in several cases to design the blocking mechanism such that the value of $(\det[\mathbf{R}_{\mathbf{xx}}(0)])$ in eq. (3.1) is reduced. The question now is what are the constraints which the new blocking/unblocking mechanism has to satisfy?

Consider the system shown in Fig. 4.1 with $J_i = J, \quad i = 1, \ldots, M-1$. Hence, we have used a generalized polyphase basis comprising of the functions $z^{-iJ}, \quad i = 0, \ldots, M-1$. The matrix $\mathbf{A}$ is the polyphase matrix of the filters in a generalized sense. Since the basis can be implemented using only delay elements, this scheme is equivalent to a transform coding scheme in terms of complexity.

Fact 3.1: Consider the system shown in Fig. 4.1 with $J_i = J, \quad i = 1, \ldots, M-1$. This is a perfect reconstruction system if and only if $gcd(J, M) = 1$.

Proof: One proof of this fact appears in [9]. We present here a proof based on GPP for the sake of completeness. Let $\mathbf{u}(z)$ be the vector with elements $u_i(z) = z^{-iJ}, \quad i = 0, \ldots, M-1$. Let $\mathbf{u}(z) = \mathbf{V}(z^M)\mathbf{e}(z)$, where $\mathbf{e}(z)$ is as defined above. If $gcd(J, M) = 1$, then it can be verified that the matrix $\mathbf{V}(z)$ has only one entry per column, and this entry is a delay. Hence $\det[\mathbf{V}(z)]$ is a delay, implying that this is a valid GPP. Furthermore, $\mathbf{V}(z)$ is paraunitary [3], i.e., it satisfies $\mathbf{V}(z)\widetilde{\mathbf{V}}(z) = \mathbf{I}$, where $\widetilde{\mathbf{V}}(z)$ is obtained from $\mathbf{V}(z)$ by transposition, followed by conjugation of coefficients followed by replacing z by z^{-1}. The polyphase matrix of the unblocking mechanism is $\widetilde{\mathbf{V}}(z)$, and hence the system is a perfect-reconstruction system. Conversely, if $gcd(J, M) \neq 1$, it can be verified that at least one of the columns of $\mathbf{V}(z)$ will have all zeros, and hence the system cannot have perfect reconstruction property. $\diamond$

Comment: Suppose M is fixed. There are several choices of J which satisfy Fact

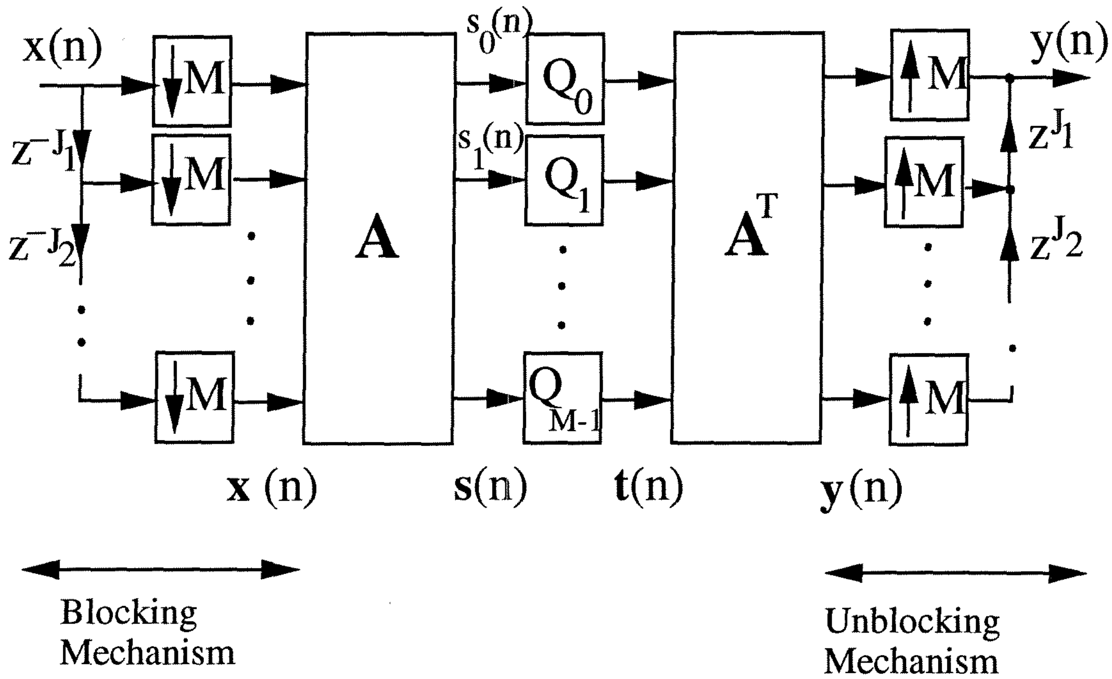


Fig. 4.1. A generalized transform coding scheme

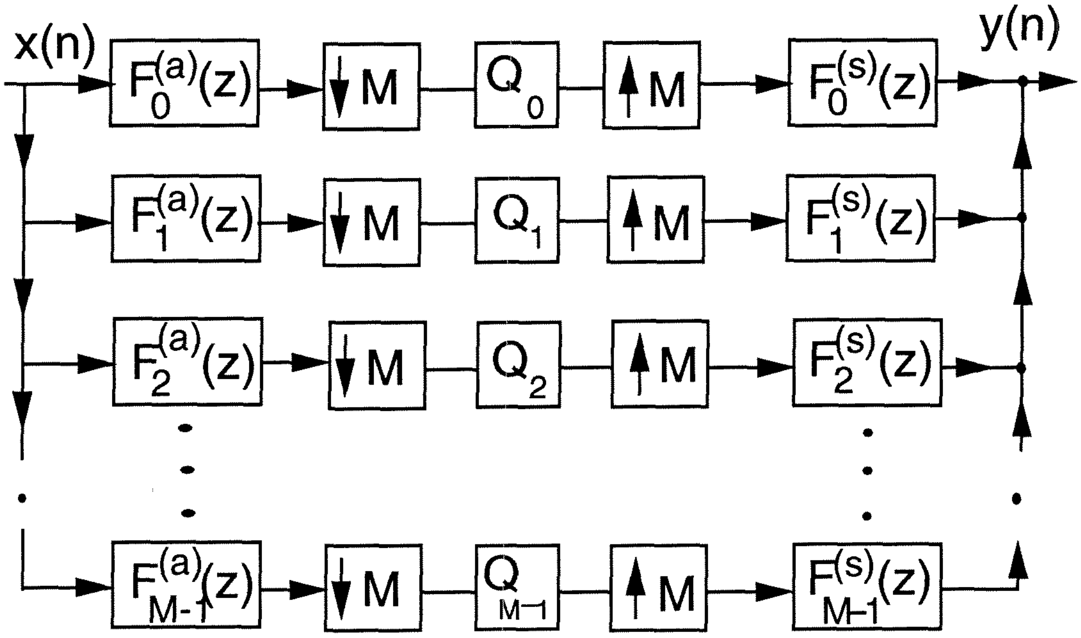


Fig. 4.2. A typical subband coding scheme

3.1. In practice, we choose J such that the correlation between samples distance J apart is high. If the selection of both J and M is up to the designer, J is first chosen as above, and then M is chosen so as to satisfy Fact 3.1. However, we have not proved theoretically the optimality of such an approach.

Coding gain example: As an example of a process where the coding gain of the new system is better than the transform coding system, consider a process with the autocorrelation function

$$\begin{aligned} \rho(k) &= \rho_1(k) + \rho_2(k) \quad where \\ \rho_1(k) &= (0.1)^{|k|} \quad and \\ \rho_2(k) &= (0.9)^{|k|/4} \quad if\ |k|\ is\ a\ multiple\ of\ 4,\ and\ 0\ otherwise. \end{aligned} \tag{3.2}$$

Such an autocorrelation could arise where, for example, the correlation between non-adjacent samples is high. If we used a traditional transform coding scheme on such an input, the coding gain would only be 0.029 db, whereas using $J = 4$ gives a gain of 1.63 db (in both cases, $M = 3$).

Transform coding is often used to encode images. Data from images normally shows high correlation between adjacent samples and is often modelled as an AR(1) process. For such data, the choice of J and M is simplified by the following lemma.

Lemma 3.2: For an AR(1) process, the value of J which maximizes the coding gain of the system is $J = 1$ (i.e., traditional transform coding) for all M.

Proof: Consider running the Linear Predictive Algorithm (LPC) [6] on the AR(1) input. Let ϵ_i denote the prediction error variances for the ith order optimal predictor. Then it can be shown that $\epsilon_0 = 1$, and $\epsilon_i = (1 - \rho^2), \quad for\ all\ i \geq 1$. Here, ρ is the correlation coefficient of the AR(1) process. Now consider the autocorrelation matrix $\mathbf{R_{xx}}$ of size $M \times M$ corresponding to the vector input sequence $\mathbf{x}(n)$. It can be shown [6] that the determinant of this matrix is given by $\det[\mathbf{R_{xx}}] = \prod_{i=0}^{M-1} \epsilon_i = (1 - \rho^2)^{M-1}$. If we use $J \neq 1$, it can be verified that the autocorrelation matrix of the new vector process is similar to $\mathbf{R_{xx}}$, but with correlation coefficient equal to ρ^J. The determinant of the new autocorrelation matrix is $\det[\mathbf{R_{xx}}'] = (1-\rho^{2J})^{M-1}$. If $|\rho| \leq 1$, $\det[\mathbf{R_{xx}}'] \geq \det[\mathbf{R_{xx}}]$. Hence, from eq. (3.1), the coding gain of the new system can be no better than the traditional transform coding system. $\diamond$

Monotonicity: In the case of traditional transform coders, the optimal coding gain can be shown (Appendix C of [10]) to be a monotonic function of the number of channels M for arbitrary inputs. In systems such as in Fig. 4.1, however, the optimal coding gain is a function of M as well as J. For a given input, and a certain number of channels, there exists an optimal J satisfying Fact 3.1, which maximizes the coding gain of the system. Let $G_{opt}(J_{opt}, M)$ denote the maximum gain after having chosen the optimal J for a particular input. It can be shown that $G_{opt}(J_{opt}, M)$ is *not* a monotonic function of M. To see this, consider the following autocorrelation function

$$R(k) = (\rho)^{|k|} \quad if\ k\ is\ a\ multiple\ of\ 6,\ and\ 0\ otherwise. \tag{3.3}$$

If $M=5$, and if we use $J=6$, we get a coding gain of $-log(1-\rho^2)^{4/5}db$. If $\rho = 0.95$, for example, this value is 8.08db. However, if $M=6$, it is not possible to get a coding gain greater than 0 db.

Fact 3.3: Consider the system in Fig. 4.1. Let $P_k = \sum_{i=1}^{k} J_i$, with $P_0 = 0$. Then the system is a perfect-reconstruction system if and only if the numbers $(P_k)\ mod\ M$ are distinct.

Proof: As in the proof of fact 3.1, let $\mathbf{u}(z)$ be the vector with elements $u_i(z) = z^{-P_i}$ $i = 0, \ldots, M-1$, and let $\mathbf{u}(z) = \mathbf{V}(z^M)\mathbf{e}(z)$. It can again be verified that under the condition that the numbers $(P_k)\ mod\ M$ are distinct, $\det[\mathbf{V}(z)]$ is a delay, implying that this is a valid GPP. Furthermore, $\mathbf{V}(z)$ is also paraunitary, thereby implying perfect reconstruction. Conversely, if the P_k do not satisfy the stated property, $\mathbf{V}(z)$ will be singular, implying that perfect-reconstruction is not possible. $\diamond$

The important point to note in this new scheme is that the autocorrelation matrix of the vector $\mathbf{x}(n)$, i.e., $\mathbf{R_{xx}}$ is no longer Toeplitz. Hence it is in general difficult to find the J_i which maximize the coding gain for a given process. This would involve minimization of the determinant of a general positive definite matrix under the constraints imposed by Fact 3.3.

One can, however, construct examples to demonstrate an improvement in the coding gain by using systems such as those in Fig. 4.1.

Example 1. Let $M = 3$, and consider an AR(5) process at the input whose first six autocorrelation coefficients are $\rho_0 = 1.0,\ \rho_1 = 0.2,\ \rho_2 = -0.45,\ \rho_3 = 0.38,\ \rho_4 = 0.7,\ \rho_5 = -0.4$. Traditional transform coding would give a gain of 0.5 db, whereas using $J_1 = 4$ and $J_2 = 1$ in Fig. 4.1 would give a coding gain of 2.3 db.

Example 2. Let $M = 4$, and consider an AR(6) process at the input whose first seven autocorrelation coefficients are $\rho_0 = 1.0,\ \rho_1 = -0.2,\ \rho_2 = -0.2,\ \rho_3 = 0.5,\ \rho_4 = -0.46,\ \rho_5 = 0.39\ \rho_6 = 0.76$. Traditional transform coding would give a gain of 0.512 db, whereas using $J_1 = 1,\ J_2 = 2$ and $J_3 = 3$ in Fig. 4.1 would give a coding gain of 3.19 db.

NOTE: One can verify that the above two examples present valid autocorrelation sequences. This can be done by verifying that the relevant Toeplitz autocorrelation matrices (of size 6×6 in example 1, and of size 7×7 in example 2) are positive definite.

In the denominator of coding gain expressions, $\det(\mathbf{R_{xx}})$ plays a crucial role. So it is important to explore the meaning of $\det(\mathbf{R_{xx}}) = 0$. In the traditional case, we know that the $M \times M$ matrix $\mathbf{R_{xx}}(0)$ is singular if the input process $x(n)$ is harmonic with atmost M frequencies. In the case of the system shown in Fig. 4.1, the following result holds:

Lemma 3.4: Consider the system in Fig. 3, and let $P_k = \sum_{i=1}^{k} J_i$ with $P_0 = 0$. Let the $M \times M$ autocorrelation matrix $\mathbf{R_{xx}}$ be singular. Then, the input process $x(n)$ is harmonic with atmost P_{M-1} frequencies.

Proof: Let $\mathbf{R}_B$ be the autocorrelation matrix of size $(P_{M-1}+1) \times (P_{M-1}+1)$ corresponding to the input sequence $x(n)$, i.e., $\mathbf{R}_{B_{i,j}} = [\rho(i-j)]$. We know [7] that if $\mathbf{R}_B$ is singular, the input process is harmonic with atmost P_{M-1} frequencies. We now show that $\det[\mathbf{R}_B] \leq \det[\mathbf{R}_{\mathbf{xx}}]$. Since both autocorrelation matrices are positive semi-definite, singularity of $\mathbf{R}_{\mathbf{xx}}$ would guarantee the singularity of $\mathbf{R}_B$.

For a suitable choice of permutation matrix $\mathbf{P}$, we have

$$\mathbf{P}\mathbf{R}_B\mathbf{P} = \mathbf{Q} = \begin{pmatrix} \mathbf{R}_{\mathbf{xx}} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Y} \end{pmatrix}. \tag{3.4}$$

Hence (pg. 404 of [12]),

$$\det[\mathbf{R}_B] = \det[\mathbf{Q}] \leq \det[\mathbf{R}_{\mathbf{xx}}]. \quad \diamond \tag{3.5}$$

IV COMMENTS

In this paper, we have developed a characterization of generalized polyphase representations (GPP). The GPP allows us a greater freedom in designing multirate systems. We studied a particular application of GPP, namely in enhancing the coding gain of transform coding systems. The advantage of using GPP was demonstrated for several inputs. Moreover the additional complexity of the new system is only slightly greater than the transform coding system, the difference being the higher number of delay elements used. We also proved several properties of the new system.

REFERENCES

[1] M. Bellanger, G. Bonnerot, M. Coudreuse, "Digital Filtering by Polyphase Network: Application to Sample Rate Alteration and Filter Banks," IEEE Trans. on ASSP, vol. 24, pp. 109-114, Apr. 1976.

[2] R. Crochiere and L. Rabiner, *Multirate Digital Signal Processing*, Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1983.

[3] P. P. Vaidyanathan, "Multirate Digital Filters, Filter Banks, Polyphase Networks, and Applications: A Tutorial," Proc. of IEEE, Vol.78, No.1, Jan. 1990.

[4] S. K. Mitra, "Some Unconventional Signal Processing Applications of Multirate

Techniques," Proc. of ISCAS, pp. 13-16, Singapore, June 1991.

[5] J. J. Huang and P. M. Schultheiss, "Block Quantization of Correlated Gaussian Random Variables," IEEE Trans. on Comm. Syst., pp. 289-296, Sept. 1963.

[6] N. S. Jayant and P. Noll, *Digital Coding of Waveforms- Principles and Applications to Speech and Video,* Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1984.

[7] S. Kay and S. Marple, "Spectrum Analysis: A Modern Perspective," Proc. of IEEE, vol. 69, pp. 1380-1419, Nov. 1981.

[8] Y. Neuvo, C. Dong, S. K. Mitra, "Interpolated Finite Impulse Response Filters," IEEE Trans. on ASSP, pp. 563-570, June 1984.

[9] T. Nguyen and P. P. Vaidyanathan, "Maximally Decimated Perfect-Reconstruction Filter Banks with Pairwise Mirror-Image Analysis (and Synthesis)Frequency Responses," IEEE Trans. on ASSP, pp. 693-706, May 1988.

[10] P. P. Vaidyanathan, *Multirate Systems and Filter Banks, Prentice-Hall,* 1993.

[11] M. Vetterli, "A Theory of Multirate Filter Banks," IEEE Trans. on ASSP, pp. 356-372, Mar. 1987.

[12] R. Horn and C. Johnson *Matrix Analysis*, Cambridge Univ. Press, 1991.

Chapter 5

LINEAR PHASE PARAUNITARY FILTER BANKS: THEORY, FACTORIZATIONS AND APPLICATIONS

Abstract[1]

M channel maximally decimated filter banks have been used in the past to decompose signals into subbands. The theory of perfect-reconstruction filter banks has also been studied extensively. Non-paraunitary systems with linear phase filters have also been designed. In this paper, we study paraunitary systems in which each individual filter in the analysis and synthesis banks has linear phase. Specific instances of this problem have been addressed by other authors, and linear phase paraunitary systems have been shown to exist. This property is often desirable for several applications, particularly in image processing.

We begin by answering several theoretical questions pertaining to linear phase paraunitary systems. Next, we develop a minimal factorization for a large class of such systems. This factorization will be proved to be complete for even M. Further, we structurally impose the additional condition that the filters satisfy pairwise mirror-image symmetry in the frequency domain. This significantly reduces the number of parameters to be optimized in the design process. We then demonstrate the use of these filter banks in the generation of M-band orthonormal wavelets. Finally, we use the linear phase paraunitary system to encode signals and provide a comparison with

[1]To appear, Special Issue on Wavelets, IEEE Trans. on Signal Processing, Sept. 1993.

traditional techniques. Several design examples are also given to validate the theory.

I INTRODUCTION

Digital filter banks have been used in the past to decompose a signal into frequency subbands [1]-[12]. The signals in different subbands are then coded and transmitted. Such schemes are popular for encoding data from speech and image signals. The process of decomposition and eventual reconstruction are done by what is termed as the 'analysis-synthesis' filter bank system shown in Fig. 5.1. In this scheme, the $H_i(z)$ are the analysis filters and $F_i(z)$ are the synthesis filters. The boxes with $\downarrow M$ denote the decimators, or the subsampling devices, whereas the boxes with $\uparrow M$ denote the expanders, which increase the sampling rate. Their definitions are as in [1], [3].

Fig. 5.2 is a representation of the subband coding scheme in terms of the polyphase matrices [3]. $\mathbf{E}(z)$ is the polyphase matrix corresponding to the analysis filters, and $\mathbf{R}(z)$ is the polyphase matrix corresponding to the synthesis filters. The decimators and expanders have been moved across the polyphase matrices using the noble identities [3]. It has been shown that it is indeed possible to perfectly reconstruct the original signal using such analysis-synthesis systems [5]-[12]. In particular, this can be done by filters that have finite impulse response (FIR), and are hence guaranteed to be stable. One way to do this is to let $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$, and then choose the matrix $\mathbf{E}(z)$ so that both matrices are FIR. Such a system is called a biorthonormal system [13].

