OPERATOR DIFFERENTIAL EQUATIONS

IN HILBERT SPACE

Thesis by

Louis A. Lopes

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1964

(Submitted May 19, 1964)

ACKNOWLEDGMENTS

I am indebted to Professor Charles De Prima and to Professor Arthur Erdélyi for their guidance and encouragement. My gratitude is also extended to the Naval Ordnance Test Station for the fellowship which made it possible for me to return to my studies, and to the California Institute of Technology for their support through a scholarship grant.

ABSTRACT

In this paper the theory of dissipative linear operators in Hilbert space developed by R. S. Phillips has been applied in the study of the Cauchy problem

$$\dot{x}(t) + A(t)x(t) = f(t), \; x(o) = x_o$$

where $A(t)$, $t \in [o, \mathcal{T}]$, is a family of unbounded linear operators with a common dense domain D in a Hilbert space H, $f \in \mathcal{H}$, the Hilbert space of measurable functions on $[o, \mathcal{T}]$ with values in H which have square integrable norm, and $x_o \in H$. It is assumed that for each $t \in [o, \mathcal{T}]$ $A(t)$ is maximal dissipative, satisfying for each $x \in D$, Re $(A(t)x,x) \geq \alpha |x|^2$, $\alpha > o$, and $A(t)x$ is strongly continuous and has a bounded measurable strong derivative on J. Let A_o be any maximal dissipative linear operator with domain D satisfying Re $(A_o x,x) \geq \alpha |x|^2$ for all $x \in D$. Then $B(t) = A(t)A_o^{-1}$ is a one-to-one continuous linear transformation of H onto itself. It is assumed that $B^{-1}(t)$ is bounded on $[o, \mathcal{T}]$. Under these conditions it is shown that, first, there exists a weak solution to the Cauchy problem, and, second, that the weak solution is a unique strong solution which is the limit of a sequence of classical solutions. The theory is applied to a time-dependent hyperbolic system of partial differential equations.

I. INTRODUCTION

This paper is concerned with a class of operator differential equations in Hilbert space. An operator differential equation is an equation of the form

$$\dot{x}(t) + A(t)\, x(t) = f(t) \tag{1.1}$$

where $A(t)$, $t \in [0, \tau]$, is a family of operators whose domains and ranges are in a Hilbert space H[1], and f is a function on $[0, \tau]$ with values in H. A classical solution of such an equation is a strongly differentiable function x on $[0, \tau]$ such that $x(t) \in \mathcal{D}(A(t))$[2], $\dot{x}$ is the strong derivative of x with respect to t, and (1.1) is satisfied for $t \in [0, \tau]$. The Cauchy problem consists in finding a solution to (1.1) satisfying the initial condition $x(0) = x_0 \in \mathcal{D}(A(0))$.

The prototype of equation (1.1) is a partial differential equation, or a system of partial differential equations, where A(t) is the differential operator with respect to space variables in some region Ω, and t corresponds to the time variable. The domain of A(t) is determined by boundary conditions on Ω, and A(t) is a closed unbounded operator with dense domain in $L_2(\Omega)$. Accordingly, we shall

[1] The inner product of two elements x, y, in H will be denoted by (x,y). The norm in H is denoted by $|x| = (x,x)^{1/2}$.

[2] $\mathcal{D}(A)$ denotes the domain of the operator A.

assume in the abstract problem that A(t) is a closed unbounded operator with dense domain in H. The case in which A(t) is a bounded operator on a Banach space has been studied at length by Massera and Schäffer[16]

It often happens that the definition of a solution to (1.1) given above is too restrictive. It is necessary to broaden the definition of a solution if many reasonable Cauchy problems are to have solutions. To this end the concept of a <u>weak</u> <u>solution</u> has been introduced. This concept takes many forms. A classification is given in Lions ([10] , Chapter I). The form of the weak solution which we shall define here is related to the notion of a <u>formal</u> <u>adjoint</u>.

<u>Definition</u>. Let T be a linear operator with dense domain in a Hilbert space $\mathcal{H}$, with inner product $< x, y >$. A linear operator S, $\mathcal{D}(S) \subset \mathcal{H}$, is said to be a <u>formal</u> <u>adjoint</u> of T if for all $x \in \mathcal{D}(T)$ and all $y \in \mathcal{D}(S)$ $< T x, y > = < x, S y >$.

An operator T with dense domain has at least one formal adjoint with dense domain if and only if it is closeable. If T is closed and $S = T^*$, and if $< z, y > = < x, S y >$ for all $y \in \mathcal{D}(S)$ then x is in the domain of T and $T x = z$. Generally $S \subset T^*$ and we define a <u>weak</u> <u>solution</u> of the equation $T x = z$ as follows:

<u>Definition</u>. If T is a linear operator with dense domain in the Hilbert space $\mathcal{H}$, and if S is a formal adjoint of T we say that $x \in \mathcal{H}$ is a weak solution with respect to S of the equation $T x = z$, $z \in \mathcal{H}$, if for all $y \in \mathcal{D}(S)$ we have $< x, S y > = < z, y >$.

The case where $A(t) = A$, independent of t, leads to the theory of semi-group solutions to the homogeneous equation

$$\dot{x} + A\,x = 0 \tag{1.2}$$

The development of the theory of semi-groups is due largely to the efforts of E. Hille [1] . If $-A$ is the infinitesimal generator of a strongly continuous semi-group of bounded linear operators $U(t)$ then the solution to (1.2) with $x(0) = x_0 \in \mathcal{D}(A)$ is $U(t)\,x_0$. Necessary and sufficient conditions that a closed linear operator with dense domain be the infinitesimal generator of a semi-group of bounded linear operators on a Banach space are given in a theorem bearing the names of Hille, Yosida, and Phillips (see e.g. [12] VIII. 1. 13).

Starting with the semi-group solution Kato [6] constructed solutions of the homogeneous equation (1.1) by means of a Riemann product integral. He assumed that, for each t, $-A(t)$ is the infinitesimal generator of a strongly continuous semi-group of bounded linear operators on a Banach space, that the domain of $A(t)$ is independent of t, and that the bounded operator $B(t, s) = [I + A(t)][I + A(s)]^{-1}$ is uniformly bounded, i.e. there exists a constant $M > 0$ such that $\| B(t, s) \| \leq M$ for all s, t. Other assumptions were made in order to obtain convergence of the product integral and strong solutions to (1.1). The product integral leads to an operator-valued function $U(t, s)$ such that $U(t, s) = U(t, \tau)\,U(\tau, s)$ for $t \geq \tau \geq s$, and $U(t, t) = I$ for all $t \in [0, T]$. Such a function is sometimes called a "fundamental solution."

In order to study weak solutions to (1.1) in Hilbert space, it is more convenient to start with the theory of dissipative operators rather than the semi-group theory. The operator T with domain in the Hilbert space $\mathcal{H}$ is said to be dissipative[3] if for all $x \in \mathcal{D}(T)$ $\mathrm{Re} \langle T x, x \rangle \geq 0$, and to be maximal dissipative if it is not the restriction of any other dissipative operator. Phillips [7] has shown that an operator is the infinitesimal generator of a strongly continuous semi-group of contraction operators on a Hilbert space if and only if it is maximal dissipative with dense domain. Phillips' theory of dissipative operators is developed in his paper on dissipative operators and hyperbolic systems of partial differential equations, [7]. In the following we shall lean heavily on this theory, using such facts as the following:

<u>Theorem 1.1.</u> Every dissipative operator T_0 with dense domain in $\mathcal{H}$ has a (closed) maximal dissipative extension T. The range of $T + \alpha I$ is all of $\mathcal{H}$ for any $\alpha > 0$. [7] (Theorem 1.1.1)

<u>Theorem 1.2.</u> If T is maximal dissipative and closed it has dense domain. [7] (Lemma 1.1.3)

<u>Theorem 1.3.</u> If T is a maximal dissipative operator with dense domain then so is its adjoint, T^*. [7] (Theorem 1.1.2).

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[3] The term "dissipative" has also been applied to A if $\mathrm{Im} \langle A x, x \rangle \geq 0$ (≤ 0) or $\mathrm{Re} \langle A x, x \rangle \leq 0$. These usages correspond respectively to consideration of the equations $\frac{1}{\pm i} \dot{x} + A x = 0$, or $\dot{x} = A x$. Our usage is governed by the form of the differential equation $\dot{x} + A x = 0$.

<u>Theorem 1.4.</u> If T_o is a dissipative operator with dense domain, and $S_1 = T_o^*$, then there exists maximal dissipative operators T and S, adjoints of each other, such that $T \supset T_o$ and $S \subset S_1$.[4] [7] (Corollary to Theorem 1.1.2)

Our assumptions on A (t) in the following will be similar to those of Kato, inasmuch as we shall assume that A (t) is maximal dissipative with dense domain for each t (Re $(A(t)x, x) \geq \alpha \|x\|^2$, $\alpha > 0$), and that $\mathcal{D}(A(t))$ is independent of t. Moreover, we introduce assumptions equivalent to the uniform boundedness of Kato's B (t, s). We introduce the Hilbert space $\mathcal{H}$ consisting of measurable functions on $[0, \tau]$ with values in H and having square integrable norms. The inner product on $\mathcal{H}$ is given by

$$\langle x, y \rangle = \int_0^{\tau} (x(t), y(t)) \, dt.$$

We seek solutions to (1.1) for $f \in \mathcal{H}$. We have placed sufficient conditions on the A (t) to enable us to define a maximal dissipative linear operator $\mathcal{a}$ with dense domain in $\mathcal{H}$, such that $\mathcal{a}\, x(t) = A(t)\, x(t)$ a.e. for each $x \in \mathcal{D}(\mathcal{a})$. Section II is devoted to the definition of $\mathcal{a}$ and to a discussion of its properties. In Section III we consider the Cauchy problem for x (0) = 0. The linear operator T_o, with dense domain in $\mathcal{H}$, is defined by $T_o \varphi(t) = \dot{\varphi}(t) + A(t)\varphi(t)$ where $\varphi \in \mathcal{D}(T_o)$ is strongly

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[4] The notation $T \supset T_o$ ($T_o \subset T$) means that T is an extension of T_o.

continuously strongly differentiable with values in D, the common domain of the A (t), and $\varphi(0) = 0$. T_0 is then a dissipative linear operator, satisfying $\mathrm{Re} < T_0 \varphi, \varphi > \ \geq \alpha \| \varphi \|^2$. A formal adjoint S_0 of T_0 is defined by $S_0 \psi(t) = -\dot{\psi}(t) + A^*(t) \psi(t)$ for $\psi \in \Psi$, where Ψ is a class of weakly differentiable functions such that $\psi(t) \in \mathcal{D}(A^*(t))$, $\psi(\tau) = 0$, and $A^*(\cdot)\psi(\cdot) \in \mathcal{H}$. Then $T_0 x = f$ will have a weak solution, with respect to S_0, for any $f \in \mathcal{H}$.

