MICROWAVE INTERACTION
WITH BOUNDED GYROELECTRIC PLASMAS

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ABSTRACT

In the following we investigate, theoretically, the interaction of microwaves with gyroelectric plasmas of finite extent, particularly those having cylindrical or spherical boundaries. Within the latter class of problems, only those involving the axially magnetized column with circular cross section are amenable to rigorous analysis. We find that one of the important effects of the anisotropy is to induce changes in the polarization of the scattered field resulting from interaction with an obliquely incident plane wave.

As a means of solving problems which involve uniform but arbitrarily directed magnetization, we develop the perturbation theory of microwave interaction in which the static magnetic field is regarded as a small perturbation of the isotropic plasma. The field equations are derived for all orders but only those of first order, linear in the magnitude of the static magnetic field, are solved. This solution is carried out in general, the only restriction being that the fields for the isotropic problem are assumed to be known.

The first order theory is then applied to cylindrical and spherical problems. When the approximate solution for the axially magnetized column is compared with the exact result, agreement is obtained provided that the static magnetic field is weak, as expected. Finally we consider the problem of a magnetic dipole radiating from within a weakly magnetized plasma sphere.

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I. INTRODUCTION

In the following report we present the results of a theoretical investigation into certain aspects of microwave interaction with a bounded gyroelectric plasma, i.e., a plasma in which there is maintained a static magnetic field*, Such a medium may be characterized, within the framework of a phenomenological theory, by a relative permeability equal to unity and a relative permittivity given by a second rank tensor, the latter reducing to a scalar tensor in the limit of vanishing magnetic field.

The interaction of microwaves with gyroelectric plasmas has been the subject of many investigations. These have been directed toward, for example, the explanation of various geophysical phenomena such as ionospheric double refraction and whistler propagation and toward the use of microwaves to analyze fundamental processes in gases. However, because of the complexity of Maxwell's equations for such a medium, most of the theoretical work has been limited to one-dimensional problems which involve planar boundaries. It should be recognized, however, that there are many practical situations in which the plasma is finite with curved boundaries. We are thus motivated to consider several two- and three-dimensional problems, specifically ones involving cylindrical or spherical geometry.

It appears that the early work concerning the interaction of microwaves with cylindrical gyrotropic structures was for a gyromagnetic

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*We regard as outside the scope of this work a discussion of the manner by which the gas is contained or the origin of the static field.

rather than a gyroelectric medium. Motivated by interest in a device known as the ferrite isolator, Suhl and Walker (1), and later Epstein (2), presented the relevant equations for the field components in an axially magnetized cylinder. Papas (3) showed that the permittivity of a magnetically biased plasma is also a tensor, mathematically analogous to the permeability of a ferrite. Hence the work previously referred to may be applied to this type of medium as well.

Several years later Platzman and Ozaki (4) solved the problem of scattering of a normally incident plane wave by a cylindrical, axially magnetized plasma with circular cross section. Wilhelmsson (5), interested in electron beam interaction, discussed the more general case of oblique incidence. However, in both instances, emphasis was placed on the effect of the magnetic field on the scattering cross section. In the present work we shall be interested in the change in polarization of the scattered field rather than in its intensity.

Examples involving spherical boundaries, on the other hand, have received little attention. These are of interest to radio astronomers since celestial microwave sources are plasmas which may, as a first approximation, be considered spherical in shape. In addition, there is some evidence of Faraday rotation of the waves emitted from such bodies which suggests that they have gyroelectric properties (6). This effect might be exploited in the determination of their physical parameters, such as electron density, magnetic field, etc., so that theoretical work in this area is of more than academic interest.

The report is divided into five main parts. In the first we examine the properties of a gyroelectric plasma and derive an expression for the tensor permittivity. The second part contains a rigorous analysis of the fields inside a cylindrical plasma with application to the problem of scattering of an obliquely incident plane wave by an axially magnetized column. The third consists of a development of the perturbation theory of microwave propagation in gyroelectric plasmas in which we regard the static magnetic field as a small perturbation of the isotropic medium. In the fourth part perturbation techniques are applied to cylindrical systems with the axially magnetized column considered as a special case. As anticipated, the results compare favorably to those of the exact analysis provided that the validity criterion of weak magnetic field is satisfied. Finally we employ a perturbation method to find the field of a radiating magnetic dipole immersed in a gyroelectric sphere. This structure is suggested as a crude model for a celestial microwave source.

The rationalized MKS system of units and a time dependence of the form $\exp(-i\omega t)$ will be used throughout.

II. PROPERTIES OF A GYROELECTRIC PLASMA

A. Introduction

We use the term plasma to denote a gas which is at least partially ionized. There are therefore some free electrons which are free to move under the influence of an ambient electromagnetic field and, in the case of a gyroelectric plasma, a static magnetic field $\underline{B}_o$ as well. Maxwell's equations for such a medium are

$$\nabla \times \underline{E} = i\omega\mu_o \underline{H}$$

$$\nabla \times \underline{H} = -i\omega \epsilon_o \underline{E} - Ne\underline{v} \qquad (2.1)$$

where $\underline{v}$ is the average induced velocity of the electrons, of which there are assumed to be N per cubic meter.

We confine this investigation to the so-called "cold" plasma, which is a lossless medium since collisions between electrons and other particles are ignored. In addition it will be assumed that the frequency of the electromagnetic field is high enough that the induced motion of the positive ions can be neglected. Thus the only effect of these heavier particles is to make the total average charge density zero.

Since $\underline{v}$ will turn out to be a linear vector function of $\underline{E}$, it is convenient to write the second of Maxwell's equations as

$$\nabla \times \underline{H} = - i\omega \epsilon_o \underline{\underline{\kappa}} \cdot \underline{E} \qquad (2.2)$$

thereby defining an apparent relative dielectric tensor $\underline{\underline{\kappa}}$. Our objective in this chapter is to present a formula for $\underline{\underline{\kappa}}$ in terms of the physical parameters of the medium.

B. The Equation of Motion for Electrons

In order to derive the expression for the dielectric tensor we must examine the motion of electrons in a combination of time dependent electric and static magnetic fields. From Newton's law of motion and the Lorentz force equation we have that

$$\frac{m d\underline{v}}{dt} = - e\underline{E} - e\underline{v} \times \underline{B}_o \quad . \tag{2.3}$$

Note that the effect of the alternating magnetic field on the motion is ignored. This is a valid approximation provided that the magnitude of the electron velocity is small compared to the speed of light. We prefer to write 2.3 in the form

$$\frac{d\underline{v}}{dt} - \omega_g \, \underline{e}_B \times \underline{v} = - \frac{e}{m} \underline{E} \tag{2.4}$$

where ω_g is the gyro-frequency of the electron and $\underline{e}_B$ is a unit vector in the direction of the static magnetic field. If it is assumed that the latter is uniform, then the equation for $\underline{v}$ is a linear equation with constant coefficients.

A more convenient relation can be found by exploiting the algebraic properties of the linear differential operator. Using the

Heaviside notation $p = d/dt$ we write 2.4 as

$$\underline{\underline{L}}(p) \cdot \underline{v} = -\frac{e}{m}\underline{E} \tag{2.5}$$

where

$$\underline{\underline{L}}(p) = p\underline{\underline{U}} - \omega_g \, \underline{e}_B \times \tag{2.6}$$

and $\underline{\underline{U}}$ is the unit dyadic. The next step is to invert $\underline{\underline{L}}(p)$. This may be done by using a matrix representation in rectangular coordinates, treating p as an algebraic variable. The result, cast back into vector form, is

$$\underline{\underline{L}}^{-1}(p) = \frac{p^2\underline{\underline{U}} + \omega_g p \, \underline{e}_B \times + \omega_g^2 \, \underline{e}_B \, \underline{e}_B}{p^3 + \omega_g^2 \, p} \quad . \tag{2.7}$$

The interpretation of 2.7 is that $\underline{v}$ satisfies the differential equation

$$\frac{d^3\underline{v}}{dt^3} + \omega_g^2 \frac{d\underline{v}}{dt} = -\frac{e}{m}\frac{d^2\underline{E}}{dt^2} - \frac{e}{m}\omega_g \, \underline{e}_B \times \frac{d\underline{E}}{dt} - \frac{e}{m}\omega_g^2 \, \underline{e}_B(\underline{e}_B \cdot \underline{E}) \quad . \tag{2.8}$$

Under an assumed time dependence $\exp(-i\omega t)$, the steady state solution to 2.8 is

$$\underline{v} = \frac{e}{m\omega(\omega^2 - \omega_g^2)}\left[-i\omega^2\underline{E} + \omega\omega_g \, \underline{e}_B \times \underline{E} + i\omega_g^2 \, \underline{e}_B(\underline{e}_B \cdot \underline{E})\right] \quad . \tag{2.9}$$

C. The Dielectric Tensor

From the solution for the electron velocity $\underline{v}$ in terms of the electric field $\underline{E}$, the expression for the dielectric tensor may be derived by making use of 2.1 and the defining equation 2.2. Using 2.9 in 2.1 we obtain, for the right hand side of the equation,

$$- i\omega\epsilon_o \left[\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_g^2}\right) \underline{E} - i \frac{\omega_p^2 \omega_g}{\omega(\omega^2 - \omega_g^2)} \underline{e}_B \times \underline{E} + \frac{\omega_p^2 \omega_g^2}{\omega^2(\omega^2 - \omega_g^2)} \underline{e}_B (\underline{e}_B \cdot \underline{E}) \right] \tag{2.10}$$

leading directly to

$$\underline{\underline{\kappa}} = \left[\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_g^2}\right) \underline{\underline{U}} - \frac{i\omega_p^2 \omega_g}{\omega(\omega^2 - \omega_g^2)} \underline{e}_B \times \; + \frac{\omega_p^2 \omega_g^2}{\omega^2(\omega^2 - \omega_g^2)} \underline{e}_B \, \underline{e}_B \right], \tag{2.11}$$

where ω_p denotes the so-called plasma frequency

$$\omega_p \doteq \left(\frac{Ne^2}{m\epsilon_o} \right)^{1/2} . \tag{2.12}$$

As anticipated, the relative permittivity is a tensor. 2.11 is a convenient representation which is invariant under transformation of coordinates and, if we set $\omega_g = 0$, reduces to the well-known result for an isotropic plasma

$$\lim_{\omega_g \to 0} \underline{\underline{\kappa}} = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \underline{\underline{U}} . \tag{2.13}$$

In order to simplify the subsequent equations we introduce the following symbols:

$$\kappa_1 = 1 - \frac{\omega_p^2}{\omega^2 - \omega_g^2}$$

$$\kappa_3 = \frac{\omega_p^2 \omega_g^2}{\omega^2(\omega^2 - \omega_g^2)}$$

$$\kappa_2 = \kappa_1 + \kappa_3 = 1 - \frac{\omega_p^2}{\omega^2}$$

$$g = \frac{\omega_p^2 \omega_g}{\omega(\omega^2 - \omega_g^2)} \quad , \qquad (2.14)$$

The inverse of the dielectric tensor, which is also a useful quantity, is found most easily by employing a matrix representation in a rectangular coordinate system oriented with its z-axis parallel to the static magnetic field. In this system $\underline{\underline{\kappa}}$ is given by

$$\kappa = \begin{bmatrix} \kappa_1 & ig & 0 \\ -ig & \kappa_1 & 0 \\ 0 & 0 & \kappa_2 \end{bmatrix} \quad . \qquad (2.15)$$

The inverse of $\underline{\underline{\kappa}}$, denoted by the symbol $\underline{\underline{\eta}}$, is

$$\underline{\underline{\eta}} = \begin{bmatrix} \eta_1 & -ih & 0 \\ ih & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{bmatrix} , \tag{2.16}$$

where

$$\eta_1 = \frac{\kappa_1}{\kappa_1 - g^2}$$

$$\eta_2 = \frac{1}{\kappa_2}$$

$$h = \frac{g}{\kappa_1^2 - g^2} . \tag{2.17}$$

It will be useful to represent $\underline{\underline{\eta}}$ in a form analogous to the expression 2.11. This representation is

$$\underline{\underline{\eta}} = \eta_1 \underline{\underline{U}} + ih \, \underline{e}_B \, x + \eta_3 \, \underline{e}_B \, \underline{e}_B \tag{2.18}$$

where

$$\eta_3 = \eta_2 - \eta_1 .$$

III. THE FIELDS IN A GYROELECTRIC CYLINDER-RIGOROUS THEORY

A. The Fields in an Isotropic Dielectric

Before proceeding to the main problem of this chapter we discuss the more familiar one of solving Maxwell's equations in homogeneous, isotropic materials. Having done this will make the effect of the anisotropy more discernible.

For a source-free region Maxwell's equations are, assuming no induced magnetic dipole moment,

$$\nabla \times \underline{E} = i\omega \mu_o \underline{H}$$

$$\nabla \times \underline{H} = - i\omega \epsilon_o \kappa \underline{E} \quad . \tag{3.1}$$

When there is a boundary separating two media with different dielectric constants, the technique generally used is that of obtaining solutions in each medium separately, but in such a form that boundary conditions may be satisfied. There is some arbitrariness in the choice of conditions; we will require that the components of $\underline{E}$ and $\underline{H}$ tangent to the interface be continuous.

Eliminating $\underline{E}$ from 3.1 and using the fact that $\underline{H}$ is solenoidal, we obtain the equation for the magnetic field,

$$\nabla^2 \underline{H} + k^2 \underline{H} = 0 \quad . \tag{3.2}$$

Boundary value problems associated with the vector wave equation 3.2 are difficult to solve for several reasons. First, with the exception of rectangular coordinates, the Laplacian operates on the unit vectors as well as the components. The result is that the scalar equations corresponding to 3.2 usually contain more than one of the components of $\underline{H}$. For example, in cylindrical coordinates these equations are (13):

$$\nabla^2 H_\rho - \frac{2}{\rho^2}\frac{\partial H_\phi}{\partial \phi} - \frac{H_\rho}{\rho^2} + k^2 H_\rho = 0$$

$$\nabla^2 H_\phi + \frac{2}{\rho^2}\frac{\partial H_\rho}{\partial \phi} - \frac{H_\phi}{\rho^2} + k^2 H_\phi = 0$$

$$\nabla^2 H_z + k^2 H_z = 0 \quad . \tag{3.3}$$

Observe that only in the third equation does a component appear by itself, this being because z is also a rectangular coordinate. Furthermore, we would like the solutions to be in such a form that boundary conditions may be satisfied. This not only dictates the choice of coordinate system but also requires separability with respect to behavior along the coordinates of the interface; a very stringent requirement in all but a few select cases.

A number of solutions to the vector wave equation may be found by exploiting two important commutative properties of the Laplacian operator. Let $\underline{u}$ denote any of the rectangular unit vectors $\underline{e}_x$,

$\underline{e}_y$, $\underline{e}_z$ or the (normalized) radius vector $k_o\underline{r}$, k_o being the vacuum wave number ω/c . Then the following are valid identities.

$$\nabla^2 \left[\nabla \times \underline{u}\, W_1 \right] = \nabla \times \underline{u} \left[\nabla^2 W_1 \right] \tag{3.4}$$

$$\nabla^2 \left[\nabla \times (\underline{u} \times \nabla\, W_2) \right] = \nabla \times \left[\underline{u} \times \nabla(\nabla^2 W_2) \right] \tag{3.5}$$

where W_1 and W_2 are arbitrary scalars. The proofs are given in Appendix A.

These identities are used in the following way (7): Assume that a solution to 3.2 may be written as*

$$\underline{H} = \frac{1}{k_o} \nabla \times \underline{u}\, W_1 + \frac{1}{k_o^2} \nabla \times (\underline{u} \times \nabla\, W_2) \quad . \tag{3.6}$$

Substituting back into the vector wave equation and employing 3.4 and 3.5 we find that 3.6 will be a solution provided that W_1 and W_2 each obey the scalar wave equation

$$\nabla^2 W + k^2 W = 0 \tag{3.7}$$

which is considerably easier to solve than 3.2. The process of "scalarization", i.e., determining scalar functions such as W_1 and

*This contrived form is employed so that W_1 and W_2 will turn out to have the same dimensions, namely those of magnetic field.

W_2 which will generate solutions to Maxwell's equations, is a very powerful technique in solving boundary value problems.

For the case of cylindrical geometry with generatrix in the z direction we let $\underline{u} = \underline{e}_z$. In addition we assume that the z dependence is $\exp(i\gamma k_o z)$ so that using some vector identities (equations A.3, A.4) we can transform 3.6 into

$$\underline{H} = -\left[\underline{e}_z \times \frac{1}{k_o} \nabla_t W_1 + \frac{i\gamma}{k_o} \nabla_t W_2 + \underline{e}_z \nu^2 W_2\right] \tag{3.8}$$

where ∇_t is the transverse gradient,

$$\nabla_t = \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y}$$

and ν is the transverse index of refraction

$$\nu = \frac{1}{k_o} (k^2 - (\gamma k_o)^2)^{1/2} \quad . \tag{3.9}$$

Furthermore, because of the exponential dependence on z , the functions W_1 and W_2 now satisfy the two-dimensional Helmholtz equation

$$(\nabla_t^2 + \beta^2) W = 0 \tag{3.10}$$

where $\beta = \nu k_o$.

The corresponding expression for $\underline{E}$ can be found from 3.8 and Maxwell's equations, with the result that

$$\underline{E} = iZ_o\eta\left[\kappa\underline{e}_z \times \frac{1}{k_o}\nabla_t W_2 + \frac{i\Upsilon}{k_o}\nabla_t W_1 + \underline{e}_z \nu^2 W_1\right] \quad (3.11)$$

From inspection of 3.8 and 3.11 we conclude that the function W_1 generates waves in which the magnetic field is transverse to $\underline{e}_z$ while W_2 generates waves in which the electric field is transverse to $\underline{e}_z$. Referred to as TM (transverse-magnetic) and TE (transverse-electric) modes, they are linearly independent solutions of the vector wave equation for cylindrical geometry with z-directed generatrix*. As will be demonstrated later, similar results are obtained for spherical problems.

B. The Fields in a Gyroelectric Medium

In order to determine the fields in a gyroelectric medium we follow a procedure similar to the one used by Epstein for ferrites. The details are given in Appendix B.

Maxwell's equations for a gyroelectric plasma are

$$\nabla \times \underline{E} = i\omega\mu_o \underline{H}$$

$$\nabla \times \underline{H} = i\omega\epsilon_o \underline{\underline{\kappa}}\, \underline{E} \quad (3.12)$$

from which it is straightforward to show that $\underline{H}$ satisfies

$$\nabla \times (\underline{\underline{\eta}} \nabla \times \underline{H}) = k_o^2 \underline{H} \quad (3.13)$$

*In general, the designation TE and TM will be used for waves with electric and magnetic fields, respectively, transverse to the vector $\underline{u}$ in 3.6. Thus W_1 is said to generate a TM wave and W_2 a TE wave.

A rigorous analysis is possible only when the two preferred directions, i.e., those of the magnetic field and the generatrix of the cylinder, coincide. We shall therefore be limited, in what follows, to this particular case.

