THEORY OF THE SCATTERING OF ELECTROMAGNETIC WAVES BY IRREGULAR INTERFACES

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Kenneth Martin Mitzner

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ABSTRACT

Two problems involving electromagnetic scattering from irregular interfaces are treated, both deterministic and statistical irregularities being considered.

First, reflection of a partially polarized plane wave from a plane interface with large irregularities is studied using geometrical optics. Matrix transformations relating incident and reflected waves are obtained for reflection from a single specular point and from an extended area containing many independent reflectors. The properties of a wave reflected from a diffusely illuminated rough interface are found, and these results are used to study reflection noise reduction when a polarization-sensitive detector viewing near the Brewster angle is used in infrared temperature measurements.

Second, the method of small perturbations is used to study scattering of an arbitrary completely polarized wave from an irregular interface of general underlying shape. The irregularities are replaced by equivalent surface currents and then the field in space is found using the dyadic Green's functions of the unperturbed problem. The results obtained are valid when the irregularity has small slope and amplitude small compared to the wavelength and local radii of curvature. To facilitate applications, the theory of dyadic Green's functions is developed, and the necessary functions are evaluated for simple geometries. As an example, the first perturbation is calculated for scattering from a perfectly conducting cylinder with sinusoidal irregularities.

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PART I

GENERAL INTRODUCTION

The theory of scattering of waves from irregular interfaces* has been studied continuously since the late 19th century, with interest especially intense during the last twenty-five years. A great amount of work has been published, mostly in journal articles, and a book is expected shortly (1).

The best bibliography on the subject is contained in a survey paper by Lysanov (2). Additional references may be found in Parts II and III of the present paper and in papers by Twersky (3), Feinberg (4), Rense (5), and Aksenov (6). Articles of interest not cited in any of the above places include those of Hufford (7), Jacobson (8), Lysanov (9), Grasyuk (10), Lapin (11), Barkhatov et al. (12), Gulin (13), Lippmann (14), Marsh (15), Urusovskii (16), and Barantsev (17).

Despite the great amount of work done on scattering from irregular interfaces, there are still many problems of immediate practical interest which have not been solved satisfactorily. One reason for this situation is that a unified approach is not possible; there are four distinctly different methods of solution, each being the best for some problems but inapplicable or undesirable for others. Furthermore, these basic methods appear in various modifications depending on the specifics of the problem and the results desired.

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*By an irregular interface is meant an interface of somewhat complicated geometrical form which can be considered as a distortion of a simpler geometrical form. The irregularities may be "regular" in the sense of being periodic.

Finally, most problems have a scalar (acoustical) and a vector (electromagnetic) form, the latter being more difficult to treat.

In order to put the material of Parts II and III in context, let us consider here the four basic methods of solution. First we have the geometrical optics (or acoustics) approach, useful when the wavelength is sufficiently small. The usual procedure is to assume that the effect of the interface curvature can be neglected in calculating the reflected and transmitted fields right at the interface; then the fields everywhere in space can be found by well-known techniques. In many cases the interface may be considered as made up of small specular areas which scatter independent beams; this assumption simplifies the problem considerably.

In a modified form of the geometrical method, the field is expanded in an asymptotic series in the wavelength, the usual geometrical optics solution being the zero order term. Reference 8 gives an example of this approach.

At the other end of the spectrum, for wavelengths large compared to the interface irregularities, the method of small perturbations is useful. Here the change in field due to the irregularities is assumed to be small and is calculated by expanding the field in a series and requiring that each term satisfy appropriate boundary conditions. The vector form of this technique is treated in detail in Part III.

In problems where neither the assumptions of geometrical optics nor those of the method of small perturbations hold, an integral equation approach is usually necessary. That is, an integral equation formulation of the problem must be considered directly and solved by some approximate technique. Variations of this method are given in References 14-18.

In the fourth method, the problem is attacked from a different point of view. Surface corrugations of simple shape are considered and a boundary value problem is solved--exactly or approximately--for the particular shape. This approach is discussed by Lysanov (2) under the two headings, "The Method of Images" and "The Method of Matching Fields." The approach is of course adapted only to very special problems. However, in cases where it can be used, it facilitates a study of the transition from short wavelength to long wavelength conditions.

The problems treated in this paper involve applying the first two methods--geometrical optics and perturbation theory--to vector problems. In Part II, the reflection of a partially polarized plane wave from a plane interface with large irregularities is studied using geometrical optics with the simplifying assumption of independent scattering from small specular areas. In Part III, the method of small perturbations is used to study scattering of an arbitrary completely polarized electromagnetic wave from an interface of general shape with small irregularities.

The problem of Part II has no scalar counterpart. The scalar problem analogous to the problem of Part III has not been treated because it lies outside the realm of electromagnetic theory; it will be discussed in a forthcoming paper.

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(These papers are independent developments of the same technique.)

PART II

REFLECTION OF A PARTIALLY POLARIZED WAVE FROM A ROUGH PLANE INTERFACE—GEOMETRICAL OPTICS THEORY

1. Introduction

As already noted, in recent years a great amount of study has been devoted to optical reflection from irregular plane interfaces. The extended geometrical optics of Luneberg (1), has been used by Primakoff, J. Keller, and H. Keller (2,3) to develop formulas for optical reflection and transmission of field components at an arbitrary curved interface. Longuet-Higgins (4,5) has studied the reflection of a scalar wave from a plane with Gaussian random roughness of small slope. Beckmann (6), using techniques similar to those to be developed here, has calculated the rotation in polarization when a completely polarized plane wave is reflected from a rough plane. Many other workers have treated the rough interface problem, but all have either used scalar representations of light or have considered only completely polarized light. There has been no treatment of unpolarized or partially polarized light.

In this paper we shall give such a treatment. Specifically, we shall study the intensity and polarization properties of light specularly reflected from a rough plane interface between two linear, homogeneous, isotropic media when a partially polarized plane wave is incident. Unpolarized and completely polarized incident waves will appear in our formulation as special cases of partially polarized waves.

The properties of a light wave will be described in terms of its coherency matrix J (7) defined by

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \left\langle \begin{bmatrix} E_{x'} \\ E_{y'} \end{bmatrix} \begin{bmatrix} E^*_{x'} & E^*_{y'} \end{bmatrix} \right\rangle =$$

$$\left\langle \begin{bmatrix} E_{x'} E^*_{x'} & E_{x'} E^*_{y'} \\ E_{y'} E^*_{x'} & E_{y'} E^*_{y'} \end{bmatrix} \right\rangle \quad , \tag{1.1}$$

or, equivalently, in terms of its power matrix

$$W = \frac{1}{2} \frac{\epsilon_1}{\mu_1} J \text{ (watts/m}^2\text{)} . \tag{1.2}$$

Here E is the analytic signal belonging to the electric field, ϵ_1 and μ_1 are the material parameters of the medium of propagation, and $< \cdots >$ indicates time averaging. The direction of propagation is identified with the z'-direction, and the (x',y',z') coordinate system is right-handed. The matrix J or W describes completely the intensity and polarization state of the wave; convenience determines which of the two is used.

Two closely related problems will be studied: reflection from the neighborhood of a single specular point and reflection from an area containing many specular points. In treating the latter problem, it will be assumed that we can neglect interference effects, shadowing, and multiple reflection and refraction. Emphasis will be placed on far (Fraunhofer) zone calculations.

It will be shown that the coherency matrix of the wave reflected in a given direction is related to the coherency matrix of the incident wave by a linear matrix transformation of form

$$\boxed{J^{refl} = \eta \, P \, J^{inc} \, \tilde{P}^*} \tag{1.3}$$

The 2x2 transformation matrix P is a function of the material parameters of the media and of the directions of incidence and reflection; P is the same whether we consider a single specular point or an extended area. The scalar η is the product of a function depending on the interface geometry with a function of the directions of incidence and reflection.

In Sections 2 and 3 we calculate P and η . In Section 4 we consider the important case in which the roughness is described statistically. In Section 5 we apply our results to a problem, of practical interest in infrared temperature measurement, involving the polarization properties of the field reflected from a diffusely illuminated interface.

2. An Auxiliary Problem--Reflection from a Tilted Plane

2.1 Analysis in Angular Coordinates

We shall now solve an auxiliary problem, evaluating a matrix P which in Section 3 will be identified with the P of Equation 1.3. Figure 1 gives the geometry of the auxiliary problem. Two linear, homogeneous, isotropic media, M_1 and M_2, are separated by the boundary plane, which is characterized by the unit normal vector $\hat{n}$ (from M_1 to M_2). Medium M_1 is a lossless dielectric ($\sigma_1 = 0$) but M_2 may be lossy.

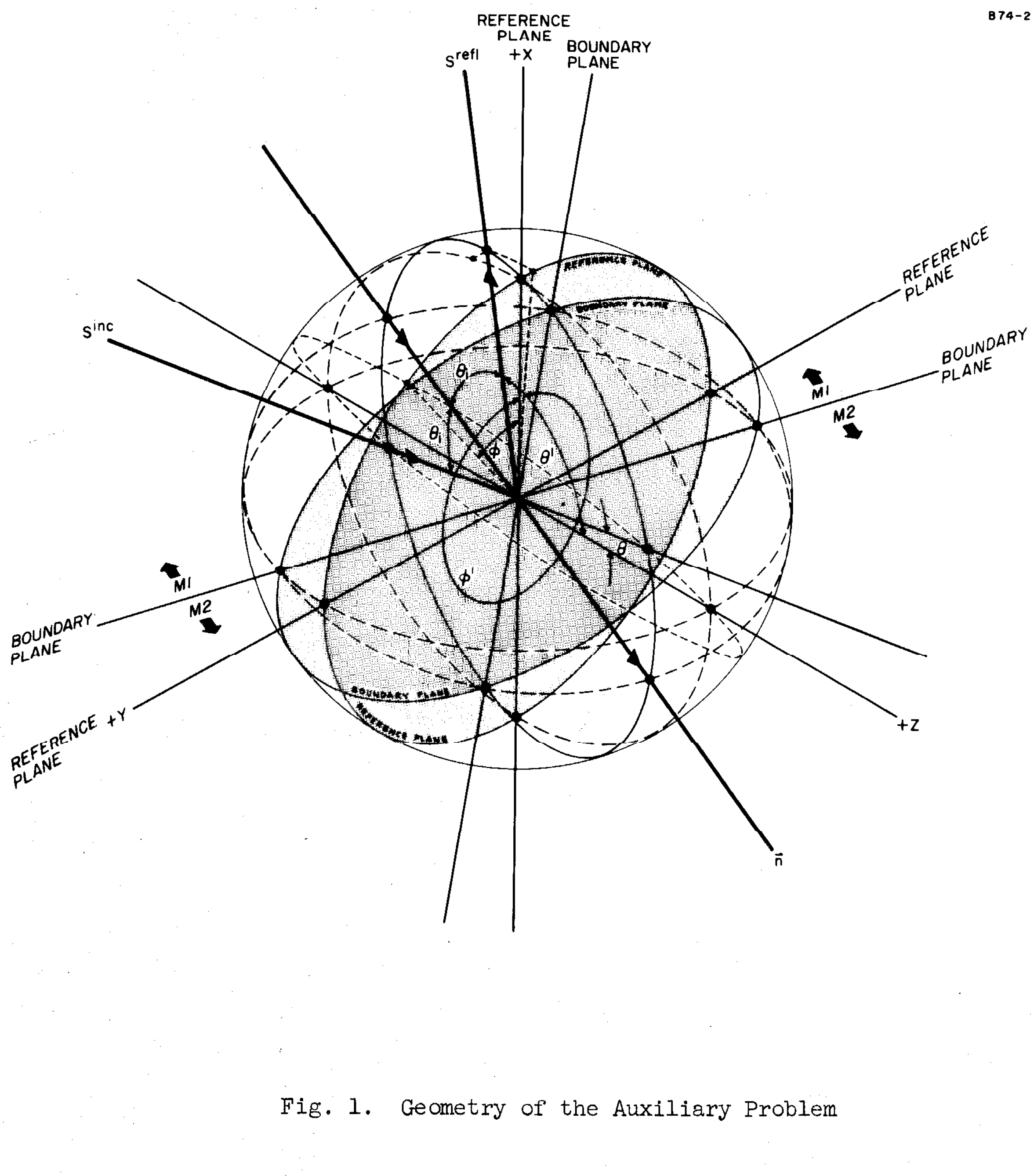

Fig. 1. Geometry of the Auxiliary Problem

We establish a Cartesian reference coordinate system C, characterized by the unit vectors $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, $\underset{\sim}{e}_z$, and designate the plane $z = 0$ as the reference plane. In general, the boundary plane is tilted with respect to the reference plane.

Let a homogeneous plane wave S^{inc} be incident on the boundary plane from M_1. This wave shall be described by the unit vector $\underset{\sim}{s}^{inc}$ in the direction of propagation and the coherency matrix

$$J^{inc} = \left\langle \begin{bmatrix} E_{x_o} \\ \\ E_{y_o} \end{bmatrix} \begin{bmatrix} E^*_{x_o} & E^*_{y_o} \end{bmatrix} \right\rangle . \tag{2.1}$$

The components of E are measured in the incidence coordinate system C_o characterized by the unit vectors

$$\underset{\sim}{e}_{x_o} = \frac{(\underset{\sim}{e}_z \times \underset{\sim}{s}^{inc}) \times \underset{\sim}{s}^{inc}}{|\underset{\sim}{e}_z \times \underset{\sim}{s}^{inc}|} , \quad \underset{\sim}{e}_{y_o} = \frac{\underset{\sim}{e}_z \times \underset{\sim}{s}^{inc}}{|\underset{\sim}{e}_z \times \underset{\sim}{s}^{inc}|} , \quad \underset{\sim}{e}_{z_o} = \underset{\sim}{s}^{inc} . \tag{2.2}$$

The vector $\underset{\sim}{e}_{z_o}$ can be expressed in the C system by

$$\underset{\sim}{e}_{z_o} = \underset{\sim}{s}^{inc} = \cos \phi \sin \Theta \, \underset{\sim}{e}_x + \sin \phi \sin \Theta \, \underset{\sim}{e}_y + \cos \Theta \, \underset{\sim}{e}_z \quad ; \tag{2.3}$$

here Θ is the angle between $\underset{\sim}{s}^{inc}$ and the positive z-axis, and ϕ is the azimuth of $\underset{\sim}{s}^{inc}$. If the wave S^{inc} is considered incident on the reference plane, then E_{x_o} is the field component parallel to the plane of incidence and E_{y_o} is the component perpendicular to the plane of incidence.

We designate the reflected wave as S^{refl}, described by the unit propagation vector $\underset{\sim}{s}^{refl}$ and the coherency matrix

$$J^{refl} = \left\langle \begin{bmatrix} E_{x_2} \\ E_{y_2} \end{bmatrix} \begin{bmatrix} E^*_{x_2} & E^*_{y_2} \end{bmatrix} \right\rangle . \qquad (2.4)$$

The field components are measured in the reflection coordinate system C_2 , characterized by

$$\underset{\sim}{e}_{x_2} = \frac{(\underset{\sim}{e}_z \times \underset{\sim}{s}^{refl}) \times \underset{\sim}{s}^{refl}}{|\underset{\sim}{e}_z \times \underset{\sim}{s}^{refl}|} \; ; \; \underset{\sim}{e}_{y_2} = \frac{\underset{\sim}{e}_z \times \underset{\sim}{s}^{refl}}{|\underset{\sim}{e}_z \times \underset{\sim}{s}^{refl}|} \; ; \; \underset{\sim}{e}_{z_2} = \underset{\sim}{s}^{refl} . \qquad (2.5)$$

The vector $\underset{\sim}{e}_{z_2}$ can be expressed in the C system by

$$\underset{\sim}{e}_{z_2} = \underset{\sim}{s}^{refl} = \cos\phi'\sin\Theta' \, \underset{\sim}{e}_x + \sin\phi'\sin\Theta' \, \underset{\sim}{e}_y + \cos\Theta' \, \underset{\sim}{e}_z . \qquad (2.6)$$

Here Θ' is the angle between $\underset{\sim}{s}^{refl}$ and the positive z-axis, and ϕ' is the azimuth of $\underset{\sim}{s}^{refl}$. If the wave S^{refl} is considered reflected from the reference plane, then E_{x_2} is the field component parallel to the plane of reflection and E_{y_2} is the component perpendicular to the plane of reflection.

Both S^{inc} and S^{refl} will also be represented in the boundary coordinate system C_1 , characterized by

$$\underset{\sim}{e}_{x_1} = \frac{\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}|} \; ; \; \underset{\sim}{e}_{y_1} = \frac{\underset{\sim}{s}^{inc} \times \underset{\sim}{s}^{refl}}{|\underset{\sim}{s}^{inc} \times \underset{\sim}{s}^{refl}|} \; ; \; \underset{\sim}{e}_{z_1} = \frac{\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}}{|\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} . \qquad (2.7)$$

To understand the significance of C_1 , we write the law of reflection in the vector form

$$\underset{\sim}{n} = \frac{\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}}{|\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} = \underset{\sim}{e}_{z_1} , \qquad (2.8)$$

and note that

$$\frac{\hat{n} \times \hat{s}^{inc}}{|\hat{n} \times \hat{s}^{inc}|} = \frac{\hat{n} \times \hat{s}^{refl}}{|\hat{n} \times \hat{s}^{refl}|} = \frac{\hat{s}^{inc} \times \hat{s}^{refl}}{|\hat{s}^{inc} \times \hat{s}^{refl}|} = \underset{\sim}{e}_{y_1} \quad . \tag{2.9}$$

Now it is clear that when S^{inc} and S^{refl} are considered respectively incident on and reflected from the boundary plane, then E_{x_1} lies parallel to the boundary plane and in the plane of incidence (and reflection), E_{y_1} lies parallel to the boundary plane and perpendicular to the plane of incidence, and E_{z_1} is perpendicular to the boundary and thus in the plane of incidence.

The problem of reflection from a smooth plane is usually solved in the boundary coordinate system. The incident and reflected fields at the surface are then related by the simple equation

$$\underset{\sim}{E}_1^{refl} = R\, \underset{\sim}{E}_1^{inc} , \tag{2.10}$$

where R is the diagonal matrix

$$R = \begin{bmatrix} -R_{||} & 0 & 0 \\ 0 & R_{\perp} & 0 \\ 0 & 0 & R_{||} \end{bmatrix} . \tag{2.11}$$

Here $R_{||}$ and $R_{\perp}$ are the Fresnel reflection coefficients, given by

$$R_{||} = \frac{(\mu_1/\mu_2)n \cos\theta_i - \cos\theta_t}{(\mu_1/\mu_2)n \cos\theta_i + \cos\theta_t} , \tag{2.12}$$

$$R_{\perp} = \frac{\cos\theta_i - (\mu_1/\mu_2)n \cos\theta_t}{\cos\theta_i + (\mu_1/\mu_2)n \cos\theta_t} ; \tag{2.13}$$

and the index of refraction n is given by

$$n = \left[\frac{\omega^2\, \epsilon_2\, \mu_2 + i\omega\, \sigma_2\, \mu_2}{\omega^2\, \epsilon_1\, \mu_1}\right]^{1/2} . \tag{2.14}$$

The angle θ_i is the angle of incidence with respect to the boundary plane, the angle between $\underset{\frown}{s}^{inc}$ and $\underset{\frown}{n}$; it lies in the first quadrant and is given by

$$\cos^2\theta_i = \frac{1}{2}\,[1 - \cos\theta\cos\theta' - \cos(\phi' - \phi)\sin\theta\sin\theta'] . \tag{2.15}$$

The angle θ_t (in general complex) is given by Snell's law,

$$\sin\theta_i = n\sin\theta_t . \tag{2.16}$$

Now let us introduce the three-dimensional coherency matrix in the C_1 system,

$$J_1 = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} = \left\langle \begin{bmatrix} E_{x_1} \\ E_{y_1} \\ E_{z_1} \end{bmatrix} \begin{bmatrix} E^*_{x_1} & E^*_{y_1} & E^*_{z_1} \end{bmatrix} \right\rangle . \tag{2.17}$$

Then from Equation 2.10 we obtain

$$J_1^{refl} = R\, J_1^{inc}\, R^* . \tag{2.18}$$

Furthermore, we note that

$$J_1^{inc} = M\, J^{inc}\, \tilde{M} , \tag{2.19}$$

$$J^{refl} = \tilde{N}\, J_1^{refl}\, N , \tag{2.20}$$

where M is the real 3 x 2 matrix which transforms the incident field from the C_o system into the C_1 system and N is the real 3 x 2 matrix which transforms the reflected field from the C_2 system into the C_1 system.

Combining the last three equations yields the important result

$$J^{refl} = P\, J^{inc}\, \tilde{P}^* , \tag{2.21}$$

where P is the 2 x 2 matrix

$$P = \tilde{N}\, R\, M . \tag{2.22}$$

The matrices M and N can be expressed in terms of simpler matrices by

$$M = TL_o \ , \ N = TL_2 \ , \tag{2.23}$$

where T is the 3 x 3 matrix which transforms a vector from the C system into the C_1 system, L_o is the 3 x 2 matrix which transforms the incident field from the C_o system to the C system, and L_2 is the 3 x 2 matrix which transforms the reflected field from the C_2 system to the C system. Invoking the rules of orthogonal coordinate transformations (8), we obtain

$$\tilde{T} = [\, \underset{\sim}{e}_{x_1} \ \ \underset{\sim}{e}_{y_1} \ \ \underset{\sim}{e}_{z_1} \,] , \tag{2.24}$$

$$L_o = [\, \underset{\sim}{e}_{x_o} \ \ \underset{\sim}{e}_{y_o} \,] \qquad , \tag{2.25}$$

$$L_2 = [\, \underset{\sim}{e}_{x_2} \ \ \underset{\sim}{e}_{y_2} \,] \qquad , \tag{2.26}$$

where the matrix elements are all to be expressed in the reference system C .

We can now calculate P in terms of the Fresnel coefficients and of the two pairs of angles, (θ,ϕ) and (θ',ϕ'), which designate the directions of incidence and reflection in the reference system. The computations, which are rather tedious, are outlined in Appendix 1. The result is

$$\boxed{P = \frac{1}{K}\begin{bmatrix} B_1B_2R_{\parallel} + B_3B_4R_{\perp} & B_1B_3R_{\parallel} - B_2B_4R_{\perp} \\ \\ B_2B_4R_{\parallel} - B_1B_3R_{\perp} & B_3B_4R_{\parallel} + B_1B_2R_{\perp} \end{bmatrix}} \quad , \quad (2.27)$$

where

$$K = 1 - [\cos\theta\cos\theta' + \cos(\phi' - \phi)\sin\theta\sin\theta']^2 \quad (2.28)$$

$$B_1 = \cos\theta\sin\theta' - \cos(\phi' - \phi)\sin\theta\cos\theta' \quad (2.29)$$

$$B_2 = \cos(\phi' - \phi)\cos\theta\sin\theta' - \sin\theta\cos\theta' \quad , \quad (2.30)$$

$$B_3 = \sin(\phi' - \phi)\sin\theta' \quad (2.31)$$

$$B_4 = \sin(\phi' - \phi)\sin\theta \quad (2.32)$$

and the Fresnel coefficients are given by Equations 2.12-2.16. Thus P can be expressed as a function of two material parameters--n and μ_1/μ_2--and three independent angles--θ, θ', and $(\phi' - \phi)$.

The five quantities in Equations 2.28-2.32 are related by

$$\boxed{K = B_1^2 + B_4^2 = B_2^2 + B_3^2} \quad ; \quad (2.33)$$

the fact that only three can be chosen independently corresponds to the fact that the five quantities depend on three independent angles.

2.2 Representation in Mixed Coordinate Systems

Thus far we have considered P as a function of the directions of incidence and reflection, these directions uniquely determining the slope of the boundary plane. However, as we shall see in Section 5, there are problems in which it is desirable to express P in terms of the slope of the boundary plane and the direction of incidence (reflection), considering the direction of reflection (incidence) as thus uniquely determined. Equation 2.27 will still hold, but we must now express K, the four B_j, and θ_i (which determines $R_{||}$ and $R_{\perp}$) in terms of the new independent variables.

Let us write the equation of the boundary plane as

$$F = z - (f_x x + f_y y) = 0 . \tag{2.34}$$

Then

$$\underset{\sim}{n} = \nabla F / |\nabla F| = [\underset{\sim}{e}_z - (f_x \underset{\sim}{e}_x + f_y \underset{\sim}{e}_y)] / |\nabla F| . \tag{2.35}$$

Comparing this with Equations 2.3, 2.6, and 2.8, we find

$$f_x = - \frac{\cos\phi' \sin\theta' - \cos\phi \sin\theta}{\cos\theta' - \cos\theta} ,$$

$$f_y = - \frac{\sin\phi' \sin\theta' - \sin\phi \sin\theta}{\cos\theta' - \cos\theta} . \tag{2.36}$$

Now we define the quantities

$$\lambda = f_x \cos\phi + f_y \sin\phi , \qquad \lambda' = f_x \cos\phi' + f_y \sin\phi' ;$$

$$\kappa = f_x \sin\phi - f_y \cos\phi , \qquad \kappa' = f_x \sin\phi' - f_y \cos\phi' ;$$

$$\mu^2 = \lambda^2 + \kappa^2 = \lambda'^2 + \kappa'^2 = f_x^2 + f_y^2 . \tag{2.37}$$

Then it can be shown by the method illustrated in Appendix 2 that, in terms of (f_x, f_y) and (θ, ϕ), we have

$$\boxed{\begin{aligned}
\cos^2\theta_i &= (\lambda \sin\theta - \cos\theta)^2/(1+\mu^2) \;, \\
B_1 &= -\frac{2(\lambda \sin\theta - \cos\theta)\left\{(1+\mu^2) - (\lambda\sin\theta - \cos\theta)[2\lambda\sin\theta - (1-\mu^2)\cos\theta]\right\}}{(1+\mu^2)\left\{(1+\mu^2)^2 - [2\lambda\sin\theta - (1-\mu^2)\cos\theta]^2\right\}^{1/2}} \;, \\
B_2 &= -2(\lambda\sin\theta - \cos\theta)(\lambda\cos\theta + \sin\theta)/(1+\mu^2) \;, \\
B_3 &= 2\kappa\,(\lambda\sin\theta - \cos\theta)/(1+\mu^2) \;, \\
B_4 &= 2\kappa\sin\theta(\lambda\sin\theta - \cos\theta)\left\{(1+\mu^2)^2 - [2\lambda\sin\theta - (1-\mu^2)\cos\theta]^2\right\}^{-1/2} , \\
K &= 4(\lambda\sin\theta - \cos\theta)^2\,[(\lambda\cos\theta + \sin\theta)^2 + \kappa^2]/(1+\mu^2)^2 \,.
\end{aligned}} \tag{2.38}$$

In terms of (f_x, f_y) and (θ', ϕ'), we have

$$\boxed{\begin{aligned}
\cos^2\theta_i &= (\lambda'\sin\theta' - \cos\theta')^2/(1+\mu^2) \;, \\
B_1 &= 2(\lambda'\sin\theta' - \cos\theta')(\lambda'\cos\theta' + \sin\theta')/(1+\mu^2) \;, \\
B_2 &= \frac{2(\lambda'\sin\theta' - \cos\theta')\left\{(1+\mu^2) - (\lambda'\sin\theta' - \cos\theta')[2\lambda'\sin\theta' - (1-\mu^2)\cos\theta']\right\}}{(1+\mu^2)\left\{(1+\mu^2)^2 - [2\lambda'\sin\theta' - (1-\mu^2)\cos\theta']^2\right\}^{1/2}} \;, \\
B_3 &= -2\kappa'\sin\theta'(\lambda'\sin\theta' - \cos\theta')\left\{(1+\mu^2)^2 - [2\lambda'\sin\theta' - (1-\mu^2)\cos\theta']^2\right\}^{-1/2} , \\
B_4 &= -2\kappa'\,(\lambda'\sin\theta' - \cos\theta')/(1+\mu^2) \;, \\
K &= 4(\lambda'\sin\theta' - \cos\theta')^2\,[(\lambda'\cos\theta' + \sin\theta')^2 + \kappa'^2]/(1+\mu^2)^2 .
\end{aligned}} \tag{2.39}$$

2.3 Simplification for Unpolarized Incident Light

In many important practical problems such as that of Section 5, the incident light is completely unpolarized. In this case

$$W^{inc} = W_o \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \tag{2.40}$$

where W_o is a constant. Then Equation 2.21 becomes

$$W^{refl} = W_o \, P \, \tilde{P}^* = \frac{W_o}{B_1^2+B_4^2} \begin{bmatrix} |R_{||}|^2 \, B_1^2 + |R_\perp|^2 \, B_4^2 & (|R_{||}|^2 - |R_\perp|^2) B_1 B_4 \\ (|R_{||}|^2 - |R_\perp|^2) B_1 B_4 & |R_{||}|^2 \, B_4^2 + |R_\perp|^2 \, B_1^2 \end{bmatrix} . \tag{2.41}$$

The reflected power density is given by

$$\mathcal{W}_r = W_o \, (|R_{||}|^2 + |R_\perp|^2) \ . \tag{2.42}$$

If we specialize further to the case of perfect reflection, then

$$|R_{||}|^2 = |R_\perp|^2 = 1 \ , \tag{2.43}$$

and Equation 2.41 becomes

$$W^{refl} = W_o \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = W^{inc} \ ; \tag{2.44}$$

this is in accord with the well-known result that reflection from a perfectly conducting plane does not change the nature of unpolarized light.

3. The Rough Plane Problem

3.1 Preliminary Remarks

We shall now give a complete delineation of the problem of reflection from a rough plane as considered in this paper. In this statement we shall adhere closely to the conventions and nomenclature of Section 2 and Figure 1, for indeed this formalism has been devised primarily for use in the rough plane problem.

Again we consider two linear, homogeneous, isotropic media, M_1 and M_2, where M_1 is lossless and M_2 may be lossy. These media are separated by a rough boundary, the profile of which approximates the plane $z = 0$; the equation of this boundary will be written

$$F = z - f(x,y) = 0 \; . \tag{3.1}$$

The unit normal $\hat{n}$ to the boundary is still given by Equation 2.35, but now f_x and f_y are functions of x and y .

The plane $z = 0$ is again designated as the reference plane and plays the same important role as before. The incident field is, as before, a plane wave S^{inc} with arbitrary coherency matrix, traveling through M_1 in direction $\hat{s}^{inc}$; the wave will again be described by the angles (θ,ϕ) and the coherency matrix in the C_0 system. The wave S^{refl} reflected in the direction $\hat{s}^{refl}$ will be described by the angles (θ',ϕ') and the coherency matrix in the C_2 system.

It is assumed that the interface geometry is such that we are justified in using the ideas of Luneberg's extended geometrical optics (1) (this essentially means that the boundary curvature cannot be too great near a point of interest).

As in classical optics, the rays travel in straight lines and obey the laws of reflection and refraction. Furthermore, the incident, reflected, and refracted fields at any point on the boundary are related by the Fresnel formulas calculated for a plane interface; in other words, the reflected (and refracted) field at a given point of the boundary is found by replacing the rough interface by a tangential plane interface through the point and then performing the calculation.

The fictional tangential plane plays the same role as the boundary plane in Section 2. Fixing the direction (θ,ϕ) of S^{inc} and the slope (f_x,f_y) at a point of the boundary determines the direction (θ',ϕ') in which energy is reflected from the neighborhood of the point. Likewise, if we know the energy is reflected in the (θ',ϕ') direction, we may determine $(f_x,f_y)[(\theta,\phi)]$ if $(\theta,\phi)[(f_x,f_y)]$ is given. The coordinate transformation Equation 2.36 still holds.

From the above, it is clear that <u>at every point on the boundary</u> the coherency matrices of the incident and reflected wave are related by Equation 2.21, with P the same as in Section 2. However, now P is a function of position on the boundary. Furthermore, because of the curvature of the boundary surface, the coherency matrix of the reflected wave changes in magnitude with distance away from the boundary.

3.2 Reflection from a Single Specular Point

Consider the light reflected from the neighborhood of a single specular point on the rough surface. Let J_o^{refl} be the coherency matrix of the reflected wave measured at the boundary and let J_D^{refl} be the coherency matrix measured at distance D further along the wave. Then

$$J_D^{refl} = I\, J_o^{refl} = I\, P\, J^{inc}\, \tilde{P}^* \,, \qquad (3.2)$$

where I is a scalar given by Equation 10 or 14 of Reference 3 [where it is designated by the notation $J(\frac{p'}{p})$]. Comparing Equation 3.2 with Equation 1.3, we see that for a single specular point $\eta = I$.

The most important special case is observation in the far zone. There we readily find from Equation 14, Reference 3, that

$$I = (4D^2|G_g|)^{-1} , \tag{3.3}$$

where G_g is the Gaussian curvature at the reflection point and is given (9) by

$$G_g = \frac{\Omega}{(1+f_x^2 + f_y^2)^2} = \frac{(\cos\theta - \cos\theta')^4 \, \Omega}{4[1 - \cos\theta\cos\theta' - \cos(\phi'-\phi)\sin\theta\sin\theta']^2} \tag{3.4}$$

and

$$\Omega = f_{xx} f_{yy} - f_{xy}^2 . \tag{3.5}$$

For problems involving only the far zone, it is desirable to suppress the range factor $1/D^2$. Thus we define the far zone normalized coherency matrix

$$J^{refl'} = \lim_{D \to \infty} (D^2 J_D^{refl}) \tag{3.6}$$

and the associated normalized power matrix

$$W^{refl'} = \frac{1}{2}\sqrt{\frac{\epsilon_1}{\mu_1}} \, J^{refl'} . \tag{3.7}$$

The trace of $W^{refl'}$ is the average power per unit solid angle crossing the wavefront.

We can now write the very important result

$$J^{refl'} = \frac{1}{4\,|G_g|}\, P\, J^{inc}\, \tilde{P}^* =$$

$$\boxed{\frac{[1 - \cos\Theta\cos\Theta' - \cos(\phi' - \phi)\sin\Theta\sin\Theta']^2}{(\cos\Theta - \cos\Theta')^4}\, |\Omega|^{-1}\, P\, J^{inc}\tilde{P}^*}\ , \quad (3.8)$$

where the quantity Ω is to be determined from Equation 3.5.

3.3 Reflection from an Extended Area

Two complications arise in passing from reflection from a single specular point to reflection from an extended area. The first is the possibility that multiple point geometric effects are important, i.e., that a significant amount of light reaches the observer by multiple reflection or refraction or fails to reach him because specular points are shadowed. These effects are important when the angle of incidence or of reflection is close to grazing and when the roughness is steep. We shall exclude such situations from our analysis and assume that no specular points lie in shadow and that every ray once reflected or refracted travels in a straight line to infinity.