We note that if the range of T_0 is dense in $\mathcal{H}$ then its closure T (which exists in consequence of the general theory of dissipative operators) is maximal dissipative, and the equation $T x = f$ has a unique solution $x \in \mathcal{D}(T)$. Imposition of the condition that A (t) x have a bounded strong derivative $(\dot{A}(t) x)$, which is measurable on $[0, \tau]$, for each $x \in D$ is sufficient for T to be maximal dissipative. Then $\dot{A}(\cdot) A^{-1}(\cdot)$ will be bounded linear operator on $\mathcal{H}$. Formally differentiating (1.1) and writing $\dot{x} = y$ we have

$$\dot{y} + A y + \dot{A} x = \dot{f} \tag{1.3}$$

Then, substituting from (1.1), $\dot{x} = A^{-1}(f - y)$ we have

$$\dot{y} + A y - \dot{A} A^{-1} y = \dot{f} - \dot{A} A^{-1} f \tag{1.4}$$

We then show that there is a weak solution to (1.4). That is, there is a continuous function y such that

$$< y, -\dot{\psi} + \mathcal{a}^* \psi - (\dot{A} A^{-1})^* \psi > = < \dot{f} - \dot{A} A^{-1} f, \psi > \tag{1.5}$$

for all $\psi \in \Psi$, $f \in C_0^1(J, H)$[5]. Now replace $\psi(t)$ by $A^{*-1}(t) \int_t^\tau \varphi(\tau)\, d\tau$, $\varphi \in \Psi$. Then (1.5) reduces to

$$< \mathcal{A}^{-1}(f - y), \varphi > = - < y, \int_t^\tau \varphi(\tau)\, d\tau > \tag{1.6}$$

Let $x(t) = \int_0^t y(\tau)\, d\tau$. Then, integrating by parts,

$- < y, \int_t^\tau \varphi(\tau)\, d\tau > = < x, \varphi >$. Hence

$$< \mathcal{A}^{-1}(f - y), \varphi > = < x, \varphi > . \tag{1.7}$$

Since $\varphi \in \Psi$ is arbitrary, and Ψ is dense in $\mathcal{H}$ we have $x = \mathcal{A}^{-1}(f-y)$, or $\dot{x} + \mathcal{A}x = f$. And this for any $f \in C_0^1(J, H)$. Hence T is maximal dissipative and generalized solutions, called "strong" solutions, to (1.1) will exist for any $f \in \mathcal{H}$, and they are unique.

As motivation for the method by which we introduce non-zero initial conditions in Section IV we take $\varphi \in \mathcal{D}(T_0)$, $x_0 \in D$. Then in (1.1), setting $x = x_0 + \varphi$ we have $\dot{\varphi}(t) + A(t)(x_0 + \varphi(t)) = f(t)$. Or, considering x_0 as the constant function on $[0, \tau]$, $x_0 \in \mathcal{D}(\mathcal{A})$, and $T_0\varphi + \mathcal{A}x_0 = f$. This suggests introducing the Hilbert space $\mathcal{K} = \mathcal{H} \times H$. We define the linear operator $\mathcal{T}_0$ on pairs $[x, x_0] \in \mathcal{K}$, such that

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[5] By $C^n(J, H)$, $0 \le n \le \infty$, we denote the set of all strongly continuous functions on $J = [0, \tau]$ with values in H which have n strongly continuous strong derivatives. $C_0^n(J, H)$ denotes the subclass of functions with compact support on J. We also write $C^0(J, H) = C(J, H)$, $C_0^0(J, H) = C_0(J, H)$.

$x = x_0 + \varphi$ where $x_0 \in D$ and $\varphi \in \mathcal{D}(T_0)$, by $\mathcal{T}_0 [x, x_0] = [T_0 (x - x_0) + \mathcal{A} x_0, x_0]$. $\mathcal{T}_0$ is then a dissipative linear operator with dense domain in $\mathcal{H}$ satisfying $\mathrm{Re} < \mathcal{T}_0 \boldsymbol{x}, \boldsymbol{x} >_{\mathcal{H}} \geq \beta \|\boldsymbol{x}\|^2$, $\beta > 0$, for all $\boldsymbol{x} \in \mathcal{D}(\mathcal{T}_0)$. The closure $\mathcal{T}$ of $\mathcal{T}_0$ will be maximal dissipative in $\mathcal{H}$ if and only if T, the closure in $\mathcal{H}$ of T_0, is maximal dissipative in $\mathcal{H}$. It follows that if T is maximal dissipative then for arbitrary $[f, x_0] \in \mathcal{H}$ there are sequences $x_{0n} \in D$, $\varphi_n \in \mathcal{D}(T_0)$, such that $x_n(t) = x_{0n} + \varphi_n(t)$ converges uniformly to a continuous function $x(t)$ while $\dot{x}_n(t) + A(t)\, x_n(t)$ converges to $f(t)$ a.e. and $x_n(0) = x_{0n}$ converges to $x_0 \in H$.

In Section V we define the fundamental solution operator $U(t, s)$ and represent the generalized solution in terms of it. Namely,

$$x(t) = U(t, 0)\, x_0 + \int_0^t U(t, s)\, f(s)\, ds$$

In Section VI we apply the theory of the operator differential equation to a time-dependent hyperbolic system of differential equations.

II. THE OPERATOR $\mathcal{A}$.

Let J be the interval $0 \leq t \leq \mathcal{T} < \infty$, and let $\{A(t): t \in J\}$ be a family of unbounded closed linear operators, each with the same domain D, dense in the Hilbert space H. We shall assume the following:

(A) There exists a constant $\alpha > 0$ such that for each $x \in D$, $t \in J$, we have

$$\operatorname{Re}\,(A(t)x,x) \geq \alpha |x|^2 \tag{2.1}$$

(B) For each $t \in J$ $A(t)$ is maximal dissipative.

(C) For each $x \in D$ $A(\cdot)x$ is strongly continuous on J and has a bounded measurable(6) derivative on J.

Let A_0 be a maximal dissipative linear operator with domain D, satisfying $\operatorname{Re}\,(A_0x,x) \geq \alpha |x|^2$ for all $x \in D$. (A_0 could be, for example, one of the A(t).)

It follows from Theorem 1.1 that A_0, or A(t), is a one-to-one mapping of D onto H. A_0^{-1} is then bounded linear transformation mapping H onto D. Let $B(t) = A(t)A_0^{-1}$, $t \in J$. Since A(t) is closed and A_0^{-1} is bounded B(t) is closed. It follows from the closed graph theorem that B(t) is bounded, since it is defined on all of H. We state this result as

Lemma 2.1 B(t) is a one-to-one continuous linear transformation of H onto itself.

We introduce here an additional assumption.

6The term measurable is used in the sense of Dunford-Schwartz [12], Definition III.2.10.

(D) $|B^{-1}(t)|$ is bounded on J.

Let $B_h(t) = (1/h)\,(B(t+h) - B(t))$, t, $t + h \in J$. By assumption (C) $\lim_{h\to 0} B_h(t)x$ exists for each $t \in J$, $x \in H$. Denoting this limit by $\dot{B}(t)\,x$ we have

<u>Lemma 2.2</u> $\dot{B}(t)$ is a continuous linear operator on H for each $t \in J$. $|\dot{B}(t)|$ is bounded on J.

Proof: Since $\lim_{h\to 0} B_h(t)x$ exists for each $x \in H$ it follows from [12], Theorem II.1.17, that $\dot{B}(t)$ is a continuous linear operator on H. $\dot{B}(t)x$ is bounded on J for each $x \in H$. It follows from the uniform boundedness principle ([12], Corollary II.3.21) that $|\dot{B}(t)|$ is bounded on J.

<u>Lemma 2.3</u> $B^{-1}(t)x$ is strongly differentiable on J for each $x \in H$ and

$$\frac{d}{dt} B^{-1}(t)x = -B^{-1}(t)\dot{B}(t)B^{-1}(t)x$$

Proof: First we show that $B^{-1}(t)x$ is strongly continuous on J for each $x \in H$. Let $B^{-1}(t)x = \psi(t)$. Then for t, $t + h \in J$ we have $B(t+h)\,[\psi(t+h) - \psi(t)] = x - B(t+h)B^{-1}(t)x$. As $h \to 0$ $B(t+h)B^{-1}(t)x \to x$ strongly for fixed t. Hence, since $|B^{-1}(t)|$ is bounded

$$|\psi(t+h) - \psi(t)| \leq |B^{-1}(t+h)|\ |x - B(t+h)B^{-1}(t)x|$$
$$\to 0 \quad \text{as } h \to 0.$$

Now, for t, $t + h \in J$, we have

$(1/h)\,[B^{-1}(t+h)\,x - B^{-1}(t)x] = -\,B^{-1}(t+h)B_h(t)B^{-1}(t)x$. Since $B_h(t)B^{-1}(t)x = \dot{B}(t)B^{-1}(t)x + Z_h$ where $Z_h \to 0$ as $h \to 0$, the statement of the lemma then follows from the strong continuity and uniform boundedness of $B^{-1}(t)$.

Remark: It follows by a direct computation that $B(t)\varphi(t)$ and $B^{-1}(t)\varphi(t)$ are strongly differentiable for each $\varphi \in C'(J,H)$ with respective derivatives

$$\frac{d}{dt}(B(t)\varphi(t)) = B(t)\dot{\varphi}(t) + \dot{B}(t)\varphi(t)$$

$$\frac{d}{dt}(B^{-1}(t)\varphi(t)) = B^{-1}(t)\dot{\varphi}(t) - B^{-1}(t)\dot{B}(t)B^{-1}(t)\varphi(t)$$

$B^*(t)\varphi(t)$ and $B^{*-1}(t)\varphi(t)$ are weakly differentiable for each $\varphi \in C'(J,H)$. That is, for all $Z \in H$

$$\frac{d}{dt}(B^*(t)\varphi(t), Z) = (\dot{B}^*(t)\varphi(t) + B^*(t)\dot{\varphi}(t), Z)$$

$$\frac{d}{dt}(B^{*-1}(t)\varphi(t), Z) = (B^{*-1}(t)\dot{\varphi}(t) - B^{*-1}(t)\dot{B}^*(t)B^{*-1}(t)\varphi(t), Z)$$

Since $B_h(t)x$ converges as $h \to o$ for each x, $\sup_h |B_h(t)| < \infty$ ([12], Theorem II.3.6). Thus there exists a constant $M > o$ such that $|B(t+h) - B(t)| < M|h|$. That is, $B(t)$ is continuous in the uniform operator topology. We then have the following theorem, which is a special case of one due to Bartle and Graves, [15] (Theorem 4).

Theorem 2.1 The mapping $\varphi \to B(\cdot)\varphi(\cdot)$ is one-to-one on $C(J,H)$ onto $C(J,H)$.

Lemma 2.4 If $U(t)$, $t \in J$, is a family of bounded linear operators on H such that $U(\cdot)x$ is measurable and bounded on J for each $x \in H$, then $U(\cdot)f(\cdot)$ is measurable on J for every measurable function f on J with values in H.