Under the assumption of a z dependence $\exp(i\gamma\, k_o z)$ the linearly independent solutions to the wave equation 3.13 are derived from scalar functions V_1 and V_2 which satisfy the two-dimensional Helmholtz equations

$$\nabla_t^2 V_1 + \beta_1^2 V_1 = 0 \tag{3.14}$$

$$\nabla_t^2 V_2 + \beta_2^2 V_2 = 0 \quad . \tag{3.15}$$

The transverse propagation constants β_1 and β_2 are related to the physical parameters of the plasma as follows:

$$\beta_1^2 = \frac{k_o^2}{2}\left(\alpha + \sqrt{\alpha^2 - 4\Gamma}\right) \tag{3.16}$$

$$\beta_2^2 = \frac{k_o^2}{2}\left(\alpha - \sqrt{\alpha^2 - 4\Gamma}\right) \tag{3.17}$$

where

$$\alpha = \frac{\eta_1 + \eta_2 + \gamma^2[h^2 - \eta_1(\eta_1 + \eta_2)]}{\eta_1\, \eta_2} \tag{3.18}$$

$$\Gamma = \frac{(1 - \eta_1\gamma^2)^2 - h^2\gamma^4}{\eta_1\,\eta_2} \quad . \tag{3.19}$$

The magnetic field is then given by

$$\underline{H} = -\left[\underline{e}_z \times \frac{1}{k_o}\nabla_t(\tau_1 V_1 + \tau_2 V_2) + \frac{i\gamma}{k_o}\nabla_t(V_1 + V_2) + \underline{e}_z(\nu_1^2\, V_1 + \nu_2^2\, V_2)\right] \tag{3.20}$$

using the notation

$$\tau_1 = \frac{\eta_1(\nu_1^2 + \gamma^2) - 1}{h\gamma}$$

$$\tau_2 = \frac{\eta_1(\nu_2^2 + \gamma^2) - 1}{h\gamma} \quad . \tag{3.21}$$

In order to determine the corresponding expression for the electric field we employ Maxwell's equation,

$$\underline{E} = \frac{iZ_o}{k_o}\,\underline{\underline{\eta}}\,\nabla \times \underline{H} \tag{3.22}$$

together with the vector form for $\underline{\underline{\eta}}$ which is given by 2.19. The result is

$$\underline{E} = iZ_o \left\{ \underline{e}_z \times \frac{1}{k_o} \nabla_t \left[(\eta_1(\nu_1^2 + \gamma^2) - \gamma\tau_1 h)V_1 + (\eta_1(\nu_2^2 + \gamma^2) - \gamma\tau_2 h)V_2 \right] \right.$$

$$+ \frac{1}{k_o} \nabla_t \left[(i\gamma\tau_1\eta_1 - ih(\nu_1^2 + \gamma^2))V_1 + (i\gamma\tau_2\eta_1 - ih(\nu_2^2 + \gamma^2))V_2 \right]$$

$$\left. + \underline{e}_z \left[\tau_1\nu_1^2\eta_2 V_1 + \tau_2\nu_2^2\eta_2 V_2 \right] \right\} \quad . \qquad (3.23)$$

Some insight into the effect of the static magnetic field can be gained by finding the limiting forms taken by 3.20 and 3.23 as B_o vanishes. For the relevant parameters we find that

$$\begin{aligned} \tau_1 &\to -\sqrt{\kappa} \\ \tau_2 &\to \sqrt{\kappa} \\ h &\to 0 \\ \eta_1 &\to \eta \\ \eta_2 &\to \eta \end{aligned} \qquad (3.24)$$

where the right hand sides are the isotropic quantities which have been previously defined. Having established these limits it is straightforward to show that in the limit of vanishing static magnetic field 3.20 and 3.23 become

$$\underline{H} \to - \left[\frac{i\gamma}{k_o} \nabla_t (V_1 + V_2) + \sqrt{\kappa}\ \underline{e}_z \times \frac{1}{k_o} \nabla_t (V_2 - V_1) + \underline{e}_z \nu^2 (V_1 + V_2) \right] \qquad (3.25)$$

and

$$\underline{E} \rightarrow iZ_o\eta\left[\kappa\, \underline{e}_z \times \frac{1}{k_o}(V_1 + V_2) + \sqrt{\kappa}\,\frac{i\gamma}{k_o}\,\nabla_t(V_2 - V_1) + \underline{e}_z \sqrt{\kappa}\,\nu^2(V_2 - V_1)\right] \quad (3.26)$$

respectively.

Comparing 3.25 and 3.26 with 3.8 and 3.11 we observe that the solutions in the gyroelectric medium reduce, in the limit of vanishing static magnetic field, to linear combinations of TE and TM modes. This is to be expected since the latter are the fundamental solutions in an isotropic medium. The generating functions V_1 and V_2 are related to W_1 and W_2, in the limiting case, by the transformation

$$V_1 = \frac{1}{2\sqrt{\kappa}}\,(\sqrt{\kappa}\,W_2 - W_1)$$

$$V_2 = \frac{1}{2\sqrt{\kappa}}\,(\sqrt{\kappa}\,W_2 + W_1) \quad . \quad (3.27)$$

Thus one effect of the static magnetic field is to "couple" the TE and TM modes so that these no longer are the independent solutions to Maxwell's equations. More will be said about this point of view when we discuss the problem within the framework of perturbation theory. Furthermore, the combinations so formed are "split" in the sense that they have associated with them different transverse propagation constants. This splitting has been observed in the ionosphere where it is referred to as ionospheric double refraction.

C. Formal Solution of the Scattering Problem

Having obtained an analytical description of the fields inside an arbitrary cylindrical plasma, we proceed to the problem of scattering of an obliquely incident plane wave by a homogeneous, axially magnetized column with circular cross section. There are several reasons for considering this particular example. First, since the Helmholtz equation is separable in this coordinate system, we may calculate rigorously the effect of mode coupling on certain polarization properties of the scattered field. In this way our objectives are somewhat different than those of other investigators, who were primarily interested in the effect of the static magnetic field on the scattering cross section (4),(5). Furthermore, the exact results serve as a check for those which will be obtained using a perturbation procedure.

The coordinate system is shown in Figs. 1 and 2. It is assumed that the propagation vector of the incident wave lies in the x-z plane at an angle θ with the x axis. The choice of polarization with $\underline{E}^{(inc)}$ perpendicular to the plane of incidence is made to emphasize the effect of the static magnetic field.

The expression for the magnetic field of the plane wave is, under the assumption of unit amplitude,

$$\underline{H}^{(inc)} = \left[\exp(ik_o\rho \cos\theta \cos\phi + ik_o z \sin\theta)\right] \cdot (\underline{e}_z \cos\theta - \underline{e}_x \sin\theta) \quad (3.28)$$

and for the electric field

$$\underline{E}^{(inc)} = \frac{iZ_o}{k_o} \underline{e}_y \exp(ik_o\rho \cos\theta \cos\phi + ik_o z \sin\theta) \quad (3.29)$$

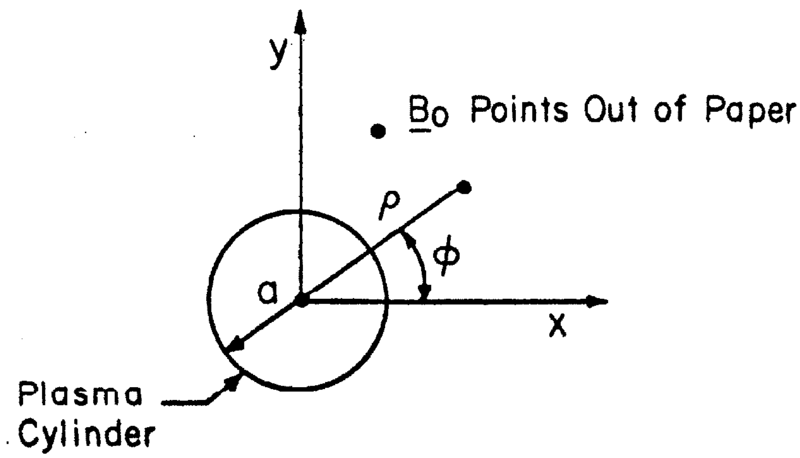

Figure 3.1

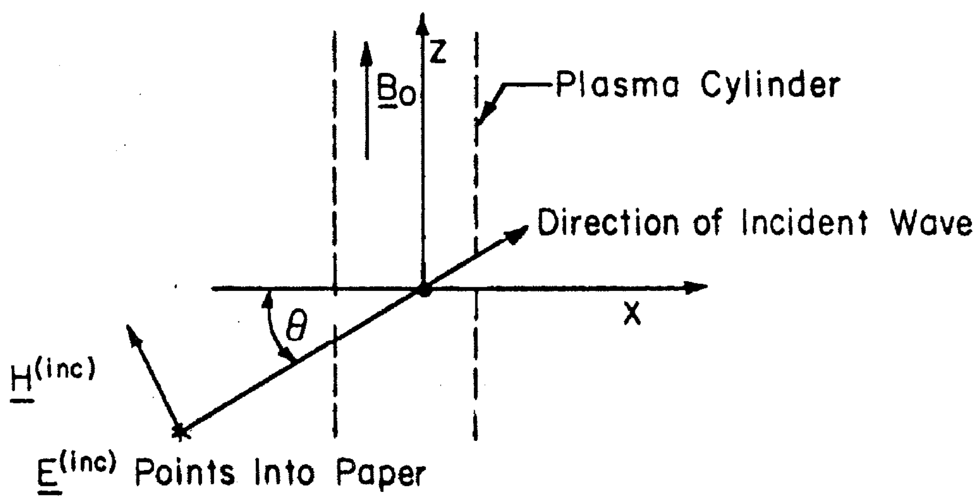

Figure 3.2

Coordinate system for scattering problem from which it is straightforward to show that the appropriate generating functions are

$$W_1^{(inc)} = 0 \tag{3.30}$$

$$W_2^{(inc)} = -\frac{1}{\cos\Theta}\exp(ik_o\rho\cos\Theta\cos\phi + ik_oz\sin\Theta) \quad . \tag{3.31}$$

In order to represent the field components in terms of cylindrical functions we employ the identity

$$\exp(i\alpha\cos\phi) = \sum_{n=-\infty}^{+\infty} i^n J_n(\alpha)e^{in\phi} \tag{3.32}$$

where J_n stands for the Bessel function of the first kind of order n . The expression for W_2 is then

$$W^{(inc)} = -\frac{1}{\cos\Theta}\sum_{n=-\infty}^{+\infty} i^n J_n(k_o\rho\cos\Theta)F_n(\phi,z) \tag{3.33}$$

with

$$F_n(\phi,z) = \exp(in\phi + ik_oz\sin\Theta) \quad . \tag{3.34}$$

The cylindrical components are calculated from 3.8 and 3.11 using the representation for the gradient

$$\nabla_t = \underline{e}_\rho\frac{\partial}{\partial\rho} + \underline{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} \quad . \tag{3.35}$$

The results are, for the nth terms,

$$ {}^{n}H_{\phi}^{(inc)} = -i^{n}\left[\frac{n \sin\theta}{k_o\rho\cos\theta} J_n(k_o\rho\cos\theta)F_n\right] $$

$$ {}^{n}H_{z}^{(inc)} = i^{n}\left[\cos\theta\, J_n(k_o\rho\cos\theta)F_n\right] $$

$$ {}^{n}E_{\phi}^{(inc)} = -iZ_o\, i^{n}\left[J_n'(k_o\rho\cos\theta)F_n\right] $$

$$ {}^{n}E_{z}^{(inc)} = 0 \quad . \tag{3.36} $$

The ρ components are omitted since they do not enter into what follows.

We proceed to expand the fields inside the column in a similar way. Observe that in order to satisfy the boundary conditions at the surface of the cylinder it is necessary to have

$$ \gamma = \sin\theta \tag{3.37} $$

for both the internal and scattered fields.

Let the generating functions V_1 and V_2 be written as a sum of solutions to the Helmholtz equation in cylindrical coordinates,

$$ V_1 = \sum_{n=-\infty}^{+\infty} a_n\, J_n(\beta_1\rho)F_n $$

$$ V_2 = \sum_{n=-\infty}^{+\infty} b_n\, J_n(\beta_2\rho)F_n \tag{3.38} $$

where β_1 and β_2 are given by 3.16 and 3.17 with $\gamma = \sin\theta$. The ϕ and z components are calculated using 3.20 and 3.23 and are found to be

$$^{n}H_{\phi}^{(tr)} = \left\{\left[\frac{n\gamma}{k_o\rho} J_n(\beta_1\rho) - \tau_1\nu_1 J_n'(\beta_1\rho)\right] a_n + \left[\frac{n\gamma}{k_o\rho} J_n(\beta_2\rho) - \tau_2\nu_2 J_n'(\beta_2\rho)\right]\right\} F_n$$

$$^{n}H_{z}^{(tr)} = -\left\{\left[\nu_1^2 J_n(\beta_1\rho)\right] a_n + \left[\nu_2^2 J_n(\beta_2\rho)\right] b_n\right\} F_n \tag{3.39}$$

$$\begin{aligned} ^{n}E_{\phi}^{(tr)} = iZ_o \Big\{ &(\gamma^2 + \nu_1^2)\left[\eta_1\nu_1 J_n'(\beta_1\rho) + \frac{nh}{k_o\rho} J_n(\beta_1\rho)\right] a_n + i\gamma\tau_1\left[\frac{in\eta_1}{k_o\rho} J_n(\beta_1\rho) + \right. \\ &\left. ih\nu_1 J_n'(\beta_1\rho)\right] a_n + (\gamma^2 + \nu_2^2)\left[\eta_1\nu_2 J_n'(\beta_2\rho) + \frac{nh}{k_o\rho} J_n(\beta_2\rho)\right] b_n + \\ &i\gamma\tau_2\left[\frac{in\eta_1}{k_o\rho} J_n(\beta_2\rho) + ih\nu_2 J_n'(\beta_2\rho)\right] b_n \Big\} \cdot F_n(\phi, z) \end{aligned}$$

$$^{n}E_{z}^{(tr)} = iZ_o \left\{\left[\eta_2\tau_1\nu_1^2 J_n(\beta_1\rho)\right] a_n + \left[\eta_2\tau_2\nu_2^2 J_n(\beta_2\rho)\right] b_n\right\} \cdot F_n(\phi, z) \tag{3.40}$$

The superscript (tr) denotes that it is the field transmitted into the column.

Finally, the same procedure is carried out for the scattered field which is assumed to be generated by $W_1^{(sc)}$ and $W_2^{(sc)}$ given by

$$W_1^{(sc)} = \sum_{n=-\infty}^{+\infty} c_n H_n(k_o\rho\cos\theta) F_n(\phi, z)$$

$$W_2^{(sc)} = \sum_{n=-\infty}^{+\infty} d_n H_n(k_o\rho\cos\theta) F_n(\phi, z) \quad . \tag{3.41}$$

H_n , denoting the Hankel function of the first kind of order n , is used to give outgoing waves at infinity under the assumed time dependence $\exp(-i\omega t)$, in accordance with the Sommerfeld radiation condition. The components of the scattered field which are important for the subsequent calculations are:

$$
\begin{aligned}
{}^{n}H_{\phi}^{(sc)} &= -\left\{\left[\cos\theta\, H_n'(k_o\rho\cos\theta)\right]c_n - \left[\frac{n\sin\theta}{k_o\rho} H_n(k_o\rho\cos\theta)\right]d_n\right\}F_n \\
{}^{n}H_{z}^{(sc)} &= -\left[\cos^2\theta\, H_n(k_o\rho\cos\theta)\right]d_nF_n
\end{aligned}
\qquad (3.42)
$$

$$
\begin{aligned}
{}^{n}E_{\phi}^{(sc)} &= iZ_o\left\{\left[\frac{-n\sin\theta}{k_o\rho} H_n(k_o\rho\cos\theta)\right]c_n + \left[\cos\theta\, H_n'(k_o\rho\cos\theta)\right]d_n\right\}F_n \\
{}^{n}E_{z}^{(sc)} &= iZ_o\left[\cos^2\theta\, H_n(k_o\rho\cos\theta)\right]c_nF_n
\end{aligned}
\qquad (3.43)
$$

In order to determine the unknown coefficients a_n, b_n, c_n and d_n we invoke the conditions that at the surface of the cylinder the tangential components of the electric and magnetic fields be continuous. Thus, at $\rho = a$

$$
\begin{aligned}
H_{\phi}^{(inc)} + H_{\phi}^{(sc)} &= H_{\phi}^{(tr)} \\
H_{z}^{(inc)} + H_{z}^{(sc)} &= H_{z}^{(tr)}
\end{aligned}
\qquad (3.44)
$$

$$
\begin{aligned}
E_{\phi}^{(inc)} + E_{\phi}^{(sc)} &= E_{\phi}^{(tr)} \\
E_{z}^{(inc)} + E_{z}^{(sc)} &= E_{z}^{(tr)} \quad .
\end{aligned}
\qquad (3.45)
$$

Using the previous results we obtain four equations in the four unknown coefficients. They are most conveniently represented in the matrix form

$$\underline{\underline{A}}_n \, \underline{v}_n = \underline{f}_n \tag{3.46}$$

where $\underline{v}_n$ denotes the column vector

$$\begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} \tag{3.47}$$

and $\underline{f}_n$ the column vector formed from the H_ϕ, H_z, E_ϕ and E_z components of the incident wave, respectively.

$$\begin{bmatrix} - \dfrac{n \sin \theta}{k_o a \cos \theta} J_n(k_o a \cos \theta) \\ \cos \theta \, J_n(k_o a \cos \theta) \\ - J_n'(k_o a \cos \theta) \\ 0 \end{bmatrix} \tag{3.48}$$

The matrix of the coefficients $\underline{\underline{A}}_n$ is given on the following page.

$$\underline{\underline{A}}_n = \begin{bmatrix} \frac{n\gamma}{k_o a} J_n(\beta_1 a) - \tau_1 \nu_1 J_n'(\beta_1 a) & \frac{n\gamma}{k_o a} J_n(\beta_2 a) - \tau_2 \nu_2 J_n'(\beta_2 a) & \cos\theta\, H_n'(k_o a \cos\theta) & \frac{-n\gamma}{k_o a} H_n(k_o a \cos\theta) \\ -\nu_1^2 J_n(\beta_1 a) & -\nu_2^2 J_n(\beta_2 a) & 0 & \cos^2\theta\, H_n(k_o a \cos\theta) \\ (\gamma^2+\nu_1^2)\left[\eta_1 \nu_1 J_n'(\beta_1 a) + \frac{nh}{k_o a} J_n(\beta_1 a)\right] - \gamma\tau_1\left[\frac{n\eta_1}{k_o a} J_n(\beta_1 a) + h\nu_1 J_n'(\beta_1 a)\right] & (\gamma^2+\nu_2^2)\left[\eta_1 \nu_2 J_n'(\beta_\nu a) + \frac{nh}{k_o a} J_n(\beta_2 a)\right] - \gamma\tau_2\left[\frac{n\eta_1}{k_o a} J_n(\beta_2 a) + h\nu_2 J_n'(\beta_2 a)\right] & \frac{n\gamma}{k_o a} H_n(k_o a \cos\theta) & -\cos\theta\, H_n'(k_o a \cos\theta) \\ \eta_2 \tau_1 \nu_1^2 J_n(\beta_1 a) & \eta_2 \tau_2 \nu_2^2 J_n(\beta_2 a) & -\cos^2\theta\, H_n(k_o a \cos\theta) & 0 \end{bmatrix} \tag{3.49}$$

D. Polarization of the Scattered Field

That property of the scattered field of particular interest is its polarization in the plane of incidence, since an important difference between the isotropic and gyroelectric cylinder is here in evidence. Assume that a TE wave, such as considered in the previous section, is obliquely incident on the column. Then, whereas in the case of the isotropic scatterer the field in the plane of incidence is also TE, if the column is gyroelectric it will be a combination of TE and TM, the magnitude of the TM component depending on the strength of the static magnetic field. After considering the problem in mathematical detail we will investigate the physical origin of this effect.

