LOW REYNOLDS NUMBER FLOW PAST FINITE CYLINDERS OF LARGE ASPECT RATIO

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ABSTRACT

Low Reynolds number flow past finite cylinders of large aspect ratio is considered in this thesis. The first cylinder under consideration consists of a finite cylinder of constant radius λ with two hemispherical caps attached to each end. The axis of the cylinder is perpendicular to uniform flow at infinity and the half length of the cylinder is L. Therefore, two Reynolds numbers can be formed in the present study, namely, $Re = \frac{U\lambda}{\nu}$ and $\overline{Re} = \frac{UL}{\nu}$. The low Reynolds number flow is studied in the limit $Re \to 0$ for $\overline{Re}$ = fixed. This clearly shows that the body is of large aspect ratio.

The other cylinder under consideration is an ellipsoid of revolution whose half-axis parallel to the flow is λ and whose half-axis perpendicular to the flow is L. The same limiting case as that for the first finite cylinder is studied.

Asymptotic expansions of the solution valid for the limiting case $Re \to 0$ are obtained by applying singular perturbation procedures with proper use of the idea of the intermediate solution. The nature of the end source variation and the order of magnitude of various effects are clarified in the present study. In addition, certain general remarks have been made about the difference between the end effects for "tapered" and "untapered" bodies. It is found that the "taper" at the ends plays an essential role.

LIST OF PRINCIPAL SYMBOLS

1. Dimensional Variables and Parameters

(x, y, z)	=	Cartesian coordinates
$\vec{i}, \vec{j}, \vec{k}$	=	Unit vector along x, y, z direction respectively
p	=	pressure
p_∞	=	pressure at infinity
$\vec{q}$	=	flow velocity
$U\vec{i}$	=	velocity at infinity
ρ	=	density = constant
μ	=	viscosity = constant
ν	=	kinematical viscosity = $\frac{\mu}{\rho}$ = constant
T	=	temperature
T_∞	=	temperature at infinity
T_c	=	temperature on the surface of the cylinder
λ	=	maximum radius of the cylinders or the sphere
L	=	half length of the finite cylinder on the major axis of the ellipsoid
D	=	drag force on the cylinders
R^2	=	$x^2 + y^2 + z^2$
r^2	=	$x^2 + y^2$
p^1	=	$p - p_\infty$
$\vec{a}$	=	constant vector
k	=	$\frac{U}{2\nu}$
(x_i)	=	(x, y, z)

2) General non-dimensional variables and parameters

$\tilde{x} = \frac{Ux}{\nu}$, $\tilde{y} = \frac{Uy}{\nu}$, $\tilde{z} = \frac{Uz}{\nu}$

$x^{(f)} = \frac{\tilde{x}}{f(Re)}$, $y^{(f)} = \frac{\tilde{y}}{f(Re)}$, $z^{(f)} = \frac{\tilde{z}}{f(Re)}$

$\vec{q}^{\,*} = \frac{\vec{q}}{U}$

$p^{*} = \frac{p-p_{\infty}}{\rho U^{2}}$

$p^{+} = \frac{\lambda(p-p_{\infty})}{\mu U} = Re\, p^{*}$

$p^{++} = Re^{2} p^{*} = Re\, p^{+}$

$p^{(f)} = f(Re)\, p^{*}$

$Re = \frac{U\lambda}{\nu}$

$\overline{Re} = \frac{UL}{\nu}$

C_D = non-dimensional drag coefficient

$\vec{q}_o$ = leading term of uniformly valid expansion

$\vec{q}_n$ = terms in uniformly valid expansion or composite expansion

$\epsilon_j(Re)$ = asymptotic sequence of function of Re

$\vec{s}_o$ = an expansion uniformly valid to order unity near the body

3) Variables and parameters for the Laplace problem

φ = solution of the exact Laplace problem

φ_A = solution of the approximate Laplace problem

$S(\lambda)$ = surface of the finite cylinder

φ_o = outer limit

φ_s = shank limit

$x^* = \frac{x}{\lambda}$, $y^* = \frac{y}{\lambda}$, $z^* = \frac{z+L}{\lambda}$

$x^+ = \frac{x}{\lambda^2}$, $y^+ = \frac{y}{\lambda^2}$, $z^+ = \frac{z+L}{\lambda^2}$

φ_e = constant source distribution solution for the ellipsoid

φ_c = constant source distribution solution for the finite cylinder

$a = L - \lambda$

$z_\alpha = \frac{z+L}{\lambda^\alpha}$ and $0 \leq \alpha \leq 1$

$$\epsilon(\lambda) = \frac{1}{\log \frac{1}{\lambda} + \log 2L}$$

4) Variables and parameters for the low Reynolds number flow past the finite cylinder

$\vec{g}_n$ = flow velocity term in outer expansion

p^*_n = pressure term in outer expansion

$\vec{u}_n$ = flow velocity term in the intermediate shank expansion

p_n = pressure term in the intermediate shank expansion

$\vec{v}_n$ = flow velocity term in the intermediate left end expansion

p^+_n = pressure term in the intermediate left end expansion

$\vec{w}_n$ = flow velocity term in the intermediate right end expansion

p_n^{++} = pressure term in the intermediate right end expansion

$x^* = \frac{\tilde{x}}{Re}$, $y^* = \frac{\tilde{y}}{Re}$, $z^* = \frac{\tilde{z} + \overline{Re}}{Re}$

$\overline{z}^* = \frac{\tilde{z} - \overline{Re}}{Re}$

$\epsilon(Re) = \frac{1}{\log \frac{1}{Re}}$

5) Variables and parameters for the low Reynolds number flow past the ellipsoid cylinder

$\vec{g}_n$ p_n^* $\vec{u}_n$ p_n $\vec{v}_n$ $\vec{w}_n$ are the same as for the finite cylinder case

p_n^l = pressure term in the intermediate left end expansion

p_n^{ll} = pressure term in the intermediate right end expansion

$x^+ = \frac{\tilde{x}}{Re^2}$, $y^+ = \frac{\tilde{x}}{Re^2}$, $z^+ = \frac{\tilde{z} + \overline{Re}}{Re^2}$

$\overline{z}^+ = \frac{\tilde{z} - \overline{Re}}{Re^2}$

$\tau = \frac{1}{2}\{\sqrt{r^{+2} + (z^+ - \tau_o)^2} - (z^+ - \tau_o)\}$

$\tau_o = \frac{1}{2\overline{Re}}$

$\tau^+ = \frac{1}{2}\{\sqrt{(\overline{z}^+ + \tau_o)^2 + r^{+2}} + (\overline{z}^+ + \tau_o)\}$

$\epsilon(Re) = \frac{1}{\log \frac{4}{Re} + \gamma - \frac{1}{2}}$

6) Mathematical symbols

$\underline{\underline{\Gamma}}$ = fundamental solution tensor for $\vec{q}$

$\vec{\pi}$ = fundamental solution vector for p

$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

K_o = modified Bessel function of the second kind of the zeroth order

$E_1(x) = -Ei(-x)$ and Ei is exponential integral

$\dot{\varphi}(\eta) = \frac{d\varphi}{d\eta}$

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I. INTRODUCTION

Viscous flow at low Reynolds numbers past two dimensional or three dimensional objects has been studied extensively in the literature with the aid of the Stokes equations and Oseen equations. Recent work by Kaplun, Lagerstrom and others (Ref. 1, 2, 3 and 4) has clarified the relation of these solutions to asymptotic expansions of solutions of the Navier-Stokes equations. These solutions exhibit a marked difference between the two dimensional and three dimensional case. The purpose of the present investigation is to clarify this difference, in particular, to study in detail the transition from the three dimensional case to the two dimensional case. For this purpose we consider flow past bodies of large aspect ratio, i.e., bodies whose extension transverse to the flow is much larger than that parallel to the flow. As a typical example, we consider a body (see Fig. 4.1) consisting of a finite cylinder of radius λ and axis perpendicular to the flow which has a hemispherical cap at each end, the extension transverse to the flow is 2L. This body will be called a finite circular cylinder. Two Reynolds numbers may be formed, namely, $Re = \frac{U\lambda}{\nu}$ and $\overline{Re} = \frac{UL}{\nu}$. We study the limit of Re tending to zero, $\overline{Re}$ being fixed. Clearly this is a body of very large aspect ratio; the case $\overline{Re}$ = infinity corresponds to two dimensional flow. As another example, we study an ellipsoid of revolution whose half-axis parallel to the flow is λ and whose half-axis perpendicular to the flow is L. The same limiting case as for the finite circular cylinder is studied. The generality of the results and difference between the two cases is discussed; it is found that the "taper" at the ends plays an essential role.

It is assumed that the flow is viscous incompressible and stationary. The classical Navier-Stokes equations are thus the governing equations. The domain of the fluid is infinite and it is assumed that there are no other boundaries except that of the given cylinder or ellipsoids.

The mathematical method used is that of singular perturbations as discussed in Ref. 1, 2, 3 and 4. This involves finding leading terms of expansions valid in different regions, using appropriately scaled variables. From these expansions one may form a composite expansion which is asymptotically valid uniformly in the entire flow field as Re tends to zero.

In Chapter 2 a resume is given of the appropriate fundamental solutions of the Stokes equations and the Oseen equations. In addition, a method for generating solutions of Stokes equations from the corresponding solution of Laplace equations is discussed; this method will later prove to be very useful. The asymptotical solution for typical cases in two and three dimensions are reviewed in Chapter 3 essentially following the work of Lagerstrom and Kaplun. Chapter 4 discusses solutions for the three dimensional Laplace equation with boundary conditions given on high aspect ratio bodies. These examples are mathematical models in the sense that the essential ideas regarding high aspect bodies are exhibited clearly for the case of a relatively simple equation. Actually, however, it will be seen later that for the same high aspect ratio body, the asymptotical Laplace solution furnishes an essential element in constructing the asymptotical Navier-Stokes solutions. These solutions are discussed in Chapter 5 (finite cylinder) and Chapter 6 (ellipsoid). The essential results are discussed in Chapter 7.

The present problem is more complicated that that of a three dimensional sphere or two dimensional cylinder because the classical inner limit is not uniform even near the body. Thus, we have to introduce several (more than two) simultaneous expansions i. e., an "outer expansion", a "shank expansion" and two "end expansions." The proper choice of variables for each expansion is discussed in detail in the geometrical matching condition established in Chapter 4. The details of expansion procedure and the matching between them present a certain interest, although certainly no new principles need to be introduced. In Chapter 5, the expansion procedures for the ellipsoid has been exhibited in detail and higher order terms are obtained. It is worth mention that the idea of an intermediate (rather than an inner) expansion is intimately involved and quite helpful in the present case.

II. OSEEN AND STOKES EQUATIONS AND THEIR FUNDAMENTAL SOLUTIONS

The role of the Oseen and Stokes equations for the study of flow at low Reynolds numbers will be discussed in subsequent chapters. For future reference some relevant formulas relating to these equations are given in the present chapter.

2.1 Oseen Equations and Their Fundamental Solutions

The Oseen equations may be derived by linearizing the Navier-Stokes equations (cf. 3-1) about the free stream velocity $U\vec{i}$. The resulting equations are then

$$U \frac{\partial \vec{q}}{\partial x} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \vec{q} \tag{2-1a}$$

$$\nabla \cdot \vec{q} = 0 \tag{2-1b}$$

A very useful concept in the study of linear differential equations is that of fundamental solutions. The fundamental solution may be defined as the response to a singular force. The fundamental solution for Oseen equations is a solution of Oseen equations (2-1a, b) with a force term $\vec{f}$ added on the right side of (2-1a). No external boundaries are present and the force per unit mass is a delta function. More precisely, let $\vec{f}$ be concentrated at a point Q and equal to $\delta(P;Q)\vec{a}$ where $\vec{a}$ is a given constant vector and the delta function $\delta(P;Q)$ is a function of the point P which is zero for $P \neq Q$, infinity for $P = Q$ and whose integral over any domain including Q is unity. Then the resultant velocity field at P, in the absence of boundaries, is determined from the fundamental tensor $\underline{\underline{\Gamma}}(P;Q)$ by

$$\vec{q}\,(P) = \underline{\underline{\Gamma}}\,(P;Q)\,\vec{a} \tag{2-3}$$

and the perturbation pressure $p' = p - p_\infty$ from the fundamental vector $\vec{\pi}$ (P;Q) by

$$p'(p) = \vec{\pi}\,(P;Q)\,\vec{a} \tag{2-4}$$

one may consider either the two dimensional or three dimensional case.

From the linearity of the Oseen equations it follows easily that $\underline{\underline{\Gamma}}$ and $\vec{\pi}$ are linear functions of $\vec{a}$; i.e., actually a tensor and a vector respectively. If a system of coordinates is chosen $\underline{\underline{\Gamma}}$ may then be represented as a matrix (Γ_{ij}). In cartesian coordinates $(\Gamma_{11}, \Gamma_{21}, \Gamma_{31})$ is the velocity field due to a unit force directed along the x-axis $(\vec{a} = \vec{i})$.

Furthermore, due to the linearity of Oseen equations, superposition of the fundamental solutions can be used to determine the effect of distributed forces. This idea of superposition will be frequently used in the subsequent discussion.

The following are the summarized results of the fundamental solution for three dimensional Oseen equations.

$$\vec{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = -\frac{\rho}{4\pi}\,\text{grad}\,\frac{1}{R} \tag{2-5}$$

and

$$\underline{\underline{\Gamma}} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} & \frac{\partial A}{\partial z} \\ \frac{\partial A}{\partial y} & 0 & 0 \\ \frac{\partial A}{\partial z} & 0 & -\frac{\partial A}{\partial x} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial B}{\partial y} & \frac{\partial B}{\partial z} \\ 0 & \frac{\partial B}{\partial z} & -\frac{\partial B}{\partial y} \end{pmatrix} + \frac{e^{-k(R-x)}}{4\pi\nu R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2\text{-}6)$$

Here

$$k = \frac{U}{2\nu}\ , \quad R^2 = x^2 + y^2 + z^2$$

$$A = \frac{1}{4\pi U}\left[\frac{1}{R} - \frac{e^{-k(R-x)}}{R}\right]$$

$$B = -\frac{1}{4\pi U}\left[(1-e^{-k(R-x)})\frac{\partial}{\partial y}\log(R-x)\right] \qquad (2\text{-}7)$$

The corresponding two dimensional case can be obtained by the method of descent and is summarized as follows:

$$\vec{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{\rho}{2\pi}\,\mathrm{grad}\,(\log r) \qquad (2\text{-}8)$$

where

$$r^2 = x^2 + y^2,$$

$$\underline{\underline{\Gamma}} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \\ \frac{\partial A}{\partial y} & -\frac{\partial A}{\partial x} \end{pmatrix}$$

$$+ \frac{1}{2\pi\nu}\ e^{kx}\,K_o(kr)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad (2\text{-}9)$$

and

$$A = \frac{1}{2\pi U}[\log r + e^{kx}K_o(kr)]$$

2.2 Stokes Equations and their Fundamental Solutions.

The Stokes equation may be derived formally by linearizing the Navier-Stokes equation about the value $\vec{q} = 0$. The resulting equations are:

$$\nabla p = \mu \nabla^2 \vec{q} \qquad \text{(2-10a)}$$

$$\nabla \cdot \vec{q} = 0 \qquad \text{(2-10b)}$$

Similarly the fundamental solutions for three dimensional Stokes equations can be summarized as follows:

$$\underline{\underline{\Gamma}} = \frac{1}{8\pi\mu} \begin{pmatrix} \frac{1}{R} + \frac{x^2}{R^3} & \frac{xy}{R^3} & \frac{xz}{R^3} \\ \frac{xy}{R^3} & \frac{1}{R} + \frac{y^2}{R^3} & \frac{yz}{R^3} \\ \frac{xz}{R^3} & \frac{yz}{R^3} & \frac{1}{R} + \frac{z^2}{R^3} \end{pmatrix} \qquad \text{(2-11)}$$

and

$$\vec{\pi} = -\frac{1}{4\pi} \operatorname{grad} \frac{1}{R} \qquad \text{(2-12)}$$

The corresponding two dimensional fundamental solution may be obtained by the method of descent as follows:

$$\underline{\underline{\Gamma}} = -\frac{1}{4\pi\mu} \begin{pmatrix} \log r - \frac{x^2}{r^2} & -\frac{xy}{r^2} \\ -\frac{xy}{r^2} & \log r - \frac{y^2}{r^2} \end{pmatrix} \qquad \text{(2-13)}$$

and

$$\vec{\pi} = \frac{1}{2\pi} \text{ grad } (\log r) \qquad (2\text{-}14)$$

It is worth noting that the above solution in (2-13,14) does not die out as $r \to \infty$ and thus the meaning of the fundamental solution is not clear. But we can still define it by use of three dimensional solutions and calculate it by assuming an infinite line singular force lying along the z-axis but acting perpendicular to the z-axis. Thus the fundamental solutions for two dimensions are obtained as in equations (2-13) and (2-14).

2.3 Method of Generating Solutions for Stokes Equations from Laplace Solutions.

Let $\varphi(x_i)$ be a solution of the Laplace equation

$$\nabla^2 \varphi = 0 \qquad (2\text{-}15)$$

Then the following solution generated by $\varphi(x_i)$

$$\vec{q} = \vec{i}\, \varphi(x_i) - x\, \nabla\varphi(x_i) \qquad (2\text{-}16a)$$

$$p = -2\mu \frac{\partial \varphi}{\partial x} \qquad (2\text{-}16b)$$

is a solution of Stokes equations (2-10). This result, valid for two or three dimensions, can easily be proved by substituting (2-16) into the Stokes equations. The fundamental solutions of the Stokes equations as given above may be derived from the corresponding fundamental solutions of the Laplace equation with the aid of (2-16) and formulas obtained from (2-16) by replacing x by y or z.

A stronger theorem is the following: Assume that the surface of a solid is given by

$$\eta(x_i) = \eta_o = \text{constant} \tag{2-17}$$

and that φ_1 and φ_2 are functions of one variable such that

$$\nabla^2 \varphi_1(\eta) = 0 \tag{2-18a}$$

$$\nabla^2 [x\, \varphi_2(\eta)] = 0 \tag{2-18b}$$

Then constants A and B may be found such that

$$\vec{q} = \vec{i} - A\{\vec{i}\, \varphi_1(\eta) - x \quad \varphi_1(\eta)\}$$

$$-B\, \nabla\, [x\, \varphi_2(\eta)] \tag{2-19a}$$

$$p = 2A\mu \frac{\partial \varphi_1}{\partial x} \tag{2-19b}$$

is a solution of Stokes equations such that $\vec{q} = 0$ on $\eta(x_i) = \eta_o$ = constant.

Proof:

a) $\vec{q}$ satisfies the Stokes equations (2-10a, b)

b) on the body $\eta = \eta_o$ = constant, we have

$$\vec{q} = \vec{i} - \vec{i}\, [A\varphi_1(\eta_o) + B\varphi_2(\eta_o)]$$

$$-[A\, \dot{\varphi}_1(\eta_o) - B\dot{\varphi}_2(\eta_o)]x\, \nabla\, \eta \tag{2-20}$$

The boundary condition $\vec{q} = 0$ on the body can be satisfied by putting

$$A\varphi_1(\eta_o) + B\varphi_2(\eta_o) = 1 \tag{2-21a}$$

$$A\dot{\varphi}_1(\eta_o) - B\dot{\varphi}_2(\eta_o) = 0 \qquad (2\text{-}21b)$$

A and B can be determined independently if we have

$$\begin{vmatrix} \varphi_1(\eta_o) & \varphi_2(\eta_o) \\ \dot{\varphi}_1(\eta_o) & - \dot{\varphi}_2(\eta_o) \end{vmatrix} \neq 0 \qquad (2\text{-}22)$$

If this condition is satisfied and $\varphi_1(\eta)$ and $x\varphi_2(\eta)$ can be found, the Stokes solution $\vec{q}$ can be generated from the Laplace solution. In general, for a finite three dimensional body we have $\vec{q} = \vec{i}$, $p = 0$ at infinity.

Example 1) For a sphere, we have

$$\varphi_1(R) = \frac{1}{R} \qquad (2\text{-}23a)$$

and

$$x\varphi_2(R) = \frac{\partial\varphi_1}{\partial x} = -\frac{x}{R^3} \qquad (2\text{-}23b)$$

Then by the above method, we obtain

$$\vec{q} = \vec{i} - \frac{3}{4}\left[\frac{\vec{i}}{R} - x\nabla\frac{1}{R}\right] + \frac{1}{4}\nabla\left(-\frac{x}{R^3}\right) \qquad (2\text{-}24a)$$

$$p = \frac{3}{2}\mu\frac{\partial}{\partial x}\frac{1}{R} \qquad (2\text{-}24b)$$

and it is obvious that

$$\vec{q} = 0 \qquad \text{at } R = 1 \qquad (2\text{-}24c)$$

$$\vec{q} = \vec{i}, \; p = 0 \qquad \text{at infinity} \qquad (2\text{-}24d)$$

Example 2) Similarly for an ellipsoid (cf. ref. 7), we have

$$\varphi_1(\eta) = abc \int_{\eta}^{\infty} \frac{ds}{w(s)} \tag{2-25a}$$

and

$$x\varphi_2(\eta) = \frac{\partial \Omega}{\partial x} = 2\pi abc \int_{\eta}^{\infty} \frac{xds}{(a^2+s)w(s)} \tag{2-25b}$$

where

$$\Omega = \pi abc \int_{\eta}^{\infty} \left(\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{c^2+s} - 1\right) \frac{ds}{w(s)} \tag{2-25c}$$

$$w(s) = \sqrt{(a^2+s)\,(b^2+s)\,(c^2+s)} \tag{2-25d}$$

and η is the positive square root of

$$\frac{x^2}{a^2+\eta} + \frac{y^2}{b^2+\eta} + \frac{z^2}{c^2+\eta} = 1 \tag{2-25e}$$

But in general for a semi-infinite body or a two dimensional body, the boundary condition $\vec{q} = \vec{i}$ at infinity is no longer satisfied and in fact $\vec{q}$ grows to infinity at infinity. Even if it is so it will be seen that the generated Stokes solution can be used to form the so-called intermediate solution for various bodies.

Example 1) For a two dimensional cylinder, we have

$$\varphi_1(r) = \log r \tag{2-26a}$$

and

$$x\varphi_2(r) = \frac{\partial \varphi_1}{\partial x} = \frac{x}{r^2} \tag{2-26b}$$

Then we obtain

$$\vec{q} = \vec{i}\,(\log r + \frac{1}{2}) - \frac{x}{r}\nabla r - \frac{1}{2}\nabla(\frac{x}{r^2}) \tag{2-26c}$$

and

$$\vec{q} = 0 \qquad \text{at } r = 1 \tag{2-26d}$$

$$\vec{q} \to \infty \qquad \text{as } r \to \infty \tag{2-26e}$$

This solution has been used in Ref. 2 to form the so-called intermediate solutions.

Example 2) For a semi-infinite paraboloid $\tau = \sqrt{z^2+r^2} - z = 1$, we have

$$\phi_1(\tau) = \log \tau = \log(\sqrt{z^2+r^2} - z) \tag{2-27a}$$

and

$$x\varphi_2(\tau) = \frac{x}{\sqrt{z^2+r^2} - z} = \frac{x}{\tau} \tag{2-27b}$$

It will be shown in Chapter 6 that the following solution

$$\vec{q} = \vec{i}\,(\log\tau + 1) - x\nabla\log\tau - \nabla(\frac{x}{\tau}) \tag{2-27c}$$

generated from (2-27a, b) will be used to form the intermediate solution. Thus we have shown that there is a certain relation between the Laplace and the Stokes solution.

III. LOW REYNOLDS NUMBER FLOW PAST A CIRCULAR CYLINDER AND A SPHERE

3.1 Introduction

Ref. 1 and Ref. 2 developed certain methods and ideas, such as the use of various limits and their associated expansions and matching between expansions, for the purpose of obtaining asymptotic solutions for low Reynolds number flow past solids in two or three dimensions. The idea will be briefly reviewed here, and it will be seen later that these methods may be generalized and adopted to the problems studied in the present thesis.

We consider viscous incompressible flow in two or three dimensions, past a body characterized by one length (for instance the radius of a circular cylinder or a sphere.) The governing equations are the Navier-Stokes equations which in dimensional form are

$$\vec{q} \cdot \nabla \vec{q} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \vec{q} \qquad (3\text{-}1a)$$

$$\nabla \cdot \vec{q} = 0 \qquad (3\text{-}1b)$$

with the boundary conditions

$$\vec{q} = 0 \qquad \text{at body} \qquad (3\text{-}2a)$$

$$\vec{q} = U\vec{i}\,;\; p = p_\infty \qquad \text{at infinity} \qquad (3\text{-}2b)$$

where $\vec{i}$ is the unit vector in the x-direction. Equation (3-1a, b) may be non-dimensionalized by use of proper non-dimensional variables such as x_i^*, x_i, p^*, p^+, $\vec{q}^{\,*}$ and Re (see the List of Symbols). This may for instance be written

$$\vec{q}^{\,*} \cdot \tilde{\nabla} \vec{q}^{\,*} + \tilde{\nabla} p^* = \tilde{\nabla}^2 \vec{q}^{\,*} \qquad (3\text{-}3a)$$

$$\tilde{\nabla} \cdot \vec{q}^{\,*} = 0 \qquad (3\text{-}3b)$$

Note that, in the present case, there is only one non-dimensional parameter, namely Re. If the $(\tilde{x}_i)$ variables are used as above, Re does not appear in the equations but it occurs in the boundary conditions since the body dimension is of order Re.

The problem is to find an asymptotic expansion

$$\vec{q}^{\,*} \sim \sum_{j=0} \vec{q}_j\,(\tilde{x}_i, \mathrm{Re}) \qquad (3\text{-}4)$$

which is uniformly valid for the entire flow field for small value of Re. Here $\vec{q}^{\,*}$ is the exact solution of equations (3-1) and (3-2). The degree of approximation is measured with a suitable asymptotic sequence of function of Re, $\epsilon(\mathrm{Re})$ having the property

$$\epsilon_o = 1, \ \lim_{\mathrm{Re} \to 0} \frac{\epsilon_{j+1}}{\epsilon_j} = 0 \qquad (3\text{-}5)$$

we require the n^{th} partial sum to be valid to order ϵ_n, i.e.

$$\lim_{\mathrm{Re} \to 0} \frac{\vec{q}^{\,*} - \sum_{j=0}^{n} \vec{q}_j(\tilde{x}_i, \mathrm{Re})}{\epsilon_n} = 0 \ \text{ uniformly in space} \qquad (3\text{-}6)$$

The method used in Refs. 1 and 2 consists of constructing two expansions which are not uniformly valid in the entire flow field but which overlap and which may be combined into one uniformly valid expansion. A principal idea is that of limits. Consider a non-dimensional flow quantity F (such as $\vec{q}^{\,*}$) which depends on $\tilde{x}_i$ and Re (or equivalently on x_i and Re). Let f(Re) be a function of Reynolds number. We define

$\lim_f F = \lim F$ as $Re \to 0$,

$$x_i^{(f)} = \frac{\tilde{x}_i}{f(Re)} = \text{constant} \tag{3-7}$$

The limit obtained by choosing $f(Re) = 1$ is called the Oseen or outer limit

$$\lim_o F = \lim F \text{ as } Re \to 0,\ \tilde{x}_i = \text{fixed} \tag{3-8}$$

If $f(Re) = Re$ we obtain the Stokes or inner limit

$$\lim_s F = \lim F \text{ as } Re \to 0,\ x_i^* = \text{fixed} \tag{3-9}$$

If $Re << f(Re) << 1$ the corresponding limit is called intermediary.