The second complication is the possibility of interference among the returns from the various specular points. This effect may be neglected when the roughness is sufficiently random and the scale is large compared to the coherence length (10) of the light. We shall assume such roughness here.

Given these assumptions the coherency matrix of the light reflected in direction (Θ',ϕ') from illuminated area A is found simply by summing the coherency matrices of the reflections from each contributing specular point. In the far zone, which is of major interest, we obtain the normalized coherency matrix

$$\boxed{J_{\Sigma}^{\text{refl}'} = \eta_{\Sigma} \, P \, J^{\text{inc}} \, \tilde{P}^{*}} \quad , \tag{3.9}$$

with

$$\boxed{\eta_{\Sigma} = \frac{[1 - \cos\theta\cos\theta' - \cos(\phi' - \phi)\sin\theta\sin\theta']^2}{(\cos\theta - \cos\theta')^4} \quad \Sigma_A |\Omega_j|^{-1}} \; ; \tag{3.10}$$

the summation is taken over all appropriate specular points in A .

Equation 3.10 can be converted to an alternate form which is usually more useful, especially when the interface is described statistically. Consider the two-dimensional incremental

$$p(f_x,f_y)A \, df_x \, df_y$$

which we define as the sum of the projections onto the reference plane of all illuminated areas of the interface with slopes in the range

$$(f_x \pm \tfrac{1}{2} \, df_x \; , \; f_y \pm \tfrac{1}{2} \, df_y) \; .$$

If we represent the projection of an infinitesimal area of appropriate slope by $dx_j \, dy_j$, then we can express the definition mathematically as

$$p(f_x,f_y) \, A \, df_x \, df_y = \Sigma_A \, dx_j \, dy_j \; . \tag{3.11}$$

It is clear from the above that

$$\int_{-\infty}^{\infty} df_x \int_{-\infty}^{\infty} df_y \, p(f_x,f_y) = 1 \; . \tag{3.12}$$

An incremental area $dx_j\ dy_j$ can also be expressed in terms of slope incrementals by

$$dx_j\ dy_j = |U_j|^{-1}\ df_x\ df_y\ , \tag{3.13}$$

where U_j is the Jacobian

$$U_j = \frac{\partial(f_x,f_y)}{\partial(x_j,y_j)} = \begin{vmatrix} \frac{\partial}{\partial x_j} f_x & \frac{\partial}{\partial x_j} f_y \\ \frac{\partial}{\partial y_j} f_x & \frac{\partial}{\partial y_j} f_y \end{vmatrix} = \Omega_j\ . \tag{3.14}$$

Comparing this with Equation 3.11, we find

$$\Sigma_A\ |\Omega_j|^{-1} = p(f_x,f_y)A\ . \tag{3.15}$$

Thus Equation 3.10 can be written

$$\boxed{\eta_\Sigma = \frac{[1 - \cos\theta\cos\theta' - \cos(\phi' - \phi)\sin\theta\sin\theta']^2}{(\cos\theta - \cos\theta')^4}\ p(f_x,f_y)A}\ . \tag{3.16}$$

3.4 An Alternate Notation

Thus far we have defined η and P in Equation 1.3 in a manner which is very convenient from the point of view of derivation. However, in application it will often be desirable to use an alternate factorization

$$J^{refl} = \eta'\ P'J^{inc}\ \tilde{P}'^{*} \tag{3.17}$$

in which η' does not depend explicitly on the angles of incidence and reflection. Then P' may be expressed in terms of P as

$$P' = \frac{1 - \cos\Theta \cos\Theta' - \cos(\phi' - \phi) \sin\Theta \sin\Theta'}{(\cos\Theta - \cos\Theta')^2} P . \qquad (3.18)$$

For a single specular point, η' is given in the far zone by

$$\eta' = I' = (D^2 |\Omega|)^{-1} ; \qquad (3.19)$$

and for an extended area A,

$$\eta'_\Sigma = \Sigma_A |\Omega_j|^{-1} = p(f_x,f_y) A . \qquad (3.20)$$

4. Statistical Roughness

4.1 Introductory Remarks

In many problems the equation of the interface is not known but a statistical description of it is available. Such problems may involve the light reflected from a single specular point--for example, a glitter on the sea surface--or the light reflected from an extended area--for example, an illuminated patch on the sea.

In Equation 3.8 we see that $|\Omega|^{-1}$ is the only random quantity on the right-hand side. Thus the statistical properties of $J^{refl'}$ can be found in a straightforward manner from the corresponding properties of $|\Omega|^{-1}$. Likewise, from Equations 3.9 and 3.16, we see that the statistics of $J_\Sigma^{refl'}$ can be found directly from the statistics of $p(f_x,f_y)$. The most important statistical quantities are the ensemble averages of $|\Omega|^{-1}$ and $p(f_x,f_y)$, which will now be stated.

4.2 Ensemble Average of $|\Omega|^{-1}$

The ensemble average $\{|\Omega|^{-1}\}_{av}$ will in general be a function of f_x and f_y and thus of the angles of incidence and reflection. However, the average is independent of slope in the very important case where $f(x,y)$ [or the instantaneous values of $f(x,y)$ for a fluctuating surface] can be considered as a sample function of a Gaussian process. This case has been studied by Longuet-Higgins (4,5,11,12,13). He finds

$$\{|\Omega|^{-1}\}_{av} = (3H)^{-1/2}\, N_\lambda \; ; \tag{4.1}$$

here

$$3H = m_{40}m_{04} - 4m_{31}m_{13} + 3m_{22}^2 \; , \tag{4.2}$$

$$m_{pq} = \int_{-\infty}^{\infty} E(u,v)u^p\, v^q\, du\, dv \; , \tag{4.3}$$

and $E(u,v)$ is the spectrum of $f(x,y)$,

$$E(u,v) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dxdy\, \cos(ux+vy)\{f(x+x',y+y')f(x',y')\}_{av} \; . \tag{4.4}$$

The quantity N_λ is a function of

$$\lambda = \frac{2\Delta}{(3H)^{3/2}} \; , \tag{4.5}$$

where

$$\Delta = \begin{vmatrix} m_{40} & m_{31} & m_{22} \\ m_{31} & m_{22} & m_{13} \\ m_{22} & m_{13} & m_{04} \end{vmatrix} . \tag{4.6}$$

Longuet-Higgins has shown that

$$2(3)^{-3/2} \geq \lambda \geq 0 \tag{4.7}$$

and that over this range N_λ (simply N in his notation) decreases monotonically from 1.571 to 1.500, a change of less than 5%; a table of N_λ versus λ is included in Reference 4. For an isotropic surface--one with statistics independent of the choice of x-, y-axes--λ takes on its maximum value and

$$N_{\lambda,\mathrm{Iso.}} = 1.5 \ . \tag{4.8}$$

The quantities H and Δ have physical significance, for it can be shown that*

$$\begin{aligned} 3H &= \left\{\Omega^2\right\}_{av} , \\ 6\Delta &= \left\{\Omega^3\right\}_{av} . \end{aligned} \tag{4.9}$$

4.3 Ensemble Average of p

The ensemble average of $p(f_x,f_y)$ is readily found for any homogeneous probability distribution--i.e., any distribution which is independent of x and y-- for it is just the value of the probability density function $\hat{p}(f_x,f_y)$ for the appropriate values of f_x and f_y. If the distribution is not homogeneous, we have the slightly more complicated expression

$$\left\{p(f_x,f_y)\right\}_{av} = \frac{1}{A} \int_A dA \ \hat{p} \ (f_x,f_y;x,y) \ ; \tag{4.10}$$

*Equation 4.5.14 of Reference 11 gives $6H = \left\{\Omega^2\right\}_{av}$, but this is a typographical error. The correct relationship appears everywhere else in Longuet-Higgins's work.

that is, the ensemble average of p is found by averaging $\hat{p}$ over the area A .

If the slope distribution is homogeneous and Gaussian, then

$$\left\{p(f_x,f_y)\right\}_{av} = \hat{p}(f_x,f_y) =$$

$$(2\pi\Lambda^{1/2})^{-1} \exp\left\{-(m_{02}f_x^2 - 2m_{11}f_xf_y + m_{20}f_y^2)/2\Lambda\right\}, \tag{4.11}$$

where

$$\Lambda = \begin{vmatrix} m_{20} & m_{11} \\ m_{11} & m_{02} \end{vmatrix} . \tag{4.12}$$

For an isotropic surface, this simplifies to

$$\left\{p(f_x,f_y)\right\}_{av} = (2\pi m_{02})^{-1} \exp\left\{-(f_x^2 + f_y^2)/2m_{02}\right\} =$$

$$(2\pi m_{02})^{-1} \exp\left\{-\frac{\sin^2\theta + \sin^2\theta' - 2\cos(\phi' - \phi)\sin\theta\sin\theta'}{2m_{02}(\cos\theta - \cos\theta')^2}\right\}. \tag{4.13}$$

5. Example--Reflection of Diffuse Illumination

5.1 Analysis

In this section we shall extend our previous work to find the matrix $\hat{W}_{\Sigma}^{refl'}$ which describes the power per steradian reflected in a given direction from an area A on a diffusely illuminated rough interface. Then, using this result, we shall study the noise introduced in an infrared temperature measurement by reflections from the surface of the body being observed.

Diffuse illumination will be defined as completely unpolarized and independent of direction (isotropic) for all directions with $\theta \leq \pi/2$. Then the power per unit area of wavefront incident in solid angle $d\omega$ is

$$W^{inc}\, d\omega = \frac{1}{2} W_o \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d\omega \qquad , \tag{5.1}$$

where the constant W_o gives the power level.

Using Equations 3.7, 3.8, 2.41, and 3.16, and integrating over the appropriate range of $d\omega$ we get (on the average)

$$\hat{W}_{\Sigma}^{refl'}(\theta',\phi') = \left(\frac{W_o A}{2}\right) \int_0^{\pi/2}\int_0^{2\pi} \sin\theta\, d\theta\, d\phi\, \hat{p}(f_x,f_y)$$

$$\times\, g(\theta,\phi;\theta',\phi') P\, \tilde{P}^*, \tag{5.2}$$

where

$$g = \frac{[1 - \cos\theta\cos\theta' - \cos(\phi' - \phi)\sin\theta\sin\theta']^2}{(\cos\theta - \cos\theta')^4} \tag{5.3}$$

and $P\tilde{P}^*$ is given by Equation 2.41. In deriving Equation 5.2, we have assumed negligible the errors due to shadowing and multiple reflection at near-grazing incidence; this assumption is satisfactory for a moderate degree of roughness.

Usually it will be simplest to evaluate Equation 5.2 if we eliminate (θ,ϕ) coordinates and integrate over an appropriate range of (f_x,f_y). Thus we write

$$d\omega = \sin\theta\, d\theta\, d\phi = |Y|^{-1} \sin\theta\, df_x df_y \ , \tag{5.4}$$

where Y is the Jacobian

$$Y = \frac{\partial(f_x, f_y)}{\partial(\theta, \phi)} \ . \tag{5.5}$$

Evaluating Y from Equation 2.36 and combining 5.3 and 5.4, we find

$$g \sin\theta \, d\theta \, d\phi = g \, |Y|^{-1} \sin\theta \, df_x \, df_y =$$

$$\frac{1 - \cos\theta \cos\theta' - \cos(\phi' - \phi)\sin\theta \sin\theta'}{\cos\theta - \cos\theta'} \, df_x df_y \ . \tag{5.6}$$

Further applying Equations 2.15, 2.39, and A2.3, we can reduce this to

$$g \sin\theta \, d\theta \, d\phi = (\lambda' \sin\theta' - \cos\theta') df_x df_y \ . \tag{5.7}$$

The integration must be taken over all (f_x, f_y) for which $\cos\theta \geq 0$. Reference to Equation A2.3, shows that the area of integration is the disc $\mathcal{R}$ in (f_x, f_y) space given by

$$2\lambda' \sin\theta' - (1 - \mu^2)\cos\theta' \geq 0 \ , \tag{5.8}$$

or equivalently,

$$(f_x \cos\theta' + \cos\phi' \sin\theta')^2 + (f_y \cos\theta' + \sin\phi' \sin\theta')^2 \leq 1 \ . \tag{5.9}$$

In interpreting this equation it must be remembered that $\cos\theta'$ is never positive.

Now we can rewrite Equation 5.2 as

$$\hat{W}_{\Sigma}^{\text{refl}'}(\theta', \phi') = W_o A Q \ ;$$

$$\boxed{Q = \frac{1}{2} \iint_{\mathcal{R}} df_x df_y (\lambda' \sin\theta' - \cos\theta') \, \hat{p}(f_x, f_y) P\tilde{P}^*} \ . \tag{5.10}$$

In evaluating $P\tilde{P}^*$ we use the expressions of Equation 2.39 for the B_j. For the usual situation in which $\mu_1 = \mu_2$, the Fresnel coefficients are readily found to be

$$R_{||} = \frac{n^2(\lambda'\sin\theta' - \cos\theta') - [(n^2-1)(1+\mu^2) + (\lambda'\sin\theta' - \cos\theta')^2]^{1/2}}{n^2(\lambda'\sin\theta' - \cos\theta') + [(n^2-1)(1+\mu^2) + (\lambda'\sin\theta' - \cos\theta')^2]^{1/2}}$$

and

$$R_{\perp} = \frac{(\lambda'\sin\theta' - \cos\theta') - [(n^2-1)(1+\mu^2) + (\lambda'\sin\theta' - \cos\theta')^2]^{1/2}}{(\lambda'\sin\theta' - \cos\theta') + [(n^2-1)(1+\mu^2) + (\lambda'\sin\theta' - \cos\theta')^2]^{1/2}} \,. \tag{5.11}$$

5.2 Application to Infrared Measurements

Consider a body composed of homogeneous lossy dielectric material and having a rough plane surface. It is assumed that the other boundaries of the body are sufficiently remote that they have no appreciable effect on either reflection or emission from the rough surface. A standard method of determining the temperature of such a body is through measurement of the infrared radiation emitted across the surface.

One form of noise limiting the accuracy of the measurement is infrared energy from external sources which is reflected from the surface and enters the detector. Often it is not possible to control the external sources, and thus any suppression of reflection noise must be accomplished at the detector. We shall now discuss a suppression technique.

If the surface were perfectly plane, the medium nonconducting and nondispersive, and the detector field of view very narrow, then complete elimination of reflection noise could be obtained by viewing the surface at the Brewster angle with a polarization-sensitive detector which rejects the E-field component perpendicular to the plane of reflection. This fact suggests that even when roughness, conductivity, dispersion, and detector field of view must be considered, it may be possible to reduce reflection noise by viewing the surface at or near the Brewster angle* and rejecting the perpendicular-polarized E-field.

When the surface is illuminated diffusely--a situation frequently encountered in practice--the noise reduction can be studied quantitatively by use of Equation 5.10. If we take into account roughness and conductivity, consider observation at angles other than the Brewster angle, but still assume a nondispersive medium and a very narrow field of view, then the reflection noise polarized parallel to the plane of reflection is proportional to Q_{11} and the perpendicular-polarized reflection noise is proportional to Q_{22}. The effect of a finite field of view may be found by comparing values of the Q_{ii} for slightly different angles of observation; dispersion may be studied by comparing values calculated for different frequencies.

Some calculations of practical interest are displayed in Figures 2-6. An isotropic Gaussian slope distribution is assumed; that is, we set

$$\hat{p}\,(f_x,f_y) = \frac{1}{2\pi\sigma^2} \exp\left\{-\,(f_x^2 + f_y^2)/2\sigma^2\right\}. \tag{5.12}$$

---

*For a lossy material the Brewster angle is defined as that angle for which $R_{||}$ is a minimum. For losses of the order to be considered here, a good approximation to the angle is $\Theta^* = (180° - \Theta') \approx \tan^{-1}[\mathrm{Re}(n)]$.

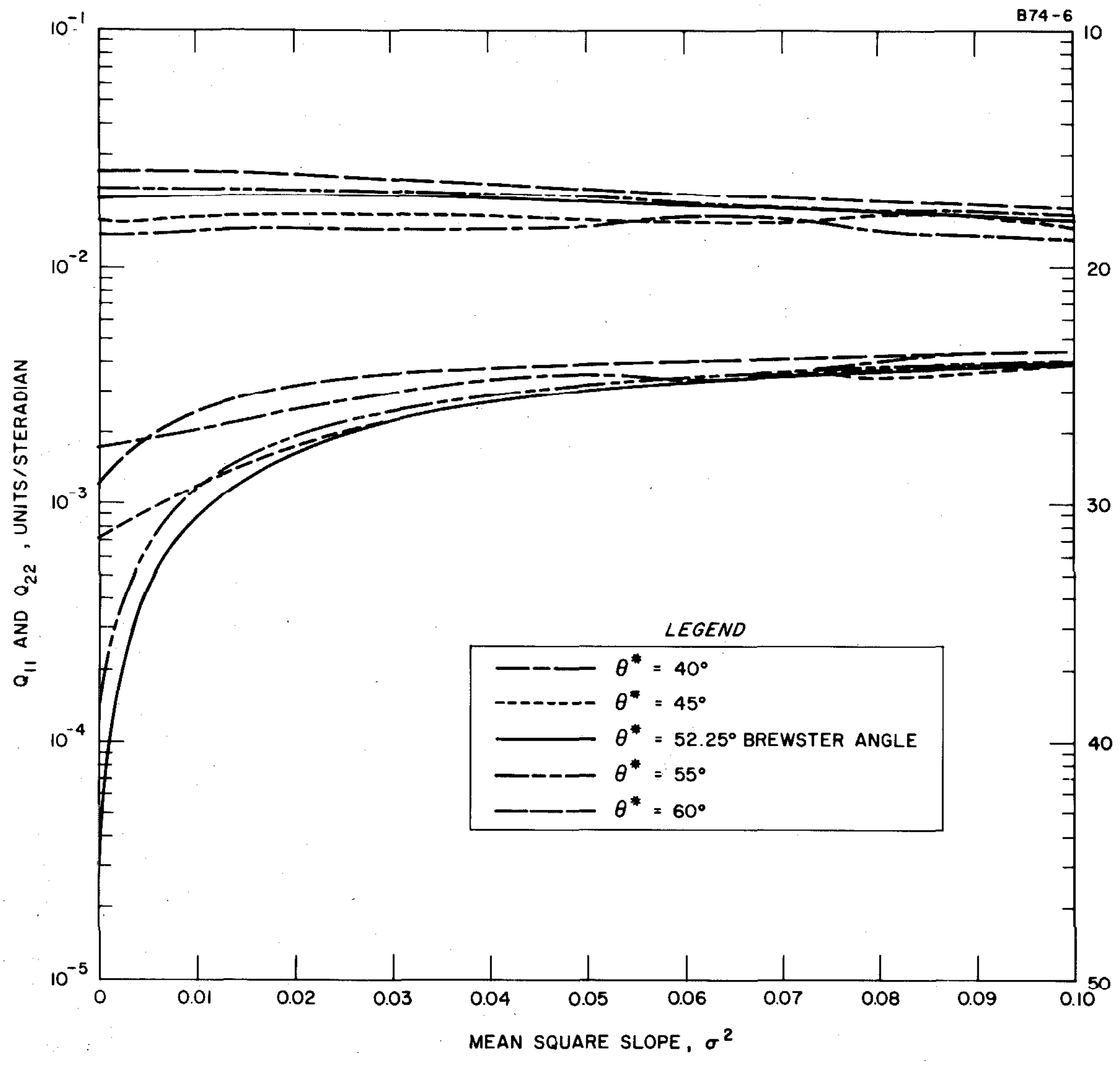

Fig. 2. Values of Q_{11} and Q_{22} versus σ^2 for various θ^* at $n = 1.292 + i\ 0.0472$ $(\lambda = 8\mu)$.

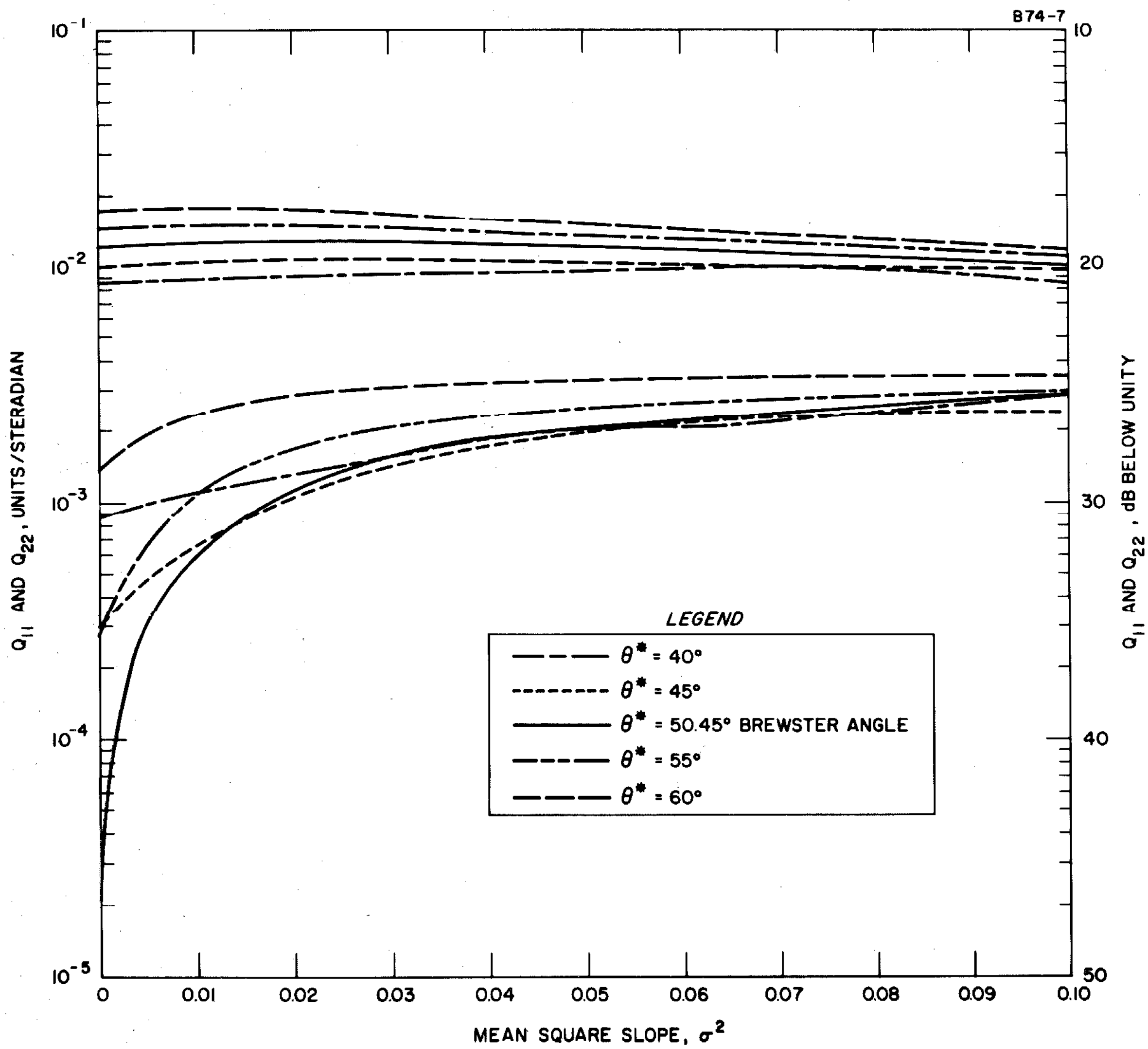

Fig. 3. Values of Q_{11} and Q_{22} versus σ^2 for various θ^* at $n = 1.212 + 0.0601$ ($\lambda = 10\mu$).

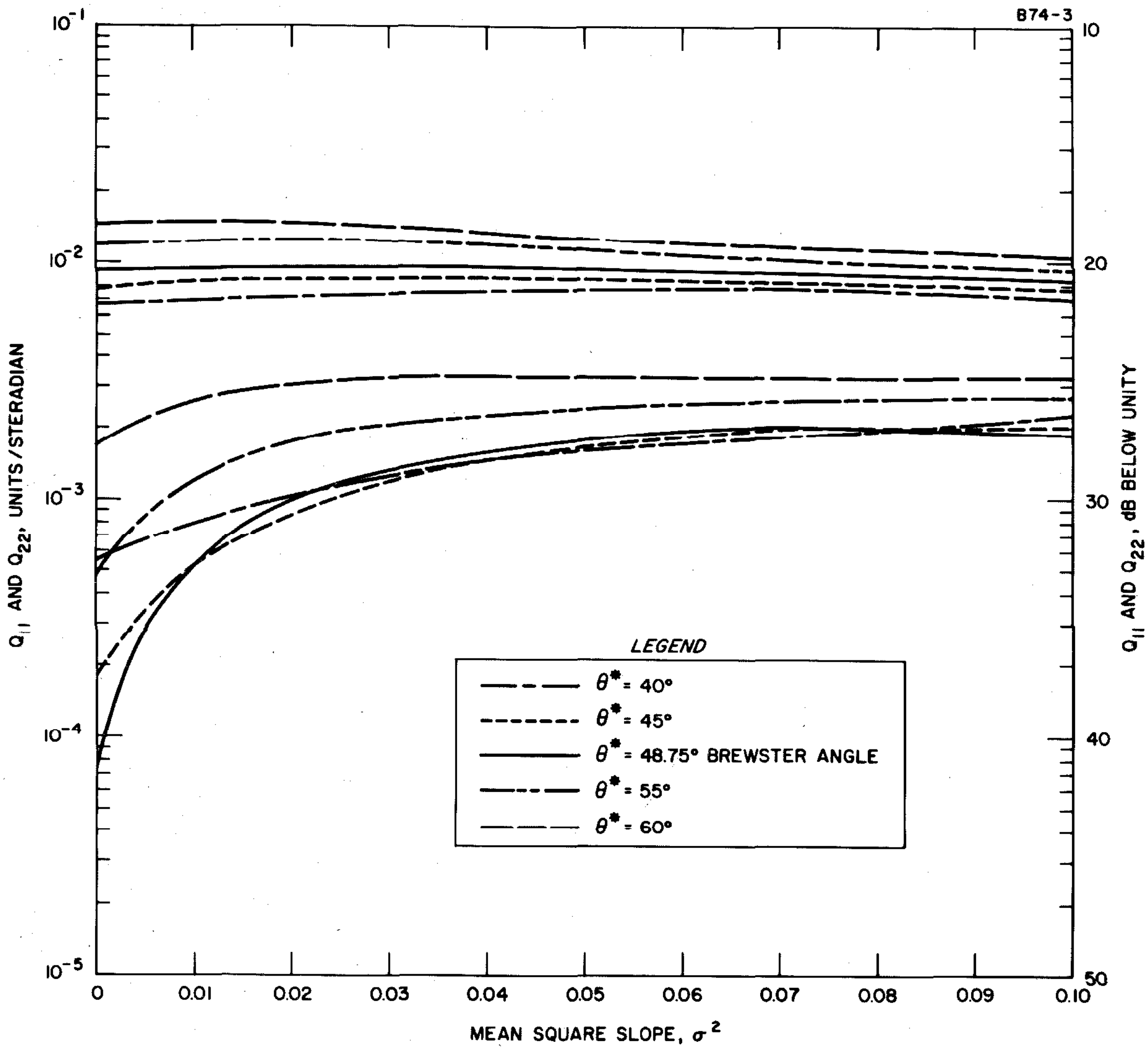

Fig. 4. Values of Q_{11} and Q_{22} versus σ^2 for various θ^* at $n = 1.143 + i\ 0.1138$ ($\lambda = 11\mu$).

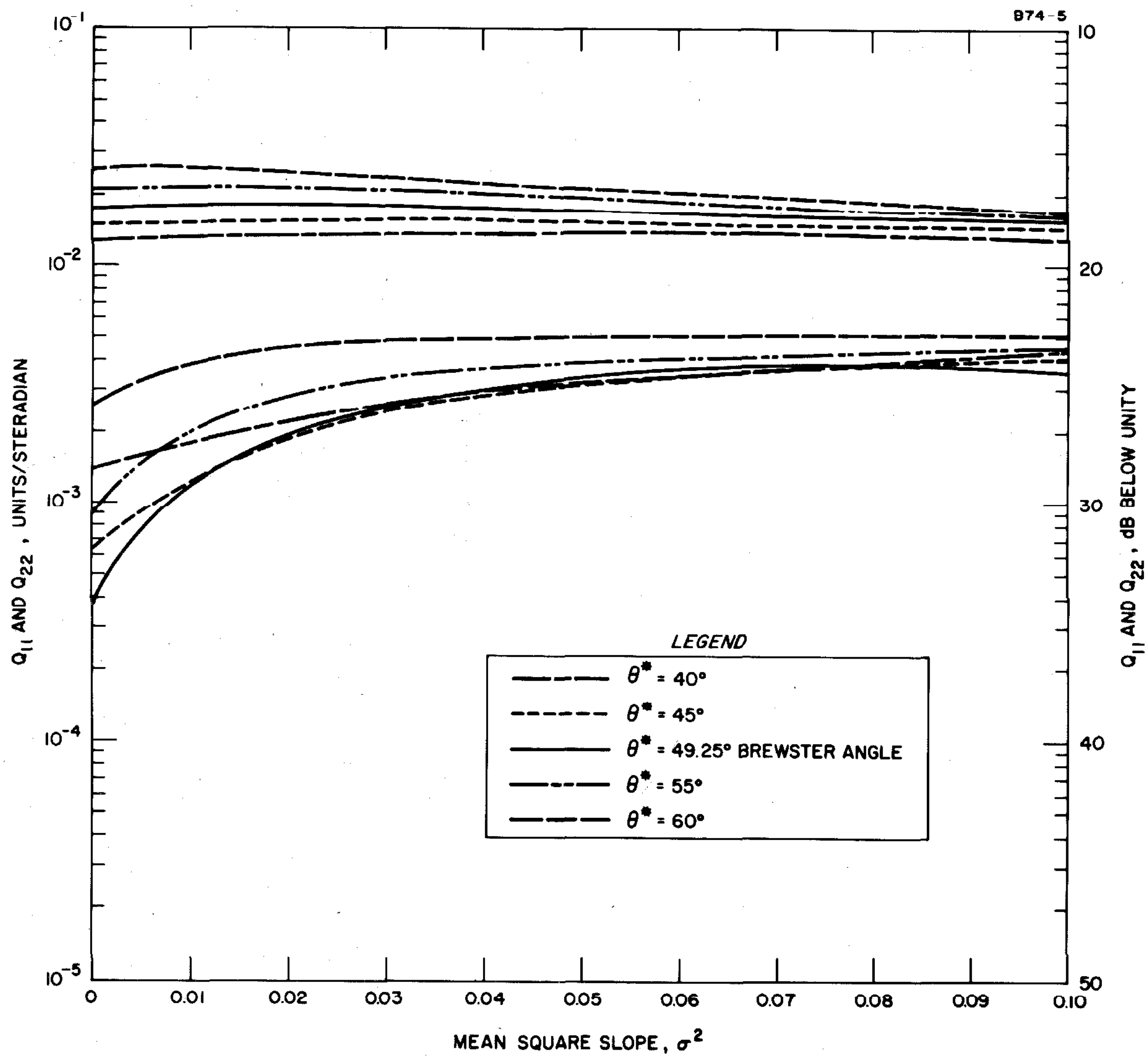

Fig. 5. Values of Q_{11} and Q_{22} versus σ^2 for various θ^* at $n = 1.165 + i\ 0.2058$ ($\lambda = 12\mu$).

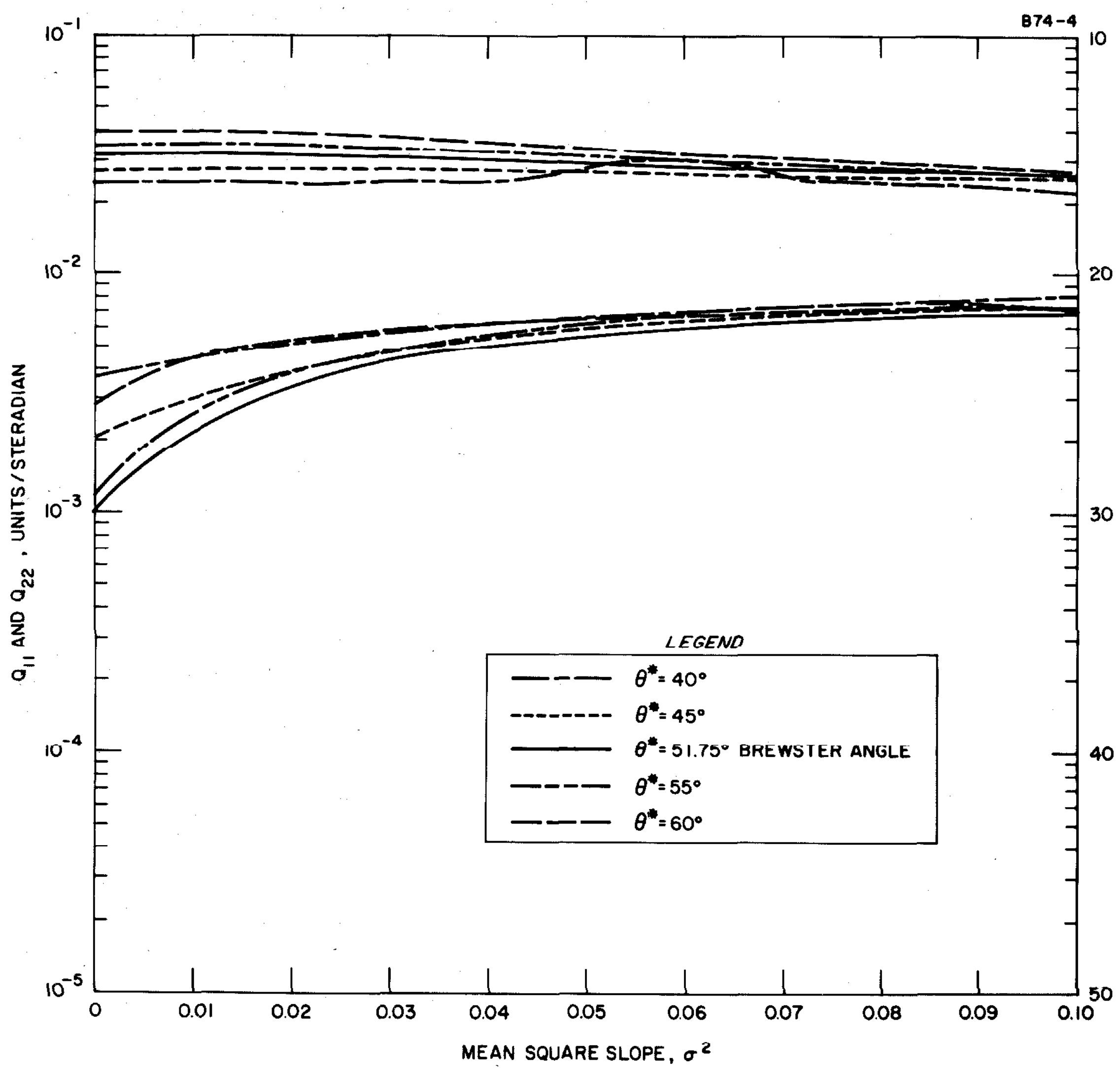

Fig. 6. Values of Q_{11} and Q_{22} versus σ^2 for various θ^* at $n = 1.270 + i\ 0.2918$ ($\lambda = 13\mu$).