I.(a) Preliminaries

Paraunitary systems: Another approach to design a perfect reconstruction system is to choose the matrix $\mathbf{E}(z)$ to be a FIR 'paraunitary' matrix. A matrix is said to be paraunitary [8] if it satisfies the equation

$$\widetilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I}, \tag{1.1}$$

where $\widetilde{\mathbf{E}}(z) = \mathbf{E}^{\dagger}(1/z^*)$. The system can be guaranteed to have the perfect recon-

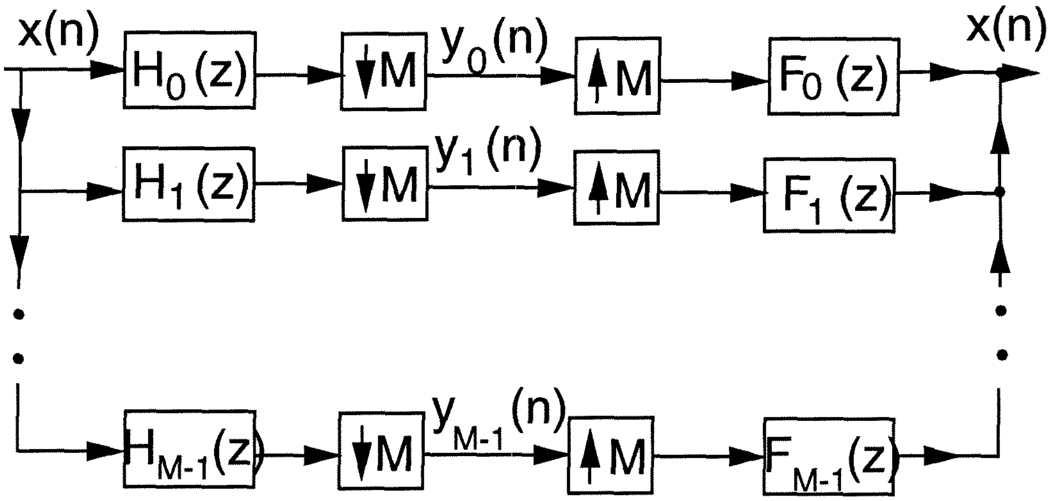


Fig. 5.1. An M-channel uniform filter bank

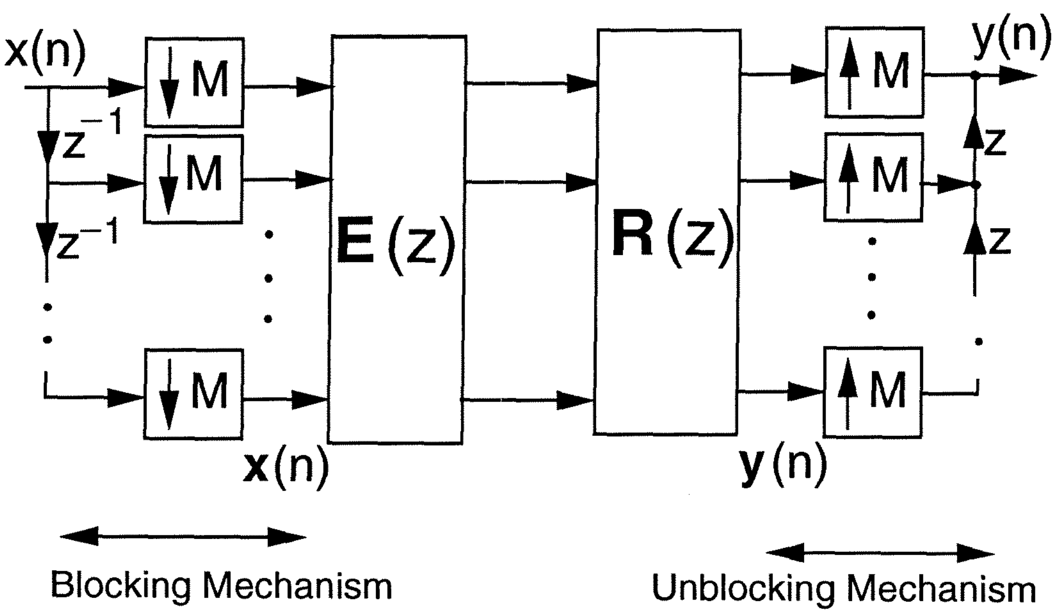


Fig. 5.2. A filter bank drawn in terms of polyphase matrices

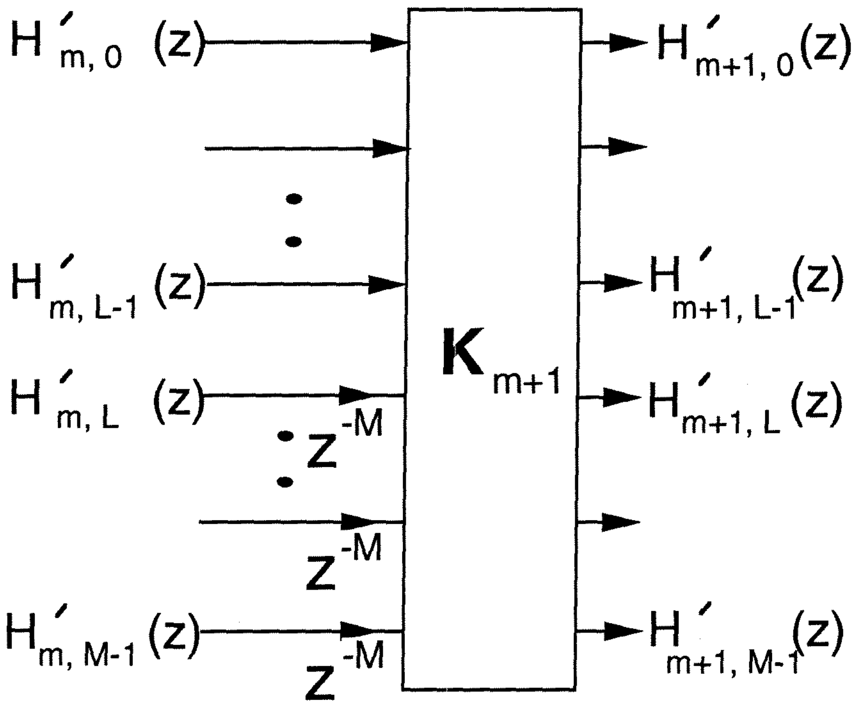


Fig. 5.3 . One stage of the filter bank developed in section III

struction property by having $\mathbf{R}(z) = \widetilde{\mathbf{E}}(z)$. This paraunitary property can be traced back to classical network synthesis [14].

Consider the synthesis bank of Fig. 5.1. The original signal can be written in terms of the subband signals as

$$x(n) = \sum_{k=0}^{M-1} \sum_{m} y_k(m) f_k(n - Mm). \tag{1.2}$$

This can be viewed as a representation of the original signal in terms of a doubly indexed set of basis functions $\eta_{km}(n) = f_k(n - Mm)$. It is known [10], [13], [15] that this set of basis functions is orthonormal if and only if the polyphase matrix $\mathbf{R}(z)$ corresponding to these filters is paraunitary.

Another feature of the paraunitary analysis-synthesis system is that the analysis and synthesis filters are simply time-reversed conjugate versions of each other and, in particular, therefore, they are of the same length.

Quantization: In a practical subband coding system, both the filter coefficients, as well as the subband signals, are quantized. It has been shown [3], [16] that there exist structures which retain the paraunitary property in spite of coefficient quantization. The perfect-reconstruction property is, however, lost when the signals in each subband are quantized. A paraunitary system still has some important features in the presence of subband quantization:

1) We can obtain bounds on the overall reconstruction error in terms of the quantization errors in each subband, no matter what the frequency responses of the filters are [17].

2) We are assured that the only error is due to signal quantization.

The coding gain [18] is often used as a criterion for judging the performance of these practical subband coding schemes.

***M*-band orthogonal wavelets:** The relation between the M-channel paraunitary system and M-band orthogonal wavelets has been shown recently in [19], [20]. M-band wavelets have also been shown to provide a more compact representation of signals than the traditional binary wavelets [21]. The M-band wavelet is obtained by cascading the M-channel paraunitary system in an infinite tree-structure. Using a linear phase paraunitary system therefore gives us an orthonormal basis of linear phase wavelets. This will be demonstrated later in this paper.

I.(b) Previous work on linear phase perfect reconstruction systems:

In several applications, and particularly in image coding, it is desirable to have each filter in the system be a linear phase filter. This would not be necessary if there were no subband quantization, which is not a case of practical interest. The problem of designing two-channel linear phase, *non-paraunitary*, perfect reconstruction systems has been discussed in the past [22], [23]. However, for the two-channel case, it can be shown that if a *paraunitary* system has linear phase filters, it is degenerate, i.e., the filters can be no better than a sum of two delays [3]. For M-channel paraunitary systems, linear phase property has been demonstrated in certain special cases, by Princen and Bradley in [24] and by Malvar in [25]. In [25], the author gives examples of linear phase Lapped Orthogonal Transforms (LOT), which have been shown to be order one paraunitary systems of a specific form. In [24], too, the filters mentioned correspond to a special type of paraunitary systems of order one. The more general case of linear phase paraunitary systems of larger degrees was addressed for the first time by Vetterli and Le Gall in [26]. The authors derive systems of higher degree from those of smaller degree by multiplication with certain types of paraunitary matrices, when the number of channels is even. For the four-channel case, the authors give judicious examples of such building blocks.

A structure is said to be *minimal* [10] if it uses the minimum number of delay elements to implement the particular transfer function. *Completeness* of a structure, on the other hand, implies being able to factorize a given linear phase paraunitary system in terms of the proposed structure. Another important consideration while designing filters by optimization is being able to characterize the building blocks in terms of a minimal number of free parameters. None of the earlier works addresses any of the above three issues. We shall address them in this paper. We will also present, for the first time, design examples of linear phase paraunitary systems of higher degrees.

In [12], Nayebi, et al. have proposed a time-domain approach in which filter banks are designed by formulating a system of equations. In this formalism, one can incorporate time-domain constraints, and it should also be possible to impose the linear phase property therein. The purpose of our paper is, however, different, as outlined below.

I.(c) Aim of the paper

This paper attempts a thorough study of *linear phase paraunitary filter banks.* In particular, the following is the new contribution of this work:

- Section II: We develop the theory of linear-phase paraunitary systems, and prove several new results.

- Section III: For the case where the number of channels M is even, we present a factorization of the linear-phase paraunitary filter bank that it is minimal as well as complete for a large class of filter banks important from a practical standpoint.

- Section IV: We further structurally impose the constraint that the filters be pairwise symmetric around $\pi/2$ in the frequency domain. This significantly reduces the number of variables to be optimized in the design.

- Section V: We provide a cascade structure for linear-phase paraunitary systems when M is odd, and prove that it is minimal.

- Section VI: We apply the above ideas to generate symmetric, orthonormal, M-band wavelets. The issue of regularity [18], [12] is addressed.

- Section VII: We study the performance of linear phase paraunitary systems in subband coding. In particular, we provide comparison for coding gains obtained using linear phase systems with those obtained using general paraunitary coding schemes.

We also present several design examples to validate the above theory.

I.(d) Notations

Bold-faced quantities denote matrices and vectors, as in $\mathbf{A}$ and $\mathbf{x}$. $\mathbf{A}^T$, $\mathbf{A}^{-1}$ and $Tr(\mathbf{A})$ denote the transpose, the inverse, and the trace of the matrix $\mathbf{A}$, respectively. A subscript on a matrix indicates its size, when the size is not clear from the context. Reserved symbols for special matrices are as follows: $\mathbf{I}$ is the identity matrix. The matrix $\mathbf{J}_N$ is the anti-diagonal matrix of size $N \times N$. For example, the anti-diagonal matrix of size 4 is

$$\mathbf{J}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{1.3}$$

$\mathbf{0}$ will denote the null matrix, whose size will be clear from the context. $\mathbf{V}_K$ will denote a special diagonal matrix of size $K \times K$, with alternating $\pm 1's$ on the diagonal, starting with $+1$.

A superscript asterisk as in $f^*(n)$ denotes conjugation. Consider a transfer function $A(z)$. It can be written in terms of its M polyphase components [27] as follows:

$$A(z) = a_0(z^M) + z^{-1}a_1(z^M) + \ldots + z^{-(M-1)}a_{M-1}(z^M). \tag{1.4}$$

This is known as Type I polyphase. Let $H_i(z), \quad i = 0, \ldots, M-1$, be a set of analysis filters. They can be written as

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{kl}(z^M) \qquad k = 0, \ldots, M-1.$$

The matrix $\mathbf{E}(z) = [E_{k,l}(z)]$ is called the polyphase matrix of the analysis filters. A set of filters $H_k(z)$ whose polyphase matrix is paraunitary are said to form a paraunitary system (eq. (1.1)). Throughout this paper, we will deal with *real, causal, FIR systems.* Given such a system $\mathbf{E}(z)$ of order N, we can write it explicitly as

$$\mathbf{E}(z) = \mathbf{e}(0) + \mathbf{e}(1)z^{-1} + \mathbf{e}(2)z^{-2} + \ldots + \mathbf{e}(N)z^{-N}, \qquad \mathbf{e}(N) \neq \mathbf{0}. \tag{1.5}$$

II THEORY OF LINEAR PHASE PARAUNITARY SYSTEMS

In order to obtain factorizations of linear phase paraunitary systems, we first need to obtain a characterization of their polyphase matrix which reflects the linear phase property of the individual filters. Consider a set of M paraunitary transfer functions whose polyphase matrix $\mathbf{E}(z)$ satisfies the property [26]

$$\mathbf{D}z^{-N}\mathbf{E}(z^{-1})\mathbf{J}_M = \mathbf{E}(z), \tag{2.1}$$

where N is the order of the paraunitary matrix $\mathbf{E}(z)$. Such a polyphase matrix corresponds to a set of filters which have linear phase. The matrix $\mathbf{D}$ is a diagonal matrix whose entries are $\pm 1's$, the $+1's$ in those rows which correspond to symmetric filters and $-1's$ in those that correspond to antisymmetric filters. The filters described by this equation have the same center of symmetry $((N+1)M-1)/2$.

It is conceivable that there are linear phase paraunitary systems which cannot be characterized as in eq. (2.1). One example is that of the 'delay chain,' wherein the analysis filters are simply $H_i(z) = z^{-i}, \quad i = 0, \ldots, M-1$. However, as said earlier, obtaining factorizations requires us to impose constraints on the polyphase matrix of the filters, and eq. (2.1) represents a large class of filter banks important from a practical standpoint. In this paper, we will consider only those systems that can be described by eq. (2.1). We will also show several good design examples based on such systems.

The linear phase constraint in conjunction with the paraunitary property imposes interesting conditions on the filters. The paraunitary property implies orthonormality of the impulse response to its own shifted versions [10],[15], and the linear phase property implies that the filters are time-reversed versions of themselves (up to a factor of ± 1). This, for example, imposes a restriction on the length of the filters.

Fact 2.1: Let $F_i(z)$ be a set of M linear phase paraunitary filters of length L each with $f_i(0) \neq 0$. Then, $L \neq lM + 1$ for any integer $l \geq 1$.

Proof: The orthonormality condition on the filters [10] in particular implies,

$$\sum_{n=-\infty}^{\infty} f_j(n) f_j^*(n - lM) = \delta(i). \tag{2.2}$$

If the length of the filters is $L = lM + 1$, in view of linear phase property this means that

$$f_j(0) f_j^*(0) = 0, \tag{2.3}$$

implying that $f_j(0) = 0$. Hence the length $L \neq lM + 1$ for any integer $l \geq 1$. ◇◇◇

The perfect reconstruction condition also imposes a constraint on the number of symmetric and antisymmetric functions in the filter bank. This is stated in the following theorem:

Theorem 1: Consider an M-channel linear phase perfect reconstruction system.

i) If M is even, there are $M/2$ symmetric, and $M/2$ antisymmetric filters.

ii) If M is odd, there are $(M+1)/2$ symmetric and $(M-1)/2$ antisymmetric filters.

This result has been proved in [28] for the special case where the order of the paraunitary matrix $\mathbf{E}(z)$ is one. The proof therein is based on subspace techniques and, moreover, does not extend to the case where $\mathbf{E}(z)$ has an arbitrary order. The result has been stated explicitly as an assumption in [26]. We provide below a formal proof that this is indeed true. Note that the result is not restricted to paraunitary filter banks.

Proof: Consider eq. (2.1). The trace of the matrix $\mathbf{D}$ holds the key to the number of symmetric and antisymmetric filters in the system. Using the fact that the matrix

$\mathbf{E}(z)$ is invertible, we have,

$$\begin{aligned} Tr(\mathbf{D}) &= Tr(z^N \mathbf{E}(z)\mathbf{J}_M \mathbf{E}^{-1}(z^{-1})) && (2.4) \\ &= Tr(z^N \mathbf{E}^{-1}(z^{-1})\mathbf{E}(z)\mathbf{J}_M). && (2.5) \end{aligned}$$

We have used the fact that $Tr(\mathbf{AB}) = Tr(\mathbf{BA})$. The left hand side of this equation is constant. Hence its value can be found by evaluating the right hand side for one value of the variable z. Putting $z = 1$ in the above equation we get,

$$Tr(\mathbf{D}) = Tr(\mathbf{E}^{-1}(1)\mathbf{E}(1)\mathbf{J}_M) = Tr(\mathbf{J}_M), \quad (2.6)$$

with the anti-diagonal matrix $\mathbf{J}_M$ as in eq. (1.3). Therefore, it can be verified that $Tr(\mathbf{D}) = 0$ if M is even, and $Tr(\mathbf{D}) = 1$ if M is odd. Hence there are an equal number of symmetric and antisymmetric functions if M is even, whereas if M is odd, there is one extra symmetric function. $\diamond\diamond\diamond$

In particular, the above theorem implies that all the filters cannot be zero phase. The proof of the above theorem also implies an interesting constraint on the order of the linear phase polyphase matrix $\mathbf{E}(z)$ when the number of filters M is odd.

Corollary 1: If the number of channels M is odd, the order N of the polyphase matrix $\mathbf{E}(z)$ cannot be odd.

Proof: Consider eq. (2.5), and let N be odd. If one evaluates the right hand side of this equation at $z = -1$ instead of $z = 1$, we get

$$Tr(\mathbf{D}) = Tr((-1)^N \mathbf{E}^{-1}(-1)\mathbf{E}(-1)\mathbf{J}_M) = -Tr(\mathbf{J}_M). \quad (2.7)$$

This, along with eq. (2.6), would imply that $Tr(\mathbf{J}_M) = 0$, but this is not possible since M is odd. Hence we get a contradiction, proving that N cannot be odd. $\diamond\diamond\diamond$

An interesting consequence of imposing the paraunitary constraint on an M-channel filter bank is that it guarantees that if the first $M-1$ filters are linear phase, the last filter is also linear phase. This is formally stated as follows:

Theorem 2: Let a set of filters $F_i(z), \quad i = 0, \ldots, M-1$ be paraunitary, and let the first $M-1$ of them have linear phase. Then the last one is guaranteed to have linear phase.

Before we prove the theorem, we will prove a lemma which will help us in the proof.

Lemma 1: If $M-1$ functions of an FIR paraunitary system are known, the last one is uniquely determined (up to a factor of the form $(e^{j\theta})z^{lM}$).

Proof: Let $F_i(z), \quad i=0,\ldots,M-1$ form an FIR paraunitary system. Let, if possible, $u(z)$ be another FIR function, which along with $F_i(z), \quad i=0,\ldots,M-2$ forms a paraunitary system. Let $\mathbf{E}'(z)$, the polyphase matrix corresponding to this modified set of paraunitary functions, be partitioned as

$$\mathbf{E}'(z)=\begin{bmatrix}\mathbf{E}'_1(z)\\ \mathbf{u}(z)\end{bmatrix}. \tag{2.8}$$

This means that the row vector $\mathbf{u}(z)$ has as its elements the polyphase components of the filter $u(z)$. Since $\mathbf{E}'(z)$ is unitary on the unit circle, $\mathbf{u}(e^{j\omega})$ is uniquely determined up to a scale factor of the form $e^{j\Phi(\omega)}$. Hence, by analytic continuation, $u(z)=A(z^M)F_{M-1}(z)$, where $A(z)$ is all pass. It can be verified that the condition $det(\mathbf{E}'(z))=delay$, which is necessary for paraunitariness, implies that $A(z)=(e^{j\theta})z^{lM}$. Hence, given $M-1$ functions of an FIR paraunitary system, the last function is determined up to a factor of the form $(e^{j\theta})z^{lM}$. ◇◇◇

Using this lemma, we can now prove Theorem 2.