With each limit and a suitably chosen sequence (ϵ_j) we may associate an expansion obtained by repeated application of the limit. For example, the Oseen expansion of $\vec{q}^*$ is

$$\vec{q}^* \sim \sum_{j=0} \epsilon_j \vec{g}_j (\tilde{x}_i) \tag{3-10a}$$

where $\vec{g}_o = \lim_o \vec{q}^*$

$$\vec{g}_{n+1} = \lim_o \frac{\vec{q}^* - \sum_{j=1}^{n} \epsilon_j \vec{g}_j}{\epsilon_{n+1}} \tag{3-10b}$$

Similarly by a repeated application of the Stokes limit gives a Stokes expansion

$$\vec{q}^* \sim \sum_{j=0}^{n} \epsilon_j(Re)\, \vec{h}_j(x_i^*) \tag{3-11}$$

By insertion of these expansions into equation (3-1) (written in Oseen and Stokes variables respectively) one obtains the equation for the

$\vec{g}_j$ and $\vec{h}_j$. For a large class of bodies, including all finite bodies $\vec{g}_o = \vec{i}$ and $\vec{g}_j$, then satisfies the Oseen equation. The Oseen expansion should then satisfy the outer boundary condition (3-2b). The Stokes expansion should satisfy the inner boundary condition (3-2a). The inner boundary condition for the $\vec{g}_j$ and the outer boundary condition for the $\vec{h}_j$ are provided by matching principles which will be discussed below in connection with specific examples.

3.2 Low Reynolds Number Solution for a Two Dimensional Circular Cylinder

Let the radius of the cylinder be λ. In the Oseen limit the non-dimensional radius is then equal to Re and the cylinder shrinks to a line with no influence on the flow. Hence, $\lim_o \vec{q}^* = \vec{i}$; however, because of the boundary condition (3-2a) this limit is non-uniform near the body where $\vec{q}^* = 0$. Similarly one can see that $\lim_s \vec{q}^* = 0$ is a limit which is non-uniform at infinity. Obviously these two limits cannot be matched. To overcome this difficulty Kaplun observes that if $\lim_f$ is applied to the Navier-Stokes equations, then for any $f(Re) \ll 1$, the Stokes equations result. We then say that the Stokes equations are formally valid for $f(Re) \ll 1$. Kaplun then assumes that there exists a solution (which need not be a limit) of the Stokes equations which is valid in the same domain as the equations. This means that there exists a function $\vec{u}_o(x_i^*, Re)$ such that if $\vec{q}^*$ is the exact solution of the Navier-Stokes equations, then for any $f(Re) \ll 1$

$$\lim_{Re \to 0} \left| \vec{q}^* - \vec{u}_o \right| = 0 \qquad \text{uniformly for } Re \leq \tilde{r} < f_1(Re) \qquad (3\text{-}12)$$

We note that $\tilde{r} = Re$ is the surface of the circular cylinder and that hence $\vec{u}_o$ must satisfy the boundary condition on the body.

Since $\vec{q}_o = \vec{i}$ is obtained by the outer limit it is expected that it is valid outside of any constant value of $\tilde{r}$, i.e.

$$\lim_{Re \to 0} \left| \vec{q}^{\,*} - \vec{g}_o \right| = 0 \qquad \text{uniformly for } c \leq \tilde{r} < \infty \tag{3-13}$$

where c is a constant > 0. With the aid of Kaplun's extension theorem one can show that the domain of validity of $\vec{g}_o$ is larger than that indicated by (3-13). There must exist a function $f_2(Re) \ll 1$ such that

$$\lim_{Re \to 0} \left| \vec{q}^{\,*} - \vec{g}_o \right| = 0 \qquad \text{uniformly for } f_2(Re) < \tilde{r} < \infty \tag{3-14}$$

Since the only requirement on $f_1(Re)$ in (3-12) was that $f_1(Re) \ll 1$ we may choose it such that $f_2(Re) \ll f_1(Re)$. The approximations $\vec{u}_o$ and $\vec{g}_o$ have then a domain of overlap. In particular there exists a function f(Re) such that $f_2(Re) \ll f(Re) \ll f_1(Re)$ such that

$$\lim_f \left| \vec{q}^{\,*} - \vec{u}_o \right| = 0 \tag{3-15a}$$

$$\lim_f \left| \vec{q}^{\,*} - \vec{g}_o \right| = 0 \tag{3-15b}$$

and hence

$$\lim_f \left| \vec{g}_o - \vec{u}_o \right| = 0 \tag{3-15c}$$

Thus, while the outer limit cannot be matched with the inner limit it can be matched with an intermediate solution. The matching is expressed by (3-15c).

The function $\vec{u}_o$ can be found easily. Let

$$\vec{h}_1 = \vec{i}\,(\log r^* + \tfrac{1}{2}) - \frac{x^*}{r^*}\,\nabla^* r^* - \tfrac{1}{2}\nabla^* \frac{x^*}{r^{*2}} \tag{3-16}$$

This satisfies the inner boundary condition

$$\vec{h}_1 = 0 \qquad \text{for } r^* = 1 \tag{3-17}$$

Actually $\vec{h}_1$ is generated from the Laplace solution (cf. 2-26)

$$\varphi_1 (r^*) = \log r^* \tag{3-18a}$$

$$x^* \varphi_2(r^*) = \frac{\partial \varphi_1}{\partial x^*} \tag{3-18b}$$

We now put

$$\vec{u}_o = \epsilon(Re)\, \vec{h}_1(x_i^*) \tag{3-19}$$

where $\epsilon(Re)$ is to be determined from (3-15c). Obviously $\vec{u}_o(Re, x_i^*)$ satisfies the Stokes equations and the inner boundary conditions. To apply (3-15c) we write $\vec{u}_o$ in terms of $x_i^{(f)}$ and we use the fact $\vec{g}_o = \vec{i}$. Then

$$\vec{u}_o - \vec{i} = \vec{i}\left(\epsilon \log \frac{1}{Re} - 1\right) + \epsilon\Big\{\vec{i}\big[\log r^{(f)} + \tfrac{1}{2} + \log f(Re)\big] - \frac{x^{(f)}}{r^{(f)}} \nabla^{(f)} r^{(f)} - \frac{1}{2}\frac{f^2(Re)}{Re^2} \nabla^{(f)} \frac{x^{(f)}}{r^{(f)}}\Big\} \tag{3-20}$$

We can see that (3-15) is satisfied if $\epsilon(Re)$ is chosen such that

$$\lim_{Re \to 0} (-\epsilon \log Re) = 1 \tag{3-21}$$

Hence, $\epsilon(Re)$ may be assumed to satisfy a relation

$$-\epsilon \log Re = 1 + b_1\epsilon + b_2\epsilon^2 + \ldots\ldots \tag{3-22}$$

Where the b_n can be normalized later. Note that in this case the overlap domain is very small. A sufficiently slow limit is obtained by taking

$$f(Re) = \frac{1}{\log \frac{1}{Re}} \qquad (3-23)$$

The higher order approximations are obtained by a similar argument. We assume that there is an intermediate expansion

$$\vec{q}^{\,*} \sim \sum_{n=0}^{\infty} \epsilon^n \vec{u}_n (x_i^*, Re) \qquad (3-24)$$

which overlaps with the outer expansions. For instance, matching to order ϵ requires

$$\lim_f \frac{\vec{i} + \epsilon \vec{g}_1 - (\vec{u}_o + \epsilon \vec{u}_1)}{\epsilon} = 0 \qquad (3-25)$$

for f(Re) in some overlap domain. For further discussion the reader is referred to Ref. 2. Here we only give the results. The function $\vec{g}_1$ must satisfy the Oseen equation and is

$$\vec{g}_1 = -2\vec{i}\; e^{\frac{1}{2}\tilde{x}} K_o(\tfrac{1}{2}\tilde{r}) + 2\tilde{\nabla}\{e^{\frac{1}{2}\tilde{x}} K_o(\tfrac{1}{2}\tilde{r}) + \log \tilde{r}\} \qquad (3-26)$$

The function $\vec{u}_1$ must satisfy the Stokes equations and have the form

$$\vec{u}_1 = c_1 \vec{u}_o \qquad (3-27)$$

We can make the constant c_1 equal to zero by choosing $b_1 = \gamma - \log 4 - \frac{1}{2}$, $b_n = 0$ for $n > 1$. This gives

$$\epsilon = \frac{1}{\log \frac{4}{Re} + \frac{1}{2} - \gamma} \qquad (3-28)$$

3.3 Low Reynolds Number Solution for a Sphere.

We now consider flow past a sphere of radius λ and center at origin. In this case there exists a solution of the Stokes equations satisfying the boundary conditions at the body as well as at infinity. The velocity can easily be generated from Laplace solutions as discussed in the preceding chapter and is given by a function $\vec{A}$

$$\vec{A} = \vec{i} - \frac{3}{2}\vec{A}_1 + \frac{1}{4}\vec{A}_2 \tag{3-29}$$

where

$$\vec{A}_1 = \frac{\vec{i}}{R^*} - \nabla^* \frac{t^*}{R^*}, \qquad t^* = \frac{1}{2}(x^* - R^*)$$

$$\vec{A}_2 = \nabla^* \frac{\partial}{\partial x^*}\left(\frac{1}{R^*}\right)$$

This solution is the first term of the intermediate expansions and also of the Stokes expansion, i.e.

$$\vec{u}_o = \vec{h}_o = \vec{A}_o \tag{3-30}$$

Due to the simple structure of $\vec{u}_o$ the matching condition (3-15c) reduces to the boundary condition $\vec{u}_o = \vec{i}$ at $R^* = \infty$. The fact that $\vec{u}_o$ satisfies the ''correct'' boundary condition at infinity, i.e. the same condition as the solution of the Navier-Stokes equations, should be regarded as an exceptional coincidence. This shows that in the present case, there exists an overlap domain between inner and outer expansion and thus in the present case the inner expansion and the intermediate expansions are the same.

$\vec{g}_1$ can be determined by the condition that it should cancel the unbounded term of $\lim_f \frac{\vec{u}_o - \vec{i}}{\epsilon_1}$, i.e. the term $-\frac{3}{2}\frac{A_1}{\epsilon_1}$. The governing equation for $\vec{g}_1$ is the homogeneous Oseen equation. This gives

$$\vec{g}_1 = \frac{3}{2}\left[-\vec{i}\ \frac{e^{\tilde{t}}}{\tilde{R}} + \nabla\left(\frac{e^{\tilde{t}}-1}{\tilde{R}}\right)\right] \tag{3-31}$$

and also

$$\epsilon_1 = Re \tag{3-32}$$

Then $\vec{h}_1$ or $\vec{u}_1$ can be determined by the matching condition

$$\lim_f \frac{\vec{i} + Re\,\vec{g}_1 - (\vec{u}_o + Re\,\vec{u}_1)}{Re} = 0 \tag{3-33}$$

for f(Re) in some overlap domain.

3.4 Intermediate Solutions and Stokeslets.

A main underlying idea in finding the solution for the finite cylinder under the present study is that the so-called ''Stokeslets'' form ''intermediate solutions'' for various types of bodies, e.g. the circular cylinder, the semi-infinite cylinder, the paraboloids of various types, etc. and hence the ''Stokeslets'' will form the intermediate solutions needed in our expansion. This idea is best explained by considering the circular cylinder at low Reynolds number discussed above. The governing approximate equations for small r are then the two dimensional Stokes equations. As is well known there exists no solution of these equations which satisfies both the conditions at the body and at infinity. However the slowly growing solution

$$\vec{h}_1 = \vec{i}\ (\log r^* + \tfrac{1}{2}) - \frac{x^*}{r^*}\nabla^* r^* - \tfrac{1}{2}\nabla^* \frac{x^*}{r^{*2}} \tag{3-34}$$

is often considered as a solution for the circular cylinder and this may be justified in several senses. $\vec{h}_1$ satisfies the Stokes equations and the exact boundary conditions at the body, i.e.

$$\vec{h}_1 (x_i^*) = 0 \qquad \text{at } r^* = 1 \tag{3-35}$$

and multiplying by proper ϵ(Re), an intermediate solution is found and matches to the outer solution $\vec{g}_o = \vec{i}$ in some overlap domain. The part

$$\vec{h}_1^1 (x_i^*) = \vec{i} \log r^* - x^* \nabla^* \log r^* \tag{3-36}$$

of $h_1(x_i^*)$ is called a ''Stokeslet''. Note now that the ''Stokeslet'' itself is a solution for the circular cylinder in another sense: when multiplied by ϵ(Re) the ''Stokeslet'' forms an intermediate solution $\vec{u}_o^1$

$$\vec{u}_o^1 = \epsilon(Re)\{\vec{i} \log r^* - x^* \nabla^* \log r^*\} \tag{3-37}$$

which satisfies the boundary in the limit Re $\to$ 0 and it also matches to the outer solution. It is a solution of Stokes equations and is uniformly valid to order unity, i.e.

$$\lim_f (\vec{q}^* - \vec{u}_o) = 0 \tag{3-38}$$

for Re $\leq$ f(Re) $<<$ 1. This is evident from the fact that omitted terms in $\vec{h}_1$ are bounded and hence small after multiplication by ϵ. Equivalently the ''Stokeslet'' $\vec{h}_1^1$ may be described as a Stokes solution which grows slowly at infinity and instead of the exact boundary conditions is bounded along the body; the solution $\vec{u}_o^1 = \epsilon \vec{h}_1^1$ therefore satisfies the condition at the body to order unity; $\vec{u}_o^1 \to 0$ at

$r^* = 1$ as $Re \to 0$. We see therefore that the exact conditions at the body produce effects of higher orders only and that "intermediate solutions" may be formed with "Stokeslets" which are bounded at the body. It will be shown later that the idea of "Stokeslets" is very useful in the present study.

IV. A RELATED PROBLEM FOR LAPLACE EQUATIONS

This thesis is concerned with asymptotic expansions of the solutions to certain boundary value problems for the Navier-Stokes equations. In this chapter, we will discuss similar problems for the Laplace equations. The purpose of this study is two-fold. First of all the Laplace equation will serve as a model equation. The problem of end effects occurs for certain boundary value problems of the Laplace equation. Since this equation is considerably simpler than the Navier-Stokes equations, the nature of the end effect may be studied in detail and various limits may be discussed. Secondly, as seen in the preceding chapter, Laplace solutions may be used for generating certain important Stokes solutions.

We shall formulate a certain boundary value problem for a Laplace solution outside a finite cylinder, the "exact" Laplace problem. An approximation to the solution of this problem will be obtained by solving a corresponding "approximate" problem. For comparison, the somewhat simpler case of an ellipsoid of revolution will also be discussed. Also, certain general remarks will be made about the difference between the end effects for "tapered" and "untapered" bodies.

4.1 The Exact Laplace Problem.

The exact Laplace problem we wish to study is to construct a solution satisfying the three dimensional Laplace equations

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{4-1}$$

outside of the finite cylinder shown in Fig. 4-1 and satisfying the following boundary conditions

$\varphi = 0$ at infinity (4-2a)

and

$\varphi = 1$ on the surface $S(\lambda)$ of the finite cylinder (4-2b)

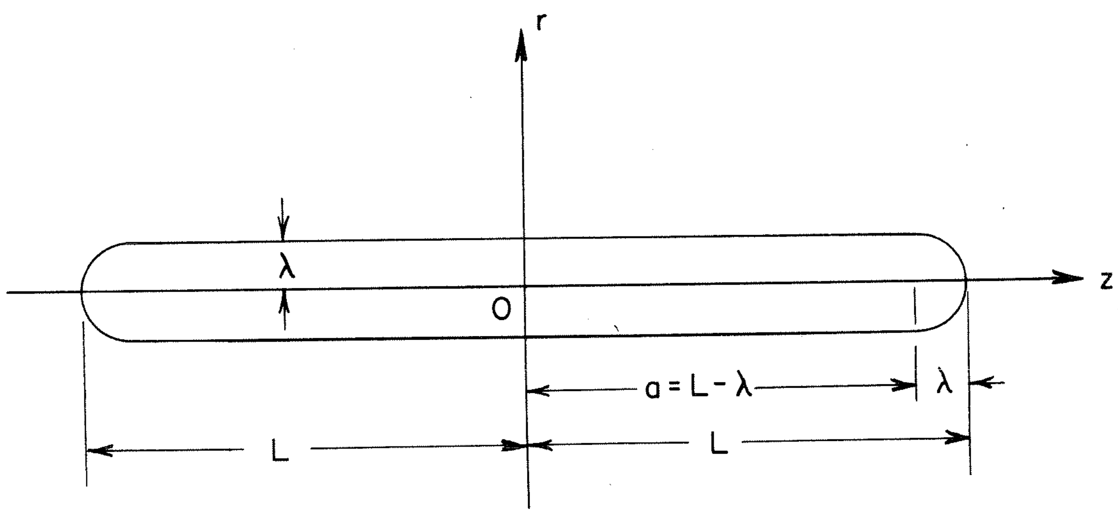

Fig. 4-1

The finite cylinder is symmetric with respect to the plane $z = 0$ and the surface $S(\lambda)$ of the cylinder is defined as follows

$r = \lambda$ for $-a \leq z \leq a$

$r^2 + (z \pm a)^2 = \lambda^2$ for z in the interval $[-L, -a]$ or $[a, L]$ (4-3)

where $a = L - \lambda$.

The exact problem posed above is a well defined problem to which both existence and uniqueness theorems can be proved.

The exact problem is the so-called exterior Dirichlet problem and the existence theorem of this problem has been discussed by I. Petrovsky in Ref. 6. The existence of a unique solution is also physically plausible. One possible physical interpretation is that the exterior of the cylinder is heat conducting, the temperature at infinity is T_∞, the temperature on the cylinder is T_c and $\varphi = \frac{T - T_\infty}{T_c - T_\infty}$. The solution of the Laplace problem then represents the steady-state temperature distribution.

4.2 The Approximate Laplace Problem.

While the exact solution to the problem just formulated may be found in principle, we shall be concerned only with the following approximation. We assume that the quantity L is fixed and that λ tends to zero. The exact solution then has an asymptotic expansion. We seek the uniformly valid leading term of this expansion, i.e. a function φ_A such that φ is the exact solution

$$\varphi - \varphi_A = o(1) \tag{4-4}$$

uniformly on and outside the cylinder. Here o(1) denotes a function which tends to zero as λ tends to zero, uniformly in the domain described. We note that φ_A may depend on λ. Furthermore, it is of course not uniquely determined. We are only interested in finding one φ_A which satisfies (4-4).

An approximation to the exact solution of (4-1, 2) may be found by solving the following approximate problem

$$\nabla^2 \varphi_A = 0 \tag{4-5}$$

and

$$\varphi_A = 0 \qquad \text{at infinity} \tag{4-6a}$$

$$\varphi_A = 1 + o(1) \qquad \text{on the surface } S(\lambda) \text{ of the cylinder} \tag{4-6b}$$

Here o(1) denotes a function whose value on the surface of the cylinder tends to zero uniformly as λ tends to zero.

It can be seen from the well-known maximum principle for the Laplace equations that the solution of the approximate problem gives the desired approximation to the solution of the exact problem. The maximum or minimum value of a Laplace solution must occur on the boundary. Let, as before, φ be the solution of (4-1, 2) and φ_A the solution of (4-5, 6). By assumption, for any $\epsilon > 0$ there exists a λ_ϵ such that for $\lambda \leq \lambda_\epsilon$, $|\varphi - \varphi_A| < \epsilon$ on the boundary. Since $\varphi - \varphi_A$ satisfies the Laplace equation it follows that $|\varphi - \varphi_A|$ is also $< \epsilon$ in the domain outside the boundary. This is exactly the property required by the approximation φ_A according to (4-4).

4.3 Outer Limit and Shank Limit.

Outer limit: If we keep r and z fixed, the cylinder tends to a line segment as λ tends to zero. We call this limit the outer limit. It is to be expected that for any point at a finite distance from this cylinder ($r \neq 0$ or $|z| > a$) the influence of the cylinder is negligible and that hence the outer limit is zero. Thus we have

Outer limit: r and z fixed for $\lambda \to 0$. (4-7a)

Obviously we have the outer solution

$$\varphi_o = 0 \tag{4-7b}$$

Shank limit: Obviously the solution φ_o is not uniformly valid near the cylinder since $\varphi_A = 1 + o(1)$ on the surface of the cylinder. To study the behavior of φ_A in the neighborhood of the cylinder we scale the coordinate r with λ. The coordinates obtained will be called shank coordinates. The shank coordinates are

$$r^* = \frac{r}{\lambda} \text{ and } z \tag{4-8}$$

The corresponding limit is

Shank limit: r^*, z fixed for $\lambda \to 0$ (4-9)

In the shank limit, the boundary conditions at the surface of the body should be satisfied as far as possible. However we observed that the radius of cylinder is λ for $|z| \leq a$ but decreases to zero for $|z| \geq a$. Thus, the scaling with the cylinder expressed by (4-8) is correct only for $|z| \leq a$ and we expect the method to lead to difficulties near the ends of the cylinders.

Disregarding these difficulties for the moment, we find that in the shank limit the three dimensional Laplace equation becomes two dimensional. A solution of this equation satisfies the boundary conditions (4-6b) as follows

$$\varphi_s = 1 + A \log r^* \tag{4-10}$$

where A so far is arbitrary. The factor A may be determined by matching φ_s and φ_o as follows. We consider the limits which are intermediate between the inner and outer solution, i.e. the limits

$$r_\eta = \frac{r}{\eta(\lambda)} \text{ and } z \text{ fixed for } \lambda \to 0 \tag{4-11a}$$

where

$$\lambda << \eta(\lambda) << 1 \tag{4-11b}$$

and requires that for some range of intermediate limits $\varphi_o - \varphi_s$ tend to zero. This gives

$$1 + A\,[\log r_\eta - \log \lambda + \log \eta(\lambda)] \to 0 \tag{4-12}$$

This is satisfied by taking

$$A = \frac{1}{\log \frac{1}{\lambda} + c} = \epsilon\,(\lambda) \tag{4-13}$$

provided $\eta(\lambda)$ is such that $\epsilon << \log \eta$ and $\eta << 1$. Here c is any constant. Hence we obtain

$$\varphi_s = 1 - \epsilon\,(\lambda) \log r^* \tag{4-14}$$

The term containing $\log r^*$ is a two dimensional source. We have obtained a constant source distribution for $|z| < a$. A solution which contains the shank solution and the outer solution is obtained by putting three dimensional sources of the same strength along the z-axis for $|z| \leq a$. Thus we have

$$\varphi_c = \frac{1}{2}\,\epsilon\,(\lambda) \int_{-a}^{a} \frac{d\zeta}{\sqrt{(z-\zeta)^2 + r^2}} \tag{4-15}$$

Obviously this satisfies the three dimensional Laplace equation

$$\nabla^2 \varphi_c = 0 \tag{4-16a}$$

and

$$\varphi_c = 0 \quad \text{at infinity} \tag{4-16b}$$

A statement equivalent to (4-16b) is that in the outer limit it tends to zero.

To verify the boundary conditions at the surface of the body for $|z| < a$ we express φ_c in shank variables

$$\varphi_c = 1 - \epsilon(\lambda) \log r^* + \frac{1}{2}\epsilon(\lambda) \log 4(a^2 - z^2) + O(\lambda^2) \tag{4-17}$$

This shows that the boundary conditions are satisfied for $r^* = 1$, $|z| < a$.

4.4 Failure of Constant Source Distribution, End Limits.

We note that the first two terms of (4-17) represent the solution of the two dimensional Laplace equation obtained earlier. The third term is a constant as far as the two dimensional equation is concerned and the inclusion of this term does not change anything in the preceding argument for $|z| < a$. However we note that for $|z| \to a$ this third term becomes infinite and the boundary condition is not satisfied. This shows indeed that the method used is not applicable near the ends of the cylinder. A discussion of the solution for $|z|$ near a is thus needed.

End limit. Because of symmetry it is sufficient to consider the left end only. We introduce

End coordinates: $r^* = \frac{r}{\lambda}, \; z^* = \frac{z+L}{\lambda}$ (4-18)

and the corresponding limit is

End limit: r^* and z^* fixed for $\lambda \to 0$ (4-19)

Note that in the end limit the left end retains its shape. It is a hemisphere of radius unity with center at $r^* = 0$, $z^* = 1$. The coordinate of the right end is $z^* = \frac{2L+\lambda}{\lambda}$ which tends to $+\infty$ as $\lambda \to 0$. Thus we obtain the semi-infinite cylinder shown in Fig. 4-2.