Then Q_{11} and Q_{22} (Q_{22} is always the upper curve) are plotted against σ^2 with the angle

$$\theta^* = (180^\circ - \theta') \tag{5.13}$$

as a parameter. Each figure corresponds to a different value of n, the values chosen corresponding to the values for water at 8, 10, 11, 12, and 13 microns wavelength as given by Centeno (14). The range of σ^2 is greater than that encountered in water waves in the absence of whitecaps. In interpreting the curves in terms of improvement in signal-to-noise ratio, it should be remembered that the thermal radiation is approximately unpolarized and thus a polarizer excludes half the useful signal power.

Inspection of the data shows that, over a fairly wide range of θ^* around the Brewster angle, rejection of Q_{22} improves the signal-to-noise ratio by at least 3db.. As would be expected, the improvement is greatest for small values of σ^2; an increase of 10 to 25db. in signal-to-noise ratio is possible for surfaces with $\sigma^2 < 0.01$ provided a small enough field of view and a narrow enough bandwidth are used. A more detailed discussion of the data is contained in Reference 15.

6. Concluding Remarks

Let us now review what has been accomplished. Within the framework of geometrical optics, expressions have been derived giving the intensity and polarization of the light reflected from a single specular point on a rough interface when a plane wave of arbitrary polarization is incident. Similar expressions have been found for the properties of the light reflected from an extended area on a rough interface; here, however, it has been necessary to require that the roughness not be too steep, that the angles of incidence and reflection not be too near grazing, and that the return from each specular point be an independent beam.

The above-mentioned expressions constitute our most important results. It is significant to note that despite the great amount of algebraic manipulation necessary to derive the expressions, the final forms are fairly simple and compact, especially so in angular coordinates. The expressions, besides enabling us to find numbers, aid us in visualizing the phenomenon.

Special attention has been given to the case in which the interface is a sample function of a Gaussian random process. The average statistics of the reflected field have been found through use of some results of Longuet-Higgins.

The field reflected from a diffusely illuminated interface has been analyzed, and the results have been used to study the suppression of reflection noise in infrared measurements. In the specific case considered, we find appreciable suppression is obtained by filtering out the perpendicular-polarized E-field and viewing the surface at an angle not too far from the Brewster angle.

No attempt has been made to extend the results to rough curved interfaces. Such an extension is straightforward in many individual cases, but a general treatment would be somewhat messy.

In closing, we remark that the matrix P of Equation 2.27 is potentially useful in the synthesis and analysis of optical systems involving tilted plane reflectors. For example, it can be used in studying the effect of varying the angle of a beam-splitter in an interferometer.

APPENDIX 1

COMPUTATION OF THE MATRIX P

The derivation of Equations 2.27-2.32 requires the multiplication together of five matrices and simplification of the results. The process involves repeated use of standard vector and trigonometric identities. No attempt will be made to include these details here, but the calculations will be outlined in sufficient depth to enable the reader to reconstruct them.

Combining Equations 2.2, 2.7, and 2.23-2.25 gives

$$M = \frac{1}{|\underset{\sim}{e}_z \times \hat{s}^{inc}|}$$

$$\times \begin{bmatrix} \dfrac{(\hat{s}^{inc} + \hat{s}^{refl}) \cdot [(\underset{\sim}{e}_z \times \hat{s}^{inc}) \times \hat{s}^{inc}]}{|\hat{s}^{inc} + \hat{s}^{refl}|} & \dfrac{(\hat{s}^{inc} + \hat{s}^{refl}) \cdot (\underset{\sim}{e}_z \times \hat{s}^{inc})}{|\hat{s}^{inc} + \hat{s}^{refl}|} \\ \dfrac{(\hat{s}^{inc} \times \hat{s}^{refl}) \cdot [(\underset{\sim}{e}_z \times \hat{s}^{inc}) \times \hat{s}^{inc}]}{|\hat{s}^{inc} \times \hat{s}^{refl}|} & \dfrac{(\hat{s}^{inc} \times \hat{s}^{refl}) \cdot (\underset{\sim}{e}_z \times \hat{s}^{inc})}{|\hat{s}^{inc} \times \hat{s}^{refl}|} \\ \dfrac{(\hat{s}^{inc} - \hat{s}^{refl}) \cdot [(\underset{\sim}{e}_z \times \hat{s}^{inc}) \times \hat{s}^{inc}]}{|\hat{s}^{inc} - \hat{s}^{refl}|} & \dfrac{(\hat{s}^{inc} - \hat{s}^{refl}) \cdot (\underset{\sim}{e}_z \times \hat{s}^{inc})}{|\hat{s}^{inc} - \hat{s}^{refl}|} \end{bmatrix} =$$

$$\frac{1}{|\underset{\sim}{e}_z \times \underset{\sim}{s}^{inc}|} \begin{bmatrix} \dfrac{\beta_i}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}|} & \dfrac{\alpha}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}|} \\ \\ \dfrac{-2\alpha}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}||\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} & \dfrac{2\beta_i}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}||\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} \\ \\ \dfrac{-\beta_i}{|\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} & \dfrac{-\alpha}{|\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} \end{bmatrix}, \qquad (A1.1)$$

where the second form is obtained by using various vector identities and defining

$$\alpha = \underset{\sim}{e}_z \cdot (\underset{\sim}{s}^{inc} \times \underset{\sim}{s}^{refl}) = \sin(\phi' - \phi)\sin\Theta \sin\Theta' \qquad (A1.2)$$

and

$$\beta_i = \underset{\sim}{e}_z \cdot [\underset{\sim}{s}^{inc} \times (\underset{\sim}{s}^{inc} \times \underset{\sim}{s}^{refl})] \quad . \qquad (A1.3)$$

By an analogous procedure, we obtain

$$\tilde{N} = \frac{1}{|\underset{\sim}{e}_z \times \underset{\sim}{s}^{refl}|}$$

$$\times \begin{bmatrix} \dfrac{-\beta_r}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}|} & \dfrac{-2\alpha}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}||\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} & \dfrac{-\beta_r}{|\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} \\ \\ \dfrac{-\alpha}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}|} & \dfrac{2\beta_r}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}||\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} & \dfrac{-\alpha}{|\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|} \end{bmatrix},$$

$$(A1.4)$$

where

$$\beta_r = \underset{\sim}{e}_z \cdot [\underset{\sim}{s}^{refl} \times (\underset{\sim}{s}^{inc} \times \underset{\sim}{s}^{refl})] \,. \tag{A1.5}$$

Then Equation 2.22 gives

$$P = \frac{1}{|\underset{\sim}{e}_z \times \underset{\sim}{s}^{inc}||\underset{\sim}{e}_z \times \underset{\sim}{s}^{refl}|}$$

$$\times \begin{bmatrix} \delta_1\beta_i\beta_r R_{||} + \delta_2 \alpha^2 R_{\perp} & \delta_1\alpha\beta_r R_{||} - \delta_2\alpha\beta_i R_{\perp} \\ \delta_1\alpha\beta_i R_{||} - \delta_2\alpha\beta_r R_{\perp} & \delta_1\alpha^2 R_{||} + \delta_2\beta_i\beta_r R_{\perp} \end{bmatrix}, \tag{A1.6}$$

where

$$\delta_1 = \frac{1}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}|^2} + \frac{1}{|\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|^2} \,,$$

$$\delta_2 = \frac{4}{|\underset{\sim}{s}^{inc} + \underset{\sim}{s}^{refl}|^2 \, |\underset{\sim}{s}^{inc} - \underset{\sim}{s}^{refl}|^2} \,. \tag{A1.7}$$

Now we define γ by

$$\gamma = \underset{\sim}{s}^{inc} \cdot \underset{\sim}{s}^{refl} - (\underset{\sim}{e}_z \cdot \underset{\sim}{s}^{inc})(\underset{\sim}{e}_z \cdot \underset{\sim}{s}^{refl}) = \cos(\phi' - \phi)\sin\theta \sin\theta' \,. \tag{A1.8}$$

Then it is easily shown that

$$\begin{aligned} \beta_i &= \gamma \cos\theta - \sin^2\theta \cos\theta' \\ \beta_r &= \cos\theta \sin^2\theta' - \gamma \cos\theta' \end{aligned} \tag{A1.9}$$

$$\delta_1 = \delta_2 = [\, 1 - (\cos\theta \cos\theta' + \gamma)^2 \,]^{-1} \,.$$

Also,

$$\frac{1}{|\underset{\sim}{e}_z \times \underset{\sim}{s}^{inc}||\underset{\sim}{e}_z \times \underset{\sim}{s}^{refl}|} = \frac{1}{\sin\theta \sin\theta'} \quad . \tag{A1.10}$$

The next step is to substitute Equations A1.9 and A1.10 into Equation A1.6 and express P as a function of α, γ, θ and θ' . Then we use Equations A1.2 and A1.8 to convert this expression to the form given by Equations 2.27-2.32. The details can readily be supplied by the interested reader.

APPENDIX 2

TRANSFORMATION TO MIXED COORDINATE SYSTEM

We shall calculate B_2 in terms of (f_x, f_y) and (θ', ϕ'). The other calculations leading to Equations 2.38 and 2.39 are very similar.

First Equation 2.36 is written as

$$\cos\phi \sin\theta = \cos\phi' \sin\theta' - f_x(\cos\theta - \cos\theta') ,$$

$$\sin\phi \sin\theta = \sin\phi' \sin\theta' - f_y(\cos\theta - \cos\theta') . \qquad (A2.1)$$

Squaring and adding these two equations we obtain

$$1 - \cos^2\theta = \sin^2\theta' + \mu^2(\cos\theta - \cos\theta')^2 - 2\lambda' \sin\theta'(\cos\theta - \cos\theta') ; \qquad (A2.2)$$

this is a quadratic equation for $\cos\theta$. Solving and rejecting the extraneous root $\cos\theta'$, we find

$$\cos\theta = [2\lambda' \sin\theta' - (1 - \mu^2)\cos\theta']/(1 + \mu^2) ,$$

$$\cos\theta - \cos\theta' = 2(\lambda' \sin\theta' - \cos\theta')/(1 + \mu^2) . \qquad (A2.3)$$

From Equation 2.30 we have

$$B_2 = \cos\phi' \sin\theta' \cos\phi \cos\theta + \sin\phi' \sin\theta' \sin\phi \cos\theta - \cos\theta' \sin\theta . \qquad (A2.4)$$

Using Equations A2.1 and 2.37, we obtain

$$B_2 = \left\{\cos\theta \sin^2\theta' - \cos\theta' \sin^2\theta - \lambda' \sin\theta' \cos\theta\,(\cos\theta - \cos\theta')\right\}/\sin\theta =$$

$$(\cos\theta - \cos\theta')\,[1 - (\lambda' \sin\theta' - \cos\theta')\cos\theta]/\sin\theta . \qquad (A2.5)$$

Substituting from Equation A2.3 into Equation A2.5 gives the result cited in Equation 2.39.

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PART III

SCATTERING OF ELECTROMAGNETIC WAVES FROM AN INTERFACE WITH SMALL IRREGULARITIES

1. Introduction

We present here a theoretical study of the scattering of an arbitrary time harmonic electromagnetic field from an irregular interface between two linear homogeneous isotropic media. More specifically, we treat the very important case in which the interface can be represented as the mathematical superposition on a smooth underlying (unperturbed) interface of small scale irregularity. By small-scale irregularity, we mean irregularity of amplitude small compared to the wavelengths of interest and compared to the local radii of curvature of the unperturbed surface, and of slope small compared to unity. For practical application of the results, it is further necessary that the underlying interface be of simple enough shape so that the unperturbed scattering problem can be treated adequately. Problems of the type described occur in connection with radio wave propagation over the ocean, background clutter in radar observations, reflection of radar signals from natural and artificial bodies in space, scattering of light by polished but slightly irregular mirrors and lenses, and many other situations of practical importance.

Our treatment is based on the method of small perturbations, a technique first developed by Lord Rayleigh (1) for a similar problem, reflection of a scalar wave from an irregular plane wall. In this method, the irregularity of the surface is characterized by a small displacement parameter ϵ, and the field is calculated as a power series in ϵ, the constant term being the field in the unperturbed case. Usually only the first or first few coefficients of the power series are actually calculated. In many discussions of the method, including Lord Rayleigh's, the small parameter ϵ is not expressed explicitly.

The method of small perturbations has already been applied successfully to scattering of a vector wave at an irregular plane interface. Bass and Bocharov (2) have solved this problem for an arbitrary wave incident on a perfectly conducting interface; Rice (3) has solved it for a plane wave incident from the dielectric side on an interface between a dielectric and an arbitrary medium. Quite different representations of the results are obtained in the two solutions.

There do not appear to be in the literature any satisfactory treatments of vector problems in which the underlying interface is not a plane. Two interesting papers on the analogous scalar (acoustical) problem have been published recently by Kur'yanov (4) and Lapin (5), but their approaches are not as general as that to be presented here for the vector problem.

The approach introduced in this paper can best be understood against the background of the work of Bass and Bocharov and of Rice. Their work is therefore recapitulated in Section 2, with emphasis on the features common to both analyses.

In order to proceed to a more general analysis, we adopt a point of view different from that of either of the background papers. We look upon the perturbation technique as a method of mathematically replacing the surface irregularities by appropriate electric and magnetic surface currents imposed on the unperturbed interface. Once these currents have been determined, the fields anywhere in space can be found by using dyadic Green's functions (henceforth abbreviated d.G.f.'s).

These ideas are developed in Sections 3 and 4. In Section 3 we treat the theory of d.G.f.'s, including both material which is used directly in later sections and material of general interest. Some of this material has not appeared in the literature previously or has appeared in incorrect form.

Section 4 contains the most important results: general expressions to second order in ϵ for the effective surface currents and for the perturbed fields. In an appendix to this section, it is shown that the results of Rice and of Bass and Bocharov are in agreement with the more general theory.

In Section 5, we extend the analysis to problems in which the interface irregularity is described statistically. Specifically, we calculate various significant averages of the perturbation fields in terms of the mean and correlation function of the interface irregularity.

To illustrate the application of the theory, in Section 6 we calculate to first order the field scattered when a plane wave is incident on a perfectly conducting cylinder with sinusoidal surface irregularities.

Harmonic time dependence $e^{-i\omega t}$ is to be understood everywhere. Since Rice and Bass and Bocharov assumed a time factor $e^{+i\omega t}$, the results cited here will be the complex conjugates of the original forms.

2. A Survey of Previous Results

2.1 Method of Bass and Bocharov

Bass and Bocharov's derivation (2) is rather short and shall be repeated here in full, with somewhat modified notation. Let the equation of a perfectly conducting irregular plane interface S be

$$Q = z' - z_o(x', y') = z' - z_o(\underset{\sim}{r}'_o) = 0 \ , \tag{2.1.1}$$

and let the propagation constant in the upper half-space be k_2 . The function z_o is constrained to take on values small compared to the wavelength of the incident field. The small parameter ϵ may be identified with the maximum absolute value of z_o , the RMS value of z_o , or any other convenient measure of the magnitude of the irregularity.

The perturbed electric field is written

$$\underset{\sim}{E}^{\epsilon}(\underset{\sim}{r}') = \underset{\sim}{E}^{o}(\underset{\sim}{r}') + \triangle \underset{\sim}{E}(\underset{\sim}{r}') = \underset{\sim}{E}^{o}(\underset{\sim}{r}') + \delta \underset{\sim}{E}(\underset{\sim}{r}') + \delta^2 \underset{\sim}{E}(\underset{\sim}{r}') + \cdots \ , \tag{2.1.2}$$

where $\underset{\sim}{E}^{o}$ is the total unperturbed field--incident plus reflected--and $\delta^n \underset{\sim}{E}$ is the perturbation field of order z_o^n , i.e., of order ϵ^n . The field at the irregular interface is also expressed as

$$\underset{\sim}{E}^{\epsilon}(\underset{\sim}{r}'_o + \underset{\sim}{e}_{z'}\, z_o) =$$

$$\underset{\sim}{E}^{\epsilon}(\underset{\sim}{r}'_o) + \frac{\partial}{\partial z'} \underset{\sim}{E}^{\epsilon}(\underset{\sim}{r}'_o)\, z_o + \frac{1}{2}\left(\frac{\partial}{\partial z'}\right)^2 \underset{\sim}{E}^{\epsilon}(\underset{\sim}{r}'_o) z_o^2 + \cdots \ . \tag{2.1.3}$$

Furthermore, since $\underset{\sim}{E}^{\epsilon}$ must vanish tangent to the interface, we have

$$E^{\epsilon}_{x'} + E^{\epsilon}_{z'} \frac{\partial}{\partial x'} z_o = 0 \ , \quad E^{\epsilon}_{y'} + E^{\epsilon}_{z'} \frac{\partial}{\partial y'} z_o = 0 \text{ on } S \ . \tag{2.1.4}$$

Combining the above equations, we obtain for the effective first and second order fields at the unperturbed interface

$$\delta E_{x',y'}(\underset{\sim}{r}'_o) = - z_o \frac{\partial}{\partial z'} E^o_{x',y'} - E^o_{z'} \frac{\partial}{\partial x',y'} z_o \quad ,$$

$$\delta^2 E_{x',y'}(\underset{\sim}{r}'_o) = - z_o \frac{\partial}{\partial z'} \delta E_{x',y'} - \delta E_{z'} \frac{\partial}{\partial x',y'} z_o \quad . \qquad (2.1.5)$$

These fields correspond to effective magnetic surface currents

$$\delta \underset{\sim}{K}_m = - \underset{\sim}{e}_{z'} \times \delta \underset{\sim}{E} \quad , \quad \delta^2 \underset{\sim}{K}_m = - \underset{\sim}{e}_{z'} \times \delta^2 \underset{\sim}{E} \quad . \qquad (2.1.6)$$

The field everywhere in the upper half-space can be found by applying to Equation 2.1.5 the Kirchhoff formulas

$$E_{x,y}(\underset{\sim}{r}) = - 2 \frac{\partial}{\partial z} \int_{S_o} dS' \, E_{x',y'}(\underset{\sim}{r}'_o) \, G_f \, (\underset{\sim}{r};\underset{\sim}{r}'_o) \quad , \qquad (2.1.7)$$

$$E_z(\underset{\sim}{r}) = 2 \int_{S_o} d\,S' \left[\frac{\partial}{\partial x'} E_{x'}(\underset{\sim}{r}'_o) + \frac{\partial}{\partial y'} E_{y'}(\underset{\sim}{r}'_o)\right] G_f \, (\underset{\sim}{r};\underset{\sim}{r}'_o) \quad . \qquad (2.1.8)$$

Here S_o is the unperturbed interface $z' = 0$, and G_f is the scalar Green's function for an unbounded region.

$$G_f \, (\underset{\sim}{r};\underset{\sim}{r}') = \frac{1}{4\pi|\underset{\sim}{r} - \underset{\sim}{r}'|} \exp \, (i \, k_2 |\underset{\sim}{r} - \underset{\sim}{r}'|) \quad . \qquad (2.1.9)$$

2.2 Rice's Method

Rice's derivation (3) is quite different in form. It is a direct extension to vector problems of Rayleigh's scalar solution (1) and is thus based on a mode representation of the scattered field. Details of the analysis are lengthy, so we shall only outline the procedure and cite the results here.

The interface is again described by Equation 2.1.1. A plane wave is incident from the upper medium M_2, characterized by permeability μ_2 and real propagation constant k_2. The lower medium M_1 is characterized by μ_1 and k_1, where k_1 may be complex. The irregularity function z_o is assumed periodic in x and y with very long period L so that we can write the Fourier series expansion

$$z_o(x',y') = \Sigma_{m,n} P(m,n) \exp [- i a (mx' + ny')] , \tag{2.2.1}$$

where

$$a = 2\pi/L \tag{2.2.2}$$

and the sums run from $-\infty$ to $+\infty$. The angle of the incident plane wave is constrained to be such that the incident field has period L in both x and y. The periodicity requirements are important to the derivation, but in practical applications we can treat non-periodic roughness and arbitrary angle of incidence by letting L approach infinity.

The perturbation field is assumed to be of the form

$$\Delta \underset{\sim}{E}(\underset{\sim}{r}) = \Sigma_{m,n} [\underset{\sim}{e}_x A_{mn} + \underset{\sim}{e}_y B_{mn} + \underset{\sim}{e}_z C_{mn}] E(m,n,z) , \quad z > z_o ;$$

$$\Delta \underset{\sim}{E}(\underset{\sim}{r}) = \Sigma_{m,n} [\underset{\sim}{e}_x G_{mn} + \underset{\sim}{e}_y H_{mn} + \underset{\sim}{e}_z I_{mn}] F(m,n,z) , \quad z < z_o . \tag{2.2.3}$$

Here the mode functions are the plane waves

$$E(m,n,z) = \exp\left\{i a (mx + ny) + i b(m,n)z\right\} ,$$

$$F(m,n,z) = \exp\left\{i a (mx + ny) - i c(m,n)z\right\} , \tag{2.2.4}$$

where

$$b(m,n) = + [k_2^2 - a^2 (m^2 + n^2)]^{1/2}, \; c(m,n) = + [k_1^2 - a^2 (m^2 + n^2)]^{1/2}. \qquad (2.2.5)$$

The first step in determining the six sets of unknown coefficients is to expand each coefficient as a power series in ϵ ; thus, for example,

$$A_{mn} = A_{mn}^{(1)} + A_{mn}^{(2)} + A_{mn}^{(3)} + \cdots \quad , \qquad (2.2.6)$$

where $A_{mn}^{(\ell)}$ is of order ϵ^{ℓ}. Next, the $E(m,n,z)$ and $F(m,n,z)$ are expanded in power series in z. Then the tangential field boundary conditions are written to second order and the expansion of $\Delta \underset{\sim}{E}$ is inserted. Terms of the same order are equated, and we thus obtain a set of linear algebraic equations for the first and second order approximations to the coefficients of Equation 2.2.3. The solution of these equations is straightforward.

Rice gives the results for a vertically polarized (no H_z) plane wave incident on a perfectly conducting interface and for a horizontally polarized (no E_z) plane wave incident on a medium M_1 with k_1 arbitrary but $\mu_1 = \mu_2$.

In the vertical polarization case, the unperturbed field in Medium M_2 is, for an incident wave of unit amplitude,

$$\underset{\sim}{E}^{o}(\underset{\sim}{r}) = 2 \exp\{i\, k_2\, \alpha\, x\}[- i\, \gamma \sin (k_2\, \gamma\, z)\, \underset{\sim}{e}_x + \alpha \cos (k_2\, \gamma\, z)\, \underset{\sim}{e}_z\,]; \qquad (2.2.7)$$

here

$$\alpha = \sin \theta\ ,\ \gamma = \cos \theta \quad , \qquad (2.2.8)$$

and θ is the angle between the positive z-axis and the direction of propagation of the incident wave. The coefficients are

$$A_{mn}^{(1)} = -2\,i\,(\alpha\,a\,m - k_2)\,P\,(m-\nu,n)\;,\; B_{mn}^{(1)} = -2\,i\,\alpha\,a\,n\,P(m-\nu,n),$$

$$C_{mn}^{(1)} = 2\,i\,b^{-1}\,(m,n)\,[a\,(\nu-m)\,k_2 - \alpha\,b^2\,(m,n)]\,P(m-\nu,n)\;;$$

$$A_{mn}^{(2)} = 2\;\Sigma_{k\,\ell}[a^2\,(m-k)(\nu-k)\,k_2 + (k_2-\alpha\,a\,m)\,b^2\,(k,\ell)]\,Q(m,n,k,\ell)\;,$$

$$B_{mn}^{(2)} = 2a\;\Sigma_{k\ell}[a\,(n-\ell)(\nu-k)\,k_2 - \alpha\,n\,b^2\,(k,\ell)]\,Q(m,n,k,\ell)\;,$$

$$C_{mn}^{(2)} = 2b^{-1}\,(m,n)\;\Sigma_{k\,\ell}\Big\{a^3\,(k-\nu)\,(m^2+n^2-m\,k-n\,\ell)\,k_2$$

$$+\,a\,[\alpha\,a\,(m^2+n^2) - m\,k_2]\,b^2\,(k,\ell)\Big\}Q(m,n,k,\ell)\;;$$

$$G_{mn}^{(1)} = G_{mn}^{(2)} = H_{mn}^{(1)} = H_{mn}^{(2)} = I_{mn}^{(1)} = I_{mn}^{(2)} = 0\;. \qquad (2.2.9)$$

Here

$$a\,\nu = k_2\,\alpha\;,\; Q(m,n,k,\ell) = b^{-1}\,(k,\ell)\,P\,(k-\nu,\ell)\,P\,(m-k,n-\ell)\,. \qquad (2.2.10)$$

In the horizontal polarization case, a plane wave of unit amplitude incident at angle θ gives rise to an unperturbed field

$$\underset{\sim}{E}_1^{o} = T\,\exp\{i\,k_1\,\alpha'\,x - i\,k_1\,\gamma'\,z\}\underset{\sim}{e}_y\;,\; z<0\;;$$

$$\underset{\sim}{E}_2^{o} = \exp\{i\,k_2\,\alpha\,x\}[\exp\{-\,i\,k_2\,\gamma\,z\} + R\,\exp\{i\,k_2\,\gamma\,z\}]\,\underset{\sim}{e}_y,\; z>0\;;$$

$$(2.2.11)$$

here

$$R = \left(1-\frac{k_1\gamma'}{k_2\gamma}\right)\left(1+\frac{k_1\gamma'}{k_2\gamma}\right)^{-1}\;,\quad T = 2\left(1+\frac{k_1\gamma'}{k_2\gamma}\right)^{-1},\; 1+R=T\;, \qquad (2.2.12)$$

and

$$\kappa = k_1\,\alpha' = k_2\,\alpha\;,\quad \gamma' = [1-(\alpha')^2]^{1/2}\;. \qquad (2.2.13)$$

The first order coefficients are

$$A_{mn}^{(1)} = G_{mn}^{(1)} = -\frac{2\ i\ U\ a^2\ m\ n\ P(m-\nu,n)}{d(m,n)\ D_{mn}}$$

$$B_{mn}^{(1)} = H_{mn}^{(1)} = -\frac{2\ i\ U\ P(m-\nu,n)}{d(m,n)}\left[\frac{a^2n^2}{D_{mn}} - 1\right] ,$$

$$C_{mn}^{(1)} = -\frac{2\ i\ U\ a\ n\ c(m,n)\ P(m-\nu,n)}{d(m,n)\ D_{mn}} ,\ I_{mn}^{(1)} = -\frac{b(m,n)}{c(m,n)}\ C_{mn}^{(1)} . \qquad (2.2.14)$$

The second order coefficients can be determined from the six equations

$$a\ m\ A_{mn}^{(2)} + a\ n\ B_{mn}^{(2)} + b(m,n)\ C_{mn}^{(2)} = 0 ,$$

$$a\ m\ G_{mn}^{(2)} + a\ n\ H_{mn}^{(2)} - c(m,n)\ I_{mn}^{(2)} = 0 ,$$

$$A_{mn}^{(2)} - G_{mn}^{(2)} = h_1 ,\ B_{mn}^{(2)} - H_{mn}^{(2)} = h_2 ,$$

$$a\ n\ (C_{mn}^{(2)} - I_{mn}^{(2)}) - b(m,n)\ B_{mn}^{(2)} - c(m,n)\ H_{mn}^{(2)} = h_3 ,$$

$$a\ m\ (C_{mn}^{(2)} - I_{mn}^{(2)}) - b(m,n)\ A_{mn}^{(2)} - c(m,n)\ G_{mn}^{(2)} = h_4 . \qquad (2.2.15)$$

Here

$$U = \frac{1}{2}\ T\ (k_1^2 - k_2^2) ; \qquad (2.2.16)$$

$$d(m,n) = b(m,n) + c(m,n) ,\quad D_{mn} = a^2(m^2 + n^2) + b(m,n)\ c(m,n) ; \qquad (2.2.17)$$

and

$$h_1 = -\ i\ a\ m\ \Sigma_{k\ell}(C_{k\ell}^{(1)} - I_{k\ell}^{(1)})\ P(m-k,n-\ell) ,$$

$$h_2 = \Sigma_{k\ell}[U\ P\ (k-\nu,\ell) - i\ a\ n\ (C_{k\ell}^{(1)} - I_{k\ell}^{(1)})]P(m-k,n-\ell) ,$$

$$h_3 = -\ \Sigma_{k\ell}[U\ k_1\ \gamma'\ P(k-\nu,\ell) + i\ (k_1^2 - k_2^2)\ B_{k\ell}^{(1)}]\ P(m-k,n-\ell) ,$$

$$h_4 = -\ i\ (k_1^2 - k_2^2)\ \Sigma_{k\ell}A_{k\ell}^{(1)}\ P(m-k,n-\ell) . \qquad (2.2.18)$$

Rice's paper contains the solutions to Equation 2.2.15, but we shall not repeat them here.

2.3 Similarities of the Two Methods

In both methods, the perturbed field near the interface is expanded in a double series in ϵ and in the coordinate normal to S_o. Then this expansion is inserted into the tangential boundary condition at the perturbed interface, and the resulting equations are solved to give an intermediate result from which the perturbed field can be calculated. In the method of Bass and Bocharov, the intermediate result is the effective tangential field at the underlying interface, and the field everywhere is found using Kirchhoff's formulas. In Rice's method, the intermediate result is the six sets of mode coefficients, which are readily seen to be the coefficients of a Fourier series expansion of the effective field at the underlying interface. The field everywhere is found by multiplying these coefficients by the appropriate mode functions and summing.

The similarity of the two methods suggest that they could be generalized to more complicated problems if a general technique were available for expressing the field everywhere in terms of the tangential field at an interface or, equivalently, in terms of surface currents on the interface. In the next section we shall discuss just such a technique, the use of dyadic Green's functions.

3. The Dyadic Green's Function

3.1 A Short Review of Electromagnetic Field Theory

In this paper we are interested in problems involving two linear homogeneous isotropic media M_1 and M_2 separated by an interface S. When the order of the media is not important, we shall designate one by M_q and the other by M_p. Medium M_q will be characterized by the material parameters μ_q and k_q, and the volume filled by M_q will be called V_q. Maxwell's equations in MKS units are then

$$\nabla \times \underset{\sim}{E}(\underset{\sim}{r}) - i\,\omega\,\mu_q\,\underset{\sim}{H}(\underset{\sim}{r}) = -\underset{\sim}{J}_m(\underset{\sim}{r}) \,,$$
$$\nabla \times \underset{\sim}{H}(\underset{\sim}{r}) - \frac{k_q^2}{i\,\omega\,\mu_q}\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{J}_e(\underset{\sim}{r}) \,,\ \underset{\sim}{r} \text{ in } V_q \,, \tag{3.1.1}$$

where $\underset{\sim}{J}_e$ is an electric current distribution and $\underset{\sim}{J}_m$ is a magnetic current distribution. The magnetic current has no physical existence, but effective magnetic currents appear frequently in the mathematics of electromagnetic theory.

The two parts of Equation 3.1.1 can be combined to give the second order forms

$$\nabla \times \nabla \times \underset{\sim}{E} - k_q^2\,\underset{\sim}{E} = i\,\omega\,\mu_q\,\underset{\sim}{J}_e - \nabla \times \underset{\sim}{J}_m \,,$$
$$\nabla \times \nabla \times \underset{\sim}{H} - k_q^2\,\underset{\sim}{H} = -\frac{k_q^2}{i\,\omega\,\mu_q}\underset{\sim}{J}_m + \nabla \times \underset{\sim}{J}_e \,. \tag{3.1.2}$$

In a source-free region, the operator $(\nabla \times \nabla \times)$ can be replaced by $(-\nabla^2)$.

If neither medium is a perfect electric or magnetic conductor, then the boundary conditions at S are

$$\underset{\sim}{n} \times (\underset{\sim}{H}_2 - \underset{\sim}{H}_1) = \underset{\sim}{K}_e \,, \quad n \times (\underset{\sim}{E}_2 - \underset{\sim}{E}_1) = -\underset{\sim}{K}_m \,; \tag{3.1.3}$$

here $\underset{\sim}{n}$ is the unit normal from V_1 to V_2 , and $\underset{\sim}{K}_e$ and $\underset{\sim}{K}_m$ are respectively electric and magnetic surface current source distributions on the interface. When one medium, say M_1 , is a perfect electric conductor $(\frac{k_1^2}{\mu_1}$ infinite$)$, then electric surface current <u>sources</u> induce equal and opposite surface currents and thus have no net effect. The field in V_1 is zero, the boundary condition for $\underset{\sim}{E}$ becomes

$$\underset{\sim}{n} \times \underset{\sim}{E}_2 = - \underset{\sim}{K}_m \ , \tag{3.1.4}$$

and the electric surface current is then uniquely determined. Similarly, when M_1 is a perfect magnetic conductor (μ_1 infinite) , then magnetic surface current sources have no effect, the field in V_1 is again zero, the boundary condition for $\underset{\sim}{H}$ is

$$\underset{\sim}{n} \times \underset{\sim}{H}_2 = \underset{\sim}{K}_e \ , \tag{3.1.5}$$

and the magnetic surface current is uniquely determined.

It is important to note that in all cases the fields are unchanged (except infinitesimally close to the interface) if a surface current source is shifted an infinitesimal distance into V_2 ; by making such a shift we can always eliminate $\underset{\sim}{K}_e$ and $\underset{\sim}{K}_m$ in Equations 3.1.3-5. In some situations this is the most convenient viewpoint, whereas in others it is desirable to retain the sources on the interface.

3.2 Dyadic Notation

The theory of dyadics has been treated adequately by Gibbs (6) and by Morse and Feshbach (7), and we shall not reproduce the material here. It is necessary, however, to say a few words about notation.

The letter Φ will be used for a general dyadic, I will be used for the unit dyadic, and Γ will be used for a d.G.f.; other dyadics will be identified in context. The transpose of Φ will be written Φ^T, the complex conjugate will be written Φ^*, and the Hermitian conjugate $(\Phi^*)^T$ will be written $\Phi^\dagger$.