As discussed previously, the scattered field is derived from two scalar functions $W_1^{(sc)}$ and $W_2^{(sc)}$ by employing 3.8 and 3.11 with $\gamma = \sin\Theta$. The components of the electric field are, in cylindrical coordinates,

$$E_\rho^{(sc)} = iZ_o \left[i \sin\Theta \frac{\partial W_1^{(sc)}}{\partial(k_o\rho)} - \frac{1}{k_o\rho}\frac{\partial W_2^{(sc)}}{\partial\phi} \right]$$

$$E_\phi^{(sc)} = iZ_o \left[\frac{i\sin\Theta}{k_o\rho}\frac{\partial W_1^{(sc)}}{\partial\phi} + \frac{\partial W_2^{(sc)}}{\partial(k_o\rho)} \right]$$

$$E_z^{(sc)} = iZ_o \left[\cos^2\Theta \; W_1^{(sc)} \right] \qquad (3.50)$$

Consider the far zone field. Employing the asymptotic formulas for the Hankel function of the first kind, we find that the generating functions may be written as

$$W_1^{(sc)} = \sqrt{\frac{2}{\pi k_o \rho \cos\Theta}} \exp(ik_o\rho \cos\Theta + i\gamma k_o z \sin\Theta) g_1(\phi) + \mathcal{O}(\rho^{-3/2})$$

$$W_2^{(sc)} = \sqrt{\frac{2}{\pi k_o \rho \cos\Theta}} \exp(ik_o\rho \cos\Theta + i\gamma k_o z \sin\Theta) g_2(\phi) + \mathcal{O}(\rho^{-3/2})$$

(3.51)

where $g_1(\phi)$ and $g_2(\phi)$ are the functions which are left in 3.41 after the radial and z dependences have been factored out. They are

$$g_1(\phi) = \sum_{n=-\infty}^{+\infty} c_n \exp\left[+in(\phi - \pi/2) - i\frac{\pi}{4}\right]$$

$$g_2(\phi) = \sum_{n=-\infty}^{+\infty} d_n \exp\left[in(\phi - \pi/2) - i\frac{\pi}{4}\right] \quad . \qquad (3.52)$$

Neglecting the terms of order $\rho^{-3/2}$ we determine that in the far zone of the cylinder the electric field components are related to the generating functions in the following way:

$$E_\rho^{(sc)} = iZ_o\left[-\sin\Theta\cos\Theta\, W_1^{(sc)}\right]$$

$$E_\phi^{(sc)} = iZ_o\left[i\cos\Theta\, W_2^{(sc)}\right]$$

$$E_z^{(sc)} = iZ_o\left[\cos^2\Theta\, W_1^{(sc)}\right] \quad . \qquad (3.53)$$

In order to obtain a better physical description of the field we make a transformation to the ξ, ϕ, δ coordinate system indicated in Fig. 3.

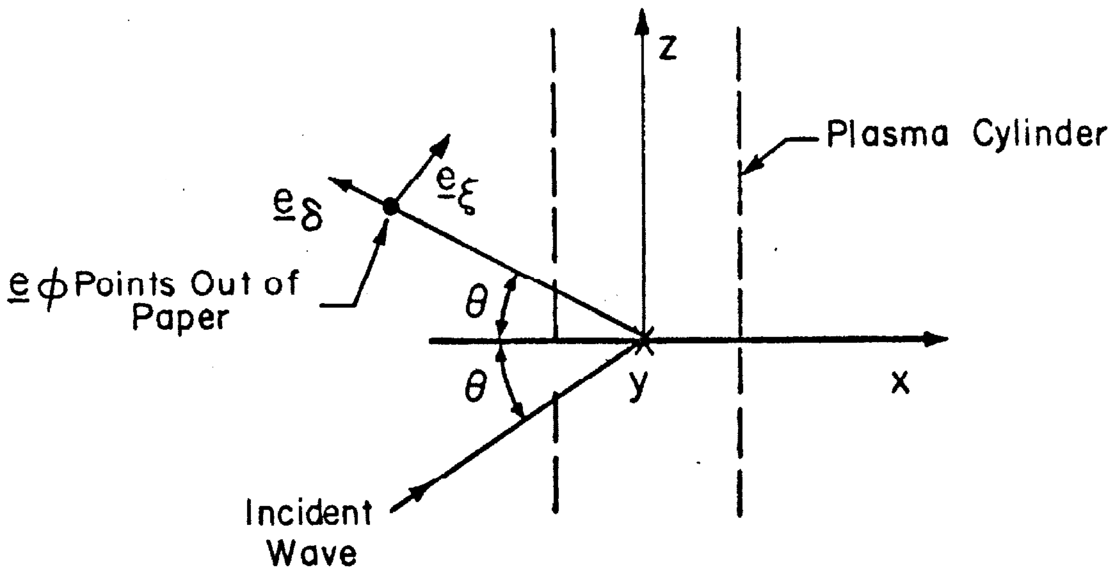

Figure 3.3 Transformation of coordinates

This is a right-handed system with the unit vectors $\underline{e}_\xi$, $\underline{e}_\phi$ and $\underline{e}_\delta$ where $\underline{e}_\delta$ is in the direction of specular reflection. The unit vectors of the original cylindrical coordinate system are related to the new unit vectors by the equations

$$\underline{e}_\rho = -\underline{e}_\xi \sin\theta + \underline{e}_\delta \cos\theta$$

$$\underline{e}_\phi = \underline{e}_\phi$$

$$\underline{e}_z = \underline{e}_\xi \cos\theta + \underline{e}_\delta \sin\theta \qquad . \tag{3.54}$$

It follows that the field components in the new system are related to those in the old by

$$E_\xi = -\sin\theta\, E_\rho + \cos\theta\, E_z$$

$$E_\phi = E_\phi$$

$$E_\delta = \cos\theta\, E_\rho + \sin\theta\, E_z \qquad . \tag{3.55}$$

Using 3.53 together with the above we arrive at the simple result that in the far zone

$$E_\xi^{(sc)} = iZ_o[\cos\theta\, W_1^{(sc)}]$$

$$E_\phi^{(sc)} = iZ_o[i\cos\theta\, W_2^{(sc)}]$$

$$E_\delta^{(sc)} = 0 \qquad . \tag{3.56}$$

From Maxwell's equations it can be shown that the electric and magnetic fields are related by

$$\underline{E}^{(sc)} = -Z_o \, \underline{e}_\delta \times \underline{H}^{(sc)} \quad , \tag{3.57}$$

again neglecting higher order terms. Thus the far zone scattered field consists of a TEM wave propagating in the δ direction. This is the cylindrical version of the familiar law of reflection. We shall be concerned with the polarization in the plane transverse to the direction of propagation.

In general, for a monochromatic TEM wave, the tip of the electric vector, in the transverse plane, traces out an ellipse as shown in Fig. 4. The parameters of this ellipse are functions of the magnitudes and relative phase of the components E_ϕ and E_ξ. For example, the angle ψ which gives the orientation of the ellipse may be calculated from (8)

$$\tan 2\psi = \frac{2|E_\xi| \cdot |E_\phi|}{|E_\xi|^2 - |E_\phi|^2} \cos[\arg E_\phi - \arg E_\xi] \tag{3.58}$$

where arg denotes the argument of the complex number. In terms of generating functions the above may be written as

$$\tan 2\psi = \frac{2|W_1| \cdot |W_2|}{|W_1|^2 - |W_2|^2} \cos[\arg W_2 - \arg W_1 + \tfrac{\pi}{2}] \ . \tag{3.59}$$

Using this formula, it is straightforward to show that for the far

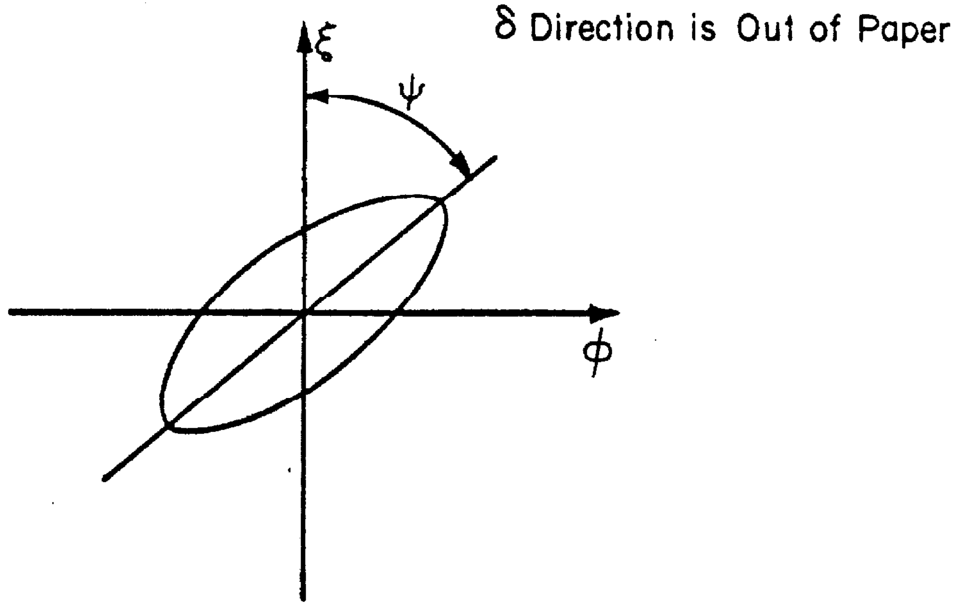

Figure 3.4 Polarization ellipse

zone scattered field in the plane of incidence $(\emptyset = \pi)$, the angle ψ is determined by the expression

$$\tan 2\psi = \frac{2|\sigma_1| \cdot |\sigma_2|}{|\sigma_1|^2 - |\sigma_2|^2} \cos[\arg \sigma_2 - \arg \sigma_1 + \frac{\pi}{2}] \tag{3.60}$$

where σ_1 and σ_2 denote the series

$$\sigma_1 = \sum_{n=-\infty}^{+\infty} i^n c_n \tag{3.61}$$

$$\sigma_2 = \sum_{n=-\infty}^{+\infty} i^n d_n \quad .$$

Another important property of the ellipse is its eccentricity, i.e., the ratio of the lengths of the minor and major axes. In terms of σ_1 and σ_2 the eccentricity, denoted by e, may be calculated from

$$e = \tan \lambda \tag{3.62}$$

where the angle λ is defined by

$$\sin 2\lambda = \frac{2|\sigma_1| \cdot |\sigma_2|}{|\sigma_1|^2 + |\sigma_2|^2} \sin[\arg \sigma_2 - \arg \sigma_1 + \frac{\pi}{2}] \quad . \tag{3.63}$$

We observe that the eccentricity may be either positive or negative. If e is positive then the electric vector is rotating in the counter-clockwise direction and if e is negative the electric vector rotates in the clockwise direction.

E. Numerical Results and Conclusions

Numerical results were obtained for the angle of orientation ψ and the eccentricity e of the polarization ellipse as a function of the parameter ω_g/ω, which is a measure of the strength of the static magnetic field. The calculations were performed on a digital computer, the IBM 7090, making use of a version of FORTRAN which permits the direct use of complex arithmetic. The procedure was straightforward, consisting essentially of an inversion of the system of linear equations for the coefficients c_n and d_n followed by summation in accordance with 3.61. The subroutines for calculating the Bessel and Hankel functions involved a series summation, with the values compared to existing tables as a check. The over-all results were checked by comparing them in the limit $\omega_g/\omega \to 0$, to those obtained for the isotropic problem which was solved and programmed independently (see Appendix C).

The results are indicated in Figs. 5 and 6, where values are given corresponding to the parameters $\omega_p/\omega = .67$, $k_o a = 2$ and $\theta = 20^o$.

These diagrams exhibit an interesting effect of the gyroelectric character of the plasma. In the isotropic case $\omega_g/\omega = 0$ the far zone scattered field, in the plane of incidence, is linearly polarized in the same direction as the incident wave, but as the static magnetic field increases the scattered wave becomes elliptically polarized. Both the orientation and eccentricity of the ellipse depend on the strength of the magnetic field, the relations being linear to first order in the parameter ω_g/ω.

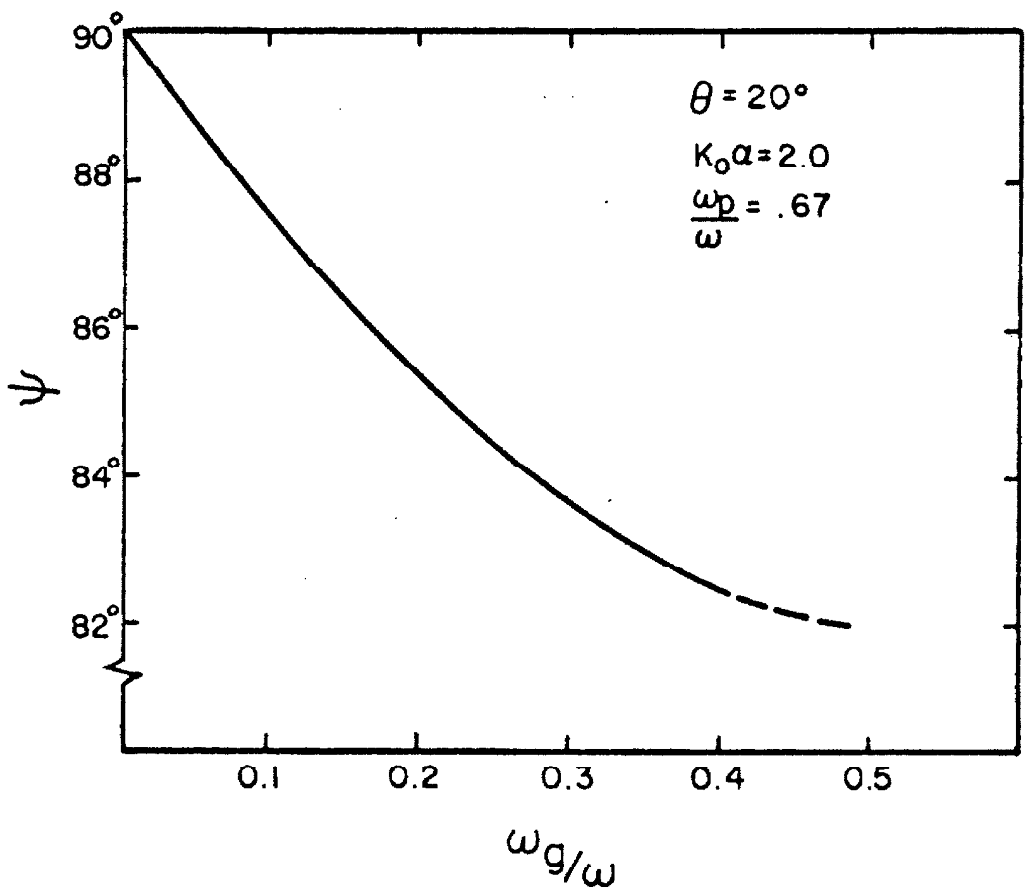

Figure 3.5 Orientation angle, ψ , of the polarization ellipse as a function of ω_g/ω

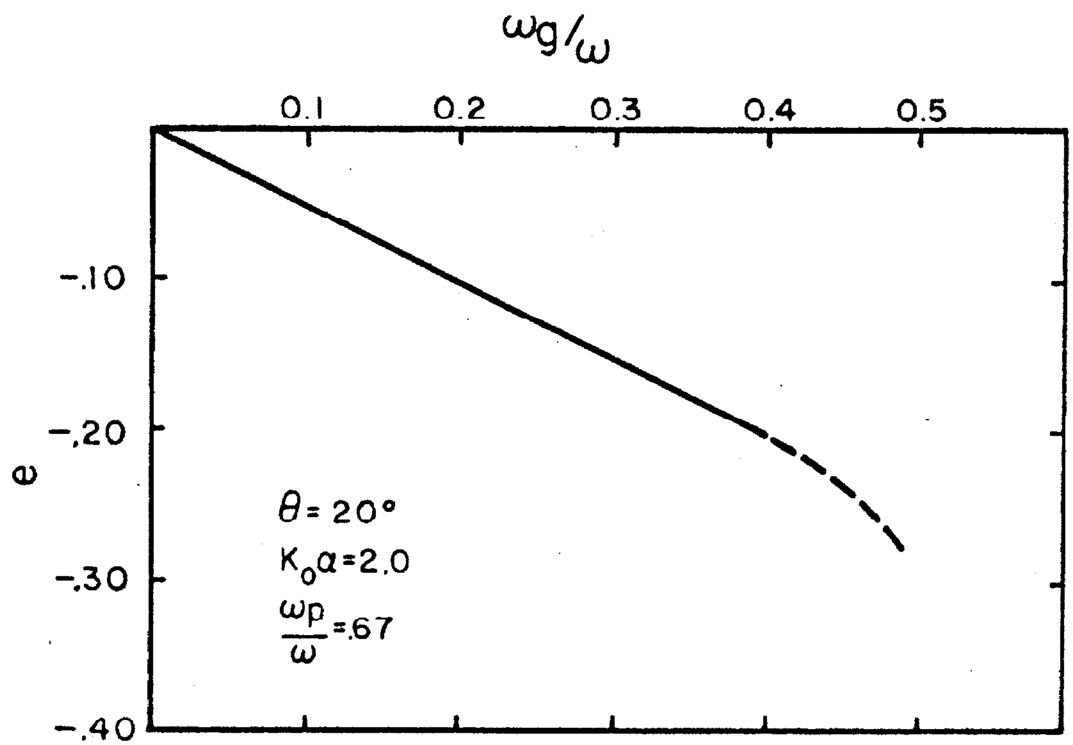

Figure 3.6 Eccentricity, e , of the polarization ellipse as a function of ω_g/ω

Although the interactions which take place inside the cylinder between the electron, the transmitted wave and the static magnetic field are very complicated, we may give a plausible physical explanation by considering the cylinder as a secondary source and focussing our attention on its electric dipole component, as viewed from the plane of incidence. In the isotropic case we have seen that in this plane under the assumed polarization of the incident wave the field of the scattered wave is polarized in the horizontal direction. The effect of the incident wave, then, is to induce y-directed dipoles which in turn radiate. The situation is as depicted in Fig. 7. Observe also that due to the oblique incidence the dipoles are out of phase. The picture changes when a longitudinal magnetic field is applied; the electrons, which were (apparently) moving along the y direction, interact with $\underline{B}_o$ and acquire a component of motion in the x direction. The result is to produce a component of dipole moment in the x direction, as shown in Fig. 8. These moments are also out of phase and may combine to produce a vertical component of electric field.