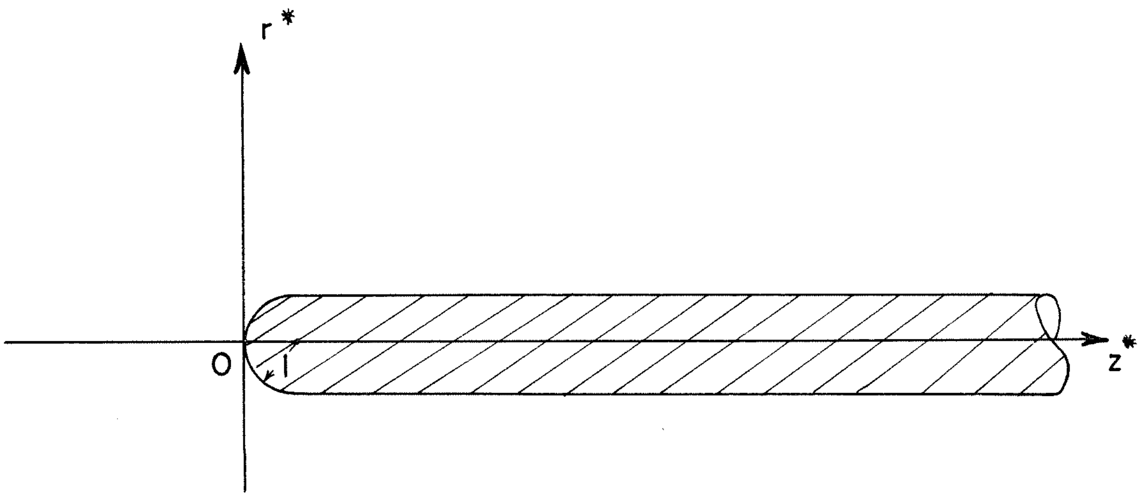

Fig. 4-2

We now study the end limit of (4-15), i.e. the potential due to a constant source distribution of strength $\frac{1}{2}\epsilon(\lambda)$ for $|z| \leq a$. Introducing end coordinates into (4-15) one finds

$$\varphi_c = \tfrac{1}{2}\epsilon(\lambda)\log 4L - \tfrac{1}{2}\epsilon(\lambda)\log\lambda + \tfrac{1}{2}\epsilon(\lambda)\log\left[\sqrt{(z^*-1)^2 + r^{*2}} + (z^*-1)\right] + O(\lambda^2) \qquad (4\text{-}20)$$

On the hemisphere $\sqrt{(z^*-1)^2 + r^{*2}} = 1$ and $0 \leq z^* \leq 1$ or on the cylinder $r^* = 1$ and $z^* =$ finite, we have

$$\varphi_c = \frac{1}{2} + o(1) \tag{4-21}$$

Thus we can see that φ_c does not satisfy the required boundary condition $\varphi_c = 1 + o(1)$ on the surface of the semi-infinite cylinder. In addition, for $r^* =$ fixed and $z^* \to \infty$ we have

$$\varphi_c = \frac{1}{2} + \frac{1}{2}\epsilon(\lambda) \log 2z^* + o(1) \tag{4-22}$$

Since $\log 2z^*$ is not bounded for $z^* \to \infty$ the limit is not uniform for $z^* \to \infty$. It will be shown that there exists a transition region where φ_c varies from $\frac{1}{2}$ to 1. This can be seen by introducing the intermediate variables (i.e. variables which are intermediate between the end variables and shank variables).

Intermediate limits. If we define the intermediate variables as $z_\alpha = \frac{z + L}{\omega(\lambda)}$ where $\lambda \leq \omega(\lambda) \leq 1$ and $r^* = \frac{r}{\lambda}$, φ_c may be expressed in terms of intermediate variables as follows

$$\varphi_c = 1 + \frac{1}{2}\epsilon(\lambda) \log \omega(\lambda)\, z_\alpha + o(1) \tag{4-23a}$$

$$= 1 - \frac{\alpha}{2} + o(1) \tag{4-23b}$$

for $\omega(\lambda) = \lambda^\alpha$ and $0 \leq \alpha \leq 1$. Thus in this region, the limit is not uniform and varies from $\frac{1}{2}$ to 1 as α varies from zero to 1. It is also interesting to know that

$$\varphi_c = 1 + o(1) \tag{4-24}$$

for all $\omega(\lambda)$ such that

$$\epsilon(\lambda) \log \omega(\lambda) \to 0 \qquad (4\text{-}25)$$

For instance $\omega(\lambda) = \epsilon^n$ satisfies equation (4-25). Thus the region where $\varphi_c = 1 + o(1)$ satisfies the required boundary condition can be extended from the shank region to the following larger region

$$-L + O(\epsilon^n) \leq z \leq L - O(\epsilon^n) \qquad (4\text{-}26)$$

To summarize: the constant source distribution gives a constant value for φ on the left hemisphere cap of the finite cylinder. At the junction between the cap and the shank part of the cylinder, this value changes discontinuously to a constant which is twice the constant value of the cap. Actually a gradual transition can be found by using the appropriate intermediate limit (i.e. the limit which is intermediate between the end limit and shank limit). What is needed is a source distribution which gives a constant value on the cap without discontinuity at the junction. We shall therefore look for a modification of the constant source distribution near $|z| \leq a$. To clarify the nature of this end source distribution is one of the principal objectives of this thesis. As will be seen later, the results obtained may be carried over from the Laplace to Navier-Stokes case.

4.5 The Nature of End Source Distribution for the Finite Cylinder.

This can be clarified by studying a rather general Laplace solution

$$\varphi_L = \gamma(\lambda) \int_{-a}^{a} f(\zeta;\lambda) \frac{d\zeta}{\sqrt{(z-\zeta)^2+r^2}} \tag{4-27}$$

where $\gamma(\lambda)$ is a function of λ to be determined and $\gamma(\lambda)$ tends to zero as λ tends to zero. By symmetry, it is sufficient to discuss the left end only. Since in the end limit the cylinder becomes semi-infinite as shown in Fig. 4-2 and the radius of the cylinder is unity and independent of λ we require that the source distribution function $f(\zeta,\lambda)$ when expressed in terms of end variables is independent of λ. Then we have

$$\varphi_L = \gamma(\lambda) \int_{-a}^{a} f(\frac{L+\zeta}{\lambda}) \frac{d\zeta}{\sqrt{(z-\zeta)^2+r^2}} \tag{4-28}$$

Now if we express φ_L in terms of end variables, we obtain

$$\varphi_L = \gamma(\lambda) \int_{1}^{\frac{2L}{\lambda}-1} f(\zeta^*) \frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} \tag{4-29}$$

We rewrite φ_L as follows:

$$\varphi_L = \gamma(\lambda) \int_{1}^{\frac{2L}{\lambda}-1} f(\zeta^*) \left[\frac{1}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} - \frac{1}{\zeta^*}\right] d\zeta^*$$

$$+ \gamma(\lambda) \int_{1}^{\frac{2L}{\lambda}-1} \frac{f(\zeta^*)}{\zeta^*} d\zeta^* \tag{4-30}$$

We shall look for a function $f(z^*)$ such that

$$\int_{1}^{\frac{2L}{\lambda}-1} \frac{f(\zeta^*)}{\zeta^*} d\zeta^* = F(\lambda) \rightarrow \infty \text{ as } \lambda \rightarrow 0 \tag{4-31}$$

and such that

$$\psi = \int_{1}^{\frac{2L}{\lambda}-1} f(\zeta^*) \left[\frac{1}{\sqrt{(z^*-\zeta^*)^2 + r^{*2}}} - \frac{1}{\zeta^*} \right] d\zeta^* \qquad (4\text{-}32)$$

is bounded on the surface of the semi-infinite cylinder for r^* = fixed and $z^* \to \infty$.

If we put

$$\gamma(\lambda) = \frac{1}{F(\lambda)} \qquad (4\text{-}33)$$

we see that by definition the second term in (4-30) tends to unity. Since $F(\lambda) \to \infty$ $\gamma(\lambda) \to 0$ and the first term $\gamma(\lambda)\psi$ tends to zero for finite z^*, that is the end limit. Since furthermore ψ is bounded for $z^* \to \infty$ and r^* = fixed, we expect φ_L to be valid not only on the left cap but also on the shank part of the cylinder.

The constant source does not satisfy these requirements. If $f(\zeta^*)$ = constant, then (4-31) is satisfied but ψ is unbounded for $z^* \to \infty$. In fact, in this case

$$\psi = - c \log \left[\sqrt{(z^*-1)^2 + r^{*2}} - (z^* - 1)\right] \qquad (4\text{-}34)$$

which is bounded for z^* = finite and r^* = 1 on the surface of the cylinder but for $z^* \to \infty$ we have

$$\psi \to c \log 2z^* \qquad (4\text{-}35)$$

Since the ψ corresponding to $f(z^*)$ = constant is infinite for z^* large, $f(z^*)$ must decrease to zero for z^* large if we want ψ to be bounded. On the other hand, $f(z^*)$ can not decrease too rapidly. If we assume, for example, $f(z^*) = \frac{1}{z^{*n}}$, $n > 0$, then

$$F(\lambda) = \int_1^{\frac{2L}{\lambda}-1} \frac{1}{\zeta^{*1+n}} \, d\zeta^* = \frac{1}{n} - \frac{\lambda}{n(2L-\lambda)} \tag{4-36}$$

and $F(\lambda)$ does not tend to ∞ as $\lambda \to 0$.

In order to obtain the correct $f(z^*)$ we study the behavior of ψ for $z^* \to \infty$ and assume only that $f(z^*) \to 0$ as $z^* \to \infty$. We divide the range of integration in (4-32) into intervals $[1, z_0]$, $[z_0, z_1]$, $[z_1, z_2]$ $[z_2, z_3]$ and $[z_3, \frac{2L}{\lambda} - 1]$ where z_0, z_1, z_2 and z_3 are defined as follows

$$\begin{aligned} z_0 &= \text{constant} > 1 \\ z_1 &= \sqrt{z^*} \\ z_2 &= (1-\beta)z^* \\ z_3 &= (1+\beta)z^* \end{aligned} \tag{4-37}$$

where $\beta << 1$, $\beta z^* \to \infty$ and $f(\sqrt{z^*}) \log \beta \to 0$ as $z^* \to \infty$. Then for $z^* \to \infty$, r^* = fixed we have the asymptotic representation

$$\psi(z^*, r^*) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \tag{4-38a}$$

and

$$I_1 = \int_1^{z_0} f(\zeta^*) \frac{dz^*}{\sqrt{(z^*-\zeta^*)^2 + r^{*2}}} = \frac{1}{z^*} \int_1^{z_0} f(\zeta^*) d\zeta^* + O(\frac{1}{z^{*2}})$$

$$= \frac{A}{z^*} + O(\frac{1}{z^{*2}}) \tag{4-38b}$$

$$I_2 = \int_{z_0}^{z_1} f(\zeta^*) \frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} \leqslant f(z_0) \int_{z_0}^{z_1} \left| \frac{1}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} \right| d\zeta^*$$

$$= \frac{f(z_0)z_1}{z^*} + O(\frac{1}{z^*}) = \frac{B}{\sqrt{z^*}} + O(\frac{1}{z^*}) \qquad (4\text{-}38c)$$

$$I_3 = \int_{z_1}^{z_2} \frac{f(\zeta^*)d\zeta^*}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} = f(z_1) \int_{z_1}^{z_2} \left| \frac{1}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} \right| d\zeta^*$$

$$= f(\sqrt{z^*}) \log \frac{z^*-z_1+\sqrt{(z^*-z_1)^2+r^{*2}}}{z^*-z_2+\sqrt{(z^*-z_2)^2+r^{*2}}}$$

$$= f(\sqrt{z^*}) \log \frac{z^*-z_1}{z^*-z_2} + O\left(\frac{f(\sqrt{z^*})}{\beta^2 z^{*2}}\right)$$

$$= f(\sqrt{z^*}) \log \beta + O\left(\frac{f(\sqrt{z^*})z_1}{z^*}\right) \qquad (4\text{-}38d)$$

$$I_4 = \int_{z_2}^{z_3} f(\zeta^*) \frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} \qquad (4\text{-}38e)$$

By putting $t = z^*-\zeta^*$ and by use of mean value theorem, we obtain

$$I_4 = \int_{z-z_3}^{z-z_2} \frac{f(z^*-t)}{\sqrt{t^2+r^{*2}}} dt = f(z^*+\bar{t}) \int_{z-z_3}^{z-z_2} \frac{dt}{\sqrt{t^2+r^{*2}}}$$

$$= f(z^*+\bar{t}) \log \frac{z^*-z_2+\sqrt{(z^*-z_2)^2+r^{*2}}}{z^*-z_3+\sqrt{(z^*-z_3)^2+r^{*2}}} \qquad z^*-z_2 \leqslant \bar{t} \leqslant z^2-z_3$$

$$= f[(1+\eta)z^*]\,[2\log z^* + 2\log 2\beta - 2\log r^*]$$

$$-\beta \leqslant \eta \leqslant \beta$$

$$+ O[\frac{f(z^*)}{\beta z^*}] \qquad (4\text{-}38f)$$

$$I_5 = \int_{z_3}^{\frac{2L}{\lambda}-1} f(\zeta^*)\,[\frac{1}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} - \frac{1}{\zeta^*}]\,d\zeta^*$$

$$\leqslant f(z_3)\int_{z_3}^{\frac{2L}{\lambda}-1} \left|\frac{1}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} - \frac{1}{\zeta^*}\right| d\zeta^*$$

$$= f(z_3)\left|\log(\frac{2L}{\lambda}-1-z^*) - \log(\frac{2L}{\lambda}-1)\right.$$

$$\left. + \log(z_3-z^*) - \log z_3\right| + O[\frac{f(z_3)}{(z_3-z^*)^2}] \qquad (4\text{-}38g)$$

$$= f(z_3)\left|\log[1-\frac{\lambda}{2L}(z^*-1)] + \log\beta\right|$$

$$+ O[\frac{f(z^*)}{\beta^2 z^{*2}}] \qquad (4\text{-}38h$$

$$I_6 = \int_1^{z_3}\frac{f(\zeta^*)}{\zeta^*}\,d\zeta^* = \int_1^{z^*}\frac{f(\zeta^*)}{\zeta^*}\,d\zeta^* + \int_{z^*}^{z_3}\frac{f(\zeta^*)}{\zeta^*}\,d\zeta^*$$

since $\int_{z^*}^{z_3}\frac{f(\zeta^*)}{\zeta^*}\,d\zeta^* \leqslant f(z^*)\left|\log z_3 - \log z^*\right|$

$$= f(z^*)\log(1+\beta)$$

$$= O[f(z^*)] \qquad (4\text{-}38i)$$

therefore

$$I_6 = \int_1^{z^*} \frac{f(\zeta^*)}{\zeta^*}\, d\zeta^* + O[f(z^*)] \tag{4-38j}$$

From the above equations, if we choose β such that

$$f(\sqrt{z^*}) \log \beta = o(1) \text{ as } z^* \to \infty \tag{4-39}$$

we find that terms which may not be of order o(1) come only from I_4 and I_6. Then we have

$$\psi(z^*, r^*) = 2f[(1+\eta)z^*] \log z^* - \int_1^{z^*} \frac{f(\zeta^*)}{\zeta^*}\, d\zeta^* + o(1)$$

$$\text{for } z^* \to \infty \text{ and } -\beta \leq \eta \leq \beta \tag{4-40}$$

In fact, we may choose $\beta = f(\sqrt{z^*}) \to 0$ as $z^* \to \infty$ and even with this small β the condition $f(\sqrt{z^*}) \log \beta = o(1)$ is also satisfied. We can say that (4-40) is true for any β such that $f(\sqrt{z^*}) \leq \beta < 1$ for $z^* \to \infty$. If $f(z^*)$ is smooth enough we may write

$$(\psi(z^*, r^*) = 2f(z^*) \log z^* - \int_1^{z^*} \frac{f(\zeta^*)}{\zeta^*}\, d\zeta^* + o(1) \tag{4-41}$$

The sum of the first two terms should be bounded for z^* large. This can be easily obtained by letting $f(z^*)$ decrease to zero as $\frac{1}{z^{*n}}$. However as we have seen in this case the corresponding $F(\lambda)$ does not tend to infinity. Thus we shall try to find a $f(z^*)$ for which the sum of the first two terms are bounded without actually going to zero for z^* large. We therefore consider the equation

$$2f(z^*) \log z^* - \int_1^{z^*} \frac{f(\zeta^*)}{\zeta^*}\, d\zeta^* = \text{constant} \tag{4-42}$$

By differentiation, we obtain

$$\frac{f(z^*)}{z^*} + 2\frac{df}{dz^*} \log z^* = 0 \tag{4-43}$$

This equation has a solution

$$f(z^*) = \frac{A}{\sqrt{\log z^*}} \tag{4-44}$$

The constant A has no significance because it will be absorbed into $\gamma(\lambda)$. Thus we can put $A = 1$ and

$$f(z^*) = \frac{1}{\sqrt{\log z^*}} \tag{4-45}$$

Substituting into (4-31), we have

$$F(\lambda) = \int_1^{\frac{2L}{\lambda}-1} \frac{1}{\sqrt{\log \zeta^*}} \frac{d\zeta^*}{\zeta^*} = 2\sqrt{\log\left(\frac{2L}{\lambda} - 1\right)} \tag{4-46}$$

tends to infinity as $\lambda \to 0$ as required. Then from (4-33), we have

$$\gamma(\lambda) = \frac{1}{2} \frac{1}{\sqrt{\log\left(\frac{2L}{\lambda} - 1\right)}} \tag{4-47}$$

For simplicity, we can take

$$\gamma(\lambda) = \frac{1}{2} \epsilon(\lambda)^{\frac{1}{2}} + O(\lambda) \tag{4-48a}$$

or

$$\epsilon(\lambda)^{\frac{1}{2}} = \frac{1}{\sqrt{\log \frac{1}{\lambda} + \log 2L}} \tag{4-48b}$$

Thus the constant c in (4-13) is chosen to be log 2L. Therefore φ_L

is determined as

$$\varphi_L = \frac{1}{2}\,\epsilon^{\frac{1}{2}} \int_1^{\frac{2L}{\lambda}-1} \frac{1}{\sqrt{\log\zeta^*}} \frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} \tag{4-49}$$

As expected, the $f(z^*)$ obtained does indeed tend to zero more slowly than $\frac{1}{z^n}$ for any $n > 0$. For comparison we shall also consider the effect of choosing

$$f(z^*) = \frac{1}{[\log z^*]^n}\ , \quad n > 0 \tag{4-50}$$

This gives according to (4-41) for $z^* \to \infty$ and r^* = fixed

$$\psi = 2(\log z^*)^{1-n} - \int_1^{z^*} \frac{d(\log\zeta^*)}{[\log\zeta^*]^n} + o(1) \tag{4-51}$$

The integral diverges for $n \geq 1$. For $0 < n < 1$ one finds

$$\psi = \frac{1-2n}{n-1}(\log z^*)^{1-n} + o(1) \tag{4-52}$$

which tends to infinity with z^* except for the case $n = \frac{1}{2}$. This justifies the solution of $f(z^*)$ in (4-45).

The heuristic argument used in finding φ_L as given in (4-49) would end up that $\varphi_L = 1 + o(1)$ on the left end cap as well as in the shank region. We shall now investigate in detail how it behaves on the surface of the finite cylinder. From (4-30), (4-45) and (4-48) it is obvious that $\varphi_L = 1 + o(1)$ on the left end cap and on the surface of the cylinder where z^* is finite. For z^* large, we can, in addition, easily show that $\varphi_L = 1 + o(1)$ is satisfied in the shank region too. The terms in (4-38) depending on how $z^* \to \infty$ with respect to the upper limit of integration as $\lambda \to 0$ are I_4, I_5, and I_6. As long as z^* is in the shank region and far from the right end, we have $z^* = \frac{2L-\lambda}{\lambda} - \frac{A}{\lambda}$ where $0 < A < 2L$. Then I_4, I_5 and I_6 are unchanged. By substituting $f(z^*)$ obtained in (4-45) into (4-38), it is obvious $\varphi_L = 1 + o(1)$ as $\lambda \to 0$ on the surface of the cylinder. Thus $\varphi_L = 1 + o(1)$ is satisfied on the surface of the finite cylinder, not only on the left end but also in the shank region. But when z^* is in the right end region, $\varphi_L = 1 + o(1)$ is no longer true. In the right end region $z^* = \frac{2L-\lambda}{\lambda} \pm b$ where b is any finite constant. In this case, we have to put $z_3 = \frac{2L}{\lambda} - 1$ in (4-38) and we find I_5 is no longer needed and

$$I_4 = f(z^*) \log z^* + o(1) \tag{4-53}$$

and

$$I_6 = \int_1^{\frac{2L}{\lambda}} \frac{f(\zeta^*)}{\zeta^*} \, d\zeta^* \tag{4-54}$$

Then from (4-30) and (4-41), we obtain

$$\begin{aligned} \varphi_L &= 1 + \gamma(\lambda)[I_4 - I_6] + o(1) \\ &= 1 - \frac{1}{2}\gamma(\lambda) I_6 + o(1) \\ &= \frac{1}{2} + o(1) \end{aligned} \tag{4-55}$$

The same result will be obtained in the right end region if $\varphi_L = \varphi_c$ where φ_c is defined in (4-15). Thus in the right end region, φ_L has the same boundary value as if the source distribution is constant. This can be further justified by expanding the obtained φ_L (cf. 4-49) for r^* = fixed and $z > -L$. Thus we obtain

$$\varphi_L = I_a + I_b \tag{4-56}$$

and

$$I_a = \frac{1}{2}\epsilon^{\frac{1}{2}} \int_{-a}^{z-\epsilon^{\frac{1}{2}}} \frac{1}{\sqrt{\log\frac{L+\zeta}{\lambda}}} \frac{d\zeta}{\sqrt{(z-\zeta)^2+\lambda^2 r^{*2}}}$$

$$= \frac{1}{2}\epsilon^{\frac{1}{2}} \int_{-a}^{z-\epsilon^{\frac{1}{2}}} \frac{1}{\sqrt{\log\frac{L+\zeta}{\lambda}}} \frac{d\zeta}{z-\zeta} + o(1)$$

$$< \frac{1}{2} \int_{-a}^{z-\epsilon^{\frac{1}{2}}} \frac{d\zeta}{\sqrt{\log\frac{L+\zeta}{\lambda}}} = o(1) \tag{4-57a}$$

$$I_b = \frac{1}{2}\epsilon^{\frac{1}{2}} \int_{z-\epsilon^{\frac{1}{2}}}^{a} \frac{1}{\sqrt{\log\frac{L+\zeta}{\lambda}}} \frac{d\zeta}{\sqrt{(z-\zeta)^2+\lambda^2 r^{*2}}}$$

$$= \frac{1}{2}\epsilon \int_{z-\epsilon^{\frac{1}{2}}}^{a} \frac{d\zeta}{\sqrt{(z-\zeta)^2+\lambda^2 r^{*2}}} + o(1)$$

$$= \frac{1}{2}\epsilon \log \frac{\epsilon^{\frac{1}{2}} + \sqrt{\epsilon + \lambda^2 r^{*2}}}{z-a+\sqrt{(z-a)^2+\lambda^2 r^{*2}}} + o(1) \tag{4-57b}$$

$$= 1 + \frac{1}{2}\epsilon \log 2\,(z-a) + o(1) \tag{4-57c}$$

Thus in the shank region, we have

$$\varphi_L = 1 + o(1) \tag{4-58}$$

as discussed previously. It is worth noting that $\varphi_c = 1 + o(1)$ in the shank region too.

In the right end region, if we choose the right end variables as

$$\bar{z}^* = \frac{z-L}{\lambda}, \qquad r^* = \frac{r}{\lambda} \tag{4-59}$$

then from (4-60b) we have

$$\varphi_L = -\frac{1}{2}\epsilon \log \lambda[\bar{z}^* + 1 + \sqrt{(\bar{z}^*+1)^2 + r^{*2}}\,] + o(1)$$

$$= \frac{1}{2} - \frac{1}{2}\epsilon \log [(\bar{z}^* + 1) + \sqrt{(\bar{z}^* + 1)^2 + r^{*2}}\,] + o(1)$$

$$= \frac{1}{2} + o(1) \quad \text{for } \bar{z}^* = \text{finite}$$

$$= \frac{1}{2} + \frac{1}{2}\epsilon \log 2(-\bar{z}^*) + o(1) \quad \text{for } \bar{z}^* \rightarrow -\infty \tag{4-60}$$

Thus $\varphi_L = \frac{1}{2} + o(1)$ in the right end region and (4-60) is the same as (4-22) due to φ_c. In the transition region near the right end, if we put $L - z = \lambda^{\alpha} z_{\alpha}$ we have

$$\varphi_L = 1 - \frac{\alpha}{2} + o(1) \tag{4-61}$$

This is the same as equation (4-23) for φ_c in the transition region. Since both φ_L and φ_c are equal to $1 + o(1)$ in the shank region, we can say that on the right half of the surface of the cylinder we have $\varphi_L = \varphi_c + o(1)$ as $\lambda \rightarrow 0$. From this result, we can easily construct $\varphi_A(P;\lambda)$ defined in (4-5) and (4-6) for the finite cylinder in Fig. 4-1.

4.6 The Solution $\varphi_A(P;\lambda)$ for the Finite Cylinder.

From the preceding discussion, we can conclude

$$\varphi_L = 1 + o(1) \qquad \text{on the left half of } S(\lambda) \tag{4-62}$$

and

$$\varphi_L = \varphi_c + o(1) \qquad \text{on the right half of } S(\lambda) \tag{4-63}$$

By symmetry, a new Laplace solution symmetric to φ_L with respect to the plane z = 0 can be constructed and it is

$$\varphi_R = \frac{1}{2}\,\epsilon^{\frac{1}{2}}\int_{-a}^{a} \frac{1}{\sqrt{\log\frac{L-\zeta}{\lambda}}}\,\frac{d\zeta}{\sqrt{(z-\zeta)^2+r^2}} \tag{4-64}$$

The symmetry can be shown as follows:

$$\varphi_L(-z) = \frac{1}{2}\,\epsilon^{\frac{1}{2}}\int_{-a}^{a} \frac{1}{\sqrt{\log\frac{L+\zeta}{\lambda}}}\,\frac{d\zeta}{\sqrt{(z+\zeta)^2+r^2}}$$

$$= \frac{1}{2}\,\epsilon^{\frac{1}{2}}\int_{-a}^{a} \frac{1}{\sqrt{\log\frac{L-\zeta}{\lambda}}}\,\frac{d\zeta}{\sqrt{(z-\zeta)^2+r^2}} = \varphi_R(z) \tag{4-65}$$

Therefore it is obvious that

$$\varphi_R = 1 + o(1) \qquad \text{on the right half of } S(\lambda) \tag{4-66}$$

and

$$\varphi_R = \varphi_c + o(1) \qquad \text{on the left half of } S(\lambda) \tag{4-67}$$

Therefore $\varphi_A(P;\lambda)$ is easily obtained as follows

$$\varphi_A(P;\lambda) = \varphi_L + \varphi_R - \varphi_c$$

$$= \frac{1}{2}\,\epsilon^{\frac{1}{2}}\int_{-a}^{a}\left\{\frac{1}{\sqrt{\log\frac{L+\zeta}{\lambda}}} + \frac{1}{\sqrt{\log\frac{L-\zeta}{\lambda}}} - \epsilon^{\frac{1}{2}}\right\}$$

$$\times \frac{d\zeta}{\sqrt{(z-\zeta)^2+r^2}} \tag{4-68}$$

Obviously $\varphi_A(P;\lambda)$ satisfies the required boundary conditions

$$\varphi_A = 1 + o(1) \quad \text{as } \lambda \to 0 \tag{4-69}$$

uniformly everywhere on the surface $s(\lambda)$ of the finite cylinder shown in Fig. 4-1.

The solution $\varphi_A(P;\lambda)$ obtained is essentially constructed by superposition of source distribution. The variation of source strength can be sketched as shown in Fig. 4-3.