Proof: By the uniform boundedness principle $|U(t)|$ is bounded on J, say $|U(t)| \leq M$. Let f_n be a sequence of simple functions ([12], Definition III.2.9) such that $\lim_{n\to\infty} f_n(t) = f(t)$ a.e. on J. Since $U(\cdot)y$ is measurable for each $y \in H$, $U(\cdot)f_n(\cdot)$ is measurable for each simple function f_n. We then have

$|U(t)f_n(t) - U(t)f(t)| \leq M|f_n(t) - f(t)|$. Hence $U(t)f_n(t)$ converges a.e. to $U(t)f(t)$. It follows from [12], Corollary III.6.14, that $U(\cdot)f(\cdot)$ is measurable.

On the Hilbert space $\mathcal{H}$ of measurable functions on J into H with square integrable norms ($\langle x,y\rangle = \int_0^T (x(t),y(t))dt$, $\|x\| = \langle x,x\rangle^{1/2}$) we define the linear transformation $\mathcal{B}$ by $\mathcal{B}x(t) = B(t)x(t)$ a.e.

<u>Lemma 2.5</u> $\mathcal{B}$ is a one-to-one bounded linear transformation of $\mathcal{H}$ onto itself.

Proof: By Lemma 2.4 $\mathcal{B}x(\cdot)$ is measurable for each $x \in \mathcal{H}$. Moreover, $\|\mathcal{B}x\| \leq M\|x\|$ where M is the bound on J of $|B(t)|$. If $\mathcal{B}x = 0$ then $B(t)x(t) = 0$ a.e.; hence $x(t) = 0$ a.e. (Lemma 2.1). Thus $\mathcal{B}$ is a one-to-one continuous linear transformation on $\mathcal{H}$ into $\mathcal{H}$. Since C(J,H) is dense in $\mathcal{H}$ we see from Theorem 2.1 that the range of $\mathcal{B}$ is dense in $\mathcal{H}$. If $y = \mathcal{B}x$ $x(t) = B^{-1}(t)y(t)$ a.e.; hence $\|\mathcal{B}^{-1}y\| \leq m\|y\|$ where m is the bound of $|B^{-1}(t)|$ assumed in (D). Hence the range of $\mathcal{B}$ is closed, and this proves the lemma.

We define the linear transformation $\dot{\mathcal{B}}$ on $\mathcal{H}$ by $\dot{\mathcal{B}}x(t) = \dot{B}(t)x(t)$ a.e.

<u>Lemma 2.6</u> $\dot{\mathcal{B}}$ is a bounded linear operator on $\mathcal{H}$.

Proof: By Lemma 2.4 and assumption (C) $\dot{B}(\cdot)x(\cdot)$ is measurable. Since $|\dot{B}(t)|$ is bounded on J (Lemma 2.2), say $|\dot{B}(t)| \leq M$, we have $\|\dot{\mathcal{B}}x\| \leq M\|x\|$.

We now define the linear transformation $\mathcal{C}$ on $\mathcal{H}$ by $\mathcal{C}y(t) = A_0^{-1}y(t)$ a.e. for $y \in \mathcal{H}$. Then $\mathcal{C}y \in \mathcal{H}$. Let $\mathcal{D}$ be the range of $\mathcal{C}$.

<u>Lemma 2.7</u> $\mathcal{C}$ is a one-to-one bounded linear transformation of $\mathcal{H}$ onto $\mathcal{D}$. $\mathcal{D}$ is dense in $\mathcal{H}$.

Proof: $\| \mathcal{C} y \| = \left[\int_0^T |A_0^{-1} y(t)|^2 dt \right]^{1/2} \leq |A_0^{-1}| \, \|y\|$

If $\mathcal{C} y = 0$ then $A_0^{-1} y(t) = 0$ a.e. Hence $y(t) = 0$ a.e., and thus $\mathcal{C}$ is one-to-one. To show that $\mathcal{D}$ is dense in $\mathcal{H}$, suppose $y \in \mathcal{D}^{\perp}$ so that $\langle x,y \rangle = 0$ for all $x \in \mathcal{D}$. In particular, $\langle \mathcal{C} y,y \rangle = 0 =$ Re $\langle \mathcal{C} y,y \rangle =$ Re $\int_0^T (A_0^{-1} y(t),y(t))\, dt \geq \alpha \|\mathcal{C} y\|^2$. Hence $\mathcal{C} y = 0$, and $y = 0$ since $\mathcal{C}$ is one-to-one. This proves the statement of the lemma.

Let $\mathcal{A}_0 = \mathcal{C}^{-1}$. $\mathcal{A}_0$ is closed, since $\mathcal{C}$ is closed, and has the dense domain $\mathcal{D}$ and range $\mathcal{H}$. $\mathcal{A}_0$ is maximal dissipative:

Re $\langle \mathcal{A}_0 x,x \rangle \geq \alpha \|x\|^2$ for all $x \in \mathcal{D}$.

Define the linear transformation $\mathcal{A}$ on $\mathcal{D}$ by $\mathcal{A} = \mathcal{B} \mathcal{A}_0$, i.e. $\mathcal{A} x(t) = A(t)x(t)$ a.e. The following lemma shows that $\mathcal{A}$ is a maximal dissipative linear operator on $\mathcal{H}$.

Lemma 2.8 $\mathcal{A}$ is a one-to-one closed linear transformation of $\mathcal{D}$ onto $\mathcal{H}$. For all $x \in \mathcal{D}$ Re $\langle \mathcal{A} x,x \rangle \geq \alpha \|x\|^2$. $\mathcal{A}^{-1}$ is bounded.

Proof: Let x_n be a sequence of elements in $\mathcal{D}$, $x_n \rightarrow x \in \mathcal{H}$, and $\mathcal{A} x_n \rightarrow z \in \mathcal{H}$. Then $\mathcal{A}_0 x_n = \mathcal{B}^{-1} \mathcal{A} x_n \rightarrow \mathcal{B}^{-1} z$ since $\mathcal{B}^{-1}$ is continuous. Hence, since $\mathcal{A}_0$ is closed, $x \in \mathcal{D}$ and $\mathcal{A}_0 x = \mathcal{B}^{-1} z$. Hence $\mathcal{A} x = \mathcal{B} \mathcal{A}_0 x = z$, and $\mathcal{A}$ is closed. Since $\mathcal{A}_0$ is one-to-one on $\mathcal{D}$ onto $\mathcal{H}$, and $\mathcal{B}$ is one-to-one on $\mathcal{H}$ onto $\mathcal{H}$, $\mathcal{A}$ is one-to-one on $\mathcal{D}$ onto $\mathcal{H}$. Hence $\mathcal{A}^{-1}$ is a bounded linear transformation on $\mathcal{H}$. For all $x \in \mathcal{D}$

$$\text{Re } \langle \mathcal{A} x,x \rangle = \text{Re } \int_0^T (A(t)x(t),\, x(t))\, dt$$
$$\geq \alpha \|x\|^2$$

Since $\mathcal{A}$ ($\mathcal{A}_0$) is maximal dissipative with dense domain $\mathcal{D}$ in $\mathcal{H}$ its adjoint $\mathcal{A}^*$ ($\mathcal{A}_0^*$) is maximal dissipative with dense domain $\mathcal{D}^*$($\mathcal{D}_0^*$) in $\mathcal{H}$, as was stated in Theorem 1.3.

$(\mathcal{A}^*)^{-1} = (\mathcal{A}^{-1})^*$ ([12], Lemma XII.1.6).

$\mathcal{B}^* = (\mathcal{A}\mathcal{A}_0^{-1})^* \supset \mathcal{A}_0^{*-1}\mathcal{A}^*$, i.e. $\mathcal{A}_0^{*-1}\mathcal{A}^*$, defined on $\mathcal{D}^*$, has the bounded extension $\mathcal{B}^*$.

In succeeding sections we shall be concerned with functions in $\mathcal{D}$ which are differentiable with respect to t. We define the class of functions Φ by

$$\Phi = \{ \varphi : \varphi = \mathcal{A}_0^{-1}\psi \ , \ \psi \in C'(J,H) \text{ and } \psi(o) = 0 \} \tag{2.2}$$

Each function in Φ has a strongly continuous strong derivative with values in D. For let $\varphi(t) = A_0^{-1}\ \psi(t)$, $\psi \in C'(J,H)$. Then, since A_0^{-1} is continuous, $\dot\varphi(t) = A_0^{-1}\ \dot\psi\ (t)$ is the strongly continuous strong derivative of $\varphi(t)$.

<u>Theorem 2.2</u> Φ is dense in $\mathcal{H}$.

Proof: For any $x \in \mathcal{H}$ we can construct a sequence $x_n \in C'(J,H)$, by means of the Friedrichs mollifier, such that $x_n \to x$. Let ρ be a continuously differentiable positive function on R^1 with $\rho(t) = 0$ for $|t| > 1$, $\int_{-\infty}^{\infty}\rho(t)dt = 1$. Define $\rho_\epsilon(t) = \frac{1}{\epsilon}\rho(t/\epsilon)$, $\epsilon > o$. x is defined a.e. on J. Let its domain of definition be extended to R^1 by setting $x(t) = 0$, $t \notin J$. The convolution of ρ_ϵ and x is defined by

$$\rho_\epsilon * x(t) = \int_{-\infty}^{\infty}\rho_\epsilon(t-\tau)\ x(\tau)d\tau. \quad \text{Clearly} \quad \rho_\epsilon * x(t) \in C'(J,H) \tag{2.3}$$

Moreover, $\rho_\epsilon * x \to x$ in $\mathcal{H}$ as $\epsilon \to o$. (See e.g. [17] p. 367), and $\|\rho_\epsilon * \ x\| \leq \|x\|$.

Since $\mathcal{D}$ is dense in $\mathcal{H}$ there exists for any $x \in \mathcal{H}$ a sequence $y_n \in \mathcal{D}$ such that $\|x-y_n\| \to o$ as $n \to +\infty$. Since $y_n \in \mathcal{D}$ there exists $Z_n \in \mathcal{H}$ with $y_n = \mathcal{A}_0^{-1}\ Z_n$ (Lemma 2.4). Now

$$\rho_\epsilon * y_n(t) = \int_{-\infty}^{\infty}\rho_\epsilon(t-\tau)\ A_0^{-1}\ Z_n\ (\tau)d\tau$$
$$= A_0^{-1}\int_{-\infty}^{\infty}\ \rho_\epsilon\ (t-\tau)\ Z_n(\tau)d\tau$$

Hence $\rho_\epsilon * y_n \in C'(J,H) \cap \mathcal{D}$, and $\rho_\epsilon * y_n \to y_n$ as $\epsilon \to o$. Given $\delta > o$ there exists, for each n, $\epsilon(n) > o$ such that $\| \rho_{\epsilon(n)} * y_n - y_n \| < \delta/2$. Then $\| \rho_{\epsilon(n)} * y_n - x \| \leq \| \rho_{\epsilon(n)} * y_n - y_n \| + \| y_n - x \|$. For sufficiently large n $\| y_n - x \| < \delta/2$. Hence $\| \rho_{\epsilon(n)} * y_n - x \| < \delta$ if n is sufficiently large. Let θ_n be a positive continuously differentiable function on J with $\theta_n(o) = 0$, $\theta_n(t) = 1$ for $t \geq \frac{1}{n}$ and $o \leq \theta_n(t) \leq 1$ for all $t \in J$. Then

$$\theta_n \rho_{\epsilon(n)} * y_n \in \Phi \text{ and } \theta_n \rho_{\epsilon(n)} * y_n \to x \text{ as } n \to +\infty,$$

which proves the theorem.