The standard notations for a dyadic are

$$\Phi = \sum_{i,j=1}^{3} \underset{\sim}{e}_i \, \Phi_{ij} \, \underset{\sim}{e}'_j = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} ; \tag{3.2.1}$$

here $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ and $(\underset{\sim}{e}'_1, \underset{\sim}{e}'_2, \underset{\sim}{e}'_3)$ are two sets of orthogonal unit vectors, not necessarily both corresponding to the same coordinate system. A row vector of Φ will be designated by

$$\underset{\sim}{\Phi}_i = \Sigma_j \, \Phi_{ij} \, \underset{\sim}{e}'_j , \tag{3.2.2}$$

and a column vector by

$$\underset{\sim}{\Phi}^C_j = \Sigma_i \, \underset{\sim}{e}_i \, \Phi_{ij} . \tag{3.2.3}$$

These vectors are related to Φ by

$$\Phi = \Sigma_i \, \underset{\sim}{e}_i \, \underset{\sim}{\Phi}_i = \Sigma_j \, \underset{\sim}{\Phi}^C_j \, \underset{\sim}{e}'_j . \tag{3.2.4}$$

The concept of the curl operating on a dyadic from the right will prove useful. This operation is defined by

$$\Phi \times \nabla \equiv -(\nabla \times \Phi^T)^T = \frac{\partial}{\partial x} \Phi \times \underset{\sim}{e}_x + \frac{\partial}{\partial y} \Phi \times \underset{\sim}{e}_y + \frac{\partial}{\partial z} \Phi \times \underset{\sim}{e}_z . \tag{3.2.5}$$

The definition is consistent with the standard practice of treating the operator ∇ as a vector, for it is analogous to

$$\Phi \times \underset{\sim}{A} = -(\underset{\sim}{A} \times \Phi^T)^T . \tag{3.2.6}$$

3.3 Theory of the Dyadic Green's Function

The theory of the d.G.f. in a homogeneous region with perfectly conducting boundaries has been developed by Schwinger (8,9). Tai (10) has attempted to extend the theory to problems involving two media, but some of his results are incorrect; the errors appear to be due to his using an incorrect analysis by Morse and Feshbach (11).

It is not difficult to find the correct extension to two media and we shall do so here. In order to identify sources directly with currents, we use a normalization of the d.G.F. different from that of Schwinger. Thus the $\Gamma^{(1)}$ of Reference 9 is $\frac{1}{i\omega\mu}\,\Gamma_e$ in our notation, and $\Gamma^{(2)}$ is $\left(-\frac{i\omega\mu}{k^2}\right)\Gamma_m$.

Let us first consider a problem in which all sources are electric currents. We take as a set of canonical sources, from which all others can be constructed by superposition, three mutually perpendicular unit δ-function currents at each point in space. These currents are chosen to lie along the axes of some convenient orthogonal coordinate system $(\underset{\sim}{e}'_1, \underset{\sim}{e}'_2, \underset{\sim}{e}'_3)$.

Now let the column vector $\underset{\sim}{\Gamma}^C_{e,j}\,(\underset{\sim}{r};\underset{\sim}{r}')$ denote the electric field at $\underset{\sim}{r}$ due to the canonical electric current at $\underset{\sim}{r}'$ parallel to $\underset{\sim}{e}'_j$. The electric d.G.f., $\Gamma_e\,(\underset{\sim}{r};\underset{\sim}{r}')$, will be defined by

$$\Gamma_e\,(\underset{\sim}{r};\underset{\sim}{r}') = \Sigma_j\,\underset{\sim}{\Gamma}^C_{e,j}\,(\underset{\sim}{r};\underset{\sim}{r}')\,\underset{\sim}{e}'_j \quad . \qquad (3.3.1)$$

It then follows by the principle of superposition that the electric field due to an arbitrary electric current source distribution $\underset{\sim}{J}_e(\underset{\sim}{r}')$ in volume V is

$$\underset{\sim}{E}\,(\underset{\sim}{r}) = \int_V dV'\;\Gamma_e\,(\underset{\sim}{r};\underset{\sim}{r}')\cdot\underset{\sim}{J}_e\,(\underset{\sim}{r}') \quad ; \qquad (3.3.2)$$

if the source distribution extends to infinity, it may be necessary to interpret the integral as a limit.

A partial differential equation and boundary conditions for Γ_e are found by applying Equations 3.1.2-5 to the $\underset{\sim}{\Gamma}{}^{c}_{e,j}$. We obtain

$$\nabla \times \nabla \times \Gamma_e\,(\underset{\sim}{r};\underset{\sim}{r}') - k^2\,\Gamma_e\,(\underset{\sim}{r};\underset{\sim}{r}') = i\,\omega\,\mu\,I\,\delta\,(\underset{\sim}{r}-\underset{\sim}{r}'); \tag{3.3.3}$$

$$\underset{\sim}{n}(\underset{\sim}{r}) \times \Gamma_{e,2}(\underset{\sim}{r};\underset{\sim}{r}') = 0, \ \underset{\sim}{r} \text{ on } S,\ M_1 \text{ perfect electric conductor;} \tag{3.3.4}$$

$$\underset{\sim}{n}(\underset{\sim}{r}) \times [\nabla \times \Gamma_{e,2}(\underset{\sim}{r};\underset{\sim}{r}')] = 0, \ \underset{\sim}{r} \text{ on } S,\ M_1 \text{ perfect magnetic conductor;} \tag{3.3.5}$$

$$\underset{\sim}{n}(\underset{\sim}{r}) \times [\Gamma_{e,2}\,(\underset{\sim}{r};\underset{\sim}{r}') - \Gamma_{e,1}\,(\underset{\sim}{r};\underset{\sim}{r}')] = 0\,,$$

$$\underset{\sim}{n}\,(\underset{\sim}{r}) \times \left[\frac{1}{i\omega\mu_2}\,\nabla \times \Gamma_{e,2}\,(\underset{\sim}{r};\underset{\sim}{r}') - \frac{1}{i\omega\mu_1}\,\nabla \times \Gamma_{e,1}\,(\underset{\sim}{r};\underset{\sim}{r}')\right] = 0\,,$$

$$\underset{\sim}{r} \text{ on } S, \text{ no perfect conductors.} \tag{3.3.6}$$

Here k and μ are evaluated at $\underset{\sim}{r}$ and it is assumed for simplicity that M_2 is never a perfect conductor.

In deriving Equations 3.3.3-6, the source point $\underset{\sim}{r}'$ is assumed to lie in one medium or the other but not exactly astride S . This assumption is necessary because $\Gamma_e\,(\underset{\sim}{r};\underset{\sim}{r}')$ is discontinuous in $\underset{\sim}{r}'$ at S . However, the dyadic

$$\Gamma_e^{\parallel}\,(\underset{\sim}{r};\underset{\sim}{r}') = -\,[\Gamma_e\,(\underset{\sim}{r};\underset{\sim}{r}') \times \underset{\sim}{n}\,(\underset{\sim}{r}')] \times \underset{\sim}{n}\,(\underset{\sim}{r}')\ ,\ \underset{\sim}{r}' \text{ on } S\,, \tag{3.3.7}$$

which represents the response to sources parallel to the boundary, is unambiguously defined.

The dyadic $\Gamma_e^{\parallel}$ plays an important part in the perturbation theory, for it gives the field due to a canonical electric surface current.

We can calculate $\Gamma_e^{\parallel}$ from Equations 3.3.3-6, but it is simpler and more natural to use a formulation in which the source is indeed treated as a surface current. Thus, using Equations 3.1.2-5, we obtain

$$\nabla \times \nabla \times \Gamma_e^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}') - k^2\, \Gamma_e^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}') = 0\ ,\quad \underset{\sim}{r} \text{ not in } V'\ ; \tag{3.3.8}$$

$$\Gamma_e^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}') \equiv 0\ ,\quad M_1 \text{ perfect electric conductor}\ ; \tag{3.3.9}$$

$$\underset{\sim}{n}(\underset{\sim}{r}) \times \frac{1}{i\omega\mu_2} \nabla \times \Gamma_{e,2}^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}') = (\underset{\sim}{e}_{\xi'}\underset{\sim}{e}_{\xi'} + \underset{\sim}{e}_{\eta'}\underset{\sim}{e}_{\eta'})(h_{\xi'}h_{\eta'})^{-1}\, \delta(\xi-\xi')\delta(\eta-\eta'), \tag{3.3.10}$$

$$\underset{\sim}{r} \text{ on } S\ ,\quad M_1 \text{ perfect magnetic conductor}\ ;$$

$$\underset{\sim}{n}(\underset{\sim}{r}) \times [\Gamma_{e,2}^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}') - \Gamma_{e,1}^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}')] = 0\ ,$$

$$\underset{\sim}{n}(\underset{\sim}{r}) \times [\frac{1}{i\omega\mu_2} \nabla \times \Gamma_{e,2}^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}') - \frac{1}{i\omega\mu_1} \nabla \times \Gamma_{e,1}^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}')] = \tag{3.3.11}$$

$$(\underset{\sim}{e}_{\xi'}\underset{\sim}{e}_{\xi'} + \underset{\sim}{e}_{\eta'}\underset{\sim}{e}_{\eta'})(h_{\xi'}h_{\eta'})^{-1}\, \delta(\xi-\xi')\delta(\eta-\eta')\ ,\quad \underset{\sim}{r} \text{ on } S, \text{ no perfect conductors.}$$

Here V' is a small sphere around $\underset{\sim}{r}'$; (ξ,η,ζ) and (ξ',η',ζ') are coordinates in a righthanded orthogonal system with metrics $h_{\xi'}, h_{\eta'}, h_{\zeta'}$;

and
$$\underset{\sim}{n} = \underset{\sim}{e}_{\zeta'}\ . \tag{3.3.12}$$

Now let us consider the magnetic d.G.f. $\Gamma_m(\underset{\sim}{r};\underset{\sim}{r}')$. Its definition is the same as that of Γ_e except that magnetic currents replace electric currents. The equations analogous to Equations 3.3.2-11 are:

$$H(\underset{\sim}{r}) = \int_V d\,V'\, \Gamma_m(\underset{\sim}{r};\underset{\sim}{r}') \cdot \underset{\sim}{J}_m(\underset{\sim}{r}')\ , \tag{3.3.13}$$

where J_m is a magnetic current source distribution;

$$\nabla \times \nabla \times \underset{\sim}{\Gamma}_m (\underset{\sim}{r};\underset{\sim}{r}') - k^2 \underset{\sim}{\Gamma}_m (\underset{\sim}{r};\underset{\sim}{r}') = - \frac{k^2}{i\omega\mu} \underset{\sim}{I}\, \delta (\underset{\sim}{r}-\underset{\sim}{r}') \; ; \qquad (3.3.14)$$

$$\underset{\sim}{n} (\underset{\sim}{r}) \times \underset{\sim}{\Gamma}_{m,2} (\underset{\sim}{r};\underset{\sim}{r}') = 0, \; \underset{\sim}{r} \text{ on S, } M_1 \text{ perfect magnetic conductor;} \qquad (3.3.15)$$

$$\underset{\sim}{n} (\underset{\sim}{r}) \times [\nabla \times \underset{\sim}{\Gamma}_{m,2} (\underset{\sim}{r};\underset{\sim}{r}')] = 0, \; \underset{\sim}{r} \text{ on S, } M_1 \text{ perfect electric conductor;} \qquad (3.3.16)$$

$$\underset{\sim}{n} (\underset{\sim}{r}) \times [\underset{\sim}{\Gamma}_{m,2} (\underset{\sim}{r};\underset{\sim}{r}') - \underset{\sim}{\Gamma}_{m,1} (\underset{\sim}{r};\underset{\sim}{r}')] = 0 \; ,$$

$$\underset{\sim}{n} (\underset{\sim}{r}) \times \left[\frac{i\omega\mu_2}{k_2^2} \nabla \times \underset{\sim}{\Gamma}_{m,2} (\underset{\sim}{r};\underset{\sim}{r}') - \frac{i\omega\mu_1}{k_1^2} \nabla \times \underset{\sim}{\Gamma}_{m,1} (\underset{\sim}{r};\underset{\sim}{r}') \right] = 0 \; ,$$

$$\underset{\sim}{r} \text{ on S , no perfect conductors ;} \qquad (3.3.17)$$

$$\underset{\sim}{\Gamma}_m^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}') = - [\underset{\sim}{\Gamma}_m (\underset{\sim}{r};\underset{\sim}{r}') \times \underset{\sim}{n} (\underset{\sim}{r}')] \times \underset{\sim}{n} (\underset{\sim}{r}'), \underset{\sim}{r}' \text{ on S ;} \qquad (3.3.18)$$

and

$$\nabla \times \nabla \times \underset{\sim}{\Gamma}_m^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}') - k^2 \underset{\sim}{\Gamma}_m^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}') = 0 \; , \; \underset{\sim}{r} \text{ not in V}' \; ; \qquad (3.3.19)$$

$$\underset{\sim}{\Gamma}_m^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}') \equiv 0, \; M_1 \text{ perfect magnetic conductor ;} \qquad (3.3.20)$$

$$\underset{\sim}{n}(\underset{\sim}{r}) \times \frac{i\omega\mu_2}{k_2^2} \nabla \times \underset{\sim}{\Gamma}_{m,2}^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}') = - (\underset{\sim}{e}_{\xi'}\underset{\sim}{e}_{\xi'} + \underset{\sim}{e}_{\eta'}\underset{\sim}{e}_{\eta'})(h_{\xi'}h_{\eta'})^{-1} \delta(\xi-\xi')\delta(\eta-\eta'),$$

$$\underset{\sim}{r} \text{ on S, } M_1 \text{ perfect electric conductor ;} \qquad (3.3.21)$$

$$\underset{\sim}{n}(\underset{\sim}{r}) \times [\underset{\sim}{\Gamma}_{m,2}^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}') - \underset{\sim}{\Gamma}_{m,1}^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}')] = 0 \; ,$$

$$\underset{\sim}{n}(\underset{\sim}{r}) \times \left[\frac{i\omega\mu_2}{k_2^2} \nabla \times \underset{\sim}{\Gamma}_{m,2}^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}') - \frac{i\omega\mu_1}{k_1^2} \nabla \times \underset{\sim}{\Gamma}_{m,1}^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}') \right] =$$

$$- (\underset{\sim}{e}_{\xi'}\underset{\sim}{e}_{\xi'} + \underset{\sim}{e}_{\eta'}\underset{\sim}{e}_{\eta'}) (h_{\xi'}h_{\eta'})^{-1} \delta (\xi-\xi') \, \delta (\eta-\eta') \; ,$$

$$\underset{\sim}{r} \text{ on S, no perfect conductors .} \qquad (3.3.22)$$

In a general problem, both electric and magnetic current sources are present, so that the total field is given by

$$\boxed{\begin{aligned} \underset{\sim}{E}(\underset{\sim}{r}) &= \int_V dV' \, \Gamma_e(\underset{\sim}{r};\underset{\sim}{r}') \cdot \underset{\sim}{J}_e(\underset{\sim}{r}') + \frac{i\omega\mu}{k^2} \nabla \times \int_V dV' \, \Gamma_m(\underset{\sim}{r};\underset{\sim}{r}') \cdot \underset{\sim}{J}_m(\underset{\sim}{r}') && (3.3.23) \\ \underset{\sim}{H}(\underset{\sim}{r}) &= \int_V dV' \, \Gamma_m(\underset{\sim}{r};\underset{\sim}{r}') \cdot \underset{\sim}{J}_m(\underset{\sim}{r}') + \frac{1}{i\omega\mu} \nabla \times \int_V dV' \, \Gamma_e(\underset{\sim}{r};\underset{\sim}{r}') \cdot \underset{\sim}{J}_e(\underset{\sim}{r}') && (3.3.24) \end{aligned}}$$

The curl operator and the integral sign may be interchanged for $\underset{\sim}{r}$ anywhere but exactly at the interface or at a source discontinuity. Equations 3.3.23,24 are the basic equations for determining the field when the d.G.f.'s are known.

The d.G.f. may also be used in formulating integral equations relating the field in a homogeneous region to its value at the boundary of the region. The appropriate equations, which are derived in Appendix 1, are

$$\underset{\sim}{E}(\underset{\sim}{r}) = \int_{V_i} dV' \left[\underset{\sim}{J}_e(\underset{\sim}{r}') - \frac{1}{i\omega\mu} \nabla' \times \underset{\sim}{J}_m(\underset{\sim}{r}') \right] \cdot \Gamma_e(\underset{\sim}{r}';\underset{\sim}{r})$$

$$- \frac{1}{i\omega\mu} \int_{S_i} dS' \left\{ [\underset{\sim}{n}(\underset{\sim}{r}') \times \underset{\sim}{E}(\underset{\sim}{r}')] \cdot \nabla' \times \Gamma_e(\underset{\sim}{r}';\underset{\sim}{r}) + [\underset{\sim}{n}(\underset{\sim}{r}') \times \nabla' \times \underset{\sim}{E}(\underset{\sim}{r}')] \cdot \Gamma_e(\underset{\sim}{r}';\underset{\sim}{r}) \right\} \tag{3.3.25}$$

$$\underset{\sim}{H}(\underset{\sim}{r}) = \int_{V_i} dV' \left[\underset{\sim}{J}_m(\underset{\sim}{r}') - \frac{i\omega\mu}{k^2} \nabla' \times \underset{\sim}{J}_e(\underset{\sim}{r}') \right] \cdot \Gamma_m(\underset{\sim}{r}';\underset{\sim}{r})$$

$$(3.3.26)$$

$$- \frac{i\omega\mu}{k^2} \int_{S_i} dS' \left\{ [\underset{\sim}{n}(\underset{\sim}{r}') \times \underset{\sim}{H}(\underset{\sim}{r}')] \cdot \nabla' \times \Gamma_m(\underset{\sim}{r}';\underset{\sim}{r}) + [\underset{\sim}{n}(\underset{\sim}{r}') \times \nabla' \times \underset{\sim}{H}(\underset{\sim}{r}')] \cdot \Gamma_m(\underset{\sim}{r}';\underset{\sim}{r}) \right\} .$$

Here V_i is an arbitrary volume throughout which the medium is linear, homogeneous, and isotropic, and S_i is the boundary of V_i . The d.G.f.'s Γ_e and Γ_m are required to satisfy Equations 3.3.3 and 3.3.14 respectively everywhere in V_i , but no specific boundary condition is imposed at S_i . In practice, S_i usually coincides with the interface S , and the d.G.f.'s for unbounded space are usually employed. The formulation of Equations 3.3.25,26 is primarily useful in finding the field due to a specific source distribution or incident wave without calculating the d.G.f. of the problem.

3.4 Properties of the Dyadic Green's Function

In Appendix 1, we outline the proof of the three important relations

$$\Gamma(\underset{\sim}{r};\underset{\sim}{r}') = \Gamma^T(\underset{\sim}{r}';\ \underset{\sim}{r}) \ , \tag{3.4.1}$$

$$\frac{1}{i\omega\mu}\ \nabla \times \Gamma_e\ (\underset{\sim}{r};\underset{\sim}{r}') = \frac{i\omega\mu'}{(k')^2}\ \Gamma_m\ (\underset{\sim}{r};\underset{\sim}{r}') \times \nabla' \ , \tag{3.4.2}$$

$$\frac{i\omega\mu}{k^2}\ \nabla \times \Gamma_m\ (\underset{\sim}{r};\underset{\sim}{r}') = \frac{1}{i\omega\mu'}\ \Gamma_e\ (\underset{\sim}{r};\underset{\sim}{r}') \times \nabla' \ . \tag{3.4.3}$$

Equation 3.4.1 indicates that Equation 3.15 of Reference 9 can be extended to two medium problems. Equations 3.4.2,3 are the extension, in our notation, of Equation 3.16 of Reference 9; these relations can also be written in the explicit form

$$\frac{k^2}{i\omega\mu}\ \Gamma_e\ (\underset{\sim}{r};\underset{\sim}{r}') = \frac{i\omega\mu'}{(k')^2}\ \nabla \times \Gamma_m\ (\underset{\sim}{r};\underset{\sim}{r}') \times \nabla' - I\ \delta(\underset{\sim}{r}-\underset{\sim}{r}') \ , \tag{3.4.4}$$

$$i\omega\mu\ \Gamma_m\ (\underset{\sim}{r};\underset{\sim}{r}') = \frac{1}{i\omega\mu'}\ \nabla \times \Gamma_e\ (\underset{\sim}{r};\underset{\sim}{r}') \times \nabla' + I\ \delta(\underset{\sim}{r}-\underset{\sim}{r}') \ . \tag{3.4.5}$$

Using the above formulas, we can determine both Γ_e and Γ_m completely if the six independent dyadic elements of either one are known for all $\underset{\sim}{r}$ and $\underset{\sim}{r}'$. This, however, is much more information than we actually need. Let us therefore now consider the question of determining the Γ everywhere from more limited information.

To this end, we write Γ_e as the sum of an incident and a scattered field; thus

$$\Gamma_e = \Gamma_e^{inc} + \Gamma_e^{scat} , \tag{3.4.6}$$

where

$$\begin{aligned} \Gamma_c^{inc}(\underset{\sim}{r};\underset{\sim}{r}') &= \Gamma_e^{0}(\underset{\sim}{r};\underset{\sim}{r}') \text{ for } \underset{\sim}{r},\underset{\sim}{r}' \text{ in the same medium ,} \\ &= 0 \text{ for } \underset{\sim}{r},\underset{\sim}{r}' \text{ not in the same medium .} \end{aligned} \tag{3.4.7}$$

Here Γ_e^0 is the electric d.G.f. of an unbounded medium, which is given by Equation 3.5.1. Equation 3.3.6 now becomes

$$\begin{aligned} &\underset{\sim}{n}(\underset{\sim}{r}'') \times \left[\Gamma_{e,2}^{scat}(\underset{\sim}{r}'';r') - \Gamma_{e,1}^{scat}(\underset{\sim}{r}'';\underset{\sim}{r}')\right] = -\underset{\sim}{n}'(\underset{\sim}{r}'') \times \Gamma_e^0(\underset{\sim}{r}'';\underset{\sim}{r}') , \\ &\underset{\sim}{n}(\underset{\sim}{r}'') \times \left[\frac{1}{i\omega\mu_2} \nabla'' \times \Gamma_{e,2}^{scat}(\underset{\sim}{r}'';\underset{\sim}{r}') - \frac{1}{i\omega\mu_1} \nabla'' \times \Gamma_{e,1}^{scat}(\underset{\sim}{r}'';\underset{\sim}{r}')\right] = \\ &\qquad - \underset{\sim}{n}'(\underset{\sim}{r}'') \times \left[\frac{1}{i\omega\mu'} \nabla'' \times \Gamma_e^0(\underset{\sim}{r}'';\underset{\sim}{r}')\right], \end{aligned} \tag{3.4.8}$$

where $\underset{\sim}{r}''$ lies on S , $\underset{\sim}{n}'(\underset{\sim}{r}'')$ is the unit normal at $\underset{\sim}{r}''$ into the medium in which $\underset{\sim}{r}'$ is located, and Γ_e^0 is evaluated for the medium in which $\underset{\sim}{r}'$ is located.

Comparing these results with Equation 3.1.3, we see that the field Γ_e^{scat} is the field due to effective dyadic surface current distributions

$$\begin{aligned} K_e(\underset{\sim}{r}'') &= -\underset{\sim}{n}'(\underset{\sim}{r}'') \times \frac{1}{i\omega\mu'} \nabla'' \times \Gamma_e^0(\underset{\sim}{r}'';\underset{\sim}{r}') , \\ K_m(\underset{\sim}{r}'') &= \underset{\sim}{n}'(\underset{\sim}{r}'') \times \Gamma_e^0(\underset{\sim}{r}'';\underset{\sim}{r}') . \end{aligned} \tag{3.4.9}$$

Thus Γ_e^{scat} can be found from Equation 3.3.23, with the volume integrals reducing to surface integrals. Adding Γ_e^{inc} to this result, we find

$$\begin{aligned} \Gamma_e(\underset{\sim}{r};\underset{\sim}{r}') = \Gamma_e^{inc}(\underset{\sim}{r};\underset{\sim}{r}') &+ \int_S dS''\, \Gamma_e(\underset{\sim}{r},\underset{\sim}{r}'')\cdot[-\underset{\sim}{n}'(\underset{\sim}{r}'') \times \frac{1}{i\omega\mu'} \nabla'' \times \Gamma_e^0(\underset{\sim}{r}'';\underset{\sim}{r}')] \\ &+ \frac{i\omega\mu}{k^2} \nabla \times \int_S dS''\, \Gamma_m(\underset{\sim}{r};\underset{\sim}{r}'')\cdot[\underset{\sim}{n}'(\underset{\sim}{r}'') \times \Gamma_e^0(\underset{\sim}{r}'';\underset{\sim}{r}')] , \end{aligned} \tag{3.4.10}$$

which can be written in terms of the $\Gamma^{\parallel}$ as

$$\Gamma_e(\underset{\sim}{r};\underset{\sim}{r}') = \Gamma_e^{inc}(\underset{\sim}{r};\underset{\sim}{r}') - \frac{1}{i\omega\mu'} \int_S dS'' \, [\Gamma_e^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}'') \times \underset{\sim}{n}'(\underset{\sim}{r}'')] \cdot \nabla'' \times \Gamma_e^{o}(\underset{\sim}{r}'';\underset{\sim}{r}')$$

$$+ \frac{i\omega\mu}{k^2} \nabla \times \int_S dS'' \, [\Gamma_m^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}'') \times \underset{\sim}{n}'(\underset{\sim}{r}'')] \cdot \Gamma_e^{o}(\underset{\sim}{r}'';\underset{\sim}{r}') \, . \qquad (3.4.11)$$

Thus Γ_e is determined if we know $\Gamma_e^{\parallel}$ and $\Gamma_m^{\parallel}$; that is, if we can evaluate the d.G.f.'s for sources on the interface parallel to the interface, then we can evaluate the d.G.f.'s everywhere.

Furthermore, in Equation 3.4.10, we can replace $\Gamma_e(\underset{\sim}{r};\underset{\sim}{r}'')$ and $\Gamma_m(\underset{\sim}{r};\underset{\sim}{r}'')$ by $\Gamma_e^T(\underset{\sim}{r}'';\underset{\sim}{r})$ and $\Gamma_m^T(\underset{\sim}{r}'';\underset{\sim}{r})$ respectively. The latter two quantities can then be expanded in terms of the $\Gamma^{\parallel}(\underset{\sim}{r}'';\underset{\sim}{r}''')$ where $\underset{\sim}{r}''$ and $\underset{\sim}{r}'''$ both lie on S . Thus we obtain the interesting result:

If $\underset{\sim}{n}(\underset{\sim}{r}) \times \Gamma_e^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}')$ and $\underset{\sim}{n}(\underset{\sim}{r}) \times \Gamma_m^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}')$ are known for all $\underset{\sim}{r}$ and $\underset{\sim}{r}'$ on S , then explicit expressions can be found for $\Gamma_e(\underset{\sim}{r};\underset{\sim}{r}')$ and $\Gamma_m(\underset{\sim}{r};\underset{\sim}{r}')$ everywhere. The expressions in question are lengthy and will not be recorded here.

3.5 Some Important Dyadic Green's Functions

In this section we shall compile some of the more important d.G.f.'s and related quantities. To avoid unnecessary repetition, we note that an expression for Γ_m can be obtained by replacing $i\omega\mu$ by $-\frac{k^2}{i\omega\mu}$ in an expression for Γ_e . As noted before, k^2/μ is infinite in a perfect electric conductor and μ is infinite in a perfect magnetic conductor.

Where the d.G.f. is of complicated form, we shall tabulate $\Gamma^{\parallel}$ and $\nabla \times \Gamma^{\parallel}$ rather than the full d.G.f. These quantities are sufficient for our purposes, being the functions that actually appear in the perturbation equations to be derived; furthermore, we have already shown that Γ can be calculated everywhere when the $\Gamma^{\parallel}$ are known.

The d.G.f.'s for an unbounded region and for a half-space with perfectly conducting boundary have been taken from Reference 9. The $\Gamma_e^{\parallel}$ and $\nabla \times \Gamma_e^{\parallel}$ for the general plane, cylindrical, and spherical boundaries have been calculated by using known mode expansions of the field (see Ref. 12) and matching the discontinuity in tangential $\underset{\sim}{H}$ at the interface; the straightforward but tedious details will not be given here. Results are presented in order of increasing geometrical complexity.

A. In an unbounded medium

$$\Gamma_e(\underset{\sim}{r};\underset{\sim}{r}') = \Gamma_e^0(\underset{\sim}{r};\underset{\sim}{r}') = i\omega\mu\left(I - \frac{1}{k^2}\nabla\nabla'\right) G_f(\underset{\sim}{r};\underset{\sim}{r}') , \qquad (3.5.1)$$

where the scalar Green's function G_f is given by Equation 2.1.9.

B. In the half-space $z > 0$, when the plane $z = 0$ is a perfect conductor,

$$\Gamma_e(\underset{\sim}{r};\underset{\sim}{r}') = \Gamma_e^0(\underset{\sim}{r};\underset{\sim}{r}') - \Gamma_e^0(\underset{\sim}{r};\underset{\sim}{r}' - 2\,\underset{\sim}{e}_z\cdot\underset{\sim}{r}'\underset{\sim}{e}_z)\cdot(I - 2\,\underset{\sim}{e}_z\underset{\sim}{e}_z) ,$$

$$\Gamma_m(\underset{\sim}{r},\underset{\sim}{r}') = \Gamma_m^0(\underset{\sim}{r};\underset{\sim}{r}') + \Gamma_m^0(\underset{\sim}{r};\underset{\sim}{r}' - 2\,\underset{\sim}{e}_z\cdot\underset{\sim}{r}'\underset{\sim}{e}_z)\cdot(I - 2\,\underset{\sim}{e}_z\underset{\sim}{e}_z) . \qquad (3.5.2)$$

In accordance with Equation 3.3.9,

$$\Gamma_e^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}') \equiv 0 ; \qquad (3.5.3)$$

we also find

$$\Gamma_m^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}') = 2\,\Gamma_m^0(\underset{\sim}{r};\underset{\sim}{r}')\cdot(\underset{\sim}{e}_x\underset{\sim}{e}_x + \underset{\sim}{e}_y\underset{\sim}{e}_y) , \; z > 0 . \qquad (3.5.4)$$

The image method used to obtain Equation 3.5.2 can be used to find in closed form the d.G.f. for any wedge of angle π/n with a perfectly conducting boundary.

C. When the half-space $z < 0$ is filled with Medium M_1 and the half space $z > 0$ is filled with Medium M_2, then for $\underset{\sim}{r}$ in Medium M_q we have

$$\Gamma^{\parallel}_{e,q}(\underset{\sim}{r};\underset{\sim}{r}') = L_q\, T_q \,, \quad \nabla \times \Gamma^{\parallel}_{e,q}(\underset{\sim}{r};\underset{\sim}{r}') = L_q\, W_q \,. \tag{3.5.5}$$

Here L_q is the operator

$$L_q = \frac{\omega\mu_1\mu_2}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} du\,dv\, \exp\left\{iu(x-x')+iv(y-y')+ih_q z\right\}\frac{[\cdots]}{(\mu_2h_1-\mu_1h_2)(\mu_2h_2k_1^2-\mu_1h_1k_2^2)}, \tag{3.5.6}$$

and

$$T_q = \begin{bmatrix} [\mu_2h_2(k_1^2-u^2)-\mu_1h_1(k_2^2-u^2)] & uv\,(\mu_1h_1-\mu_2h_2) & 0 \\ uv\,(\mu_1h_1-\mu_2h_2) & [\mu_2h_2(k_1^2-v^2)-\mu_1h_1(k_2^2-v^2)] & 0 \\ -\dfrac{uh_1h_2}{h_q}(\mu_2h_1-\mu_1h_2) & -\dfrac{vh_1h_2}{h_q}(\mu_2h_1-\mu_1h_2) & 0 \end{bmatrix},$$

$$W_q = \begin{bmatrix} i\,u\,v\,\mu_q\,(k_2^2-k_1^2) & -\dfrac{i}{h_q}[k_q^2h_1h_2(\mu_2h_1-\mu_1h_2)+\mu_qh_qu^2(k_2^2-k_1^2)] & 0 \\ \dfrac{i}{h_q}[k_q^2h_1h_2(\mu_2h_1-\mu_1h_2)+\mu_qh_qv^2(k_2^2-k_1^2)] & -\,i\,u\,v\,\mu_q\,(k_2^2-k_1^2) & 0 \\ i\,v\,(\mu_1h_1k_2^2-\mu_2h_2k_1^2) & -\,i\,u\,(\mu_1h_1k_2^2-\mu_2h_2k_1^2) & 0 \end{bmatrix}, \tag{3.5.7}$$

with

$$(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3) = (\underset{\sim}{e}'_1, \underset{\sim}{e}'_2, \underset{\sim}{e}'_3) = (\underset{\sim}{e}_x, \underset{\sim}{e}_y, \underset{\sim}{e}_z) , \tag{3.5.8}$$

$$h_1 = - [k_1^2 - (u^2 + v^2)]^{1/2} \quad , \quad h_2 = + [k_2^2 - (u^2 + v^2)]^{1/2} . \tag{3.5.9}$$

The expression for $\Gamma^{\parallel}_{e,q}$ is not quite correct at $\underset{\sim}{r} = \underset{\sim}{r}'$. Here it is necessary to add a term equal to the irrotational part of the source function. This term, however, has no effect in the perturbation theory, and we shall thus not consider it further. The same comment applies to the $\Gamma^{\parallel}_{e}$ for the cylinder and sphere.