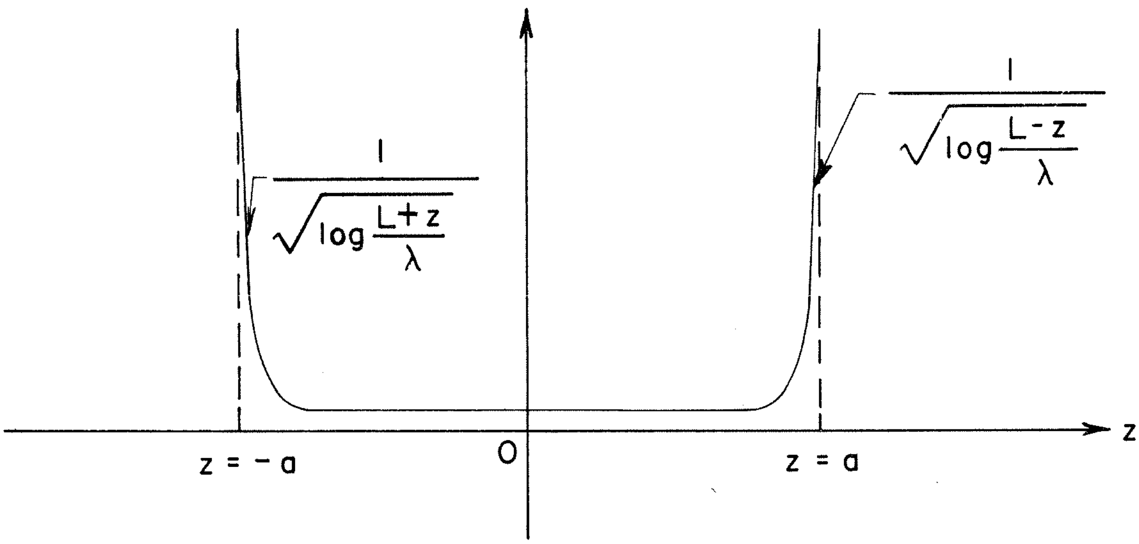

Fig. 4-3

The source strength has an inverse logarithmic variation near each end and in the limit $\lambda \to 0$ the source strength between the two ends is approximately of constant strength $\frac{1}{2}\epsilon$. This shows by φ_C gives the correct boundary value in the shank region.

In addition to source distribution, it is also interesting to know the total source strength of the solution $\varphi_A(P;\lambda)$. The total source strength can be obtained by studying the following integral for $\lambda \to o$

$$I = \int_{-a}^{a} \frac{d\zeta}{\sqrt{\log \frac{L-\zeta}{\lambda}}} \tag{4-70}$$

If we put $y = \sqrt{\log \frac{L-\zeta}{\lambda}}$ and $\eta = \sqrt{\log \frac{2L-\lambda}{\lambda}}$ we get

$$I = 2\lambda \int_{0}^{\eta} e^{y^2} \, dy \tag{4-71a}$$

and $\eta \to \infty$ as $\lambda \to 0$. The following asymptotic expansion is obtained for $\eta \to \infty$

$$\int_{0}^{\eta} e^{y^2} \, dy \sim \frac{e^{\eta^2}}{2\eta} + \frac{e^{\eta^2}}{4\eta^3} + \ldots\ldots \tag{4-71b}$$

The mathematical proof of this asymptotic formula is given in Appendix A. Therefore we obtain

$$I = (2L-\lambda) \left\{ \frac{1}{\sqrt{\log \frac{2L-\lambda}{\lambda}}} + \frac{1}{2(\log \frac{2L-\lambda}{\lambda})^{3/2}} + \ldots\ldots \right\}$$

$$= 2L\epsilon^{\frac{1}{2}} + L\epsilon^{\frac{3}{2}} + O(\epsilon^{\frac{5}{2}}) \tag{4-72}$$

The same results can be obtained by expanding the integrand in terms of series asymptotically

$$I = \int_{-a}^{a} \left\{ \epsilon^{\frac{1}{2}} - \epsilon^{\frac{3}{2}} \frac{1}{2} \log \left(\frac{L-\zeta}{2L}\right) + \ldots\ldots \right\} d\zeta$$

$$= 2L\epsilon^{\frac{1}{2}} + L\,\epsilon^{\frac{3}{2}} + O(\epsilon^{\frac{5}{2}}) \tag{4-73}$$

This justifies the expansion of the integrand in terms of series. The total source strength is then easily obtained from (4-68) as

$$s = \epsilon L + \epsilon^2 L + O(\epsilon^3) \tag{4-74}$$

4.7 <u>The Solution for an Ellipsoid Cylinder.</u>

In the above, we have shown that the constant source distribution is not correct for the finite cylinder shown in Fig. 4-1. In this section, we want to show that the solution for an ellipsoid cylinder can be obtained by use of constant source distribution.

Let the ellipsoid be defined as follows:

$$\frac{r^2}{\lambda^2} + \frac{z^2}{L^2} = 1 \tag{4-75}$$

Then the following constant source distribution

$$\varphi_e = \tfrac{1}{2}\,\epsilon(\lambda) \int_{-c}^{c} \frac{d\zeta}{\sqrt{(z-\zeta)^2 + r^2}} \tag{4-76}$$

where $c = \sqrt{L^2 - \lambda^2}$ and $\epsilon(\lambda) = \dfrac{1}{\log \frac{1}{\lambda} + \log 2L}$

satisfies

$$\nabla^2 \varphi_e = 0 \tag{4-77}$$

and

$$\varphi_e = 0 \quad \text{at infinity} \tag{4-78}$$

On the surface of the ellipsoid cylinder φ_e can be studied by introducing the following proper limits:

a) Shank limit. This limit is defined as the limit $\lambda \to o$ for $r^* = \frac{r}{\lambda}$ and z fixed and $-L < z < L$. As λ tends to zero with L fixed, the body shrinks to a line $r = 0$, $-L \leq |z| \leq L$. However, in the shank limit a point in space (z, r) approaches $(z, 0)$ in such a way that $\frac{r}{\lambda}$ retains the same value. Thus in shank coordinates, the body remains an ellipsoid as can be seen from (4-75) which may be written

$$\frac{r^{*2}L^2}{L^2 - z^2} = 1 \tag{4-79}$$

Under this limit, we obtain

$$\varphi_e = \frac{1}{2}\epsilon \log \frac{z + c + \sqrt{(z+c)^2 + \lambda^2 r^{*2}}}{(z - c) + \sqrt{(z-c)^2 + \lambda^2 r^{*2}}}$$

$$= \frac{1}{2}\epsilon \log 2(z+c) - \frac{1}{2}\epsilon \log \frac{\lambda^2 r^{*2}}{2(c-z)} + O(\lambda^2)$$

$$= \epsilon \log \frac{1}{\lambda} - \frac{1}{2}\epsilon \log \frac{r^{*2}}{4(c^2 - z^2)} + O(\lambda^2)$$

$$= 1 - \frac{1}{2}\epsilon \log \frac{r^{*2}L^2}{L^2 - z^2} + O(\lambda^2) \tag{4-80}$$

A comparison of (4-80) and the equation for the ellipsoid in shank coordinates shows that on the surface of the body

$$\varphi_e = 1 + O(\lambda^2) \tag{4-81}$$

b) End limits. An inspection of (4-80) shows that the term in $\epsilon(\lambda)$ is not uniformly small for r^* large. Furthermore the argument for finite fixed r^* in this term also tends to infinity as $|z|$ tends to L. Thus even for finite r^* we need a different approximation near the ends of the cylinder. We shall therefore introduce appropriate end limits. By symmetry, we will discuss the left end only. The left end limit is defined as $\lambda \to 0$ for $z^+ = \frac{z+L}{\lambda^2}$ and $r^+ = \frac{r}{\lambda^2}$ fixed. Under this limit, we have

$$\varphi_e = \frac{1}{2} \epsilon \log \frac{\sqrt{(z-c)^2 + r^2} - (z-c)}{\sqrt{(z+c)^2 + r^2} - (z+c)}$$

$$= \frac{1}{2} \epsilon \log \frac{\sqrt{(\lambda^2 z^+ + \frac{\lambda^2}{2L} - 2L)^2 + \lambda^4 r^{+2}} - (\lambda^2 z^+ + \frac{\lambda^2}{2L} - 2L)}{\sqrt{\lambda^4 (z^+ - \frac{1}{2L})^2 + \lambda^4 r^{+2}} - \lambda^2 (z^+ - \frac{1}{2L})}$$

$$= \frac{1}{2} \epsilon \log 4L - \frac{1}{2} \epsilon \log \lambda^2 [\sqrt{(z^+ - \frac{1}{2L})^2 + r^{+2}} - (z^+ - \frac{1}{2L})]$$

$$+ O(\lambda^2)$$

$$= 1 - \frac{1}{2} \epsilon \log L - \frac{1}{2} \epsilon \log [\sqrt{(z^+ - \frac{1}{2L})^2 + r^{+2}} - (z^+ - \frac{1}{2L})]$$

$$+ O(\lambda^2) \tag{4-82}$$

Under this end limit, the body in the end coordinates becomes

$$r^{+2} = \frac{2z^+}{L} + O(\lambda^2) \tag{4-83a}$$

Thus to $O(\lambda^2)$, the end body is a paraboloid defined by

$$r^{+2} = \frac{2z^+}{L} \tag{4-83b}$$

or

$$\sqrt{r^{+2} + (z^+ - \frac{1}{2L})^2} - (z^+ - \frac{1}{2L}) = \frac{1}{L} \tag{4-83c}$$

A comparison of (4-82) and (4-83c) shows that on the surface of the body

$$\varphi_e = 1 + O(\lambda^2) \qquad (4\text{-}83d)$$

Therefore from (4-81) and (4-83d) we have shown that $\varphi_e = 1 + O(\lambda^2)$ on the surface of the ellipsoid cylinder. Thus φ_e is the approximate solution for the case of an ellipsoid cylinder. This kind of cylinder with paraboloid ends is classified as "tapered" body. The cylinder shown in Fig. 4-1 is classified as "untapered" body. The definition of "tapered" and "untapered" body will be discussed in the following section.

Finally, from (4-76), the source distribution for the ellipsoid of revolution is constant and the total source strength is

$$s_e = L\,\epsilon + O(\lambda^2) \qquad (4\text{-}84)$$

4.8 Discussion of the Geometrical Matching Condition.

In this chapter, solutions have been obtained for the finite cylinder shown in Fig. 4-1 and the ellipsoid cylinder defined in (4-75). In each case several limits have been introduced. From a purely geometric point of view, the body under both the shank limit and the end limits must be non-degenerate. In this section the geometrical matching condition between the shank and the end limits will be discussed.

1) Shank limit. The shank variables are defined as $r^* = \frac{r}{\lambda}$, $z = z$ where λ is the maximum radius of the cylinders. The shank

limit is the limit $\lambda \to 0$ for r^* and z fixed and $-L < z < L$. Under this limit, the finite cylinder with hemispherical caps and the ellipsoid cylinder becomes $r^* = 1$ and $r^* = \sqrt{\frac{L^2-z^2}{L^2}}$ respectively. Both of them are non-degenerate bodies and this shows that the shank limit is correctly introduced. Now if we define $r = r_s$ under the shank limit, we have

$$r_s = \lambda f_s(z) \tag{4-85}$$

where $f_s(z) = 1$ for the finite cylinder (4-85a)

$$f_s(z) = \sqrt{\frac{L^2-z^2}{L^2}} \text{ for the ellipsoid} \tag{4-85b}$$

2) <u>The end limit:</u> In order that proper three dimensional equations can be obtained, the end variables can, in general, be defined as $r_\alpha = \frac{r}{\lambda^\alpha}$, $z_\alpha = \frac{z+L}{\lambda^\alpha}$ for the left end. By symmetry, it is sufficient to discuss the left end only. The end limit is then the limit $\lambda \to 0$ for r_α, z_α fixed where α will be determined from the pure geometrical consideration that the end body under this limit is non-degenerate.

a) <u>The finite cylinder with hemispherical caps:</u> From (4-3) and $\bar{z} = z + L$, we have

$$r^2 + (\bar{z} - \lambda)^2 = \lambda^2 \qquad \text{for } \bar{z} \leq \lambda \tag{4-86a}$$

$$r = \lambda \qquad \text{for } \bar{z} \geq \lambda \tag{4-86b}$$

Expressing in terms of z_α and r_α, (4-86) gives

$$(r_\alpha^{\ 2} + z_\alpha^{\ 2})\ \lambda^{2(\alpha-1)} - 2z_\alpha\ \lambda^{(\alpha-1)} = 0 \quad \text{for } z_\alpha \leqslant \lambda^{(1-\alpha)} \tag{4-87a}$$

$$r_\alpha = \lambda^{(1-\alpha)} \qquad \text{for } z_\alpha \geqslant \lambda^{(1-\alpha)} \tag{4-87b}$$

Now the only non-degenerate end body which can be obtained under this limit is for $\alpha = 1$ and (4-87) gives

$$r_\alpha^2 + (z_\alpha - 1)^2 = 1 \qquad \text{for } z_\alpha \leqslant 1 \tag{4-88a}$$

$$r_\alpha = 1 \qquad \text{for } z_\alpha \geqslant 1 \tag{4-88b}$$

Then the end variables for the finite cylinder shown in Fig. 4-1 are

$$r_\alpha = \frac{r}{\lambda}, \qquad z_\alpha = \frac{z}{\lambda} \tag{4-89}$$

b) <u>The ellipsoid cylinder.</u> The ellipsoid is defined in (4-75) and if we express it in terms of r_α and z_α, we have

$$r_\alpha^{\ 2}\ \lambda^{2(\alpha-1)} = \frac{2}{L}\ z_\alpha \lambda^\alpha - \frac{z_\alpha^{\ 2}}{L^2}\ \lambda^{2\alpha} \tag{4-90}$$

The only non-degenerate end body which can be obtained is for

$$2(\alpha-1) = \alpha \qquad \text{or } \alpha = 2 \tag{4-91}$$

which gives

$$r_\alpha^{\ 2} = \frac{2}{L}\ z_\alpha \tag{4-92}$$

a semi-infinite paraboloid. Thus the proper end variables in this case are

$$r_\alpha = \frac{r}{\lambda^2}, \qquad z_\alpha = \frac{\bar{z}}{\lambda^2} = \frac{z+L}{\lambda^2} \tag{4-93}$$

If we define $r = r_e$ under the end limit, then in general we can express it in the following form

$$r_e = \lambda^{\alpha} f_e (z_{\alpha}) \tag{4-94}$$

Then for the finite cylinder, we have $\alpha = 1$ and

$$f_e(z_{\alpha}) = \sqrt{1 - (z_{\alpha}-1)^2} \qquad \text{for } z_{\alpha} \leq 1 \tag{4-95a}$$

$$f_e (z_{\alpha}) = 1 \qquad \text{for } z_{\alpha} \geq 1 \tag{4-95b}$$

and for the ellipsoid, we have $\alpha = 2$

$$f_e = \sqrt{\frac{2}{L} \; z_{\alpha}} \tag{4-96}$$

3) Geometrical matching: From pure geometrical consideration, the following matching condition must be fulfilled between the shank and end limit

$$\lim_{int} \frac{r_s}{r_e} = \lim_{\substack{\lambda \to 0 \\ z_{\eta}=\text{fixed}}} \frac{\lambda f_s(z_{\eta} \eta(\lambda))}{\lambda^{\alpha} f_e(\frac{z_{\eta}\eta(\lambda)}{\lambda^{\alpha}})} = 1 \tag{4-97}$$

where $z_{\eta} = \frac{\bar{z}}{\eta(\lambda)} = \frac{z+L}{\eta(\lambda)}$ and $\lambda^{\alpha} << \eta(\lambda) << 1$.

This limit is the intermediate limit between the end limit and the shank limit and is defined as the limit $\lambda \to 0$ for z_{η} fixed. This matching condition also helps to check whether the end limit is properly chosen. In the present case, we can easily show that (4-9) is fulfilled for both cylinders.

a) The finite cylinder. (4-85) and (4-95) give

$$f_s(z) = f_s [z_{\eta} \; \eta(\lambda)] = 1 \tag{4-98a}$$

and

$$f_e(z_\alpha) = f_e\left[\frac{z_\eta\,\eta(\lambda)}{\lambda}\right]$$

$$= f_e\,(\infty) = 1 \tag{4-98b}$$

Then it is obvious that we have

$$\lim_{int} \frac{r_s}{r_e} = \lim_{\substack{\lambda\to 0 \\ z_\eta = \text{fixed}}} \frac{\lambda \times 1}{\lambda \times 1} = 1 \tag{4-99}$$

It can easily be shown that this matching condition can not be satisfied if we choose $\alpha \neq 1$ for the end limit.

b) <u>The ellipsoid cylinder.</u> (4-85) and (4-96) give

$$\lim_{int} \frac{r_s}{r_e} = \frac{\lambda \sqrt{2\,\frac{z_\eta\,\eta(\lambda)}{L} - \frac{z_\eta^{\,2}\,\eta(\lambda)^2}{L^2}}}{\lambda^2 \sqrt{\frac{2z_\eta\,\eta(\lambda)}{\lambda^2 L}}} \tag{4-100}$$

$$= \lim_{\substack{\lambda\to 0 \\ z_\eta = \text{fixed}}} \sqrt{1 - \frac{1}{2}\,\frac{\eta(\lambda)\,z_\eta}{L}} = 1 \tag{4-101}$$

Thus the geometrical matching is also fulfilled. It can also easily be seen that this geometrical matching condition can not be satisfied if we choose $\alpha \neq 2$ for the ellipsoid cylinder.

4) <u>The "tapered" body and the "untapered" body.</u> From the two spécific finite cylinders under study here we can see that under the proper end limits, the cylinders, in general, become semi-infinite cylinders and furthermore there are only two kinds of such semi-infinite cylinders, i.e.

a) The semi-infinite cylinders have the character that $r_\alpha \to \infty$ as $z_\alpha \to \infty$. For instance, the semi-infinite paraboloid obtained from the end limit for the ellipsoid cylinder belongs to this kind. In this thesis, we will define those finite cylinders which have this kind of end bodies under the proper end limit as the "tapered" body. For instance, the ellipsoid cylinder is a "tapered" body.

b) The semi-infinite cylinders have the character that r_α = finite as $z_\alpha \to \infty$. For instance, the semi-infinite cylinder shown in Fig. 4-2, obtained from the end limit for the finite cylinder with hemispherical caps belongs to this kind. In this thesis, we will define those finite cylinders which have this kind of end bodies under the proper end limit as the "untapered" body. From this definition, it is obvious that the solution φ_A obtained for the finite cylinder shown in Fig. 4-1 is in fact valid for any kind of "untapered" bodies.

V. LOW REYNOLDS NUMBER FLOW PAST A FINITE CIRCULAR CYLINDER OF LARGE ASPECT RATIO WITH TWO SPHERICAL CAPS ATTACHED TO EACH END

5.1 Introduction

In this chapter, the problem of viscous incompressible flow past a circular finite cylinder of large aspect ratio will be discussed. The finite cylinder under study is assumed to have circular cross section with two hemispherical caps attached to each end. The axis of the cylinder is perpendicular to the uniform flow at infinity. The cylinder is shown in Fig. 4-1.

In the present study, two non-dimensional parameters, namely, $Re = \frac{U\lambda}{\nu}$ and $\overline{Re} = \frac{UL}{\nu}$ where λ is the radius of the cylinder and L is the half length of the given cylinder, can be formed. The low Reynolds number is understood in the sense $Re \rightarrow 0$, $\overline{Re}$ = fixed. This may be thought of as letting $\lambda \rightarrow 0$ with all other parameters (i.e. U, L, ν) fixed. Thus $\frac{L}{\lambda} \rightarrow \infty$ and this implies the finite cylinder is of large aspect ratio. Under this limit, the cylinder shrinks to a needle of zero radius and the principal limit (i.e. outer limit) is obviously $\vec{g}_o = \vec{i}$. Thus the outer limit is the Oseen limit discussed in Chapter 3. The principal limit does not satisfy the boundary condition near the body. Therefore different limits must be introduced near the body. As we know that in the limit $Re \rightarrow 0$, there is a very viscous region near the body and Stokes flow are expected.

The present problem is more difficult than the corresponding problem for a sphere in three dimensions or a two dimensional circular cylinder. The reason is that in the present case, the inner limit is not uniform even near the body. This has been discussed in detail in the Laplace problem in Chapter 4. A similar shank limit will be introduced for flow far from the two ends. This limit corresponds to the inner limit for a two dimensional cylinder and the two dimensional Stokes equations are expected. Two end limits will be introduced for flow near two ends and they are similar to the corresponding limits in the Laplace case. These limits are introduced with the help of the fact that under these limits, three dimensional Stokes equations are expected and the cylinder becomes a non-degenerate semi-infinite cylinder. The proper end variables which will be introduced have been discussed in the geometrical matching in Chapter 4. In the present case, the following limits will be introduced

1) outer limit
2) shank limit
3) left end limit
4) right end limit

The necessity for introducing the above limits are also discussed in detail in the related Laplace problem in Chapter 4. The main underlying idea for obtaining solution in the present case is the idea of "Stokeslet" discussed in Chapter 3. The same idea should apply to other shapes, i.e. the semi-infinite cylinder, etc. For the finite cylinder the solution in the shank region (i.e. r small and $-L < z < L$) is simply given by the simplest "Stokeslet"

$$\vec{h}_1{}^1 = \vec{i} \log r^* - x^* \nabla^* \log r^* \qquad (5\text{-}1)$$

This is simply the fundamental solution as shown in equation (2-13). The shank solution fails to be valid near the end. This is obvious from the fact that near the end, the flow is three dimensional. Therefore the end limit process must be introduced and under this limit the end body becomes a non-degenerate semi-infinite body. One obvious solution for semi-infinite body is the Stokes solution for a paraboloid and is found as

$$\vec{\ell}_1 = \vec{i}\ \tfrac{1}{2}\left\{\log \frac{\tau^*}{\tau_o} + 1\right\} - \tfrac{1}{2} x^* \nabla^* \log \tau^* - \frac{\tau_o}{2} \nabla^* \frac{x^*}{\tau^*} \tag{5-2a}$$

where

$$\tau^* = \tfrac{1}{2}\left\{\sqrt{r^{*2} + (z^* - \tau_o)^2} - (z^* - \tau_o)\right\}$$

This solution satisfies the exact boundary condition.

$$\vec{\ell}_1 = 0 \qquad \text{on } \tau = \tau_o = \text{constant}$$

and grows slowly at infinity. This shows that the idea of Stokeslets can be applied for a semi-infinite body and a "Stokeslet" is easily found to be

$$\vec{\ell}_1^{\,1} = \vec{i}\ \tfrac{1}{2} \log \tau^* - \tfrac{1}{2} x^* \nabla^* \log \tau^* \tag{5-2b}$$

This is a correct "Stokeslet" for the end body of an ellipsoid but is not the correct "Stokeslet" for the finite circular cylinder shown in Fig. 4-1. For the finite cylinder the end body is a semi-infinite cylinder with a hemispherical cap attached to the end. It can easily be shown that $\vec{\ell}_1^1$ is not the correct Stokeslet for this semi-infinite cylinder because it is not bounded on the surface of the cylinder. For the finite cylinder, it will be shown that the "Stokeslets" can be generated from the corresponding solution for the related Laplace problem. The nature of source distribution for the semi-infinite end cylinder can easily be clarified with the help of the solution for the Laplace case. Then a uniformly valid expansion to order unity can be obtained.

5.2 Limits, Expansions and Associated Equations:

In this section, we will define the various limits needed in various regions discussed above. Then proper expansions will be introduced and the associated equations or the governing equations will be established in the corresponding regions.

a) Outer. As independent variables (outer variables) we use

$$\tilde{x}_i = \frac{Ux_i}{\nu} \tag{5-3a}$$

The dependent variables are $\vec{q}^* = \frac{\vec{q}}{U}$ and $p^* = \frac{p-p_\infty}{\rho U^2}$ (see list of symbols). The outer limit is defined as the limit $Re \to 0$ for $\overline{Re}$, $\tilde{x}_i$ fixed. Thus Navier-Stokes equations can be written in terms of outer variables as

$$(\vec{q}^* \cdot \tilde{\nabla})\vec{q}^* + \tilde{\nabla} p^* = \tilde{\nabla}^2 \vec{q}^* \tag{5-3b}$$

$$\tilde{\nabla} \cdot \vec{q}^* = 0 \tag{5-3c}$$

where $\tilde{\nabla}$ and $\tilde{\nabla}^2$ are in terms of outer variables. The outer expansion of velocity and pressure are assumed to have the form

$$\vec{q}^{\,*} = \vec{g}_0 + \epsilon\, \vec{g}_1 + \epsilon^2\, \vec{g}_2 + \ldots\ldots \tag{5-3d}$$

$$p^* = \epsilon\, p_1^* + \epsilon^2\, p_2^* + \ldots\ldots \tag{5-3e}$$

The outer limit is $\vec{g}_0 = \vec{i}$ in the present study. Therefore the governing equations for $\vec{g}_1$ are the Oseen equations

$$\frac{\partial \tilde{g}_1}{\partial \tilde{x}} + \tilde{\nabla}\, p_1^* = \tilde{\nabla}^2\, \vec{g}_1 \tag{5-4a}$$

$$\tilde{\nabla} \cdot \vec{g}_1 = 0 \tag{5-4b}$$

and the governing equations for $\vec{g}_n$ $(n \geq 2)$ are

$$\frac{\partial \vec{g}_n}{\partial \tilde{x}} + \tilde{\nabla}\, p_n^* = \nabla^2\, \vec{g}_n + f_n(\tilde{x}_i) \tag{5-5a}$$

$$\tilde{\nabla} \cdot \vec{g}_n = 0 \tag{5-5b}$$

where

$$f_n(\tilde{x}_i) = \sum_{i=1}^{n-1} (\vec{g}_i \cdot \tilde{\nabla})\, \vec{g}_{n-i}$$

b) <u>Shank</u>. In the shank region $-\overline{Re} < \tilde{z} < \overline{Re}$, we use the following independent variables.

$$x^* = \frac{x}{\lambda} = \frac{\tilde{x}}{Re}\,, \quad y^* = \frac{\tilde{y}}{Re}\,, \quad \tilde{z} = \frac{Uz}{\nu} \tag{5-6a}$$

The shank limit is defined as the limit $Re \to 0$ for x^*, y^*, $\tilde{z}$ and $\overline{Re}$ fixed.