We shall also be concerned in succeeding sections with functions in $\mathcal{D}^*$ which are weakly differentiable with respect to t. We define the class of functions Ψ by

$$\Psi = \{ \psi : \psi = \mathcal{A}^{*-1}\varphi, \ \varphi \in C'(J,H), \varphi(T) = 0 \} \tag{2.4}$$

<u>Lemma 2.9</u> Ψ is dense in $\mathcal{H}$.

Proof: Suppose there exists $z \in \mathcal{H}$ such that $\langle \mathcal{A}^{*-1}\varphi, z \rangle = 0$ for all $\varphi \in C'(J,H)$ with $\varphi(T) = 0$. Then, since this class is dense in $\mathcal{H}$, $\mathcal{A}^{-1} z = 0$. Hence $z = 0$, and this proves the lemma.

III. CAUCHY PROBLEM WITH VANISHING INITIAL CONDITION

We now consider the Cauchy problem

$$\dot{x} + \mathcal{A} x = f, \; x(0) = 0 \tag{3.1}$$

where $\dot{x}$ is in some sense the derivative of x with respect to t. The differentiation operator applied to Φ is a dissipative linear operator since if $x \in \Phi$ and $\dot{x}$ is the continuous strong derivative of x we have $\mathrm{Re} < \dot{x}, x > = \frac{1}{2} |x(\mathcal{T})|^2 \geq 0$. It follows that the differentiation operator with the dense domain Φ is closeable in $\mathcal{H}$. Let D_t be its closure. Now consider the linear operator Σ defined on $\mathcal{H}$ by

$$(\Sigma f)(t) = \int_0^t f(\tau)\, d\tau \tag{3.2}$$

Since $|(\Sigma f)(t)|^2 \leq t \int_0^t |f(\tau)|^2 d\tau \leq \mathcal{T} \| f \|^2$

we have $\| \Sigma f \| \leq \mathcal{T} \| f \|$. Thus Σ is a bounded linear operator on $\mathcal{H}$. Moreover $f = D_t \Sigma f = \Sigma D_t f$, hence $D_t = \Sigma^{-1}$, and D_t is maximal dissipative. We now define the linear operator Σ^* on $\mathcal{H}$ by

$$(\Sigma^* f)(t) = \int_t^{\mathcal{T}} f(\tau)\, d\tau \tag{3.3}$$

Then for all $f, g \in \mathcal{H}$ we have

$$- < \Sigma f, g > + < f, \Sigma^* g > = \int_0^{\mathcal{T}} \frac{d}{dt} (\Sigma f(t), \Sigma^* g(t))\, dt = 0$$

Thus, justifying the notation, Σ^* is the adjoint operator of Σ, and $D_t^* = \Sigma^{*-1}$. It is observed that $\Psi \subset \mathcal{D}(D_t^*)$ and $D_t^* \psi = -\dot{\psi}$ where $\dot{\psi}$ is the weak derivative of $\psi \in \Psi$.

If $x \in \mathcal{D}(D_t) \cap \mathcal{D}(\mathcal{A})$ and $D_t x + \mathcal{A} x = f$ we shall say that x is a classical solution to (3.1). In general we cannot expect a classical solution of (3.1) for every $f \in \mathcal{H}$. In order to broaden the concept of a solution we define the linear operator T_0 on Φ by $T_0 \varphi = \dot{\varphi} + \mathcal{A} \varphi$. We have

$$\operatorname{Re} < T_0 \varphi, \varphi > = \tfrac{1}{2} \, | \varphi(T) |^2 + \operatorname{Re} < \mathcal{A} \varphi, \varphi > \geq \alpha \|\varphi\|^2$$

Hence T_0 is dissipative with dense domain in $\mathcal{H}$, and it has a closure, which we shall denote by $\overline{T_0} = T$. Clearly T also satisfies $\operatorname{Re} < T x, x > \geq \alpha \| x \|^2$ for all $x \in \mathcal{D}(T)$. We shall say that x is a strong solution to (3.1) if $x \in \mathcal{D}(T)$ and $T x = f$.

We shall see below that with the conditions we have imposed on $A(t)$ there will be a strong solution to (3.1) for every $f \in \mathcal{H}$. First, however, we shall show that there are always solutions in the following broader sense. We define the operator S_0 on the class of functions by $S_0 \psi = -\dot{\psi} + \mathcal{A}^* \psi = D_t^* \psi + \mathcal{A}^* \psi$. It is seen that S_0 is a formal adjoint of T. We shall say that x is a weak solution of (3.1) with respect to S_0 if for all $\psi \in \Psi$ we have

$$< x, S_0 \psi > = < f, \psi > \qquad (3.4)$$

The following theorem shows that there is always a weak solution to (3.1) in this sense.

Theorem 3.1. If S_0 is a dissipative linear operator with (not necessarily dense) domain in a Hilbert space $\mathcal{H}$ which satisfies $\operatorname{Re} < S_0 \Psi, \Psi > \geq \alpha \| \Psi \|^2$ for some $\alpha > 0$, then for any $f \in \mathcal{H}$ there exists $x \in \mathcal{H}$ satisfying $< x, S_0 \Psi > = < f, \Psi >$ for all $\Psi \in \mathcal{D}(S_0)$

Proof: $\| S_0 y \| \| y \| \geq \operatorname{Re} < S_0 y, y > \geq \alpha \| y \|^2$ for all $y \in \mathcal{D}(S_0)$. Hence S_0 is one-to-one on $\mathcal{D}(S_0)$ onto $\mathcal{R}(S_0)$, and S_0^{-1}, defined on $\mathcal{R}(S_0)$, is bounded. S_0^{-1} has a unique extension to a bounded linear operator R_0 with domain $\overline{\mathcal{R}(S_0)}$, and R_0 may be extended to a bounded linear operator R on $\mathcal{H}$, by setting $R y = 0$ for $y \in \mathcal{R}(S_0)^{\perp}$ for example. Then if $y \in \mathcal{D}(S_0)$ and $S_0 y = z$, $y = R z$ and we have $< f, y > = < f, R z > = < R^* f, z > = < R^* f, S_0 y >$. Thus $x = R^* f$ satisfies the statement of the theorem.

The next theorem reveals some of the properties of a strong solution.

Theorem 3.2. T is a one-to-one mapping of $\mathcal{D}(T)$ onto a closed subspace of $\mathcal{H}$. If $x \in \mathcal{D}(T)$, $T x = f$, then x is equivalent to a strongly continuous function on J and

$$| x(t) | \leq \int_0^t e^{-\alpha (t - \tau)} | f(\tau) | \, d\tau$$

Proof: For each $x \in \mathcal{D}(T)$ we have

$$\| x \| \| T x \| \geq \operatorname{Re} < T x, x > \geq \alpha \| x \|^2 \text{ or } \| T x \| \geq \alpha \| x \| .$$

Hence T is one-to-one onto $\mathcal{R}(T)$ and T^{-1}, defined on $\mathcal{R}(T)$, is bounded. Since T is closed so is T^{-1}, and it follows that $\mathcal{R}(T)$ is closed.

For $x \in \Phi$, $T_0\, x = f \in \mathcal{H}$,

$$(T_0\, x(t), x(t)) = (\dot{x}(t), x(t)) + (\mathcal{A}\, x(t), x(t))$$
$$= (f(t), x(t)) \text{ a.e.}$$

$$\text{or } \tfrac{1}{2}\frac{d}{dt}|x(t)|^2 = -\operatorname{Re}(A(t)\, x(t), x(t)) + \operatorname{Re}(f(t), x(t))$$
$$\leq -\alpha\, |x(t)|^2 + |f(t)|\;|x(t)|$$

If $|x(t)| \neq 0$ we conclude that

$$\frac{d}{dt}|x(t)| \leq -\alpha\, |x(t)| + |f(t)| \quad \text{a.e.}$$

(This also holds if $|x(t)| = 0$ since we then have

$$\frac{d}{dt}|x(t)| \leq |\dot{x}(t)| = |f(t)| \;.)$$

By Gronwall's lemma, then,

$$|x(t)| \leq \int_0^t e^{-\alpha(t-\tau)}\, |f(\tau)|\, d\tau \;.$$

Now if $x \in \mathcal{D}(T)$, $T\, x = f$, there exists a sequence $x_n \in \Phi$ such that $x_n \longrightarrow x$, $T\, x_n = f_n \longrightarrow f$. Then

$$|x_n(t) - x_m(t)| \leq \int_0^t e^{-\alpha(t-\tau)}\, |f_n(\tau) - f_m(\tau)|\, d\tau$$
$$\leq \mathcal{T}^{\frac{1}{2}}\, \| f_n - f_m \|$$

Thus the convergence of x_n to x is uniform in t, and hence x is equivalent to a continuous function which satisfies the inequality

$$|x(t)| \leq \int_0^t e^{-\alpha(t-\tau)}\, |f(\tau)|\, d\tau$$

It is clear that a classical solution to (3.1) is also a strong solution. It follows that if there is a classical solution of (3.1) for every f in a dense linear subspace of $\mathcal{H}$ then the range of T is dense in $\mathcal{H}$, and hence $\mathcal{R}(T) = \mathcal{H}$ since $\mathcal{R}(T)$ is closed (Theorem 3.2). We state this result as

<u>Lemma 3.1.</u> If there exists a classical solution to the Cauchy problem (3.1) for every f in a dense subspace of $\mathcal{H}$ then T is maximal dissipative, and there exists a unique solution to the equation $T x = f$ for any $f \in \mathcal{H}$.

Let $C(t)$, $t \in J$, be a family of bounded linear operators on H such that $C(\cdot)\, x(\cdot) \in \mathcal{H}$ for every $x \in \mathcal{H}$. Define the linear transformation $\mathcal{C}$ on $\mathcal{H}$ by $\mathcal{C} x = C(\cdot)\, x(\cdot)$. The following theorem shows that if T is maximal dissipative we do not lose strong solutions to (3.1) by perturbing it with $\mathcal{C}$.

<u>Theorem 3.3.</u> If T is maximal dissipative then for any $f \in \mathcal{H}$ there is a unique $x \in \mathcal{D}(T)$ such that $T x + \mathcal{C} x = f$.

Proof: Let $\mathcal{C}_1 = \mathcal{C} + \|\mathcal{C}\| I$. Then $\mathcal{C}_1$ is a bounded dissipative linear operator. We now show that $T + \mathcal{C}_1$ is maximal dissipative. We have $\mathrm{Re} < (T + \mathcal{C}_1)\, x, x > \geq \alpha \|x\|^2$ for all $x \in \mathcal{D}(T)$. Since T is closed and $\mathcal{C}_1$ is bounded it is clear that $T + \mathcal{C}_1$ is closed, moreover $\mathcal{R}(T + \mathcal{C}_1)$ is a closed subspace of $\mathcal{H}$. Suppose $z \in \mathcal{R}(T + \mathcal{C}_1)^{\perp}$.