D. When the interior of the circular cylinder $\rho = a$ is filled with Medium M_1 and the exterior is of Medium M_2, then for $\underset{\sim}{r}$ in Medium M_q we have

$$\Gamma^{\parallel}_{e,q} (\underset{\sim}{r};\underset{\sim}{r}') = L \, [\underset{\sim}{s}_q(\rho)\underset{\sim}{c}_q + \frac{1}{k_q} \underset{\sim}{t}_q(\rho)\underset{\sim}{d}_q] \quad ,$$

$$\nabla \times \Gamma^{\parallel}_{e,q}(\underset{\sim}{r};\underset{\sim}{r}') = L \, [k_q \underset{\sim}{s}_q(\rho)\underset{\sim}{d}_q + \underset{\sim}{t}_q(\rho)\underset{\sim}{c}_q] \quad . \tag{3.5.10}$$

Here L is the operator

$$L = \frac{i\omega k_1 k_2 \mu_1 \mu_2}{4\pi^2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh \, \exp\left\{i \, n(\theta-\theta')+i \, h(z-z')\right\}[v_1^2 v_2^2 J_n(v_1)H_n^{(1)}(v_2)]^{-2}$$

$$\left\{n^2h^2\left(\frac{1}{v_1^2} - \frac{1}{v_2^2}\right)^2 - \left[\frac{\mu_1}{v_1} \frac{J_n'(v_1)}{J_n(v_1)} - \frac{\mu_2}{v_2} \frac{H_n^{(1)\prime}(v_2)}{H_n^{(1)}(v_2)}\right]\left[\frac{k_1^2}{\mu_1 v_1} \frac{J_n'(v_1)}{J_n(v_1)} - \frac{k_2^2}{\mu_2 v_2} \frac{H_n^{(1)\prime}(v_2)}{H_n^{(1)}(v_2)}\right]\right\}^{-1}$$

$$[\cdots] \; ; \tag{3.5.11}$$

and

$$\underset{\sim}{s}_q (\rho) = \frac{i \, n}{\rho} Z_n^q (\lambda_q \rho) \, \underset{\sim}{e}_\rho - \frac{\partial}{\partial\rho} Z_n^q (\lambda_q \rho) \, \underset{\sim}{e}_\theta \; ; \tag{3.5.12}$$

$$\underset{\sim}{t}_q(\rho) = i\,h\,\frac{\partial}{\partial\rho}\,Z_n^q\,(\lambda_q\rho)\,\underset{\sim}{e}_\rho - \frac{nh}{\rho}\,Z_n^q\,(\lambda_q\rho)\,\underset{\sim}{e}_\theta + \lambda_q^2\,Z_n^q\,(\lambda_q\rho)\,\underset{\sim}{e}_z\ ; \qquad (3.5.13)$$

$$\underset{\sim}{c}_1 = \left\{\frac{n^2h^2(v_1^2-v_2^2)}{k_1k_2\mu_2}\,J_n\,(v_1)[H_n^{(1)}(v_2)]^2 - \frac{k_2v_1^2v_2^2}{k_1\mu_2}\,J_n\,(v_1)\,[H_n^{(1)\prime}(v_2)]^2\right.$$

$$\left.+\frac{k_1v_1v_2^3}{k_2\mu_1}\,J_n'(v_1)H_n^{(1)}(v_2)H_n^{(1)\prime}(v_2)\right\}a\,\underset{\sim}{e}_\theta, + \frac{nhv_2^2(v_2^2-v_1^2)}{k_1k_2\mu_2}\,J_n(v_1)[H_n^{(1)}(v_2)]^2\underset{\sim}{e}_z,$$

$$\underset{\sim}{c}_2 = \left\{\frac{n^2h^2(v_1^2-v_2^2)}{k_1k_2\mu_1}\,[J_n(v_1)]^2H_n^{(1)}(v_2) + \frac{k_1v_1^2v_2^2}{k_2\mu_1}\,[J_n'(v_1)]^2\,H_n^{(1)}(v_2)\right. \qquad (3.5.14)$$

$$\left.-\frac{k_2v_1^3v_2}{k_1\mu_2}\,J_n(v_1)J_n'(v_1)H_n^{(1)\prime}(v_2)\right\}a\,\underset{\sim}{e}_\theta, + \frac{nhv_1^2(v_2^2-v_1^2)}{k_1k_2\mu_1}[J_n(v_1)]^2H_n^{(1)}(v_2)\underset{\sim}{e}_z\ ;$$

$$\underset{\sim}{d}_1 = \frac{v_2^2}{k_2}\,H_n^{(1)}(v_2)\,\underset{\sim}{d}_o\ ,\ \underset{\sim}{d}_2 = \frac{v_1^2}{k_1}\,J_n\,(v_1)\,\underset{\sim}{d}_o\ ; \qquad (3.5.15)$$

$$\underset{\sim}{d}_o = [\frac{v_1}{\mu_2}\,J_n'\,(v_1)\,H_n^{(1)}\,(v_2) - \frac{v_2}{\mu_1}\,J_n\,(v_1)\,H_n^{(1)\prime}\,(v_2)]\ a\,n\,h\ \underset{\sim}{e}_\theta,$$

$$+ [\frac{v_1}{\mu_1}\,J_n\,(v_1)\,H_n^{(1)\prime}\,(v_2) - \frac{v_2}{\mu_2}\,J_n'(v_1)\,H_n^{(1)}(v_2)]\ v_1v_2\underset{\sim}{e}_z\ ; \qquad (3.5.16)$$

$$v_q = a\lambda_q = +a\sqrt{k_q^2 - h^2}\ ; \qquad (3.5.17)$$

$$Z_n^1 = J_n\ (\text{Bessel Function}),\ Z_n^2 = H_n^{(1)}\ (\text{Hankel Function})\ ; \qquad (3.5.18)$$

$$Z'\,(v) = \frac{d}{dv}\,Z(v)\ . \qquad (3.5.19)$$

E. When the interior of the sphere $r = a$ is filled with Medium M_1 and the exterior is of Medium M_2, then for $\underset{\sim}{r}$ in Medium M_q we have

$$\Gamma^{\parallel}_{e,q}(\underset{\sim}{r};\underset{\sim}{r}') = L\,[c_o^{-1}\underset{\sim}{m}_{q,\ell,m}(\underset{\sim}{r})\underset{\sim}{m}_{p,\ell,-m}(\underset{\sim}{r}') + d_o^{-1}\underset{\sim}{n}_{q,\ell,m}\,(\underset{\sim}{r})\hat{\underset{\sim}{n}}_{p,\ell,-m}(\underset{\sim}{r}')]\ , \qquad (3.5.20)$$

$$\nabla \times \Gamma^{\parallel}_{e,q}(\underset{\sim}{r};\underset{\sim}{r}') = L[c_o^{-1}\underset{\sim}{n}_{q,\ell,m}(\underset{\sim}{r})\underset{\sim}{m}_{p,\ell,-m}(\underset{\sim}{r}') + d_o^{-1}k_q^2\underset{\sim}{m}_{q,\ell,m}(\underset{\sim}{r})\hat{\underset{\sim}{n}}_{p,\ell,-m}(\underset{\sim}{r}')]\ . \qquad (3.5.21)$$

Here L is the operator

$$L = \frac{i\omega\mu_1\mu_2}{a} \Sigma_{\ell=0}^{\infty} \Sigma_{m=-\ell}^{\ell} \frac{(-1)^m}{\ell(\ell+1)} [\cdots] ; \tag{3.5.22}$$

and

$$\underset{\sim}{m}_{q,\ell,m}(\underset{\sim}{r}) = z_\ell^q(\rho) [\underset{\sim}{e}_\Theta \frac{im}{\sin\Theta} Y_{\ell,m} (\Theta,\phi) - \underset{\sim}{e}_\phi \frac{d}{d\Theta} Y_{\ell,m} (\Theta,\phi)] , \tag{3.5.23}$$

$$\hat{\underset{\sim}{n}}_{q,\ell,m}(\underset{\sim}{r}) = \frac{1}{r} [\rho z_\ell^q(\rho)]' [\underset{\sim}{e}_\Theta \frac{d}{d\Theta} Y_{\ell,m} (\Theta,\phi) + \underset{\sim}{e}_\phi \frac{im}{\sin\Theta} Y_{\ell,m} (\Theta,\phi)], \tag{3.5.24}$$

$$\underset{\sim}{n}_{q,\ell,m}(\underset{\sim}{r}) = \underset{\sim}{e}_r \frac{\ell(\ell+1)}{r} z_\ell^q (\rho) Y_{\ell,m} (\Theta,\phi) + \hat{\underset{\sim}{n}}_{q,\ell,m}(\underset{\sim}{r}) , \tag{3.5.25}$$

$$c_o = \mu_2 h_\ell^{(1)}(\rho_2)[\rho_1 j_\ell (\rho_1)]' - \mu_1 j_\ell (\rho_1) [\rho_2 h_\ell^{(1)}(\rho_2)]' , \tag{3.5.26}$$

$$d_o = \mu_1 k_2^2 h_\ell^{(1)}(\rho_2)[\rho_1 j_\ell(\rho_1)]' - \mu_2 k_1^2 j_\ell(\rho_1)[\rho_2 h_\ell^{(1)}(\rho_2)]' , \tag{3.5.27}$$

$$\rho = k_q r , \rho_q = k_q a , \tag{3.5.28}$$

$$[\rho z_\ell(\rho)]' = \frac{\partial}{\partial\rho} [\rho z_\ell (\rho)] . \tag{3.5.29}$$

The radial functions are

$$z_\ell^1 = j_\ell \text{ (Spherical Bessel Function)} , z_\ell^2 = h_\ell^{(1)} \text{(Spherical Hankel Function)}, \tag{3.5.30}$$

and the $Y_{\ell,m}$ are the spherical harmonics of angular momentum theory (13), which can be expressed in terms of the more common associated Legendre functions by

$$Y_{\ell,m}(\Theta,\phi) = (-1)^m \left[\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}\right]^{1/2} P_\ell^m (\cos \Theta) e^{im\phi} . \tag{3.5.31}$$

4. Calculation of the Perturbation Field to Second Order

4.1 Introductory Remarks

The perturbation in the scattered field caused by interface irregularity can be reproduced approximately by a distribution of electric and magnetic surface currents on the underlying interface. In this section we shall determine these effective surface currents to second order. Then the field will be found to second order everywhere in space by using the results of Section 3.

The approach to be used is general, valid for underlying interfaces of any shape and not dependent on any particular representation of the solution to the unperturbed problem. However, in order to avoid obscuring the basic simplicity of the approach by the details necessary for generality, we precede the general derivation with a treatment of the simple case of an irregular plane interface.

4.2 The Plane Interface

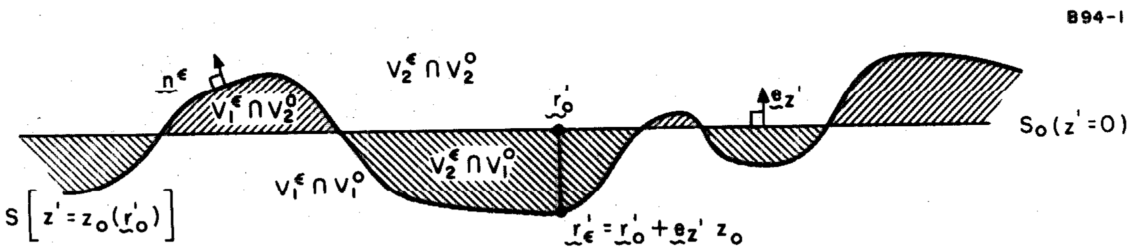
The configuration of interest is shown in Figure 1a. As in Section 2, the upper Medium M_2 is divided from the lower Medium M_1 by the almost plane interface S with equation

$$Q = z' - z_o(x',y') = z' - z_o(\underset{\sim}{r}'_o) = 0 \; . \tag{4.2.1}$$

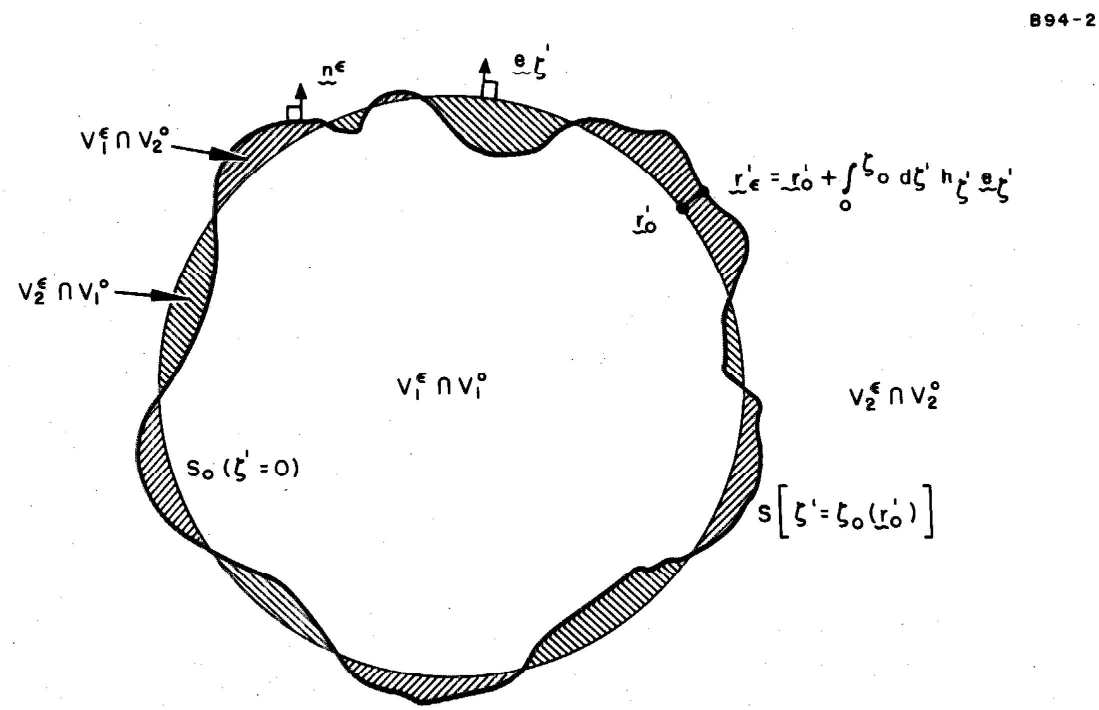
That is, a point $\underset{\sim}{r}'_o$ on the unperturbed surface S_o is perturbed into a point

$$\underset{\sim}{r}'_\epsilon = \underset{\sim}{r}'_o + \underset{\sim}{e}_z \, z_o \tag{4.2.2}$$

on S. The volume below S is designated by V_1^ϵ, the volume above S



a) Plane Interface



b) General Interface

Fig. 1. Perturbed Interface Configuration

by V_2^{ϵ} . Similarly, the volume below S_o is designated by V_1^o , that above by V_2^o . The points common to two regions will be denoted using the standard mathematical symbol for an intersection $\cap$; thus, e.g., $V_1^{\epsilon} \cap V_2^o$ is the set of points common to V_1^{ϵ} and V_2^o .

It is convenient to introduce the small parameter ϵ by setting

$$z_o(\underset{\sim}{r}_o') = w_o(\underset{\sim}{r}_o')\epsilon \ , \tag{4.2.3}$$

where w_o is a specified function with the properties

$$\mathrm{Max}\left\{|w_o(\underset{\sim}{r}_o')|\right\} \leq 1 \ , \ \mathrm{Max}\left\{|\nabla_T' w_o(\underset{\sim}{r}_o')|\right\} \leq B \tag{4.2.4}$$

(here B is a positive constant, and ∇_T' is the two-dimensional gradient tangent to S_o). Then the single parameter ϵ describes the magnitude of the perturbation.

Now let $\underset{\sim}{F}_q^o$ be a field vector, either $\underset{\wedge}{E}$ or $\underset{\wedge}{H}$, evaluated in medium M_q when the interface is <u>unperturbed</u>. Then, for z' of order ϵ, we write formally

$$\underset{\sim}{F}_q^o(\underset{\wedge}{r}') = \underset{\sim}{F}_q^o(\underset{\sim}{r}_o') + z' \frac{\partial}{\partial z'} \underset{\sim}{F}_q^o(\underset{\sim}{r}_o') + \frac{1}{2}(z')^2 \left(\frac{\partial}{\partial z'}\right)^2 \underset{\sim}{F}_q^o(\underset{\sim}{r}_o') + O(\epsilon^3), \tag{4.2.5}$$

where $\underset{\wedge}{r}'$ and $\underset{\sim}{r}_o'$ differ only in their z' coordinate. If $\underset{\wedge}{r}'$ lies in V_q^o , then Equation 4.2.5 is a statement of fact. Otherwise, the equation <u>defines</u> the continuation of the field $\underset{\sim}{F}_q^o$ outside V_q^o . The conditions under which the continuation is meaningful are discussed in Appendix 2.

Next, let us formally define the perturbed field $\underset{\sim}{F}_q^{\epsilon}$ by

$$\underset{\sim}{F}_q^{\epsilon}(\underset{\wedge}{r}') = \underset{\sim}{F}_q^o(\underset{\wedge}{r}') + \Delta \underset{\sim}{F}_q(\underset{\wedge}{r}') = \underset{\sim}{F}_q^o(\underset{\wedge}{r}') + \delta \underset{\sim}{F}_q(\underset{\wedge}{r}') + \delta^2 \underset{\sim}{F}_q(\underset{\wedge}{r}') + O(\epsilon^3) \ . \tag{4.2.6}$$

Here $\delta^n \underset{\sim}{F}_q$ is the perturbation field of order ϵ^n and satisfies Maxwell's equations for Medium M_q. Assuming that $\delta \underset{\sim}{F}_q$ and $\delta^2 \underset{\sim}{F}_q$ also have expansions of the form of Equation 4.2.5, we can rewrite Equation 4.2.6 as

$$\underset{\sim}{F}^{\epsilon}_q(\underset{\sim}{r}') = \underset{\sim}{F}^{o}_q(\underset{\sim}{r}'_o) + [z' \frac{\partial}{\partial z'} \underset{\sim}{F}^{o}_q(\underset{\sim}{r}'_o) + \delta \underset{\sim}{F}_q(\underset{\sim}{r}'_o)] + [\tfrac{1}{2}(z')^2 (\frac{\partial}{\partial z'})^2 \underset{\sim}{F}^{o}_q(\underset{\sim}{r}'_o)$$

$$+ z' \frac{\partial}{\partial z'} \delta \underset{\sim}{F}_q(\underset{\sim}{r}'_o) + \delta^2 \underset{\sim}{F}_q(\underset{\sim}{r}'_o)] + O(\epsilon^3) \qquad (4.2.7)$$

for z' of order ϵ. The conditions under which this equation is valid depend on the concept of continuation, for the discussion of which the reader is again referred to Appendix 2.

The boundary conditions on S are

$$\underset{\sim}{n}^{\epsilon}(\underset{\sim}{r}'_{\epsilon}) \times \Delta \underset{\sim}{F}^{\epsilon}(\underset{\sim}{r}'_{\epsilon}) = 0 \ . \qquad (4.2.8)$$

Here $\underset{\sim}{n}^{\epsilon}$ is the unit normal to S from V_1^{ϵ} to V_2^{ϵ} and is given by

$$\underset{\sim}{n}^{\epsilon}(\underset{\sim}{r}'_{\epsilon}) = \nabla' Q / |\nabla' Q| , \ \nabla' Q = \nabla' [z' - z_o(\underset{\sim}{r}'_o)] = \underset{\sim}{e}_z - \nabla'_T z_o(\underset{\sim}{r}'_o) \ ; \qquad (4.2.9)$$

$\Delta \underset{\sim}{F}$ is the discontinuity in field across the interface,

$$\Delta \underset{\sim}{F}(\underset{\sim}{r}'_o) = \underset{\sim}{F}_2(\underset{\sim}{r}'_o) - \underset{\sim}{F}_1(\underset{\sim}{r}'_o) \ ; \qquad (4.2.10)$$

and $\underset{\sim}{F}^{\epsilon}$ takes on the values $\underset{\sim}{E}^{\epsilon}$ and $\underset{\sim}{H}^{\epsilon}$.

The effective surface currents on S_o are found by setting Equations 4.2.7 and 4.2.9 into Equation 4.2.8, multiplying the resulting equation by $|\nabla' Q|$, and equating terms of the same order. Thus we find

$$\delta \underset{\sim}{K}_m(\underset{\sim}{r}'_o) = - \underset{\sim}{e}_z \times \Delta(\delta \underset{\sim}{E}) = \underset{\sim}{e}_z \times \left[(\Delta E^o_{z'}) \nabla'_T z_o + z_o \Delta(\frac{\partial}{\partial z'} \underset{\sim}{E}^o) \right]; \qquad (4.2.11)$$

$$\delta^2 \underset{\sim}{K}_m(\underset{\sim}{r}'_o) = - \underset{\sim}{e}_z \times \Delta(\delta^2 \underset{\sim}{E}) = \underset{\sim}{e}_z \times \left\{ \frac{1}{2} z_o^2 \Delta\left[(\frac{\partial}{\partial z'})^2 \underset{\sim}{E}^o \right] + [\Delta(\delta E_{z'})] \nabla'_T z_o + z_o \Delta\left[\frac{\partial}{\partial z'} (\delta \underset{\sim}{E}) \right] \right\}; \qquad (4.2.12)$$

$$\delta \underset{\sim}{K}_e(\underset{\sim}{r}'_o) = \underset{\sim}{e}_z \times \Delta(\delta \underset{\sim}{H}) = - \underset{\sim}{e}_z \times \left[(\Delta H^o_{z'}) \nabla'_T z_o + z_o \Delta(\frac{\partial}{\partial z'} \underset{\sim}{H}^o) \right]; \qquad (4.2.13)$$

$$\delta^2 \underset{\sim}{K}_e(\underset{\sim}{r}'_o) = \underset{\sim}{e}_z \times \Delta(\delta^2 \underset{\sim}{H}) = - \underset{\sim}{e}_z \times \left\{ \frac{1}{2} z_o^2 \Delta\left[(\frac{\partial}{\partial z'})^2 \underset{\sim}{H}^o \right] + [\Delta(\delta H_{z'})] \nabla'_T z_o + z_o \Delta\left[\frac{\partial}{\partial z'} (\delta \underset{\sim}{H}) \right] \right\}. \qquad (4.2.14)$$

Here all quantities on the right hand side are evaluated at $\underset{\sim}{r}'_o$. The identity

$$\Delta \frac{\partial}{\partial z'} F^o_{z'} = - \Delta \left(\frac{\partial}{\partial x'} F^o_{x'} + \frac{\partial}{\partial y'} F^o_{y'} \right) \equiv 0 \qquad (4.2.15)$$

is used in deriving the second order expressions.

The perturbation fields are found by applying Equations 3.3.23 and 24. Thus we find

$$\delta \underset{\sim}{E}(\underset{\sim}{r}) = \int_{S_o} dS' \, \Gamma^{\parallel}_e(\underset{\sim}{r};\underset{\sim}{r}'_o) \cdot \delta \underset{\sim}{K}_e(\underset{\sim}{r}'_o) + \frac{i\omega\mu}{k^2} \nabla \times \int_{S_o} dS' \, \Gamma^{\parallel}_m(\underset{\sim}{r};\underset{\sim}{r}'_o) \cdot \delta \underset{\sim}{K}_m(\underset{\sim}{r}'_o),$$

$$\delta^2 \underset{\sim}{E}(\underset{\sim}{r}) = \int_{S_o} dS' \, \Gamma^{\parallel}_e(\underset{\sim}{r};\underset{\sim}{r}'_o) \cdot \delta^2 \underset{\sim}{K}_e(\underset{\sim}{r}'_o) + \frac{i\omega\mu}{k^2} \nabla \times \int_{S_o} dS' \, \Gamma^{\parallel}_m(\underset{\sim}{r};\underset{\sim}{r}'_o) \cdot \delta^2 \underset{\sim}{K}_m(\underset{\sim}{r}'_o); \qquad (4.2.16)$$

$$\delta \underset{\sim}{H}(\underset{\sim}{r}) = \int_{S_o} dS' \, \Gamma^{\parallel}_m(\underset{\sim}{r};\underset{\sim}{r}'_o) \cdot \delta \underset{\sim}{K}_m(\underset{\sim}{r}'_o) + \frac{1}{i\omega\mu} \nabla \times \int_{S_o} dS' \, \Gamma^{\parallel}_e(\underset{\sim}{r};\underset{\sim}{r}'_o) \cdot \delta \underset{\sim}{K}_e(\underset{\sim}{r}'_o),$$

$$\delta^2\underset{\sim}{H}(\underset{\sim}{r}) = \int_{S_o} dS' \,\Gamma_m^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}_o') \cdot \delta^2\underset{\sim}{K}_m(\underset{\sim}{r}_o') + \frac{1}{i\omega\mu} \nabla \times \int_{S_o} dS' \,\Gamma_e^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}_o') \cdot \delta^2\underset{\sim}{K}_e(\underset{\sim}{r}_o') . \quad (4.2.17)$$

The $\Gamma^{\parallel}$ are those of Section 3.5C or, if M_1 is a perfect conductor, of Section 3.5B.

Let us now consider the conditions under which these results are valid. An obvious constraint is that no source may lie in $V_1^{\epsilon} \cap V_2^{o}$ or $V_2^{\epsilon} \cap V_1^{o}$, the volumes through which the surface is perturbed. Also, in order for the difference in field at corresponding points on S and S_o to be small, ϵ must be small compared to the wavelengths in M_1 and M_2.* Finally, the slope between corresponding points on S and S_o must be small, for otherwise the field lines would be distorted too severely. The discussion in Appendix 2 does not introduce any additional conditions, but reinforces the ones cited.

The material presented above can be applied readily to verify the results of Bass and Bocharov and of Rice. This is done in Appendix 3.

4.3 The General Interface

The general case is illustrated (with the irregularity exaggerated) in Figure 1b. An orthogonal curvilinear coordinate system (ξ', η', ζ') is chosen such that S_o is a coordinate surface $\zeta' = 0$ and $\zeta' > 0$ in V_2^{o}. The metrics of the system, $(h_{\xi'}, h_{\eta'}, h_{\zeta'})$, are in general functions of position. The coordinate system need not be defined everywhere in space, but the perturbed interface must lie within the volume in which the system is defined.

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*If, however, one medium--say M_1--is quite lossy, then the wavelength in M_1 is of no consequence in determining whether the perturbation theory can be used in M_2.

The equation of the perturbed interface S is

$$Q = \zeta' - \zeta_0(\xi',\eta') = \zeta' - \zeta_0(\underset{\sim}{r}'_0) = \zeta' - w_0(\underset{\sim}{r}'_0)\,\epsilon = 0 \,. \qquad (4.3.1)$$

Here w_0 (or ζ_0) is to be considered as defined in a volume but independent of ζ'; it satisfies the inequalities

$$\mathrm{Max}\left\{h_{\zeta'}(\underset{\sim}{r}')|w_0|\right\} \le 1 \,, \ \mathrm{Max}\left\{h_{\zeta'}(\underset{\sim}{r}')|\nabla'_T w_0|\right\} \le B \,, \qquad (4.3.2)$$

where the maximum is taken over the volumes $V_1^\epsilon \cap V_2^0$ and $V_2^\epsilon \cap V_1^0$. A point on S will be denoted by $\underset{\sim}{r}'_\epsilon$.

Now consider a point

$$\underset{\sim}{r}' = \underset{\sim}{r}'_0 + \Delta\underset{\sim}{r}' = \underset{\sim}{r}'_0 + \delta\underset{\sim}{r}' + O(\epsilon^2) = \underset{\sim}{r}'_0 + \int_0^{\zeta'} d\zeta'\, h_{\zeta'}\, \underset{\sim}{e}_{\zeta'} \qquad (4.3.3)$$

which differs from $\underset{\sim}{r}'_0$ only in its ζ' coordinate and is such that the curvilinear distance

$$d = \int_0^{\zeta'} d\zeta'\, h_{\zeta'} \qquad (4.3.4)$$

between the points is of order ϵ. It is readily shown that

$$\delta\underset{\sim}{r}' = h_{\zeta'}(\underset{\sim}{r}'_0)\,\zeta'\,\underset{\sim}{e}^0_{\zeta'} \,, \qquad (4.3.5)$$

where

$$\underset{\sim}{e}^0_{\zeta'} = \underset{\sim}{e}_{\zeta'}(\underset{\sim}{r}'_0) \,. \qquad (4.3.6)$$

Thus to first order $\Delta\underset{\sim}{r}'$ is a straight line segment.

The extension of Equation 4.2.5 is now found to be

$$\underset{\sim}{F}_q^0 (\underset{\sim}{r}') = \underset{\sim}{F}_q^0 (\underset{\sim}{r}_o') + \zeta' \frac{\partial}{\partial \zeta'} \underset{\sim}{F}_q^0 (\underset{\sim}{r}_o') + \frac{1}{2} (\zeta')^2 \left(\frac{\partial}{\partial \zeta'}\right)^2 \underset{\sim}{F}_q^0 (\underset{\sim}{r}_o') + O(\epsilon^3). \quad (4.3.7)$$

Combining this with Equation 4.2.6, which still applies in the general case, we obtain

$$\underset{\sim}{F}_q^\epsilon (\underset{\sim}{r}') = \underset{\sim}{F}_q^0 (\underset{\sim}{r}_o') + \left[\zeta' \frac{\partial}{\partial \zeta'} \underset{\sim}{F}_q^0 (\underset{\sim}{r}_o') + \delta \underset{\sim}{F}_q (\underset{\sim}{r}_o')\right]$$

$$+ \left[\frac{1}{2} (\zeta')^2 \left(\frac{\partial}{\partial \zeta'}\right)^2 \underset{\sim}{F}_q^0 (\underset{\sim}{r}_o') + \zeta' \frac{\partial}{\partial \zeta'} \delta \underset{\sim}{F}_q (\underset{\sim}{r}_o') + \delta^2 \underset{\sim}{F}_q (\underset{\sim}{r}_o')\right] + O(\epsilon^3); \quad (4.3.8)$$

this equation is the generalization of Equation 4.2.7. As before, the legitimacy of the formal result depends on the existence of the appropriate continuations of $\underset{\sim}{F}_q^0$, $\delta \underset{\sim}{F}_q$, and $\delta^2 \underset{\sim}{F}_q$, a matter discussed in Appendix 2.

As in the plane case, we designate by $\hat{n}^\epsilon(\underset{\sim}{r}_\epsilon')$ the unit normal to S from V_1^ϵ to V_2^ϵ. The boundary conditions of Equation 4.2.8 are still valid, but now the expression for $\hat{n}^\epsilon$ is more complicated. We must take account of the fact that the operator ∇' depends on $\hat{r}'$ and thus write

$$\hat{n}^\epsilon (\underset{\sim}{r}_\epsilon') = (\nabla' Q / |\nabla' Q|)_{\hat{r}' = \underset{\sim}{r}_\epsilon'} \,. \quad (4.3.9)$$

It is convenient to replace $\hat{n}^\epsilon$ by the vector

$$\hat{X} (\underset{\sim}{r}_\epsilon') = (h_{\zeta'} \nabla' Q)_{\hat{r}' = \underset{\sim}{r}_\epsilon'} \,, \quad (4.3.10)$$

which has the same direction as $\hat{n}^\epsilon$ everywhere on S. Expanding $\hat{X}$ in the same way as $\underset{\sim}{F}_q^0$, we obtain

$$\underset{\sim}{X}(\underset{\sim}{r}'_{\epsilon}) = (\underset{\sim}{e}_{\zeta'} - h_{\zeta'} \nabla'_T \zeta_o)_{\underset{\sim}{r}' = \underset{\sim}{r}'_{\epsilon}} = \Big\{ \underset{\sim}{e}_{\zeta'} + \zeta_o \frac{\partial}{\partial \zeta'} \underset{\sim}{e}_{\zeta'} - h_{\zeta'} \nabla'_T \zeta_o$$

$$+ \frac{1}{2} \zeta_o^2 \left(\frac{\partial}{\partial \zeta'}\right)^2 \underset{\sim}{e}_{\zeta'} - \zeta_o \frac{\partial}{\partial \zeta'} (h_{\zeta'} \nabla'_T \zeta_o) \Big\}_{\underset{\sim}{r}' = \underset{\sim}{r}'_o} + O(\epsilon^3) \; ; \qquad (4.3.11)$$

here ∇'_T is to be interpreted as the two-dimensional gradient normal to $\underset{\sim}{e}_{\zeta'}$. From Equation 1.3.6 of Reference 7, we find

$$\frac{\partial}{\partial \zeta'} \underset{\sim}{e}_{\zeta'} = -\nabla'_T h_{\zeta'} \; . \qquad (4.3.12)$$

Use of this in Equation 4.3.11 leads, after some simplification, to

$$\underset{\sim}{X}(\underset{\sim}{r}'_{\epsilon}) = \Big\{ \underset{\sim}{e}_{\zeta'} - \nabla'_T (h_{\zeta'} \zeta_o) - \frac{1}{2} \frac{\partial}{\partial \zeta'} \nabla'_T (h_{\zeta'} \zeta_o^2) \Big\}_{\underset{\sim}{r}' = \underset{\sim}{r}'_o} + O(\epsilon^3) \; . \qquad (4.3.13)$$

Setting Equations 4.3.8 and 4.3.13 into the boundary conditions, we determine the effective surface currents on S_o to be

$$\delta \underset{\sim}{K}_m(\underset{\sim}{r}'_o) = - \underset{\sim}{e}^o_{\zeta'} \times \Delta [\delta \underset{\sim}{E}(\underset{\sim}{r}'_o)] = \underset{\sim}{e}^o_{\zeta'} \times \Big\{ [\Delta E^o_{\zeta'}(\underset{\sim}{r}'_o)] \nabla'_T (h_{\zeta'} \zeta_o)$$

$$+ \zeta_o [\Delta \frac{\partial}{\partial \zeta'} \underset{\sim}{E}^o (\underset{\sim}{r}'_o)] \Big\}, \qquad (4.3.14)$$

$$\delta^2 \underset{\sim}{K}_m(\underset{\sim}{r}'_o) = - \underset{\sim}{e}^o_{\zeta'} \times \Delta [\delta^2 \underset{\sim}{E}(\underset{\sim}{r}'_o)] = \underset{\sim}{e}^o_{\zeta'} \times \Big\{ \frac{1}{2} \zeta_o^2 [\Delta \left(\frac{\partial}{\partial \zeta'}\right)^2 \underset{\sim}{E}^o (\underset{\sim}{r}'_o)]$$

$$+ \frac{1}{2} [\Delta E^o_{\zeta'} (\underset{\sim}{r}'_o)] \frac{\partial}{\partial \zeta'} \nabla'_T (h_{\zeta'} \zeta_o^2) + [\Delta \frac{\partial}{\partial \zeta'} E^o_{\zeta'} (\underset{\sim}{r}'_o)] \zeta_o \nabla'_T (h_{\zeta'} \zeta_o)$$

$$+ [\Delta \, \delta \, E_{\zeta'} (\underset{\sim}{r}'_o)] \nabla'_T (h_{\zeta'} \zeta_o) + \zeta_o [\Delta \frac{\partial}{\partial \zeta'} \delta \, \underset{\sim}{E}(\underset{\sim}{r}'_o)] \Big\} \qquad (4.3.15)$$

$$\delta \underset{\sim}{K}_e(\underset{\sim}{r}'_o) = \underset{\sim}{e}^o_{\zeta'} \times \Delta [\delta \underset{\sim}{H}(\underset{\sim}{r}'_o)] = - \underset{\sim}{e}^o_{\zeta'} \times \Big\{ [\Delta H^o_{\zeta'} (\underset{\sim}{r}'_o)] \nabla'_T (h_{\zeta'} \zeta_o)$$

$$+ \zeta_o [\Delta \frac{\partial}{\partial \zeta'} \underset{\sim}{H}^o (\underset{\sim}{r}'_o)] \Big\}, \qquad (4.3.16)$$

$$\delta^2 \underset{\sim}{K}_e (\underset{\sim}{r}'_o) = \underset{\sim}{e}^o_{\zeta'} \times \Delta [\delta^2 \underset{\sim}{H} (\underset{\sim}{r}'_o)] = - \underset{\sim}{e}^o_{\zeta'} \times \Big\{ \frac{1}{2} \zeta_o^2 [\Delta (\frac{\partial}{\partial \zeta'})^2 \underset{\sim}{H}^o (\underset{\sim}{r}'_o)]$$

$$+ \frac{1}{2} [\Delta H^o_{\zeta'} (\underset{\sim}{r}'_o)] \frac{\partial}{\partial \zeta'} \nabla'_T (h_{\zeta'} \zeta_o^2) + [\Delta \frac{\partial}{\partial \zeta'} H^o_{\zeta'} (\underset{\sim}{r}'_o)] \zeta_o \nabla'_T (h_{\zeta'} \zeta_o)$$

$$+ [\Delta \, \delta H_{\zeta'} (\underset{\sim}{r}'_o)] \nabla'_T (h_{\zeta'} \zeta_o) + \zeta_o [\Delta \frac{\partial}{\partial \zeta'} \delta \underset{\sim}{H} (\underset{\sim}{r}'_o)] \Big\} , \qquad (4.3.17)$$

where $\Delta \underset{\sim}{F}$ is defined by Equation 4.2.10 and the ∇'_T terms are evaluated at $\underset{\sim}{r}'_o$.