Proof of Theorem 2: Let $\mathbf{E}(z)$ be a paraunitary polyphase matrix corresponding to a set of filters that have linear phase. Let $\mathbf{E}_1(z)$ be the polyphase matrix of size $(M-1)\times M$ corresponding to the first $M-1$ filters in the system, and let $\mathbf{u}(z)$ be the row vector whose elements are the polyphase components corresponding to the last filter $u(z)$ of the system. Now, eq. (2.1) can be rewritten as

$$\mathbf{D}z^{-n}\begin{bmatrix}\mathbf{E}_1(z^{-1})\\ \mathbf{u}(z^{-1})\end{bmatrix}\mathbf{J}_M=\begin{bmatrix}\mathbf{E}_1(z)\\ \mathbf{v}(z)\end{bmatrix}. \tag{2.9}$$

This means that the row vector $\mathbf{v}(z)$ has as its elements the polyphase components of the filter $v(z)$, which is the time-reversed version of $u(z)$ (up to ± 1). Now, since all matrices on the left hand side of this equation are paraunitary, the matrix on the right hand side of this equation is also paraunitary. But the first block of this matrix is $\mathbf{E}_1(z)$. This means by lemma 1, that $v(z) = \pm z^{lM} u(z)$. But since $v(z)$ is also the time-reversed version of the filter $u(z)$, it implies that $u(z)$ has linear phase. ◇◇◇

III FACTORIZATION OF LINEAR PHASE PARAUNITARY SYSTEMS FOR EVEN M

In this section, we will first derive a cascade-form structure for synthesizing linear phase paraunitary systems. Our theory will provide an interpretation for the condition mentioned in [26]. We will then prove the main result of this section, namely, *every linear phase paraunitary system described by eq. (2.1) can be factored in terms of the proposed structure.*

The synthesis procedure consists of two steps. In the first step, we propagate the property that the set of filters generated be pairwise time-reversed versions of one another. This means that they are related as $h'_k(n) = h'_{M-1-k}(n), \quad k = 0, \ldots, M-1.$

Notice that the sum of two sequences related as above is symmetric, and their difference is antisymmetric. Furthermore, any linear combination of symmetric (antisymmetric) sequences is symmetric (antisymmetric). In the second step, we add an orthogonal block which performs these operations on the pairwise symmetric sequences to obtain filters that have linear phase.

The reason for this two step approach is that it can be shown that it is not possible to propagate the linear-phase property itself by addition of further building blocks.

Consider fig. 5.3. The pairwise time-reversed property implies the following relation between the filters:

$$H'_{m,M-1-k}(z) = z^{-((m+1)M-1)} H'_{m,k}(z^{-1}), \quad k = 0, \ldots, \; L-1, \tag{3.1}$$

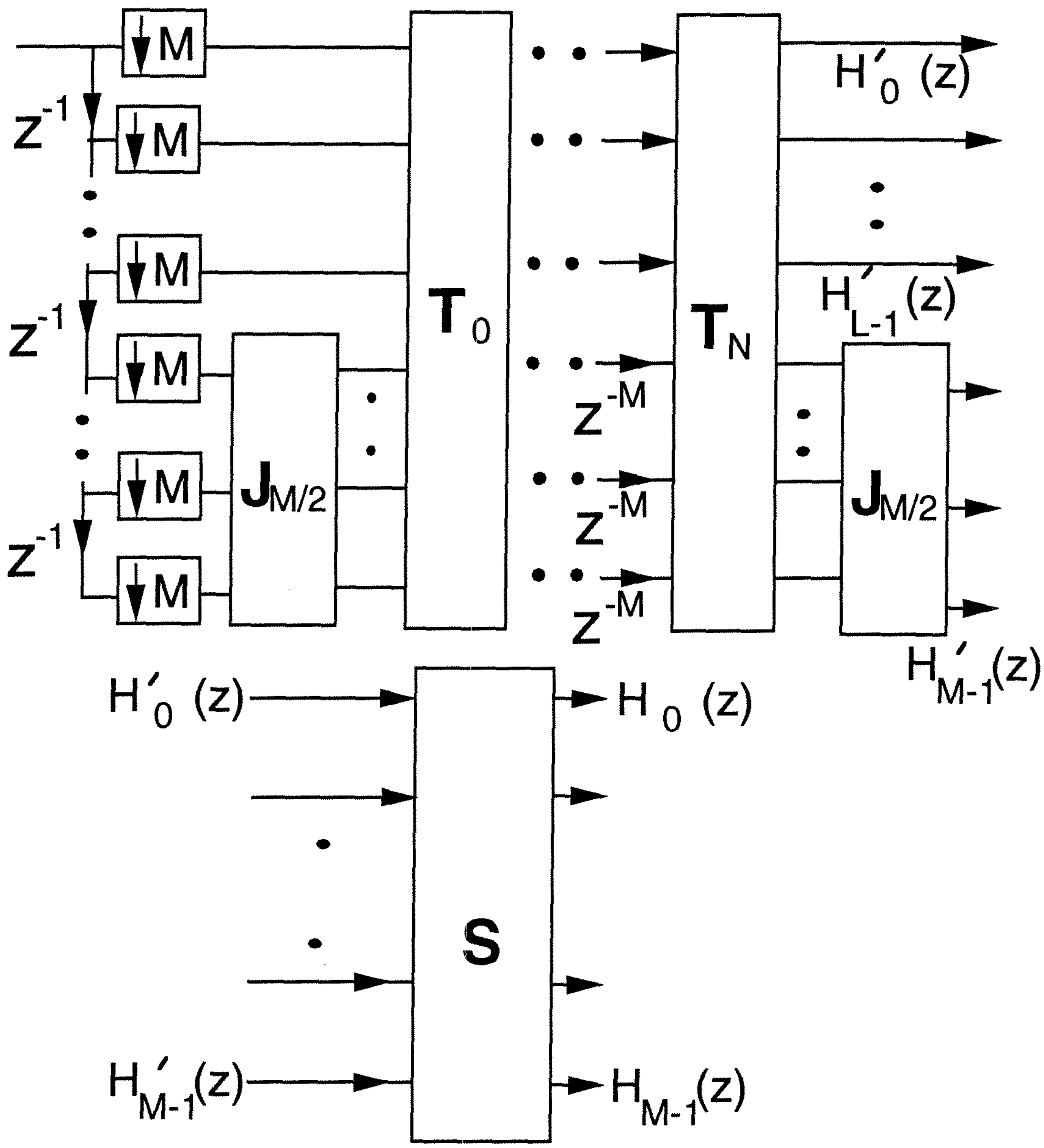


Fig. 5.4. An equivalent structure for the linear phase paraunitary system

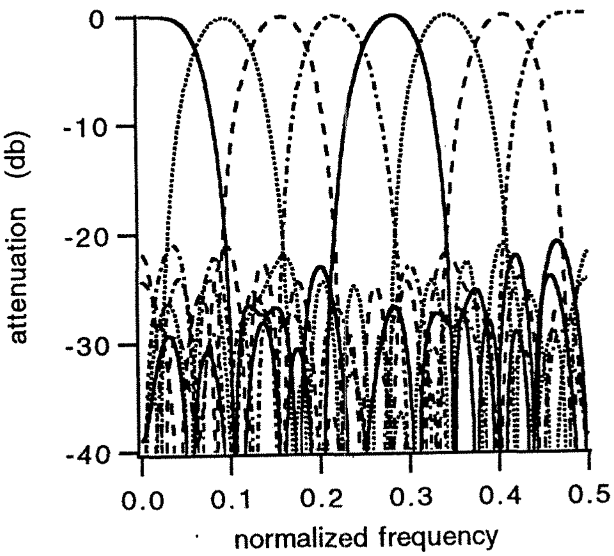


Fig. 5.5. Eight-channel linear phase paraunitary system

$H_0(z)$	$H_1(z)$	$H_2(z)$	$H_3(z)$
2.7063633740569D-02	-1.6091969205871D-02	-2.8456950643002D-02	1.4960492098433D-02
-1.2136967794206D-04	4.5104219698022D-02	1.9019503506337D-03	4.4678162875469D-02
1.2169911500658D-04	5.7067555274487D-03	4.0561845270800D-02	-4.0347878743950D-03
-1.0939438116179D-02	1.1596091494340D-02	2.9503201934660D-03	-1.2928325761109D-02
-3.9006836245961D-02	1.4102613035988D-02	-2.2991473562361D-02	-1.5109881590786D-02
-1.7129046826691D-02	-1.1144644575007D-02	3.4018237002281D-03	-4.9444834516179D-02
-6.2580468711733D-02	4.0975324850524D-02	9.2311984186006D-02	-3.7266345907583D-02
-1.5970643737541D-02	0.12106010509825	8.6349676608556D-03	8.4900080405164D-02
-8.0085817350597D-02	0.12469607197029	-7.7589420570820D-02	0.11866646696795
1.7124808276196D-02	0.15395817207336	-0.22772066314211	-0.10980819448873
1.0067593331610D-01	1.6326854841075D-02	-0.19287393785391	-0.26124185969832
0.17114723136171	-0.14257162890702	0.10867205234840	-1.3044768601487D-03
0.24072321382144	-0.34043725682195	0.34879540903907	0.37437930463834
0.32076121985196	-0.41375436862609	0.30145304416569	0.12094472643289
0.35502595590299	-0.33699605405521	-3.4833482443945D-02	-0.38952044617052
0.40018576621481	-0.14275844384806	-0.40959004960062	-0.29227460355523

Table 5.1. Coefficients of an eight-channel filter bank. Only the coefficients $h_i(0)$ through $h_i(15)$ are tabulated for the first four filters. Linear phase implies $h_i(n)=h_i(31-n)$, and mirror image symmetry implies $H_i(z)=H_{7-i}(-z)$.

where $L = M/2$. If $\mathbf{F}_m(z)$ is the polyphase matrix corresponding to these filters, then

$$z^{-m}\mathbf{J}_M\mathbf{F}_m(z^{-1})\mathbf{J}_M = \mathbf{F}_m(z). \tag{3.2}$$

Consider Fig. 6.3, and let

$$\mathbf{F}_{m+1}(z) = \mathbf{K}_{m+1}\mathbf{\Lambda}(z)\mathbf{F}_m(z), \tag{3.3}$$

where $\mathbf{\Lambda}(z) = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_{M/2} \end{pmatrix}$, and $\mathbf{F}_{m+1}(z)$ is the polyphase matrix corresponding to the filters at the next stage. If $\mathbf{K}_{m+1}$ is an orthogonal matrix, we have

$$\mathbf{F}_m(z) = \mathbf{\Lambda}(z^{-1})\mathbf{K}_{m+1}^T\mathbf{F}_{m+1}(z). \tag{3.4}$$

For the filters at the next stage to retain the pairwise time-reversed property, we need

$$z^{-(m+1)}\mathbf{J}_M\mathbf{F}_{m+1}(z^{-1})\mathbf{J}_M = \mathbf{F}_{m+1}(z). \tag{3.5}$$

Using eq. (3.4) in eq. (3.2), we obtain the following equation

$$z^{-m}\mathbf{J}_M\mathbf{\Lambda}(z)\mathbf{K}_{m+1}^T\mathbf{F}_{m+1}(z^{-1})\mathbf{J}_M = \mathbf{\Lambda}(z^{-1})\mathbf{K}_{m+1}^T\mathbf{F}_{m+1}(z).$$

By imposing (3.5), and using the identity $\mathbf{\Lambda}(z)\mathbf{J}_M\mathbf{\Lambda}(z) = z^{-1}\mathbf{J}_M$, we see that the necessary and sufficient condition on $\mathbf{K}_{m+1}$ is, $\mathbf{K}_{m+1}\mathbf{J}_M\mathbf{K}_{m+1}^T = \mathbf{J}_M$. By partitioning $\mathbf{K}_{m+1}$ as $\begin{pmatrix} \mathbf{A}' & \mathbf{C}' \\ \mathbf{B}' & \mathbf{D}' \end{pmatrix}$, we can verify that the necessary and sufficient condition for eq. (3.5) to hold is that the matrix $\mathbf{K}_{m+1}$ be of the form

$$\mathbf{K}_{m+1} = \begin{pmatrix} \mathbf{A}' & \mathbf{C}' \\ \mathbf{J}_{M/2}\mathbf{C}'\mathbf{J}_{M/2} & \mathbf{J}_{M/2}\mathbf{A}'\mathbf{J}_{M/2} \end{pmatrix}. \tag{3.6}$$

Thus $\mathbf{K}_{m+1}$ can be rewritten as

$$\mathbf{K}_{m+1} = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{M/2} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{m+1} & \mathbf{C}_{m+1} \\ \mathbf{C}_{m+1} & \mathbf{A}_{m+1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{M/2} \end{pmatrix}, \tag{3.7}$$

where $\mathbf{A}_{m+1} = \mathbf{A}'$, and $\mathbf{C}_{m+1} = \mathbf{C}'\mathbf{J}_{M/2}$.

It can be verified that a matrix $\mathbf{K}_0$ with a form similar to that described above can be used to initialize the process.

Given a paraunitary polyphase matrix $\mathbf{F}(z)$ of order N, corresponding to a set of filters that are pair-wise time-reversed versions of one another, i.e.,

$$z^{-N}\mathbf{J}_M\mathbf{F}(z^{-1})\mathbf{J}_M = \mathbf{F}(z), \tag{3.8}$$

we are interested in obtaining a paraunitary polyphase matrix $\mathbf{E}(z)$ of order N corresponding to a set of linear phase paraunitary filters, i.e., satisfying eq. (2.1). Let $\mathbf{E}(z) = \mathbf{SF}(z)$, where $\mathbf{S}$ is an orthogonal matrix. Under the constraint (3.8), it can be shown (by substitution and simplification) that $\mathbf{E}(z)$ satisfies eq. (2.1) if and only if $\mathbf{S}^T\mathbf{DS} = \mathbf{J_M}$.

Hence the following product gives linear phase paraunitary polyphase matrix

$$\mathbf{E}(z) = \mathbf{SPT}_N\mathbf{P\Lambda}(z)\mathbf{PT}_{N-1}\mathbf{P\Lambda}(z)\mathbf{P}\ldots\mathbf{P\Lambda}(z)\mathbf{PT}_0\mathbf{P}, \tag{3.9}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{M/2} \end{pmatrix}, \quad \text{and} \quad \mathbf{T}_i = \begin{pmatrix} \mathbf{A}_i & \mathbf{C}_i \\ \mathbf{C}_i & \mathbf{A}_i \end{pmatrix}. \tag{3.10}$$

Noting that $\mathbf{P\Lambda}(z)\mathbf{P} = \mathbf{\Lambda}(z)$, we obtain,

$$\mathbf{E}(z) = \mathbf{SPT}_N\mathbf{\Lambda}(z)\mathbf{T}_{N-1}\mathbf{\Lambda}(z)\ldots\mathbf{\Lambda}(z)\mathbf{T}_0\mathbf{P}, \tag{3.11}$$

which is shown in Fig. 5.4.

Minimal Characterization: The matrix $\mathbf{K}_{m+1}$ is parametrized completely by parametrizing all orthogonal matrices of the form $\mathbf{T}_{m+1} = \begin{pmatrix} \mathbf{A}_{m+1} & \mathbf{C}_{m+1} \\ \mathbf{C}_{m+1} & \mathbf{A}_{m+1} \end{pmatrix}$. Now, such a matrix can always be factored as

$$\begin{pmatrix} \mathbf{A}_{m+1} & \mathbf{C}_{m+1} \\ \mathbf{C}_{m+1} & \mathbf{A}_{m+1} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{pmatrix} \begin{pmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{U} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{pmatrix}, \tag{3.12}$$

where $\mathbf{W} = (\mathbf{A}_{m+1} + \mathbf{C}_{m+1})/2$ and $\mathbf{U} = (\mathbf{A}_{m+1} - \mathbf{C}_{m+1})/2$. Thus $\mathbf{T}_{m+1}$ is orthogonal if and only if the two matrices $\mathbf{W}$ and $\mathbf{U}$ are orthogonal. The orthogonal matrices $\mathbf{W}$ and $\mathbf{U}$ can be completely characterized by $\begin{pmatrix} M/2 \\ 2 \end{pmatrix}$ rotations each [29].

On the other hand, it can be shown that a unitary matrix $\mathbf{S}$ satisfies the condition $\mathbf{S}^T\mathbf{D}\mathbf{S} = \mathbf{J_M}$ if and only if it can be written as

$$\mathbf{S} = (1/\sqrt{2})\begin{pmatrix} \mathbf{S}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 \end{pmatrix}\begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{J}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{J}_{M/2} \end{pmatrix}, \tag{3.13}$$

where $\mathbf{S}_0$ and $\mathbf{S}_1$ are orthogonal matrices of size $M/2 \times M/2$, (Partition $\mathbf{S}$ into four blocks, sustitute in $\mathbf{S}^T\mathbf{D}\mathbf{S} = \mathbf{J_M}$ and simplify.) $\mathbf{S}$ can hence be parametrized by $2\begin{pmatrix} M/2 \\ 2 \end{pmatrix}$ rotations.

We now come to the main result of this section, which is the converse of the previous result.

Theorem 3: Let $\mathbf{E}(z)$ be an FIR linear phase paraunitary matrix (satisfying eq. (2.1)). Then it can always be factored as in eq. (3.11), where $\mathbf{\Lambda}(z) = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_{M/2} \end{pmatrix}$, and $\mathbf{T}_i$ and $\mathbf{P}$ are as in eq. (3.10).

Proof of Theorem 3: The rest of this section deals with the proof of the above theorem. The reader may skip over to the next section without loss of continuity.

The proof of the theorem will use the definition of 'balanced vectors' which we now propose:

Definition: A vector $\mathbf{y}$ is said to be 'balanced' if it is orthogonal to its own flipped version, i.e., it saisfies the equation

$$\mathbf{y}^T\mathbf{J}_M\mathbf{y} = 0.$$

The significance of balanced vectors has been explained in Appendix A.

Proof of the Theorem: In this case we are given a matrix $\mathbf{E}(z)$ satisfying eq. (2.1). The first step is to show that from this linear phase paraunitary matrix, we can always get a polyphase matrix $\mathbf{F}(z)$ whose filters are pairwise time-reversed versions of one another (satisfying eq. (3.8)). For this, let $\mathbf{S}$ be any matrix of the form given in eq. (3.13), where $\mathbf{S}_0$ and $\mathbf{S}_1$ are arbitrary orthogonal matrices. Then it can be shown by substitution that the product $\mathbf{F}(z) = \mathbf{S}^T\mathbf{E}(z)$ satisfies eq. (3.8).

Now we need to show that the matrix $\mathbf{F}(z)$ can always be factored into the required form. This is achieved by performing the 'order-reduction' process as outlined below. Let

$$\mathbf{F}_{m+1}(z) = \mathbf{f}_{m+1}(0)+\mathbf{f}_{m+1}(1)z^{-1}+\mathbf{f}_{m+1}(2)z^{-2}+\ldots+\mathbf{f}_{m+1}(m+1)z^{-(m+1)}, \ \mathbf{f}_{m+1}(m+1) \neq \mathbf{0}. \tag{3.14}$$

We will show that there exists $\mathbf{F}_m(z)$ of the form

$$\mathbf{F}_m(z) = \mathbf{f}_m(0) + \mathbf{f}_m(1)z^{-1} + \mathbf{f}_m(2)z^{-2} + \ldots + \mathbf{f}_m(m)z^{-m}, \qquad \mathbf{f}_m(m) \neq \mathbf{0}, \tag{3.15}$$

and satisfying the required properties. Let $\mathbf{F}_{m+1}(z)$ satisfy eq. (3.5). Specifically, we will now show that it can always be written as

$$\mathbf{F}_{m+1}(z) = \mathbf{P}\mathbf{T}_{m+1}\mathbf{P}\mathbf{\Lambda}(z)\mathbf{F}_m(z), \tag{3.16}$$

where $\mathbf{F}_m(z)$ satisfies eq. (3.2), and the matrices $\mathbf{P}, \mathbf{T}_{m+1}$, and $\mathbf{\Lambda}(z)$ have the form described earlier. Paraunitariness of $\mathbf{F}_m(z)$ follows by noting that

$$\mathbf{F}_m(z) = \mathbf{\Lambda}(z^{-1})\mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{F}_{m+1}(z), \tag{3.17}$$

where all matrices on the right hand side of this equation are paraunitary.