Then $z \in \mathcal{D}((T + \mathcal{C}_1)^*) = \mathcal{D}(T^*)$, and $T^*z + \mathcal{C}_1^* z = 0$. But T^* is maximal dissipative (Theorem 1.3) and satisfies $\mathrm{Re} < T^* z, z > \geq \alpha \| z \|^2$, hence $\mathrm{Re} < (T^* + \mathcal{C}_1^*) z, z > \geq \alpha \| z \|^2$, and $z = 0$. Thus for any $g \in \mathcal{H}$ there is a unique $y \in \mathcal{D}(T)$ such that $(T + \mathcal{C}_1) y = g$. Define the bounded linear transformation σ on $\mathcal{H}$ by $\sigma g(t) = e^{\|\mathcal{C}\| t} g(t)$ a.e. Obviously σ is one-to-one on $\mathcal{H}$ onto itself. Let $x = \sigma y$, and let y_n be a sequence of elements of Φ such that $y_n \longrightarrow y$ and $(T + \mathcal{C}_1) y_n = g_n \longrightarrow g$.
Then $(T + \mathcal{C}) \sigma y_n = \sigma (T + \mathcal{C}_1) y_n = \sigma g_n$. Set $g = \sigma^{-1} f$.
As $n \longrightarrow +\infty$ $\sigma y_n \longrightarrow x$, $\sigma g_n \longrightarrow f$, and $(T + \mathcal{C}) x = f$. Since y is uniquely determined so is x.

Even if T is not maximal dissipative there is a weak solution to the equation $(T + \mathcal{C}) x = f$ in the following sense.

Lemma 3.2. For any $f \in \mathcal{H}$ there exists $x \in \mathcal{H}$ such that

$$< x, (S_0 + \mathcal{C}^*) \Psi > = < f, \Psi > \quad \text{for all } \Psi \in \mathbf{\Psi}$$

Proof: $S_0 + \mathcal{C}^* + \| \mathcal{C} \| I$ satisfies the conditions of Theorem (3.1). Hence for any $g \in \mathcal{H}$ there exists $y \in \mathcal{H}$ satisfying

$$< y, (S_0 + \mathcal{C}^* + \| \mathcal{C} \| I) \Psi > = < g, \Psi > \text{ for all } \Psi \in \mathbf{\Psi} . \tag{3.5}$$

Now, as in the proof of Theorem 3.3, let the bounded linear operator σ be defined on $\mathcal{H}$ by $\sigma f(t) = e^{\| \mathcal{C} \| t} f(t)$ a.e. on J. Take $\varphi \in \mathbf{\Psi}$. Then $\sigma \varphi = \Psi \in \mathbf{\Psi}$ and $S_0 \Psi = \sigma (S_0 \varphi - \| \mathcal{C} \| \varphi)$.

Thus $< y, \sigma (S_0 \varphi + \mathcal{C}^* \varphi) > \; = \; < g, \sigma \varphi >$

or $< \sigma y, (S_0 + \mathcal{C}^*) \varphi > \; = \; < \sigma g, \varphi >$ for all $\varphi \in \Psi$. Letting $g = \sigma^{-1} f$, $x = \sigma y$, we have the statement of the lemma.

We are now in a position to state conditions for the existence of unique strong solutions to (3.1).

Theorem 3.4. If A (t) satisfies assumptions (A), (B), (C), and (D) then T is maximal dissipative.

Proof: Take $f \in C_0^1 (J, H)$. Since $\dot{\mathcal{B}} \mathcal{B}^{-1} x(t) = \dot{B}(t) B^{-1}(t) x(t)$ a.e. $-\dot{\mathcal{B}} \mathcal{B}^{-1}$ satisfies the conditions on the operator $\mathcal{C}$ of Lemma 3.2. Hence for any $f \in C_0^1 (J, H)$ there exists $y \in \mathcal{H}$ such that

$$< y, S_0 \Psi - (\dot{\mathcal{B}} \mathcal{B}^{-1})^* \Psi > \; = \; < \dot{f} - \dot{\mathcal{B}} \mathcal{B}^{-1} f, \Psi > \text{ for all } \Psi \in \Psi \quad . (3.6)$$

For any $\varphi \in \mathcal{D}(D_t^*)$ we have

$$D_t^* \mathcal{A}^{*^{-1}} \varphi = D_t^* (\mathcal{B}^{*^{-1}} \mathcal{A}_0^{*^{-1}} \varphi)$$

$$= \mathcal{B}^{*^{-1}} \mathcal{A}_0^{*^{-1}} D_t^* \varphi + \mathcal{B}^{*^{-1}} \dot{\mathcal{B}}^* \mathcal{B}^{*^{-1}} \mathcal{A}_0^{*^{-1}} \varphi$$

or $(D_t^* - \mathcal{B}^{*^{-1}} \dot{\mathcal{B}}^*) \mathcal{A}^{*^{-1}} \varphi \; = \; \mathcal{A}^{*^{-1}} D_t^* \varphi$

Since $\mathcal{A}^{-1*} \varphi \in \Psi$ for any $\varphi \in \mathcal{D}(D_t^*)$ substitution of $\Psi = \mathcal{A}^{-1*} \varphi$ in (3.6) gives

$$< y, (D_t^* - \mathcal{B}^{*^{-1}} \dot{\mathcal{B}}^*) \mathcal{A}^{*^{-1}} \varphi + \varphi > \; = \; < \dot{f} - \dot{\mathcal{B}} \mathcal{B}^{-1} f, \mathcal{A}^{*^{-1}} \varphi > \qquad (3.7)$$

or $< y, \mathcal{A}^{*^{-1}} D_t^* \varphi + \varphi > \; = \; < \mathcal{A}^{-1} \dot{f} - \mathcal{A}^{-1} \dot{\mathcal{B}} \mathcal{B}^{-1} f, \varphi >$

Now $D_t \mathcal{A}^{-1} f = \mathcal{A}^{-1} \dot{f} - \mathcal{A}^{-1} \dot{\mathcal{B}} \mathcal{B}^{-1} f$. Hence (3.7) becomes

$$< y, \varphi > \; = \; - < \mathcal{A}^{-1} y, D_t^* \omega > + < D_t \mathcal{A}^{-1} f, \varphi > \qquad (3.8)$$

or $< y, \varphi > \; = \; < \mathcal{A}^{-1} (- y + f), D_t^* \omega >$ for all $\varphi \in \mathcal{D}(D_t^*)$.

Let $x = \Sigma y$. Then $D_t x = y$ and

$$< x, D_t^* \varphi > \; = \; < \mathcal{A}^{-1} (- y + f), D_t^* \varphi > \qquad (3.9)$$

for all $\varphi \in \mathcal{D}(D_t^*)$. Since $\mathcal{R}(D_t^*) = \mathcal{H}$ we have $x = \mathcal{A}^{-1}(- y + f)$ or $D_t x + \mathcal{A} x = f$. That is, x is a classical solution to the Cauchy problem (3.1). It follows then from Lemma 3.1 that T is maximal dissipative since $C_0^1 (J, H)$ is dense in $\mathcal{H}$.

IV. GENERAL CAUCHY PROBLEM

This section deals with the Cauchy problem with non vanishing initial conditions:

$$\dot{x} + \mathcal{A} x = f \quad , \quad x(o) = x_o \tag{4.1}$$

It is convenient to introduce the Hilbert space $\mathcal{G} = \mathcal{H} \times H$ with the inner product

$$\langle [f,x_o], [g,y_o] \rangle_{\mathcal{G}} = \langle f,g \rangle + (x_o,y_o) \tag{4.2}$$

where $f,g \in \mathcal{H}$, $x_o,y_o \in H$. We define the linear transformation U by

$$\mathcal{D}(U) = \{x : x = x_o + y,\ y \in \mathcal{D}(T),\ x_o \in D\} \tag{4.3}$$

and $\qquad Ux = [x,x_o] \in \mathcal{G}$

(Since the functions in $\mathcal{D}(T)$ are only equivalent to continuous functions we define U by means of the continuous representative of x)

Lemma 4.1 $\mathcal{R}(U)$ is dense in $\mathcal{G}$.

Proof: Let $[g,y_o]$ be an element of $\mathcal{G}$. Since Φ is dense in $\mathcal{H}$ there is a sequence $\varphi_n \in \Phi$ such that $\varphi_n \longrightarrow g - y_o \in \mathcal{H}$.
Since D is dense in H there is a sequence $y_n \in D$ such that $y_n \longrightarrow y_o$.
Let $x_n = y_n + \varphi_n \in \mathcal{D}(U)$. Then $Ux_n = [y_n + \varphi_n, y_n] \longrightarrow [g,y_o]$ as $n \rightarrow +\infty$. Since $[g,y_o]$ is an arbitrary element of $\mathcal{G}$ it follows that $\mathcal{R}(U)$ is dense in $\mathcal{G}$.

We define the linear operator $\mathcal{T}_o$ on $\mathcal{R}(U)$ by $\mathcal{T}_o [x,x_o] = [T(x-x_o) + \mathcal{A} x_o, x_o]$. Writing $Ux = \chi \in \mathcal{D}(\mathcal{T}_o)$ we have

Lemma 4.2 Re $\langle \mathcal{T}_o \chi , \chi \rangle_{\mathcal{G}} \geq \beta \|\chi\|^2_{\mathcal{G}}$ for all $\chi \in \mathcal{D}(\mathcal{T}_o)$, where $\beta = \min(\frac{1}{2}, \alpha)$.

Proof: If $x = x_o + \varphi$, $x_o \in D$, $\varphi \in \Phi$, $\mathcal{X} = [x_o + \varphi, x_o]$ and

$$\text{Re} \langle \mathcal{T}_o \mathcal{X}, \mathcal{X} \rangle_{\mathcal{H}} = \text{Re} \langle \dot{\varphi} + \mathcal{A}(\varphi + x_o), \varphi + x_o \rangle + |x_o|^2$$

$$= \text{Re} \langle \mathcal{A}x, x \rangle + \tfrac{1}{2} (|x(\tau)|^2 + |x_o|^2)$$

$$\geq \alpha \| x \|^2 + \tfrac{1}{2} |x_o|^2$$

$$\geq \beta \| \mathcal{X} \|^2_{\mathcal{H}}$$

It follows from Theorem 1.1 that $\mathcal{T}_o$ is closable. Let its closure be denoted by $\overline{\mathcal{T}}_o = \mathcal{T}$.

<u>Theorem 4.1</u> $\mathcal{T}$ is a maximal dissipative linear operator with dense domain in $\mathcal{H}$ if and only if T is a maximal dissipative linear operator in H.

Proof: Suppose first that T is maximal dissipative in $\mathcal{H}$. Then if $[f, x_o] \in \mathcal{H}$ with $x_o \in D$, let $y = T^{-1}(f - \mathcal{A} x_o)$, $x = x_o + y$, and we then have $\mathcal{T} Ux = [f, x_o]$. It is clear that $\mathcal{H} \times D$ is dense in $\mathcal{H}$, and since the range of $\mathcal{T}$ is closed $\mathcal{R}(\mathcal{T}) = \mathcal{H}$.