The situation is different when there is no dependence on z, i.e., the case of normal incidence. Here the vertical component of the electric field vanishes due to symmetry and the original polarization is retained. This case was discussed by Platzman and Ozaki (4).

In addition to its theoretical interest, the existence of a cross-polarized component in the far zone electric field may prove to have some practical value. Since the properties of this component depend on the parameters of the plasma as well as the operating

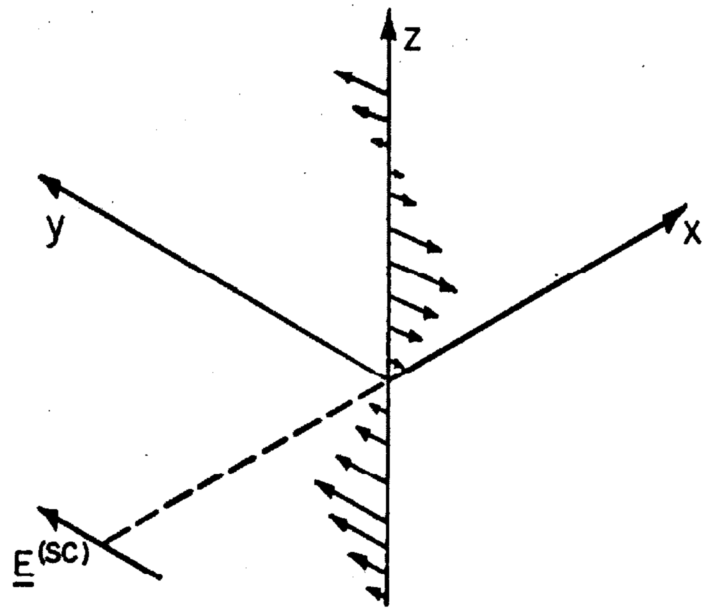

Figure 3.7 Induced dipoles as viewed from plane of incidence; $B_o = 0$

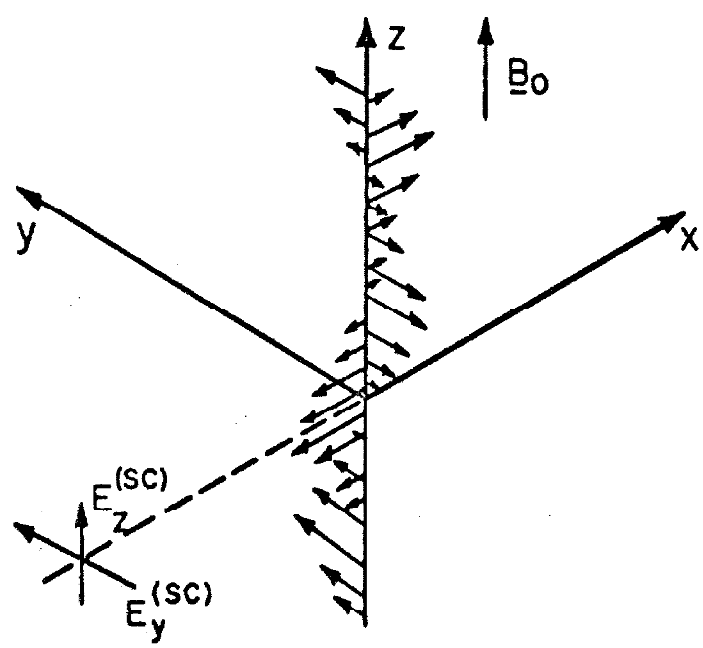

Figure 3.8 Induced dipoles as viewed from plane of incidence; $B_o \neq 0$

frequency and the strength of the magnetic field, the results of an experiment in which the cross field is measured as a function of, say, ω_g/ω might yield information concerning the plasma frequency, assuming that all other parameters were known.

IV. THE PERTURBATION THEORY OF MICROWAVE INTERACTION WITH GYROELECTRIC PLASMAS

A. Introduction

The absence of a general technique, such as the one outlined in Section III.A for isotropic media, for determining rigorous solutions to Maxwell's equations in arbitrarily magnetized plasmas prompts us to search for approximate methods. In this chapter we shall develop one such method, applicable when the static magnetic field is "small" in a sense to be defined more precisely later on. To use the vernacular, we shall regard the static magnetic field as a perturbation of the isotropic plasma.

Perturbation theory has enjoyed a rather remarkable history as a technique for solving physical problems not amenable to rigorous analysis. For example, in quantum mechanics it has been used successfully to determine both stationary and non-stationary states when the Schroedinger equation involves a complicated Hamiltonian. The Born approximation of atomic scattering theory is representative of this class of problems.

The perturbation which is of interest here is more complicated than usual because it is a vector perturbation of a vector field. Nevertheless, this approach is useful here as well since it yields, within the limits of applicability, solutions to problems which cannot be analyzed rigorously using presently available techniques.

B. Power Series Expansion of the Dielectric Tensor

In this section we shall derive an expansion of the dielectric tensor in powers of the parameter $i\,\frac{\omega_g}{\omega}$, which is a measure of the strength of the static magnetic field. The series is somewhat unconventional in that the terms are matrices rather than scalars. However, this presents no essential mathematical difficulty since the theory of functions of a matrix is analogous in many respects to the corresponding theory for scalar variables.

To begin, we recall the equation of motion for an electron acted upon simultaneously by a harmonic electric field $\underline{E}\,e^{-i\omega t}$ and an arbitrarily directed static magnetic field $\underline{B}_o$,

$$i\omega m\underline{v} = e\,\underline{E} - e\,\underline{B}_o \times \underline{v} \tag{4.1}$$

or, alternatively,

$$\left[\underline{\underline{u}} - i\,\frac{\omega_g}{\omega}\,\underline{e}_B \times\right] \cdot \underline{v} = \frac{e\,E}{i\omega m} \quad . \tag{4.2}$$

We may then, by inverting 4.2, write that the velocity vector is given by

$$\underline{v} = \frac{e}{i\omega m}\left[\underline{\underline{u}} - i\,\frac{\omega_g}{\omega}\,\underline{e}_B \times\right]^{-1} \cdot \underline{E} \tag{4.3}$$

which is equivalent to 2.9.

We now regard the operator $\left(\underline{\underline{U}} - i \frac{\omega_g}{\omega} \underline{e}_B \, x\right)^{-1}$ as a function of the matrix $i \frac{\omega_g}{\omega} \underline{e}_B \, x$ in analogy with the function $(1-s)^{-1}$ of the ordinary complex variable $s = \alpha + i\beta$.

There exists a matrix-scalar correspondence principle (9) which states that if $f(s)$ has a Taylor series expansion

$$f(s) = \sum_{n=0}^{\infty} a_n s^n \tag{4.4}$$

then the same function, but with a matrix $\underline{\underline{S}}$ substituted for the scalar, will have the power series expansion

$$\underline{\underline{f}}(\underline{\underline{s}}) = \sum_{n=0}^{\infty} a_n \underline{\underline{S}}^n \tag{4.5}$$

which converges provided that the eigenvalues of $\underline{\underline{S}}$ all lie inside the circle of convergence of 4.4.

Since the circle of convergence of the series

$$(1-s)^{-1} = \sum_{n=0}^{\infty} s^n \tag{4.6}$$

is the unit circle centered about the origin, it follows that the representation

$$\left[\underline{\underline{U}} - i \frac{\omega_g}{\omega} \underline{e}_B \, x\right]^{-1} = \sum_{n=0}^{\infty} \left(i \frac{\omega_g}{\omega} \underline{e}_B \, x\right)^n \tag{4.7}$$

is valid provided that the eigenvalues λ_i $(i=1,2,3)$ of the matrix

$i \frac{\omega_g}{\omega} \underline{e}_B \times$ satisfy the inequality

$$|\lambda_i| < 1 \qquad i = 1,2,3 \; . \tag{4.8}$$

In calculating the eigenvalues we take advantage of their invariance under rotation of coordinates and use a system in which the z axis is in the direction of the static magnetic field. In such a system the matrix representation is

$$i \frac{\omega_g}{\omega} \underline{e}_B \times = \begin{bmatrix} 0 & -i \frac{\omega_g}{\omega} & 0 \\ i \frac{\omega_g}{\omega} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.9}$$

for which the secular equation is

$$\lambda(\lambda^2 - \frac{\omega_g^2}{\omega^2}) = 0 \tag{4.10}$$

with roots $\lambda_1 = 0$, $\lambda_2 = \frac{\omega_g}{\omega}$, $\lambda_3 = -\frac{\omega_g}{\omega}$. The criterion for the validity of 4.7 is thus that $\omega_g/\omega < 1$, and it will be assumed in what follows that this condition is satisfied.

Employing 4.2 in a manner similar to the calculation in Section II.C we obtain the following series expansion for the dielectric tensor

$$\underline{\underline{\epsilon}} = \epsilon \left[\underline{\underline{U}} - \zeta \sum_{n=1}^{\infty} g^n (\underline{e}_B \times)^n \right] \tag{4.11}$$

where

$$\epsilon = \epsilon_o (1 - \frac{\omega_p^2}{\omega^2}) \quad , \quad g = i \frac{\omega_g}{\omega} \quad , \quad \zeta = \frac{\omega_p^2}{\omega^2 - \omega_p^2} \quad .$$

It is straightforward to show that, in a coordinate system aligned with the z axis along $\underline{e}_B$, the first few terms of 4.11 give the same results as an expansion, element by element, of the formula derived in Section II.C.

C. Expansion for the Field in a Gyroelectric Plasma in Terms of Partial Fields

Using the result of the previous section we write Maxwell's equations in the medium as

$$\nabla \times \underline{E} = i \omega \mu_o \underline{H} \tag{4.12}$$

$$\nabla \times \underline{H} = -i\omega \, \epsilon \underline{E} + i \, \omega \epsilon \zeta \left[\sum_{n=1}^{\infty} g^n (\underline{e}_B \times)^n \right] \cdot \underline{E}$$

We next assume that there exists an expansion for the electric and magnetic fields of the form

$$\underline{E} = \sum_{m=0}^{\infty} g^m \, \underline{E}^{(m)}$$

$$\underline{H} = \sum_{m=0}^{\infty} g^m \, \underline{H}^{(m)} \tag{4.13}$$

where $\underline{E}^{(m)}$ and $\underline{H}^{(m)}$ are referred to as the mth order partial fields. Substituting 4.13 into 4.12 and equating powers of the expansion parameter, we conclude that the partial fields satisfy the following equations:

$$\nabla \times \underline{E}^{(o)} = i\omega\mu_o \underline{H}^{(o)}$$

$$\nabla \times \underline{H}^{(o)} = -i\omega\epsilon\, \underline{E}^{(o)} \qquad (4.14)$$

$$\nabla \times \underline{E}^{(n)} = i\omega\mu_o \underline{H}^{(n)}$$

$$\nabla \times \underline{H}^{(n)} = -i\omega\epsilon\, \underline{E}^{(n)} + i\omega\epsilon\zeta \sum_{m=0}^{n-1} (\underline{e}_B \times)^{n-m}\, \underline{E}^{(m)}, \; n > 0 \qquad (4.15)$$

The formulation in terms of partial fields differs from the conventional statement of Maxwell's equations both mathematically and physically. From a mathematical point of view it represents a change from the problem of solving a pair of homogeneous partial differential equations which involve a tensor operator to a problem of solving an infinite sequence of inhomogeneous equations where the source terms depend on solutions to the equations of lower order. Physically, we have introduced a description of the electromagnetic field in the plasma in which we regard the total field as being composed of a sum of fields. These are arranged in a hierarchy of complexity in which those of lower order "interact" with the static magnetic field to produce the ones of higher order.

The present formulation has the advantage that if the static magnetic field is weak, the more complex fields may be ignored since they are of higher order in g . In this work we will consider in detail only the zero order field which, as can be seen from 4.14, is that which would exist if there were no static magnetic field, and the first order component which is linear in g . It is thus assumed that B_o is small enough so that terms of order $(\omega_g/\omega)^2$ are negligible. Before specializing, however, we will say more about the general problem.

D. A Note on Boundary Conditions

In what follows we shall be considering problems which involve a boundary between a plasma and vacuum, as indicated in Fig. 1. According to 4.13 the fields inside the plasma will be given by

$$\underline{E}_i = \underline{E}_i^{(o)} + g\,\underline{E}_i^{(1)} + g^2\underline{E}_i^{(2)} + \cdots$$

$$\underline{H}_i = \underline{H}_i^{(o)} + g\,\underline{H}_i^{(1)} + g^2\underline{H}_i^{(1)} + \cdots \qquad (4.16)$$

Because of the boundary conditions there must exist a corresponding field on the vacuum side of the interface of the form

$$\underline{E}_e = \underline{E}_e^{(0)} + g\,\underline{E}_e^{(1)} + g^2\underline{E}_e^{(2)} + \cdots$$

$$\underline{H}_e = \underline{H}_e^{(0)} + g\underline{H}_e^{(1)} + g^2\underline{H}_e^{(2)} + \cdots \qquad (4.17)$$

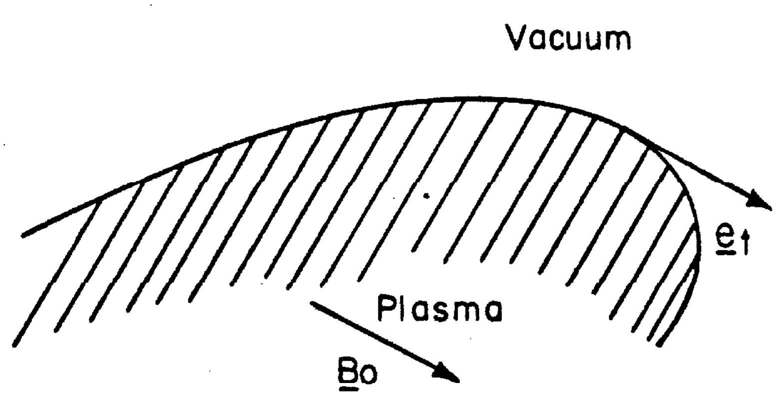

Figure 4.1 Boundary between plasma and vacuum

and equating tangential components we obtain the result that

$$E_{e_t}^{(n)} = E_{i_t}^{(n)} \qquad \text{all } n \qquad (4.18)$$
$$H_{e_t}^{(n)} = H_{i_t}^{(n)}$$

Hence the boundary conditions must be satisfied at each step of the perturbation procedure. Physically this means that the internal interactions between the fields of a given order and $\underline{B}_o$ produce waves which are, in turn, partially transmitted and partially reflected at the plasma-vacuum boundary.

E. On the Criteria for Validity of the Perturbation Expansion

In general the solution to a physical or mathematical problem depends on several parameters and hence an asymptotic form is rarely uniformly valid, i.e., applicable in a range of one of the parameters regardless of the values of the others. For example, the conventional asymptotic formulas for the Bessel functions with large argument do not apply when the order is of comparable magnitude.

It has been implied that the perturbation theory for gyroelectric plasmas is applicable provided that the magnetic field is weak in the sense that $\omega_g/\omega \ll 1$. We now explore the question of validity criteria more thoroughly in order to determine how the other parameters, namely the plasma frequency and physical dimensions of the interaction zone affect the convergence of the expansion 4.13.

Dependence on Plasma Frequency

In order to determine the effect of plasma frequency we eliminate $\underline{H}^{(n)}$ in 4.15 and obtain as the equation for $\underline{E}^{(n)}$

$$\nabla \times \nabla \times \underline{E}^{(n)} - k^2 \underline{E}^{(n)} = - k^2 \zeta \sum_{m=0}^{n-1} (\underline{e}_B \times)^{n-m} \underline{E}^{(m)} \quad . \qquad (4.19)$$

Using mathematical induction we can show that the particular solution for the nth order field $(n > 0)$ will be of the form

$$\underline{E}_p^{(n)} = \sum_{m=1}^{n} \zeta^m \underline{f}_m^n \qquad (4.20)$$

where $\zeta = \omega_p^2 / \omega^2 - \omega_p^2$, so that in the expression for $\underline{E}^{(n)}$ there will be a term proportional to ζ^n . This parameter becomes large at frequencies very close to the plasma frequency and we reason that here the convergence of the perturbation expansion will be poor. However, since the original model proposed for a cold plasma is itself inaccurate near the plasma frequency, this limitation should not be regarded as serious.

Effect of Physical Dimensions of the Interaction Zone

Because of the difficulty of attacking a completely general problem we demonstrate the effect of physical dimensions by considering the specific example of a TEM wave propagating in the direction of a static magnetic field. The conclusions which are extrapolated nevertheless seem plausible on physical grounds.

The coordinate system and the unperturbed wave $\underline{E}^{(0)}, \underline{H}^{(0)}$ are indicated in Fig. 2.

The equations for the first order fields are, from 4.15

$$\nabla \times \underline{E}^{(1)} = i\omega\mu_o \underline{H}^{(1)}$$

$$\nabla \times \underline{H}^{(1)} = -i\omega\epsilon \underline{E}^{(1)} + i\omega\epsilon\zeta \underline{e}_B \times \underline{E}^{(0)} \tag{4.21}$$

and setting $\underline{e}_B = \underline{e}_z$ with $\underline{E}^{(0)} = \underline{e}_x E_o \exp(ikz)$, 4.21 becomes

$$\nabla \times \underline{E}^{(1)} = i\omega\mu_o \underline{H}^{(1)}$$

$$\nabla \times \underline{H}^{(1)} = -i\omega\,\epsilon \underline{E}^{(1)} + i\omega\epsilon E_o \zeta\, \underline{e}_y e^{ikz} \tag{4.22}$$

Eliminating $\underline{H}^{(1)}$ we obtain the inhomogeneous equation for $\underline{E}^{(1)}$

$$\nabla^2 \underline{E}^{(1)} + k^2 \underline{E}^{(1)} = \zeta k^2 E_o \underline{e}_y e^{ikz} \tag{4.23}$$

which has the particular solution

$$\underline{E}_p^{(1)} = \underline{e}_y \frac{\zeta E_o kz}{2i} e^{ikz} \quad . \tag{4.24}$$

The corresponding first order change in the electric field is obtained

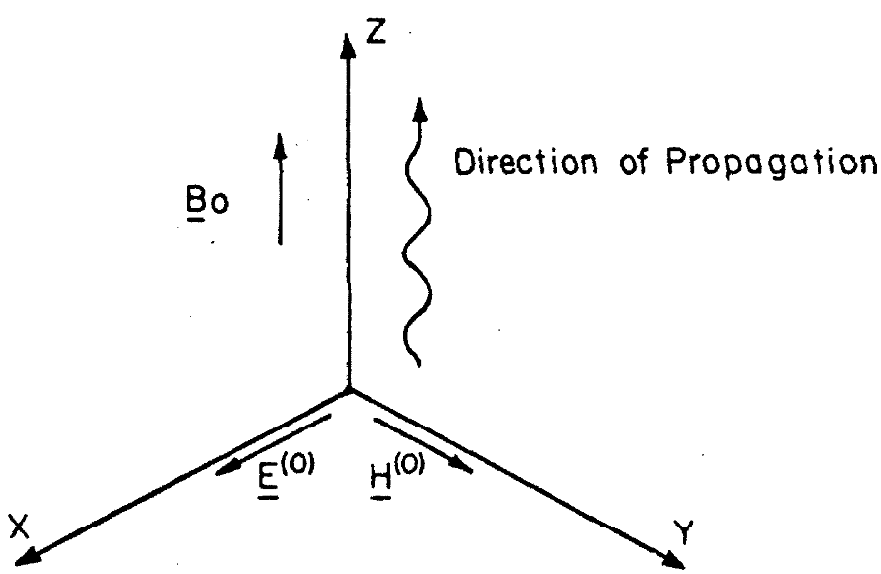

Figure 4.2 Unperturbed TEM wave propagating in the direction of the static magnetic field

after multiplication by $i \frac{\omega_g}{\omega}$ and is given by

$$\underline{e}_y \frac{\omega_g \zeta E_o kz}{2\omega} e^{ikz} \quad . \tag{4.25}$$

This term, proportional to kz, produces a rotation of the electric vector in the transverse plane, an effect known as Faraday rotation.