Linear phase property: We want to show that $\mathbf{F}_m(z)$ satisfies eq. (3.2). Substituting eq. (3.16) into eq. (3.5), we get

$$z^{-(m+1)}\mathbf{J}_M\mathbf{P}\mathbf{T}_{m+1}\mathbf{P}\mathbf{\Lambda}(z^{-1})\mathbf{F}_m(z^{-1})\mathbf{J}_M = \mathbf{P}\mathbf{T}_{m+1}\mathbf{P}\mathbf{\Lambda}(z)\mathbf{F}_m(z). \tag{3.18}$$

Since $\mathbf{P}^{-1} = \mathbf{P}$ and $\mathbf{F}_m(z)$ is paraunitary, we get,

$$z^{-(m+1)}\mathbf{\Lambda}(z^{-1})\mathbf{F}_m(z^{-1})\mathbf{J}_M\widetilde{\mathbf{F}}_m(z) = \mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{J}_M\mathbf{P}\mathbf{T}_{m+1}\mathbf{P}\mathbf{\Lambda}(z). \tag{3.19}$$

If $\mathbf{T}_{m+1}$ is an orthogonal matrix of the form described in eq. (3.10), and $\mathbf{P}$ has the form described in eq. (3.10), then it can be verified that $\mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{J}_M\mathbf{P}\mathbf{T}_{m+1}\mathbf{P} = \mathbf{J}_M$. Hence we get

$$z^{-(m+1)}\mathbf{\Lambda}(z^{-1})\mathbf{F}_m(z^{-1})\mathbf{J}_M\widetilde{\mathbf{F}}_m(z) = \mathbf{J}_M\mathbf{\Lambda}(z), \tag{3.20}$$

i.e.,

$$z^{-m}[z^{-1}\mathbf{\Lambda}(z^{-1})\mathbf{J}_M\mathbf{\Lambda}(z^{-1})]\mathbf{F}_m(z^{-1})\mathbf{J}_M\widetilde{\mathbf{F}}_m(z) = \mathbf{I}. \tag{3.21}$$

It can be verified that $[z^{-1}\mathbf{\Lambda}(z^{-1})\mathbf{J}_M\mathbf{\Lambda}(z^{-1})] = \mathbf{J}_M$. Sustituting this into eq. (3.21), and rearranging the terms, we get eq. (3.2).

Causality: It only remains to show that there exists a matrix $\mathbf{T}_{m+1}$ such that $\mathbf{F}_m(z)$ obtained from eq. (3.17) is causal. Both the linear phase property, and the paraunitary property continue to hold for the reduced system as long as the matrix $\mathbf{T}_{m+1}$ is *any* orthogonal matrix of the required form (eq. (3.10)). Indeed, it is the causality condition on the reduced system which determines the particular choice of the matrix $\mathbf{T}_{m+1}$.

From eq. (3.17) we get,

$$\mathbf{F}_m(z) = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{F}_{m+1}(z) + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & z\mathbf{I}_{M/2} \end{pmatrix} \mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{F}_{m+1}(z). \tag{3.22}$$

The second term on the right hand side of this equation is responsible for the noncausality. In particular, the noncausal part of the second term is given by

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & z\mathbf{I}_{M/2} \end{pmatrix} \mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{f}_{m+1}(0). \tag{3.23}$$

We have to show that there exists a matrix $\mathbf{T}_{m+1}$ of the form in eq. (3.10) which makes this term equal to zero. Let

$$\mathbf{T}_{m+1} = \begin{pmatrix} \mathbf{A}_{m+1} & \mathbf{C}_{m+1} \\ \mathbf{C}_{m+1} & \mathbf{A}_{m+1} \end{pmatrix}. \tag{3.24}$$

Simplifying eq. (3.23), we find that $\mathbf{T}_{m+1}$ should be such that

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{M/2} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{m+1}^T & \mathbf{C}_{m+1}^T\mathbf{J}_{M/2} \\ \mathbf{C}_{m+1}^T & \mathbf{A}_{m+1}^T\mathbf{J}_{M/2} \end{pmatrix} \mathbf{f}_{m+1}(0) = \mathbf{0}. \tag{3.25}$$

Hence, it is sufficient to find $\mathbf{A}_{m+1}$ and $\mathbf{C}_{m+1}$ such that

$$\begin{pmatrix} \mathbf{C}_{m+1}^T & \mathbf{A}_{m+1}^T\mathbf{J}_{M/2} \end{pmatrix} \mathbf{f}_{m+1}(0) = \mathbf{0}. \tag{3.26}$$

Now, eq. (3.5) in particular means that

$$\mathbf{J}_M \mathbf{f}_{m+1}(0)\mathbf{J}_M = \mathbf{f}_{m+1}(m+1). \tag{3.27}$$

The paraunitary condition in the time domain implies $\mathbf{f}_{m+1}^T(m+1)\mathbf{f}_{m+1}(0) = \mathbf{0}$. Hence we have

$$\mathbf{f}_{m+1}^T(0)\mathbf{J}_M \mathbf{f}_{m+1}(0) = \mathbf{0}. \tag{3.28}$$

By Sylvester's rank inequality [30], therefore, we get $rank(\mathbf{f}_{m+1}(0)) = r \leq M/2$.

Equation (3.28) implies that the columns of the matrix $\mathbf{f}_{m+1}(0)$ are balanced. Hence it can be shown (Appendix B) that there exists a set of orthonormal balanced vectors $\mathbf{x}_i, \quad i = 1, \ldots, M/2$ such that if $\mathbf{X}^T$ is the matrix of size $M/2 \times M$ whose rows are these vectors, this matrix satisfies the following properties:

1) $\mathbf{X}^T\mathbf{X} = \mathbf{I}_{M/2}$ (from the fact that $\mathbf{x}_i$ are orthogonal).

2) $\mathbf{X}^T\mathbf{J}_M\mathbf{X} = \mathbf{0}$ (from the fact that $\mathbf{x}_i$ are balanced).

3) $\mathbf{X}^T\mathbf{J}_M\mathbf{f}_{m+1}(0) = \mathbf{0}$ (by the construction outlined in Appendix B).

It can be verified that the matrix

$$\mathbf{T}_{m+1} = \begin{pmatrix} \mathbf{X}^T \\ \mathbf{X}^T\mathbf{J}_M \end{pmatrix} \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{M/2} \end{pmatrix} \tag{3.29}$$

can be written in the form as in eq. (3.24). Moreover, with this choice of the matrix $\mathbf{T}_{m+1}$, eq. (3.25) is satisfied. This proves that $\mathbf{F}_m(z)$ is causal.

Order reduction: Given the fact that $\mathbf{F}_m(z)$ is causal, and that it satisfies eq. (3.2), we can see that the order of $\mathbf{F}_m(z)$ is m. Thus there is a reduction in order by 1 Hence for a system of order N, the factorization process is guaranteed to terminate in N steps.

This concludes the proof of theorem 3. $\diamond\diamond\diamond$

The above theorem guarantees the factorization of all linear phase paraunitary systems satisfying eq. (2.1). Such a linear phase paraunitary filter bank of order N can hence be characterized by $2(N)\begin{pmatrix} M/2 \\ 2 \end{pmatrix}$ rotation angles.

The degree of a causal rational system is defined as (sec.13.8 [10]) the minimum number of delays required for its implementation. A structure is said to be minimal if the number of delays used is equal to the degree of the transfer function. For a paraunitary system, we know that (Thm. 14.7.1 [10])

$$deg[det[\mathbf{E}(z)]] = deg[\mathbf{E}(z)]. \tag{3.30}$$

In our case,

$$deg[det[\mathbf{E}(z)]] = deg[det[\mathbf{SPT}_N\mathbf{\Lambda}(z)\mathbf{T}_{N-1}\mathbf{\Lambda}(z)\ldots\mathbf{\Lambda}(z)\mathbf{T}_0\mathbf{P}]] = NM/2, \tag{3.31}$$

which is equal to the number of delays used. Hence the factorization is minimal.

IV LINEAR PHASE PARAUNITARY FILTERS WITH PAIRWISE MIRROR-IMAGE FREQUENCY RESPONSES FOR EVEN M

In the previous section, we factorized a linear phase paraunitary system into a product of orthogonal building blocks each of which can be implemented with $2\binom{M/2}{2}$ rotation angles. These angles can be made the variables in the design process. The number of angles can become fairly large when the number of channels M increases. It would be useful to cut down the number of optimization variables by structurally imposing some other additional constraints on the filters. One of the constraints that can be imposed is that of pairwise mirror image symmetry in the frequency domain around $\pi/2$. Such a condition had been imposed on general paraunitary systems in [31]. One way to impose the condition that the filters satisfy the pairwise mirror image condition in the frequency domain is to ensure that the filters are related as

$$H_{M-1-k}(z) = H_k(-z), \qquad k = 0, \ldots,\ L-1, \tag{4.1}$$

where $L = M/2$. If M is even, in terms of the polyphase matrix of the filters, this becomes

$$\mathbf{J}_M\mathbf{E}(z) = \mathbf{E}(z)\mathbf{V}_M. \tag{4.2}$$

As mentioned earlier, the matrix $\mathbf{V}_M$ is a diagonal matrix of size $M \times M$ with alternate $\pm 1's$ on the diagonal, starting with $+1$. This symmetry condition is in addition to the conditions of linear phase (eq. (2.1)) and paraunitariness (eq. (1.1)).

To develop a cascade structure which generates such filters, we will assume that we have a paraunitary matrix $\mathbf{E}_{m-1}(z)$ of order $m-1$ satisfying the conditions of paraunitariness (eq. (1.1)), linear phase (eq. (2.1)), and pairwise mirror-image symmetry of frequency responses (eq. (4.2)). From it, we will show how a paraunitary matrix $\mathbf{E}_m(z)$ of order m can be obtained satisfying the above three properties. We will do this by *post multiplying* the given matrix $\mathbf{E}_{m-1}(z)$ by a paraunitary matrix $\mathbf{R}(z)$ of order one. [1]

Let

$$\mathbf{E}_m(z) = \mathbf{E}_{m-1}(z)\mathbf{R}(z). \tag{4.3}$$

Clearly, $\mathbf{E}_m(z)$ is paraunitary. Also,

$$\mathbf{E}_{m-1}(z) = \mathbf{E}_m(z)\widetilde{\mathbf{R}}(z). \tag{4.4}$$

Propagating the Linear Phase Property: From the fact that $\mathbf{E}_{m-1}(z)$ satisfies the linear phase property, we have

$$z^{-m}\mathbf{D}\mathbf{E}_m(z^{-1})\widetilde{\mathbf{R}}(z^{-1})\mathbf{J}_M = \mathbf{E}_m(z)\widetilde{\mathbf{R}}(z), \tag{4.5}$$

i.e.,

$$z^{-m}\mathbf{D}\mathbf{E}_m(z^{-1})\widetilde{\mathbf{R}}(z^{-1})\mathbf{J}_M\mathbf{R}(z) = \mathbf{E}_m(z). \tag{4.6}$$

Hence for $\mathbf{E}_m(z)$ to satisfy the linear phase property, $\mathbf{R}(z)$ should satisfy

$$\widetilde{\mathbf{R}}(z^{-1})\mathbf{J}_M\mathbf{R}(z) = z^{-1}\mathbf{J}_M. \tag{4.7}$$

[1] This derivation could also be made by pre-multiplying an existing matrix by an extra block. This was the approach followed in section III, because it simplifies the proof of theorem 3 to some extent. In proving the results of this section, the post-multiplication strategy will lead to slightly simpler derivations. The reader must note that preference for one strategy over the other has been dictated purely by simplicity of presentation.

It can be verified that if $\mathbf{R}(z) = \mathbf{\Lambda}(z)\mathbf{PTP}$ with the matrices $\mathbf{\Lambda}(z)$, $\mathbf{P}$ and $\mathbf{T}$ as in the previous section, $\mathbf{R}(z)$ satisfies eq. (4.7).

Propagating the Pairwise Mirror-Image Property in the Frequency Domain: Assuming that eq. (4.2) holds for $\mathbf{E}_{m-1}(z)$, and using eq. (4.4) we get

$$\mathbf{J}_M\mathbf{E}_m(z)\widetilde{\mathbf{R}}(z) = \mathbf{E}_m(z)\widetilde{\mathbf{R}}(z)\mathbf{V}_M, \tag{4.8}$$

i.e.,

$$\mathbf{J}_M\mathbf{E}_m(z) = \mathbf{E}_m(z)\widetilde{\mathbf{R}}(z)\mathbf{V}_M\mathbf{R}(z). \tag{4.9}$$

Hence, $\mathbf{R}(z)$ should satisfy the property

$$\widetilde{\mathbf{R}}(z)\mathbf{V}_M\mathbf{R}(z) = \mathbf{V}_M. \tag{4.10}$$

We now have two cases:

Case 1: M/2 is even: In this case, $\mathbf{V}_M = \begin{pmatrix} \mathbf{V}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{M/2} \end{pmatrix}$. Substituting this, and the fact that $\mathbf{R}(z) = \mathbf{\Lambda}(z)\mathbf{PTP}$ with $\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C} & \mathbf{A} \end{pmatrix}$, in eq. (4.10) and simplifying, we get

$$\begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{C}^T & \mathbf{A}^T \end{pmatrix}\begin{pmatrix} \mathbf{V}_{M/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{V}_{M/2} \end{pmatrix}\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C} & \mathbf{A} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_{M/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{V}_{M/2} \end{pmatrix}. \tag{4.11}$$

Using a factorization for $\mathbf{T}$ similar to eq. (3.12) and simplifying, we get

$$\begin{pmatrix} \mathbf{0} & \mathbf{U}^T\mathbf{V}_{M/2}\mathbf{W} \\ \mathbf{W}^T\mathbf{V}_{M/2}\mathbf{U} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{V}_{M/2} \\ \mathbf{V}_{M/2} & \mathbf{0} \end{pmatrix}. \tag{4.12}$$

The above equation is satisfied if $\mathbf{U}$ is taken to be an arbitrary orthogonal matrix of size $M/2 \times M/2$, and the matrix $\mathbf{W}$ is chosen as

$$\mathbf{W} = \mathbf{V}_{M/2}\mathbf{U}\mathbf{V}_{M/2}. \tag{4.13}$$

Hence, in this case, we have $\begin{pmatrix} M/2 \\ 2 \end{pmatrix}$ degrees of freedom to optimize per stage.

Case 2: M/2 is odd: In this case, $\mathbf{V}_M = \begin{pmatrix} \mathbf{V}_{M/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{V}_{M/2} \end{pmatrix}$, unlike the case where $M/2$ is even. However, if we use the relation $\mathbf{R}(z) = \mathbf{\Lambda}(z)\mathbf{PTP}$ with $\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C} & \mathbf{A} \end{pmatrix}$, and perform the simplifications as before, we get eq. (4.12) once again, proving that there are $\binom{M/2}{2}$ degrees of freedom to be optimized in this case also.

Thus all three properties have been satisfied.

Initialization: It only remains to find a degree zero paraunitary matrix $\mathbf{E}_0(z)$ (i.e., a constant orthogonal matrix $\mathbf{S}$), which will initialize the above process. From the discussion in section III, it can be verified that the matrix $\mathbf{S}$ satisfies the linear phase property ($\mathbf{DSJ}_M = \mathbf{S}$), if it is of the form

$$\mathbf{S} = (1/\sqrt{2}) \begin{pmatrix} \mathbf{S}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 \end{pmatrix} \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{J}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{J}_{M/2} \end{pmatrix} \mathbf{Q}, \tag{4.14}$$

where $\mathbf{Q}$ is a symmetric permutation matrix. This is because $\mathbf{QJ}_M\mathbf{Q} = \mathbf{J}_M$ for any such permutation matrix. Let $\mathbf{Q}$ be so chosen that $\mathbf{QV}_M\mathbf{Q} = \mathbf{D}$, where $\mathbf{D} = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{M/2} \end{pmatrix}$. Now, let $\mathbf{S}' = \mathbf{SQ}$. For the matrix $\mathbf{S}$ to satisfy the pairwise mirror-image property ($\mathbf{J}_M\mathbf{S} = \mathbf{SV}_M$), it can be verified that the matrix $\mathbf{S}'$ should satisfy $\mathbf{S}'\mathbf{DS}'^T = \mathbf{J}_M$. Substituting the forms of various matrices and simplifying, we get

$$\begin{pmatrix} \mathbf{0} & \mathbf{S}_0\mathbf{S}_1^T \\ \mathbf{S}_1\mathbf{S}_0^T & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{J}_{M/2} \\ \mathbf{J}_{M/2} & \mathbf{0} \end{pmatrix}. \tag{4.15}$$

This equation can be satisfied by letting $\mathbf{S}_0$ be an arbitrary orthogonal matrix, and choosing $\mathbf{S}_1 = \mathbf{J}_{M/2}\mathbf{S}_0$. Thus the matrix $\mathbf{S}$ can be realized with $\binom{M/2}{2}$ rotations [29].

The foregoing discussion can be summarized in the following theorem:

Theorem 4: A linear phase paraunitary matrix satisfying eq. (2.1) whose filters satisfy the additional pairwise mirror-image property in the frequency domain (eq. (4.2)) can be realized as

$$\mathbf{E}(z) = \mathbf{S}\mathbf{\Lambda}(z)\mathbf{PT}_0\mathbf{\Lambda}(z)\ldots\mathbf{T}_N\mathbf{P}, \tag{4.16}$$

where

$$\mathbf{S} = (1/\sqrt{2}) \begin{pmatrix} \mathbf{S}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{M/2}\mathbf{S}_0 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{J}_{M/2} \\ \mathbf{I} & -\mathbf{J}_{M/2} \end{pmatrix} \mathbf{Q}, \tag{4.17}$$

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{pmatrix} \begin{pmatrix} \mathbf{V}_{M/2}\mathbf{U}_i\mathbf{V}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_i \end{pmatrix} \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{pmatrix}, \tag{4.18}$$

and the matrix $\mathbf{P}$ is as in eq. (3.10). ◇◇◇

The fact that the structure continues to be minimal is easily verified, though we have not shown it to be complete.

Fig. 5.5 shows an example of an eight-channel system where a four stage lattice was used. The filters are linear phase, paraunitary, and satisfy the pairwise mirror-image symmetry in the frequency domain. The impulse response coefficients of the eight-channel system have been tabulated in Table 5.1.

V LINEAR PHASE PARAUNITARY FILTERS FOR ODD M

While the existence of linear phase paraunitary filter banks had been indicated in [26] for an even number of channels M, for an odd number of channels, the existence of non-degenerate filter banks has not been shown so far. In this section we shall synthesize linear phase paraunitary filter banks for an odd number of channels. There are two ways to design such systems. One way is to develop a cascade structure as we did in the previous sections. The second way is to obtain linear phase systems for a certain odd M by suitably combining linear phase systems of size $(M-1)/2$ and $(M+1)/2$, while maintaining the paraunitary property. We will consider both of these approaches in this section.

A cascade based approach: In this sub-section we proceed as we did in section III, i.e., first design a set of filters which satisfy the property that the filters are pairwise flipped versions of each other in the time domain, and then suitably combine these to get a linear phase system.

Fact 4.1 Consider a polyphase matrix of size $M \times M$, M odd, which is obtained as the following product:

$$\mathbf{F}(z) = \mathbf{P}\mathbf{T}_N\mathbf{\Lambda}(z)\mathbf{T}_{N-1}\mathbf{\Lambda}(z)\ldots\mathbf{\Lambda}(z)\mathbf{T}_0\mathbf{P}, \tag{5.1}$$

where,

$$\mathbf{P} = \begin{pmatrix} \mathbf{I}_{(M+1)/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{(M-1)/2} \end{pmatrix}, \tag{5.2}$$

$\mathbf{T}_i$ are orthogonal matrices of the form

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{A}_i & \mathbf{0} & \mathbf{C}_i \\ \mathbf{0}^T & 1 & \mathbf{0}^T \\ \mathbf{C}_i & \mathbf{0} & \mathbf{A}_i \end{pmatrix}, \tag{5.3}$$

and

$$\mathbf{\Lambda}(z) = \begin{pmatrix} \mathbf{I}_{(M+1)/2} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_{(M-1)/2} \end{pmatrix}. \tag{5.4}$$

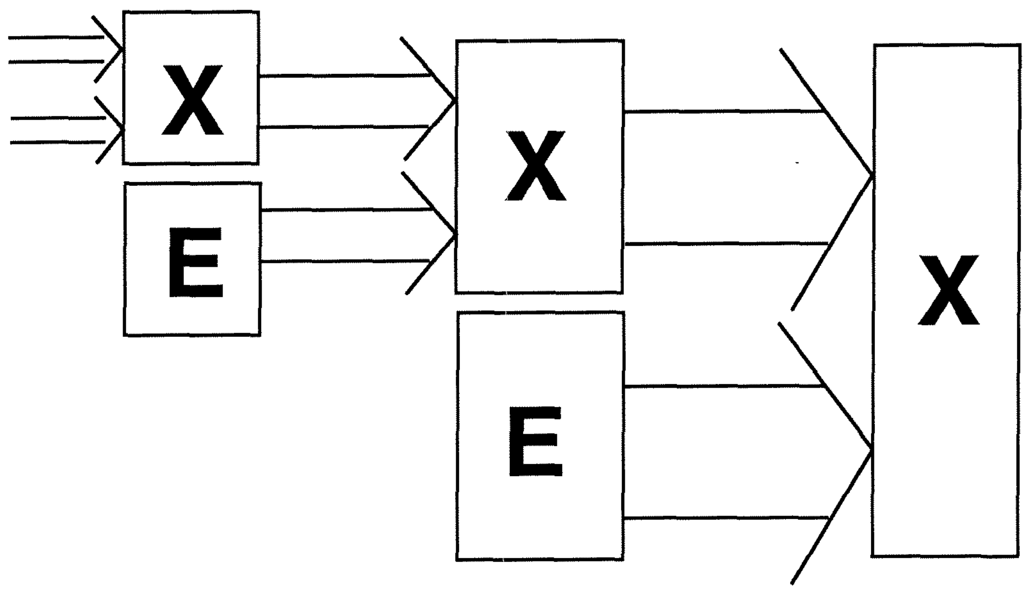
Then this structure generates a paraunitary filter bank in which $H_k(z)$ is the time-reversed version of $H_{M-1-k}(z)$.