If T is not maximal dissipative then $\mathcal{R}(\mathcal{T})$ lacks all elements of the form $[f, o]$, $f \in \mathcal{R}(T)^{\perp}$, and the theorem is proved.

A function $x \in C(J,H)$ satisfying $\mathcal{T} Ux = [f, x_o] \in \mathcal{H}$ will be called a <u>strong solution</u> to the Cauchy problem (4.1). We may define weak solutions with respect to the linear operator $\mathcal{S}_o$ defined on pairs $[\psi, \psi(o)]$, $\psi \in \Psi$ by $\mathcal{S}_o [\psi, \psi(o)] = [S_o \psi, o]$. $\mathcal{S}_o$ is a formal adjoint of $\mathcal{T}$, for if $Ux \in \mathcal{D}(\mathcal{T}_o)$ we have

$$\langle \mathcal{T}_o Ux, [\psi, \psi(o)] \rangle_{\mathcal{H}} = \langle T(x - x(o)) + \mathcal{A} x(o), \psi \rangle + (x(o), \psi(o))$$

$$= \langle x, -\dot{\psi} + \mathcal{A}^* \psi \rangle + \langle x(o), \dot{\psi} \rangle + (x(o), \psi(o))$$

$$= \langle Ux, \mathcal{S}_o [\psi, \psi(o)] \rangle_{\mathcal{H}}$$

Let x_n be a sequence in $\mathcal{D}(U)$ such that $Ux_n \longrightarrow \chi \in \mathcal{Y}$, $\sigma_o Ux_n \to \sigma\chi$. Then $\langle \sigma\chi, [\psi, \psi(o)]\rangle_{\mathcal{Y}} = \langle \chi, \delta_o[\psi, \psi(o)]\rangle_{\mathcal{Y}}$ for all $\psi \in \Psi$.

A function $x \in C(J,H) \cap \mathcal{D}(\mathcal{A})$ is a <u>classical solution</u> to (4.1) if its distribution derivative $\dot{x}$ is in $\mathcal{H}$, $x(o) = x_o \in D$ and $\dot{x} + \mathcal{A}x = f$. A classical solution is a strong solution since then

$$\sigma Ux = [T(x-x_o) + \mathcal{A}x_o, x_o]$$

$$= [\dot{x} + \mathcal{A}x, x_o]$$

$$= [f, x_o]$$

<u>Theorem 4.2</u> If x is a strong solution to the Cauchy problem (4.1) then x is equivalent to a strongly continuous function, $x(o) = x_o$, and

$$|x(t)| \leq e^{-\alpha t}|x_o| + \int_0^t e^{-\alpha(t-\tau)}|f(\tau)|\,d\tau$$

Proof: Let $x_n \in D$ be a sequence such that $x_n \longrightarrow x_o$ as $n \to +\infty$. Let φ_n be a sequence of elements of Φ such that $\varphi_n \longrightarrow x - x_o$. Then

$$\sigma U(x_n + \varphi_n) = [T\varphi_n + \mathcal{A}x_n, x_n]$$

$$= [\dot{Z}_n + \mathcal{A}Z_n, x_n]$$

where $Z_n = x_n + \varphi_n$. We have $[Z_n, x_n] \longrightarrow [x, x_o]$ in $\mathcal{Y}$, and $\dot{Z}_n + \mathcal{A}Z_n = f_n \longrightarrow f$ in $\mathcal{H}$. In a manner similar to that used in Theorem 3.2 we may show that

$$|Z_n(t)| \leq e^{-\alpha t}|x_n| + \int_0^t e^{-\alpha(t-\tau)}|f_n(\tau)|\,d\tau \tag{4.4}$$

and that

$$|Z_n(t) - Z_m(t)| \leq |x_n - x_m| + T^{1/2}\|f_n - f_m\| \tag{4.5}$$

It follows from (4.5) that Z_n converges uniformly to x; hence x is continuous and $x(o) = \lim_{n\to+\infty} Z_n(o) = \lim_{n\to+\infty} x_n = x_o$. From (4.4) we obtain, in taking the limit as $n \to +\infty$,

$$|x(t)| \leq e^{-\alpha t}|x_o| + \int_0^t e^{-\alpha(t-\tau)}|f(\tau)|\,d\tau$$

As in the case of vanishing initial conditions we may perturb (4.1) with a bounded operator $\mathcal{C}$ on $\mathcal{H}$ which is defined through a family of bounded operators $C(t)$, $t \in J$, by $\mathcal{C}x = C(\cdot)x(\cdot)$. For convenience we introduce the bounded linear operator $\mathcal{B}$ defined on $\mathcal{G}$ by $\mathcal{B}\,[x,x_o] = [\,\mathcal{C}x + \|\mathcal{C}\|\,x, o]$ for $[x,x_o] \in \mathcal{G}$. $\mathcal{B}$ is then dissipative linear operator on $\mathcal{G}$, and $\mathcal{T} + \mathcal{B}$ is a dissipative linear operator with dense domain with the formal adjoint $\mathcal{S}_o + \mathcal{B}^*$ which satisfies $\mathrm{Re}\,\langle \mathcal{S}_o y + \mathcal{B}^* y, y\rangle_{\mathcal{G}} \geq \beta\|y\|^2_{\mathcal{G}}$ for all $y = [\psi, \psi(o)], \psi \in \Psi$. Hence (Theorem 3.1) the equation $(\mathcal{T} + \mathcal{B})\,x = f$ has a weak solution for any $f \in \mathcal{G}$ in the sense that there exists $x \in \mathcal{G}$ satisfying $\langle x, \mathcal{S}_o y + \mathcal{B}^* y\rangle_{\mathcal{G}} = \langle f, y\rangle_{\mathcal{G}}$. Writing $x = [y, y_o]$, $y = [\psi, \psi(o)]$ $f = [g, x_o]$ this becomes $\langle y, S_o\psi + \mathcal{C}^*\psi + \|\mathcal{C}\|\psi\rangle = \langle g, \psi\rangle + (x_o, \psi(o))$ Now, as in Theorem 3.3, we let $\psi(t) = \sigma\,\varphi(t) = e^{\|\mathcal{C}\|t}\,\varphi(t)$. Then $\langle y, \sigma(S_o\varphi + \mathcal{C}^*\varphi)\rangle = \langle g, \sigma\varphi\rangle + (x_o, \psi(o))$ or $\langle \sigma y, S_o\varphi + \mathcal{C}^*\varphi\rangle = \langle \sigma g, \varphi\rangle + (x_o, \varphi(o))$. We state this result as

<u>Lemma 4.3</u> For any $f \in \mathcal{H}$, $x_o \in H$, there exists $x \in \mathcal{H}$ satisfying
$$\langle x, S_o\psi + \mathcal{C}^*\psi\rangle = \langle f, \psi\rangle + (x_o, \psi(o)) \text{ for all } \psi \in \Psi .$$

We can now state conditions for the existence of a unique classical solution to the Cauchy problem (4.1).

<u>Theorem 4.3</u> If $A(t)$ satisfies assumptions (A), (B), (C), and (D) there is a unique classical solution to the Cauchy problem (4.1) for every $f \in C(J,H)$ whose distribution derivative $\dot{f}$ is in $\mathcal{H}$ and $x_o \in D$.

Proof: By Lemma (4.3) there exists $y \in \mathcal{H}$ such that

$$\langle y, -\dot{\psi} + \mathcal{A}^*\psi - (\dot{\mathcal{B}}\mathcal{B}^{-1})^*\psi\rangle \tag{4.6}$$

$$= \langle \dot{f} - \dot{\mathcal{B}}\mathcal{B}^{-1} f, \psi\rangle + (f(o)-A(o)x_o, \psi(o))$$

for all $\psi \in \Psi$. Take $\varphi \in \mathcal{D}(D_t^*)$. Then $\mathcal{A}^{*-1}\varphi \in \Psi$ and, setting $\psi = \mathcal{A}^{*-1}\varphi$, $\dot{\psi} = \mathcal{A}^{*-1}\dot{\varphi} - \mathcal{B}^{*-1}\dot{\mathcal{B}}^* \mathcal{A}^{*-1}\varphi = \mathcal{A}^{*-1}\dot{\varphi} - (\dot{\mathcal{B}}\mathcal{B}^{-1})^*\psi$

Substitution in (4.6) then gives

$$\langle y, -\mathcal{A}^{*-1}\dot{\varphi} + \varphi\rangle = \langle \mathcal{A}^{-1} (f - \dot{\mathcal{B}}\mathcal{B}^{-1} f), \varphi\rangle \tag{4.7}$$

$$+ (f(o) - A(o)x_o, A^{-1}(o)\ \varphi(o))$$

Set $x(t) = x_o + \int_o^t y(\tau)d\tau$. Then y is the distribution derivative of x. Moreover, $\mathcal{A}^{-1}(f - \dot{\mathcal{B}}\mathcal{B}^{-1} f)$ is the distribution derivative of $\mathcal{A}^{-1}f$. Integration of (4.7) by parts then gives

$$\langle x, D_t^*\varphi\rangle = \langle \mathcal{A}^{-1} (f-y), D_t^*\varphi\rangle \tag{4.8}$$

Since the range of D_t^* is $\mathcal{H}$ we have $x = \mathcal{A}^{-1}(f-y)$. Hence x is a classical solution to the Cauchy problem since y = x, the distribution derivative of x, x(o) = 0, and $\dot{x} + \mathcal{A}x = f$.

V. THE FUNDAMENTAL SOLUTION

In the previous argument we could have considered an interval $[s, \mathcal{T}]$, $o \leq s < \mathcal{T}$, instead of $[o, \mathcal{T}]$. With assumptions (A), (B), (C), and (D) we then obtain a unique continuous solution to the Cauchy problem

$$x(t) + A(t)x(t) = 0, \quad x(s) = x_o \in H. \tag{5.1}$$

If $x_o \in D$ the solution $x(t)$ is a classical solution. Otherwise, for $x_o \notin D$, it is a strong solution in the sense that $\mathcal{O}[x,x(s)] = [o,x_o]$. It is clear that the mapping $x_o \longrightarrow x(t)$ is a linear continuous mapping on H since we have $|x(t)| \leq e^{-\alpha(t-s)}|x_o|$ from Theorem 4.2. We shall write $x(t) = U(t,s)x_o$; $|U(t,s)| \leq e^{-\alpha(t-s)}$. Now suppose that $o \leq \tau < s < \mathcal{T}$, $x_o \in D$. $U(t,\tau)x_o$, restricted to $[s,\mathcal{T}]$ is a classical solution to (5.1) and is equal to $U(s,\tau)x_o \in D$ for $t = s$. Hence $U(t,s)U(s,\tau)x_o = U(t,\tau)x_o$ for all $t \in [s,\mathcal{T}]$, $x_o \in D$, since both sides are solutions to (5.1) with the same initial condition on $[s,\mathcal{T}]$. Since the operators are continuous and D is dense in H we have

$$U(t,\tau) = U(t,s)U(s,\tau), \tag{5.2}$$

for $o \leq \tau \leq s \leq t \leq \mathcal{T}$.