The perturbation fields are calculated using Equations 4.2.16 and 17 with the $\Gamma^{\parallel}$ appropriate to the unperturbed problem. The conditions under which the results are valid are the same as for the plane interface with one addition: the irregularity must be of amplitude small compared to the local radii of curvature of the underlying surface. This condition is necessary, among other reasons, because the field varies rapidly with position near a region of large curvature. At a sharp edge, the field is usually infinite, and no perturbation at all is allowed.

5. Scattering from Random Interface Irregularities

5.1 Problems of Interest

In many practical problems, the objective is to relate the statistics of random interface irregularities and the statistics of the field scattered from the irregularities. Such problems are of two types: those in which surface statistics are used to determine field statistics and those in which field statistics are used to determine surface statistics.

The first type of problem is straightforward; explicit equations for the important statistical averages of the field are readily derived from the results of Section 4. On the other hand, problems of the second type tend to be difficult. In general, the observed data must be supplemented with additional facts or assumptions about the nature of the surface, and then integral equations must be solved for the surface statistics. When possible, it is desirable to work with statistics of the field which, at least to lowest order, involve only $\delta \underset{\wedge}{F}$ and not $\delta^2 \underset{\wedge}{F}$. Other details of the approach depend strongly on the individual problem.

Each of the two types of problems may be further divided into three classes according to the nature of the interfaces being observed. First, there are problems involving observations on various fixed irregular interfaces* that are all samples of the same statistical ensemble; clearly ensemble statistics is the appropriate tool in such problems.

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*Note that observation of the instantaneous configuration of a slowly fluctuating interface is equivalent to observation of a fixed interface.

Second, there are problems involving observation over a long period of time of a single interface with slowly fluctuating irregularities (the fluctuations must be slow enough so that the scattered field can be considered monochromatic). Here time statistics are obviously appropriate; however, in many cases ergodicity may be invoked to replace certain of the time statistics by equivalent ensemble statistics.

The third class comprises problems involving a single fixed interface for which the following three statements are valid:*

1. The intersection of the underlying interface with any surface ξ' = const. is the same curve except for displacement and rotation.

2. For L sufficiently large but still small compared to the total range L_o of ξ', the integral

$$\mathcal{R}(\xi_1^o;\chi;\eta_1',\eta_2') = \frac{1}{L}\int_{\xi_1^o}^{\xi_1^o+L} d\xi_1' \; \zeta_o\,(\xi_1',\eta_1')\zeta_o(\xi_1'+\chi,\eta_2') \qquad (5.1.1)$$

is approximately independent of ξ_1^o and equal to

$$\mathcal{R}(\chi;\eta_1',\eta_2') = \lim_{L\to L_o} \frac{1}{L}\int_o^L d\xi_1' \; \zeta_o\,(\xi_1',\eta_1')\zeta_o(\xi_1'+\chi,\eta_2') . \qquad (5.1.2)$$

3. If $\chi > \chi_o$, where χ_o is very small compared to L_o, then

$$\mathcal{R}(\chi;\eta_1',\eta_2') \approx 0 . \qquad (5.1.3)$$

---

*In the following discussion, ξ' and η' can be interchanged, provided we do so throughout.

Under these circumstances, $\mathcal{R}(\chi;\eta_1',\eta_2')$ is a correlation function describing the surface irregularity. Furthermore, for all practical purposes we can assume

$$\mathcal{R}(\chi;\eta_1',\eta_2') = R(\chi;\eta_1',\eta_2') , \qquad (5.1.4)$$

where $R(\chi;\eta_1',\eta_2')$ is the correlation function of a statistical ensemble the sample functions of which include all the shifts of the irregularity function $\zeta_0(\xi'+c,\eta')$. Thus there exists a situation quite similiar to ergodicity; indeed, if L_0 is infinite, true ergodicity may occur.

5.2 Relaxation of the Boundedness Condition

The results of Section 4 have been derived under the assumption that the set of admissible interface perturbation functions is uniformly bounded in both amplitude and slope. This assumption is necessary for a rigorous theory of small perturbations.

On the other hand, it is often desirable from a computational point of view to consider an ensemble of perturbation functions which is not uniformly bounded, perhaps not even bounded, in amplitude and slope; an important example of such an ensemble is that associated with a Gaussian process.

The resulting dilemma, serious from the theoretical point of view, is usually resolved easily in practice. Consider a statistical process for which the probability is almost unity that a randomly chosen sample function has satisfactorily small amplitude and slope over a very large proportion of the underlying interface. This process is a mathematical model of a physical situation and tends to be least accurate at extreme values; indeed, the mathematical model usually exaggerates the occurrence of extreme values because limiting non linearities have been neglected. Therefore, little accuracy should be lost by replacing the original statistical process by a similar process with truncated amplitude and slope distributions. Then the analysis of Section 4 can be applied to the new process. But results obtained in this manner will usually be almost unchanged from those obtained by formally applying the analysis of Section 4 to the original process. Thus calculations can be done in exactly the same way as when dealing with a bounded process.

5.3 Ensemble Statistics of the Scattered Field

Having pointed out in Section 5.1 the importance of ensemble statistics in all classes of problems, we shall now discuss such statistics. The first order averages* of the perturbation fields are $\delta\bar{\underset{\sim}{E}}(\underset{\sim}{r})$ and $\delta\bar{\underset{\sim}{H}}(\underset{\sim}{r})$; the second order averages are $\delta^2\bar{\underset{\sim}{E}}(\underset{\sim}{r})$, $\delta^2\bar{\underset{\sim}{H}}(\underset{\sim}{r})$, and the matrices

$$\delta\mathcal{E}_{av}(\underset{\sim}{r}_1;\underset{\sim}{r}_2) = [\delta\underset{\sim}{E}(\underset{\sim}{r}_1)\delta\underset{\sim}{E}^*(\underset{\sim}{r}_2)]_{av} \quad , \quad \delta\mathcal{H}_{av}(\underset{\sim}{r}_1;\underset{\sim}{r}_2) = [\delta\underset{\sim}{H}(\underset{\sim}{r}_1)\delta\underset{\sim}{H}^*(\underset{\sim}{r}_2)]_{av} \; ,$$

$$\delta\mathcal{G}_{av}(\underset{\sim}{r}_1;\underset{\sim}{r}_2) = [\delta\underset{\sim}{E}(\underset{\sim}{r}_1)\delta\underset{\sim}{H}^*(\underset{\sim}{r}_2)]_{av} \; ; \qquad (5.3.1)$$

$$\delta E_{av}(\underset{\sim}{r}_1;\underset{\sim}{r}_2) = [\delta\underset{\sim}{E}(\underset{\sim}{r}_1)\delta\underset{\sim}{E}(\underset{\sim}{r}_2)]_{av} \quad , \quad \delta H_{av}(\underset{\sim}{r}_1;\underset{\sim}{r}_2) = [\delta\underset{\sim}{H}(\underset{\sim}{r}_1)\delta\underset{\sim}{H}(\underset{\sim}{r}_2)]_{av} \; ,$$

$$\delta G_{av}(\underset{\sim}{r}_1;\underset{\sim}{r}_2) = [\delta\underset{\sim}{E}(\underset{\sim}{r}_1)\delta\underset{\sim}{H}(\underset{\sim}{r}_2)]_{av} \; . \qquad (5.3.2)$$

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*Ensemble averages will be denoted by a bar over the quantity averaged or by the subscript av .

The matrices of Equation 5.3.1 are analogous to Wolf's time correlation matrices (14) and will thus be called ordinary ensemble correlation matrices; the matrices of Equation 5.3.2 will be called modified ensemble correlation matrices. Following the terminology of Born and Wolf (15), in the special case $\hat{r}_1=\hat{r}_2$ we refer to coherency matrices instead of correlation matrices. Coherency matrices arise in connection with observations by a single antenna,* correlation matrices in connection with interferometer observations.

The field averages enumerated above are not all independent; some can be calculated from others using Maxwell's equations. A convenient basic set of field averages comprises $\delta\hat{\overline{E}}$, $\delta^2\hat{\overline{E}}$, $\delta\,\mathcal{E}_{av}$, and $\delta\boldsymbol{E}_{av}$; these are the quantities which appear in analyses in which the receiver is considered sensitive to the electric field. Since only six terms each in $\delta\,\mathcal{E}_{av}$ and $\delta\boldsymbol{E}_{av}$ are independent, the first and second order field statistics are determined by eighteen independent functions.

Knowing these functions enables us to calculate to second order such important statistics as the mean and variance of the observed field and the mean and variance of the observed power. Indeed, we can calculate to second order the average of any function of form

$$f = \sum_i \prod_j L_i\, F_{ij}(\hat{r}_{ij}) \,, \tag{5.3.3}$$

where L_i is a linear operator and the F_{ij} are fields, complex conjugates of fields, and real and imaginary parts of fields.

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*The term "antenna" here refers to not only those devices usually thought of as antennas, but also to optical receivers such as telescopes.

In practical problems, the number of independent functions to be considered is usually many fewer then eighteen. We usually deal with irregularities which have zero mean:

$$\bar{\zeta}_o(\underset{\sim}{r}'_o) \equiv 0 \ , \ \text{all } \underset{\sim}{r}'_o \ ; \tag{5.3.4}$$

it then follows from Equations 4.2.16 and 4.3.14 and 16 that

$$\delta\bar{\underset{\wedge}{E}}(\underset{\wedge}{r}) \equiv 0 \ , \ \text{all } \underset{\wedge}{r} \ . \tag{5.3.5}$$

In addition, it is often possible to choose a coordinate system in which the components of $\delta\underset{\wedge}{E}$ and $\delta^2\underset{\wedge}{E}$ along one coordinate are negligible or zero. Under these conditions, the numbers of non-zero independent functions is eight. Furthermore, the information desired in a given problem may involve only one or a few of these eight functions over part of their range. A case in point is the example of Section 5.4.

Using the results of Section 4, we can express the field averages in terms of the mean surface displacement $\bar{\zeta}_o$ (which, as noted previously, is usually zero) and the displacement correlation function

$$R = R(\underset{\sim}{r}'_o, \underset{\sim}{r}''_o) = [\zeta_o(\underset{\sim}{r}'_o)\zeta_o(\underset{\sim}{r}''_o)]_{av} \ . \tag{5.3.6}$$

We obtain for the first order average

$$\delta\bar{\underset{\wedge}{E}}(\underset{\wedge}{r}) = \int_{S_o} dS' \ \Gamma^{\parallel}_e(\underset{\wedge}{r};\underset{\sim}{r}'_o)\cdot\delta\bar{\underset{\sim}{K}}_e(\underset{\sim}{r}'_o) + \frac{i\omega\mu}{k^2}\nabla \times \int_{S_o} dS' \ \Gamma^{\parallel}_m(\underset{\wedge}{r};\underset{\sim}{r}'_o)\cdot\delta\bar{\underset{\sim}{K}}_m(\underset{\sim}{r}'_o) \ , \tag{5.3.7}$$

where

$$\delta\bar{\underset{\sim}{K}}_m(\underset{\sim}{r}_o') = \underset{\sim}{e}^o_{\zeta'} \times \left\{[\Delta E^o_{\zeta'}(\underset{\sim}{r}_o')]\nabla_T'(h_{\zeta'}\bar{\zeta}_o) + \bar{\zeta}_o\, [\Delta \frac{\partial}{\partial\zeta'}\underset{\sim}{E}^o(\underset{\sim}{r}_o')]\right\},$$

$$\delta\bar{\underset{\sim}{K}}_e(\underset{\sim}{r}_o') = -\underset{\sim}{e}^o_{\zeta'} \times \left\{[\Delta H^o_{\zeta'}(\underset{\sim}{r}_o')]\nabla_T'(h_{\zeta'}\bar{\zeta}_o) + \bar{\zeta}_o\, [\Delta \frac{\partial}{\partial\zeta'}\underset{\sim}{H}_o(\underset{\sim}{r}_o')]\right\}. \qquad (5.3.8)$$

The ordinary ensemble correlation matrix is given by

$$\delta\mathcal{E}_{av}(\underset{\sim}{r}_1;\underset{\sim}{r}_2) = \int_{S_o} dS' \int_{S_o} dS'' \Big\{\Gamma_e^{\parallel}(\underset{\sim}{r}_1;\underset{\sim}{r}_o')\cdot[\delta\underset{\sim}{K}_e(\underset{\sim}{r}_o')\delta\underset{\sim}{K}_e^*(\underset{\sim}{r}_o'')]_{av}\cdot[\Gamma_e^{\parallel}(\underset{\sim}{r}_2;\underset{\sim}{r}_o'')]^\dagger$$

$$+ \left(\frac{i\omega\mu}{k^2}\right)^*\Gamma_e^{\parallel}(\underset{\sim}{r}_1;\underset{\sim}{r}_o')\cdot[\delta\underset{\sim}{K}_e(\underset{\sim}{r}_o')\delta\underset{\sim}{K}_m^*(\underset{\sim}{r}_o'')]_{av}\cdot[\nabla_2\times\Gamma_m^{\parallel}(\underset{\sim}{r}_2;\underset{\sim}{r}_o'')]^\dagger$$

$$+ \frac{i\omega\mu}{k^2}\nabla_1\times\Gamma_m^{\parallel}(\underset{\sim}{r}_1;\underset{\sim}{r}_o')\cdot[\delta\underset{\sim}{K}_m(\underset{\sim}{r}_o')\delta\underset{\sim}{K}_e^*(\underset{\sim}{r}_o'')]_{av}\cdot[\Gamma_e^{\parallel}(\underset{\sim}{r}_2;\underset{\sim}{r}_o'')]^\dagger$$

$$+ \left(\frac{i\omega\mu}{k^2}\right)\left(\frac{i\omega\mu}{k^2}\right)^*\nabla_1\times\Gamma_m^{\parallel}(\underset{\sim}{r}_1;\underset{\sim}{r}_o')\cdot[\delta\underset{\sim}{K}_m(\underset{\sim}{r}_o')\delta\underset{\sim}{K}_m^*(\underset{\sim}{r}_o'')]_{av}\cdot[\nabla_2\times\Gamma_m^{\parallel}(\underset{\sim}{r}_2;\underset{\sim}{r}_o'')]^\dagger\Big\},$$

$$(5.3.9)$$

where

$$[\delta\underset{\sim}{K}_e(\underset{\sim}{r}_o')\delta\underset{\sim}{K}_e^*(\underset{\sim}{r}_o'')]_{av} =$$

$$\underset{\sim}{e}^o_{\zeta'} \times\Big\{[\Delta H^o_{\zeta'}(\underset{\sim}{r}_o')][\Delta H^o_{\zeta''}(\underset{\sim}{r}_o'')]^*\nabla_T'\nabla_T''(h_{\zeta'}h_{\zeta''}R)$$

$$+ [\Delta H^o_{\zeta'}(\underset{\sim}{r}_o')]\nabla_T'(h_{\zeta'}R)[\Delta\frac{\partial}{\partial\zeta''}\underset{\sim}{H}^o(\underset{\sim}{r}_o'')]^*$$

$$+ [\Delta\frac{\partial}{\partial\zeta'}\underset{\sim}{H}^o(\underset{\sim}{r}_o')][\Delta H^o_{\zeta''}(\underset{\sim}{r}_o'')]^*\nabla_T''(h_{\zeta''}R)$$

$$+ [\Delta\frac{\partial}{\partial\zeta'}\underset{\sim}{H}^o(\underset{\sim}{r}_o')][\Delta\frac{\partial}{\partial\zeta''}\underset{\sim}{H}^o(\underset{\sim}{r}_o'')]^*R\Big\}\times\underset{\sim}{e}^o_{\zeta''},$$

$$[\delta\underset{\sim}{K}_e(\underset{\sim}{r}_o')\delta\underset{\sim}{K}_m^*(\underset{\sim}{r}_o'')]_{av} =$$

$$- \underset{\sim}{e}^{o}_{\zeta'} \times \Big\{ [\Delta H^{o}_{\zeta'}(\underset{\sim}{r}'_{o})][\Delta E^{o}_{\zeta''}(\underset{\sim}{r}''_{o})]^{*} \nabla'_{T} \nabla''_{T} (h_{\zeta'} h_{\zeta''} R)$$

$$+ [\Delta H^{o}_{\zeta'}(\underset{\sim}{r}'_{o})] \nabla'_{T}(h_{\zeta'} R)[\Delta \frac{\partial}{\partial \zeta''} \underset{\sim}{E}^{o}(\underset{\sim}{r}''_{o})]^{*} + [\Delta \frac{\partial}{\partial \zeta'} \underset{\sim}{H}^{o}(\underset{\sim}{r}'_{o})][\Delta E^{o}_{\zeta''}(\underset{\sim}{r}''_{o})]^{*} \nabla''_{T}(h_{\zeta''} R)$$

$$+ [\Delta \frac{\partial}{\partial \zeta'} \underset{\sim}{H}^{o}(\underset{\sim}{r}'_{o})][\Delta \frac{\partial}{\partial \zeta''} \underset{\sim}{E}^{o}(\underset{\sim}{r}'_{o})]^{*} R \Big\} \times \underset{\sim}{e}^{o}_{\zeta''} ,$$

$$[\delta \underset{\sim}{K}_{m}(\underset{\sim}{r}'_{o}) \delta \underset{\sim}{K}^{*}_{m}(\underset{\sim}{r}''_{o})]_{av} =$$

$$\underset{\sim}{e}^{o}_{\zeta'} \times \Big\{ [\Delta E^{o}_{\zeta'}(\underset{\sim}{r}'_{o})][\Delta E^{o}_{\zeta''}(\underset{\sim}{r}''_{o})]^{*} \nabla'_{T} \nabla''_{T}(h_{\zeta'} h_{\zeta''} R)$$

$$+ [\Delta E^{o}_{\zeta'}(\underset{\sim}{r}'_{o})] \nabla'_{T}(h'_{\zeta} R)[\Delta \frac{\partial}{\partial \zeta''} \underset{\sim}{E}^{o}(\underset{\sim}{r}''_{o})]^{*} + [\Delta \frac{\partial}{\partial \zeta'} \underset{\sim}{E}^{o}(\underset{\sim}{r}'_{o})][\Delta E^{o}_{\zeta''}(\underset{\sim}{r}''_{o})]^{*} \nabla''_{T}(h''_{\zeta} R)$$

$$+ [\Delta \frac{\partial}{\partial \zeta'} \underset{\sim}{E}^{o}(\underset{\sim}{r}'_{o})][\Delta \frac{\partial}{\partial \zeta''} \underset{\sim}{E}^{o}(\underset{\sim}{r}''_{o})]^{*} R \Big\} \times \underset{\sim}{e}^{o}_{\zeta''} . \qquad (5.3.10)$$

In interpreting these equations, the four position vectors $\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}'_o$, and $\underset{\sim}{r}''_o$ must be distinguished from each other even when two or more coincide. All ∇ operators have been tagged to indicate with respect to which coordinates they operate.

The modified ensemble correlation matrix $\delta \mathcal{E}_{av}$ can be determined from Equations 5.3.9 and 10 by removing all complex conjugate symbols (*) and replacing the Hermitian conjugate $(\Gamma^{\parallel})^{\dagger}$ with the transpose $(\Gamma^{\parallel})^{T}$.

The second order average is

$$\delta^2 \overline{\underset{\sim}{E}}(\underset{\sim}{r}) = \int_{S_o} dS' \, \Gamma^{\parallel}_{e}(\underset{\sim}{r};\underset{\sim}{r}'_{o}) \cdot \delta^2 \overline{\underset{\sim}{K}}_{e}(\underset{\sim}{r}'_{o}) + \frac{i\omega\mu}{k^2} \nabla \times \int_{S_o} dS' \, \Gamma^{\parallel}_{m}(\underset{\sim}{r};\underset{\sim}{r}'_{o}) \cdot \delta^2 \overline{\underset{\sim}{K}}_{m}(\underset{\sim}{r}'_{o}) , \qquad (5.3.11)$$

where

$$\delta^2 \overline{K}_{\sim m}(\underset{\sim}{r}_o') = \underset{\sim}{e}^o_{\zeta'} \times \Big\{ \frac{1}{2} [\Delta (\frac{\partial}{\partial \zeta'})^2 \underset{\sim}{E}^o(\underset{\sim}{r}_o')] R_{oo} + \frac{1}{2} [\Delta E^o_{\zeta'}(\underset{\sim}{r}_o')] \frac{\partial}{\partial \zeta'} \nabla_T'(h_{\zeta'} R_{oo})$$

$$+ [\Delta \frac{\partial}{\partial \zeta'} E^o_{\zeta'}(\underset{\sim}{r}_o')][\nabla_T'(h_{\zeta'} R_{oo}) - \frac{1}{2} h_{\zeta'} \nabla_T' R_{oo}]$$

$$+ \Delta \int_{S_o} dS'' \underset{\sim}{e}^o_{\zeta'} \cdot \Gamma^{\parallel}_e(\underset{\sim}{r}_o';\underset{\sim}{r}_o'') \cdot \Big\{ \delta \underset{\sim}{K}_e(\underset{\sim}{r}_o'') \nabla_T'[h_{\zeta'} \zeta_o(\underset{\sim}{r}_o')] \Big\}_{av}$$

$$+ \Delta \frac{i\omega\mu}{k^2} \int_{S_o} dS'' \underset{\sim}{e}^o_{\zeta'} \cdot \nabla' \times \Gamma^{\parallel}_m(\underset{\sim}{r}_o';\underset{\sim}{r}_o'') \cdot \Big\{ \delta \underset{\sim}{K}_m(\underset{\sim}{r}_o'') \nabla_T'[h_{\zeta'} \zeta_o(\underset{\sim}{r}_o')] \Big\}_{av}$$

$$+ \Delta \int_{S_o} dS'' \Big\{ \frac{\partial}{\partial \zeta'} \Gamma^{\parallel}_e(\underset{\sim}{r}_o';\underset{\sim}{r}_o'') \cdot [\delta \underset{\sim}{K}_e(\underset{\sim}{r}_o'') \zeta_o(\underset{\sim}{r}_o')]_{av}$$

$$+ \frac{i\omega\mu}{k^2} \frac{d}{d\zeta'} [\nabla' \times \Gamma^{\parallel}_m(\underset{\sim}{r}_o';\underset{\sim}{r}_o'')] \cdot [\delta \underset{\sim}{K}_m(\underset{\sim}{r}_o'') \zeta_o(\underset{\sim}{r}_o')]_{av} \Big\} \Big\},$$

$$\delta^2 \overline{K}_{\sim e}(\underset{\sim}{r}_o') = - \underset{\sim}{e}^o_{\zeta'} \times \Big\{ \frac{1}{2} [\Delta (\frac{\partial}{\partial \zeta'})^2 H^o(\underset{\sim}{r}_o')] R_{oo}$$

$$+ \frac{1}{2}[\Delta H^o_{\zeta'}(\underset{\sim}{r}_o')] \frac{\partial}{\partial \zeta'} \nabla_T' (h_{\zeta'} R_{oo}) + [\Delta \frac{\partial}{\partial \zeta'} H^o_{\zeta'} (\underset{\sim}{r}_o')][\nabla_T'(h_{\zeta'} R_{oo}) - \frac{1}{2} h_{\zeta'} \nabla_T' R_{oo}]$$

$$+ \Delta \int_{S_o} dS'' \underset{\sim}{e}^o_{\zeta'} \cdot \Gamma^{\parallel}_m (\underset{\sim}{r}_o';\underset{\sim}{r}_o'') \cdot \Big\{ \delta \underset{\sim}{K}_m(\underset{\sim}{r}_o'') \nabla_T' [h_{\zeta'} \zeta_o(\underset{\sim}{r}_o')] \Big\}_{av}$$

$$+ \Delta \frac{1}{i\omega\mu} \int_{S_o} dS'' \underset{\sim}{e}^o_{\zeta'} \cdot \nabla' \times \Gamma^{\parallel}_e(\underset{\sim}{r}_o';\underset{\sim}{r}_o'') \cdot \Big\{ \delta \underset{\sim}{K}_e(\underset{\sim}{r}_o'') \nabla_T'[h_{\zeta'} \zeta_o(\underset{\sim}{r}_o')] \Big\}_{av}$$

$$+ \Delta \int_{S_o} dS'' \Big\{ \frac{\partial}{\partial \zeta'} \Gamma^{\parallel}_m(\underset{\sim}{r}_o';\underset{\sim}{r}_o'') \cdot [\delta \underset{\sim}{K}_m(\underset{\sim}{r}_o'') \zeta_o(\underset{\sim}{r}_o')]_{av} + \frac{1}{i\omega\mu} \frac{\partial}{\partial \zeta'} [\nabla' \times \Gamma^{\parallel}_e(\underset{\sim}{r}_o';\underset{\sim}{r}_o'')] \cdot$$

$$[\delta \underset{\sim}{K}_e(\underset{\sim}{r}_o'') \; \zeta_o(\underset{\sim}{r}_o')]_{av} \Big\} \Big\}. \qquad (5.3.12)$$

Here

$$R_{oo} = R_{oo}(\underset{\sim}{r}_o') = R(\underset{\sim}{r}_o';\underset{\sim}{r}_o') \qquad (5.3.13)$$

(R_{oo} to be interpreted as a function of a single position vector) and

$$\left\{\delta\, \underset{\sim}{K}_m(\underset{\sim}{r}_o'')\ \nabla_T'[h_{\zeta'}\zeta_o(\underset{\sim}{r}_o')]\right\}_{av} =$$

$$\underset{\sim}{e}^o_{\zeta''} \times \left\{[\Delta E^o_{\zeta''}(\underset{\sim}{r}_o'')]\nabla_T''\nabla_T'\ (h_{\zeta'}h_{\zeta''}R) + [\Delta \frac{\partial}{\partial \zeta''}\ \hat{E}^o(\underset{\sim}{r}_o'')]\nabla_T'(h_{\zeta'}R)\right\},$$

$$\left\{\delta\, \underset{\sim}{K}_e(\underset{\sim}{r}_o'')\nabla_T'[h_{\zeta'}\zeta_o(\underset{\sim}{r}_o')]\right\}_{av} =$$

$$-\ \underset{\sim}{e}^o_{\zeta''} \times \left\{[\Delta H^o_{\zeta''}(\underset{\sim}{r}_o'')]\nabla_T''\nabla_T'\ (h_{\zeta'}h_{\zeta''}R) + [\Delta \frac{\partial}{\partial \zeta''}\ \hat{H}^o(\underset{\sim}{r}_o'')]\nabla_T'(h_{\zeta'}R)\right\},$$

$$[\delta\, \underset{\sim}{K}_m(\underset{\sim}{r}_o'')\zeta_o(\underset{\sim}{r}_o')]_{av} = \underset{\sim}{e}^o_{\zeta''} \times \left\{[\Delta E^o_{\zeta''}(\underset{\sim}{r}_o'')]\nabla_T''(h_{\zeta''}R) + [\Delta \frac{\partial}{\partial \zeta''}\ \hat{E}^o(\underset{\sim}{r}_o'')]R\right\},$$

$$[\delta\, \underset{\sim}{K}_e(\underset{\sim}{r}_o'')\zeta_o(\underset{\sim}{r}_o')]_{av} = -\ \underset{\sim}{e}^o_{\zeta''} \times \left\{[\Delta H^o_{\zeta''}(\underset{\sim}{r}_o'')\nabla_T''(h_{\zeta''}R) + [\Delta \frac{\partial}{\partial \zeta''}\ \hat{H}^o(\underset{\sim}{r}_o'')]R\right\}.$$

(5.3.14)

It is obvious from these formidable expressions that it is highly desirable to be able to neglect $\delta^2\overline{\hat{E}}$ when determining surface statistics from field measurements.

5.4 An Important Special Case

We now discuss briefly a simple problem of great practical importance. Consider a power-measuring antenna at point $\underset{\sim}{r}_L$ in the radiation zone which is observing the field scattered from a finite irregular interface illuminated by a known source. The observation point $\underset{\sim}{r}_L$ is chosen so that if the interface were unperturbed, then the scattered electric field at $\underset{\sim}{r}_L$ would be linearly polarized along $\underset{\sim}{e}_\eta$. The antenna is polarization sensitive, passing only the ξ-component of the electric field.

To second order, the average power observed by the antenna is

$$P_{av} = [E_{\xi}(\underset{\sim}{r}_L)E^*_{\xi}(\underset{\sim}{r}_L)] = [\delta E_{\xi}(\underset{\sim}{r}_L)\delta E^*_{\xi}(\underset{\sim}{r}_L)] = \delta \mathcal{J}_{\xi\xi,av}(\underset{\sim}{r}_L) , \qquad (5.4.1)$$

where $\delta \mathcal{J}$ is the coherency matrix corresponding to $\delta \mathcal{E}$. Thus in this case the average power observed is equal to a single element of the ordinary coherency matrix.

This simple problem arises frequently in connection with noise suppression. Furthermore, the configuration is a convenient one for making measurements to determine surface statistics. A similar convenient interferometer configuration can be set up along a line of linear polarization.

6. Example--Scattering of a Plane Wave from a Perfectly Conducting Cylinder with Sinusoidal Irregularities

In most cases, application of the theory developed above leads to rather complicated expressions for the field perturbations and even more complicated expressions for the statistics of the field. Although such expressions can often be treated by existing analytic and numerical methods, a great deal of effort may be required to obtain a useful form.

On the other hand, there are some problems of considerable interest in which it is fairly simple to calculate at least $\delta \underset{\frown}{E}$. The plane interface problem is one example; we shall now consider another.

The geometry of the unperturbed problem is shown in Figure 2. A plane wave

$$\underset{\frown}{E}^{inc} = \underset{\sim}{e}_z E_o e^{-ikx} = \underset{\sim}{e}_z E_o \sum_{n=-\infty}^{+\infty} J_n(k\rho)\, e^{in(\theta-\pi/2)} , \qquad (6.1)$$

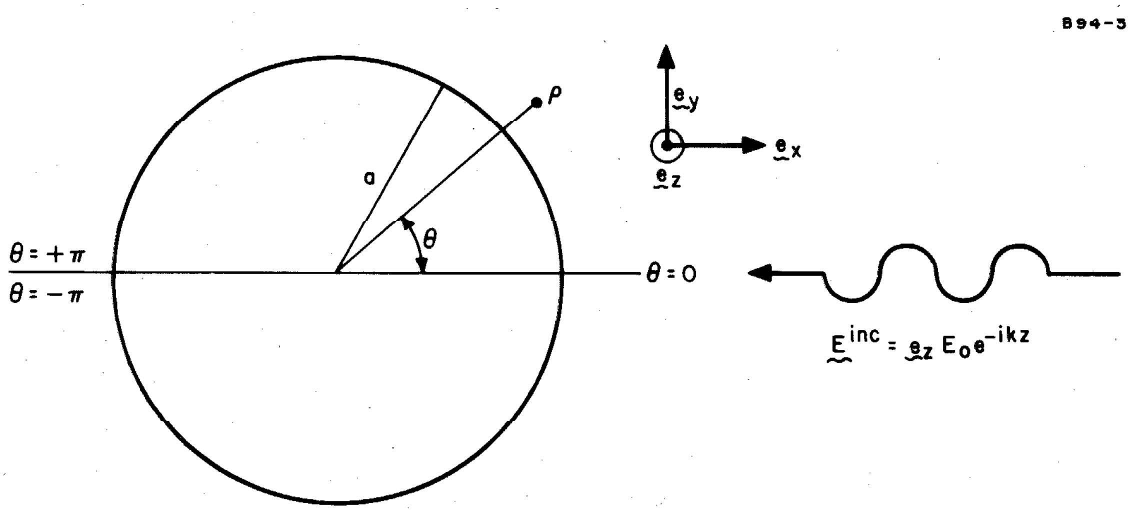


Fig. 2. Geometry of the Unperturbed Problem

travelling in the negative x-direction, is incident on a perfectly conducting circular cylinder of radius a. The scattered field is readily calculated to be

$$\underset{\sim}{E}^{scat} = - \underset{\sim}{e}_z E_o \sum_{n=-\infty}^{+\infty} [H_n^{(1)}(ka)]^{-1} H_n^{(1)}(k\rho) J_n(ka) e^{in(\theta-\pi/2)} . \tag{6.2}$$

In the far zone this can be rewritten as

$$\underset{\sim}{E}^{scat} = \underset{\sim}{e}_z E_o (k\rho)^{-1/2} e^{i(k\rho-\pi/4)} M(\theta,ka) , \tag{6.3}$$

where M is the field pattern

$$M(\theta,ka) = - (\tfrac{2}{\pi})^{1/2} \sum_{n=-\infty}^{+\infty} [H_n^{(1)}(ka)]^{-1} J_n(ka) e^{in(\theta-\pi)} . \tag{6.4}$$

In Figure 3, the magnitude and phase of M are plotted for $ka=6$ and

$$0° \leq \theta \leq 180° ;$$

only half the pattern is needed because M is an even function of angle. The apparent discontinuities in phase angle are caused by telescoping a continuous phase curve into a range of 360°.