In a similar manner it may be shown that $\underline{E}^{(2)}$ will have a term proportional to $(kz)^2$, $\underline{E}^{(3)}$ to $(kz)^3$ and so on. Thus the larger the value of kz the more significant will be the higher order terms. Consequently another limitation on the validity of the perturbation procedure has to do with the physical size of the plasma. For a slab the characteristic dimension would be its width, and for cylinders and spheres it would be their respective radii.

On the basis of these results we reason that it is not sufficient that $\omega_g/\omega \ll 1$; the static magnetic field, operating frequency, plasma frequency and physical dimension L must be such that

$$\zeta \frac{\omega_g}{\omega} kL \ll 1 \quad . \tag{4.26}$$

If this inequality is satisfied then the perturbation theory should yield good results.

F. Solution for the First Order Fields

Having established what appear to be sufficient conditions for validity of the perturbation theory, we proceed to determine the general solutions for the first order fields, which satisfy

$$\nabla \times \underline{E}^{(1)} = i\omega\mu_o \underline{H}^{(1)}$$

$$\nabla \times \underline{H}^{(1)} = -i\omega\epsilon \underline{E}^{(1)} + i\omega\epsilon\zeta\, \underline{e}_B \times \underline{E}^{(0)} \tag{4.27}$$

The general solution to 4.27 consists of two parts, a particular integral and a complementary integral. For problems involving bounded plasmas, which are of interest here, both parts have physical significance. The particular solution is the field which arises directly from the interaction between the zero order wave and the static magnetic field. On the other hand the complementary solution which satisfies the homogeneous equations

$$\nabla \times \underline{E}_c^{(1)} = i\omega\mu_o \underline{H}_c^{(1)}$$

$$\nabla \times \underline{H}_c^{(1)} = -i\omega\epsilon\, \underline{E}_c^{(1)}$$

is superimposed in order to satisfy boundary conditions and thus may be interpreted as that part of the particular solution which is reflected from the boundary back into the plasma. Although the required complementary solutions may be obtained by methods already discussed (Section III.A), the particular integral presents a more formidable problem.

We partition the field $\underline{E}_p^{(1)}$ into two parts, one accounting for the divergence and a remainder which is solenoidal. The first is found by using the divergence operator on the second equation of 4.27, from which we find that

$$\nabla \cdot \underline{E}_p^{(1)} = \zeta \, \nabla \cdot \left[\underline{e}_B \times \underline{E}^{(0)} \right] \quad . \tag{4.28}$$

We are led to write that

$$\underline{E}_p^{(1)} = \zeta \, \underline{e}_B \times \underline{E}^{(0)} + \underline{E}_a^{(1)} \tag{4.29}$$

where $\underline{E}_a^{(1)}$ is solenoidal. From substitution of 4.29 into 4.27 and elimination of $\underline{H}^{(1)}$ it is determined that $\underline{E}_a^{(1)}$ satisfies

$$\nabla^2 \underline{E}_a^{(1)} + k^2 \underline{E}_a^{(1)} = -i \, Z_o \zeta k_o^2 (\underline{e}_B \cdot \frac{\nabla}{k_o}) \underline{H}^{(0)} \quad . \tag{4.30}$$

Assume now that the fields for the isotropic problem are known and that, in accordance with Section III.A , $\underline{H}^{(0)}$ is given by

$$\underline{H}^{(0)} = \left[\frac{\nabla}{k_o} \times \underline{u} \, W_1^{(0)} + \frac{\nabla}{k_o} \times (\underline{u} \times \frac{\nabla}{k_o} W_2^{(0)} \right] \tag{4.31}$$

where $\underline{u} = \underline{e}_x, \underline{e}_y, \underline{e}_z$ or $k_o \underline{r}$. Furthermore, let $\underline{E}_a^{(1)}$ be represented as

$$\underline{E}_a^{(1)} = iZ_o \, \zeta (\underline{e}_B \cdot \frac{\nabla}{k_o}) \left[\frac{\nabla}{k_o} \times \underline{u} \, V_1^{(1)} + \frac{\nabla}{k_o} \times (\underline{u} \times \frac{\nabla}{k_o} V_2^{(1)} \right] \tag{4.32}$$

where $V_1^{(1)}$ and $V_2^{(1)}$ are scalar functions. It then follows from the fact that $\underline{e}_B$ is a constant vector, and from the commutative properties of the Laplacian given in Appendix A, that 4.32 will be the solution to 4.30 provided that the scalar functions $V_1^{(1)}$ and $V_2^{(1)}$ are related to the functions $W_1^{(1)}$ and $W_2^{(1)}$ by the differential equations

$$\nabla^2 V_1^{(1)} + k^2 V_1^{(1)} = -k_o^2\, W_1^{(0)}$$

$$\nabla^2 V_2^{(1)} + k^2 V_2^{(1)} = -k_o^2\, W_2^{(0)} \quad . \tag{4.33}$$

Combining these results we determine that the particular solution to 4.27 is

$$\underline{E}_p^{(1)} = \zeta \left\{ \underline{e}_B \times \underline{E}^{(0)} + iZ_o (\underline{e}_B \cdot \frac{\nabla}{k_o}) \left[\frac{\nabla}{k_o} \times (\underline{u}\, V_1^{(1)}) + \frac{\nabla}{k_o} \times (\underline{u} \times \frac{\nabla}{k_o}\, V_2^{(1)}) \right] \right\} \tag{4.34}$$

where $\underline{E}^{(0)}$ is the zero order electric field and $V_1^{(1)}$ and $V_2^{(1)}$ are derived from the generating functions for the zero order magnetic field as prescribed by 4.33.

It is important to note that the function $V_1^{(1)}$ generates a TE wave while $V_2^{(1)}$ generates a TM wave, the opposite from their respective zero order counterparts $W_1^{(0)}$ and $W_2^{(0)}$. This shows very clearly why TE and TM modes alone cannot, in general, be solutions to Maxwell's equations for a gyroelectric plasma; a TE mode will "interact" with $\underline{B}_o$ to produce a TM mode and vice versa. The physical origin of this interaction is of course the effect of the static magnetic field on the motion of the electrons, but to examine the field structure on the basis of the individual orbits would be prohibitively complicated. The macroscopic, or phenomenological, approach is a convenient alternative.

An example worth mentioning at this point is the case of a plane TE wave normally incident on a longitudinally magnetized column. Here $\underline{e}_B \cdot \nabla \equiv 0$ and $\underline{e}_B \times \underline{E}^{(0)}$ is in the transverse plane so that the TE character of the incident wave is retained throughout. This problem was discussed in the previous chapter.

G. Summary and Conclusions

The purpose of this chapter has been to present a theory of microwave interaction with gyroelectric plasmas in which the biasing magnetic field is regarded as a perturbation. In this approach we represent the electromagnetic field as being made up of a sum of partial fields arranged in order of increasing complexity. The fields of lower order are presumed to interact with the static magnetic field to produce those of higher order. Such a formulation has the advantage that under suitable conditions the more complex fields may be neglected. These conditions are that the ratio ω_g/ω be less than unity and that the other physical parameters, i.e., the plasma frequency and characteristic dimension L be such that the inequality

$$\left| \frac{\omega_g \, \omega_p^2 \, kL}{\omega(\omega^2 - \omega_p^2)} \right| \ll 1$$

be satisfied. Together they are equivalent to the physical requirement that the static magnetic field have a relatively small effect, i.e., that the additional fields which result from the gyroelectric character of the plasma be small compared to those which would exist if the medium were isotropic. This is to be expected since, if such were not the case, the static magnetic field could hardly be regarded as a perturbation.

By solving formally for the first order fields we find that TE and TM modes are not, in general, solutions to Maxwell's equations for a gyroelectric medium. The physical reason for this is that the effect of the static magnetic field on the electron motion induced by a wave of one type will be such as to produce a wave of the other type.

The application of first order theory to cylindrical and spherical problems will constitute the remainder of this report.

V. FIRST ORDER THEORY OF CYLINDRICAL SYSTEMS

A. Introduction

In this chapter we will apply first order perturbation theory to cylindrical plasmas with circular cross section. Such examples might arise, for example, in geophysics where one is studying radar echoes from meteor trails in the earth's magnetic field. Also, as was mentioned earlier, applying an axial magnetic field to a laboratory plasma and observing its effects may have value as a diagnostic technique.

Of the present class of problems only the one where the magnetic field is along the axis of the cylinder is amenable to exact analysis. This is because for this example there is only one "preferred" direction, i.e., the vectors $\underline{u}$ and $\underline{e}_B$ are both equal to $\underline{e}_z$ and furthermore the solution has exponential dependence in the z direction. The resulting equations are relatively simple and the problem may be scalarized. However, the situation becomes considerably more complicated when the static magnetic field is off the axis and we must resort in such a case to an approximate method.

B. Geometry of the Problem

Fig. 1 gives the coordinate system and the orientation of the plasma cylinder and static magnetic field $\underline{B}_o$. It is assumed that the vector $\underline{e}_B$ has direction cosines ℓ_x, ℓ_y and ℓ_z with

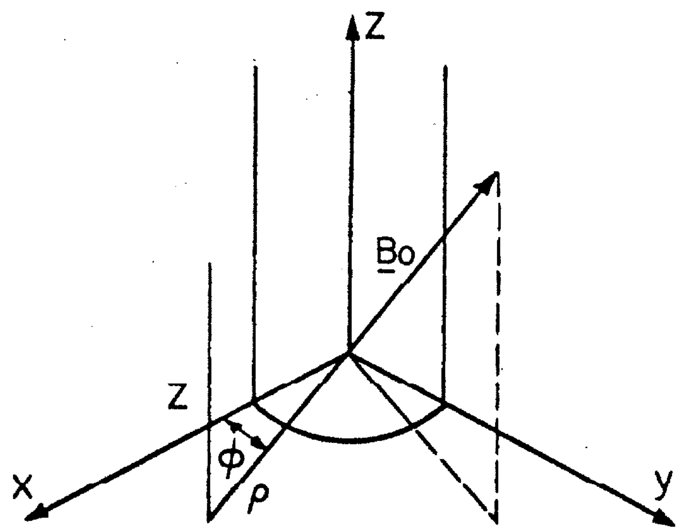

Figure 5.1 Cylindrical geometry with arbitrarily directed static magnetic field

the respective coordinate axes, so that

$$\underline{e}_B = \ell_x \underline{e}_x + \ell_y \underline{e}_y + \ell_z \underline{e}_z \tag{5.1}$$

or, in terms of the cylindrical coordinates ρ, ϕ, z

$$\underline{e}_B = (\ell_- e^{i\phi} + \ell_+ e^{-i\phi})\underline{e}_\rho + i(\ell_- e^{i\phi} - \ell_+ e^{-i\phi})\underline{e}_\phi + \ell_z \underline{e}_z \tag{5.2}$$

where

$$\ell_\pm = \ell_x \pm i\ell_y \quad . \tag{5.3}$$

The reason for this designation will become clear later on. In addition we suppose that the dependence on z is again $e^{i\gamma k_o z}$ so that the operator $\underline{e}_B \cdot \nabla/k_o$ is given by

$$\underline{e}_B \cdot \frac{\nabla}{k_o} = (\ell_- e^{i\phi} + \ell_+ e^{-i\phi})\frac{\partial}{\partial(k_o\rho)} + \frac{i(\ell_- e^{i\phi} - \ell_+ e^{-i\phi})}{k_o\rho}\frac{\partial}{\partial\phi} + i\gamma \quad . \tag{5.4}$$

C. The Expressions for the First Order Fields

As was mentioned earlier, the first order fields can be determined only after the isotropic problem is solved. Assume that this has been done (See Appendix C for an analysis of scattering by an isotropic cylinder) and that the zero order electric field is given by

$$\underline{E}^{(0)} = iZ_o\eta\left[\kappa\, \underline{e}_z x \frac{\nabla t}{k_o} W_2^{(0)} + \frac{i\gamma}{k_o} \nabla_t W_1^{(0)} + \underline{e}_z \nu^2\, W_1^{(0)}\right] \tag{5.5}$$

where

$$W_1^{(0)} = \sum_{n=-\infty}^{+\infty} a_n^{(0)}\, J_n(\beta\rho)\, F_n(\phi,z) \tag{5.6}$$

$$W_2^{(0)} = \sum_{n=-\infty}^{+\infty} b_n^{(0)}\, J_n(\beta\rho)\, F_n(\phi,z) \quad .$$

<u>Calculation of the First Order Generating Functions $V_{1,2}^{(1)}$</u>

The first order generating functions $V_1^{(1)}$ and $V_2^{(1)}$ from which we determine the field $\underline{E}_a^{(1)}$ are the particular solutions to the differential equations

$$\nabla^2 V_1^{(1)} + k^2 V_1^{(1)} = -k_o^2\, W_1^{(0)} \tag{4.33}$$

$$\nabla^2 V_2^{(1)} + k^2 V_2^{(1)} = -k_o^2\, W_2^{(0)} \quad .$$

Since both $W_1^{(0)}$ and $W_2^{(0)}$ have the same form, given by 5.6, it is sufficient to solve

$$\nabla^2 u_n + k^2 u_n = -k_o^2\, J_n(\beta\rho) e^{in\phi + i\gamma k_o z} \quad . \tag{5.7}$$

It may be shown that the ϕ and z dependence of u_n is the same as on the right hand side, so we represent the solution as

$$u_n(\beta\rho,\phi,z) = R_n(\beta\rho)\, e^{in\phi + i\gamma k_o z} \quad . \tag{5.8}$$

Substituting 5.8 into 5.7 we determine that $R_n(\beta\rho)$ must satisfy the ordinary differential equation

$$v^2 \frac{d^2R_n(v)}{dv^2} + v\,\frac{dR_n(v)}{dv} + (v^2 - n^2)\, R_n(v) = -\frac{v^2}{\nu^2}\, J_n(v) \tag{5.9}$$

where $v = \beta\rho$. The above is an inhomogeneous form of Bessel's equation and is examined in Appendix D, where it is shown that the solution is

$$R_n(v) = \frac{v}{2\nu^2}\, J'_n(v) = \frac{\beta\rho}{2\nu^2}\, J'_n(\beta\rho) \quad . \tag{5.10}$$

It follows that the expressions for $V_1^{(1)}$ and $V_2^{(1)}$ are

$$V_1^{(1)} = \frac{\beta\rho}{2\nu^2} \sum_{n=-\infty}^{+\infty} a_n^{(0)}\, J'_n(\beta\rho)\, F_n(\phi,z) \tag{5.11}$$

$$V_2^{(1)} = \frac{\beta\rho}{2\nu^2} \sum_{n=-\infty}^{+\infty} b_n^{(0)}\, J'_n(\beta\rho)\, F_n(\phi,z) \quad .$$

We next consider two specific examples designed to illustrate the application of these results. They are (1) scattering by an axially magnetized cylinder and (2) scattering of a normally incident wave by a circular cylinder with the static magnetic field perpendicular to the axis.

D. Scattering by an Axially Magnetized Cylinder

In order to ascertain the validity of the perturbation procedure, we consider once again the problem of scattering by an axially magnetized cylinder with circular cross section. As an application of the approximate method this example turns out to be particularly simple, since $\underline{e}_B = \underline{e}_z$ and therefore

$$\underline{e}_B \cdot \frac{\nabla}{k_o} = i\gamma \quad . \tag{5.12}$$

The particular solutions to the first order equations 4.22 may be obtained by applying 4.34 and Maxwell's first equation. The results for the ϕ and z components are, in terms of the generating functions,

$$E_{\phi p}^{(1)} = -iZ_o\zeta \left[\frac{1}{k_o\rho} \frac{\partial}{\partial\phi} (W_2^{(0)} - \gamma^2 V_2^{(1)}) - i\gamma \frac{\partial}{\partial(k_o\rho)} (\eta W_1^{(0)} - V_1^{(1)}) \right] .$$

$$E_{zp}^{(1)} = -iZ_o\zeta \left[i\gamma(W_2^{(0)} + \nu^2 V_2^{(1)} \right] \tag{5.13}$$

$$H^{(1)}_{\phi p} = \zeta\left[\frac{\gamma^2}{k_o\rho}\frac{\partial}{\partial\phi}(\eta W^{(0)}_1 - V^{(1)}_1) + \frac{i\gamma}{\eta}\frac{\partial V^{(1)}_2}{\partial(k_o\rho)}\right]$$

$$H^{(1)}_{zp} = i\gamma\zeta\left[\nu^2 V^{(1)}_1 + (1-\eta\nu^2)W^{(0)}_1\right] \quad . \tag{5.14}$$

Applying 5.6 and 5.11 we find that

$$ {}_nE^{(1)}_{\phi p} = iZ_o\zeta\left\{\frac{in}{k_o\rho}\left[J_n(\beta\rho) - \frac{\gamma^2 k_o\rho}{2\nu}J'_n(\beta\rho)\right]b^{(0)}_n - \left[i\gamma\nu\eta\, J'_n(\beta\rho) + \frac{i\gamma k_o\rho}{2}\left(1-\frac{n^2}{(\beta\rho)^2}\right)J_n(\beta\rho)\right]a^{(0)}_n\right\}F_n(\phi,z) \tag{5.15}$$

$$ {}_nE^{(1)}_{zp} = -iZ_o\zeta\left\{i\gamma\left[J_n(\beta\rho) + \frac{\beta\rho}{2}J'_n(\beta\rho)\right]b^{(0)}_n\right\}F_n(\phi,z)$$

$$ {}_nH^{(1)}_{zp} = \zeta\left\{\frac{in\gamma^2}{k_o\rho}\left[\eta J_n(\beta\rho) - \frac{k_o\rho}{2\nu}J'_n(\beta\rho)\right]a^{(0)}_n - \frac{i\gamma k_o\rho}{2\eta}\left[\left(1-\frac{n^2}{(\beta\rho)^2}\right)J_n(\beta\rho)\right]b^{(0)}_n\right\}F_n$$

$$ {}_nH^{(1)}_{zp} = i\gamma\zeta\left\{\left[\frac{\beta\rho}{2}J'_n(\beta\rho) + (1-\eta\nu^2)J_n(\beta\rho)\right]a^{(0)}_n\right\}F_n \quad . \tag{5.16}$$

Following what was said in Section IV.C we assume that, due to the boundary, the fields corresponding to the particular solution excite additional waves inside and outside the plasma. These secondary fields satisfy the homogeneous forms of Maxwell's equations in the respective media and so are derived from

$$
\begin{aligned}
{}^{(tr)}W_1^{(1)} &= \sum a_n^{(1)} J_n(\beta\rho)F_n \\
{}^{(tr)}W_2^{(1)} &= \sum b_n^{(1)} J_n(\beta\rho)F_n
\end{aligned}
\tag{5.17}
$$

and

$$
\begin{aligned}
{}^{(sc)}W_1^{(1)} &= \sum c_n^{(1)} H_n(k_o\rho \cos\theta)F_n \\
{}^{(sc)}W_2^{(1)} &= \sum d_n^{(1)} H_n(k_o\rho \cos\theta)F_n
\end{aligned}
\tag{5.18}
$$

as prescribed by 3.8 and 3.11. The unknown coefficients are determined by invoking the conditions that the ϕ and z components of $\underline{E}^{(1)}$ and $\underline{H}^{(1)}$ be continuous at the surface of the cylinder.