Proof: Since all matrices in the product are individually paraunitary, the product $\mathbf{F}(z)$ is also paraunitary. Now to prove that the filters are pairwise flipped versions of one another, we need to show that the matrix $\mathbf{F}(z)$ satisfies the condition

$$\begin{pmatrix} z^{-N}\mathbf{I}_{(M-1)/2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & z^{-N}\mathbf{I}_{(M-1)/2} \end{pmatrix} \mathbf{J}_M\mathbf{F}(z^{-1})\mathbf{J}_M = \mathbf{F}(z), \tag{5.5}$$

where N is the order of the polyphase matrix $\mathbf{F}(z)$. In particular, by our construction, the middle filter will be just $H'_{(M+1)/2}(z) = z^{-(M+1)/2}$. Hence it is the flipped version of itself. Substituting the forms of various matrices, we can verify that eq. (5.5) indeed holds. ◇◇◇

Now suppose we are given a paraunitary matrix $\mathbf{F}(z)$ satisfying eq. (5.5) and whose order N is even (as required by Lemma 1); we can obtain a matrix $\mathbf{E}(z)$ from it which corresponds to a set of linear phase paraunitary filters. Since the middle filter is just $H'_{(M+1)/2}(z) = z^{-(M+1)/2}$, we first multiply this filter by an appropriate delay $z^{-N/2}$. This can be done by premultiplying the matrix $\mathbf{F}(z)$ by the diagonal matrix



E = structure giving linear-phase paraunitary filters for even M

X= the interleaving mechanism

Fig. 5.6. Obtaining linear phase paraunitary filters by interleaving smaller systems

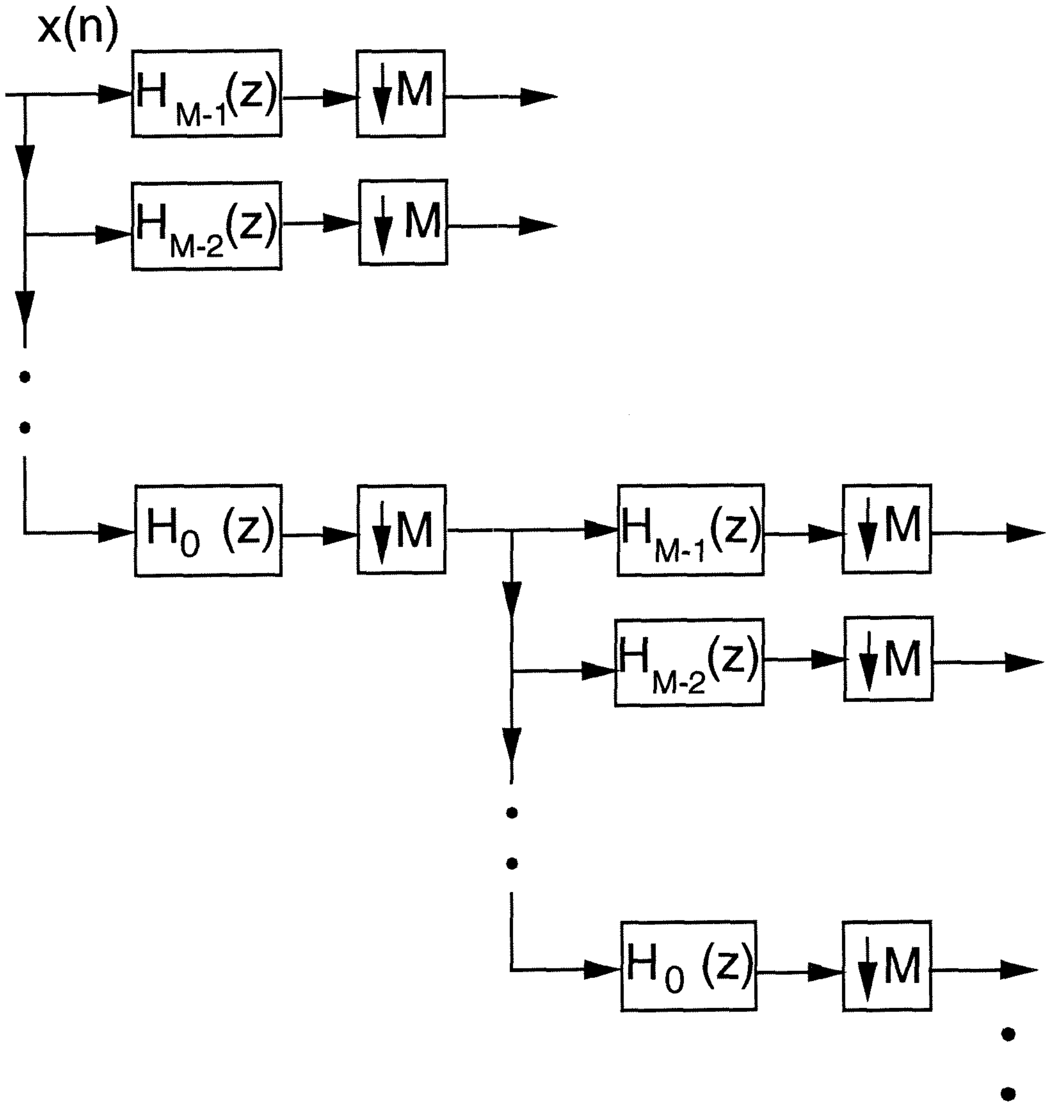


Fig. 5.7. A tree-structure using M-channel filter bank

$\mathbf{\Lambda}'(z) = diag[1\ 1\ \ldots\ z^{-N/2}\ \ldots\ 1\ 1]$ to get $\mathbf{F}'(z) = \mathbf{\Lambda}'(z)\mathbf{F}(z)$. The matrix $\mathbf{F}'(z)$ hence satisfies the equation

$$z^{-N}\mathbf{J}_M\mathbf{F}'(z^{-1})\mathbf{J}_M = \mathbf{F}'(z). \tag{5.6}$$

Now let $\mathbf{E}(z) = \mathbf{S}\mathbf{F}'(z)$ where $\mathbf{S}$ is an orthogonal matrix. Clearly, with this construction, the matrix $\mathbf{E}(z)$ is also paraunitary. For the matrix $\mathbf{E}(z)$ to satisfy the linear phase property (eq. (2.1)), it can be verified that it is both necessary and sufficient that $\mathbf{S}$ be of the form

$$\mathbf{S} = (1/\sqrt{2})\begin{pmatrix} \mathbf{U}_{(M+1)/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{(M-1)/2} \end{pmatrix}\begin{pmatrix} \mathbf{I}_{(M-1)/2} & \mathbf{0}_c & \mathbf{J}_{(M-1)/2} \\ \mathbf{0}_c^T & 1 & \mathbf{0}_c^T \\ \mathbf{J}_{(M-1)/2} & \mathbf{0}_c & -\mathbf{I}_{(M-1)/2} \end{pmatrix}, \tag{5.7}$$

where $\mathbf{U}_{(M+1)/2}$ and $\mathbf{W}_{(M-1)/2}$ are arbitrary orthogonal matrices of the sizes indicated.

The above discussions can be summarized in the following theorem:

Theorem 5: A linear phase paraunitary matrix with an odd number of channels can be realized as

$$\mathbf{E}(z) = \mathbf{S}\mathbf{\Lambda}'(z)\mathbf{P}\mathbf{T}_N\mathbf{\Lambda}(z)\mathbf{T}_{N-1}\mathbf{\Lambda}(z)\ldots\mathbf{T}_0\mathbf{P}, \tag{5.8}$$

where $\mathbf{P}$ is as in eq. (5.2), $\mathbf{T}_i$ is as in eq. (5.3) and $\mathbf{\Lambda}(z)$ is as in eq. (5.4). ◇◇◇

The fact that the structure is minimal can be verified as at the end of section III.

Matrix interleaving and linear phase filters: In this subsection, we will consider the problem of obtaining a larger linear phase paraunitary system given smaller linear phase paraunitary systems. Let M, the number of channels be odd. Let $L = (M-1)/2$. Let $\mathbf{G}(z)$ and $\mathbf{F}(z)$ be two linear phase paraunitary matrices of sizes $(L+1)\times(L+1)$ and $L \times L$ respectively, and of order N each. In particular, let us write them as

$$\mathbf{G}(z) = \begin{pmatrix} \mathbf{g}_0(z) & \mathbf{g}_1(z) & \ldots & \mathbf{g}_{L-1}(z) & \mathbf{g}_L(z) \\ g'_0(z) & g'_1(z) & \ldots & g'_{L-1}(z) & g'_L(z) \end{pmatrix}, \tag{5.9}$$

and

$$\mathbf{F}(z) = \begin{pmatrix} \mathbf{f}_0(z) & \mathbf{f}_1(z) & \ldots & \mathbf{f}_{L-2}(z) & \mathbf{f}_{L-1}(z) \end{pmatrix}. \tag{5.10}$$

In eq. (5.9) the vectors $\mathbf{g}_i(z)$ are of size L and represent the columns of the matrix $\mathbf{G}(z)$, *except for the last element in each column*, which has been written seperately as $g'_i(z)$. In eq. (5.10) the vectors $\mathbf{f}_i(z)$ are also of size L and are simply the columns of the matrix $\mathbf{F}(z)$. Hence note that vectors $\mathbf{g}_i(z)$ and $\mathbf{f}_i(z)$ are all of size L each. Now, construct the matrix $\mathbf{E}(z)$ of size $M \times M$, which is as follows:

$$\mathbf{E}(z) = \begin{pmatrix} \frac{\mathbf{g}_0(z)}{\sqrt{2}} & \frac{\mathbf{f}_0(z)}{\sqrt{2}} & \frac{\mathbf{g}_1(z)}{\sqrt{2}} & \frac{\mathbf{f}_1(z)}{\sqrt{2}} & \cdots & \frac{\mathbf{f}_{L-2}(z)}{\sqrt{2}} & \frac{\mathbf{g}_{L-2}(z)}{\sqrt{2}} & \frac{\mathbf{f}_{L-1}(z)}{\sqrt{2}} & \frac{\mathbf{g}_L(z)}{\sqrt{2}} \\ \frac{\mathbf{g}_0(z)}{\sqrt{2}} & -\frac{\mathbf{f}_0(z)}{\sqrt{2}} & \frac{\mathbf{g}_1(z)}{\sqrt{2}} & -\frac{\mathbf{f}_1(z)}{\sqrt{2}} & \cdots & -\frac{\mathbf{f}_{L-2}(z)}{\sqrt{2}} & \frac{\mathbf{g}_{L-2}(z)}{\sqrt{2}} & -\frac{\mathbf{f}_{L-1}(z)}{\sqrt{2}} & \frac{\mathbf{g}_L(z)}{\sqrt{2}} \\ g'_0(z) & 0 & g'_1(z) & 0 & \cdots & 0 & g'_{L-1}(z) & 0 & g'_L(z) \end{pmatrix}. \tag{5.11}$$

Note that the filters corresponding to this polyphase matrix are formed simply by interleaving in a particular manner the impulse response coefficients of the filters in the smaller systems $\mathbf{G}(z)$ and $\mathbf{F}(z)$.

Lemma 2: The matrix $\mathbf{E}(z)$ of size $M \times M$ in eq. (5.11) is a linear phase paraunitary matrix of order N.

Proof: The fact that $\mathbf{E}(z)$ is paraunitary is clear from the construction. It only remains to prove the linear phase property. Because the matrices $\mathbf{G}(z)$ and $\mathbf{F}(z)$ are linear phase, we have the following relations

$$\mathbf{g}_i(z) = \pm z^{-N} \mathbf{g}_{L-i}(z^{-1}), \tag{5.12}$$

$$\mathbf{f}_i(z) = \pm z^{-N} \mathbf{f}_{L-i-1}(z^{-1}), \tag{5.13}$$

and

$$g'_i(z) = \pm z^{-N} g'_{L-i}(z^{-1}). \tag{5.14}$$

Let $\mathbf{e}_i(z)$ denote the columns of the matrix $\mathbf{E}(z)$. Then, it can be seen from the construction of the matrix $\mathbf{E}(z)$ that the columns satisfy the condition

$$\mathbf{e}_i(z) = \pm z^{-N} \mathbf{e}_{M-1-i}(z^{-1}). \tag{5.15}$$

This is sufficient to prove that $\mathbf{E}(z)$ has linear phase filters. $\diamond\diamond\diamond$

Lemma 2 gives us a way to synthesize larger paraunitary systems from smaller ones. Thus, one can obtain an M channel linear phase paraunitary filter bank by using a schematic as shown in Fig. 5.6. Here, Lemma 2 is repeatedly used to synthesize the odd component on each level.

VI M-BAND ORTHONORMAL WAVELETS

The wavelet transform [32]-[34], [13] is a representation of a signal in terms of a set of basis functions which are obtained by dyadic dilations and shifts of a single function called the wavelet function. It provides a description of a signal on various levels of resolution or scale. The wavelet transform has of late found several applications in signal and image processing [34], [35]. One way of constructing the wavelet transform [32] is by using a two-channel quadrature-mirror filter bank in an infinite tree. This idea of wavelets (henceforth referred to as dyadic wavelets) has recently been extended to the more general case of M-band wavelets [19],[20],[36]. It has been shown therein that a square integrable function $f(t)$ can be represented in terms of the dilates and translates of $M-1$ functions $\psi_i(t)$, which are called the M-band wavelets. As in the case of dyadic wavelets, it has been shown [20] that M-band wavelets can be obtained by using an M-channel filter bank system in an infinite recursive tree-structure as shown in fig. 5.7. M-band wavelets often provide a more compact representation of signals, and are therefore useful in several applications [21].

It can be shown [20] that for the wavelet basis to be orthonormal, a necessary condition is that the M-channel filter bank used in fig. 5.7 should be paraunitary. The theory developed in the previous sections allows us to design symmetric and antisymmetric wavelets that are also orthonormal. This can be done simply by using the structure developed in section III to generate the M-channel system on each level of the tree. Consider the Fourier transform of an M-band wavelet function

$$\Psi_i(\omega) = (1/\sqrt{M})H_i(e^{j\omega/M}) \lim_{K\to\infty}\prod_{k=2}^{K}(1/\sqrt{M})H_0(e^{j\omega(M)^{-k}}). \tag{6.1}$$

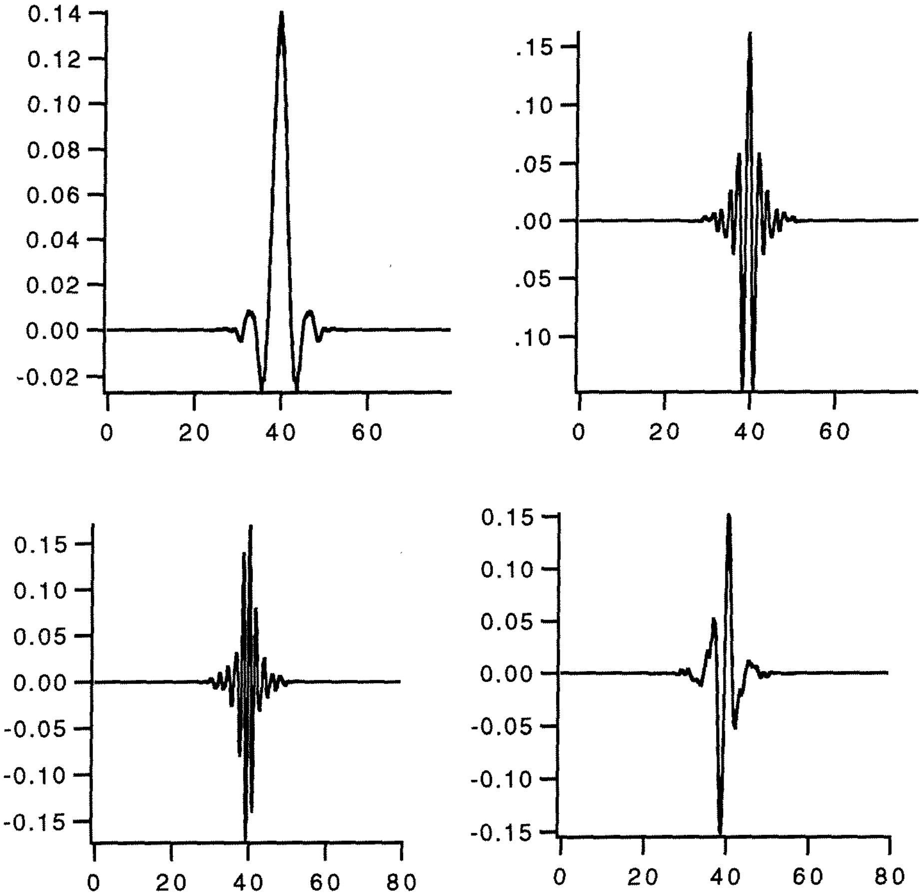


Fig. 5.8. 4-Band orthonormal linear phase wavelets

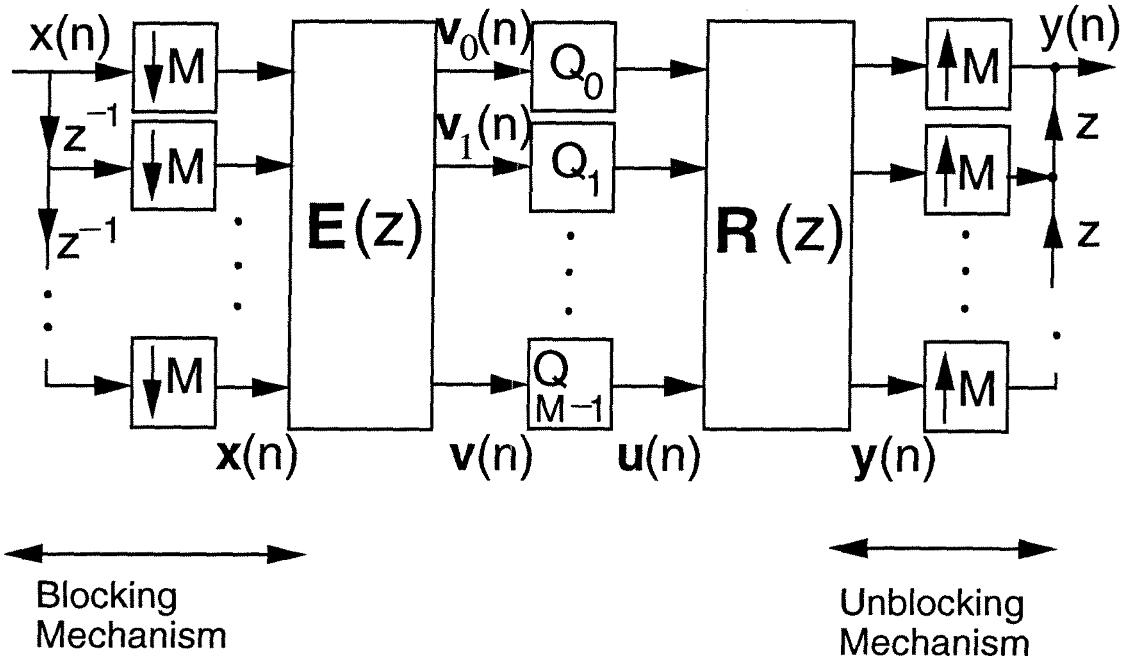


Fig. 5.9 Sub-band coding scheme showing polyphase matrices

For the linear phase paraunitary system developed in section III, the filters can be written as $H_i(e^{j\omega}) = e^{-j(N-1)\omega/2} H_{iR}(\omega)$, where $H_{iR}(\omega)$ is the real part of $H_i(e^{j\omega})$, and we have,

$$\Psi_i(\omega) = e^{-j(N-1)\omega(M^{-1}+M^{-2}+\ldots)/2}(1/\sqrt{M})H_{iR}(\omega)\lim_{K\to\infty}\prod_{k=2}^{K}(1/\sqrt{M})H_{0R}(\omega(M)^{-k}), \tag{6.2}$$

which becomes

$$\Psi_i(\omega) = e^{-j(N-1)\omega/2(M-1)}\lim_{K\to\infty}(1/\sqrt{M})H_{iR}(\omega)\prod_{k=2}^{K}(1/\sqrt{M})H_{0R}(\omega(M)^{-k}). \tag{6.3}$$

Hence assuming $\Psi_i(\omega)$ exists, it has linear phase.