Now let us introduce a sinusoidal interface perturbation so that the equation of the new interface is

$$Q = (\rho-a) - \rho_o = (\rho-a) - b \cos(p\theta+\psi) = 0 . \tag{6.5}$$

Here the perturbation amplitude b is small compared to both the wavelength and the radius (which are approximately equal at $ka=6$), the positive integer p is the number of full sine waves impressed around the cylinder, and the phase angle ψ depends on the position of the maxima of the interface perturbation.

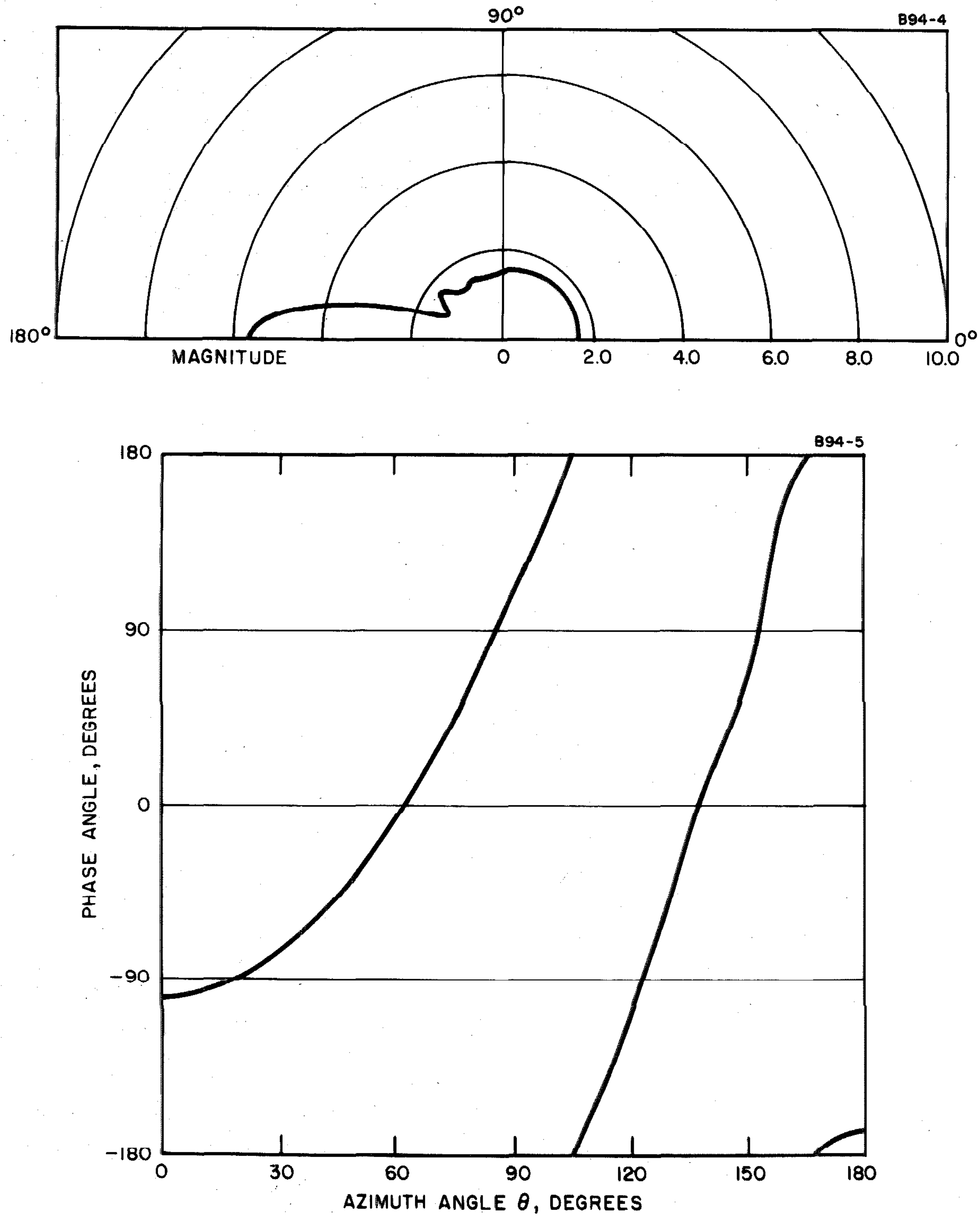


Fig. 3. Magnitude and Phase of the Unperturbed Field Pattern M(θ,6).

Using the fact that the cylinder is a perfect conductor, we find from Equation 4.2.16 that

$$\delta \underset{\sim}{E} = \frac{i\omega\mu}{k^2} \int_{S_o} dS' \nabla \times \underset{\approx}{\Gamma}_m^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}_o') \cdot \delta \underset{\sim}{K}_m (\underset{\sim}{r}_o') \ , \tag{6.6}$$

where, from Equation 4.3.14,

$$\delta \underset{\sim}{K}_m (\underset{\sim}{r}_o') = - \underset{\sim}{e}_{\theta'}^o \ \rho_o \frac{\partial}{\partial \rho'} E_z^o =$$

$$\underset{\sim}{e}_{\theta'}^o \ \frac{2i}{\pi} \frac{b}{a} E_o \cos(p\theta'+\psi) \sum_{n=-\infty}^{+\infty} [H_n^{(1)}(ka)]^{-1} e^{in(\theta'-\pi/2)} \ . \tag{6.7}$$

The dyadic $\nabla \times \underset{\approx}{\Gamma}_m^{\parallel}$ can be obtained from Section 3.5D. However, rather than insert the dyadic directly into Equation 6.6, let us rewrite that equation as

$$\delta \underset{\sim}{E} = \frac{i\omega\mu}{k^2} \int_{-\pi}^{\pi} d\theta \ a\underset{\sim}{e}_{\theta'}^o \cdot \delta \underset{\sim}{K}_m (\underset{\sim}{r}_o') \ \underset{\sim}{G}(\underset{\sim}{r};\underset{\sim}{r}_o') \ , \tag{6.8}$$

with

$$\underset{\sim}{G} (\underset{\sim}{r};\underset{\sim}{r}_o') = \int_{-\infty}^{+\infty} dz' \nabla \times \underset{\approx}{\Gamma}_m^{\parallel} (\underset{\sim}{r};\underset{\sim}{r}_o') \cdot \underset{\sim}{e}_{\theta'}^o \ . \tag{6.9}$$

In Equation 6.9, the integration over z eliminates from the Green's function all waves obliquely incident on the cylinder, the dot product with $\underset{\sim}{e}_{\theta'}^o$ reduces the dyadic to a vector, and the fact that the cylinder is a perfect conductor leads to further simplifications as compared to Section 3.5D. Thus we obtain without difficulty

$$\underset{\sim}{G} = \underset{\sim}{e}_z \frac{1}{2\pi a} \frac{k^2}{i\omega\mu} \sum_{n=-\infty}^{+\infty} [H_n^{(1)}(ka)]^{-1} H_n^{(1)}(k\rho) \ e^{in(\theta-\theta')} \ . \tag{6.10}$$

Using Equations 6.7 and 6.10 in 6.8 and performing the integration, we obtain

$$\delta \underset{\sim}{E} = \underset{\sim}{e}_z E_o \frac{b}{a} \frac{i}{\pi} \sum_{n=-\infty}^{+\infty} \frac{H_n^{(1)}(k\rho)}{H_n^{(1)}(ka)} e^{in(\theta - \pi/2)} \left[\frac{e^{i(p\pi/2+\psi)}}{H_{n-p}^{(1)}(ka)} + \frac{e^{-i(p\pi/2+\psi)}}{H_{n+p}^{(1)}(ka)} \right] . \quad (6.11)$$

Some algebraic manipulation leads to the more elegant form

$$\delta \underset{\sim}{E} = \underset{\sim}{e}_z E_o \frac{b}{a} [P_e \cos \psi + P_o \sin \psi] , \quad (6.12)$$

where

$$\left.\begin{matrix} P_e \\ \\ P_o \end{matrix}\right\} = \frac{2i}{\pi} \sum_{n=-\infty}^{+\infty} [H_n^{(1)}(ka) H_{n+p}^{(1)}(ka)]^{-1} H_n^{(1)}(k\rho) e^{-i(n+p)\pi/2} \left\{\begin{matrix} \cos n\Theta \\ \\ \sin n\Theta \end{matrix}\right. . \quad (6.13)$$

Now let us set $\psi=0$ so that $\delta \underset{\sim}{E}$ depends only on P_e , and let us consider only the far zone. We find

$$\delta \underset{\sim}{E} = \underset{\sim}{e}_z E_o (k\rho)^{-1/2} e^{i(k\rho-\pi/4)} [\frac{b}{a} M^{\epsilon} (\Theta,ka;p)] , \quad (6.14)$$

where

$$M^{\epsilon}(\Theta,ka;p) = i \left(\frac{2}{\pi}\right)^{3/2} \sum_{n=-\infty}^{+\infty} [H_n^{(1)}(ka) H_{n+p}^{(1)}(ka)]^{-1} e^{-i(n+p/2)\pi} \cos n\Theta . \quad (6.15)$$

The field pattern analogous to M of Equation 6.4 is $\frac{b}{a} M^{\epsilon}$.

In Figures 4-8, the magnitude and phase of $M^{\epsilon}(\Theta,ka;p)$ are plotted for $ka=6$ and $p=1,4,8,12,18$. Since we have set $\psi=0, M^{\epsilon}$ is an even function of angle and thus again only half the pattern is required. In calculating M^{ϵ} , the number of terms necessary for a given accuracy is reduced considerably by rearranging the series representation as follows:

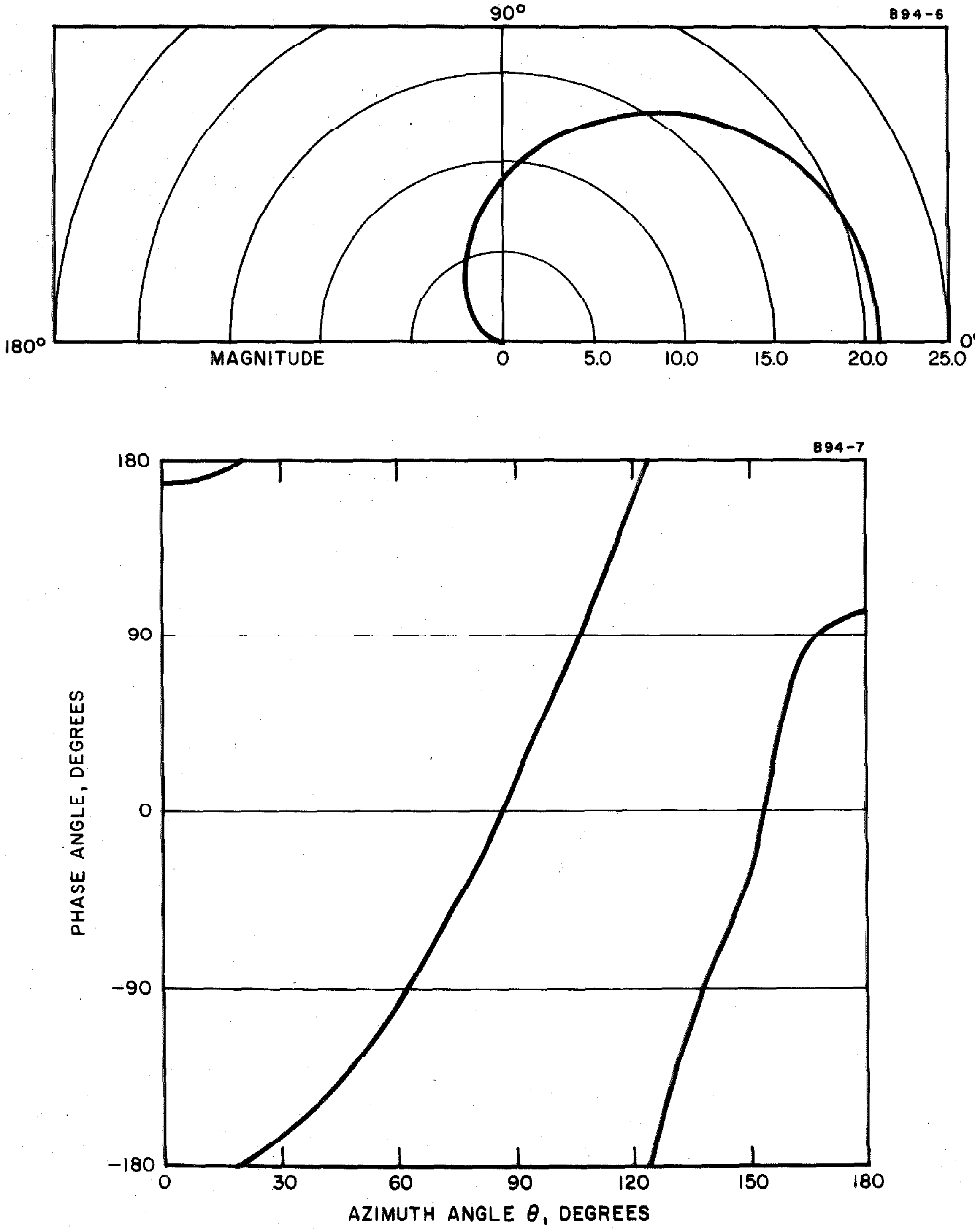
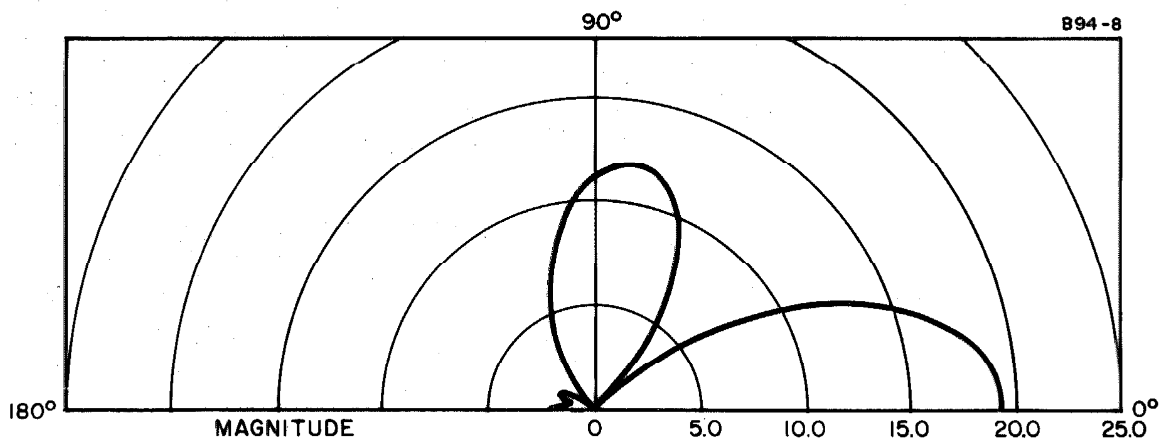


Fig. 4. Magnitude and Phase of the Perturbation Field Pattern $M^{\epsilon}(\theta,6;1)$.



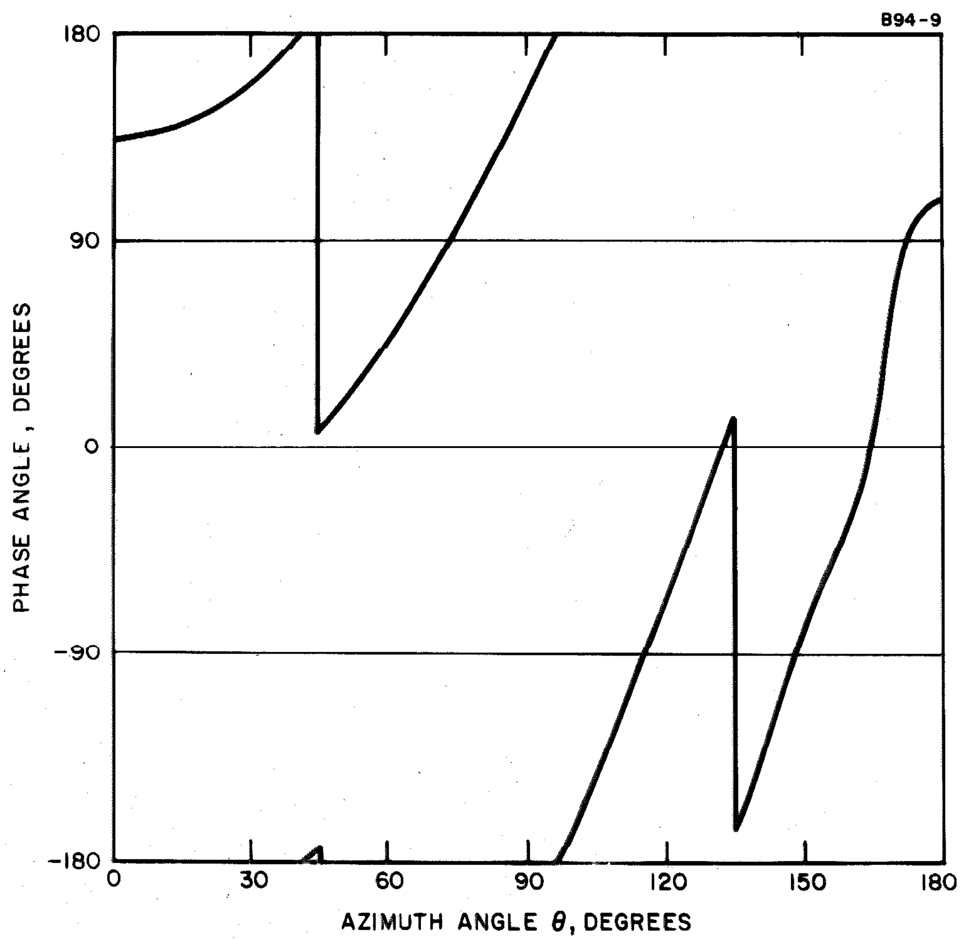


Fig. 5. Magnitude and Phase of the Perturbation Field Pattern $M^{\epsilon}(\theta,6;4)$.

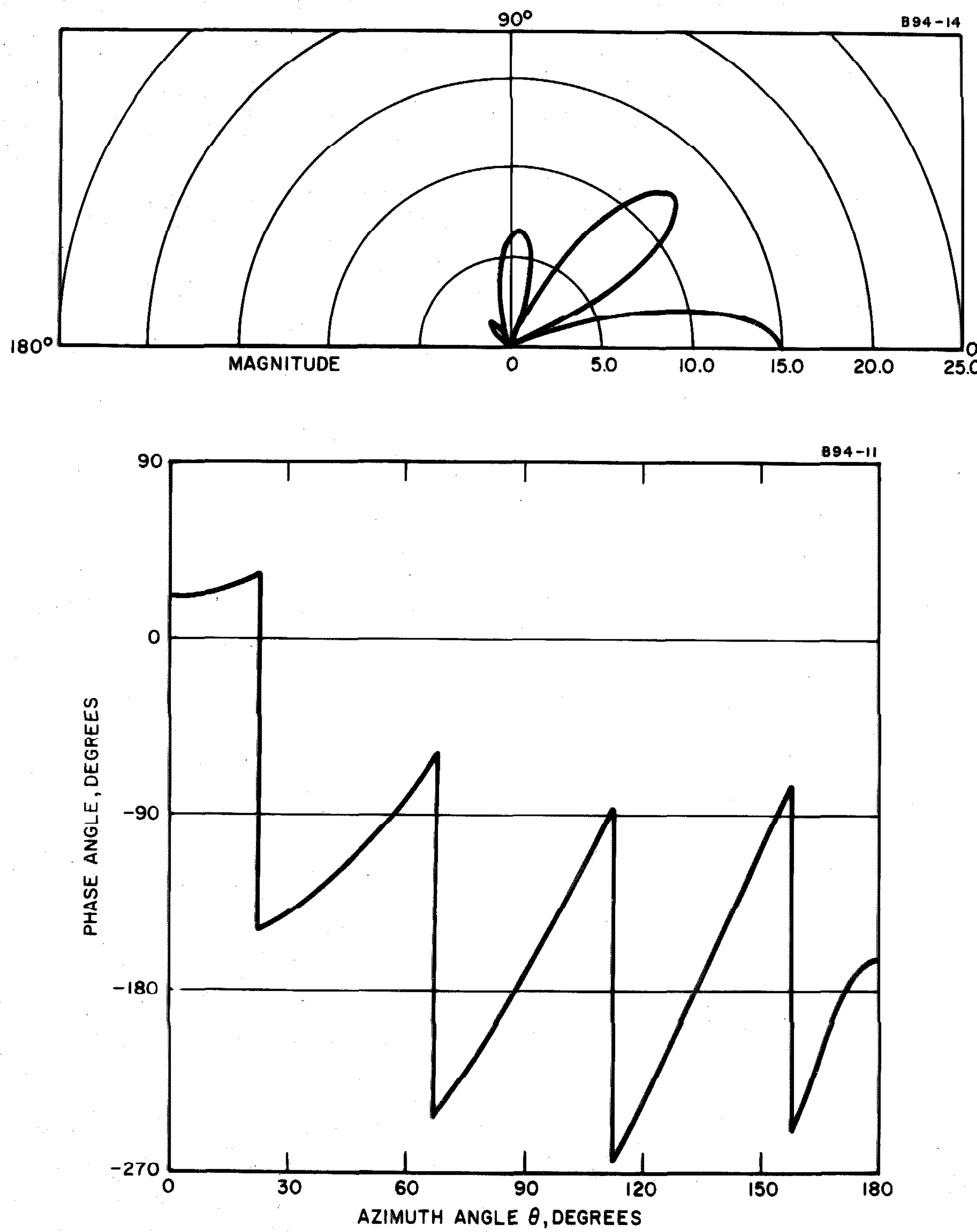
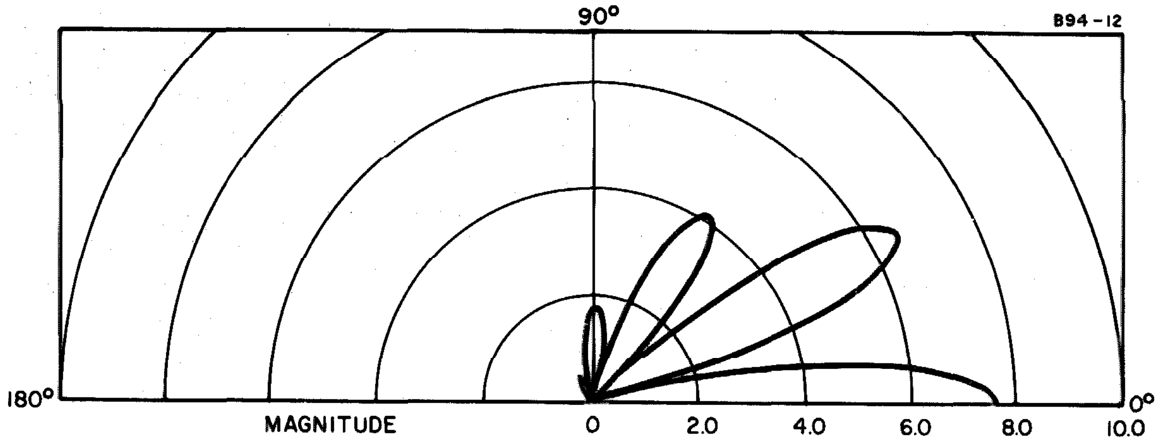


Fig. 6. Magnitude and Phase of the Perturbation Field Pattern $M^{\epsilon}(\theta,6;8)$.



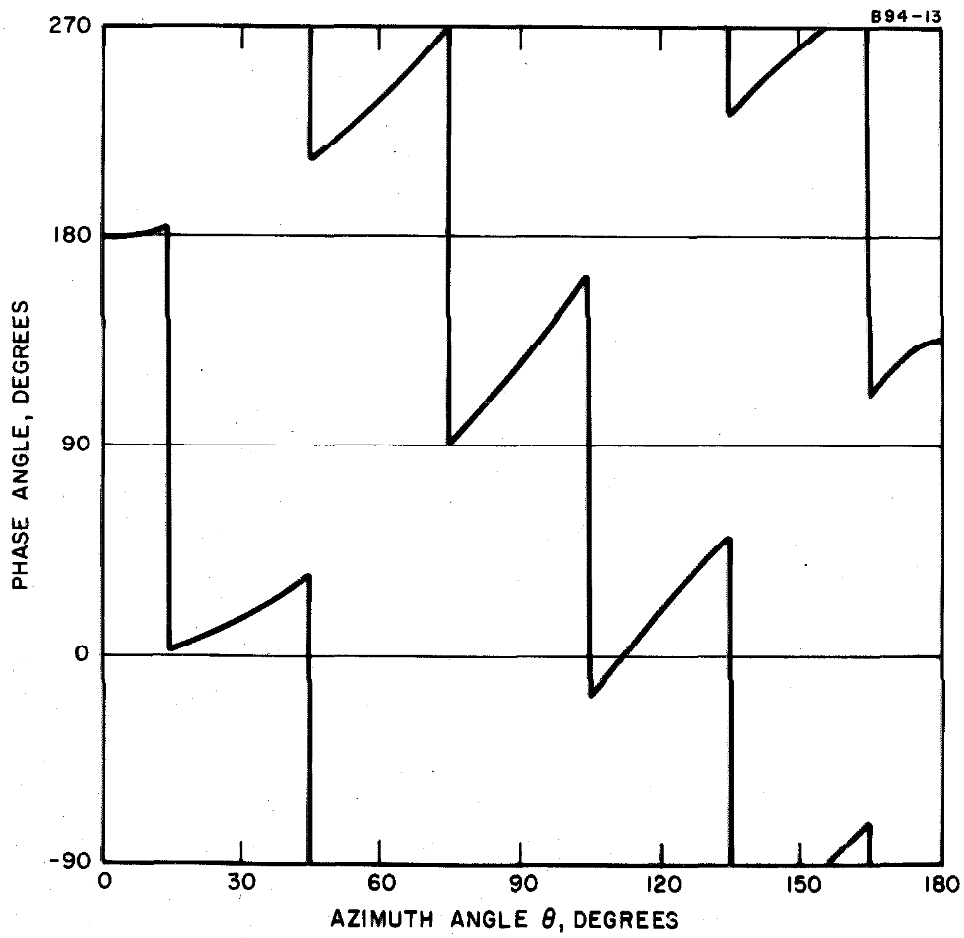
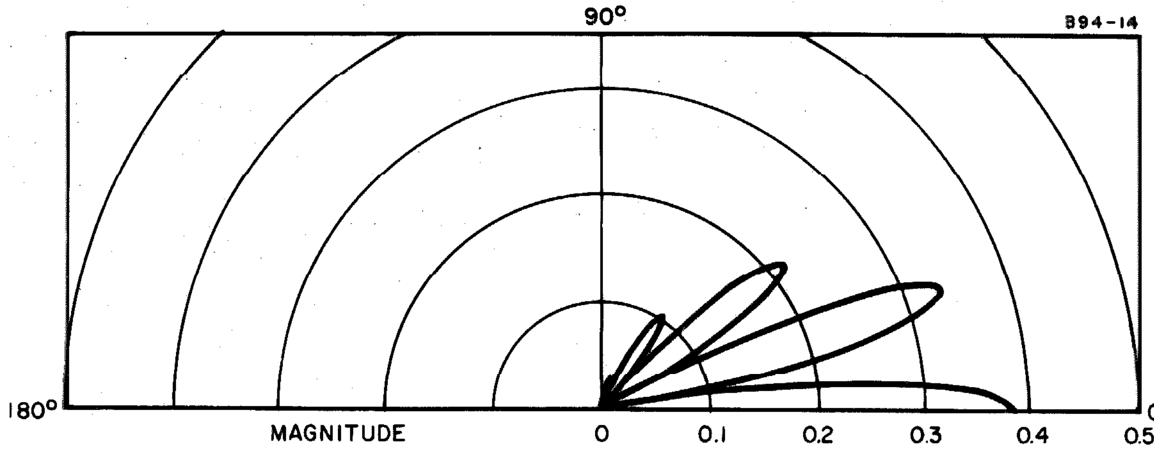


Fig. 7. Magnitude and Phase of the Perturbation Field Pattern $M^{\epsilon}(\theta,6;12)$.



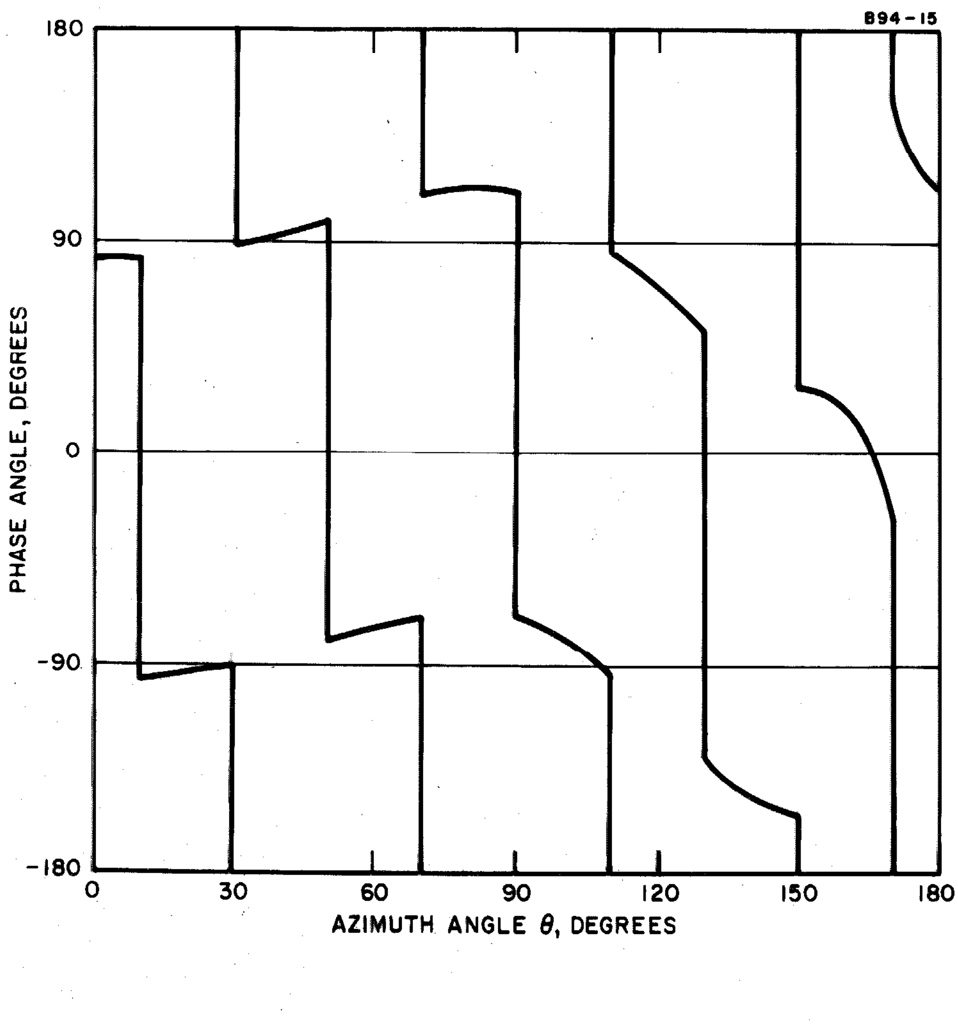


Fig. 8. Magnitude and Phase of the Perturbation Field Pattern $M^{\epsilon}(\theta,6;18)$.

$$M^{\epsilon}(\theta,ka;2s+1) = 2\left(\frac{2}{\pi}\right)^{3/2} \cos[(s+\tfrac{1}{2})\theta] \sum_{n=0}^{\infty} [A(s,n)]^{-1} \cos[(n+\tfrac{1}{2})\theta] \ , \tag{6.16}$$

with

$$\begin{aligned} A(s,n) &= (-1)^{s} H^{(1)}_{s+n+1}(ka)\, H^{(1)}_{s-n}(ka) \ , \qquad n \leq s \ , \\ &= (-1)^{n} H^{(1)}_{s+n+1}(ka)\, H^{(1)}_{n-s}(ka) \ , \qquad n \geq s+1 \ ; \end{aligned} \tag{6.17}$$

$$M^{\epsilon}(\theta,ka;2s) = (-1)^{s}\left(\frac{2}{\pi}\right)^{3/2} \cos s\theta \left\{[H^{(1)}_{s}(ka)]^{-2} + 2\sum_{n=1}^{\infty} [B(s,n)]^{-1} \cos n\,\theta\right\}, \tag{6.18}$$

with

$$\begin{aligned} B(s,n) &= H^{(1)}_{s+n}(ka)\, H^{(1)}_{s-n}(ka) \ , \qquad n \leq s \ , \\ &= (-1)^{n-s} H^{(1)}_{s+n}(ka)\, H^{(1)}_{n-s}(ka) \ , \qquad n \geq s+1 \ . \end{aligned} \tag{6.19}$$

The rearranged series exhibit clearly the facts that the perturbation field pattern has p evenly spaced nulls and that there is associated with each null--except the one which occurs at $\pm$ 180° when p is odd--a phase discontinuity of 180°. For consistency all these discontinuities have been plotted as decreases.*

Let us now compare the data for M^{ϵ} with the data for M. We see that the magnitude of M is greatest in the forward direction whereas in all cases the magnitude of M^{ϵ} is greatest for back-scattering (θ=0°). Furthermore, for all but the highest value of p considered, the magnitude of M^{ϵ} is considerably greater than that of M at θ=0°. Thus, for p

---

*More precisely, the discontinuities would appear as decreases if the curve were not telescoped.

sufficiently small, a small surface perturbation produces a relatively large change in the total back-scattered field. The nature of the change depends on the relative phases of M and M^{ϵ} . For $b/a = 0.01$, $\theta = 0°$, and $p = 1$, the change in phase is 6.9° but the change in magnitude is only 0.7%; for $b/a = 0.01$, $\theta = 0°$, and $p = 4$, the magnitude decreases by 6% and the phase changes by 5.6°.

We have already noted that as p increases the number of nulls of the perturbed field (and thus the number of lobes in the field pattern) increases. Furthermore, we see that the directivity of the perturbed field pattern increases and its maximum magnitude decreases. As p is increased above 18, the magnitude decreases very rapidly. Thus our theory indicates that the scattering process is not sensitive to perturbations of wavelength short compared to the wavelength of the field.*

The results we have cited here are of considerable interest in their own right. Much more significant, however, is the fact that we have indeed in a simple and straightforward manner obtained useful information in a problem involving a perturbed curved interface. This verifies the usefulness of our theory.

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*It may be that perturbations of very short wavelength are important, their effect being to make the cylinder scatter as though it were smooth but of radius $(a+b)$. Our theory, however, cannot be expected to predict this, for the assumptions are violated when the slope of the irregularities is steep enough to produce such an effect.