The dependent variables are

$$\vec{q}^{\,*} = \frac{\vec{q}}{U} = \vec{i}\, U^* + \vec{j}\, V^* + \vec{k}\, W^* \tag{5-6b}$$

$$p^+ = \frac{\lambda(p - p_\infty)}{\rho U} \tag{5-6c}$$

The Navier-Stokes equations in terms of shank variables are as follows.

$$Re\, U^* \frac{\partial U^*}{\partial x^*} + Re\, V^* \frac{\partial U^*}{\partial y^*} + Re^2\, W^* \frac{\partial U^*}{\partial \tilde{z}} + \frac{\partial p^+}{\partial x^*}$$

$$= \frac{\partial^2 U^*}{\partial x^{*2}} + \frac{\partial^2 U^*}{\partial y^{*2}} + Re^2 \frac{\partial^2 U^*}{\partial \tilde{z}^2} \tag{5-7a}$$

$$Re\, U^* \frac{\partial V}{\partial x^*} + Re\, V^* \frac{\partial V^*}{\partial y^*} + Re^2\, W^* \frac{\partial V^*}{\partial \tilde{z}} + \frac{\partial p^+}{\partial y^*}$$

$$= \frac{\partial^2 V^*}{\partial x^{*2}} + \frac{\partial^2 V^*}{\partial y^{*2}} + Re^2 \frac{\partial^2 V^*}{\partial \tilde{z}^2} \tag{5-7b}$$

$$Re\, U^* \frac{\partial W^*}{\partial x^*} + Re\, V^* \frac{\partial W^*}{\partial y^*} + Re^2\, W^* \frac{\partial W^*}{\partial \tilde{z}} + Re \frac{\partial p^+}{\partial \tilde{z}}$$

$$= \frac{\partial^2 W^*}{\partial x^{*2}} + \frac{\partial^2 W^*}{\partial y^{*2}} + Re^2 \frac{\partial^2 W^*}{\partial z^{*2}} \tag{5-7c}$$

$$\frac{\partial U^*}{\partial x^*} + \frac{\partial V^*}{\partial y^*} + Re \frac{\partial W^*}{\partial \tilde{z}} = 0 \tag{5-7d}$$

The limiting equations for $Re \rightarrow 0$ are

$$\frac{\partial^2 U^*}{\partial x^2} + \frac{\partial^2 U^*}{\partial y^{*2}} = \frac{\partial p^+}{\partial x^*} \tag{5-8a}$$

$$\frac{\partial^2 V^*}{\partial x^{*2}} + \frac{\partial^2 V^*}{\partial y^{*2}} = \frac{\partial p^+}{\partial y^*} \tag{5-8b}$$

$$\frac{\partial^2 W^*}{\partial x^{*2}} + \frac{\partial^2 W^*}{\partial y^{*2}} = 0 \tag{5-8c}$$

and

$$\frac{\partial U^*}{\partial x^*} + \frac{\partial V^*}{\partial y^*} = 0 \qquad (5\text{-}8d)$$

As in the case of low Reynolds number flow past a circular cylinder discussed in Chapter 3, the intermediate limits between the outer limit and the shank limit can be introduced here and are called the intermediate shank limits. The significance of the intermediate limits have been discussed in Chapter 2 and will not be repeated here. The intermediate shank expansions of velocity and pressure are assumed to have the form

$$\vec{q}^* = \vec{u}_o + \epsilon\, \vec{u}_1 + \ldots\ldots \qquad (5\text{-}9a)$$

$$p^+ = p_o + \epsilon\, p_1 + \ldots\ldots \qquad (5\text{-}9b)$$

where $\vec{u}_o$, p_o, etc. are intermediate shank solutions which are functions of x^*, y^*, $\tilde{z}$, $\overline{Re}$ and Re. The governing equation for $\vec{u}_o$ and p_o are equations (5-8).

c) Left End. In the left end region, as independent variables, we use

$$x^* = \frac{\tilde{x}}{\overline{Re}}, \quad y^* = \frac{\tilde{y}}{\overline{Re}}, \quad z^* = \frac{\tilde{z} + \overline{Re}}{\overline{Re}} \qquad (5\text{-}10)$$

This corresponds to using the left end as origin for the rectangular cartesian coordinates x^*, y^*, z^*.

The left end limit is defined as the limit $Re \to 0$ for x^*, y^*, z^* and $\overline{Re}$ fixed. The Navier-Stokes equation can be written in terms of left end variables as

$$Re\ (\vec{q}^{\,*} \cdot \nabla^*)\ \vec{q}^{\,*} + \nabla^* p^+ = \nabla^{*2}\, \vec{q} \tag{5-11a}$$

$$\nabla^* \cdot \vec{q}^{\,*} = 0 \tag{5-11b}$$

Similarly we can introduce an intermediate limit between the outer limit and the left end limit. These intermediate limits will be called the intermediate left end limits. The intermediate left end expansions of velocity and pressure are

$$\vec{q}^{\,*} = \vec{v}_o + \epsilon\, \vec{v}_1 + \ldots\ldots \tag{5-12a}$$

$$p^+ = p_o^+ + \epsilon\, p_1^+ + \ldots\ldots \tag{5-12b}$$

where $\vec{v}_o$ and p_o are intermediate left end solutions in the same sense of the intermediate solutions discussed in Chapter 3. The governing equations for $\vec{v}_o$ and p_o^+

$$\nabla^* p_o^+ = \nabla^{*2}\, \vec{v}_o \tag{5-13a}$$

$$\nabla^* \cdot \vec{v}_o = 0 \tag{5-13b}$$

The above equations are the familiar three dimensional Stokes equations and thus the intermediate left end solution may be formed by proper use of the idea of Stokeslet discussed in Chapter 3.

d) <u>Right End.</u> In the right end region we use the right end variable

$$x^* = \frac{\tilde{x}}{Re}\ ,\quad y^* = \frac{\tilde{y}}{Re}\ ,\quad \bar{z}^* = \frac{\tilde{z} - \overline{Re}}{Re} \tag{5-14}$$

The right end limit is defined as the limit $Re \rightarrow 0$ for x^*, y^*,

$\vec{z}^*$ and $\overline{Re}$ fixed. Similarly an intermediate limit between the outer and the right end limit can be introduced and is called the intermediate right end limit. The intermediate right end solution of velocity and the pressure are assumed to have the form

$$\vec{q}^* = \vec{w}_o + \epsilon \vec{w}_1 + \ldots\ldots \tag{5-15a}$$

$$p^+ = p_o^{++} + \epsilon\, p_1^{++} + \ldots\ldots \tag{5-15b}$$

Since the cylinder has two similar ends and is symmetrical with respect to the xy-plane $\vec{w}_o$, $\vec{w}_1$ will be similar to $\vec{v}_o$ and $\vec{v}_1$ respectively. The governing equations are three dimensional Stokes equations.

5.3 Determination of Solutions

Determination of $\vec{g}_o$: In the present problem, it is obvious that the principal limit is $\vec{g}_o = \vec{i}$. The main idea is that in the limit $Re \to 0$, the cylinder shrinks to a three dimensional needle of zero radius, has no arresting power and therefore it can not cause finite disturbances. In this sense, $\vec{g}_o = \vec{i}$ is the correct principal limit. Then the intermediate shank solution $\vec{u}_o$ and the intermediate end solution $\vec{v}_o$ and $\vec{w}_o$ can be determined by matching to $\vec{g}_o = \vec{i}$.

Determination of $\vec{u}_o$: Under the shank limit, the cylinder becomes infinite. The following simplest Stokeslet

$$\vec{h}_1^1 = \vec{i} \log r^* - x^* \nabla^* \log r^* \tag{5-16}$$

can be used. In fact, this Stokes solution can be generated directly from the corresponding Laplace solution (4-14) by use of the same source distribution. Then as in the case of two dimensional low

Reynolds number flow past a circular cylinder discussed in Chapter 3, the intermediate solution $\vec{u}_o$ can be formed by multiplying $\vec{h}_1^1$ by a small parameter $\epsilon_1(Re) \to 0$ as $Re \to 0$ and is

$$\vec{u}_o = \epsilon_1(Re) \{ \vec{i} \log r^* - x^* \nabla^* \log r^* \} \tag{5-17}$$

It is obvious that $\vec{u}_o$ satisfies the governing equation (5-8) and $\vec{u}_o \to 0$ on the cylinder as $Re \to 0$. $\epsilon_1(Re)$ can be determined by the matching condition

$$\lim_{\substack{f \\ Re \to 0}} | \vec{u}_o - \vec{i} \, | = 0 \tag{5-18}$$

in the limit $Re \to 0$ for order f in some domain. The matching condition is satisfied if we choose

$$\epsilon_1 = \frac{1}{\log \frac{1}{Re}} = \epsilon(Re) \tag{5-19}$$

Therefore we have

$$\vec{u}_o = \epsilon \{ \vec{i} \log r^* - x^* \nabla^* \log r^* \} \tag{5-20}$$

<u>Determination of $\vec{v}_o$</u>: $\vec{v}_o$ may be constructed by use of the same source distribution for the function φ_L established in (4-49). From φ_L, we know that the following Laplace solution

$$F^{(1)}(z^*, r^*) = \int_1^{\frac{2\overline{Re}}{Re} - 1} \frac{1}{\sqrt{\log \zeta^*}} \left\{ \frac{1}{\sqrt{(z^* - \zeta^*)^2 + r^{*2}}} - \frac{1}{\zeta^*} \right\} d\zeta^* \tag{5-21}$$

is bounded over the surface of the semi-infinite end cylinder under the present study. Therefore by the method of generating Stokes solution from Laplace solution, the following Stokes solution

$$\vec{\ell}_o = \vec{i}\, F^{(1)}(z^*, r^*) - x^* \nabla^* F^{(1)}(z^*, r^*) \qquad (5\text{-}22)$$

is obtained. $\vec{\ell}_o$ will be a "Stokeslet" if we can show that $\vec{\ell}_o$ is bounded on the surface of the semi-infinite cylinder. This can be shown as follows:

$$\vec{\ell}_o = \vec{i}\, F^{(1)} + \vec{i} \int_1^{\frac{2\overline{Re}}{Re} - 1} \frac{1}{\sqrt{\log \zeta^*}} \frac{x^{*2}\, d\zeta^*}{[(z^* - \zeta^*)^2 + r^{*2}]^{3/2}}$$

$$+ \vec{j} \int_1^{\frac{2\overline{Re}}{Re} - 1} \frac{1}{\sqrt{\log \zeta^*}} \frac{x^* y^*\, d\zeta^*}{[(z^* - \zeta^*)^2 + r^{*2}]^{3/2}}$$

$$+ \vec{k} \int_1^{\frac{2\overline{Re}}{Re} - 1} \frac{1}{\sqrt{\log \zeta^*}} \frac{x^* (z^* - \zeta^*)\, d\zeta^*}{[(z^* - \zeta^*)^2 + r^{*2}]^{3/2}}$$

$$= \vec{i}\, F^{(1)} + \vec{i}\, A + \vec{j}\, B + \vec{k}\, C \qquad (5\text{-}23)$$

We can easily see that $\vec{\ell}_o$ is bounded on the surface of the cylinder for finite z^*. The asymptotic representation of $\vec{\ell}_o$ for $z^* \to \infty$ can be studied by the same method as in Chapter 4. We will divide the range of integration into the same intervals as before $[1, z_o]$

$[z_o, z_1]$, $[z_1, z_2]$, $[z_2, z_3]$ and $[z_3, \frac{2\overline{Re}}{Re} - 1]$

where z_o, z_1, z_2, z_3 are defined as follows

$$z_0 = \text{constant} > 1$$
$$z_1 = \sqrt{z^*}$$
$$z_2 = (1 - \beta)z^* \qquad (5\text{-}24)$$
$$z_3 = (1 + \beta)z^*$$

In fact, we can choose β to be any finite positive constant smaller than unity. Then for $z^* \to \infty$ and $r^* =$ fixed, we have

$$F^{(1)}(z^*, r^*) = -\frac{2\log r^*}{\sqrt{\log z^*}} + O\left(\frac{g(\beta)}{\sqrt{\log z^*}}\right) \qquad (5\text{-}25)$$

The detail evaluation of $F^{(1)}(z^*, r^*)$ was shown in (4-38) and will not be repeated here. $g(\beta)$ in (5-25) is a function of β and is a finite constant. Thus we can see that $F^{(1)}(z^*, r^*) = O\left(\frac{1}{\sqrt{\log z^*}}\right) \to 0$ and is bounded on the surface of the cylinder for $z^* \to \infty$. In the shank region, we have

$$\frac{1}{\sqrt{\log z^*}} = \frac{1}{\sqrt{\log \frac{1}{Re} + \log[\tilde{z} + \overline{Re}]}} \qquad (5\text{-}26)$$

$$= \epsilon^{\frac{1}{2}} - \frac{1}{2}\epsilon^{\frac{3}{2}} \log[\tilde{z} + \overline{Re}] + O(\epsilon^{\frac{5}{2}}) \qquad (5\text{-}27)$$

or

$$F^{(1)}(z^*, r^*) = -2\epsilon^{\frac{1}{2}} \log r^* + O(\epsilon^{\frac{1}{2}}) \qquad (5\text{-}28)$$

which is of the order of $\epsilon^{\frac{1}{2}} = \frac{1}{\sqrt{\log 1/Re}}$ on the surface of the cylinder in the shank region. As the only relation between r^* and z^* used in obtaining the representation of $F^{(1)}$ is $\frac{r^*}{z^*} \to 0$, we have

$$F^{(1)} = -2\epsilon^{\frac{1}{2}} \log \frac{r^{(f)} f(Re)}{Re} + O(\epsilon^{\frac{1}{2}}) \qquad (5\text{-}29)$$

for ord Re $\leq$ ord f(Re) < 1 in the intermediate shank region. The second term in the shank region and the intermediate shank region is always $O(\epsilon^{\frac{1}{2}})$ and the leading has the above expression for not only inner shank region but also all the intermediate shank regions.

It is also interesting to know that in the inner shank region, i.e. f(Re) = Re), we have

$$F^{(1)} = O(\epsilon^{\frac{1}{2}}) \qquad (5\text{-}30)$$

but in the intermediate shank region $f(Re) = Re^{\alpha}$, $0 < \alpha < 1$, we have

$$F^{(1)} = -\frac{2(1-\alpha)}{\epsilon^{\frac{1}{2}}} + O(\epsilon^{\frac{1}{2}}) \qquad (5\text{-}31)$$

This shows why the first term must separate from the second term.

Similarly for r^* = fixed and $z^* \to \infty$, we have

$$A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \qquad (5\text{-}32a)$$

and

$$A_1 = r^{*2} \int_1^{z_0} \frac{1}{\sqrt{\log \zeta^*}} \frac{d\zeta^*}{[(z^*-\zeta^*)^2 + r^{*2}]^{3/2}} = \frac{r^{*2} A_0}{z^{*3}} + O(\frac{1}{z^{*6}}) \qquad (5\text{-}32b)$$

$$A_2 = r^{*2} \int_{z_0}^{z_1} \frac{1}{\sqrt{\log \zeta^*}} \frac{d\zeta^*}{[(z^*-\zeta^*)^2 + r^{*2}]^{3/2}} \leq \frac{r^{*2} A_1}{z^{*5/2}} \qquad (5\text{-}32c)$$

$$A_3 = x^{*2}\int_{z_1}^{z_2} \frac{1}{\sqrt{\log\zeta^*}} \frac{d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}} \leqslant \frac{x^{*2}(1-\beta^2)}{\beta^2 z^{*2}\sqrt{1/2\log z^*}} \qquad (5\text{-}32d)$$

$$A_4 = x^{*2}\int_{z_2}^{z_3} \frac{1}{\sqrt{\log\zeta^*}} \frac{d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}}$$

$$= \frac{x^{*2}}{\sqrt{\log(1+\eta)z^*}} \int_{z_2}^{z_3} \frac{d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}}$$

$$-\beta \leqslant \eta \leqslant \beta$$

$$= \frac{x^{*2}}{r^{*2}\sqrt{\log(1+\eta)z^*}} \left\{ \frac{z^*-z_3}{[(z^*-z_3)^2+r^{*2}]^{1/2}} - \frac{z^*-z_2}{[(z^*-z_2)^2+r^{*2}]^{1/2}} \right\}$$

$$= \frac{x^{*2}}{r^{*2}} \frac{1}{\sqrt{\log(1+\eta)z^*}} \left\{ 2 + O\left(\frac{r^{*2}}{\beta^2 z^{*2}}\right) \right\} \qquad (5\text{-}32e)$$

$$A_5 = x^{*2}\int_{z_3}^{\frac{2\overline{Re}}{Re}-1} \frac{1}{\sqrt{\log\zeta^*}} \frac{d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}} \leqslant \frac{x^{*2}}{\beta^2 z^{*2}} \frac{1}{\sqrt{\log(1+\eta)z^*}} \qquad (5\text{-}32f)$$

$$A_6 = 0 \qquad (5\text{-}32g)$$

Therefore we have

$$A = \frac{2}{\sqrt{\log(1+\eta)z^*}} \frac{x^{*2}}{r^{*2}} + O\left[\frac{1}{\beta^2 z^{*2}\sqrt{\log z^*}}\right] \qquad (5\text{-}33)$$

$$\rightarrow 0 \quad \text{as } z^* \rightarrow \infty$$

In the shank region, we have $z^* = \frac{\tilde{z}+\overline{Re}}{Re}$ and

$$A = 2\epsilon^{\frac{1}{2}} \frac{x^{*2}}{r^{*2}} + O(\epsilon^{\frac{3}{2}}) \tag{5-34}$$

Thus we have shown that A is bounded on the surface of the semi-infinite cylinder. In addition, it is bounded on the surface of the cylinder in the shank region too.

Since we have

$$B = \frac{y^* A}{x^*} \tag{5-35}$$

Therefore for r^* = fixed and $z^* \to \infty$, we have

$$B = \frac{2}{\sqrt{\log(1+\eta)z^*}} \frac{x^* y^*}{r^{*2}} + O\left[\frac{1}{\beta^2 z^{*2}\sqrt{\log z^*}}\right] \tag{5-36}$$

$$\to 0 \quad \text{as} \quad z^* \to \infty$$

Thus B is also bounded on the surface of the semi-infinite cylinder. In the shank region, we have

$$B = 2\epsilon^{\frac{1}{2}} \frac{x^* y^*}{r^{*2}} + O(\epsilon^{\frac{3}{2}}) \tag{5-37}$$

as expected.

Lastly the asymptotical expression of C for r^* = fixed and $z^* \to \infty$, can be obtained by the same method as

$$C = C_1 + C_2 + C_3 + C_4 + C_5 + C_6 \tag{5-38a}$$

and

$$C_1 = \int_1^{z_o} \frac{1}{\sqrt{\log\zeta^*}} \frac{(z^*-\zeta^*)d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}} = \frac{A_o}{z^{*2}} + O(\frac{1}{z^{*4}}) \qquad (5\text{-}38b)$$

$$C_2 = \int_{z_0}^{z_1} \frac{1}{\sqrt{\log\zeta^*}} \frac{(z^*-\zeta^*)d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}} \leq \frac{A_1}{z^{*2}} \qquad (5\text{-}38c)$$

$$C_3 = \int_{z_1}^{z_2} \frac{1}{\sqrt{\log\zeta^*}} \frac{(z^*-\zeta^*)d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}} \leq \frac{1}{\sqrt{1/2\log z^*}} \int_{z_1}^{z_2} \frac{d\xi^*}{(z^*-\xi^*)^2}$$

$$= \frac{1}{\sqrt{1/2\log z^*}} \left| \frac{1}{z^*-z_1} - \frac{1}{z^*-z_2} \right|$$

$$= \frac{1}{\sqrt{1/2\log z^*}} \frac{1-\beta}{\beta z^*} + O\left(\frac{1}{z^{*3/2}\sqrt{\log z^*}}\right) \qquad (5\text{-}38d)$$

$$C_4 = \int_{z_2}^{z_3} \frac{1}{\sqrt{\log\zeta^*}} \frac{(z^*-\zeta^*)d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}} \leq \frac{1}{\sqrt{\log(1-\beta)z^*}} \int_{-\beta z^*}^{\beta z^*} \left| \frac{t}{[t^2+r^2]^{3/2}} \right| dt$$

$$= \frac{1}{\sqrt{\log(1-\beta)z^*}} \frac{2}{(\beta^2 z^{*2}+r^{*2})^{1/2}}$$

$$= \frac{2}{\sqrt{\log(1-\beta)z^*}} \frac{1}{\beta z^*} + O\left(\frac{1}{z^{*3}\sqrt{\log z^*}}\right) \qquad (5\text{-}38e)$$

$$C_5 = \int_{z_2}^{z_3} \frac{1}{\sqrt{\log\zeta^*}} \frac{(z^*-\zeta^*)d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2})^{3/2}} \leq \frac{1}{\sqrt{1/2\log z^*}} \frac{1}{\beta z^*} \qquad (5\text{-}38f)$$

$$C_6 = 0 \tag{5-38g}$$

Thus we obtain

$$C = O\left(\frac{1}{\beta z^* \sqrt{\log z^*}}\right) \to 0 \text{ as } z^* \to \infty \tag{5-39}$$

and C is bounded on the surface of the semi-infinite cylinder. In the shank region, we have

$$C = O\left(Re\,\epsilon^{\frac{1}{2}}\right) \tag{5-40}$$

Therefore we have shown that $\vec{\ell}_o$ is a Stokes solution which is bounded on the surface of the semi-infinite cylinder and $\vec{\ell}_o$ is a correct "Stokeslet" for the specific semi-infinite end cylinder under the present study.

The intermediate left end solution $\vec{v}_o$ can be formed by multiplying $\vec{\ell}_o$ by $\epsilon_1(Re)$. $\epsilon_1(Re)$ tends to zero as Re tends to zero and will be determined by matching with outer solution $\vec{g}_o = \vec{i}$. Now we have

$$\vec{v}_o = \epsilon_1(Re)\,\{\vec{i}\,F^{(1)}(z^*, r^*) - x^*\nabla^* F^{(1)}(z^*, r^*)\}$$

$$= \vec{i}\,\epsilon_1(Re)\, 2\sqrt{\log\left(\frac{2\overline{Re}}{Re} - 1\right)}$$

$$+ \vec{i}\,\epsilon_1(Re) \int_1^{\frac{2\overline{Re}}{Re} - 1} \frac{1}{\sqrt{\log \zeta^*}} \frac{d\zeta^*}{\sqrt{(z^* - \zeta^*)^2 + r^{*2}}}$$

$$-\epsilon_1(Re)\,x^*\nabla^* \int_1^{\frac{2\overline{Re}}{Re} - 1} \frac{1}{\sqrt{\log \zeta^*}} \frac{d\zeta^*}{\sqrt{(z^* - \zeta^*)^2 + r^{*2}}} \tag{5-41}$$

If we express v_o in terms of intermediate variables $x_i^{(f)} f(Re) = x_i^* Re$ we have

$$\vec{v}_o = -\vec{i}\,2\epsilon_1(Re)\sqrt{\log\left(\frac{2\overline{Re}}{Re}-1\right)}$$
$$+\,i\,\epsilon_1(Re)\int_{\frac{Re}{f(Re)}}^{\frac{2\overline{Re}}{f(Re)}-\frac{Re}{f(Re)}} \frac{1}{\sqrt{\log\frac{f(Re)\zeta^{(f)}}{Re}}}\;\frac{d\zeta^{(f)}}{\sqrt{(\zeta^{(f)}-z^{(f)})^2+r^{(f)2}}} \quad (5\text{-}42)$$
$$-\,x^{(f)}\nabla^{(f)}\epsilon_1(Re)\int_{\frac{Re}{f(Re)}}^{\frac{2\overline{Re}}{f(Re)}-\frac{Re}{f(Re)}} \frac{1}{\sqrt{\log\frac{f(Re)\zeta^{(f)}}{Re}}}\;\frac{d\zeta^{(f)}}{\sqrt{(z^{(f)}-\zeta^{(f)})^2+r^{(f)2}}}$$

It can easily be shown that

$$\int_{\frac{Re}{f(Re)}}^{\frac{2\overline{Re}}{f(Re)}-\frac{Re}{f(Re)}} \frac{d\zeta^{(f)}}{\sqrt{\log\frac{f(Re)\zeta^{(f)}}{Re}}} = \frac{1}{f(Re)}\int_{Re}^{2\overline{Re}-Re}\frac{d\zeta}{\sqrt{\log\frac{\zeta}{Re}}} = O\left(\frac{\epsilon^{\frac{1}{2}}}{f(Re)}\right)$$

(5-43)

Therefore the second and the third terms of $\vec{v}_o$ tend to zero as Re tends to zero for $\sqrt{(z^{(f)}-\zeta^{(f)})^2+r^{(f)2}}$ finite and ord $\epsilon^{\frac{1}{2}} \leq$ ord$f(Re) \leq 1$ ord 1. Thus the matching condition

$$\lim_{\substack{f\\Re\to o}} \left|\vec{i}-\vec{v}_o\right| = 0 \quad (5\text{-}44)$$

is satisfied for ord f in some overlap domain if we choose

$$\epsilon_1(Re) = -\frac{1}{2}\,\frac{1}{\sqrt{\log\frac{1}{Re}}} = -\frac{1}{2}\,\epsilon^{\frac{1}{2}} \quad (5\text{-}45)$$

If $\epsilon_1(Re) = -\frac{1}{2}\epsilon^{\frac{1}{2}}$ is chosen, we have

$$\vec{v}_o = -\frac{\vec{i}}{2}\epsilon^{\frac{1}{2}}F^{(1)}(z^*, r^*) + \frac{1}{2}\epsilon^{\frac{1}{2}} x^* \nabla^* F^{(1)}(z^*, r^*) \qquad (5\text{-}46)$$

and from equations (5-28, 34, 37, 40), $\vec{v}_o$ in the shank region becomes

$$\vec{v} = \vec{i}\,\epsilon \log r^* - \vec{i}\,\epsilon \frac{x^{*2}}{r^{*2}} - \vec{j}\,\epsilon \frac{x^* y^*}{r^{*2}} + O(\epsilon)$$

$$= \epsilon\{\vec{i} \log r^* - x^* \nabla^* \log r^*\} + O(\epsilon) \qquad (5\text{-}47)$$

The last term is of ord ϵ for all order of intermediate shank regions. The above representation is also valid in all intermediate shank regions (i.e. $-\overline{Re} < \tilde{z} < \overline{Re}$, and $r^* = \frac{r^{(f)}f(Re)}{Re}$ for ord Re $\leq$ ord f $<$ ord 1). Thus in the intermediate shank region, we have

$$\vec{v}_o = \vec{i}\,\epsilon \log \frac{r^{(f)}f(Re)}{Re} - \vec{i}\,\epsilon \frac{x^{(f)^2}}{r^{(f)^2}}$$

$$- \vec{j}\,\epsilon \frac{x^{(f)}y^{(f)}}{r^{(f)^2}} + O(\epsilon)$$

$$= \vec{i}\,[1 + \epsilon \log f(Re)] + \epsilon\,(\vec{i} \log r^{(f)}$$

$$- \vec{i}\,\frac{x^{(f)^2}}{r^{(f)^2}} - \vec{j}\,\frac{x^{(f)}y^{(f)}}{r^{(f)^2}} + O(\epsilon) \qquad (5\text{-}48)$$

<u>Determination of $\vec{w}_o$</u>: By symmetry, $\vec{w}_o$ is easily determined to be

$$\vec{w}_o = -\frac{\vec{i}}{2}\epsilon^{\frac{1}{2}}F^{(2)}(z^*, r^*) + \frac{1}{2}\epsilon^{\frac{1}{2}} x^* \nabla^* F^{(2)}(z^*, r^*) \qquad (5\text{-}49a)$$

and

$$F^{(2)}(z^*, r^*) = \int_{-\frac{2\overline{Re}}{Re}+1}^{-1} \frac{1}{\sqrt{\log(-\zeta^*)}} \left\{ \frac{1}{\sqrt{(z^*-\zeta^*)^2 + r^{*2}}} + \frac{1}{\zeta^*} \right\} d\zeta^* \quad (5\text{-}49b)$$

Similarly we can show that $\vec{w}_o$ is the required intermediate solution and the detail will not be repeated here. Similarly in the shank region, we have

$$\vec{w}_o = \vec{i}\,\epsilon \log \frac{r^{(f)} f(Re)}{Re} - \vec{i}\,\epsilon \frac{x^{(f)^2}}{r^{(f)^2}}$$

$$- \vec{j}\,\epsilon \frac{x^{(f)} y^{(f)}}{r^{(f)^2}} + O(\epsilon) \quad (5\text{-}50)$$

for $-\overline{Re} < \tilde{z} < \overline{Re}$ and ord Re $\leq$ ord f < ord 1. $\vec{w}_o$ can also easily be shown to match $\vec{g}_o = \vec{i}$ and satisfies the boundary condition in the limit Re $\to$ 0.