For all $x \in H$, $o \leq s < s + h \leq t$, we have

$$|U(t,s+h)x_o - U(t,s)x_o| = |U(t, s+h)(x_o - U(s+h,s)x_o)|$$
$$\leq |x_o - U(s+h,s)x_o|$$

and this tends to zero as $h \longrightarrow o$. Hence $U(t,s)x_o$ is continuous from the right as a function of s. Again, supposing $x_o \in D$ we have

$$U(t,s)x_o = x_o - \int_s^t A(\tau)U(\tau,s)x_o d\tau$$

Hence

$$U(s + h,s)x_o = x_o - hA(s)x_o + o(h)$$

since $A(\tau)U(\tau,s)$ is a strongly continuous function of τ. Then $(1/h)\,[U(t,s + h)x_o - U(t,s)x_o] = U(t,s + h)\,[A(s)x_o + o(1)]$ and this tends strongly to $U(t,s)A(s)x_o$ as $h \downarrow o$. Hence $U(t,s)x_o$ has a strong right-hand derivative $(\partial/\partial s)^{+}U(t,s)x_o = U(t,s)A(s)x_o$ for all $x_o \in D$, $o \leq s \leq t \leq \mathcal{T}$. For each $x_o \in D$ $|U(t,s)A(s)x_o|$ is bounded on $o \leq s \leq t \leq \mathcal{T}$ (Assumption (C)). Thus for each x_o D there is a positive number $M_{x_o} < \infty$ such that $|(\partial/\partial s)^{+}U(t,s)x_o| < M_{x_o}$. It follows that $U(t,s)x_o$ is uniformly continuous from the right for each $x_o \in D$. Let $x_o \in D$ and $t \in J$ be fixed, N a positive integer, $S_n = \frac{n}{N}t$, $n = 0, 1, \ldots, N$. We define the step function f_n by $f_n(s) = U(t,s_n)x_o$ for $s_n \leq s < s_{n+1}$, $n = 0, 1, \ldots, N-1$. Because of the uniform continuity from the right $f_n(s) \longrightarrow U(t,s)x_o$ as $N \to +\infty$ for each $s \in [o,t]$. Hence $U(t,s)x_o$ is measurable as a function of s on $[o,t]$ for each $x_o \in D$, $t \in J$. Moreover, if x_n is a sequence of elements of D such that $x_n \longrightarrow x_o \in H$ as $n \longrightarrow +\infty$, then $U(t,s)x_n \longrightarrow U(t,s)x_o$. Hence $U(t,s)x_o$ is measurable as a function of s for all $x_o \in H$.

<u>Theorem 5.1</u> If $f \in \mathcal{B}$ is continuous there is a unique classical solution on $[o, \mathcal{T}]$ to the Cauchy problem

$$\dot{x}(t) + A(t)\,x(t) = f(t) \quad , \quad x(o) = x_o \in D \tag{5.3}$$

given by

$$x(t) = U(t,o)x_o + \int_o^t U(t,s)f(s)ds \tag{5.4}$$

Proof: Since $U(t,s)y$ is measurable as a function of s so is $U(t,s)f(s)$

(Lemma 2.4). Thus the expression for $x(t)$ makes sense. Moreover, for $o \leq s \leq t \leq \mathcal{T}$, $\partial/\partial t\, U(t,s)f(s) = -A(t)U(t,s)f(s)$. Thus, taking the strong derivative of (5.4) we have

$$\dot{x}(t) = -A(t)U(t,o)x_o - \int_o^t A(t)U(t,s)f(s)ds + f(t) = -A(t)x(t) + f(t)$$

which proves the theorem.

<u>Theorem 5.2</u> If $f \in \mathcal{H}$, $x_o \in H$, the strong solution of Cauchy problem $\dot{x} + \mathcal{A}x = f$, $x(o) = x_o$, is given by

$$x(t) = U(t,o)x + \int_o^t U(t,s)f(s)ds \tag{5.5}$$

In particular, if $f \in C(J,H)$ with distribution derivative $\dot{f} \in \mathcal{H}$, $x_o \in D$, then (5.5) gives the classical solution.

Proof: Since the continuous functions in $\mathcal{D}$ are dense in $\mathcal{H}$ there is a sequence f_n of these functions such that $f_n \rightarrow f$ as $n \rightarrow +\infty$. Let Z_n be a sequence in D such that $Z_n \rightarrow x_o$ as $n \rightarrow +\infty$.

Then

$$x_n(t) = U(t,o)Z_n + \int_o^t U(t,s)f_n(s)ds$$

is a sequence of classical solutions such that $\mathcal{T}[x_n(\cdot), Z_n] = [f_n, Z_n]$. Now $\lim_{n \rightarrow +\infty} x_n(t) = U(t,o)x_o + \int_o^t U(t,s)f(s)ds = x(t)$

Hence $\mathcal{T}[x(\cdot), x_o] = [f, x_o]$ since $\mathcal{T}$ is closed; that is, $x(\cdot)$ is the strong solution to the Cauchy problem.

If $f \in C(J,H)$ with $\dot{f} \in \mathcal{H}$, $x_o \in D$, then the strong solution is a classical solution, by Theorem 4.2.

VI. A TIME-DEPENDENT HYPERBOLIC SYSTEM

In this section we present an application of the preceding theory to a time-dependent symmetric hyperbolic system of partial differential equations. Let Ω be an open connected set in m-dimensional real enclidean space R^m, with points $\xi = \left[\xi^1, \xi^2, \dots, \xi^m \right]$. Let $J = \left[0, \tau \right]$ be the time interval for which we are to find a solution. $x(\xi, t)$ will denote a function on $\Omega \times J$ with values in k-dimensional complex enclidean space C^k. We shall consider the initial value problem

$$\dot{x} = E^{-1} \left[(\beta A^i x)_i + B x \right], \quad x(\xi, 0) = x_0(\xi) \qquad (6.1)$$

where the usual summation convention is used for repeated indices, and the subscript i denotes differentiation with respect to ξ_i. The symbols E, A^i, and B represent $k \times k$ matrix-valued functions, E and A^i being functions of ξ alone, B being a function of ξ and t. β is a positive scalar function of ξ and t. E is positive definite and the A^i are hermitian. We assume that the elements of E are continuous on Ω; the elements of A^i are continuous and bounded on Ω and continuously differentiable with respect to ξ^i; the elements of A^i_i and B are continuous and bounded, and the elements of B have continuous derivatives with respect to t which are bounded on $\Omega \times J$. We assume that $\beta(\xi, t)$ has continuous bounded derivatives with respect to t and the ξ^i, β_i has a continuous bounded derivative with respect to t, and there is a positive constant β_0 such that $\beta(\xi, t) \geq \beta_0 > 0$ for all ξ, t. For $x, y \in C^k$ we denote the inner product $x^i \bar{y}^i$ by

$\{x, y\}$, x^i, y^i, $i = 1, 2, \ldots, k$, being the components of x, y, respectively. We assume that for all $x \in C^k$, $\xi \in \Omega$ we have

$$\{ A^i_i (\xi) x, x \} \leq 0 \tag{6.2}$$

First, however, we consider the time-independent system

$$\dot{x} = E^{-1} (A^i x)_i \tag{6.3}$$

This system has been given a physical interpretation by Wilcox [8]. The form $\eta = \frac{1}{2} \{ E (\xi) x, x \}$ is interpreted as an "energy density." The forms $\Sigma^i = -\frac{1}{2} \{ A^i (\xi) x, x \}$ are interpreted as the components of a "Poynting vector" describing the flow of power. Thus $\Sigma^i / \eta = v^i$ may be interpreted as the velocity of energy flow. The speed at which energy propagates is bounded above by

$$c = \sup \frac{- \{ A^i (\xi) x, x \} n^i}{\{ E (\xi) x, x \}}$$

where the supremum is over $x \in C^k$, $\xi \in \Omega$, and unit vectors n^i ($n^i n^i = 1$). In order to assure that c be finite we assume that there exists a constant $\rho > 0$ such that

$$\{ E (\xi) x, x \} \geq \rho \{x, x\} \tag{6.5}$$

for all $\xi \in \Omega$, $x \in C^k$. Then $c < + \infty$ since the elements of A^i are bounded.

Using the energy as a norm we are led to the Hilbert space H with inner product

$$(x, y) = \int_\Omega \{ E (\xi) x (\xi), y (\xi) \} d \xi \tag{6.6}$$

The Hilbert space of functions x on Ω with values in C^k such that

$$(x, x)_{L_2} = \int_\Omega \{ x(\xi), x(\xi) \} \, d\xi < +\infty \tag{6.7}$$

will be denoted by $L_2(\Omega)$. Because of (6.5) we have

$$(x, x) = (E x, x)_{L_2} \geq \rho (x, x)_{L_2} . \tag{6.8}$$

It follows that E^{-1} is a bounded operator on $L_2(\Omega)$. We shall denote its norm by $\| E^{-1} \|_{L_2}$. We have $\| E^{-1} \|_{L_2} \leq 1/\rho$.

Let $C^n(\Omega)$ be the set of n - times continuously differentiable functions on Ω with values in C^k. $C^n_o(\Omega)$ will be the subset of functions with compact support in Ω.

Let L_{oo} be the linear operator with domain $C^1_o(\Omega)$ defined by

$$L_{oo} \, x(\xi) = -E^{-1}(\xi) \, (A^i(\xi) \, x(\xi))_i \tag{6.9}$$

Assuming for the moment that the boundary of Ω ($\partial\Omega$) is sufficiently regular we define an outward normal $\nu = [\nu^1, \nu^2, \ldots, \nu^m]$, $\nu^i \nu^i = 1$. Then by the divergence theorem we write

$$\int_\Omega \{ A^i x, y \}_i \, d\xi = \int_{\partial\Omega} \{ A^i x, y \} \, \nu^i \, d\sigma \tag{6.10}$$

for any $x, y \in C^1(\Omega) \cap L_2(\Omega)$. Thus for $x, y \in C^1_o(\Omega)$ we have

$$\int_\Omega \{ (A^i x)_i , y \} \, d\xi = - \int_\Omega \{ x, A^i y_i \} \, d\xi \tag{6.11}$$

We observe that (6.11) holds even if $\partial\Omega$ is not regular since we can enclose the support of x or y in a cube and apply the divergence theorem to this region. Defining the linear operator M_{oo} with domain $C^1_o(\Omega)$ by

$$M_{oo} \, y = E^{-1} A^i y_i \tag{6.12}$$

we then have

$$(L_{00}\, x, y) = (x, M_{00}\, y) \tag{6.13}$$

Since $C_0^1(\Omega)$ is dense in $L_2(\Omega)$, hence a fortiori dense in H, it follows that L_{00} and M_{00} are closeable in H. Let their closures be denoted respectively by L_0 and M_0. Let $L_1 = M_0^*$, $M_1 = L_0^*$. Clearly $L_1 \supset L_0$ and $M_1 \supset M_0$.