7. Concluding Remarks

Let us now review the material which has been presented. The principal result is a method of calculating the first and second order field perturbations for scattering of an arbitrary time harmonic electromagnetic wave from an irregular interface of arbitrary underlying shape. The method is valid provided that the slope of the interface irregularities is small compared to unity and that the amplitude of the irregularities is small compared to both the wavelength and the local radii of curvature of the underlying interface. Unfortunately, we have not obtained any quantitative information on how small the amplitude and slope must be.

The method essentially consists of two steps. The first is replacement of the irregularities by a system of currents on the unperturbed interface which gives the same effect (to some order). In carrying out this step, it is necessary to know the value at the interface of the unperturbed fields and of their derivatives. The second step is determination of the field in space from the surface currents by use of the dyadics $\Gamma_e^{\parallel}$ and $\Gamma_m^{\parallel}$ for the unperturbed problem. At both steps, the required information about the unperturbed problem may be in either analytical or numerical form.

It has been shown that treatments of the irregular plane interface by Rice and by Bass and Bocharov are special cases of our method. Even in these cases the new development represents an improvement in that we have justified the assumption implicit in the earlier work that the fields have continuations across the interface.

To illustrate the workings of the method in a problem involving a curved underlying interface, we have evaluated $\delta \underset{\sim}{E}$ for scattering of a plane wave from a perfectly conducting cylinder with sinusoidal irregularities. No evaluations of $\delta^2 \underset{\sim}{E}$ have been given, for these involve considerably more work even in the simplest cases.

The basic method has been extended to problems involving statistical irregularity. The emphasis here has been on the "philosophical" aspects: delineating the types of problem which are of interest, determining a basic set of field averages in terms of which other averages can be represented, and justifying the use of statistical distributions which allow large interface perturbations. The actual derivation of expressions for the field averages is straightforward.

The field averages, especially the average of the second order field, are given by rather long and complicated expressions. Fortunately, simplifications occur in many problems. We have given one example along this line, a practical situation in which the average power received by an antenna can be calculated to second order without considering the statistics of the second-order field.

A considerable amount of material has been presented on the dyadic Green's function, some of it not directly related to the development of our method. We have included this material because, on the one hand, facility in handling d.G.f.'s is important in the application of the method and, on the other, there is little reliable information in the literature on the d.G.f. for problems involving two media. In our discussion of the d.G.f., little effort has been made to distinguish among old, new, and corrected material.

Definitions, equations, and boundary conditions have been given for Γ_e and Γ_m, and also for $\Gamma_e^{\parallel}$ and $\Gamma_m^{\parallel}$. The use of d.G.f.'s to express an arbitrary field in terms of the source distribution and to derive integral equations for the field has been discussed. Reciprocity relations for the d.G.f.'s have been derived. The dyadics $\Gamma^{\parallel}$ and $\nabla \times \Gamma^{\parallel}$ have been evaluated for some common configurations.

The most interesting result of our study of d.G.f.'s is the fact that if the two four-element dyadics

$$\underset{\frown}{n}(\underset{\frown}{r}) \times \Gamma_e^{\parallel} \ (\underset{\frown}{r},\underset{\frown}{r}') \text{ and } \underset{\frown}{n}(\underset{\frown}{r}) \times \Gamma_m^{\parallel} \ (\underset{\frown}{r},\underset{\frown}{r}')$$

are known for all $\underset{\frown}{r}$ and $\underset{\frown}{r}'$ on the interface, then by explicit operations Γ_e and Γ_m can be calculated everywhere in space. Equation 3.4.1 tells us that one of the off-diagonal terms in each $\Gamma^{\parallel}$ is redundant; thus we see that any scattering problem involving two linear homogeneous isotropic media is in principle solved by determining six scalar functions on the interface. Furthermore, it is not clear that all six functions are necessary; indeed, there are considerations (too intuitive to express here) which lead us to suspect that only four functions are needed. This matter shall be investigated in the near future.

APPENDIX 1

DERIVATIONS IN SUPPLEMENT TO SECTION 3

The technique to be used here is taken from Reference 8. It involves use of the vector Green's theorem (16) in the two equivalent forms

$$\int_V dV' (\underset{\sim}{Q} \cdot \nabla' \times \nabla' \times \underset{\sim}{P} - \underset{\sim}{P} \cdot \nabla' \times \nabla' \times \underset{\sim}{Q}) = \int_S dS' \underset{\sim}{n}' \cdot (\underset{\sim}{P} \times \nabla' \times \underset{\sim}{Q} - \underset{\sim}{Q} \times \nabla' \times \underset{\sim}{P}) \,, \tag{A1.1}$$

$$\int_V dV' (\underset{\sim}{Q} \cdot \nabla' \times \nabla' \times \underset{\sim}{P} - \nabla' \times \underset{\sim}{Q} \cdot \nabla' \times \underset{\sim}{P}) = -\int_S dS' \underset{\sim}{n}' \cdot (\underset{\sim}{Q} \times \nabla' \times \underset{\sim}{P}) \,. \tag{A1.2}$$

Here V is a volume bounded by S, $\underset{\sim}{n}$ is the outward normal to S, and $\underset{\sim}{Q}$ and $\underset{\sim}{P}$ obey continuity conditions which can be assumed to hold for field vectors in a homogeneous region.

To prove Equation 3.3.25, we use Equation A1.1 with

$$V = V_i \,, \ \underset{\sim}{P} = \underset{\sim}{E}(\underset{\sim}{r}') \,, \ \underset{\sim}{Q} = \Gamma_e(\underset{\sim}{r}';\underset{\sim}{r}) \cdot \underset{\sim}{e} \,, \tag{A1.3}$$

where $\underset{\sim}{c}$ is an arbitrary constant vector. The volume integral is then

$$\int_{V_i} dV' \Big\{ [\Gamma_e(\underset{\sim}{r}';\underset{\sim}{r}) \cdot \underset{\sim}{e}] \cdot \nabla' \times \nabla' \times \underset{\sim}{E}(\underset{\sim}{r}') - \underset{\sim}{E}(\underset{\sim}{r}') \cdot [\nabla' \times \nabla' \times \Gamma_e(\underset{\sim}{r}';\underset{\sim}{r}) \cdot \underset{\sim}{e}] \Big\} =$$

$$\int_{V_i} dV' \Big\{ [k^2 \underset{\sim}{E}(\underset{\sim}{r}') + i\omega\mu \underset{\sim}{J}_e(\underset{\sim}{r}') - \nabla' \times \underset{\sim}{J}_m(\underset{\sim}{r}')] \cdot \Gamma_e(\underset{\sim}{r}';\underset{\sim}{r})$$

$$- \underset{\sim}{E}(\underset{\sim}{r}') \cdot [k^2 \Gamma_e(\underset{\sim}{r}';\underset{\sim}{r}) + i\omega\mu I \delta(\underset{\sim}{r}'-\underset{\sim}{r})] \Big\} \cdot \underset{\sim}{e} =$$

$$\Big\{ - i\omega\mu \underset{\sim}{E}(\underset{\sim}{r}) + \int_{V_i} dV' [i\omega\mu \underset{\sim}{J}_e(\underset{\sim}{r}') - \nabla' \times \underset{\sim}{J}_m(\underset{\sim}{r}')] \cdot \Gamma_e(\underset{\sim}{r}';\underset{\sim}{r}) \Big\} \cdot \underset{\sim}{e} \,. \tag{A1.4}$$

The substitution into the surface integral can also be made so that $\underset{\sim}{e}$ is a postmultiplier. Then $\underset{\sim}{e}$ can be suppressed altogether, and Equation 3.3.25 is obtained. The derivation of Equation 3.3.26 is similar, with

$$\underset{\sim}{P} = \underset{\sim}{H}\,(\underset{\sim}{r}')\ ,\ \underset{\sim}{Q} = \Gamma_m\,(\underset{\sim}{r}';\underset{\sim}{r}) \cdot \underset{\sim}{e}\ . \qquad \text{(A1.5)}$$

To prove Equation 3.4.1 for $\Gamma = \Gamma_e$, we first consider the case in which both media are lossy. In Equation A1.1, we set

$$\underset{\sim}{P} = \frac{1}{i\omega\mu''}\,\Gamma_e\,(\underset{\sim}{r}'';\underset{\sim}{r}') \cdot \underset{\sim}{e}'\ ,\ \underset{\sim}{Q} = \frac{1}{i\omega\mu''}\,\Gamma_e\,(\underset{\sim}{r}'';\underset{\sim}{r}) \cdot \underset{\sim}{e}\ , \qquad \text{(A1.6)}$$

where $\underset{\sim}{r}''$ is the integration variable and $\underset{\sim}{e}'$ and $\underset{\sim}{e}$ are arbitrary constant vectors. Then we apply the equation to V_1 and V_2 . The surface integrals on S cancel when the expressions for V_1 and V_2 are added, and the surface integrals at infinity vanish because the media are lossy. The remaining volume integrals are readily evaluated, and the constant vectors are then suppressed to give Equation 3.4.1. The extension to lossless media follows by taking the limit of both sides of Equation 3.4.1 as the losses go to zero.

The proof for Γ_m is the same except that we set

$$\underset{\sim}{P} = \frac{i\omega\mu''}{(k'')^2}\,\Gamma_m(\underset{\sim}{r}'';\underset{\sim}{r}') \cdot \underset{\sim}{e}'\ ,\ \underset{\sim}{Q} = \frac{i\omega\mu''}{(k'')^2}\,\Gamma_m\,(\underset{\sim}{r}'';\underset{\sim}{r}) \cdot \underset{\sim}{e}\ . \qquad \text{(A1.7)}$$

Equation 3.4.2 is proved using Equation A1.2 with

$$\underset{\sim}{P} = \frac{1}{(k'')^2}\,\Gamma_m\,(\underset{\sim}{r}'';\underset{\sim}{r}) \cdot \underset{\sim}{e}\ ,\ \underset{\sim}{Q} = \nabla'' \times \Gamma_e\,(\underset{\sim}{r}'';\underset{\sim}{r}') \cdot \underset{\sim}{e}'\ . \qquad \text{(A1.8)}$$

Again we first consider the lossy case, apply the Green's theorem to V_1 and to V_2, and add the resulting expressions. Again all surface integrals cancel and the desired theorem is obtained by evaluating the volume integrals. Equation 3.4.3 is proved in the same way using

$$\underset{\sim}{P} = \frac{1}{(k'')^2} \Gamma_e (\underset{\sim}{r}'';\underset{\sim}{r}) \cdot \underset{\sim}{e} , \quad \underset{\sim}{Q} = \nabla'' \times \Gamma_m (\underset{\sim}{r}'';\underset{\sim}{r}') \cdot \underset{\sim}{e}' . \qquad \text{(A1.9)}$$

APPENDIX 2

MATHEMATICAL CONTINUATION OF FIELDS

The formal development of perturbation theory in Section 4 is valid only when the equations

$$\delta^0 \underset{\sim}{F}_q(\underset{\sim}{r}') = \underset{\sim}{F}^0_q(\underset{\sim}{r}') = \underset{\sim}{F}^0_q(\underset{\sim}{r}'_o) + \zeta' \frac{\partial}{\partial \zeta'} \underset{\sim}{F}^0_q(\underset{\sim}{r}'_o) + \frac{1}{2} (\zeta')^2 \left(\frac{\partial}{\partial \zeta'}\right)^2 \underset{\sim}{F}^0_q(\underset{\sim}{r}'_o) + 0\ (\epsilon^3), \tag{A2.1}$$

$$\delta^1 \underset{\sim}{F}_q(\underset{\sim}{r}') = \delta \underset{\sim}{F}_q(\underset{\sim}{r}') = \delta \underset{\sim}{F}_q(\underset{\sim}{r}'_o) + \zeta' \frac{\partial}{\partial \zeta'} \delta \underset{\sim}{F}_q\ (\underset{\sim}{r}'_o) + 0\ (\epsilon^3)\ , \tag{A2.2}$$

$$\delta^2 \underset{\sim}{F}_q(\underset{\sim}{r}') = \delta^2 \underset{\sim}{F}_q(\underset{\sim}{r}'_o) + 0\ (\epsilon^3)\ , \tag{A2.3}$$

are meaningful everywhere on the perturbed boundary S. That is, there must be a unique mathematical continuation through $V^\epsilon_q \cap V^o_p$ to S of the fields $\underset{\sim}{F}^o_q$, $\delta \underset{\sim}{F}_q$, and $\delta^2 \underset{\sim}{F}_q$, which exist physically only in V^o_q. Since the continuation is to be expressed in a Taylor series, there must be no singularities in $V^\epsilon_q \cap V^o_p$.

The simplest examples of continuation arise in problems which can be solved by the method of images; in such problems, the field can be continued mathematically into any volume which does not contain an image source. Also, in the famous conducting half-plane problem, the field can be continued across the plane onto another sheet of the Riemann surface, but no continuation is possible across the knife edge, at which the field is singular. When a plane wave is reflected from a plane interface, the field on either side may be continued everywhere in space; Rice (3) uses this fact in his development of the perturbation theory.

If the continuation were always as readily found as in the cases cited, then results derived formally could easily be checked. In practice, however, calculation of the continuation is often very difficult, especially for $\delta F_{\sim q}$ and $\delta^2 F_{\sim q}$. Thus it would be helpful to find some general rules which aid us in determining whether the continuation exists, but which involve no calculation.

To this end, let us consider a continuous deformation of the interface accomplished by increasing ϵ from zero in

$$Q = \zeta' - w_o(\underset{\sim}{r}'_o)\, \epsilon = 0 \ . \qquad \text{(A2.4)}$$

The upper limit on ϵ is set by the stronger of the two conditions:

(a) The field $\delta^n F_{\sim q}$ must have a regular continuation in $V_q^\epsilon \cap V_p^o$;

(b) None of the sources of the unperturbed problem lie in $V_1^\epsilon \cap V_2^o$ or $V_2^\epsilon \cap V_1^o$.

The former condition is by far the more important, and we shall consider only the case in which it is critical.

Now, for any admissible value of ϵ , it is possible to establish the field $\delta^n F_{\sim q}$ in $V_q^\epsilon \cap V_q^o$ by the combination of a volume source distribution independent of ϵ and a surface current distribution on S . Furthermore, it cannot be possible to establish the field in this manner for ϵ greater than the upper limit. Thus it follows that the critical condition determining the upper limit is the necessity of introducing a new volume source into $V_q^\epsilon \cap V_p^o$. But such a source cannot suddenly appear in the interior of $V_q^\epsilon \cap V_p^o$. Therefore, at the critical value of ϵ the field must be singular at some point on S .

We immediately obtain the very important result that any $\delta^n \underset{\sim}{F}_q$ which is not singular on S_o can be continued for some distance beyond S_o. The continuation, which can easily be shown to be unique, is the field produced in $V_q^\epsilon \cap V_p^o$, by the sources necessary to produce $\delta^n \underset{\sim}{F}_q$ in $V_q^\epsilon \cap V_q^o$.

Another important consequence of the above is that no continuation at all is possible across a knife edge or other geometrical feature at which the field is singular; it is possible, however, as already noted in the half-plane problem, to make continuations across surfaces having such features elsewhere than at the features.

Let us now ask why it eventually becomes necessary to introduce a volume source into $V_q^\epsilon \cap V_p^o$. The answer is that the spatial variation of amplitude of the field somewhere on S_o is so rapid that it cannot be maintained by sources farther removed. This point of view leads to the following important generalization:

The more rapid the spatial variation of amplitude of the field $\delta^n \underset{\sim}{F}_q$ near a point on S_o, the smaller the depth is to which the function can be continued near that point.

Since the field varies rapidly near a point of large curvature, we have the corollary that the depth of continuation tends to be small where the curvature is large. Another corollary is that the depth of continuation will tend to increase as the sources move away from S_o; this does not impose any new restriction on our theory, for we have treated the incident field as a given quantity.

In general, $\delta \underset{\sim}{F}_q$ will represent finer details of the field than will $\underset{\sim}{F}_q^o$, and thus $\delta \underset{\sim}{F}_q$ will usually have the more rapid spatial variation. Likewise, $\delta^2 \underset{\sim}{F}_q$ will tend to vary more rapidly than $\delta \underset{\sim}{F}_q$, and by extension $\delta^{n+1} \underset{\sim}{F}_q$ will tend to vary more rapidly than $\delta^n \underset{\sim}{F}_q$. Thus, as the order of perturbation increases, the depth to which the field can be continued tends to decrease. This suggests that in many cases the technique developed in Section 4 gives an asymptotic approximation to the perturbed field rather than a convergent one.

APPENDIX 3

VERIFICATION OF THE RESULTS OF BASS AND BOCHAROV AND OF RICE

A3.1 Bass and Bocharov's Results

Tangential $\underset{\sim}{E}^{o}$ is zero at the perfectly conducting unperturbed interface, and thus

$$\underset{\sim}{e}_z \times \left(\frac{\partial}{\partial z'}\right)^2 \underset{\sim}{E}_2^o = - \underset{\sim}{e}_z \times \left[\left(\frac{\partial}{\partial x'}\right)^2 + \left(\frac{\partial}{\partial y'}\right)^2 + k_2^2 \right] \underset{\sim}{E}_2^o \equiv 0 \text{ on } z' = 0 \,; \tag{A3.1}$$

furthermore, $\underset{\sim}{E}_1^o \equiv 0$. It follows that the expressions for $\delta \underset{\sim}{K}_m$ and $\delta^2 \underset{\sim}{K}_m$ in Equations 2.1.5 and 6 are equivalent to those in Equations 4.2.11 and 12. Then by comparing the Kirchhoff formulas of Equations 2.1.7 and 8 with Equation 4.2.16 and noting that $\Gamma_e^{\parallel} \equiv 0$, we reduce the verification process to a proof that the two Kirchhoff formulas are equivalent to

$$\underset{\sim}{E}(\underset{\sim}{r}) = - \frac{i\omega\mu_2}{k_2^2} \nabla \times \int_{S_o} dS' \, \Gamma_m^{\parallel}(\underset{\sim}{r};\underset{\sim}{r}_o') \cdot [\underset{\sim}{e}_z \times \underset{\sim}{E}(\underset{\sim}{r}_o')] \, . \tag{A3.2}$$

Expressing $\Gamma_m^{\parallel}$ by means of Equations 3.5.4, 3.5.1, and 2.1.9, and performing some straightforward vector and dyadic algebra,* we obtain from Equation A3.2 the equivalent expression

$$\underset{\sim}{E}(\underset{\sim}{r}) = - 2 \frac{\partial}{\partial z} \int_{S_o} dS' \underset{\sim}{E}_{tan}(\underset{\sim}{r}_o') G_f(\underset{\sim}{r};\underset{\sim}{r}_o') + \underset{\sim}{e}_z \, 2 \int_{S_o} dS' [\nabla' \cdot \underset{\sim}{E}_{tan}(\underset{\sim}{r}_o')] G_f(\underset{\sim}{r};\underset{\sim}{r}_o')$$
$$- \underset{\sim}{e}_z \, 2 \int_{S_o} dS' \nabla' \cdot [G_f(\underset{\sim}{r};\underset{\sim}{r}_o') \, \underset{\sim}{E}_{tan}(\underset{\sim}{r}_o')] \, . \tag{A3.3}$$

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*Note that the interchange of integration and differentiation is valid.

But it can be shown that

$$\int_{S_o} dS' \, \nabla' \cdot [G_f \, (\underset{\sim}{r};\underset{\sim}{r}'_o) \, \underset{\sim}{E}_{tan} \, (\underset{\sim}{r}'_o)] =$$

$$\underset{\sim}{e}_y \cdot \int_{S_o} dS' \, [\underset{\sim}{e}_z \, x \nabla'(G_f E_x)] - \underset{\sim}{e}_x \cdot \int_{S_o} dS' [\underset{\sim}{e}_z \, x \nabla'(G_f E_y)] \equiv 0 \, . \qquad (A3.4)$$

Thus Equation A3.3 reduces to two terms which can immediately be identified with the Kirchhoff formulas, and the verification is complete.

A3.2 Rice's Results

We shall consider directly only the case in which $z_o(x',y')$ has an ordinary two-dimensional Fourier transform. In this case we shall demonstrate the equivalence of the results obtained from the theory of Section 4 and the limit, as the fundamental period L goes to infinity, of Rice's results. The verification can be extended to the case in which $z_o(x',y')$ has no transform by considering $z_o(x',y')$ as the limit of a sequence of truncated functions.

The two-dimensional Fourier transform of $w(x,y)=w(\underset{\sim}{r}_o)$ will be designated by $\hat{w}$ or w_{Trans} and will be normalized in the form

$$\hat{w}(u,v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx \, dy \, \exp\left\{- \, iux-ivy\right\} w \, (\underset{\sim}{r}_o) \, ,$$

$$w(\underset{\sim}{r}_o) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} du \, dv \, \exp\left\{iux + ivy\right\} \hat{w} \, (u,v) \, . \qquad (A3.5)$$

To facilitate comparison with Equation 2.2.1, we also introduce

$$p(u,v) = \frac{1}{4\pi^2} \, \hat{z}_o(u,v) \, . \qquad (A3.6)$$

Here u is equivalent to Rice's (am) and v to (an), so that $p(u,v)$ is equivalent to $a^{-2}P(m,n)$ and

$$\int_{\hat{S}} d\hat{S} \cdots = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} du\, dv \cdots \text{ is equivalent to } a^2\Sigma_{mn} \cdots .$$

The equivalents to Equations 2.2.3-6 and 2.2.17 are

$$\Delta \underset{\sim}{E}(\underset{\sim}{r}) = \int_{\hat{S}} d\hat{S}[\underset{\sim}{e}_x A(u,v) + \underset{\sim}{e}_y B(u,v) + \underset{\sim}{e}_z C(u,v)]E(u,v,z) \;,\; z > z_o \;;$$

$$\Delta \underset{\sim}{E}(\underset{\sim}{r}) = \int_{\hat{S}} d\hat{S}[\underset{\sim}{e}_x G(u,v) + \underset{\sim}{e}_y H(u,v) + \underset{\sim}{e}_z I(u,v)]F(u,v,z) \;,\; z < z_o \;; \quad (A3.7)$$

$$E(u,v,z) = \exp\{iux + ivy + ibz\};\; F(u,v,z) = \exp\{iux + ivy - icz\};\quad (A3.8)$$

$$b = +[k_2^2-(u^2+v^2)]^{1/2} \;;\; c = +[k_1^2-(u^2+v^2)]^{1/2};\; d=b+c \;;\; D=u^2+v^2+bc \;;\quad (A3.9)$$

$$A(u,v) = A^{(1)}(u,v) + A^{(2)}(u,v) + A^{(3)}(u,v) + \cdots, \text{ etc.} \qquad (A3.10)$$

Here $A(u,v)$ is equivalent to $a^{-2}A_{mn}$, etc.

Let us consider the set $\mathcal{G}$ of all scalar functions $g(\underset{\sim}{r}-\underset{\sim}{r}_o')$, defined either in $z < 0$ or in $z > 0$, which in the region of definition can be expressed in the form

$$g(\underset{\sim}{r}-\underset{\sim}{r}_o') = \frac{1}{4\pi^2}\int_{\hat{S}} d\hat{S} \exp\{iu(x-x_o') + iv(y-y_o') + ihz\}\hat{g}(u,v) \;,\quad (A3.11)$$

where $\hat{g}(u,v)$ is the Fourier transform of $g(\underset{\sim}{r}_o)$. If $w(\underset{\sim}{r}_o)$ is a function with Fourier transform $\hat{w}(u,v)$, then it is readily shown by the convolution theorem that

$$\int_{S_o} dS' g(\underset{\sim}{r}-\underset{\sim}{r}_o') w(\underset{\sim}{r}_o') = \frac{1}{4\pi^2} \int_{\hat{S}} d\hat{S} \exp\{iux + ivy + ihz\} \hat{g}(u,v)\, \hat{w}(u,v) . \quad (A3.12)$$

From Equation 3.5.6 it is clear that the set $\mathcal{G}$ includes all scalar components of the $\Gamma^{\parallel}$ and $\nabla \times \Gamma^{\parallel}$ for the plane interface problem. Also included in $\mathcal{G}$ is $G_f(\underset{\sim}{r};\underset{\sim}{r}_o')$ of Equation 2.1.9 and all its derivatives; specifically, we note

$$\hat{G}_f(u,v) = \frac{i}{2h(u,v)} . \quad (A3.13)$$

Now we are ready to deal with the vertical polarization, perfectly conducting interface case. Using Equations A3.12 and 13, we reduce Equation 2.1.7 to the simple form

$$E_{x,y}(\underset{\sim}{r}) = \frac{1}{4\pi^2} \int_{\hat{S}} d\hat{S}' \exp\{iux + ivy + ibz\} \hat{\underset{\sim}{E}}_{x,y}(u,v) . \quad (A3.14)$$

Then, using Equation 2.1.5 with $\underset{\sim}{E}^o$ determined from Equation 2.2.7, we find

$$\delta \hat{E}_x(u,v) = A^{(1)}(u,v) = -2i(\alpha u - k_2)\, p\,(u-k_2\alpha, v) ,$$

$$\delta \hat{E}_y(u,v) = B^{(1)}(u,v) = -2i\alpha v\, p\,(u-k_2\alpha, v) ; \quad (A3.15)$$

and from the divergence condition we obtain

$$\delta \hat{E}_z(u,v) = C^{(1)}(u,v) = 2i b^{-1}(\alpha k_2^2 - uk_2 - \alpha b^2)\, p(u - k_2\alpha, v) . \quad (A3.16)$$

From the above we readily find by convolution of transforms

$$\delta^2 \hat{E}_x(u,v) = A^{(2)}(u,v) =$$

$$2\int_{\hat{S}} d\hat{S}'[(u-u')(\alpha\, k_2-u')k_2 + (k_2-\alpha u)\, b^2(u',v')]\frac{p(u'-k_2\alpha,v')p(u-u',v-v')}{b(u',v')} ,$$

$$\delta^2 \hat{E}_y(u,v) = B^{(2)}(u,v) =$$

$$2\int_{\hat{S}} d\hat{S}'[(v-v')(\alpha\, k_2-u')k_2 - \alpha\, v\, b^2(u',v')]\ \frac{p(u'-k_2\alpha,v')p(u-u',v-v')}{b(u',v')};\quad (A3.17)$$

and the divergence condition gives

$$\delta^2 \hat{E}_z(u,v) = C^{(2)}(u,v) =$$

$$\frac{2}{b(u,v)}\int_{\hat{S}} d\hat{S}'\left\{(u'-\alpha k_2)(u^2+v^2-uu'-vv')k_2 + [\alpha(u^2+v^2) - uk_2]\ b^2(u',v')\right\} \times$$

$$\frac{p(u'-k_2\alpha,v')p(u-u',v-v')}{b(u',v')}\ . \qquad (A3.18)$$

The expressions in Equations A3.15-18 are readily seen to be equal to the expressions obtained from Equation 2.2.9 through multiplying by a^{-2} and taking the limit as L goes to infinity. Thus Rice's results for vertical polarization are verified.

Now let us consider the horizontal polarization case. The unperturbed field $\underset{\frown}{E}^{o}$ is given by Equation 2.2.11. Using the constraint that μ is the same in both media and the fact that $\underset{\frown}{E}^{o}$ has only a y-component, we obtain

$$\Delta E_z^o = 0 \ , \ \Delta H_z^o = 0 \ , \ \Delta \frac{\partial}{\partial z} H_y^o = 0 \ , \ \Delta \frac{\partial}{\partial z} \underset{\frown}{E}^o = 0 \ ,$$

$$\Delta \frac{\partial}{\partial z} H_x^o = - \frac{2U}{i\omega\mu} e^{i\kappa x} \ ; \tag{A3.19}$$

here κ is defined by Equation 2.2.13 and U by Equation 2.2.16. Thus Equations 4.2.11 and 13 become

$$\delta \underset{\sim}{K}_m(\underset{\sim}{r}'_o) = 0 \ , \ \delta \underset{\sim}{K}_e(\underset{\sim}{r}'_o) = \underset{\sim}{e}_y \frac{2U}{i\omega\mu} z_o(\underset{\sim}{r}'_o) e^{i\kappa x'} \ . \tag{A3.20}$$

Equation A3.14 holds for $\delta \underset{\frown}{E}$, and with the aid of Equation A3.12 we find

$$\delta \hat{\underset{\sim}{E}}_q(u,v) = \frac{2U}{i\omega\mu} [\Gamma_e^{\|} (\underset{\sim}{r}_o;o)\cdot \underset{\sim}{e}_y]_{Trans}[z_o(\underset{\sim}{r}_o)e^{i\kappa x}]_{Trans}, \tag{A3.21}$$

where the transforms are taken with respect to $\underset{\sim}{r}_o$. Inserting the d.G.f. of Section 3.5C into Equation A3.21 gives

$$\delta \hat{\underset{\frown}{E}}_2(u,v) = \underset{\sim}{e}_x A^{(1)}(u,v) + \underset{\sim}{e}_y B^{(1)}(u,v) + \underset{\sim}{e}_z C^{(1)}(u,v) =$$

$$- \frac{2\ i\ U\ p(u-\kappa,v)}{dD} [\underset{\sim}{e}_x uv + \underset{\sim}{e}_y(v^2-D) + \underset{\sim}{e}_z v\ c\ ] \ , \tag{A3.22}$$

$$\delta \hat{\underset{\frown}{E}}_1(u,v) = \underset{\sim}{e}_x G^{(1)}(u,v) + \underset{\sim}{e}_y H^{(1)}(u,v) + \underset{\sim}{e}_z\ I^{(1)}(u,v) =$$

$$\underset{\sim}{e}_x A^{(1)}(u,v) + \underset{\sim}{e}_y B^{(1)}(u,v) - \underset{\sim}{e}_z \frac{b}{c} C^{(1)}(u,v) \ . \tag{A3.23}$$

As expected, these results are equivalent to Rice's first-order results given in Equation 2.2.14.

Instead of verifying the second-order fields directly, we shall confirm the six parts of Equation 2.2.15. The first two parts, indeed, are just the divergence conditions and need no confirmation. In confirming the other four parts, we use the equations

$$(\delta^2 \hat{\underset{\sim}{E}}_2 - \delta^2 \hat{\underset{\sim}{E}}_1)_{x,y} = \left\{ i\omega\mu \, \Delta \, [\frac{1}{k^2} \nabla \times \Gamma_m^{\parallel} \ (\underset{\sim}{r}_o ; o)]_{\text{Trans}} \cdot \delta^2 \hat{\underset{\sim}{K}}_m \right\}_{x,y} ,$$

$$(\delta^2 \hat{\underset{\sim}{H}}_2 - \delta^2 \hat{\underset{\sim}{H}}_1)_{x,y} = \left\{ \frac{1}{i\omega\mu} \Delta \, [\nabla \times \Gamma_e^{\parallel} \ (\underset{\sim}{r}_o ; o)]_{\text{Trans}} \cdot \delta^2 \hat{\underset{\sim}{K}}_e \right\}_{x,y} , \qquad (A3.24)$$

which are derived from Equations 4.2.16 and 17 by use of Equations 3.3.11, 3.3.22 and A3.12. It is readily shown that

$$i\omega\mu \, \Delta \, [\frac{1}{k^2} \nabla \times \Gamma_m^{\parallel} \ (\underset{\sim}{r}_o ; o)]_{\text{Trans}} = - \underset{\sim}{e}_x \underset{\sim}{e}_y + \underset{\sim}{e}_y \underset{\sim}{e}_x ,$$

$$\frac{1}{i\omega\mu} \Delta \, [\nabla \times \Gamma_e^{\parallel} \ (\underset{\sim}{r}_o ; o)]_{\text{Trans}} = \underset{\sim}{e}_x \underset{\sim}{e}_y - \underset{\sim}{e}_y \underset{\sim}{e}_x , \qquad (A3.25)$$

and

$$\delta^2 \hat{\underset{\sim}{K}}_m = \underset{\sim}{e}_x \int_{\hat{S}} d\hat{S}' \left\{ U \ p(u'-\kappa, v') - iv[C^{(1)}(u',v') - I^{(1)}(u',v')] \right\} p(u-u';v-v')$$

$$+ \underset{\sim}{e}_y \int_{\hat{S}} d\hat{S}' \left\{ iu[C^{(1)}(u',v') - I^{(1)}(u',v')] \right\} p(u-u',v-v') ,$$

$$\delta^2 \hat{\underset{\sim}{K}}_e = \underset{\sim}{e}_x \frac{1}{i\omega\mu} \int_{\hat{S}} d\hat{S}' (k_1^2 - k_2^2) \ A^{(1)}(u',v') p(u-u',v-v')$$

$$+ \underset{\sim}{e}_y \frac{1}{i\omega\mu} \int_{\hat{S}} d\hat{S}' [-iUk_1\gamma' p(u'-\kappa,v') + (k_1^2 - k_2^2) \ B^{(1)}(u',v')] p(u-u',v-v') . \qquad (A3.26)$$

Setting Equations A3.25 and 26 into Equation A3.24, we obtain

$$\delta^2 \hat{E}_{2,x} - \delta^2 \hat{E}_{1,x} = A^{(2)}(u,v) - G^{(2)}(u,v) =$$

$$-\,iu\int_{\hat{S}} d\hat{S}'[C^{(1)}(u',v') - I^{(1)}(u',v')]p(u-u',v-v'),$$

$$\delta^2 \hat{E}_{2,y} - \delta^2 \hat{E}_{1,y} = B^{(2)}(u,v) - H^{(2)}(u,v) =$$

$$\int_{\hat{S}} d\hat{S}'\left\{Up(u'-\kappa,v') - iv[C^{(1)}(u',v') - I^{(1)}(u',v')]\right\}p(u-u',v-v') ,$$

$$\omega\mu(\delta^2 \hat{H}_{2,x} - \delta^2 \hat{H}_{1,x}) = v[C^{(2)}(u,v) - I^{(2)}(u,v)] - bB^{(2)}(u,v) - cH^{(2)}(u,v) =$$

$$-\int_{\hat{S}} d\hat{S}'[Uk_1\gamma' p(u'-\kappa,v') + i(k_1^2-k_2^2)B^{(1)}(u',v')]p(u-u',v-v') ,$$

$$\omega\mu(\delta^2 \hat{H}_{2,y} - \delta^2 \hat{H}_{1,y}) = -\,u[C^{(2)}(u,v) - I^{(2)}(u,v)] - bA^{(2)}(u,v) - cG^{(2)}(u,v) =$$

$$\int_{\hat{S}} d\hat{S}'[i(k_1^2-k_2^2)A^{(1)}(u',v')]p(u-u',v-v') . \qquad (A5.27)$$

These results are equivalent to those of Equation 2.2.15, and thus the verification of Rice's results is complete.

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