<u>Matching between $\vec{u}_o$ and $\vec{v}_o$ (or $\vec{w}_o$)</u>: By symmetry, we will discuss only the matching between $\vec{u}_o$ and $\vec{v}_o$. Now naturally the question arises whether $\vec{u}_o$ matches with $\vec{v}_o$. But it is obvious from equations (5-47) we can see that the matching condition

$$\lim_{Re \to o} \left| \vec{u}_o - \vec{v}_o \right| \to 0 \quad (5\text{-}51)$$

holds for any $-\overline{Re} < \tilde{z} < \overline{Re}$. They match each other not only in the inner shank region (i.e. $-Re < \tilde{z} < \overline{Re}$ and $r^* = O(1)$ but also in all intermediate shank regions (i.e. $-\overline{Re} < \tilde{z} < \overline{Re}$ and $r^* = \frac{r^{(f)} f(Re)}{Re}$ for ord Re $\leq$ ord f < ord 1). Furthermore, we can see that $\vec{v}_o$

contains $\vec{u}_o$ in the shank region. Thus a uniformly valid expansion which is uniformly valid in the shank and in the left end region can be constructed and is $\vec{v}_o$. Similarly from equation (5-50), $\vec{w}_o$ matches $\vec{u}_o$ in the shank region. A uniformly valid expansion which is uniformly valid in the shank region and in the right end region can easily be constructed and is $\vec{w}_o$.

<u>An Expansion Uniformly Valid near the Body.</u> Since $\vec{w}_o$ and $\vec{v}_o$ are overlapping in the shank region, an expansion $\vec{s}_o$ which is uniformly valid near the body can be constructed in principle. We can easily see that

$$\vec{s}_o = \vec{w}_o + \vec{v}_o - \vec{c}_o \tag{5-52}$$

where $\vec{c}_o$ is going to be determined. In the shank region, $\vec{c}_o$ under shank limit must be equal to $\vec{u}_o$ in the limit $Re \rightarrow 0$. In the right end, $\vec{c}_o$ must cancel the right end limit of $\vec{v}_o$. It has been shown in the Laplace problem that $F^{(1)}$ is not bounded on the surface of the cylinder near the right end of the cylinder and therefore $\vec{v}_o$ is also not bounded near the right end of the cylinder. Thus $\vec{c}_o$ under the right end limit should cancel $\vec{v}_o$. Similarly the same condition must be satisfied in the left end. In the related Laplace problem, it was found that by use of constant source distribution, all conditions are satisfied. This suggests to us to investigate the following Stokes solution $\vec{c}_o$.

$$\vec{c}_o = \vec{i} - \frac{\vec{i}}{2}\epsilon \int_1^{\frac{2\overline{Re}}{Re}-1} \frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2 + r^{*2}}} + \frac{\epsilon}{2} x^* \nabla^* \int_1^{\frac{2\overline{Re}}{Re}-1} \frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2 + r^{*2}}}$$

$$= \vec{i} - \frac{\vec{i}}{2}\epsilon \int_1^{\frac{2\overline{Re}}{Re}-1} \frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2 + r^{*2}}} - \frac{\vec{i}}{2}\epsilon x^{*2} \int_1^{\frac{2\overline{Re}}{Re}-1} \frac{d\zeta^*}{[(z^*-\zeta^*)^2 + r^{*2}]^{3/2}}$$

$$- \vec{j}\frac{\epsilon}{2} x^* y^* \int^{\frac{2\overline{Re}}{Re}-1} \frac{d\zeta^*}{[(z^*-\zeta^*)^2 + r^{*2}]^{3/2}}$$

$$- \vec{k}\frac{\epsilon}{2} x^* \int_1^{\frac{2\overline{Re}}{Re}-1} \frac{(z^*-\zeta^*)d\zeta^*}{[(z^*-\zeta^*)^2 + r^{*2}]^{3/2}}$$

$$= \vec{i}A - \vec{i}B - \vec{i}C - \vec{i}D \qquad (5\text{-}53)$$

a) Under the shank limit (i.e. $Re \to 0$ for $\tilde{z}$, r^* = fixed) in the shank region ($-\overline{Re} < \tilde{z} < \overline{Re}$) we have

$$A = 1 - \frac{1}{2}\epsilon \int_{-\beta}^{\beta} \frac{d\tilde{\zeta}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + Re^2 r^{*2}}} \; ; \quad \beta = \overline{Re} - Re$$

$$= \epsilon\,[\log r^* - \frac{1}{2}\log(\overline{Re} + \tilde{z}) - \frac{1}{2}\log(\overline{Re} - \tilde{z})]$$

$$+ O(\epsilon\, Re^2) \qquad (5\text{-}54a)$$

$$B = \frac{1}{2}\epsilon \, Re^2 x^{*2} \int_{-\beta}^{\beta} \frac{d\tilde{\zeta}}{[(\tilde{z}-\tilde{\zeta})^2 + Re^2 r^{*2}]^{3/2}}$$

$$= \frac{1}{2}\epsilon \frac{x^{*2}}{r^{*2}} \left\{ \frac{(\tilde{z}+\beta)}{[(\tilde{z}+\beta)^2 + Re^2 r^{*2})]^{1/2}} - \frac{(\tilde{z}-\beta)}{[(\tilde{z}-\beta)^2 + Re^2 r^{*2}]^{1/2}} \right\}$$

$$= \frac{1}{2}\epsilon \frac{x^{*2}}{r^{*2}} \left\{ 1 - \frac{1}{2}\frac{Re^2 r^{*2}}{(\tilde{z}+\beta)^2} + 1 - \frac{1}{2}\frac{Re^2 r^{*2}}{(\tilde{z}-\beta)^2} + \cdots\cdots \right\}$$

$$= \epsilon \frac{x^{*2}}{r^{*2}} + O(\epsilon \, Re^2) \qquad (5\text{-}54b)$$

$$C = \frac{By^*}{x^*} = \epsilon \frac{x^* y^*}{r^{*2}} + O(\epsilon \, Re^2) \qquad (5\text{-}54c)$$

$$D = \frac{1}{2}\epsilon \, Re \, x^* \int_{-\beta}^{\beta} \frac{(\tilde{z}-\tilde{\zeta})d\tilde{\zeta}}{[(\tilde{z}-\tilde{\zeta})^2 + Re^2 r^{*2}]^{3/2}}$$

$$= \frac{1}{2}\epsilon \, Re \, x^* \left\{ \frac{1}{[(\tilde{z}+\beta)^2 + Re^2 r^{*2}]^{1/2}} - \frac{1}{[(\tilde{z}-\beta)^2 + Re^2 r^{*2}]^{1/2}} \right\}$$

$$= O(\epsilon \, Re) \qquad (5\text{-}54d)$$

Therefore

$$\vec{c}_o = \epsilon \vec{i} \log r^* - \epsilon \vec{i} \frac{x^{*2}}{r^{*2}} - \epsilon \vec{j} \frac{x^* y^*}{R^{*2}}$$

$$+ O(\epsilon) \qquad (5\text{-}55)$$

and $\vec{c}_o$ equals to $\vec{u}_o$ in the shank region in the limit $Re \to 0$, as required.

b) <u>In the right end region:</u> For $z^* \geq \frac{\overline{Re}}{Re} - \frac{1}{2}$ or the right half cylinder, and in terms of right end variables, and divide interval of integration into two parts $[-\frac{2\overline{Re}}{Re}+1, -\frac{3}{2}\frac{\overline{Re}}{Re}]$ and $[-\frac{3}{2}\frac{\overline{Re}}{Re}, -1]$, we obtain

$$\vec{c}_o = \vec{i} - \vec{i}\frac{\epsilon}{2}\int_{-\frac{3}{2}\frac{\overline{Re}}{Re}}^{-1}\frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}} - \vec{i}\frac{\epsilon}{2}x^{*2}\int_{-\frac{3\overline{Re}}{2Re}}^{-1}\frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}}$$

$$-\vec{j}\frac{\epsilon}{2}x^*y^*\int_{-\frac{3\overline{Re}}{2Re}}^{-1}\frac{d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}} - \vec{k}\frac{\epsilon}{2}x^*\int_{-\frac{3\overline{Re}}{2Re}}^{-1}\frac{(z^*-\zeta^*)d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}}$$

$$+ O(\epsilon) \qquad (5\text{-}56)$$

Now let us investigate $\vec{v}_o$ in the same region. By expressing in terms of the right end variables and dividing the interval of integration into two parts, we have

$$\vec{v}_o = \vec{i} - \frac{\vec{i}}{2}\epsilon^{\frac{1}{2}}\int_{-\frac{3\overline{Re}}{2Re}}^{-1}\frac{1}{\sqrt{\log\frac{2\overline{Re}+\zeta^* Re}{Re}}}\frac{d\zeta^*}{\sqrt{(z^*-\zeta^*)^2+r^{*2}}}$$

$$-\frac{\vec{i}}{2}\epsilon^{\frac{1}{2}}x^{*2}\int_{-\frac{3\overline{Re}}{2Re}}^{-1}\frac{1}{\sqrt{\log\frac{2\overline{Re}+\zeta^* Re}{Re}}}\frac{d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}}$$

$$-\frac{\vec{j}}{2}\epsilon^{\frac{1}{2}}x^*y^*\int_{-\frac{3\overline{Re}}{2Re}}^{-1}\frac{1}{\sqrt{\log\frac{2\overline{Re}+\zeta^* Re}{Re}}}\frac{d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}}$$

$$-\frac{\vec{k}}{2}\epsilon^{\frac{1}{2}}x^*\int_{-\frac{3\overline{Re}}{2Re}}^{-1}\frac{1}{\sqrt{\log\frac{2\overline{Re}+\zeta^* Re}{Re}}}\frac{(z^*-\zeta^*)d\zeta^*}{[(z^*-\zeta^*)^2+r^{*2}]^{3/2}} + O(\epsilon) \qquad (5\text{-}57)$$

In this new interval of integration, we can expand

$$\frac{1}{\sqrt{\log \frac{2\overline{Re}+ \zeta^* Re}{Re}}} = \epsilon^{\frac{1}{2}} - \frac{1}{2} \epsilon^{\frac{3}{2}} \log (2\overline{Re}+ \zeta^* Re) + O(\epsilon^{\frac{5}{2}}) \qquad (5\text{-}58)$$

Thus we have

$$\vec{v}_o = \vec{c}_o + O(\epsilon) \qquad (5\text{-}59)$$

in the right end region. By symmetry we can easily show that in the left end region and under the left end limit we have

$$\vec{w}_o = \vec{c}_o + O(\epsilon) \qquad (5\text{-}60)$$

Thus we have shown that $\vec{c}_o$ fulfills all the requirements and is the correct part. A uniformly valid expansion $\vec{s}_o$ which is uniformly valid to order unity everywhere near the body is found as in equation (5-52). If we express $\vec{s}_o$ in terms of outer variables, we have

$$\vec{s}_o = \vec{i} - \frac{\vec{i}}{2} \epsilon^{\frac{1}{2}} \int_{-\beta}^{\beta} \left\{ \frac{1}{\sqrt{\log \frac{\overline{Re}-\tilde{\zeta}}{Re}}} + \frac{1}{\sqrt{\log \frac{\overline{Re}+\tilde{\zeta}}{Re}}} - \epsilon^{\frac{1}{2}} \right\} \frac{d\tilde{\zeta}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}$$

$$+ \frac{\epsilon^{\frac{1}{2}}}{2} \tilde{x} \tilde{\nabla} \int_{-\beta}^{\beta} \left\{ \frac{1}{\sqrt{\log \frac{\overline{Re}+\tilde{\zeta}}{Re}}} + \frac{1}{\sqrt{\log \frac{\overline{Re}-\tilde{\zeta}}{Re}}} - \epsilon^{\frac{1}{2}} \right\} \frac{d\tilde{\zeta}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}} \qquad (5\text{-}61)$$

where $\beta = \overline{Re} - Re$

<u>Uniformly Valid Expansion:</u> In the above, we have shown that the expansions $\vec{s}_o$, $\vec{u}_o$, $\vec{v}_o$, $\vec{w}_o$ satisfy the governing equations in each region respectively and the domains of validity overlap with each

other. Thus they have covered the whole domain of interest and in principle a uniformly valid solution $\vec{q}_o$ containing $\vec{g}_o$, $\vec{u}_o$, $\vec{v}_o$ and $\vec{w}_o$ in each corresponding region such that

$$\lim_{Re \to o} |\vec{q} - \vec{q}_o| = 0 \tag{5-62}$$

uniformly valid everywhere can be obtained.

Since a uniformly valid expansion $\vec{s}_o$ near the body is obtained, $\vec{q}_o$ can in principle be obtained by use of $\vec{g}_o$ and $\vec{s}_o$. But in practice, as the governing equation in the outer region is an Oseen equation, $\vec{q}_o$ can easily be constructed by use of an Oseen solution which contains $\vec{s}_o$ for $-Re \leq \tilde{z} \leq Re$ and $\tilde{r} \to 0$. This can be done by use of the idea of source distribution. The Oseen solution constructed by use of the same source distribution for $\vec{s}_o$ will contain the Stokes solution $\vec{s}_o$ near the body. The general idea has been discussed in Ref. 5. In the present case, $\vec{q}_o$ can easily be found as follows:

$$\vec{q}_o(\tilde{x}_i, Re) = \vec{i} - \vec{i}\,\epsilon^{\frac{1}{2}} \int_{-\beta}^{\beta} f(\tilde{\zeta}, Re) \frac{e^{-\frac{1}{2}\left(\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}-\tilde{x}\right)}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + r^2}}\, d\tilde{\zeta}$$

$$+\epsilon^{\frac{1}{2}}\,\tilde{\nabla} \int_{-\beta}^{\beta} f(\tilde{\zeta}, Re)\left[\frac{e^{-\frac{1}{2}\left(\sqrt{(\tilde{z}-\tilde{\zeta})^2+r^2}-\tilde{x}\right)}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}} - \frac{1}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}\right] d\tilde{\zeta} \tag{5-63a}$$

where

$$f(\tilde{z}, Re) = \frac{1}{\sqrt{\log \frac{\widetilde{Re+z}}{Re}}} + \frac{1}{\sqrt{\log \frac{Re-\tilde{z}}{Re}}} - \frac{1}{\sqrt{\log \frac{1}{Re}}} \tag{5-63b}$$

and

$$\epsilon^{\frac{1}{2}} = \frac{1}{\sqrt{\log \frac{1}{Re}}} \quad \text{and } \beta = \overline{Re} - Re \tag{5-63c}$$

This can be justified by expanding the $\vec{q}_o$ in various different regions as follows:

1) Outer region: In the outer region, if we expand $\vec{q}_o$ in power of ϵ, we obtain

$$\vec{q}_o = \vec{i} - \vec{i} \sum_{n=1}^{\infty} a_n \epsilon^n \int_{-\overline{Re}}^{\overline{Re}} f_n(\overline{Re}, \tilde{\zeta}) \frac{e^{-\frac{1}{2}(\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2} - \tilde{x})}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}} d\tilde{\zeta}$$

$$+ \sum_{n=1}^{\infty} a_n \epsilon^n \nabla \int_{-\overline{Re}}^{\overline{Re}} f_n(\overline{Re}, \tilde{\zeta}) \left[\frac{e^{-\frac{1}{2}(\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2} - \tilde{x})}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}} - \frac{1}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}} \right] d\tilde{\zeta} \tag{5-64}$$

where a_n is generated by $\frac{1}{\sqrt{1+x}} = \sum_{n=1}^{\infty} a_n x^n$

$$f_n(\overline{Re}, \tilde{\zeta}) = [\log(\overline{Re} - \tilde{\zeta})]^{n-1} + [\log(\overline{Re} + \tilde{\zeta})]^{n+1} \tag{5-65}$$

or

$$\vec{q}_o = \vec{i} + O(\epsilon) \tag{5-66}$$

Thus in the outer region $\vec{q}_o$ contains $\vec{g}_o = \vec{i}$ and

$$\lim_{Re \to 0} |\vec{q}_o - \vec{g}_o| = 0 \tag{5-67}$$

in the outer region.

2) Near the body: For $\tilde{r} \to 0$ and $-\overline{Re} \leq \tilde{z} \leq \overline{Re}$ we have

$$\vec{q}_o = \vec{i} - \frac{\vec{i}}{2}\, \epsilon^{\frac{1}{2}} \int_{-\beta}^{\beta} f(\tilde{\zeta}, Re) \frac{d\tilde{\zeta}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}}$$

$$+ \frac{\tilde{x}}{2}\, \tilde{\nabla} \epsilon^{\frac{1}{2}} \int_{-\beta}^{\beta} f(\tilde{\zeta}, Re) \frac{d\tilde{\zeta}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}}$$

$$- \vec{i}\, \epsilon^{\frac{1}{2}} \int_{-\beta}^{\beta} f(\tilde{\zeta}, Re) \sum_{n=1}^{\infty} \left(- \frac{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}}{2}\right)^{n-1} \frac{1}{n!}\, d\tilde{\zeta}$$

$$- \vec{k}\, \epsilon^{\frac{1}{2}} \int_{-\beta}^{\beta} f(\tilde{\zeta}, Re) \frac{(\tilde{z}-\tilde{\zeta})}{[(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2]^{3/2}} \sum_{n=2}^{\infty} \left(- \frac{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}}{2}\right)^{n} \frac{d\tilde{\zeta}}{n!}$$

$$- \vec{k}\, \epsilon^{\frac{1}{2}} \int_{-\beta}^{\beta} f(\tilde{\zeta}, Re) \frac{(\tilde{z}-\tilde{\zeta})}{(z-\zeta)^2 + \tilde{r}^2} \sum_{n=1}^{\infty} \left(- \frac{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}}{2}\right)^{n} \frac{d\tilde{\zeta}}{n!}$$

$$= \vec{i} - \frac{\vec{i}}{2}\, \epsilon^{\frac{1}{2}} \int_{-\beta}^{\beta} f(\tilde{\zeta}, Re) \frac{d\tilde{\zeta}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}}$$

$$+ \frac{\epsilon^{\frac{1}{2}}}{2}\, \tilde{x}\, \tilde{\nabla} \int_{-\beta}^{\beta} f(\tilde{\zeta}, Re) \frac{d\tilde{\zeta}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}}$$

$$+ \vec{i} \sum_{n=1}^{\infty} \epsilon^n f_n(\tilde{z}) + \vec{k} \sum_{n=1}^{\infty} \epsilon^n g_n(\tilde{z}) \qquad (5\text{-}68)$$

Thus we have shown that near the body $\vec{q}_o$ equals to $\vec{s}_o$ in the limit $Re \to 0$. This justifies that $\vec{q}_o$ is a uniformly valid expansion.

Computation of the drag force: The drag force can be computed from the uniformly valid expansion $\vec{q}_o$. By comparing $\vec{q}_o$ with the fundamental solutions in equation (2-6) and the integral discussed in equation (4-71), we have

$$D = 8\pi\mu U L\, [\epsilon + (1 - \log 2\overline{Re})\, \epsilon^2 + O(\epsilon^3) \qquad (5\text{-}69)$$

or

$$C_D = \frac{4\pi}{Re}\, [\epsilon + (1 - \log 2\overline{Re})\, \epsilon^2 + O(\epsilon^3)] \qquad (5\text{-}70)$$

VI. LOW REYNOLDS NUMBER FLOW PAST AN ELLIPSOID OF REVOLUTION OF LARGE ASPECT RATIO

6.1 Introduction

In this chapter we consider the problem of viscous incompressible flow past an ellipsoid of revolution of large aspect ratio. The axis of the ellipsoid is perpendicular to the uniform flow $U\vec{i}$ at infinity.. The ellipsoid is defined as

$$\frac{r^2}{\lambda^2} + \frac{z^2}{L^2} = 1 \tag{6-1}$$

Then two similar non-dimensional parameters $Re = \frac{U\lambda}{\nu}$ and $\overline{Re} = \frac{UL}{\nu}$ can also be formed as in Chapter 5. In the present study, the low Reynolds number flow is also understood in the sense $Re \to 0$ by keeping $\overline{Re}$ = fixed. This may be thought of as letting $\lambda \to 0$ with all other parameters (i.e. U, L, ν) fixed. Under this limit, the ellipsoid cylinder shrinks to a needle of zero radius and the principal limit (i.e. outer limit) is obviously $\vec{g}_o = \vec{i}$. Thus the outer limit is the Oseen limit discussed in Chapter 3. This principal limit does not satisfy the boundary condition near the body (i.e. $\vec{q} = 0$). Therefore different limits must be introduced near the body. One limit introduced to study the inner region far from the ends is the "shank limit." This limit is similar to the boundary layer limit in the high Reynolds number flow by distorting the r-coordinate or viscous layer. Under this limit, the body becomes a finite ellipsoid and the governing equation is the Stokes equation with no derivatives with respect to z. But this "shank limit" is not valid at the both ends because near the end the flow is obviously three dimensional and the three dimensional Stokes equations are expected. Therefore two end limits have to be introduced. The proper end limit is introduced with the help of the fact that under this limit the cylinder has to be a proper semi-infinite body and the three dimensional Stokes equation can be obtained. The proper end variables have been discussed in the geometrical matching in Chapter 4. The details will not be repeated here. In the present case, not only the Stokes solutions which satisfy the exact boundary conditions in the shank regions and in the end regions can be obtained but also the solutions are in a simpler form in comparison to the solution for the finite cylinder in Chapter 5. Thus higher order terms can be obtained and the matching between them becomes interesting in detail. It is also the purpose of this chapter to exhibit in detail the expansion procedure and the matching between these expansions. The matching in the present case is much more complicated than that of two dimensional cylinders or a sphere discussed in Chapter 3 because four expansions are involved in the present case.

6.2 Limits, Expansions and Associated Equations.

Outer Expansions: The outer variables and limit are defined as the same as (5-3a). The expansions for velocity and pressure are assumed to have the same form as (5-3d,e). In terms of outer variables, the body is of the form

$$\frac{\tilde{r}^2}{Re^2} + \frac{\tilde{z}^2}{\overline{Re}^2} = 1 \qquad (6\text{-}2)$$

Thus in the limit Re tending to zero for $\overline{Re}$ fixed, the body shrinks to a finite needle of zero radius. The outer limit $\vec{g}_o = \vec{i}$ is justified in the present study. Therefore the governing equations for $\vec{g}_n (n \geq 1)$ are the same as (5-4) and (5-5).

Shank Expansion: Shank variables and shank limits are the same as (5-6). In addition the expansion and governing equations for $\vec{u}_n$ are the same as (5-7) and (5-8).

In terms of shank variables, the body can be expressed as

$$r^{*2} + \frac{\tilde{z}^2}{\overline{Re}^2} = 1 \qquad (6\text{-}3)$$

or

$$r^{*2} = \frac{\overline{Re}^2 - \tilde{z}^2}{\overline{Re}^2} \qquad (6\text{-}4)$$

Under the shank limit, the body is then a finite ellipsoid of revolution.

Left End Expansion: The left end region is much smaller than that of the finite cylinder discussed in Chapter 5. By symmetry, we will concentrate on discussing the left end. The left end variables are defined as

$$x^+ = \frac{\tilde{x}}{Re^2}, \; y^+ = \frac{\tilde{y}}{Re^2}, \; z^+ = \frac{\tilde{z}+Re}{Re^2} = \frac{\bar{z}}{Re^2} \qquad (6\text{-}5)$$

The left end limit is defined as the limit Re $\to 0$ for $\overline{Re}$, x^+, y^+ and z^+ fixed. In terms of left end variables, the body is of the form

$$r^{+2} = \frac{2z^+}{\overline{Re}} - \frac{Re^2 z^{+2}}{\overline{Re}^2} \qquad (6\text{-}6a)$$

or

$$r^{+2} = \frac{2z^+}{Re} + O(Re^2) \tag{6-6b}$$

Thus under the left end limit, the body becomes a semi-infinite paraboloid. If we define

$$\tau = \frac{1}{2}\{\sqrt{r^{+2} + (z^+ - \tau_o)^2} - (z^+ - \tau_o)\} \tag{6-7a}$$

and

$$\tau_o = \frac{1}{2Re} \tag{6-7b}$$

then to the order of Re^2, the body is

$$\tau = \tau_o = \frac{1}{2Re} \tag{6-8}$$

The intermediate left end expansion is

$$\vec{q}^{\,*} = \vec{v}_o + \epsilon\,\vec{v}_1 + \epsilon^2\,\vec{v}_2 + \ldots\ldots \tag{6-9a}$$

$$p^{++} = p_o^1 + \epsilon p_1^1 + \epsilon^2 p_2^1 + \ldots\ldots \tag{6-9b}$$

The governing equations for $\vec{v}_n$ $(n \geq 0)$ are three dimensional Stokes equations and $p^{++} = Re\, p^+ = Re^2 p^*$.