Fig. 5.8 shows an example of 4-band orthonormal, linear phase wavelets and their associated scaling function. The length of the filters was 80, and they had one zero at $\omega = \pi$. As can be seen, all of the functions are symmetric or antisymmetric.

In many signal processing applications it is desirable that the wavelets be 'smooth' or regular. This can be done by ensuring that the wavelets have sufficient number of vanishing moments. It has been shown [20] that the M-band wavelets have N vanishing moments if and only if the function $H_i(e^{j\omega}),\ \ i \neq 0$ have zeros of order N at $\omega = 0$. In particular, the condition that there be one vanishing moment simply implies that $H_i(e^{j\omega}),\ \ i \neq 0$ have one zero at $\omega = 0$. In the case of linear phase paraunitary systems, we know from Theorem 1 that half the number of filters are antisymmetric and therefore guaranteed to have a zero at $\omega = 0$. The condition that the remaining filters (all except $H_0(e^{j\omega})$) have a zero at $\omega = 0$ can be written in terms of the lattice developed in section III, as was done for the general lattice in [20]. Now, the filters can be written in terms of the polyphase matrix $\mathbf{E}(z)$ as

$$\begin{pmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{pmatrix} = \mathbf{E}(z) \begin{pmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-M+1} \end{pmatrix}. \tag{6.4}$$

At $\omega = 0$, i.e., $z = 1$, we need

$$\mathbf{E}(1)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{M} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6.5}$$

Substituting the form of the linear phase paraunitary lattice from section III and noting that $\mathbf{\Lambda}(1) = \mathbf{I}$, we have

$$\mathbf{SPT}_N\mathbf{T}_{N-1}\ldots\mathbf{T}_0\mathbf{P}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{M} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6.6}$$

Now, with $\mathbf{T}_i$ having the form as in eq. (3.10), the product $\mathbf{T} = \prod_{i=0}^{n}\mathbf{T}_i$ also has the form

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C} & \mathbf{A} \end{pmatrix}. \tag{6.7}$$

Similarly, after substituting for the form of $\mathbf{S}$ from eq. (3.13), condition (6.6) simplifies to

$$(\sqrt{2})\begin{pmatrix} \mathbf{S}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 \end{pmatrix}\begin{pmatrix} \mathbf{A}+\mathbf{C} \\ \mathbf{0} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{M} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6.8}$$

As expected, this reduces to the set of $M/2$ conditions

$$\sqrt{2}\mathbf{S}_0(\mathbf{A}+\mathbf{C})\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{M} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{6.9}$$

where the column vectors are now of size $(M/2)$.

Recall that $\mathbf{S}_0$ can be choosen to be an arbitrary orthogonal matrix for the factorization in section III. If we further wish to impose regularity, then we can exploit this freedom in the choice of $\mathbf{S}_0$, and choose it so as to satisfy eqn. (6.9). Given any vector, there exists a Houesholder matrix $\mathbf{I} - 2\mathbf{u}\mathbf{u}^{-1}$ which turns the vector into $\begin{pmatrix} \sqrt{M/2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ [30], [pp.751, 10]. So there always exists an $\mathbf{S}_0$ satisfying eq. (6.9).

VII SUBBAND CODING USING LINEAR PHASE FILTER BANKS

Subband coding is a technique often used for encoding speech and image signals. In a typical subband coding scheme, the input signal is divided into different frequency regions using a bank of filters called the analysis filters. The signals in each subband are quantized and then transmitted. At the receiver, the signals in individual subbands are combined by the bank of synthesis filters. In Fig. 5.9, the scheme has been drawn in terms of the polyphase matrices of the analysis and synthesis filters. As described earlier in this paper, the perfect reconstruction property is lost in the presence of quantizers, and so is the linear phase property of the overall transfer function. In such cases, it is sometimes important to use filters that individually have linear phase. The requirements in many image processing applications are that the filters have linear phase, and short lengths, typically less than about 20.

The coding gain [18] is often used as a figure of merit to judge the performance of various subband coding schemes. It is defined as the ratio of the reconstruction error variance of a PCM system to the reconstruction error variance of the subband coding system, i.e.,

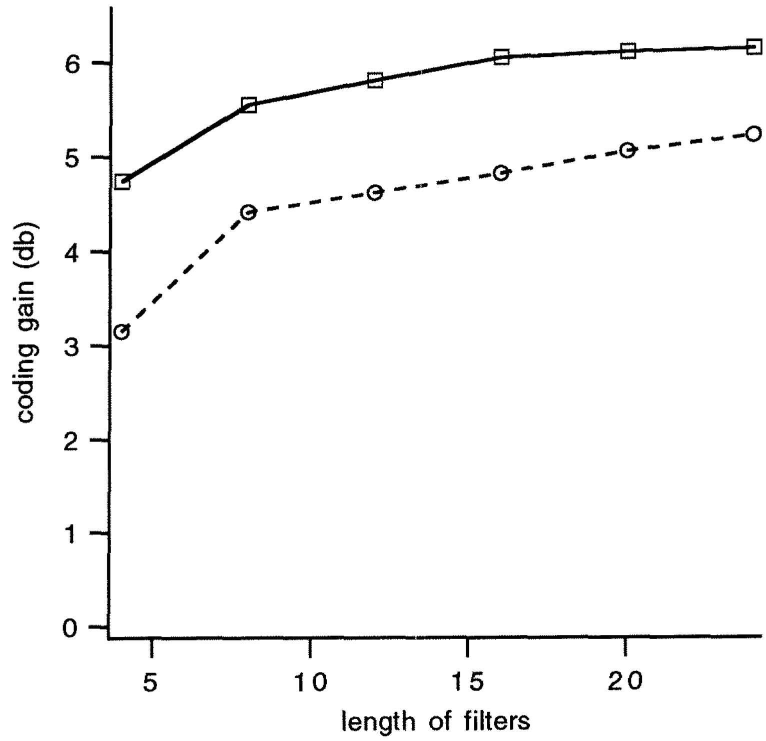
$$G_{PU} = \frac{\sigma^2_{r,PCM}}{\sigma^2_{r,PU}}. \tag{7.1}$$

Under optimal bit allocation it can be shown [17] that the coding gain of any paraunitary subband coding system becomes

$$G_{TC} = \frac{(1/M)(\sum_{i=0}^{M-1}\sigma^2_{s_k})}{(\prod_{i=0}^{M-1}\sigma^2_{s_k})^{1/M}}, \tag{7.2}$$

which is the ratio of the arithmetic mean to the geometric mean of $\sigma^2_{s_k}$, the variances of the inputs $s_k(n)$ to the quantizers.

It is of interest to compare the coding gains obtained using linear phase paraunitary systems designed in this paper with the optimum coding gain results presented in [17]. Fig. 5.10 shows this comparison for the case of a four-channel filter bank. The



____ : upper bound on coding gain of paraunitary sytems

........: coding gain of linear phase paraunitary systems
(note: not been optimized for coding gain)

Fig. 5.10. Coding gain comparisons for four-channel systems

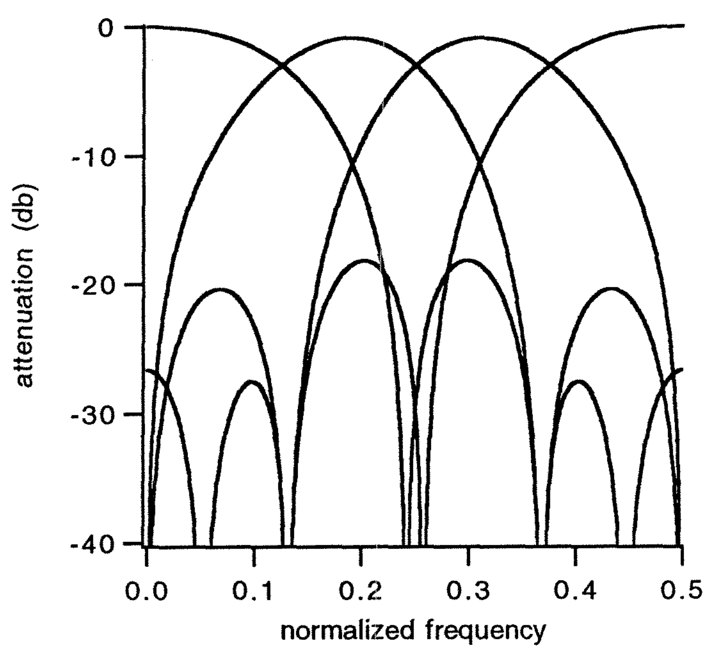


Fig. 5.11. Four-channel linear phase paraunitary system

coefficients of $H_0(z)$	coefficients of $H_1(z)$
$h_0(0)=h_0(7)=-0.091584806958951$	$h_1(0)=-h_1(7)=-0.13357390156568$
$h_0(1)=h_0(6)=0.13357390156568$	$h_1(1)=-h_1(6)=0.091584806958951$
$h_0(2)=h_0(5)=0.38923341521735$	$h_1(2)=-h_1(5)=0.56768614376856$
$h_0(3)=h_0(4)=0.567686143768 56$	$h_1(3)=-h_1(4)=0.38923341521735$

Table 5.2. Filter coefficients of a Four-channel linear phase paraunitary system with pairwise mirror-image symmetry in the frequency domain. Note that $H_3(z) = H_0(-z)$ and $H_2(z) = H_1(-z)$.

Fig. 5.12 a). "Lena" Image coded at 0.125 bpp with linear phase filters

Fig. 5.12 b). "Lena" Image coded at 0.125 bpp with nonlinear phase filters.

Fig. 5.13 a). "Peppers" Image coded at 0.125 bpp with linear phase filters

Fig. 5.13 b). "Peppers" Image coded at 0.125 bpp with nonlinear phase filters

input was low-pass speech modelled as AR process. The horizontal axis is the length of the filters, and the vertical axis is the corresponding coding gain. The length of the filters therefore is 4× (order of $\mathbf{E}(z)$ +1). The higher curve corresponds to the maximum coding gain possible (data taken from [17]), and the lower curve shows the coding gain obtained using linear phase paraunitary systems. *Note that the linear phase paraunitary systems were not designed to maximize the coding gain.* The lattice developed in section IV was used. Fig. 5.11 shows the typical frequency responses of a four-channel system where a two-stage lattice was used. This means that there are only two variables to optimize! The length of the filters is eight. The impulse response coefficients have been tabulated in Table 5.2. Notice the relation between the coefficients of the filters. Given the coefficients of one of the filters, it is possible to write the coefficients of all the filters.

The same filters were also used to encode images at low bit-rates. Figs. 5.12 (a) and 5.13 (a) show images encoded with non-linear phase filters. The artifacts are clearly visible in the form of a "brickwall" in the background. Figs. 5.12 (b) and 5.13 (b) show the same images encoded with linear phase filters. The artifacts are no longer visible.

VIII CONCLUSIONS

In this paper we studied in detail the theory, factorizations and designs of linear phase paraunitary systems. In section II we proved several results on linear phase paraunitary systems, which we used subsequently. Next we addressed the problem of designing linear phase paraunitary systems for an even number of channels M. We showed that such systems could be designed by a cascade structure which was proved to be minimal. The resulting filters are structurally linear phase paraunitary, i.e., these properties are preserved in spite of coefficient quantization. Moreover, we showed the completeness of this structure, i.e., all linear phase paraunitary systems satisfying eq. (2.1) can be generated simply by manipulating the coefficients of this cascade structure. Next we imposed the further condition on the filters that they satisfy the pairwise mirror-image property in the frequency domain. The resulting structure has much fewer multipliers, which is useful for optimization. To summarize, for these filter banks the following properties are guaranteed structurally, i.e., in spite of quantization of the multipliers (angles):

- The filter bank is paraunitary, and therefore gives perfect reconstruction.
- The analysis and synthesis filters are time-reversed versions of each other.
- The analysis and synthesis filters are all linear phase.
- The filters in the analysis and synthesis banks both satisfy the pairwise mirror-image property in the frequency domain.

Next, we extended this analysis to the case of filter banks with an odd number of channels M. In particular, we showed two ways by which such systems could be realized. One was based on a factorization approach, and the other involved designing larger systems by successively combining smaller systems in a certain manner.

It is interesting to note that the linear phase property along with the paraunitary condition implies that the analysis and synthesis banks are identical, up to a multiplier of ± 1 on some of the filters, i.e., $F_i(z) = \pm H_i(z)$.

We then considered two applications of the theory. The first was in designing symmetric and antisymmetric M-band wavelets which are also orthonormal. We also discussed the regularity condition in this context and derived conditions on the factorization proposed so that the resulting wavelets had at least one vanishing moment. The second application we considered was in subband coding. From the data presented for lowpass speech, we conclude that the linear phase paraunitary systems with filters of small length give good coding gains. With the other special features of the filters mentioned before, we conclude that this structure is a good candidate for use in practical subband coding systems.

APPENDIX A

Consider the matrix $\mathbf{J}_M$ where M is even. The eigenvalues of this matrix are ± 1, and the corresponding eigenvectors are the symmetric and antisymmetric vectors of size M. We will refer to the two eigenspaces of the matrix $\mathbf{J}_M$ as the symmetric and antisymmetric eigenspaces $\mathcal{E}_s$ and $\mathcal{E}_a$ respectively. The basis for $\mathcal{E}_s$ could be the set of vectors $\mathbf{s}_i, \quad i = 0, \ldots (M/2) - 1$, where all elements of the vectors $\mathbf{s}_i$ are zero, except $\mathbf{s}_i(i) = \mathbf{s}_i(M - 1 - i) = 1$. Similarly, a basis for $\mathcal{E}_a$ could be the set of vectors $\mathbf{a}_i, \quad i = 0, \ldots (M/2) - 1$, where all elements of the vectors $\mathbf{a}_i$ are zero, except $\mathbf{a}_i(i) = -\mathbf{a}_i(M - 1 - i) = 1$. Also, since the matrix $\mathbf{J}_M$ is symmetric, the eigenvectors span the whole space, and $\mathcal{E}_s$ and $\mathcal{E}_a$ form a direct sum for the whole space. Now, consider any vector $\mathbf{y}$. It can always be written as $\mathbf{y} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in \mathcal{E}_s$ and $\mathbf{v} \in \mathcal{E}_a$. Let $\mathbf{y}$ be orthogonal to its own flipped version, i.e.,

$$\mathbf{y}^T \mathbf{J}_M \mathbf{y} = 0. \qquad (A.1)$$

Hence we get

$$(\mathbf{u} + \mathbf{v})^T \mathbf{J}_M (\mathbf{u} + \mathbf{v}) = 0. \qquad (A.2)$$

Noting that $\mathbf{u}^T \mathbf{J}_M \mathbf{u} = \mathbf{u}^T \mathbf{u}$, $\mathbf{v}^T \mathbf{J}_M \mathbf{v} = -\mathbf{v}^T \mathbf{v}$ and $\mathbf{u}^T \mathbf{J}_M \mathbf{v} = 0$, the above equation reduces to $\mathbf{u}^T \mathbf{u} = \mathbf{v}^T \mathbf{v}$. Hence the norm of the projections in the two eigenspaces has to be equal. We say, therefore, that the vector $\mathbf{y}$ which satisfies eq. (A.1) is 'balanced' over the two eigenspaces (or simply 'balanced').

As noted above, the eigenvectors of the matrix $\mathbf{J}$ are symmetric and antisymmetric vectors (have linear phase). Furthermore, these eigenvectors are orthonormal. Hence, one would expect the eigenstructure of the $\mathbf{J}$ matrix to play a role in the synthesis of linear phase orthonormal systems.

APPENDIX B

From eq. (3.28) we have $\mathbf{W}^T\mathbf{f}_{m+1}^T(0)\mathbf{J}_M\mathbf{f}_{m+1}(0)\mathbf{W} = \mathbf{0}$, for any matrix $\mathbf{W}$. This means that the columns of $\mathbf{f}_{m+1}(0)\mathbf{W}$ are balanced. Let the matrix $\mathbf{W}$ be so chosen that the first r columns of the matrix $\mathbf{f}_{m+1}(0)\mathbf{W}$ form an orthonormal basis for the columns of matrix $\mathbf{f}_{m+1}(0)$. Denote these r vectors as $\mathbf{x}_i, \quad i = 1, \ldots, r$. Hence, the vectors $\mathbf{x}_i$ are balanced and orthonormal, i.e., $\mathbf{x}_i^T\mathbf{x}_j = 0$. Let $\mathbf{x}_i = \mathbf{u}'_i + \mathbf{v}'_i$, where $\mathbf{u}'_i \in \mathcal{E}_s$ and $\mathbf{v}'_i \in \mathcal{E}_a$. Therefore, $(\mathbf{u}'_i + \mathbf{v}'_i)^T(\mathbf{u}'_j + \mathbf{v}'_j) = 0$, which simplifies to

$$\mathbf{u}'^T_i\mathbf{u}'_j + \mathbf{v}'^T_i\mathbf{v}'_j = 0. \qquad (B.1)$$

Since the vectors $\mathbf{x}_i$ are balanced, i.e., $\mathbf{x}_i^T\mathbf{J}_M\mathbf{x}_j = 0$, we have $(\mathbf{u}'_i+\mathbf{v}'_i)^T\mathbf{J}_M(\mathbf{u}'_j+\mathbf{v}'_j) = 0$, simplifying which we get

$$\mathbf{u}'^T_i\mathbf{u}'_j = \mathbf{v}'^T_i\mathbf{v}'_j. \qquad (B.2)$$

Eqs. (B.1) and (B.2) together imply that $\mathbf{u}'^T_i\mathbf{u}'_j = 0$, and $\mathbf{v}'^T_i\mathbf{v}'_j = 0$. The vectors $\mathbf{u}'_i, \quad i = 1, \ldots, r$ and $\mathbf{v}'_i, \quad i = 1, \ldots, r$ therefore form orthonormal bases for r-dimensional subspaces of $\mathcal{E}_s$ and $\mathcal{E}_a$ respectively. In $\mathcal{E}_s$, there exist $p = M/2 - r$ orthogonal vectors $\mathbf{u}'_i, \quad i = r+1, \ldots, M/2$ which are also orthogonal to the previously mentioned set of r vectors $\mathbf{u}'_i, \quad i = 1, \ldots, r$. Similarly, in $\mathcal{E}_a$, there exist $p = M/2 - r$ orthogonal vectors $\mathbf{v}'_i, \quad i = r+1, \ldots, M/2$ which are also orthogonal to the previously mentioned set of r vectors $\mathbf{v}'_i, \quad i = 1, \ldots, r$. Now using these additional p orthonormal vectors from $\mathcal{E}_s$ and $\mathcal{E}_a$, we can form p orthonormal, balanced vectors. With this construction, it can be verified that the set of $M/2$ vectors $\mathbf{x}_i = \mathbf{u}'_i + \mathbf{v}'_i, \quad i = 1, \ldots, M/2$ satisfies the following properties:

1) They are orthonormal and balanced.

2) They are also orthonormal to the flipped versions of each other. Hence, if $\mathbf{X}^T$ is the matrix of size $M/2 \times M$ which has these vectors as its rows, this matrix satisfies the property $\mathbf{X}^T\mathbf{J}_M\mathbf{f}_{m+1}(0) = \mathbf{0}$.