Under the approximation

$$
\begin{aligned}
W_1^{(sc)} &= {}^{(sc)}W_1^{(0)} + i\,\frac{\omega_g}{\omega}\,({}^{(sc)}W_1^{(1)}) \\
W_2^{(sc)} &= {}^{(sc)}W_2^{(0)} + i\,\frac{\omega_g}{\omega}\,({}^{(sc)}W_2^{(1)})
\end{aligned}
\tag{5.19}
$$

we may proceed to calculate the first order effect of $\underline{B}_o$ on the polarization of the scattered field. The calculation was done, again by machine, yielding the results shown in Fig. 2. For purposes of

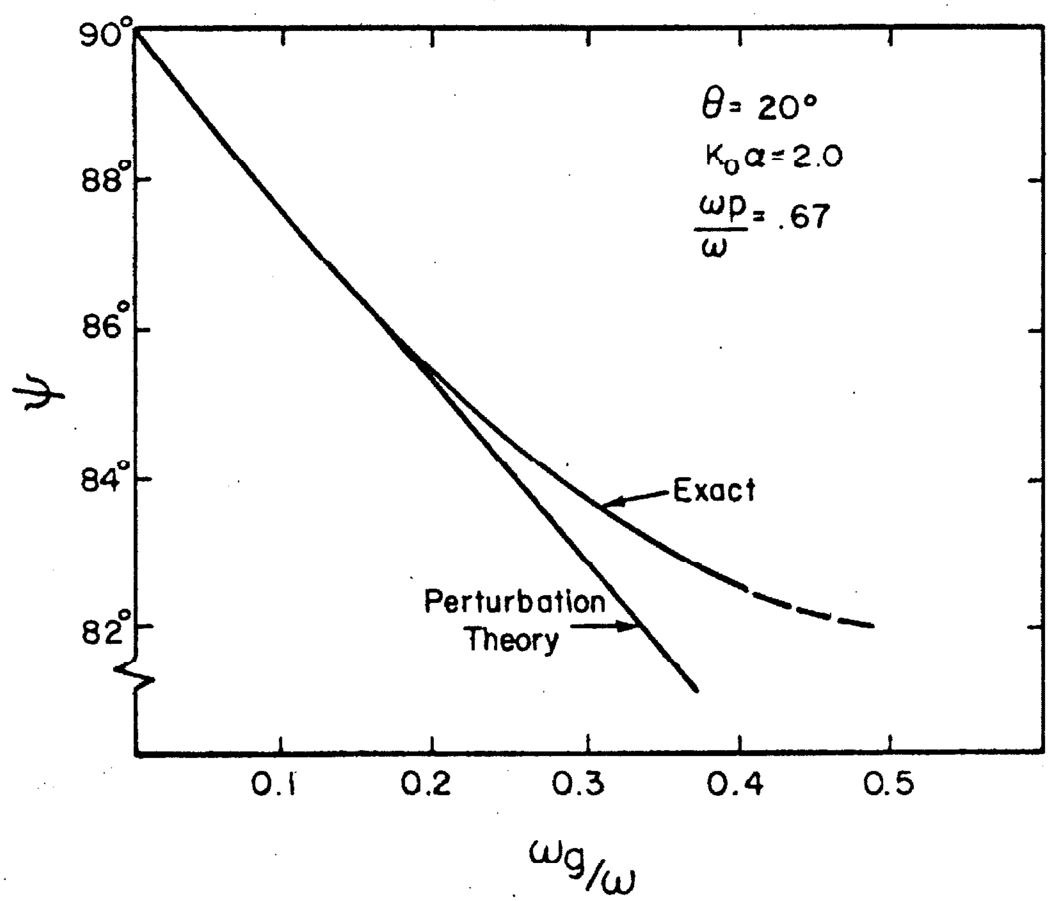

Figure 5.2 Orientation angle, ψ , of the polarization ellipse. Comparison between exact and first order results.

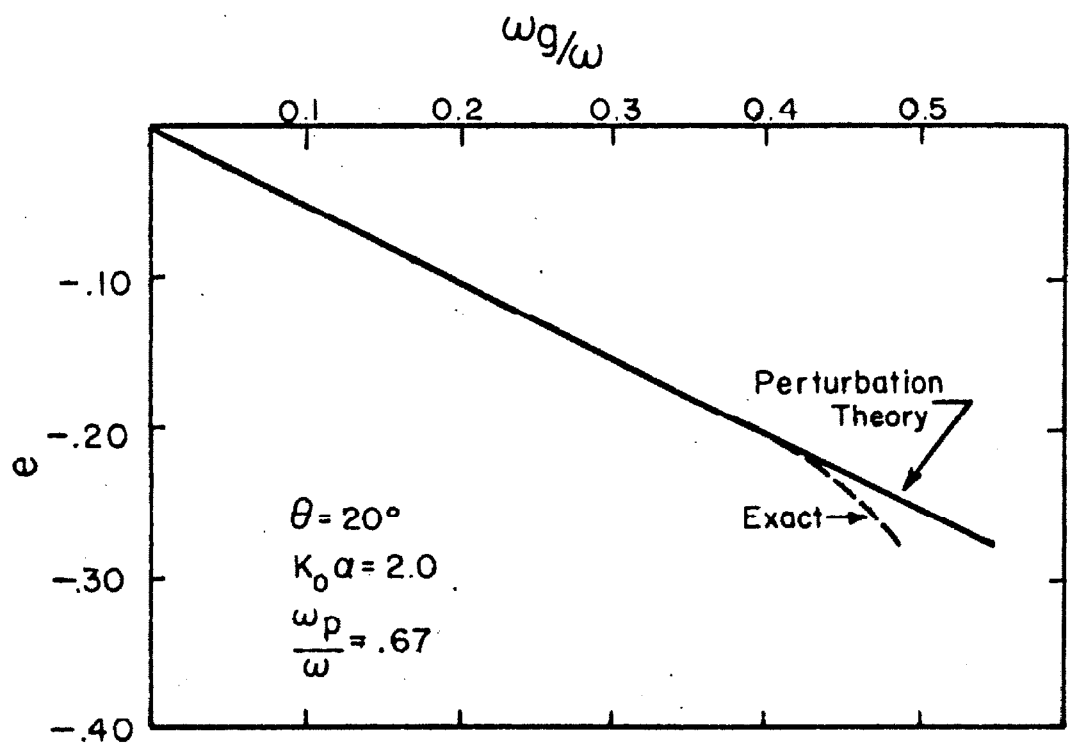

Figure 5.3 Eccentricity, e , of the polarization ellipse. Comparison between exact and first order results.

comparison, the results of the exact theory, also computed for relatively small values of magnetic field, are also indicated. As anticipated the perturbation theory correctly predicts the linear portions of the curves.

Admittedly, the use of a specific example to demonstrate the validity of the perturbation theory does not constitute a mathematical proof. The use of a vector problem, however, does make the method plausible. A rigorous analysis would require a series expansion of the exact solution in powers of $i \frac{\omega_g}{\omega}$ and subsequent comparison between the coefficient of the linear term and the first order solution calculated via the approximate method. Unfortunately, the complexity of the equations for arbitrary angle of incidence makes such a task impractical for the general case. For normal incidence, however, the algebra is manageable and, as shown in Appendix E, the series expansion and perturbation methods give the same first order results.

E. An Example Involving Off-Axis Magnetization

In order to demonstrate the effect of a static magnetic field which is not aligned with the axis of the cylinder, we consider the problem of scattering of a normally incident TM wave in which $\underline{e}_B = \underline{e}_x$.

In the absence of a magnetic field the TM character of the incident wave is retained throughout. Hence the field in the plasma, to zero order, is derived from

$$(tr)_{W_1^{(0)}} = \sum_{n=-\infty}^{+\infty} a_n^{(0)} J_n(k\rho)e^{in\phi}$$

$$(tr)_{W_2^{(0)}} \equiv 0 \quad . \tag{5.20}$$

The first order generating functions for the particular solution will then be, from 5.11

$$V_1^{(1)} = \frac{k\rho}{2\nu^2} \sum_{n=-\infty}^{+\infty} a_n^{(0)} J_n'(k\rho)e^{in\phi}$$

$$V_2^{(1)} \equiv 0 \tag{5.21}$$

from which we derive the solenoidal field

$$\underline{E}_a^{(1)} = iZ_o \zeta \left(\underline{e}_B \cdot \frac{\nabla}{k_o}\right) \left[\frac{1}{k_o} \nabla \times \left(\underline{e}_z V_1^{(1)}\right)\right] . \tag{5.22}$$

Alternatively,

$$E_a^{(1)} = -iZ_o \zeta \, \underline{e}_z \times \left[\frac{\nabla}{k_o} \left(\underline{e}_B \cdot \frac{\nabla}{k_o} V_1^{(1)}\right)\right] . \tag{5.23}$$

5.23 describes a TE wave whose apparent generating function is proportional to the expression

$$\left(\underline{e}_B \cdot \frac{\nabla}{k_o}\right) V_1^{(1)} \tag{5.24}$$

For the present example, $\underline{e}_B = \underline{e}_x$ and $\gamma = 0$ so that from 5.4

$$\underline{e}_B \cdot \frac{\nabla}{k_o} = (e^{i\phi} + e^{-i\phi}) \frac{\partial}{\partial(k_o\rho)} + i \, \frac{e^{i\phi} - e^{-i\phi}}{k_o\rho} \, \frac{\partial}{\partial\phi} \quad . \tag{5.25}$$

Combining 5.25 and 5.21 we obtain the result

$$(\underline{e}_B \cdot \frac{\nabla}{k_o}) V_1^{(1)} = \frac{1}{2\nu} \sum_{n=-\infty}^{+\infty} G_n(k\rho) e^{in\phi} \tag{5.26}$$

where

$$G_n(k\rho) = \frac{(k\rho)^2 - n(n-1)}{k\rho} \, a_{n-1}^{(0)} \, J_{n-1}(k\rho) + \frac{(k\rho)^2 - n(n+1)}{k\rho} \, a_{n+1}^{(0)} \, J_{n+1}(k\rho) \quad . \tag{5.27}$$

Thus, in addition to the coupling between TE and TM modes, an off-axis magnetic field causes an interplay between eigenfunctions of different order. This interplay results from the terms involving ℓ_+ and ℓ_- in 5.4, the former coupling to eigenfunctions of order $n+1$ and the latter to eigenfunctions of order $n-1$. The physical explanation of this behavior is in the fact that an off-axis magnetic field destroys the cylindrical symmetry of the structure. The existence of eigenfunctions whose angular dependence is $\exp(in\phi)$ is due to an invariance under rotation by an angle $2\pi/n$ and when the symmetry is removed, as in the present example, these functions are no longer appropriate.

VI. FIRST ORDER THEORY OF SPHERICAL SYSTEMS

A. Introduction

As the final application of perturbation theory, we consider the effect of a weak magnetic field on spherical waves. It appears that regardless of the relative orientation such a problem is not amenable to exact analysis.

Since the introduction of a preferred (rectangular) direction into a spherical problem destroys the original symmetry, we expect that in addition to coupling between TE and TM modes, there will be interaction among the eigenfunctions of different order such as occurred in the case of the cylinder with off-axis magnetization. This is in fact the case, as will be demonstrated later on.

The geometry is as indicated in Fig. 1 with the coordinate system oriented so that $\underline{B}_o$ is in the z direction.

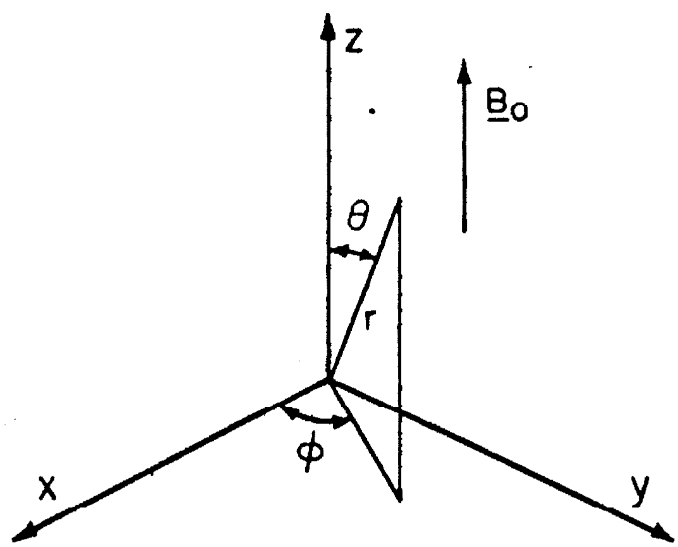

Figure 6.1 Spherical coordinates

B. The First Order Solution

In accordance with Section III.A we assume that the unperturbed alternating field inside the plasma may be derived from a set of functions $W_1^{(0)}$ and $W_2^{(0)}$ which satisfy the scalar Helmholtz equation. The zero order magnetic field is then given by

$$ {}^{i}\underline{H}^{(0)} = \frac{1}{k_o} \nabla \times (\underline{u}^{i} W_1^{(0)}) + \frac{1}{k_o^2} \nabla \times \left[\underline{u} \times \nabla^{i} W_2^{(0)} \right] \tag{6.1} $$

where $\underline{u} = k_o \underline{r}$.

In spherical coordinates these solutions take the form

$$ {}^{i}W_1^{(0)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} a_{nm}^{(0)} \; z_n(kr) \; P_n^m(\cos \theta) e^{im\phi} \tag{6.2} $$

$$ {}^{i}W_2^{(0)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} b_{nm}^{(0)} \; z_n(kr) \; P_n^m(\cos \theta) e^{im\phi} $$

where $z_n(kr)$ denotes any spherical Bessel function and $P_n^m(\cos \theta)$ an associated Legendre polynomial. As prescribed by the theory in Chapter IV we solve now for $V_1^{(1)}, V_2^{(1)}$. If, for example, we represent $V_1^{(1)}$ in the form

$$ V_1^{(1)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm}^{(0)} \; u_n^m(r,\theta,\phi) \tag{6.3} $$

then we are led to solve the equation

$$\nabla^2 u_n^m + k^2 u_n^m = -k_o^2 z_n(kr)\; P_n^m(\cos\Theta) e^{im\phi} \tag{6.4}$$

Similar results follow for $V_2^{(1)}$. 6.4 becomes, after substitution of the expression for the Laplacian in spherical coordinates,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u_n^m}{\partial r}\right) + \frac{1}{r^2 \sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta \frac{\partial u_n^m}{\partial\Theta}\right) + \frac{1}{r^2\sin^2\Theta}\frac{\partial^2 u_n^m}{\partial\phi^2} + k^2 u_n^m =$$

$$= -k_o^2 z_n(kr)\; P_n^m e^{im\phi} \tag{6.5}$$

Assume now that the Θ and ϕ dependence is the same as on the right so that

$$u_n^m(kr,\Theta,\phi) = R_n(kr)\; S_n^m(\Theta,\phi) \tag{6.6}$$

where

$$S_n^m(\Theta,\phi) = P_n^m(\cos\Theta) e^{im\phi} \tag{6.7}$$

Using the eigenfunction identity

$$\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial S_n^m}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2 S_n^m}{\partial\phi^2} = -n(n+1)S_n^m \tag{6.8}$$

we determine that R_n must satisfy the ordinary differential equation

$$v^2\frac{d^2R_n(v)}{dv^2} + 2v\frac{dR_n}{dv} + \left[v^2 - n(n+1)\right] R_n(v) = -\frac{v^2}{\nu^2} z_n(v) \tag{6.9}$$

with $v = kr$ and $\nu = k/k_o$. We next make the substitutions

$$R_n(v) = \sqrt{\frac{\pi}{2v}}\; G_n(v) \tag{6.10}$$

$$z_n(v) = \sqrt{\frac{\pi}{2v}}\, Z_{n+\frac{1}{2}}(v) \tag{6.11}$$

with 6.11 being merely the definition of the spherical Bessel function in terms of its half integral cylindrical counterpart. By substituting 6.10 and 6.11 into 6.9 we find that the function $G_n(v)$ must satisfy

$$v^2 \frac{d^2 G_n}{dv^2} + v \frac{dG_n}{dv} + \left[v^2 - (n + \tfrac{1}{2})^2 \right] G_n(v) = - \frac{v^2}{\nu^2} Z_{n+\frac{1}{2}}(v) \tag{6.12}$$

This is the same result as was obtained in the previous chapter, equation 5.9, except that the Bessel functions are now of half integral order. However, the analysis given in the appendix still applies and we write, by inspection,

$$G_n(v) = \frac{kr}{2\nu^2} Z'_{n+\frac{1}{2}}(kr) \tag{6.13}$$

and consequently, using 6.13, 6.10 and 6.6,

$$u_n^m(kr,\theta,\phi) = \frac{kr}{2\nu^2} \sqrt{\frac{\pi}{2kr}}\; Z'_{n+\frac{1}{2}}(kr)\; S_n^m(\theta,\phi) \tag{6.14}$$

C. An Application of the First Order Theory

One problem to which the first order theory might be applied is that of radiation by a celestial plasma. Investigators have found that the polarization of the electromagnetic waves emitted from these bodies has a different orientation at radio frequencies than at optical frequencies and have attributed this difference to a static magnetic field existing within the source. This explanation is plausible since at optical frequencies we would expect that gyro-electric effects are negligible while in the radio wave region the terms accounting for the change in polarization are much larger. In this section we discuss this phenomenon quantitatively.

Unfortunately, the physical models for celestial plasmas are themselves controversial. On grounds of expediency we shall use the crude model shown in Fig. 2 which, despite its simplicity, exhibits the desired effect. It is assumed that the plasma is spherical in shape and has properties which are independent of the angle ϕ . Furthermore, the magnetic field, which lies in the z direction, is taken to be uniform inside the body with the return lines somehow passing outside the source. This configuration would approximate a field set up by currents flowing in the ϕ direction.