<u>Right End Expansion:</u> By symmetry, the right end variables are defined as

$$x^+ = \frac{\tilde{x}}{Re^2}, \quad y^+ = \frac{\tilde{y}}{Re^2}, \quad \bar{z}^+ = \frac{\tilde{z} - \overline{Re}}{Re^2} \tag{6-10}$$

Then the body may be expressed in terms of right end variables as

$$r^{+2} = -\frac{2\overline{z}^{+}}{\overline{Re}} + O(Re^2) \qquad (6\text{-}11)$$

If we define

$$\tau^{+} = \frac{1}{2}\{(\overline{z}^{+} + \tau_o) + \sqrt{(\overline{z}^{+} + \tau_o)^2 + r^{+2}}\} \qquad (6\text{-}12a)$$

where

$$\tau_o = \frac{1}{2\overline{Re}} \qquad (6\text{-}12b)$$

then the body is

$$\tau^{+} = \tau_o \qquad (6\text{-}13)$$

The intermediate right end expansion is

$$\vec{q}^{\,*} = \vec{w}_o + \epsilon\,\vec{w}_1 + \epsilon^2\,\vec{w}_2 + \;.\;.\;.\;.\;. \qquad (6\text{-}14a)$$

$$p^{++} = p_o^{11} + \epsilon p_1^{11} + \epsilon^2 p_2^{11} + \;.\;.\;.\;.\;. \qquad (6\text{-}14b)$$

6.3 <u>Determination of Solutions</u>

<u>Determination of $\vec{g}_o$</u>: As discussed previously, we know that the principal limit is

$$\vec{g}_o = \vec{i} \qquad (6\text{-}15)$$

<u>Determination of $\vec{u}_o$</u>: The intermediate shank solution is determined as

$$\vec{u}_o = \epsilon(Re)\,\vec{h}_1$$

$$= \epsilon(Re)\left\{\vec{i}\left(\log r^* - \frac{1}{2}\log\frac{\overline{Re}^2 - \tilde{z}^2}{\overline{Re}^2} + \frac{1}{2}\right)\right.$$

$$\left. - x^*\nabla^*\log r^* - \frac{1}{2}\,\frac{\overline{Re}^2 - \tilde{z}^2}{\overline{Re}^2}\,\nabla^*\frac{x^*}{r^{*2}}\right\} \qquad (6\text{-}16a)$$

The corresponding pressure term p_o is obtained as

$$p_o = -\epsilon(Re)\frac{2x^*}{r^{*2}} \qquad (6\text{-}16b)$$

It is obvious that $\vec{u}_o$ satisfies the governing equations (5-8) and the boundary condition

$$\vec{h}_1 = 0 \text{ on the body } \left(\text{i.e. } r^{*2} = \frac{\overline{Re}^2 - \tilde{z}^2}{\overline{Re}^2}\right) \qquad (6\text{-}17)$$

Then by the matching condition

$$\lim_{\substack{f \\ Re\to 0}} \left|\vec{i} - \vec{u}_o\right| = 0 \qquad (6\text{-}18)$$

for ord f in some overlap domain, we obtain

$$-\epsilon \log Re = 1 + b_1\epsilon + b_2\epsilon^2 + \dots\dots \qquad (6\text{-}19)$$

<u>Determination of $\vec{v}_o$</u>: In the present case, a solution $\vec{\ell}_1(x_i^+)$ which satisfies the three dimensional Stokes equations and the boundary condition $\vec{\ell}_1(x_i^+) = 0$ on $\tau = \tau_o$, can be obtained as

$$\vec{\ell}_1(x_i^+) = \vec{i}\,\frac{1}{2}\left[\log\frac{\tau}{\tau_o} + 1\right] - \frac{1}{2}x^+\nabla^+\log\tau - \frac{1}{2}\tau_o\nabla^+\left(\frac{x^+}{\tau}\right) \qquad (6\text{-}20)$$

Then the intermediate left end solution is determined as

$$\vec{v}_o = \epsilon(Re)\,\vec{\ell}_1(x_i^+) \qquad (6\text{-}21a)$$

where the matching condition

$$\lim_{\substack{f \\ Re \to o}} \left| \vec{i} - \vec{v}_o \right| = 0 \qquad (6\text{-}21b)$$

for ord f in some overlap domain is also satisfied. Thus $\vec{v}_o$ is the correct intermediate solution. The corresponding pressure term is easily determined as

$$p_o^1 = - \epsilon \frac{\partial}{\partial x^+} \log \tau \qquad (6\text{-}22)$$

<u>Matching Between $\vec{u}_o$ and $\vec{v}_o$</u>: The matching between $\vec{u}_o$ and $\vec{v}_o$ can be studied by expressing both of them in terms of intermediate variables (i.e. r_β, z_β) which are intermediate between shank and the left end variables.

$$r^* = Re^\beta r_\beta \qquad r^+ = Re^{\beta-1}$$

$$\tilde{z} = Re^{2\beta} z_\beta \qquad z^+ = z_\beta \, Re^{2(\beta-1)} \qquad (6\text{-}23)$$

They are supposed to match at $z^+ \to \infty$ and along $\frac{r^{+2}}{z^+} = \frac{r_\beta^2}{z_\beta} = \frac{r^{*2}}{\tilde{z}} =$ constant. By expressing in terms of intermediate variables, the body is a semi-infinite paraboloid and $0 < \beta < 1$. Thus

$$\tau = \frac{1}{4} \frac{r_\beta^2}{z_\beta} + O[Re^{2(1-\beta)}] \qquad (6\text{-}24)$$

Thus

$$\vec{v}_o = \epsilon \vec{\ell}_1$$

$$= \epsilon \{ \frac{\vec{i}}{2} (\log \frac{\overline{Re}\, r_\beta^2}{2 z_\beta} + 1) - x_\beta \nabla_\beta \log r_\beta$$

$$- \frac{z_\beta}{\overline{Re}} \nabla_\beta \frac{x_\beta}{r_\beta^2} \} - \vec{k} \frac{x_\beta}{2} \{ \frac{2}{\overline{Re}\, r_\beta^2} - \frac{1}{z_\beta} \} \epsilon \, Re^{1-\beta}$$

$$+ O(\epsilon \, Re^{1-\beta}) \tag{6-25}$$

and

$$\vec{u}_o = \epsilon \{ \vec{i} \frac{1}{2} (\log \frac{\overline{Re}\, r_\beta^2}{2 z_\beta} + 1) - x_\beta \nabla_\beta \log r_\beta$$

$$- \frac{z_\beta}{\overline{Re}} \nabla_\beta \frac{x_\beta}{r_\beta^2} \} + O(\epsilon \, Re^{\beta}) \tag{6-26}$$

Therefore

$$\lim_{\substack{f \\ Re \to 0}} | \vec{u}_o - \vec{v}_o | = 0 \tag{6-27}$$

for $f(Re) = Re^{\beta}$ and $0 < \beta < 1$. In fact they are matched to $O(Re^{\beta}\epsilon, Re^{1-\beta}\epsilon)$.

<u>Determination of $\vec{w}_o$</u>: Similarly, $\vec{w}_o$ is easily determined as

$$\vec{w}_o = \epsilon \{ \frac{\vec{i}}{2} [\log \frac{\tau^+}{\tau_o} + 1] - \frac{1}{2} x^+ \nabla^+ \log \tau^+$$

$$- \frac{\tau_o}{2} \nabla^+ (\frac{x^+}{\tau^+}) \} \tag{6-28a}$$

The corresponding pressure term is obtained as

$$p_o^{11} = - \epsilon \frac{\partial}{\partial x^+} \log \tau^+ \tag{6-28b}$$

The matching between $\vec{w}_o$ and $\vec{u}_o$ is exactly the same as between $\vec{u}_o$ and $\vec{v}_o$. The details will not be repeated here.

<u>Uniformly Valid Expansion Near the Body</u>. Since $\vec{u}_o$ matches to $\vec{w}_o$ and $\vec{v}_o$, a uniformly valid expansion $\vec{s}_o$ near the body can, in principle, be obtained. In the present case, we can easily obtain $\vec{s}_o$ as follows:

$$\vec{s}_o = \vec{u}_o + \vec{v}_o + \vec{w}_o - \vec{v}_o^{\,s} - \vec{w}_o^{\,s}$$

where $\vec{v}_o^{\,s}$ is the shank limit of $\vec{v}_o$ and $\vec{w}_o^{\,s}$ is the shank limit of $\vec{w}_o$.

<u>Determination of $\vec{g}_1$</u>: $\vec{g}_1$ must satisfy the governing equation (5-4) and can be determined by the matching condition that it cancels to the unbounded terms of

$$\lim_{f_1} \frac{\vec{u}_o - \vec{i}}{\epsilon} = \vec{i}\left(\log \tilde{r} - \frac{1}{2}\log \frac{\overline{Re}^2 - \tilde{z}^2}{\overline{Re}^2} + \frac{1}{2} + b_1\right)$$
$$- \tilde{x}\,\tilde{\nabla} \log \tilde{r} \qquad (6\text{-}29)$$

for $\tilde{r} \to 0$ and $-\overline{Re} < \tilde{z} < \overline{Re}$. Similarly $\vec{g}_1$ is supposed to cancel the unbounded term of

$$\lim_{f_2} \frac{\vec{v}_o - \vec{i}}{\epsilon} = \frac{\vec{i}}{2}\left\{\log\left(\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}\right) + 2b_1 + \log 2\overline{Re} + 1\right\}$$
$$- \frac{1}{2}\tilde{x}\tilde{\nabla} \log\left(\sqrt{\tilde{r}^2 + \bar{z}^2} - \bar{z}\right) \qquad (6\text{-}30)$$

for $\tilde{r}$ and $\bar{z} + \overline{Re}$ small. Similarly $\vec{g}_1$ has to cancel the unbounded terms of $\lim_{f_2} \frac{w_o - i}{\epsilon}$ for $\tilde{r} \to 0$, and $\bar{z} = \tilde{z} - Re \to 0$. From all these matching conditions, $\vec{g}_1$ is determined as follows:

$$\vec{g}_1 = -\vec{i} \int_{-\overline{Re}}^{\overline{Re}} \frac{e^{-\frac{1}{2}\left(\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2} - \tilde{x}\right)}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}} d\tilde{\zeta}$$

$$+ \tilde{\nabla} \int_{-\overline{Re}}^{\overline{Re}} \left\{ \frac{e^{-\frac{1}{2}\left(\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2} - \tilde{x}\right)}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}} - \frac{1}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}} \right\} d\tilde{\zeta} \qquad (6\text{-}31)$$

For $\tilde{r} \to 0$, $-\overline{Re} < \tilde{z} < \overline{Re}$, we have (see Appendix B)

$$\vec{g}_1 = \vec{i}\,(\log \tilde{r} + \gamma - \log 4) - \tilde{x}\tilde{\nabla} \log \tilde{r}$$

$$+ \frac{\vec{i}}{2} \left\{ E_1\left(\frac{\overline{Re}-\tilde{z}}{2}\right) + E_1\left(\frac{\overline{Re}+\tilde{z}}{2}\right) \right\}$$

$$+ \vec{k} \left\{ \frac{e^{-\frac{1}{2}(\overline{Re}-\tilde{z})}}{\overline{Re}-\tilde{z}} - \frac{1}{\overline{Re}-\tilde{z}} + \frac{1}{\overline{Re}+\tilde{z}} - \frac{e^{-\frac{1}{2}(\overline{Re}+\tilde{z})}}{\overline{Re}+\tilde{z}} \right\}$$

$$+ O(\tilde{r} \log \tilde{r}) \qquad (6\text{-}32)$$

The function $E_1(x)$ (cf. Ref. 8) is defined as

$$E_1(x) = -Ei(-x) = \int_x^{\infty} \frac{e^{-t}}{t} dt$$

$$= -\log \gamma_o x + e^{-x} \sum_{m=1}^{\infty} \left(1 + \frac{1}{2} + \dots\dots + \frac{1}{m}\right) \frac{x^m}{m!} \qquad (6\text{-}33a)$$

where $\gamma_o = e^{\gamma} = 1.781$

or $\quad \gamma = \log \gamma_o \doteq 0.5772 =$ Euler's constant

For large value of x, an asymptotic expansion for $E_1(x)$ is

$$E_1(x) \cong \frac{e^{-x}}{x}\left(1 - \frac{1!}{x} + \frac{2!}{x^2} + \dots\dots\right) \qquad (6\text{-}33b)$$

Thus $E_1(x) \to 0$ as $x \to \infty$ and $E_1(x) \to \log \gamma_o x$ as $x \to 0$.

Therefore equation (6-32) shows $\vec{g}_1$ cancels the unbounded terms in equation (6-29) for $\tilde{r} \to 0$ and $-\overline{Re} < \tilde{z} < \overline{Re}$. If we choose all $b_n = 0$ except

$$b_1 = -\frac{1}{2} + \gamma - \log 4 \tag{6-34a}$$

we have the same ϵ as for two dimensional case and it is

$$\epsilon = \frac{1}{\log \frac{4}{\overline{Re}} - \gamma + \frac{1}{2}} \tag{6-34b}$$

In addition, for $\tilde{r} \to 0$ and $\tilde{z} \to 0$, we have (see Appendix B)

$$\vec{g}_1 = \frac{\vec{i}}{2} \{\log (\sqrt{\tilde{z}^2 + \tilde{r}^2} - \tilde{z}) + \gamma - \log 4 + E_1(\overline{Re})\}$$
$$- \frac{\tilde{x}}{2} \tilde{\nabla} \log (\sqrt{\tilde{z}^2 + \tilde{r}^2} - \tilde{z}) + \vec{k} \left(\frac{e^{-\overline{Re}}}{2\overline{Re}} - \frac{1}{2\overline{Re}}\right)$$
$$+ O(\tilde{z} \log \tilde{z}) \tag{6-35}$$

Thus it cancels the unbounded term in equation (6-30). Similarly by expanding $\vec{g}_1$ for $\tilde{r} \to 0$ and $\tilde{z} \to 0$, it will cancel the unbounded term of $\lim_{f_2} \frac{\vec{i} - \vec{w}_o}{\epsilon}$ and the details will not be repeated here. This shows that $\vec{g}_1$ is correctly determined as in equation (6-31).

The corresponding pressure term is

$$p_1^* = -\frac{\partial}{\partial \tilde{x}} \int_{-\overline{Re}}^{\overline{Re}} \frac{d\tilde{\zeta}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}} \tag{6-36}$$

<u>Determination of $\vec{u}_1$:</u> $\vec{u}_1$ can be determined by the matching condition

$$\lim_{\substack{f_1 \\ Re \to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 - (\vec{u}_o + \epsilon \vec{u}_1)}{\epsilon} = 0 \qquad (6\text{-}37)$$

for ord f_1 in some overlap domain. In the present case, $\vec{u}_1$ is then easily determined as

$$\vec{u}_1 = \vec{u}_1^{\,l} + \vec{u}_1^{\,ll} \qquad (6\text{-}38)$$

and

$$\vec{u}_1^{\,l} = f_1(\tilde{z})\, \vec{u}_o$$
$$\vec{u}_1^{\,ll} = g_1(\tilde{z})\, \vec{u}_o^{\,l} \qquad (6\text{-}39)$$

where

$$\vec{u}_o^{\,l} = \epsilon \vec{k} \left[\log r^* - \frac{1}{2} \log \frac{\overline{Re}^2 - \tilde{z}^2}{\overline{Re}^2} \right] \qquad (6\text{-}40)$$

$$f_1(\tilde{z}) = \frac{1}{2} \left[E_1 \left(\frac{\overline{Re} - \tilde{z}}{2}\right) + E_1 \left(\frac{\overline{Re} + \tilde{z}}{2}\right) + \log (\overline{Re} - \tilde{z}) \right.$$
$$\left. + \log (\overline{Re} + \tilde{z}) - 2 \log \overline{Re} \right] \qquad (6\text{-}41a)$$

and $$g_1(\tilde{z}) = \frac{e^{-\frac{1}{2}(\overline{Re} - \tilde{z})}}{\overline{Re} - \tilde{z}} - \frac{1}{\overline{Re} - \tilde{z}}$$
$$+ \frac{1}{\overline{Re} + \tilde{z}} - \frac{e^{-\frac{1}{2}(\overline{Re} + \tilde{z})}}{\overline{Re} + \tilde{z}} \qquad (6\text{-}41b)$$

It can easily be shown that $\vec{u}_o^{\,l}$ satisfies the governing equation (5-8c) and that $\vec{u}_1$ satisfies the governing equations (5-8) and $\vec{u}_1 = 0$ on the body. By expressing $\vec{u}_1$ in terms of outer variables, the matching condition is obviously satisfied and therefore $\vec{u}_1$ is correctly determined. Thus we have

$$p_1 = - \epsilon \frac{2f_1(\tilde{z})x^*}{r^{*2}} \tag{6-42}$$

<u>Determination of $\vec{v}_1$ and $\vec{w}_1$</u> : $\vec{v}_1$ can be determined by the matching condition that

$$\lim_{\substack{f_2 \\ Re \to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 - (\vec{v}_o + \epsilon \vec{v}_1)}{\epsilon^2} = 0 \tag{6-43}$$

for ord f_2 in some overlap domain. In the present case, $\vec{v}_1$ can easily be obtained as follows:

$$\vec{v}_1 = \vec{v}_1^{\,1} + \vec{v}_1^{\,11} \tag{6-44}$$

and

$$\vec{v}_1^{\,1} = c_1 \vec{v}_o = \frac{1}{2}\{E_1(\overline{Re}) - \log \overline{Re} - \gamma + \log 4\} \vec{v}_o \tag{6-45a}$$

$$\vec{v}_1^{\,11} = D_1 \vec{v}_o^{\,1} = \left(\frac{e^{-\overline{Re}}}{2\overline{Re}} - \frac{1}{2\overline{Re}}\right) \vec{v}_o^{\,1}$$

$$= \left(\frac{e^{-\overline{Re}}}{2\overline{Re}} - \frac{1}{2\overline{Re}}\right) \frac{1}{2} \epsilon \{\vec{k} \log \frac{\tau}{\tau_o} \tag{6-45b}$$

$$- \nabla^+ (\tau - \tau_o \log \tau) \}$$

It can easily be shown that $\vec{v}_1$ satisfies the three dimensional Stokes equations and the matching condition (6-43).

The corresponding pressure term is easily obtained as

$$p_1^1 = -\frac{\epsilon}{2}\,[E_1(\overline{Re}) - \log \overline{Re} - \gamma + \log 4]\,\frac{\partial}{\partial x^+}\log \tau$$

$$+\frac{\epsilon}{2}\,[\frac{1}{2\overline{Re}} - \frac{e^{-\overline{Re}}}{2\overline{Re}}]\,\frac{\partial}{\partial z^+}\log \tau \qquad (6\text{-}46)$$

Similarly by the matching condition that

$$\lim_{\substack{f_2 \\ Re \to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 - (\vec{w}_o + \epsilon\, \vec{w}_1)}{\epsilon^2} = 0 \qquad (6\text{-}47)$$

for ord f_2 in some overlap domain, $\vec{w}_1$ is easily determined as follows:

$$\vec{w}_1 = \vec{w}_1^{\,1} + \vec{w}_1^{\,11} \qquad (6\text{-}48)$$

and

$$\vec{w}_1^{\,1} = \frac{1}{2}\,\{E_1(\overline{Re}) - \log \overline{Re} - \gamma + \log 4\}\,\vec{w}_o \qquad (6\text{-}49a)$$

$$\vec{w}_1^{\,11} = -\,(\frac{e^{-\overline{Re}}}{2\overline{Re}} - \frac{1}{2\overline{Re}})\,\vec{w}_o^{\,1}$$

$$= -\frac{1}{2}\,(\frac{e^{-\overline{Re}}}{2\overline{Re}} - \frac{1}{2\overline{Re}})\,\{\vec{k}\log\frac{\tau^+}{\tau_o} - \nabla^+\,(\tau^+ - \tau_o \log \tau^+)\}$$

$$= -\,D_1\,\vec{w}_o^{\,1} \qquad (6\text{-}49b)$$

$$p_1^{11} = -\frac{\epsilon}{2}\,[E_1(\overline{Re}) - \log 2\overline{Re} - \gamma + \log 4]\,\frac{\partial}{\partial x^+}\log \tau^+$$

$$-\frac{\epsilon}{2}\,[\frac{e^{-\overline{Re}}}{2\overline{Re}} - \frac{1}{2\overline{Re}}]\,\frac{\partial}{\partial z^{++}}\log \tau^+ \qquad (6\text{-}50)$$

<u>Matching Between $\vec{u}_1$ and $\vec{v}_1$ (or $\vec{w}_1$)</u>: If we write $\vec{u}_1$ and $\vec{v}_1$ (or $\vec{w}_1$) in terms of intermediate variables, we can easily show that

they are matched to $O(Re^{\alpha}\epsilon)$ and $0 < \alpha < 1$. The details will not be repeated here. Since $\vec{u}_1$ matches to $\vec{v}_1$ (or $\vec{w}_1$), $\vec{g}_2$ can be determined by matching with them.

<u>Determination of $\vec{g}_2$</u>: It can be seen from equation (5-5) that the governing equation for $\vec{g}_2$ is a non-homogeneous Oseen equation. In general, $\vec{g}_2$ can be divided into three parts to be determined.

$$\vec{g}_2 = \vec{g}^{\,i}_2 + \vec{g}^{\,ii}_2 + \vec{g}^{\,iii}_2 \tag{6-51}$$

$\vec{g}^{\,iii}_2$ is the particular solution of Oseen equations and is

$$\vec{g}^{\,iii}_2 = \iiint_{-\infty}^{\infty} t_{ij}\,(\tilde{x}_k - \tilde{\xi}_k)\, f_j\,(\tilde{\xi}_k)\, d\tilde{\xi}_1\, d\tilde{\xi}_2\, d\tilde{\xi}_3 \tag{6-52a}$$

where

$$\vec{f}(\tilde{x}_i) = (\vec{g}_1 \cdot \tilde{\nabla})\,\vec{g}_1 = -\vec{g}_1 \times \tilde{\text{curl}}\,\vec{g}_1 + \tilde{\nabla}\,\frac{1}{2}\,\vec{g}_1^{\,2} \tag{6-52b}$$

and t_{ij} is the fundamental solution discussed in Chapter 2.

$\vec{g}^{\,iii}_2$ defined by the above integral is obviously continuous for $\tilde{r} = 0$ and $-\overline{Re} \leq \tilde{z} \leq \overline{Re}$

$$\vec{g}^{\,iii}_2 = \vec{i}\; f^{\,iii}_2\,(\tilde{z}) \pm \vec{k}\; g^{\,iii}_2\,(\tilde{z}) \tag{6-53}$$

and for $\tilde{z} = \pm\,\overline{Re}$

$$\vec{g}^{\,iii}_2 = \vec{i}\; c^{\,iii}_2 \pm \vec{k}\; D^{\,iii}_2$$

$\vec{g}^{\,i}_2$ is the solution of the homogeneous Oseen solution and is determined by the matching condition that it cancels the unbounded term of

$$\lim_{\substack{f_1 \\ \widetilde{Re}\to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 - (\vec{u}_o + \epsilon \vec{u}_1^1)}{\epsilon^2} = f_1(\tilde{z}) \{ \vec{i} \; (\log \tilde{r} - \frac{1}{2} \log \frac{\overline{Re}^2 - \tilde{z}^2}{\overline{Re}^2} + \frac{1}{2} + b_1) - \tilde{x} \tilde{\nabla} \log \tilde{r} \} \qquad (6\text{-}54)$$

for $\tilde{r} \to 0$ and $-\overline{Re} < \tilde{z} < \overline{Re}$ and ord f_1 in some overlap domain. For $\bar{z} \to 0$ and $\tilde{r} \to 0$ $\vec{g}_2^1$ has to cancel the unbounded terms of

$$\lim_{\substack{f_2 \\ Re\to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 - (\vec{v}_o + \epsilon \vec{v}_1^1)}{\epsilon^2} = c_1 \{ \frac{\vec{i}}{2} \left[\log (\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}) + 2b_1 + \log 2\overline{Re} + 1 \right] - \frac{\tilde{x}}{2} \tilde{\nabla} \log (\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}) \} \qquad (6\text{-}55)$$

Similarly for $\tilde{r} \to o$ and $\bar{z} \to o$, it cancels the unbounded terms of $\lim \frac{\vec{i} + \epsilon \vec{g}_1 - (\vec{w}_o + \epsilon \vec{w}_1^1)}{\epsilon^2}$. By these matching conditions, $\vec{g}_2^1$ is determined as follows

$$\vec{g}_2^1 = - \frac{\vec{i}}{2} \int_{-\overline{Re}}^{\overline{Re}} \left[E_1(\frac{\overline{Re} + \tilde{\zeta}}{2}) + E_1(\frac{\overline{Re} - \tilde{\zeta}}{2}) + \log (\overline{Re} - \tilde{\zeta}) + \log (\overline{Re} + \tilde{\zeta}) - 2 \log \overline{Re} \right] \frac{e^{-\frac{1}{2}(\sqrt{(\tilde{z} - \tilde{\zeta})^2 + \tilde{r}^2} - \tilde{x})}}{\sqrt{(\tilde{z} - \tilde{\zeta})^2 + \tilde{r}^2}} d\tilde{\zeta}$$

$$+ \frac{1}{2} \tilde{\nabla} \int_{-\overline{Re}}^{\overline{Re}} \left[E(\frac{\overline{Re} + \tilde{\zeta}}{2}) + E_1(\frac{\overline{Re} - \tilde{\zeta}}{2}) + \log (\overline{Re} + \tilde{\zeta}) + \log (\overline{Re} - \tilde{\zeta}) - 2 \log \overline{Re} \right] \left[\frac{e^{-\frac{1}{2}(\sqrt{(\tilde{z} - \tilde{\zeta})^2 + \tilde{r}^2} - \tilde{x})}}{\sqrt{(\tilde{z} - \tilde{\zeta})^2 + \tilde{r}^2}} - \frac{1}{\sqrt{(\tilde{z} - \tilde{\zeta})^2 + \tilde{r}^2}} \right] d\tilde{\zeta} \qquad (6\text{-}56)$$

It can be shown (see Appendix C) that $\vec{g}_2^1$ is correctly determined such that

$$\lim_{\substack{f_1 \\ Re\to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 + \epsilon^2 \vec{g}_2^{\,1} - (\vec{u}_o + \epsilon \vec{u}_1^{\,1})}{\epsilon^2} = \vec{i}\, f_2^1(\tilde{z}) + \vec{k}\, g_2^1(\tilde{z}) \qquad (6\text{-}57)$$

and

$$\lim_{\substack{f_2 \\ Re\to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 + \epsilon^2 \vec{g}_2^{\,1} - (\vec{v}_o + \epsilon \vec{v}_1^{\,1})}{\epsilon^2} = \vec{i}\, c_2^1 + \vec{k}\, D_2^1 \qquad (6\text{-}58)$$

and

$$\lim_{\substack{f_2 \\ Re\to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 + \epsilon^2 \vec{g}_2^{\,1} - (\vec{w}_o + \epsilon \vec{w}_1)}{\epsilon^2} = \vec{i}\, c_2^1 - \vec{k}\, D_2^1\,, \qquad (6\text{-}59)$$

Similarly $\vec{g}_2^{\,11}$ can be determined by the matching conditions as follows

1) For $\tilde{r} \to 0$ and $-\overline{Re} < \tilde{z} < \overline{Re}$, it satisfies

$$\lim_{\substack{f_1 \\ Re\to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 + \epsilon \vec{g}_2^{\,11} - (\vec{u}_o + \epsilon \vec{u}_1^{\,11})}{\epsilon^2} = \vec{i}\, f_2^{11}(\tilde{z}) + \vec{k}\, g_2^{11}(\tilde{z}) \qquad (6\text{-}60)$$

2) For $\bar{z} \to o$ and $\tilde{r} \to o$, it satisfies

$$\lim_{\substack{f_2 \\ Re\to 0^2}} \frac{\vec{i} + \epsilon \vec{g}_1 + \epsilon^2 \vec{g}_2^{\,11} - (\vec{v}_o + \epsilon \vec{v}_1^{\,11})}{\epsilon^2} = \vec{i}\, c_2^{11} + \vec{k}\, D_2^{11} \qquad (6\text{-}61)$$

3) For $\bar{\bar{z}} \to o$, $\tilde{r} \to o$, it satisfies

$$\lim_{\substack{f_2 \\ Re\to 0^2}} \frac{\vec{i} + \epsilon \vec{g}_1 + \epsilon^2 \vec{g}_2^{\,11} - (\vec{w}_o + \epsilon \vec{w}_1^{\,11})}{\epsilon^2} = \vec{i}\, c_2^{11} - \vec{k}\, D_2^{11} \qquad (6\text{-}62)$$

Since $\vec{u}_1^{\,11}$, $\vec{v}_1^{\,11}$, and $\vec{w}_1^{\,11}$ are symmetrical with respect to the axis of the cylinder, $\vec{g}_2^{\,11}$ will have no contribution to drag calculation. Thus the detail of determination of $\vec{g}_2^{\,11}$ will not be repeated here.