REFERENCES

[1] R. Crochiere and L. Rabiner, *Multirate Digital Signal Processing*, Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1983.

[2] J. W. Woods and S. D. O'Neil, "Sub-band coding images," IEEE Trans. on ASSP, Vol. 34, pp. 1278-1288, Oct. 1986.

[3] P. P. Vaidyanathan, "Multirate Digital Filters, Filter Banks, Polyphase Networks, and Applications: A Tutorial," Proc. of IEEE, Vol. 78, No.1, Jan. 1990.

[4] W. A. Pearlman, "Performance Bounds for Subband Coding," Ch. 1 of *Subband Image Coding*, edited by J. Woods, Kluwer, 1991.

[5] M. J. T. Smith and T. P. Barnwell, III, "A Procedure for Designing Exact Reconstruction Filter Banks for Tree-Structured Sub-band Coders," Proc. of IEEE Int. Conf. on ASSP, San Diego, March 1984.

[6] F. Mintzer, "Filters for Distortion-free Two-band Multirate Filter Banks," IEEE Trans. on ASSP, pp. 626-630, June 1985.

[7] M. Vetterli, "Filter Banks Allowing for Perfect Reconstruction," Signal Processing, Vol. 10, pp. 219-244, Apr. 1986.

[8] P. P. Vaidyanathan, "Theory and Design of M-channel Maximally Decimated Filters with Arbitrary M, having Perfect Reconstruction Property," IEEE Trans. on ASSP, pp. 476-492, Apr. 1987.

[9] M. Vetterli, "A Theory of Multirate Filter Banks," IEEE Trans. on ASSP, pp. 356-372, Mar. 1987.

[10] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1993.

[11] H. S. Malvar, "Lapped Transforms for Efficient Transform/Subband coding," IEEE Trans. on ASSP, Vol. 38, No. 6, June 1990.

[12] K. Nayebi, T. Barnwell and M. Smith, "A General Time Domain Analysis and Design Framework for Exact Reconstruction FIR Analysis/Snthesis Filter Banks," Proc. ISCAS, pp. 2022-2025, New Orleans, May 1990.

[13] O. Rioul and M. Vetterli, "Wavelets and Signal Processing," IEEE Signal Processing Magazine, pp. 14-38, Oct. 1991.

[14] V. Belevitch, *Classical Network Theory*, Holden Day, San Fransisco, 1968.

[15] A. Soman and P. P. Vaidyanathan, "Paraunitary Filter Banks and Wavelet Packets," Proc. of ICASSP, pp. 397-400, San Fransisco, 1992.

[16] P. P. Vaidyanathan and Z. Doğanata, "The role of lossless systems in modern digital signal processing: A tutorial," Special Issue on Circuits and Systems, IEEE Trans. on Education, Aug. 1989.

[17] A. Soman and P. P. Vaidyanathan, "Coding Gain in Paraunitary Analysis Synthesis Systems," to appear, IEEE Trans. on Sig. Proc., May 1993.

[18] N. S. Jayant and P. Noll, *Digital Coding of Waveforms- Principles and Applications to Speech and Video,* Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1984.

[19] I. Daubechies, *Ten Lectures on Wavelets*, SIAM CBMS series, Apr. 1992.

[20] H. Zou and A. H. Tewfik, "Discrete Orthogonal M-Band Wavelet Decompositions," Proc. ICASSP, San Fransisco, Mar. 1992.

[21] A. Tewfik and M. Kim, "Fast Multiscale Signal Processing Algorithms," Proc. of Int. Conf. on ASSP, pp. 373-376, San Fransisco, Mar. 1992.

[22] M. Vetterli and D. Le Gall, "Analysis and Design of Perfect Reconstruction Filter Banks Satisfying Symmetry Constraints," Proc. 22nd Conf. Info. Sci. Syst., Princeton, NJ, Mar. 1988.

[23] T. Nguyen and P. P. Vaidyanathan, "Two Channel Perfect Reconstruction FIR QMF Structures Which Yield Linear-Phase Analysis Filters," IEEE Trans. on ASSP, pp. 676-690, May, 1989.

[24] J. Princen and A. Bradley, "Analysis/Synthesis Filter Bank Design Based on Time-Domain Aliasing Cancellation," IEEE Trans. on ASSP, pp. 1153-1161, Oct. 1986.

[25] H. S. Malvar and D. Staelin, "The LOT: Transform Coding Without Blocking Effects," IEEE Trans. on ASSP, pp. 553-559, Apr. 1989.

[26] M. Vetterli and D. Le Gall, "Perfect Reconstruction Filter Banks: Some Properties and Factorizations," IEEE Trans. on ASSP, pp. 1057-1071, July 1989.

[27] M. Bellanger, G. Bonnerot, M. Coudreuse, "Digital Filtering by Polyphase Network: Application to Sample Rate Alteration and Filter Banks," IEEE Trans. on ASSP, vol. 24, pp. 109-114, Apr. 1976.

[28] P. Cassereau, "A New Class of Orthogonal Transforms for Image Processing," S.M. Thesis, Dept. EECS, Mass. Inst. Tech., Cambridge, May 1985.

[29] F. Murnaghan, *The Unitary and Rotation Groups*, Spartan Books, Washington D.C., 1962.

[30] R. Horn and C. Johnson *Matrix Analysis*, Cambridge Univ. Press, 1985.

[31] T. Nguyen and P. P. Vaidyanathan, "Maximally Decimated Perfect-Reconstruction Filter Banks with Pairwise Mirror-Image Analysis (and Synthesis) Frequency Responses," IEEE Trans. on ASSP, pp. 693-706, May 1988.

[32] I. Daubechies, "Orthonormal Bases of Compactly Supported Wavelets," Comm. on Pure and Appl. Math. Vol. 4, pp. 909-996, Nov. 1988.

[33] R. Coifman, Y. Meyer, S. Quake and V. Wickerhauser, "Signal Processing with Wavelet Packets," Numerical Algorithms Research Group, Yale University, 1990.

[34] S. Mallat, "Multifrequency Channel Decomposition of Images and Wavelet Models," IEEE Trans. on ASSP, Vol. 37, pp. 2091-2110, 1989.

[35] M. Antonin, M. Barlaud, P. Mathieu and I. Daubechies, "Image Coding using Vector Quantization in Wavelet Transform Domain," Proc. of Intern. Conf. on ASSP, pp. 2297-2300, Albuquerque, NM, 1990.

[36] R. Gopinath and C. Burrus, "Wavelet Transforms and Filter Banks," in *Wavelets and Applications* edited by C. H. Chui, Academic Press, 1991.

Chapter 6

A COMPLETE FACTORIZATION OF PARAUNITARY MATRICES WITH PAIRWISE MIRROR-IMAGE SYMMETRY IN THE FREQUENCY DOMAIN

Abstract [1]

The problem of designing orthonormal (paraunitary) filter banks has been addressed in the past. Several structures have been reported for implementing such systems. One of the structures reported [6] imposes a pairwise mirror-image symmetry constraint on the frequency responses of the analysis (and synthesis) filters around $\pi/2$. This structure requires fewer multipliers, and the design time is correspondingly less than most other structures. The filters designed also have much better attenuation.

In this paper, we characterize the polyphase matrix of the above filters in terms of a matrix equation. We then prove that the structure reported in [6], with minor modifications, is complete. This means that every polyphase matrix whose filters satisfy the mirror-image property can be factorized in terms of the proposed structure.

I INTRODUCTION

Digital filter banks have been used in the past to decompose a signal into frequency subbands [1]. The theory of perfect reconstruction filter banks has also been studied

[1]Under review, IEEE Trans. on Signal Processing.

extensively [2]-[5]. Fig. 6.1 shows an M-channel maximally decimated filter bank. In this scheme, the $H_i(z)$ are the analysis filters and $F_i(z)$ are the synthesis filters. Fig. 6.2 is a representation of the scheme in terms of the polyphase matrices [3],[4]. $\mathbf{E}(z)$ is the polyphase matrix corresponding to the analysis filters, and $\mathbf{R}(z)$ is the polyphase matrix corresponding to the synthesis filters. The decimators and expanders have been moved across the polyphase matrices using the noble identities [5]. In this system, one can achieve perfect reconstruction by letting $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$, and then choosing the matrix $\mathbf{E}(z)$ so that $\mathbf{R}(z)$ exists. Such a system is called a biorthonormal system. Another approach to design a perfect reconstruction system is to choose the matrix $\mathbf{E}(z)$ to be a 'paraunitary' matrix. A matrix is said to be paraunitary [3] if it satisfies the equation

$$\widetilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I}, \tag{1.1}$$

where $\widetilde{\mathbf{E}}(z) = \mathbf{E}^{\dagger}(1/z^*)$. The dagger superscript '$\dagger$' denotes conjugation followed by transposition, whereas the asterisk '$*$' denotes conjugation only. The system can be guaranteed to have the perfect reconstruction property by having $\mathbf{R}(z) = \widetilde{\mathbf{E}}(z)$. An important feature of the paraunitary system is the orthonormality property [5]. Another feature of the system is that the analysis and synthesis filters are simply time-reversed conjugate versions of each other for perfect reconstruction, and in particular, therefore, they are of the same length. Moreover, in the presence of subband quantization in such systems, one can obtain bounds on the final reconstruction error in terms of the errors introduced in each subband [10].

Several structures have been developed for implementing paraunitary systems [5]. These structures are robust to quantization, i.e., the perfect reconstruction property is retained in spite of coefficient quantization.

In [6], the authors have imposed the *additional* condition that the analysis (and synthesis) filters satisfy the pairwise mirror-image symmetry constraint in the frequency domain around $\pi/2$. The advantage of the resulting structure is that it

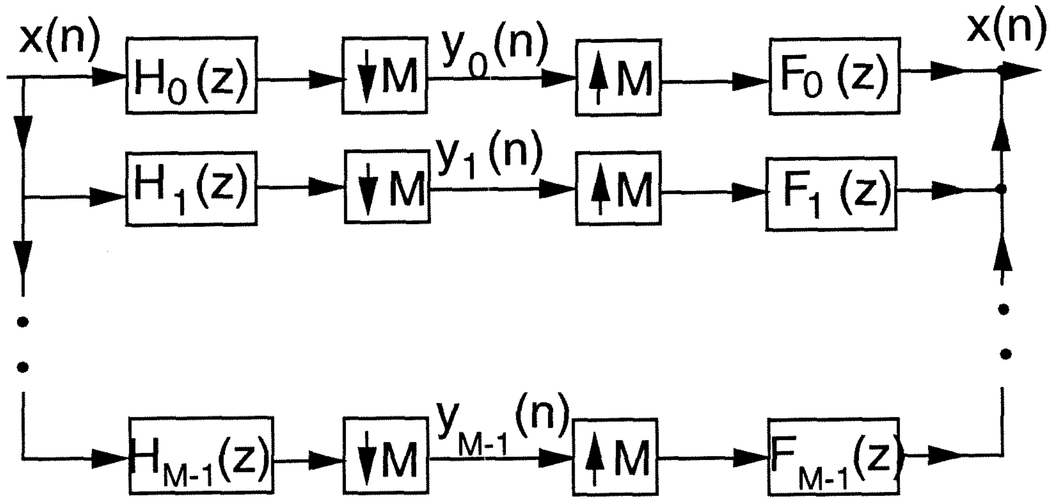


Fig. 6.1. An M-channel uniform filter bank

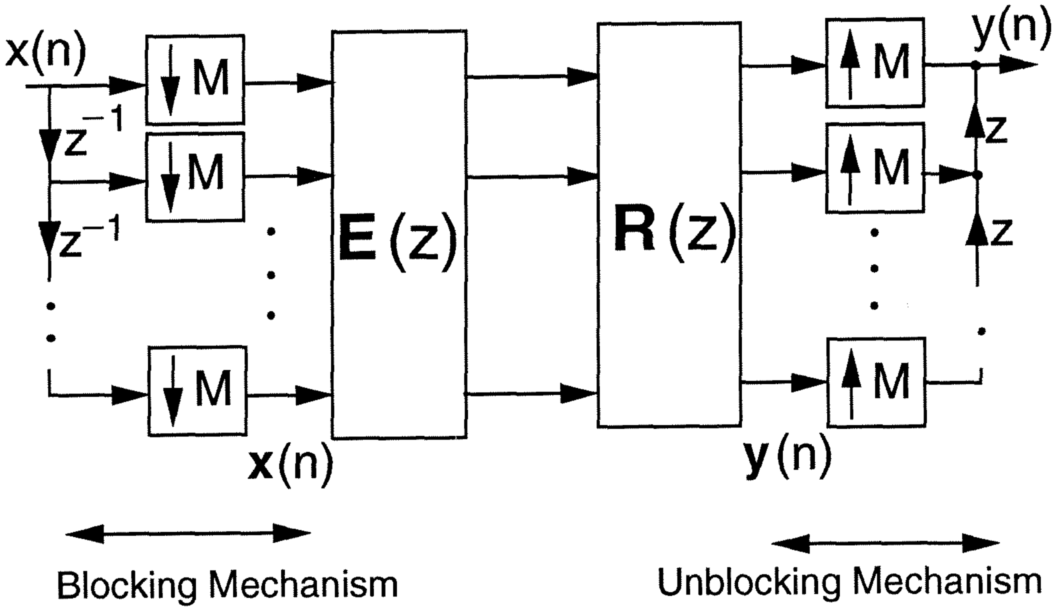


Fig. 6.2 A filter bank drawn in terms of polyphase matrices

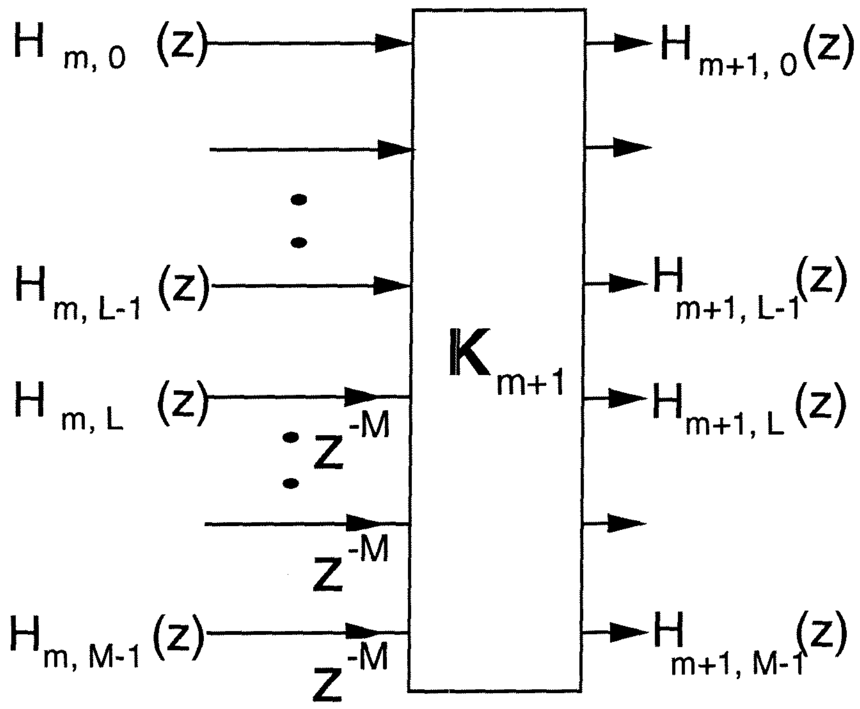


Fig. 6.3 . One stage of the filter bank developed in section II

requires fewer parameters [1]. The design time is correspondingly lower than other structures. The filter responses obtained using this structure are also better and have been widely used.

A natural question which arises is whether the structure suggested in [6] is *complete* as the structures for general paraunitary matrices have been shown to be [5]. Completeness of the structure would imply that every polyphase matrix whose filters satisfy the pairwise mirror-image symmtery property can be factorized in terms of this structure. This question has not been addressed in [6]. The purpose of this work is to prove that the factorization suggested in [6] is indeed complete. We will also show how each of the building blocks can be parametrized by a minimum number of free variables.

Another point worth mentioning is the *minimality* of the structure. A structure is said to be minimal if it uses the minimum number of delay elements necessary [3]. The minimality of our structure can be easily verified using Theorem 14.7.2 in [3].

Most of our notation will be identical to that used in [5]. The number of channels M is even. The other notation will be defined as and when required.

II FACTORIZATION OF PARAUNITARY MATRICES HAVING PAIRWISE MIRROR-IMAGE SYMMETRY

In this section we first obtain a factorization of paraunitary matrices whose filters satisfy the pairwise mirror-image symmetry. The filters have real coefficients, and hence the polyphase matrices are also real. The approach we take is based on directly manipulating the polyphase matrices, and yields a more compact derivation than [6]. Next, we prove the completeness of this structure, which is the main result of this paper.

Consider Fig. 6.3. The filters at the mth stage are denoted as $H_{m,i}(z)$. If these

[1]Subsequent to [6], the cosine modulated filter bank was reported [7]-[9], which requires even fewer parameters.

filters satisfy pairwise mirror-image symmetry, they can be related to each other as (eq. 33, [6])

$$H_{m,M-1-k}(z) = z^{-((m+1)M-1)} H_{m,k}(-z^{-1}), \qquad k = 0, \ldots, \; M-1. \tag{2.1}$$

The order of each filter is $(m+1)M-1$. It can be verified that in this case, the polyphase matrix $\mathbf{E}_m(z)$ of the filters satisfies the matrix equation

$$z^{-m}\mathbf{Q}\mathbf{J}_M\mathbf{V}_M\mathbf{E}_m(z^{-1})\mathbf{V}_M\mathbf{J}_M = \mathbf{E}_m(z). \tag{2.2}$$

In the above equation, the matrix $\mathbf{J}_M$ is the anti-diagonal matrix of size $M \times M$. For example,

$$\mathbf{J}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix $\mathbf{V}_M$ is a diagonal matrix of size $M \times M$ with alternating $\pm 1's$ on the diagonal starting with $+1$. Hence if $M/2$ is even, we can write $\mathbf{V}_M = \begin{pmatrix} \mathbf{V}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{M/2} \end{pmatrix}$, whereas if $M/2$ is odd, $\mathbf{V}_M = \begin{pmatrix} \mathbf{V}_{M/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{V}_{M/2} \end{pmatrix}$. The matrix $\mathbf{Q}$ is by definition, $\mathbf{Q} = \begin{pmatrix} \mathbf{V}_{M/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{V}_{M/2} \end{pmatrix}$ for even $M/2$, whereas $\mathbf{Q} = \begin{pmatrix} \mathbf{V}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{M/2} \end{pmatrix}$ for odd $M/2$. In either case, eq. (2.2) may be simplified to

$$z^{-m}\mathbf{W}\mathbf{E}_m(z^{-1})\mathbf{V}_M\mathbf{J}_M = \mathbf{E}_m(z), \tag{2.3}$$

where

$$\mathbf{W} = \mathbf{Q}\mathbf{J}_M\mathbf{V}_M = \begin{pmatrix} \mathbf{0} & -\mathbf{J}_{M/2} \\ \mathbf{J}_{M/2} & \mathbf{0} \end{pmatrix}. \tag{2.4}$$

Suppose we add another stage to the cascade as shown in Fig. 6.3. Then, the polyphase matrix corresponding to the filters on the next stage is given by

$$\mathbf{E}_{m+1}(z) = \mathbf{K}_{m+1}\mathbf{\Lambda}(z)\mathbf{E}_m(z), \tag{2.5}$$

where

$$\mathbf{\Lambda}(z) = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_{M/2} \end{pmatrix}. \tag{2.6}$$

$\mathbf{I}_N$ is the identity matrix of size $N \times N$ in the above. If $\mathbf{K}_{m+1}$ is an orthogonal matrix, we have

$$\mathbf{E}_m(z) = \mathbf{\Lambda}(z^{-1})\mathbf{K}_{m+1}^T \mathbf{E}_{m+1}(z). \tag{2.7}$$

For the filters at the next stage to retain the pairwise mirror-image property, we need

$$z^{-(m+1)}\mathbf{W}\mathbf{E}_{m+1}(z^{-1})\mathbf{V}_M\mathbf{J}_M = \mathbf{E}_{m+1}(z). \tag{2.8}$$

Using eq. (2.7) in eq. (2.3), we obtain the following equation

$$z^{-m}\mathbf{W}\mathbf{\Lambda}(z)\mathbf{K}_{m+1}^T\mathbf{E}_{m+1}(z^{-1})\mathbf{V}_M\mathbf{J}_M = \mathbf{\Lambda}(z^{-1})\mathbf{K}_{m+1}^T\mathbf{E}_{m+1}(z).$$

Using the identity $\mathbf{\Lambda}(z)\mathbf{W}\mathbf{\Lambda}(z) = z^{-1}\mathbf{W}$, we see that the necessary and sufficient condition on $\mathbf{K}_{m+1}$ for eq. (2.8) to hold is $\mathbf{K}_{m+1}\mathbf{W}\mathbf{K}_{m+1}^T = \mathbf{W}$. By partitioning $\mathbf{K}_{m+1}$ as $\begin{pmatrix} \mathbf{A}' & \mathbf{C}' \\ \mathbf{B}' & \mathbf{D}' \end{pmatrix}$, we can verify that the necessary and sufficient condition for eq. (2.8) to hold is that the matrix $\mathbf{K}_{m+1}$ be of the form

$$\mathbf{K}_{m+1} = \begin{pmatrix} \mathbf{A}' & \mathbf{C}' \\ -\mathbf{J}_{M/2}\mathbf{C}'\mathbf{J}_{M/2} & \mathbf{J}_{M/2}\mathbf{A}'\mathbf{J}_{M/2} \end{pmatrix}. \tag{2.9}$$

Thus $\mathbf{K}_{m+1}$ can be rewritten as

$$\mathbf{K}_{m+1} = \underbrace{\begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{M/2} \end{pmatrix}}_{\mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{A}_{m+1} & \mathbf{C}_{m+1} \\ -\mathbf{C}_{m+1} & \mathbf{A}_{m+1} \end{pmatrix}}_{\mathbf{T}_{m+1}} \underbrace{\begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{M/2} \end{pmatrix}}_{\mathbf{P}} \tag{2.10}$$

where $\mathbf{A}_{m+1} = \mathbf{A}'$, and $\mathbf{C}_{m+1} = \mathbf{C}'\mathbf{J}_{M/2}$. The result of [6] is equivalent to this result. The conditions on the matrix $\mathbf{K}_0$ which initializes the process can be worked out similarly, as has been done in [6].