In order to account for the emitted radiation we imagine that there is a distribution of electric and magnetic multipoles located at the center of the sphere. In this discussion we shall be concerned with the magnetic multipoles; analysis of the other type is similar.

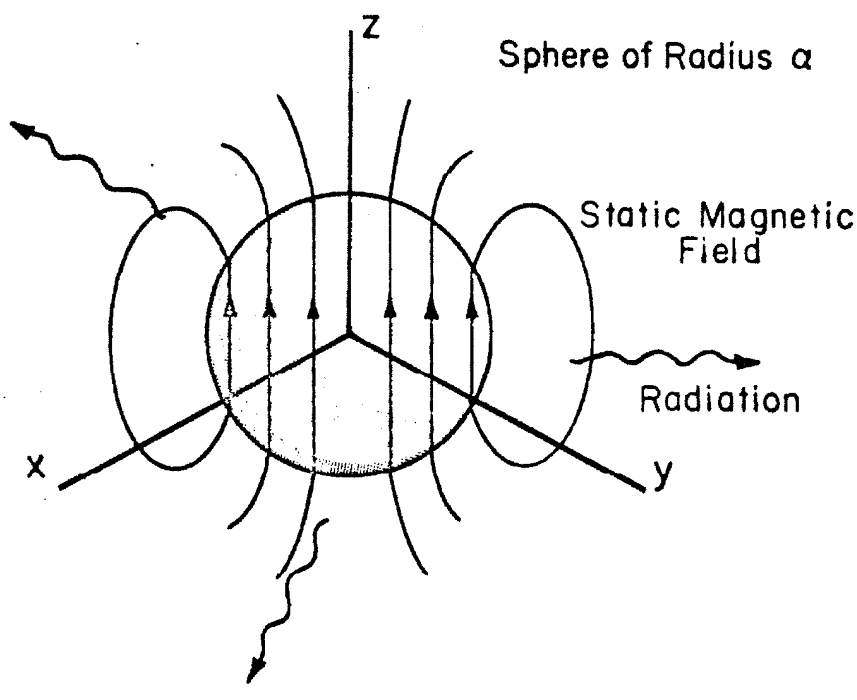

Figure 6.2 Model for celestial radio source

The field emitted by a magnetic source is derived from the scalar function

$$^{s}W_2^{(0)} = h_n(kr)\, P_n(\cos\theta) \tag{6.15}$$

where $h_n(kr)$ is a spherical Hankel function of the first kind, used to give outgoing waves under the assumed time dependence $\exp(-i\omega t)$. The components of $\underline{E}$ and $\underline{H}$ are derived from 6.1 and Maxwell's equations and are

$$^{s}H_r^{(0)} = -\frac{n(n+1)}{k_o r}\, h_n(kr)\, P_n(\cos\theta)$$

$$^{s}H_\theta^{(0)} = -\left[\nu h_n'(kr) + \frac{1}{k_o r}\, h_n(kr)\right] \frac{dP_n(\cos\theta)}{d\theta}$$

$$^{s}E_\phi^{(0)} = iZ_o h_n(kr)\, \frac{dP_n}{d\theta} \tag{6.16}$$

Due to the interface at $r = a$ there will be an additional internal field due to reflection and an external field due to transmission at the boundary. These may be derived from, respectively,

$$^{i}W_2^{(0)} = b_n^{(0)}\, j_n(kr)\, P_n(\cos\theta)$$

$$^{e}W_2^{(0)} = d_n^{(0)}\, h_n(k_o r)\, P_n(\cos\theta) \tag{6.17}$$

and have corresponding field components

$$^{i}H_r^{(0)} = -b_n^{(0)} \frac{n(n+1)}{k_o r} j_n(kr) P_n(\cos\theta)$$

$$^{i}H_\theta^{(0)} = -b_n^{(0)} \left[\nu j_n'(kr) + \frac{1}{k_o r} j_n(kr) \right] \frac{dP_n}{d\theta}$$

$$^{i}E_\phi^{(0)} = iZ_o b_n^{(0)} j_n(kr) \frac{dP_n}{d\theta} \qquad (6.18)$$

$$^{e}H_r^{(0)} = -d_n^{(0)} \frac{n(n+1)}{k_o r} h_n(k_o r) P_n(\cos\theta)$$

$$^{e}H_\theta^{(0)} = -d_n^{(0)} \left[h_n'(k_o r) + \frac{1}{k_o r} h_n(k_o r) \right] \frac{dP_n}{d\theta}$$

$$^{e}E_\phi^{(0)} = iZ_o d_n^{(0)} h_n(k_o r) \frac{dP_n}{d\theta} \qquad (6.19)$$

From the conditions that E_ϕ and H_θ be continuous across the surface of the sphere, we may solve for the constants $b_n^{(0)}$ and $d_n^{(0)}$. The results are

$$b_n^{(0)} = \frac{\nu h_n'(ka)\, h_n(k_o a) - h_n(ka)\, h_n'(k_o a)}{h_n'(k_o a)\, j_n(ka) - \nu h_n(k_o a)\, j_n'(ka)}$$

$$d_n^{(0)} = \frac{-i\nu}{(ka)^2 \left[\nu j_n'(ka)\, h_n(k_o a) - h_n'(k_o a) j_n(ka)\right]} \qquad (6.20)$$

In order to determine the first order effect of $\underline{B}_o$ on the field inside the plasma we employ 4.34 and 6.13. After considerable manipulation we can show that the first order particular solution has the form

$$E_{p\Theta}^{(1)} = iZ_o \left\{ Q_{n-1}(kr) \frac{dP_{n-1}}{d\Theta} + Q_{n+1}(kr) \frac{dP_{n+1}}{d\Theta} \right\}$$

$$E_{\phi p}^{(1)} = 0$$

$$H_{\Theta p}^{(1)} = 0$$

$$H_{\phi p}^{(1)} = Y_{n-1}(kr) \frac{dP_{n-1}}{d\Theta} + Y_{n+1} \frac{dP_{n+1}}{d\Theta} \qquad (6.21)$$

Equation 6.21 describes the field due to a combination of n-1 and n+1 order multipoles of the electric, rather than the magnetic type with which we began. Although the radial functions are different from those usually obtained, the solutions which we would assume to be transmitted outside the sphere, in order to satisfy the boundary conditions, would be those corresponding to conventional electric dipoles. We conclude that the effect of $\underline{B}_o$, as viewed by an observer outside the sphere, is to produce a field different from the original one in both character (magnetic $\rightleftarrows$ electric) and order. This fact explains the previously mentioned changes in the polarization of the radiated field since the addition of a component perpendicular to the original electric field changes the properties of the polarization ellipse.

Since the general forms of the radial functions in 6.21 are very complicated, we shall specialize to the case of n = 1 (dipole). Furthermore, it will be assumed that the radius of the sphere is much larger than the wavelength of the radiation. To insure the validity of the perturbation theory, it must also be that the magnetic field is very small and that the frequency is well above the plasma frequency. It seems appropriate at this point to present some numerical values in order to justify these assumptions. Consider the case of the Crab Nebula. Typical parameters are (6)

$$\omega = 6.3 \times 10^{10} \text{ sec}^{-1} \quad (3 \text{ cm. waves})$$

$$\omega_p^2 = .3 \times 10^{10} \text{ sec}^{-2} \quad (N \approx 1 \text{ cm}^{-3})$$

$$\omega_g = 1.8 \times 10^3 \text{ sec}^{-1} \quad (B_o \approx 10^{-8} \text{ webers/m}^2)$$

$$L \approx 5 \times 10^{14} \text{ m}$$

and we calculate for the smallness parameter

$$\left| \frac{\omega_g}{\omega} \frac{\omega_p^2}{\omega^2 - \omega_p^2} kL \right| \approx .002 \quad .$$

On the basis of these figures, the first order theory should apply.

Letting $n = 1$ and employing the asymptotic formulas for the spherical Bessel and Hankel functions, we find that the first order particular solutions near the surface of the sphere are

$$\underline{E}_p^{(1)} \sim \underline{e}_\theta \left\{ \frac{-iZ_o \zeta}{6} \left[ie^{ikr} - b_1^{(0)} \sin kr \right] \right\} \frac{dP_2}{d\theta} \qquad (6.22)$$

$$\underline{H}_p^{(1)} \sim \underline{e}_\phi \left\{ \frac{\zeta \nu}{6} \left[e^{ikr} + b_1^{(0)} \cos kr \right] \right\} \frac{dP_2}{d\theta} \quad .$$

In order to satisfy boundary conditions we assume an additional internal and radiated field derived from, respectively,

$$ {}^{i}W_1^{(1)} = a_2^{(1)} \; j_2(kr) \; P_2(\cos\Theta) $$

$$ {}^{e}W_1^{(1)} = c_2^{(1)} \; h_2(k_o r) \; P_2(\cos\Theta) \qquad (6.23) $$

The coefficients $a_2^{(1)}$ and $c_2^{(1)}$ are determined, as usual, by requiring that E_Θ and H_ϕ be continuous at $r = a$. The expression for $c_2^{(1)}$ which determines the perturbation in the radiated field, is found to be, in the limit $ka \to \infty$

$$ c_2^{(1)} \sim \frac{i\zeta ka}{6\nu} e^{i(2k - k_o)a} \left\{ \left[\nu \cos ka - i \sin ka \right] + i(\nu - 1)e^{ika} \left[\frac{\nu \cos^2 ka + \sin^2 ka}{i \cos ka + \nu \sin ka} \right] \right\} . \qquad (6.24) $$

A further simplification is possible if we assume that the frequency of interest is much higher than the plasma frequency of the body. The figures previously given indicate that this is a valid approximation. Expanding the formulas of 6.24 in powers of the ratio ω_p^2/ω^2 we obtain the result that

$$ c_2^{(1)} \sim \frac{ik_o a \, \omega_p^2}{6\omega^2} e^{-\frac{i}{2} \frac{\omega_p^2 k_o a}{\omega^2}} . \qquad (6.25) $$

To the same degree of approximation, $d_1^{(0)} = 1$ so that the total far zone fields are, to first order in the parameters ω_p^2/ω^2 and ω_g/ω ,

$$\underline{H}^e = \underline{e}_\Theta \left[\frac{e^{ik_o r}}{ik_o r} \sin\Theta \right] + \underline{e}_\phi \left[\frac{k_o a \omega_g \omega_p^2}{4\omega^3 i} \frac{e^{ik_o(r - \frac{\omega_p^2 a}{2\omega^2})}}{k_o r} \sin 2\Theta \right] \tag{6.26}$$

$$\underline{E}^e = -Z_o \underline{e}_r \times \underline{H}^e \tag{6.27}$$

from which the properties of the polarization ellipse may be determined. An interesting feature of this result is the dependence of the first order field on the angle Θ , which is of the form $\sin 2\Theta$ as compared with $\sin\Theta$ for the zero order field. Analogous results were obtained by Kuehl (13) who considered the problem of an electric dipole radiating into an infinite, anisotropic plasma.

VII. CONCLUDING REMARKS

As was stated in the introduction to this report, our purpose has been to study, within the framework of a macroscopic formulation, some of the effects of a static magnetic field on the propagation of electromagnetic waves in a plasma, with particular emphasis on the solution of boundary value problems. This, the final part, will be devoted to a summary and evaluation of some of the general results of this investigation. In addition we shall comment on possible extensions and on other applications of the ideas which have been presented.

The essential effect of introducing a static magnetic field into the plasma is to couple together the TE and TM modes, which might otherwise exist independently. The physical explanation for this is that under the influence of a TE or TM mode and the biasing field, the motion of an electron will be such as to generate a field of the dual type. This phenomenon was demonstrated rigorously in the problem of the axially magnetized column and then in general using first order perturbation theory.

The apparent value of the perturbation technique lies in the fact that it permits, with just a moderate amount of additional labor, the extrapolation from isotropic to anisotropic problems. However, such a technique may be used only when certain criteria are satisfied. These are that the operating frequency be greater than the gyro-frequency of the electrons and furthermore that the plasma frequency and size of the interaction zone be such that the inequality

$$\left| \frac{\omega_p^2}{\omega^2 - \omega_p^2} \cdot \frac{\omega_g}{\omega} \cdot kL \right| < 1$$

is satisfied. This criterion is equivalent to the physical requirement that the effect of the static magnetic field be small.

Regarding other applications we should note that since the expansion of the dielectric tensor is an algebraic point function it is valid, under the given convergence criterion, even if $\underline{e}_B$ is not a rectangular unit vector. Therefore, perturbation techniques may be applied to problems which involve curved and possibly inhomogeneous static magnetic fields, both of these having thus far received very little attention.

Finally, we might mention the propagation of electromagnetic waves in gyrotropic crystals (10). The dielectric tensor of a crystal in the presence of a magnetic field is similar to that of a gyroelectric plasma, so that the results obtained here would apply in the case of crystals as well.

APPENDIX A

SOME COMMUTATIVE PROPERTIES OF THE LAPLACIAN OPERATOR

In this section we wish to show that if $\underline{u}$ stands for any of the unit vectors $\underline{e}_x$, $\underline{e}_y$, $\underline{e}_z$ or the radius vector $\underline{r}$, then the following are valid identities:

$$\nabla^2 \left[\nabla \times (\underline{u}W) \right] = \nabla \times \left[\underline{u}(\nabla^2 W) \right] \tag{A.1}$$

$$\nabla^2 \left[\nabla \times (\underline{u} \times \nabla W) \right] = \nabla \times \left[\underline{u} \times \nabla(\nabla^2 W) \right] \tag{A.2}$$

where W is a scalar function.

Other formulas which will be useful are stated here for reference. They may be found in text books on vector analysis.

Let $\underline{f}$, $\underline{g}$ and ϕ be two arbitrary vectors and an arbitrary scalar, respectively. Then,

$$\nabla \times (\phi \underline{f}) = \phi \nabla \times \underline{f} - \underline{f} \times \nabla \phi \tag{A.3}$$

$$\nabla \times (\underline{f} \times \underline{g}) = \underline{f}(\nabla \cdot \underline{g}) - \underline{g}(\nabla \cdot \underline{f}) - 2(\underline{f} \cdot \nabla)g + \nabla(\underline{f} \cdot \underline{g}) - \underline{f} \times (\nabla \times \underline{g}) - \underline{g} \times (\nabla \times \underline{f}) \tag{A.4}$$

$$\nabla \times \nabla \times \underline{f} = \nabla(\nabla \cdot \underline{f}) - \nabla^2 \underline{f} \quad . \tag{A.5}$$

We will prove A.1 first. Since the divergence of the field generated by a curl operator vanishes, the left hand side of A.1 may be written as $-\nabla \times \nabla \times \nabla \times (\underline{u}W)$. Using A.3 and the fact that $\underline{u}$ is irrotational we have that

$$- \nabla \times \nabla \times \nabla \times (\underline{u}W) = \nabla \times \nabla \times (\underline{u} \times \nabla W) \qquad (A.6)$$

Furthermore, from A.4, using the fact that the curl of a gradient vanishes,

$$-\nabla \times \nabla \times \nabla \times (\underline{u}W) = \nabla \times \left[\underline{u}\nabla^2 W - (\nabla W)(\nabla \cdot \underline{u}) - 2(\underline{u} \cdot \nabla)\nabla W \right] \qquad (A.7)$$

But $\nabla \cdot \underline{u}$ is a constant, 0 if $\underline{u} = \underline{e}_x, \underline{e}_y, \underline{e}_z$ and 3 if $\underline{u} = \underline{r}$. In addition, it can be shown that in each case

$$\nabla \times \left[(\underline{u} \cdot \nabla) \nabla W \right] = 0 \qquad (A.8)$$

so that the proof is complete. Equation A.2 is proven in essentially the same way. We have that

$$\nabla^2 \left[\nabla \times (\underline{u} \times \nabla W) \right] = -\nabla \times \nabla \times \nabla \times (\underline{u} \times \nabla W)$$

$$= - \nabla \times \nabla \times \left[\underline{u}\, \nabla^2 W - (\nabla W)(\nabla \cdot \underline{u}) - 2(\underline{u} \cdot \nabla)\nabla W \right] \qquad (A.9)$$

$$= - \nabla \times \left[\nabla \times (\underline{u}\, \nabla^2 W) \right]$$

and employing A.3 we obtain the desired result.

APPENDIX B

FIELD RELATIONS IN A HOMOGENEOUS GYROELECTRIC PLASMA

In this section we derive the necessary relations for the components H_x, H_y and H_z in a homogeneous, gyroelectric plasma.

The vector wave equation for the magnetic field $\underline{H}$ is

$$\nabla \times \left[\underline{\underline{\eta}} \, \nabla \times \underline{H} \right] - k_o^2 \, \underline{H} = 0 \qquad \text{(B.1)}$$

where $\underline{\underline{\eta}}$ is the symbol for the inverse of the dielectric tensor. In rectangular coordinates the scalar equations corresponding to B.1 constitute a set of coupled, linear partial differential equations with constant coefficients. Hence H_x, H_y and H_z all satisfy the same equation.