Again supposing for the moment that $\partial\Omega$ is sufficiently regular we see from (6.10) that for $x \in C^1(\Omega) \cap L_2(\Omega)$ we have

$$2\,\mathrm{Re}\int_\Omega \{(A^i x)_i , x\}\, d\xi = \int_\Omega \{A^i_i x, x\}\, d\xi + \int_{\partial\Omega} \{A^i x, x\}\, \nu^i d\sigma$$

or (6.14)

$$2\,\mathrm{Re}\,(L_1 x, x) = -\int_\Omega \{A^i_i x, x\}\, d\xi - \int_{\partial\Omega} \{A^i x, x\}\, \nu^i\, d\sigma$$

Because of (6.2) we then have

$$2\,\mathrm{Re}\,(L_1 x, x) \geq \int_{\partial\Omega} \Sigma^i \nu^i\, d\sigma \tag{6.15}$$

If the flow of energy is outward at each point of $\partial\Omega$, then, $\mathrm{Re}\,(L_1 x, x) \geq 0$. This motivates the following definition.

<u>Definition</u>. A closed dissipative linear operator L such that $L_0 \subset L \subset L_1$ will be said to be locally dissipative if for each $x \in \mathcal{D}(L)$ $\phi x \in \mathcal{D}(L)$ for all continuously differentiable real ϕ with compact support in R^m.

We assume that L and its adjoint M are locally dissipative. This implies that L and M are maximal dissipative, $L_0 \subset L \subset L_1$ and $M_0 \subset M \subset M_1$.

Now, for arbitrary $t \in J$, let the linear operators $L(t)$ with domain $\mathcal{D}(L)$ and $M(t)$ with domain $\mathcal{D}(M)$ be defined respectively by

$$L(t)x = -E^{-1}(\cdot)(\beta(\cdot,t)A^i(\cdot)x(\cdot))_i \tag{6.16}$$

$$M(t)x = E^{-1}(\cdot)\beta(\cdot,t)A^i(\cdot)x_i(\cdot)$$

Let the linear operator $C(t)$ be defined by

$$C(t)x = \tfrac{1}{2}E^{-1}(\cdot)\beta_i(\cdot,t)A^i(\cdot)x(\cdot) \tag{6.17}$$

Since the elements of A^i and β_i are bounded continuous functions on Ω $C(t)$ is a bounded linear operator on H.

<u>Theorem 6.1.</u> $L(t) + C(t)$ is a maximal dissipative linear operator with dense domain $\mathcal{D}(L)$ in H, and $M(t) + C(t)$ is its (maximal dissipative) adjoint.

Proof: Because $\beta(\xi,t) \geq \beta_0 > 0$ and is continuous and bounded on $\Omega \times J$ the mapping $x \to \beta x$ is a one-to-one continuous mapping of H onto itself. We have for $x \in \mathcal{D}(L)$

$$\begin{aligned} L(t)x &= -\beta E^{-1}(A^i x)_i - E^{-1}\beta_i A^i x \\ &= \beta L x - 2C(t)x \end{aligned}$$

Hence $L(t)$ is closed. Similarly we have, for $y \in \mathcal{D}(M)$,

$$M(t)\,y = \beta M y$$

Thus $M(t)$ is closed. We note that $L^*(t) = M\beta - 2C$, $M^*(t) = L\beta$. That is, $z \in \mathcal{D}(L^*(t))$ if and only if $\beta z \in \mathcal{D}(M)$ and $z \in \mathcal{D}(M^*(t))$ if and only if $\beta z \in \mathcal{D}(L)$.

Now let ψ be a continuously differentiable positive function with compact support on $[0, +\infty)$, $\psi(t) = 1$ for $0 \le t \le 1$. For $\epsilon > 0$ define $\beta_\epsilon(\xi, t) = \psi_\epsilon(\epsilon|\xi|)\,\beta(\xi, t)$ where $|\xi| = \left[\xi^i \xi^i\right]^{\frac{1}{2}}$. Then for any $y \in H$, $x \in \mathcal{D}(L)$ we have

$$\int_\Omega \{(\beta_\epsilon A^i x)_i, y\}\, d\xi = \int_\Omega \psi\, \{(\beta A^i x)_i, y\}\, d\xi + \epsilon \int_\Omega \{\beta A^i x, y\}\, |\xi|_i\, \psi^1\, d\xi$$

Hence as $\epsilon \longrightarrow 0$

$$\int_\Omega \{(\beta_\epsilon A^i x)_i, y\}\, d\xi \longrightarrow \int_\Omega \{(\beta A^i x)_i, y\}\, d\xi$$

Now since $\beta_\epsilon x \in \mathcal{D}(L)$ for each $x \in \mathcal{D}(L)$ we have

$$-\int_\Omega \{(A^i \beta_\epsilon x)_i, y\}\, d\xi = \int_\Omega \{\beta_\epsilon x, A^i y_i\}\, d\xi$$

for each $x \in \mathcal{D}(L)$, $y \in \mathcal{D}(M)$. Hence, letting $\longrightarrow 0$ we have

$$-\int_\Omega \{(A^i \beta x)_i, y\}\, d\xi = \int_\Omega \{x, \beta A^i y_i\}\, d\xi$$

or $(L(t)x, y) = (x, M(t)y)$ for all $x \in \mathcal{D}(L)$, $y \in \mathcal{D}(M)$. Hence $L(t) \subset M^*(t)$ and $M(t) \subset L^*(t)$.

Since $\sqrt{\beta_\epsilon}\, x \in \mathcal{D}(L)$ for $x \in \mathcal{D}(L)$ we have

$$0 \le \operatorname{Re} \int_\Omega \{ - (A^i \sqrt{\beta_\epsilon}\, x)_i , \sqrt{\beta_\epsilon}\, x \} \, d\xi$$

$$= \operatorname{Re} \int_\Omega \{ - (\beta_\epsilon A^i x)_i , x \} \, d\xi$$

$$+ \tfrac{1}{2} \int_\Omega \{ A^i x , x \} \beta_{\epsilon i} \, d\xi$$

$$= \operatorname{Re} \int_\Omega \{ - (\beta_\epsilon A^i x)_i , x \} \, d\xi$$

$$+ \tfrac{1}{2} \int_\Omega \psi \{ A^i x , x \} \beta_i \, d\xi$$

$$+ \tfrac{1}{2} \epsilon \int_\Omega \{ A^i x , x \} | \xi |_i \beta \psi^1 \, d\xi$$

Taking the limit as $\epsilon \longrightarrow 0$ we have $\operatorname{Re} (L(t) x , x) + (C(t) x , x) \ge 0$

i.e. $L(t) + C(t)$ is dissipative.

Similarly, since $\sqrt{\beta_\epsilon}\, y \in \mathcal{D}(M)$ for $y \in \mathcal{D}(M)$ we have

$$0 \le \operatorname{Re} \int_\Omega \{ A^i (\sqrt{\beta_\epsilon}\, y)_i , \sqrt{\beta_\epsilon}\, y \} \, d\xi$$

$$= \operatorname{Re} \int_\Omega \{ \beta_\epsilon A^i y_i , y \} \, d\xi + \tfrac{1}{2} \int_\Omega \{ A^i y , y \} \beta_{\epsilon i} \, d\xi$$

As before, letting $\epsilon \longrightarrow 0$ we obtain

$$\operatorname{Re} (M(t) y, y) + (C(t) y, y) \ge 0$$

Thus $M(t) + C(t)$ is dissipative.

Suppose $L(t) + C(t)$ is not maximal dissipative. Then $\mathcal{R}(L(t) + C(t) + \alpha I)$ is a closed proper subspace of H for any $\alpha > 0$. (I is the identity operator on H). Thus for each $\alpha > 0$

there is a $z \in \mathcal{D}(L^*(t))$, $(z, z) = 1$ such that $(L^*(t) + C(t) + \alpha I)z = 0$. Or $M\beta z - C(t)z + \alpha z = 0$. Hence $\operatorname{Re}(M\beta z, \beta z) + \alpha(z, \beta z) = (C(t)z, \beta z)$. Now $\operatorname{Re}(M\beta z, \beta z) \geq 0$, and $\alpha(z, \beta z) \geq \alpha\beta_o(z, z) = \alpha\beta_o$ Thus $\alpha\beta_o \leq (C(t)z, \beta z)$. But the right side of the inequality is bounded. Taking sufficiently large then leads to a contradiction. Similarly, supposing $M(t) + C(t)$ not to be maximal dissipative, for $\alpha > 0$ there is $z \in \mathcal{D}(M^*(t))$ such that $(z, z) = 1$ and $(M^*(t) + C(t) + \alpha I)z = 0$. Hence $L\beta z + C(t)z + \alpha z = 0$, and $\operatorname{Re}(L\beta z, \beta z) + \alpha(z, \beta z) = -(C(t)z, \beta z)$. As before this implies $\alpha\beta_o \leq -(C(t)z, \beta z)$, leading to a contradiction. It follows that $M(t) + C(t)$ is maximal dissipative. And since $L^*(t) + C(t)$ is a dissipative extension of $M(t) + C(t)$ we have $L^*(t) + C(t) = M(t) + C(t)$. Hence $M(t) = L^*(t)$ and $L(t) = M^*(t)$.

Let $L_2(J, H)$ be the space of Lebesgue measurable functions on J with values in H whose norms are square integrable. $C^n(J, H)$ will denote the space of strongly continuous functions on J with values in H which have n strongly continuous strong derivatives. Since the elements of $B(\xi, t)$ (as in (6.1)) are bounded and continuous on $\Omega \times J$ the linear mapping $x \longrightarrow E^{-1} Bx$ is continuous on H into itself. We denote its norm by $\| E^{-1}_{(\cdot)} B(\cdot, t) \|$ Take $\alpha > 0$ and take

$$k \geq \alpha + \sup_{t \in J} \| E^{-1}_{(\cdot)} B(\cdot, t) \| + \sup_{t \in J} \| C(t) \|$$

For $x \in \mathcal{D}(L)$ we write

$$A(t)x = -E^{-1}_{(\cdot)} \left[(\beta(\cdot, t) A^i(\cdot) x(\cdot))_i + B(\cdot, t) x(\cdot) \right. + kx$$

$$= L(t)x + E^{-1} Bx + kx$$

Then $A(t)$ is maximal dissipative for each $t \in J$, with the dense domain $\mathcal{D}(L)$, and $\operatorname{Re}(A(t)x, x) \geq \alpha(x, x)$. (Theorem 3.3). Moreover, for each $x \in \mathcal{D}(L)$

$$\dot{A}(t)x = -E^{-1}_{(\cdot)} \left(\frac{\partial\beta}{\partial t}(\cdot, t) A^i(\cdot) x(\cdot) \right)_i + \frac{\partial}{\partial t} B(\cdot, t) x(\cdot)$$

Since β, β_i, and the elements of B have continuous bounded derivates with respect to t it follows that $A(t)$ satisfies assumption (C). Hence we have

<u>Theorem 6.2.</u> The Cauchy problem

$$\dot{x} + A(t)x = f(t), \quad x(0) = x_0 \tag{6.18}$$

has a unique solution $x \in C^0(J, H)$ for each $f \in L_2(J, H)$, $x_0 \in H$. In general x is a strong solution, but if $f \in C^0(J, H)$ with $f(t) \in \mathcal{D}(L)$ for each $t \in J$ and $x_0 \in \mathcal{D}(L)$, or if $f \in C^1(J, H)$ and $x_0 \in \mathcal{D}(L)$, then x is a classical solution.

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