We now address the converse.

Theorem: Let $\mathbf{E}_{m+1}(z)$ be a FIR paraunitary matrix whose filters have pairwise mirror-image symmetry in the frequency domain (i.e., $\mathbf{E}_{m+1}(z)$ satisfies eq. (2.8)). Then it can always be factored as

$$\mathbf{E}_{m+1}(z) = \mathbf{K}_{m+1}\mathbf{\Lambda}(z)\mathbf{K}_m\mathbf{\Lambda}(z)\ldots\mathbf{\Lambda}(z)\mathbf{K}_0, \tag{2.11}$$

where $\mathbf{\Lambda}(z)$ is as in eq. (2.6), and $\mathbf{K}_i$ are as in eq. (2.10).

Proof: We prove the theorem by performing the 'order-reduction' process as outlined below. Let

$$\mathbf{E}_{m+1}(z) = \mathbf{e}_{m+1}(0) + \mathbf{e}_{m+1}(1)z^{-1} + \mathbf{e}_{m+1}(2)z^{-2} + \ldots + \mathbf{e}_{m+1}(m+1)z^{-(m+1)} \quad (2.12)$$

with $\mathbf{e}_{m+1}(m+1) \neq \mathbf{0}$ and

$$\mathbf{E}_m(z) = \mathbf{e}_m(0) + \mathbf{e}_m(1)z^{-1} + \mathbf{e}_m(2)z^{-2} + \ldots + \mathbf{e}_m(m)z^{-m}, \qquad \mathbf{e}_m(m) \neq \mathbf{0}. \quad (2.13)$$

Let $\mathbf{E}_{m+1}(z)$ satisfy eq. (2.8). Specifically, we will now show that it can always be written as

$$\mathbf{E}_{m+1}(z) = \underbrace{\mathbf{P}\mathbf{T}_{m+1}\mathbf{P}}_{\mathbf{K}_{m+1}}\,\mathbf{\Lambda}(z)\mathbf{E}_m(z), \quad (2.14)$$

where $\mathbf{E}_m(z)$ satisfies eq. (2.3), and the matrices $\mathbf{P}, \mathbf{T}_{m+1}$, and $\mathbf{\Lambda}(z)$ have the form described in eq. (2.10).

Paraunitariness of $\mathbf{E}_m(z)$ follows by noting that

$$\mathbf{E}_m(z) = \mathbf{\Lambda}(z^{-1})\mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{E}_{m+1}(z), \quad (2.15)$$

where all matrices on the right hand side of this equation are paraunitary.

Pairwise Mirror-image property: We want to show that $\mathbf{E}_m(z)$ satisfies eq. (2.3). Substituting eq. (2.14) into eq. (2.8), we get

$$z^{-(m+1)}\mathbf{W}\mathbf{P}\mathbf{T}_{m+1}\mathbf{P}\mathbf{\Lambda}(z^{-1})\mathbf{E}_m(z^{-1})\mathbf{V}_M\mathbf{J}_M = \mathbf{P}\mathbf{T}_{m+1}\mathbf{P}\mathbf{\Lambda}(z)\mathbf{E}_m(z). \quad (2.16)$$

Since $\mathbf{P}^{-1} = \mathbf{P}$ and $\mathbf{E}_m(z)$ is paraunitary, and noting that $\mathbf{W}^{-1} = -\mathbf{W}$, we get

$$z^{-(m+1)}\mathbf{\Lambda}(z^{-1})\mathbf{E}_m(z^{-1})\mathbf{V}_M\mathbf{J}_M\widetilde{\mathbf{E}}_m(z) = -\mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{W}\mathbf{P}\mathbf{T}_{m+1}\mathbf{P}\mathbf{\Lambda}(z). \quad (2.17)$$

If $\mathbf{T}_{m+1}$ is an orthogonal matrix of the form described in eq. (2.10), and $\mathbf{P}$ has the form described in eq. (2.10), then it can be verified that $\mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{W}\mathbf{P}\mathbf{T}_{m+1}\mathbf{P} = \mathbf{W}$. Hence we get

$$z^{-(m+1)}\mathbf{\Lambda}(z^{-1})\mathbf{E}_m(z^{-1})\mathbf{V}_M\mathbf{J}_M\widetilde{\mathbf{E}}_m(z) = -\mathbf{W}\mathbf{\Lambda}(z). \quad (2.18)$$

It follows that

$$z^{-m}[z^{-1}\mathbf{\Lambda}(z^{-1})\mathbf{W}\mathbf{\Lambda}(z^{-1})]\mathbf{E}_m(z^{-1})\mathbf{V}_M\mathbf{J}_M\widetilde{\mathbf{E}}_m(z) = \mathbf{I}_M. \tag{2.19}$$

It can be verified that $[z^{-1}\mathbf{\Lambda}(z^{-1})\mathbf{W}\mathbf{\Lambda}(z^{-1})] = \mathbf{W}$. Sustituting this into eq. (2.19), and rearranging the terms, we get eq. (2.3).

Causality: It only remains to show that there exists a matrix $\mathbf{T}_{m+1}$ such that $\mathbf{E}_m(z)$ obtained from eq. (2.15) is causal. Both the pairwise mirror image property and the paraunitary property continue to hold for the reduced system as long as the matrix $\mathbf{T}_{m+1}$ is *any* orthogonal matrix of the required form (eq. (2.10)). Indeed, it is the causality condition on the reduced system which determines the particular choice of the matrix $\mathbf{T}_{m+1}$.

From eq. (2.15) we get

$$\mathbf{E}_m(z) = \begin{pmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{E}_{m+1}(z) + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & z\mathbf{I}_{M/2} \end{pmatrix} \mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{E}_{m+1}(z). \tag{2.20}$$

The second term on the right hand side of this equation is responsible for the non-causality. In particular, the noncausal part of the second term is given by

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & z\mathbf{I}_{M/2} \end{pmatrix} \mathbf{P}\mathbf{T}_{m+1}^T\mathbf{P}\mathbf{e}_{m+1}(0). \tag{2.21}$$

We have to show that there exists a matrix $\mathbf{T}_{m+1}$ of the form

$$\mathbf{T}_{m+1} = \begin{pmatrix} \mathbf{A}_{m+1} & \mathbf{C}_{m+1} \\ -\mathbf{C}_{m+1} & \mathbf{A}_{m+1} \end{pmatrix} \tag{2.22}$$

which makes the above non-causal term equal to zero. Simplifying eq. (2.21), we find that $\mathbf{T}_{m+1}$ should be such that

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{M/2} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{m+1}^T & -\mathbf{C}_{m+1}^T\mathbf{J}_{M/2} \\ \mathbf{C}_{m+1}^T & \mathbf{A}_{m+1}^T\mathbf{J}_{M/2} \end{pmatrix} \mathbf{e}_{m+1}(0) = \mathbf{0}. \tag{2.23}$$

Hence, it is sufficient to find $\mathbf{A}_{m+1}$ and $\mathbf{C}_{m+1}$ such that

$$\begin{pmatrix} \mathbf{C}_{m+1}^T & \mathbf{A}_{m+1}^T\mathbf{J}_{M/2} \end{pmatrix} \mathbf{e}_{m+1}(0) = \mathbf{0}. \tag{2.24}$$

Consider the matrix $\mathbf{W}$ (in eq. (2.4)). The eigenvalues of this matrix are $\pm j$, where $j = \sqrt{-1}$. The eigenvectors corresponding to the eigenvalue j are

$$\mathbf{s}_0 = \begin{pmatrix} j \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \ldots, \mathbf{s}_{M/2-1} = \begin{pmatrix} \vdots \\ 0 \\ j \\ 1 \\ 0 \\ \vdots \end{pmatrix}. \tag{2.25}$$

We will denote the space spanned by these vectors as $\mathcal{E}_1$. Similarly, the eigenvectors corresponding to the eigenvalue $-j$ are

$$\mathbf{a}_0 = \begin{pmatrix} -j \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \ldots, \mathbf{a}_{M/2-1} = \begin{pmatrix} \vdots \\ 0 \\ -j \\ 1 \\ 0 \\ \vdots \end{pmatrix}. \tag{2.26}$$

We will denote the space spanned by these vectors as $\mathcal{E}_2$.

Since the matrix $\mathbf{W}$ is skew-symmetric, these eigenvectors together span the entire space.

Now, consider any vector $\mathbf{y}$. It can always be written as $\mathbf{y} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in \mathcal{E}_1$ and $\mathbf{v} \in \mathcal{E}_2$. Moreover, if $\mathbf{y}$ is real, we have $\mathbf{u} = \mathbf{v}^*$, where the asterisk superscript denotes conjugation (See Appendix A). This therefore implies that

$$\mathbf{u}^\dagger \mathbf{u} = \mathbf{v}^\dagger \mathbf{v}, \tag{2.27}$$

where the $\dagger$ superscript denotes conjugate transpose.

Eqn. (2.8) implies in particular that $\mathbf{W}\mathbf{e}_{m+1}(0)\mathbf{V}_M\mathbf{J}_M = \mathbf{e}_{m+1}(m+1)$. Paraunitariness of $\mathbf{E}(z)$ on the other hand implies that $\mathbf{e}_{m+1}^T(m+1)\mathbf{e}_{m+1}(0) = \mathbf{0}$. Hence,

$$\mathbf{J}_M^T \mathbf{V}_M^T \mathbf{e}_{m+1}^T(0)\mathbf{W}^T \mathbf{e}_{m+1}(0) = \mathbf{0}. \tag{2.28}$$

Using the facts that $\mathbf{W}^T = -\mathbf{W}$ and $\mathbf{e}_{m+1}(0)$ is real, we get $\mathbf{e}_{m+1}^\dagger(0)\mathbf{W}\mathbf{e}_{m+1}(0) = \mathbf{0}$. We therefore have

$$\mathbf{U}^\dagger \mathbf{e}_{m+1}^\dagger(0)\mathbf{W}\mathbf{e}_{m+1}(0)\mathbf{U} = \mathbf{0}, \tag{2.29}$$

for any matrix $\mathbf{U}$. Let the matrix $\mathbf{U}$ be so chosen that the first r columns of the matrix $\mathbf{e}_{m+1}(0)\mathbf{U}$ form an orthonormal basis of real vectors $\mathbf{x}_i$ for the columns of the matrix $\mathbf{e}_{m+1}(0)$. (This is possible, since the matrix $\mathbf{e}_{m+1}(0)$ is itself real. Hence r is the rank of the matrix $\mathbf{e}_{m+1}(0)$). These vectors $\mathbf{x}_i$ being orthonormal satisfy $\mathbf{x}_i^\dagger \mathbf{x}_j = 0, \ (i \neq j)$. Let $\mathbf{x}_i = \mathbf{u}'_i + \mathbf{v}'_i$, where $\mathbf{u}'_i \in \mathcal{E}_1$ and $\mathbf{v}'_i \in \mathcal{E}_2$. Therefore, $(\mathbf{u}'_i + \mathbf{v}'_i)^\dagger (\mathbf{u}'_j + \mathbf{v}'_j) = 0$, which simplifies to

$$\mathbf{u}'^\dagger_i \mathbf{u}'_j + \mathbf{v}'^\dagger_i \mathbf{v}'_j = 0. \tag{2.30}$$

The real vectors $\mathbf{x}_i$ satisfy $\mathbf{x}_i^\dagger \mathbf{W} \mathbf{x}_i = \mathbf{0}$, since $\mathbf{W}$ is antisymmetric. Hence

$$(\mathbf{u}'_i + \mathbf{v}'_i)^\dagger \mathbf{W} (\mathbf{u}'_j + \mathbf{v}'_j) = 0, \tag{2.31}$$

i.e.,

$$\mathbf{u}_i'^\dagger \mathbf{W} \mathbf{u}'_j + \mathbf{v}_i'^\dagger \mathbf{W} \mathbf{u}'_j + \mathbf{u}_i'^\dagger \mathbf{W} \mathbf{v}'_j + \mathbf{v}_i'^\dagger \mathbf{W} \mathbf{v}'_j = 0. \tag{2.32}$$

Noting that the vectors $\mathbf{u}'_i$ and $\mathbf{v}'_j$ are orthogonal, for all i, j, and the fact that $\mathbf{W}\mathbf{v}'_i = -j\mathbf{v}'_i, \mathbf{W}\mathbf{u}'_i = j\mathbf{u}'_i$, we get

$$\mathbf{u}'^\dagger_i \mathbf{u}'_j = \mathbf{v}'^\dagger_i \mathbf{v}'_j. \tag{2.33}$$

Eqs. (2.30) and (2.33) together imply that $\mathbf{u}'^\dagger_i \mathbf{u}'_j = 0$, and $\mathbf{v}'^\dagger_i \mathbf{v}'_j = 0$. The vectors $\mathbf{u}'_i, \quad i = 1, \ldots, r$ and $\mathbf{v}'_i, \quad i = 1, \ldots, r$ therefore form orthonormal bases for r-dimensional subspaces of $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. Moreover, since the $\mathbf{x}_i, \quad i = 1, \ldots, r$ are real, we have (again referring to Appendix A)

$$\mathbf{u}'_i = \mathbf{v}'^*_i. \tag{2.34}$$

In $\mathcal{E}_1$, there exist $p = M/2 - r$ orthogonal vectors $\mathbf{u}'_i, \quad i = r+1, \ldots, M/2$ which are also orthogonal to the previously mentioned set of r vectors $\mathbf{u}'_i, \quad i = 1, \ldots, r$. Similarly, in $\mathcal{E}_2$, there exist $p = M/2 - r$ orthogonal vectors $\mathbf{v}'_i, \quad i = r+1, \ldots, M/2$ which are also orthogonal to the previously mentioned set of r vectors $\mathbf{v}'_i, \quad i = 1, \ldots, r$. Clearly, a particular choice of these p vectors in $\mathcal{E}_2$ could be the conjugates of the p vectors chosen in $\mathcal{E}_1$.

Now using these additional p orthonormal vectors from $\mathcal{E}_1$ and adding them to their conjugates from $\mathcal{E}_2$, we can form p orthonormal real vectors $\mathbf{x}_i$.

Let $\mathbf{X}^T$ be the matrix of size $(M/2) \times M$ whose rows are the vectors $\mathbf{x}_i \quad i = 1, \ldots, M/2 - 1$. This matrix satisfies the following properties:

1) $\mathbf{X}^T\mathbf{X} = \mathbf{I}_{M/2}$ (from the fact that $\mathbf{x}_i$ are orthonormal).

2) $\mathbf{X}^T\mathbf{W}\mathbf{e}_{m+1}(0) = \mathbf{0}$ (by the construction outlined above).

Partition $\mathbf{X}^T = [\mathbf{Y}^T \ \mathbf{Z}^T]$. Then with $\mathbf{C}_{m+1}^T = \mathbf{Z}^T\mathbf{J}_{M/2}$ and $\mathbf{A}_{m+1}^T = -\mathbf{Y}^T$, the matrix $\mathbf{T}_{m+1}$ defined in eq. (2.22) is orthogonal and satisfies eq. (2.23). This proves that $\mathbf{E}_m(z)$ is causal.

Order reduction: Given the fact that $\mathbf{E}_m(z)$ is causal, and that it satisfies eq. (2.3), we can see that the order of $\mathbf{E}_m(z)$ is m. Thus there is a reduction in order by one. Hence for a system of order N, the factorization process is guaranteed to terminate in N steps.

This concludes the proof of the theorem. $\diamond\diamond\diamond$

III CONCLUSION

In this correspondence we gave a proof of completeness of the structure presented in [6]. The resulting filters have pairwise mirror-image symmetry around $\pi/2$ in the frequency domain, which leads to a greater efficiency in the design process.

Appendix A

As in Section II, we denote the two eigenspaces of the matrix $\mathbf{W}$ by $\mathcal{E}_1$ and $\mathcal{E}_2$, corresponding to the eigenvalues j and $-j$ respectively. Let $\mathbf{y} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in \mathcal{E}_1$ and $\mathbf{v} \in \mathcal{E}_2$. Let

$$\mathbf{u} = \alpha_1\mathbf{s}_1 + \ldots + \alpha_{M/2-1}\mathbf{s}_{M/2-1}$$

and

$$\mathbf{v} = \beta_1 \mathbf{a}_1 + \ldots + \beta_{M/2-1} \mathbf{a}_{M/2-1},$$

where the $\mathbf{s}_i$ and the $\mathbf{a}_i$ are the basis vectors for the two subspaces as in section II. From their definition it is clear that if $\mathbf{y}$ is real, $\alpha_i \mathbf{s}_i + \beta_i \mathbf{a}_i$ is real for $i = 1, \ldots, M/2-1$. Since $\mathbf{s}_i = \mathbf{a}_i^*$, it follows that $\alpha_i = \beta_i^*$. Hence $\mathbf{u} = \mathbf{v}^*$.

REFERENCES

[1] R. Crochiere and L. Rabiner, *Multirate Digital Signal Processing*, Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1983.

[2] M. J. T. Smith and T. P. Barnwell, III, "A Procedure for Designing Exact Reconstruction Filter Banks for Tree-Structured Sub-band Coders," Proc. of IEEE Int. Conf. on ASSP, San Diego, March 1984.

[3] P. P. Vaidyanathan, "Theory and Design of M-channel Maximally Decimated Filters with Arbitrary M, having Perfect Reconstruction Property," IEEE Trans. on ASSP, pp. 476-492, Apr. 1987.

[4] M. Vetterli, "A Theory of Multirate Filter Banks," IEEE Trans. on ASSP, pp. 356-372, Mar. 1987.

[5] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[6] T. Nguyen and P. P. Vaidyanathan, "Maximally Decimated Perfect-Reconstruction Filter Banks with Pairwise Mirror-Image Analysis (and Synthesis) Frequency Responses," IEEE Trans. on ASSP, pp. 693-706, May 1988.

[7] H. S. Malvar, "Modualted QMF Filter Banks with Perfect Reconstruction," Electronic Letters, pp. 9066-907, June 1990.

[8] R. D. Koilpillai and P. P. Vaidyanathan, "Cosine-Modulated FIR Filter Banks Satisfying Perfect Reconstruction," IEEE Trans. on Sig. Proc., April 1992.

[9] T. A. Ramstad, "Cosine Modulated Analysis-Synthesis Filter Bank With Critical Sampling and Perfect Reconstruction," Proc. of ICASSP, pp. 1789-1792, Toronto, May 1991.

[10] A. Soman and P. P. Vaidyanathan, "Coding Gain in Paraunitary Analysis Synthesis Systems," to appear, IEEE Trans. on Sig. Proc., May 1993.