Assume that the direction of the static magnetic field $\underline{B}_o$ is described by the direction cosines ℓ_x, ℓ_y and ℓ_z. Then it can be shown that the matrix representation for $\underline{\underline{\eta}}$ is

$$\underline{\underline{\eta}} = \begin{bmatrix} \eta_1 + \eta_3 \ell_x^2 & \eta_3 \ell_x \ell_y - ih\ell_z & \eta_3 \ell_x \ell_z + ih\ell_y \\ \eta_3 \ell_x \ell_y + ih\ell_z & \eta_1 + \eta_3 \ell_y^2 & \eta_3 \ell_y \ell_z - ih\ell_x \\ \eta_3 \ell_x \ell_z - ih\ell_y & \eta_3 \ell_y \ell_z + ih\ell_x & \eta_1 + \eta_3 \ell_z^2 \end{bmatrix} \qquad \text{(B.2)}$$

where the parameters η_1, η_3 and h have been defined in Chapter I.

$$\underline{\underline{\mathcal{L}}} = \begin{bmatrix} \eta_{22}(\gamma k_o)^2 + i\eta_{23}\gamma k_o p_y & -\eta_{21}(\gamma k_c)^2 - i\eta_{23}\gamma k_o p_x & -i\eta_{21}\gamma k_o p_y + i\eta_{22}\gamma k_o p_x \\ +i\eta_{32}\gamma k_o p_y - \eta_{33}p_y^2 - k_o^2 & -i\eta_{31}\gamma k_o p_y + \eta_{33}p_x p_y & +\eta_{31}p_y^2 - \eta_{32}p_x p_y \\ & & \\ -\eta_{12}(\gamma k_o)^2 - i\eta_{31}(\gamma k_o)p_y & \eta_{11}(\gamma k_o)^2 + i\eta_{13}\gamma k_o p_x & i\eta_{11}\gamma k_o p_y - \eta_{12}p_x p_y \\ -i\eta_{32}\gamma k_o p_x + \eta_{33}p_x p_y & +i\eta_{31}\gamma k_o p_x - \eta_{33}p_x^2 - k_o^2 & -\eta_{31}p_x p_y + \eta_{32}p_x^2 \\ & & \\ -i\eta_{12}\gamma k_o p_y + \eta_{13}p_y^2 & i\eta_{11}\gamma k_o p_y - \eta_{13}p_x p_y & -\eta_{11}p_y^2 + \eta_{12}p_x p_y \\ +i\eta_{22}\gamma k_o p_x - \eta_{23}p_x p_y & -i\eta_{21}\gamma k_o p_x + \eta_{23}p_x^2 & +\eta_{21}p_y p_x - \eta_{22}p_x^2 - k_o^2 \end{bmatrix}$$

(B.5)

scalar equations corresponding to B.1 are

$$\eta_1 \nabla^2 H_x + k_o^2 H_x + \gamma k_o h \left[i\gamma k_o H_y - \frac{\partial H_z}{\partial y} \right] + \eta_3 \left[\frac{\partial^2 H_x}{\partial y^2} - \frac{\partial^2 H_y}{\partial x \partial y} \right] = 0 \qquad \text{(B.7a)}$$

$$\eta_1 \nabla^2 H_y + k_o^2 H_y - \gamma k_o h \left[i\gamma k_o H_x - \frac{\partial H_z}{\partial x} \right] + \eta_3 \left[\frac{\partial^2 H_y}{\partial x^2} - \frac{\partial^2 H_x}{\partial x \partial y} \right] = 0 \qquad \text{(B.7b)}$$

$$\eta_1 \nabla^2 H_z + k_o^2 H_z - \gamma k_o h \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] = 0 \qquad \text{(B.7c)}$$

Operating on B.7a with $\partial/\partial x$, on B.7b with $\partial/\partial y$ and adding,

$$\eta_1 \nabla^2 P + k_o^2 P + i\gamma^2 k_o^2 hQ = 0 \qquad \text{(B.8)}$$

Operating on B.7a with $-\partial/\partial y$, on B.7b with $\partial/\partial x$ and adding,

$$\eta_1 \nabla^2 Q + k_o^2 Q + \eta_3 \nabla_t^2 Q + i\gamma^2 k_o^2 hP - \gamma h \nabla_t^2 H_z = 0 \qquad \text{(B.9)}$$

where we have introduced the notation

$$P = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \qquad \text{(B.10)}$$

$$Q = \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \qquad \text{(B.11)}$$

$$\nabla_t = \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} \ . \qquad \text{(B.12)}$$

All the functions H_x, H_y, H_z, P and Q will satisfy the same differential equation, since they are related by linear operations in a system of differential equations with constant coefficients. It is convenient, however, to focus attention on one of them, namely H_z.

From the divergence condition $\nabla \cdot \underline{H} = 0$,

$$P \;=\; -i\gamma k_o H_z \tag{B.13}$$

Using B.8 to solve for Q in terms of H_z and then substituting into B.9 we find that H_z satisfies

$$\nabla_t^4 H_z + k_o^2 \left[\frac{\eta_2 + \eta_1 + \gamma^2(h^2 - \eta_1(\eta_1 + \eta_2))}{\eta_1 \, \eta_2} \right] \nabla_t^2 H_z + k_o^4 \, \frac{(1 - \eta_1\gamma^2)^2 - h^2\gamma^4}{\eta_1 \, \eta_2} H_z \;=\; 0 \tag{B.14}$$

or, more concisely

$$\nabla_t^4 H_z + k_o^2 \alpha \nabla_t^2 H_z + k_o^4 \Gamma H_z \;=\; 0 \quad . \tag{B.15}$$

The above may also be written in the factored form

$$(\nabla_t^2 + \beta_1^2)(\nabla_t^2 + \beta_2^2) H_z \;=\; 0 \tag{B.16}$$

where

$$\beta_1^2 = \frac{k_o^2}{2} \left(\alpha + \sqrt{\alpha^2 - 4\Gamma}\right)$$

$$\beta_2^2 = \frac{k_o^2}{2} \left(\alpha - \sqrt{\alpha^2 - 4\Gamma}\right) \tag{B.17}$$

B.15 will be satisfied by $H_z = H_z^{(1)} + H_z^{(2)}$ where $H_z^{(1)}$ and $H_z^{(2)}$ satisfy the corresponding Helmholtz equations

$$(\nabla_t^2 + \beta_1^2) H_z^{(1)} = 0$$

$$(\nabla_t^2 + \beta_2^2) H_z^{(2)} = 0 \tag{B.18}$$

Once $H_z^{(1,2)}$ is known, we can determine the components $H_x^{(1,2)}$ and $H_y^{(1,2)}$. It can be shown that the transverse field is related to H_z by the matrix equation

$$\begin{bmatrix} H_x^{(1,2)} \\ \\ H_y^{(1,2)} \end{bmatrix} = \frac{1}{k_o} \begin{bmatrix} \dfrac{i\gamma}{\nu_{1,2}^2} & -\dfrac{\tau_{1,2}}{\nu_{1,2}^2} \\ \\ \dfrac{\tau_{1,2}}{\nu_{1,2}^2} & \dfrac{i\gamma}{\nu_{1,2}^2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial H_z^{(1,2)}}{\partial x} \\ \\ \dfrac{\partial H_z^{(1,2)}}{\partial y} \end{bmatrix} \tag{B.19}$$

where

$$\tau_{1,2} = \frac{1}{h\gamma}\left[\eta_1(\nu^2_{1,2} - \gamma^2) - 1\right]$$

$$\nu^2_{1,2} = \frac{\beta^2_{1,2}}{k^2_o} \qquad \text{(B.20)}$$

In vector notation B.19 may be written as

$$\underline{H}_t^{(1,2)} = \frac{i\gamma}{k_o\nu^2_{1,2}} \nabla_t H_z^{(1,2)} + \frac{\tau_{1,2}}{k_o\nu^2_{1,2}} \underline{e}_z \times \nabla_t H_z^{(1,2)} \qquad \text{(B.21)}$$

The most general magnetic field will be some linear combination of $\underline{H}^{(1)}$ and $\underline{H}^{(2)}$, these being the independent solutions of B.1.

In order to cast the above results into a form similar to the expressions given for an isotropic medium, we define two functions V_1 and V_2 as follows:

$$H_z^{(1,2)} = -\nu^2_{1,2} V_{1,2} \qquad \text{(B.22)}$$

where ν_1 and ν_2 are the transverse indices of refraction. In terms of V_1 and V_2 the expression for the magnetic field is

$$\underline{H}^{(1,2)} = -\left[\underline{e}_z \times \frac{1}{k_o}\nabla_t V_{1,2} + \frac{i\gamma\tau_{1,2}}{k_o}\nabla_t V_{1,2} + \underline{e}_z \nu^2_{1,2} V_{1,2}\right] . \qquad \text{(B.23)}$$

APPENDIX C

SCATTERING BY AN ISOTROPIC PLASMA COLUMN

Since the results are referred to in the text, we include a section on the problem of scattering of an obliquely incident plane wave by an isotropic plasma cylinder with circular cross section. The column is assumed to be described by the constitutive parameters $\mu = \mu_o$ and $\epsilon = \epsilon_o(1 - \frac{\omega_p^2}{\omega^2})$. The geometry is indicated in Fig. 1, Chapter III.

As shown, the plane wave is incident at an angle with the normal and is polarized with the electric vector perpendicular to the plane of incidence. We assume that the incident magnetic field is of unit amplitude. It can be shown that such a wave is derived from

$$W_1^{(inc)} = 0$$

$$W_2^{(inc)} = - \frac{1}{\cos \Theta} \exp(ik_o\rho \cos \Theta \cos \phi + ik_o z \sin \Theta) \quad (C.1)$$

C.1 may be written in separated form as

$$W_2^{(inc)} = - \frac{1}{\cos \Theta} \sum_{n=-\infty}^{+\infty} i^n J_n(k_o\rho \cos \Theta) F_n(\phi, z) \quad (C.2)$$

where

$$F_n(\phi, z) = \exp(in\phi + ik_o z \sin \Theta) \quad .$$

The incident wave causes a field inside the plasma as well as a scattered field. These are derived from the respective generating functions

$$W_1^{(tr)} = \sum_{n=-\infty}^{+\infty} a_n J_n(\beta\rho) F_n(\phi, z)$$

$$W_2^{(tr)} = \sum b_n J_n(\beta\rho) F_n(\phi, z) \qquad (C.3)$$

$$W_1^{(sc)} = \sum c_n H_n(k_o \rho \cos \Theta) F_n(\phi, z)$$

$$W_2^{(sc)} = \sum d_n H_n(k_o \rho \cos \Theta) F_n(\phi, z) \qquad (C.4)$$

H_n denotes the Hankel function of the first kind, used to give outgoing waves at infinity in accordance with the Sommerfeld radiation condition.

The conditions that H_ϕ, H_z, E_ϕ and E_z be continuous across the surface of the cylinder provide, for each value of n, a set of four equations involving a_n, b_n, c_n and d_n. They are conveniently represented in matrix form by

$$\underline{\underline{A}} \cdot \underline{x}_n = \underline{f}_n \qquad (C.5)$$

where

$$\underline{x}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} \qquad (C.6)$$

$$\underline{f}_n = \begin{bmatrix} \dfrac{-n\gamma}{k_o a \cos \Theta} \; J_n(k_o a \cos \Theta) \\ \cos \Theta \, J_n(k_o a \cos \Theta) \\ - J_n'(k_o a \cos \Theta) \\ 0 \end{bmatrix} \qquad (C.7)$$

and the matrix of the coefficients of the unknowns A is given on the following page.

Since the formulas for the coefficients for arbitrary Θ are quite complex, they will not be exhibited (14). Besides, it is much simpler in using a digital computer to program the individual matrix elements and instruct the machine to invert the system of equations.

The results for the special case of normal incidence, however, are required. Using Cramer's rule it is straightforward to show that, for $\gamma = 0$

$$a_n = 0$$

$$b_n = \frac{2i^{n+1}}{\pi ka[H_n(k_o a)J_n'(ka) - H_n'(k_o a)J_n(ka)]}$$

$$c_n = 0$$

$$d_n = i^n \frac{\nu J_n(ka)J_n'(k_o a) - J_n(k_o a)J_n'(ka)}{H_n(k_o a)J_n'(ka) - \nu H_n'(k_o a)J_n(ka)} \qquad (C.9)$$

$$\underline{\underline{A}} = \begin{bmatrix} -\nu J_n'(\beta a) & \frac{n\gamma}{k_o a} J_n(\beta a) & \cos\theta\, H_n'(k_o a \cos\theta) & \frac{-n\gamma}{k_o a} H_n(k_o a \cos\theta) \\ 0 & -\nu^2 J_n(\beta a) & 0 & \cos^2\theta\, H_n(k_o a \cos\theta) \\ \frac{-n\gamma\eta}{k_o a} & \nu\, J_n'(\beta a) & \frac{n\gamma}{k_o a} H_n(k_o a \cos\theta) & -\cos\theta\, H_n'(k_o a \cos\theta) \\ \eta\nu^2 J_n(\beta a) & 0 & -\cos^2\theta\, H_n(k_o a \cos\theta) & 0 \end{bmatrix} \tag{C.8}$$

APPENDIX D

THE SOLUTION OF AN INHOMOGENEOUS FORM OF BESSEL'S EQUATION

In this section we solve the inhomogeneous, ordinary differential equation

$$x^2 \frac{d^2F_\mu(x)}{dx^2} + x\frac{dF_\mu}{dx} + (x^2 - \mu^2)F_\mu = x^2 Z_\mu(x) \qquad (D.1)$$

where $Z_\mu(x)$ is itself a Bessel function of the first, second or third kind, and thus satisfies the homogeneous form of D.1. Note that the order μ is not necessarily an integer.

We shall use the method of variation of parameters. Let $G_\mu^{(1)}$ and $G_\mu^{(2)}$ be any two linearly independent solutions to the homogeneous equation and let C be defined by the appropriate Wronskian identity

$$G_\mu^{(1)} G_\mu^{(2)'} - G_\mu^{(1)'} G_\mu^{(2)} = \frac{c}{x} \qquad (D.2)$$

Then the solution to D.1 may be represented as

$$cF_\mu(x) = -G_\mu^{(1)}\int x\, G_\mu^{(2)} Z_\mu dx + G_\mu^{(2)}\int x\, G_\mu^{(1)} Z_\mu dx \qquad (D.3)$$

Furthermore, since $G_\mu^{(1,2)}$ and Z_μ all satisfy Bessel's equation the above integrals may be evaluated. The results are (11)

$$- G_\mu^{(1)} \int x\, G_\mu^{(2)}\, Z_\mu dx = - \frac{x^2}{4} G_\mu^{(1)} \left[2G_\mu^{(2)}\, Z_\mu - G_{\mu+1}^{(2)}\, Z_{\mu-1} - G_{\mu-1}^{(2)}\, Z_{\mu+1} \right] \quad (D.4)$$

and

$$G_\mu^{(2)} \int x\, G_\mu^{(1)}\, Z_\mu dx = \frac{x^2}{4} G_\mu^{(2)} \left[2G_\mu^{(1)}\, Z_\mu - G_{\mu+1}^{(1)}\, Z_{\mu-1} - G_{\mu-1}^{(1)}\, Z_{\mu+1} \right] \quad (D.5)$$

Using these results we find that

$$cF_\mu(x) = \frac{x^2}{4} \left[Z_{\mu-1}(G_\mu^{(1)}\, G_{\mu+1}^{(2)} - G_\mu^{(2)}\, G_{\mu+1}^{(1)} + Z_{\mu+1}(G_\mu^{(1)}\, G_{\mu-1}^{(2)} - G_{\mu-1}^{(1)} G_\mu^{(2)}) \right] \quad (D.6)$$

We next employ the identities

$$G_{\mu+1}^{(1,2)} = \frac{\mu}{x} G_\mu^{(1,2)} - G_\mu^{(1,2)'} \quad (D.7)$$

$$G_{\mu-1}^{(1,2)} = \frac{\mu}{x} G_\mu^{(1,2)} + G_\mu^{(1,2)'}$$

which, when combined with D.6 give

$$F_\mu(x) = - \frac{x}{4} \left[Z_{\mu-1} - Z_{\mu+1} \right] \quad (D.8)$$

or, alternatively

$$F_\mu(x) = - \frac{x}{2} Z_\mu'(x) \quad .$$

APPENDIX E

A SPECIFIC VERIFICATION OF FIRST ORDER PERTURBATION THEORY

It was stated in Section V.D that first order perturbation theory gives the linear term of a series expansion in powers of the parameter $i\omega_g/\omega$, of the electromagnetic fields. We shall now verify this conjecture for the special case of the field scattered by an axially magnetized column when the plane wave is normally incident. The geometry is as indicated in Figs. 3.1 and 3.2, with $\theta = 0$. The limiting forms taken by the coefficients for this example are obtained by setting $\gamma = 0$. The coefficients corresponding to the scattered field then become, using the formalism of Chapter III,

$$c_n = 0$$

$$d_n = -i^n \frac{J_n(\beta_1 a)J_n'(k_o a) - J_n(k_o a)[\eta_1 \nu_1 J_n'(\beta_1 a) + \frac{nh}{k_o a} J_n(\beta_1 a)]}{-J_n(\beta_1 a)H_n'(k_o a) + H_n(k_o a)[\eta_1 \nu_1 J_n'(\beta_1 a) + \frac{nh}{k_o a} J_n(\beta_1 a)]} \tag{E.1}$$

The next step is to differentiate d_n with respect to $i\,\omega_g/\omega$. In carrying this out we observe that

$$\left.\frac{\partial(\beta_1, \nu_1, \eta_1)}{i\;\partial(\omega_g/\omega)}\right|_{\omega_g = 0} = 0 \tag{E.2}$$

and

$$\frac{\partial h}{\partial(i\,\omega_g/\omega)} = -\, i\zeta\,\eta \tag{E.3}$$

where $\zeta = \dfrac{\omega_p^2}{\omega^2 - \omega_p^2}$ and $\eta = \dfrac{\omega^2}{\omega^2 - \omega_p^2}$. The final result is

$$\left.\frac{\partial d_n}{\partial(i\omega_g/\omega)}\right|_{\omega_g = 0} = -i^n \frac{2\zeta n\, J_n^2(ka)}{\pi(k_o a)^2\left[H_n(k_o a)J_n'(ka) - \nu J_n(ka)H_n'(k_o a)\right]^2} \tag{E.4}$$

The first order theory will now be employed in an attempt to duplicate E.4. The zero order electric field is given by

$$(tr)_{\underline{E}}^{(0)} = iZ_o \left[\underline{e}_\phi \frac{\partial\, (tr)_{W_2}^{(0)}}{\partial(k_o\rho)} - \underline{e}_\rho \frac{1}{k_o\rho} \frac{\partial\, (tr)_{W_2}^{(0)}}{\partial\phi} \right] \tag{E.5}$$

from which we obtain the first order particular solutions

$$\underline{E}_p^{(1)} = \zeta\ \underline{e}_z x\, {}^{(tr)}\underline{E}^{(0)} = iZ_o\zeta \left[-\underline{e}_\rho \frac{\partial\, (tr)_{W_2}^{(0)}}{\omega(k_o\rho)} - \underline{e}_\phi \frac{1}{k_o\rho} \frac{\partial\, (tr)_{W_2}^{(0)}}{\partial\phi} \right] \tag{E.6}$$

and

$$\underline{H}_p^{(1)} = 0 \quad . \tag{E.7}$$

Furthermore, it was determined in Appendix C that

$$(tr)_{W_2}^{(0)} = \sum_{n=-\infty}^{+\infty} b_n^{(0)} J_n(k\rho)e^{in\phi} \tag{E.8}$$

where

$$b_n^{(0)} = \frac{2i^{n+1}}{\pi ka\left[H_n(k_o a)J_n'(ka) - \nu H_n'(k_o a)J_n(ka)\right]} \; . \tag{E.9}$$

Assume now that the waves generated inside the plasma due to the interaction between the zero order field and $\underline{B}_o$ are partially reflected and partially transmitted at the surface of the cylinder, giving rise to

$$(tr)_{W_2}^{(1)} = \sum_{n=-\infty}^{+\infty} b_n^{(1)} J_n(k\rho)e^{in\phi}$$

$$(sc)_{W_2}^{(1)} = \sum_{n=-\infty}^{+\infty} d_n^{(1)} H_n(k_o\rho)e^{in\phi} \tag{E.10}$$

Then in order to satisfy the boundary conditions we must have that

$$\nu^2 J_n(ka)b_n^{(1)} = d_n^{(1)} H_n(k_o a)$$

$$\nu J_n'(ka)b_n^{(1)} - \frac{i\zeta n}{k_o a} J_n(ka)b_n^{(0)} = d_n^{(1)} H_n'(k_o a) \tag{E.11}$$

from which we determine the following expression for $d_n^{(1)}$:

$$d_n^{(1)} = \frac{i\zeta n\nu \, b_n^{(0)} \, J_n^2(ka)}{k_o a \, [H_n(k_o a)J_n'(ka) - \nu J_n(ka)H_n'(k_o a)]} \qquad (E.12)$$

Substitution of E.9 into E.12 leads to the same result as given in E.4, thus verifying our conjecture concerning the interpretation of the first order solution.

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