Determination of $\vec{u}_2$, $\vec{v}_2$ and $\vec{w}_2$: They can easily be determined by the matching conditions

$$\lim_{\substack{f_1 \\ Re \to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 + \epsilon^2 \vec{g}_2 - (\vec{u}_o + \epsilon \vec{u}_1 + \epsilon^2 \vec{u}_2)}{\epsilon^2} = 0 \tag{6-63a}$$

$$\lim_{\substack{f_2 \\ Re \to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 + \epsilon^2 \vec{g}_2 - (\vec{v}_o + \epsilon \vec{v}_1 + \epsilon^2 \vec{v}_2)}{\epsilon^2} = 0 \tag{6-63b}$$

$$\lim_{\substack{f_2 \\ Re \to 0}} \frac{\vec{i} + \epsilon \vec{g}_1 + \epsilon^2 \vec{g}_2 - (\vec{w}_o + \epsilon \vec{w}_1 + \epsilon^2 \vec{w}_2)}{\epsilon^2} = 0 \tag{6-63c}$$

and are easily found as

$$\vec{u}_2 = f_2(\tilde{z})\, \vec{u}_o + g_2(\tilde{z})\, \vec{u}_o^{\,1}$$

$$\vec{v}_2 = c_2 \vec{v}_o + D_2 \vec{v}_o^{\,1}$$

and

$$\vec{w}_2 = c_2 \vec{w}_o - D_2 \vec{w}_o^{\,1}$$

where

$$f_2(\tilde{z}) = f_2^{i}(\tilde{z}) + f_2^{ii}(\tilde{z}) + f_2^{iii}(\tilde{z})$$

$$g_2(\tilde{z}) = g_2^{i}(\tilde{z}) + g_2^{ii}(\tilde{z}) + g_2^{iii}(\tilde{z})$$

$$c_2 = c_2^{i} + c_2^{ii} + c_2^{iii}$$

and

$$D_2 = D_2^{i} + D_2^{ii} + D_2^{iii}$$

The higher order solution can be obtained by use of the similar matching procedure discussed above.

Uniformly Valid Expansion: From the intermediate expansions and the outer expansions one can construct an expansion which is uniformly valid for the entire flow field. Now the first term $\vec{q}_o(\tilde{x}_i, Re)$, uniformly valid to order unity can be constructed by considering $\vec{v}_o$, $\vec{u}_o$, $\vec{w}_o$, $\vec{g}_o$ and $\vec{g}_1$. From (6-12, 20, 21) and (6-55), $\vec{q}_o(\tilde{x}_i, Re)$ can be constructed into the following simple form:

$$\vec{q}_o(\tilde{x}_i, Re) = \vec{i} - \epsilon\, \vec{i} \int_{-c}^{c} \frac{e^{-\frac{1}{2}\left(\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2} - \tilde{x}\right)}}{\sqrt{(\tilde{z} - \tilde{\zeta})^2 + \tilde{r}^2}}\, d\tilde{\zeta}$$

$$+ \epsilon \tilde{\nabla} \int_{-c}^{c} \left[\frac{e^{-\frac{1}{2}\left(\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2} - \tilde{x}\right)}}{\sqrt{(\tilde{z} - \tilde{\zeta})^2 + \tilde{r}^2}} - \frac{1}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}} \right] d\tilde{\zeta} \qquad (6\text{-}64)$$

where $c = \overline{Re} - \frac{Re^2}{2\overline{Re}}$.

The same $\vec{q}_o(\tilde{x}_i, Re)$ can be generated from the corresponding potential solution φ_e by use of the same method for obtaining $\vec{q}_o$ for the finite cylinder in Chapter 5. This justifies the method for obtaining the leading term of the uniformly valid expansion for the finite cylinder with two hemispherical caps.

It is also interesting to note that in the present case the source distribution for $\vec{q}_o$ is constant and uniformly distributed from $-\overline{Re} + \frac{Re^2}{2\overline{Re}} \leq \tilde{z} \leq \overline{Re} - \frac{Re^2}{2\overline{Re}}$. The sources are inside the ellipsoid cylinder. But in the outer limit, the source is then uniformly distributed from $-\overline{Re} \leq \tilde{z} \leq \overline{Re}$ and the source comes to the surface of the ellipsoid at two ends. This shows that the outer limit can not be valid near two ends and two end limits must be introduced.

Computation of Drag Force: The drag force on the cylinder can be obtained either by calculating viscous stress on the cylinder or by momentum integral. In the present case, it can be obtained by comparison with the fundamental solutions of Oseen equations (cf. equation 2-6). The drag force is found to be of order ϵ and

$$\epsilon = \frac{1}{\log \frac{4}{Re} - \gamma + \frac{1}{2}} \tag{6-65}$$

From $\vec{g}_1$, it can be seen that to the order of ϵ, the drag force is constant along the cylinder. The drag per unit length is the same as that of a two dimensional cylinder

$$D_1 = 8\pi\mu\, UL\epsilon \tag{6-66}$$

and

$$\frac{D_1}{8\pi\mu UL} = \epsilon \tag{6-67}$$

From $\vec{g}_2$, we can see that the drag force is no longer constant along the cylinder to the order of ϵ^2. The variation of singular force is found as

$$f_1(\tilde{z}) = \frac{1}{2}\left[E_1\left(\frac{\overline{Re}+\tilde{z}}{2}\right) + E_1\left(\frac{\overline{Re}-\tilde{z}}{2}\right) + \log(\overline{Re}+\tilde{z}) + \log(\overline{Re}-\tilde{z}) - 2\log\overline{Re}\right] \tag{6-68}$$

The drag force D_2 can be obtained by integration. Since we know

$$\int_{-\overline{Re}}^{\overline{Re}} \{E_1(\frac{\overline{Re}+\tilde{z}}{2}) + E_1(\frac{\overline{Re}-\tilde{z}}{2})\} d\tilde{z} = 2\int_{-\overline{Re}}^{\overline{Re}} E_1(\frac{\overline{Re}-\tilde{z}}{2}) d\tilde{z}$$

$$= 4\int_0^{\overline{Re}} E_1(t)dt = \left[4E_1(t)t\right]_0^{\overline{Re}} + 4\int_0^{\overline{Re}} e^{-t}\, dt$$

$$= 4E_1(\overline{Re})\,\overline{Re} + 4 - 4e^{-\overline{Re}}$$

Thus the total drag of order ϵ^2 is

$$D_2 = 8\pi\mu UL \{E_1(\overline{Re}) + (\log 2-1)$$

$$+ \frac{1}{\overline{Re}} (1 - e^{-\overline{Re}})\} \qquad (6\text{-}69)$$

Thus the total drag is

$$D = 8\pi\mu UL \{\epsilon + [E_1(\overline{Re}) + (\log 2-1)$$

$$+ \frac{1}{\overline{Re}} (1 - e^{-\overline{Re}})]\, \epsilon^2 + O(\epsilon^3)\} \qquad (6\text{-}70)$$

or

$$C_D = \frac{4\pi}{Re} \{\epsilon + \epsilon^2 [E_1(\overline{Re}) + \log 2-1$$

$$+ \frac{1}{\overline{Re}} (1 - e^{-\overline{Re}})] + O(\epsilon^3)\} \qquad (6\text{-}71)$$

The corresponding C_D for a two dimensional cylinder (cf. Ref. 2) is

$$C_D = \frac{4\pi}{Re} \{\epsilon - 0.87\, \epsilon^3 + O(\epsilon^4)\} \qquad (6\text{-}72)$$

VII. CONCLUDING REMARKS

In this section, we will briefly summarize the results and discuss their significance. From some properties of low Reynolds number flow, we will draw some additional conclusions. Finally we shall consider some related problems which might be solved by the method illustrated here.

7.1 Nature of End Source Distributions

One of the most important results obtained in this thesis is the clarification of the source distribution near the end of the cylinder. They are the same for both potential problems and Navier-Stokes problems discussed above. The source distribution for the two finite cylinders under study can be summarized as follows:

a) The finite cylinder with hemispherical ends: The obtained uniformly valid expansion to order unity for the finite cylinder is unchanged if we choose $\epsilon = \dfrac{1}{\log \frac{1}{Re} + c}$ where c is any arbitrary constant. For comparison we will choose $\epsilon = \dfrac{1}{\log \frac{4}{Re} - \gamma + \frac{1}{2}}$, the same as the ϵ for the ellipsoid and the 2-D circular cylinder. The source distributions with the origin at the left end of the cylinder in terms of left end Stokes variables is

$$s_L(z^*) = \frac{\epsilon^{\frac{1}{2}}}{2\sqrt{\log z^*}} \tag{7-1}$$

This shows that for the semi-infinite end cylinder obtained by the end limit the source distribution has an inverse logarithmic variation. The source strength $s_L(z^*)$ decreases to zero as z^* tends to infinity and the rate of decrease is $\frac{1}{\sqrt{\log z^*}}$ which is much slower than $\frac{1}{z^*}$. It is also interesting to note that the source distribution is of order $\epsilon^{\frac{1}{2}}$ near the end while the source strength for the ellipsoid cylinder or a 2-D cylinder is of order ϵ. If we express it in terms of outer variables, we have

$$s_L(\bar{z}) = \frac{1}{2} \frac{\epsilon^{\frac{1}{2}}}{\sqrt{\log \frac{\bar{z}}{\overline{Re}}}} \tag{7-2}$$

where $\bar{z} = z + \overline{Re}$. This shows that the source strength $s_L(\tilde{z})$ decreases very rapidly to order of ϵ which is the source strength of a two dimensional cylinder as $\bar{z} > 0$ in the limit $Re \rightarrow 0$. The largest variation occurs near the end. The total variation for the finite cylinder is found to be

$$s(\tilde{z}) = \frac{\epsilon^{\frac{1}{2}}}{2} \left\{ \frac{1}{\sqrt{\log \frac{\overline{Re}+\tilde{z}}{\overline{Re}}}} + \frac{1}{\sqrt{\log \frac{\overline{Re}-\tilde{z}}{\overline{Re}}}} - \epsilon^{\frac{1}{2}} \right\} \tag{7-3}$$

The variation was shown in Fig. 4-3. It is worth mentioning that $s(\tilde{z})$ is integrable. As we discussed before, the obtained holds for any "untapered" body, and we can conclude the existence of this "untapered" end causes the source distribution to have a variation near the end as defined in (7-3).

b) <u>The ellipsoid cylinder:</u> The end body for the ellipsoid is a semi-infinite paraboloid and from the variables introduced in equation (6-5), we can see the end region is much smaller than that of the finite cylinder with hemispherical ends. From $\vec{v}_o$, we can see that the source distribution is a constant semi-infinite line source distribution. Similarly, from $\vec{g}_1$ we can see that the singular force acting on the ellipsoid cylinder is constant. Thus we have clarified that to $O(\epsilon)$ the source distribution for the ellipsoid cylinder in Navier-Stokes flow is constant. For the potential case, a better result (a uniformly valid expansion to $O(\lambda^2)$) has been obtained. We can conclude the existence of this kind of end causes no variation of source distribution near the end.

7.2 The Magnitude of Various Effects

In this section, we will discuss the order of magnitude of various effects for the two finite cylinders discussed.

a) The finite cylinder with hemispherical caps: Physically the cylinder under present study is of constant radius but near the end the radius varies from $r = \lambda$ to $r = 0$. Therefore even if we can assume the source distribution in the shank region is constant near the end we must have the variation of source distributions. In the present case, the variation is shown in Fig. 4-3. But the source distribution for an infinite cylinder is constant along the cylinder (cf. Ref. 2). Thus the order of magnitude of the end effect can be determined by calculating its effect on drag.

For the Navier-Stokes flow past this finite cylinder, the drag can be obtained by comparing the outer expansion of $\vec{q}_0$ with the fundamental solution of Oseen equations and the result is obtained in equation (5-69). To the order of ϵ, the drag can be obtained by assuming that the source distribution is constant and has the same strength per unit length as that of a two dimensional cylinder. But to $O(\epsilon^2)$, the obtained result shows that the drag is no longer constant and is different from that of a two dimensional cylinder. The result also shows that the drag has a logarithmic singularity near the ends. It is obvious that this singularity is integrable and thus the drag is finite on the cylinder. From the obtained result, we can conclude that the order of magnitude of the end effect is of $O(\epsilon^2)$ and the existence of end produces a logarithmic variation near the end at this order of ϵ^2.

b) The ellipsoid cylinder: From the obtained result in Chapter 6, it is obvious that to $O(\epsilon)$, the source distribution from $\vec{g}_1$ is constant. This shows the end and the variation of radius along the cylinder have no effect on drag to this order. But to $O(\epsilon^2)$, the drag distribution is no longer constant and has the following variation from $\vec{g}_2^1$.

$$s_2(\tilde{z}) = \frac{1}{2}\left[E_1\left(\frac{\overline{Re} + \tilde{z}}{2}\right) + E_1\left(\frac{\overline{Re} - \tilde{z}}{2}\right) + \log(\overline{Re} - \tilde{z}) \right. \qquad (7\text{-}4)$$

$$\left. + \log(\overline{Re} + \tilde{z}) - \log 2\overline{Re}\right]$$

The source distribution is finite near both ends and has no large variation near the end as that of the finite cylinder. The total drag of $O(\epsilon^2)$ is obtained in equation (6-69). In Chapter 6 we can see that the same outer solutions will be obtained by matching between outer and shank only without taking the end into consideration. Thus the end has no effect to $O(\epsilon^2)$ and we may even say no effect to $O(\epsilon^m)$. The present variation of drag distribution along the cylinder is due to the variation of radius r along the cylinder. This can easily be seen from the matching between shank and outer. If the body is $r^* = f(\tilde{z})$ instead of ellipsoid but has the same type of ends as ellipsoid, we will have the singular force variation of $O(\epsilon^2)$ as

$$s_2(\tilde{z}) = \frac{1}{2}\left[E_1\left(\frac{\overline{Re} + \tilde{z}}{2}\right) + E_1\left(\frac{\overline{Re} - \tilde{z}}{2}\right) - 2f(\tilde{z})\right] \qquad (7\text{-}5)$$

Thus we can conclude that the variation of diameters of this finite cylinder causes an effect of $O(\epsilon^2)$. From equation (7-5), it is worth mentioning that the cylinder which makes the drag of $O(\epsilon^2)$ equal to zero is

$$r^* = f(\tilde{z}) = e^{-\frac{1}{2}E_1\left(\frac{\overline{Re}+\tilde{z}}{2}\right) - \frac{1}{2}E_1\left(\frac{\overline{Re}-\tilde{z}}{2}\right)} \qquad (7\text{-}6)$$

For $\tilde{z} \to -\overline{Re}$ or $\bar{z} = \overline{Re} + \tilde{z} \to 0$, we have

$$r^* = e^{-\frac{1}{2}E_1(\overline{Re})}\, e^{\frac{1}{2}\log\frac{\gamma_0 \bar{z}}{2}} + O(\bar{z}^{\frac{3}{2}}) \qquad (7\text{-}7a)$$

or

$$r^* = A\sqrt{\bar{z}} + O(\bar{z}^{\frac{3}{2}}) \qquad (7\text{-}7b)$$

or

$$r^{+2} = A^2 z^+ + O(Re^2) \qquad (7\text{-}7c)$$

where $A = \sqrt{\frac{\gamma_0}{2}}\, e^{-\frac{1}{2}E_1(\overline{Re})}$

This cylinder also has a paraboloid end. By actual numerical plotting for large $\overline{Re}$, this cylinder has a constant radius in the center region. Since the drag of $O(\epsilon^2)$ is zero for this cylinder, and from

(6-72) the drag per unit length agrees to the value of the two dimensional cylinder to $O(\epsilon^3)$. This suggests that this cylinder may be the best one for studying the nature of the passage from the typically two dimensional case to the typically three dimensional case (especially concerning the drag formula). Finally, we can conclude from equation (7-5), the variation of diameter will always cause an effect of $O(\epsilon^2)$ and the only exceptional cylinder is defined by equation (7-6).

Thus we have clarified the order of magnitude of various effects.

7.3 Expansion Procedures.

In the present study, we have introduced four expansions. It is worth mentioning that for the finite cylinder with hemispherical ends the idea of an intermediate (rather than inner) expansion is intimately involved and quite helpful in the present case. In fact, these intermediate solutions, effectively, make the solutions of our problem possible in practice. As pointed out by S. Kaplun (cf. Ref. 9), the intermediate solutions also bring out a number of "typical" low Reynolds phenomena.

The successful use of the idea of the intermediate solutions in the present study show that the similar expansion procedure can be applied to other problems of low Re flow.

For the ellipsoid cylinder, the matching between the four introduced expansions are exhibited in detail. This example also shows how to treat several expansions (more than two). Although certainly no new principles need to be introduced, it becomes interesting in detail. In the present case, we first construct a uniformly valid expansion near the body and then the outer solution is determined by matching with this uniformly valid expansion near the body or by matching each of the intermediate solutions which have already been matched with each other.

It is worth mentioning that the low Reynolds flow is understood as the flow about a very small object, that is, the flow obtained when the characteristic length (i.e. radius of the cylinder) tends to zero with the length $\frac{\nu}{U}$ fixed and the observer fixed in space. Note that this intuitive definition of low Reynolds number flow has meaning only in connection with specific problems, but then it has the advantage that the resulting limit is a unique solution of the full equations and uniform at infinity.

7.4 Related Problems

It would be useful in obtaining the overall picture of low Reynolds number flow to solve the present problem considering some of the features which have been removed for simplicity. For example, it would be interesting to consider the effect of compressibility. The effect of compressibility at low Reynolds number is discussed in Ref. 5. However this discussion is not complete and there remain many unanswered questions in the area of compressible low Reynolds number flow. Another extension of the example which might prove interesting and could be handled by the methods discussed here would be in the extension to the finite cylinder with its axis parallel to the direction of the flow or with some angle of attack by introducing certain suitable co-ordinate systems, to finite wires with different curvatures.

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APPENDIX A. AN ASYMPTOTIC INTEGRAL

The following asymptotic expression

$$\int_0^{\eta} e^{y^2}\, dy \sim \frac{e^{\eta^2}}{2\eta} + \frac{e^{\eta^2}}{4\eta^2} + \dots\dots \tag{A-1}$$

can be established for $\eta \to \infty$.

Proof: If we let

$$f = e^{-\eta^2} \int_0^{\eta} e^{s^2}\, ds = \int_0^{\eta} e^{-(\eta^2 - s^2)}\, ds \tag{A-2}$$

and let $u^2 = \eta^2 - s^2$, we have $u^2 \geq 0$ because $\eta^2 \geq s^2$ and

$$ds = -\frac{udu}{\sqrt{\eta^2 - u^2}} \tag{A-3}$$

Therefore for $\eta \to \infty$, we obtain

$$f = \int_0^{\eta} e^{-u^2} \frac{udu}{\sqrt{\eta^2 - u^2}}$$

$$= \int_0^{\eta} e^{-u^2} \left(\frac{1}{\eta} + \frac{1}{2}\,\frac{u^2}{\eta^3} + \dots\dots\right) udu$$

$$= \frac{1}{2\eta} + \frac{1}{4\eta^3} + \dots\dots \tag{A-4}$$

This concludes the proof.

APPENDIX B. ASYMPTOTIC EXPANSION OF $\vec{g}_1(\tilde{x}_i)$

Asymptotic expansions of $\vec{g}_1$ (cf. 6-31) for $\tilde{r} \to 0$, $-\overline{Re} < \tilde{z} < \overline{Re}$ and for $\tilde{r} \to 0$, $\tilde{z} + Re \to 0$ are established in detail here. From (6-31) and for $\tilde{r} \to 0$ and $\tilde{z}$ finite, we can write $\vec{g}_1$ in the following form

$$\vec{g}_1 = -\vec{i}\,\frac{1}{2}\int_{-\overline{Re}}^{\overline{Re}} \frac{e^{-\frac{1}{2}\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}\,d\zeta$$

$$-\vec{i}\,\frac{\tilde{x}^2}{2}\int_{-\overline{Re}}^{\overline{Re}} \frac{d\tilde{\zeta}}{[(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2]^{3/2}}$$

$$-\vec{j}\,\frac{\tilde{x}\tilde{y}}{2}\int_{-\overline{Re}}^{\overline{Re}} \frac{d\tilde{\zeta}}{[(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2]^{3/2}}$$

$$+\vec{k}\,\frac{\partial}{\partial\tilde{z}}\int_{-\overline{Re}}^{\overline{Re}} \left[\frac{e^{-\frac{1}{2}(\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}\;-\tilde{x}}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}} - \frac{1}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}\right]d\zeta$$

$$+\,O(\tilde{r}\log\tilde{r})$$

$$= -\frac{\vec{i}}{2}A - \vec{i}B - \vec{j}C + \vec{k}D \qquad \text{(B-1)}$$

The asymptotic expansion of A can easily be obtained by introducing $\tilde{\zeta} = \tilde{z} + \tilde{r}\sinh\sigma$.

a) For $\tilde{r} \to 0$ and $-\overline{Re} < \tilde{z} < \overline{Re}$, let us divide the integral into three intervals $[-\overline{Re}, \tilde{z}-\nu(\tilde{r})]$, $[\tilde{z}-\nu(\tilde{r}); \tilde{z}+\nu(\tilde{r})]$ and $[\tilde{z}+\nu(\tilde{r}), \overline{Re}]$ and let us consider

$$A = A_1 + A_2 + A_3 \tag{B-2}$$

for $\dfrac{\tilde{r}}{\nu(\tilde{r})} \to 0$, $\nu(\tilde{r}) \to 0$ as $\tilde{r} \to 0$.

Then we have

$$A_1 = \int_{-Re}^{\tilde{z}-\nu(\tilde{r})} \frac{e^{-\frac{1}{2}\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}\, d\tilde{\zeta} = \int_{-\overline{Re}}^{\tilde{z}-\nu(\tilde{r})} \frac{e^{-\frac{1}{2}(\tilde{z}-\tilde{\zeta})}}{(\tilde{z}-\tilde{\zeta})}\, d\tilde{\zeta} + O(\tilde{r})$$

$$= -\log \frac{\gamma_0 \nu(\tilde{r})}{2} - E_1\left(\frac{\overline{Re}+\tilde{z}}{2}\right) + O(\nu, \tilde{r}) \tag{B-3}$$

$$A_2 = \int_{\tilde{z}-\nu(\tilde{r})}^{\tilde{z}+\nu(\tilde{r})} \frac{e^{-\frac{1}{2}\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}\, d\tilde{\zeta} = \int_{\sinh^{-1}(-\frac{\nu}{\tilde{r}})}^{\sinh^{-1}(\frac{\nu}{\tilde{r}})} e^{-\frac{1}{2}\tilde{r}\sinh\sigma}\, d\sigma$$

$$= 2\log \frac{2\nu(\tilde{r})}{\tilde{r}} + O\left(\nu, \frac{\tilde{r}}{\nu}\right) \tag{B-4}$$

Similar to A_1, we get

$$A_3 = \int_{z+\nu(\tilde{r})}^{\overline{Re}} \frac{e^{-\frac{1}{2}\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}\, d\tilde{r}$$

$$= -\log \frac{\gamma_0 \nu(\tilde{r})}{2} - E_1\left(\frac{\overline{Re}-\tilde{z}}{2}\right) + O(\nu(\tilde{r}), \tilde{r}) \tag{B-5}$$

Thus for $\tilde{r} \to 0$ and $-\overline{Re} < \tilde{z} < \overline{Re}$, we have

$$A = -2\,(\log \tilde{r} - \log 4 + \gamma) - E_1\left(\frac{\overline{Re} + \tilde{z}}{2}\right)$$

$$- E_1\left(\frac{\overline{Re} + \tilde{z}}{2}\right) + O(\nu, \frac{\tilde{r}}{\nu}) \qquad \text{(B-6)}$$

b) For $\tilde{r} \to 0$ and $\bar{z} = \tilde{z} + Re \to 0$, the asymptotic expansion of A can easily be obtained by use of the same method by dividing the integral into two intervals $[0, \bar{z} + \nu(\tilde{r})]$ and $[\bar{z} + \nu(\tilde{r}), 2\overline{Re}]$ and let us consider

$$A = A_4 + A_5 \qquad \text{(B-7)}$$

now

$$A_4 = \int_0^{z+\nu(\tilde{r})} \frac{e^{-\frac{1}{2}\sqrt{(\bar{z}-\zeta)^2 + \tilde{r}^2}}}{\sqrt{(\bar{z}-\zeta)^2 + \tilde{r}^2}}\, d\zeta$$

$$= \int_{\sinh^{-1}(-\frac{\bar{z}}{\tilde{r}})}^{\sinh^{-1}(\frac{\nu}{\tilde{r}})} e^{-\frac{1}{2}\tilde{r}\sinh\sigma}\, d\sigma$$

$$= \log \frac{2\nu}{\tilde{r}} - \log\left(\sqrt{\left(\frac{\bar{z}}{\tilde{r}}\right)^2 + 1} + \frac{\bar{z}}{\tilde{r}}\right) + O(\nu, \frac{\tilde{r}}{\nu}) \qquad \text{(B-8)}$$

$$A_5 = \int_{\bar{z}+\nu(\tilde{r})}^{2\overline{Re}} \frac{e^{\frac{1}{2}(\zeta - \bar{z})}}{\zeta - \bar{z}}\, d\zeta = -\log\frac{\nu\gamma_o}{2} - E_1(\overline{Re})$$

$$+ O(\nu, \frac{\tilde{r}}{\nu}) \qquad \text{(B-9)}$$

Therefore we have for $\tilde{r} \to 0$ and $\bar{z} \to 0$

$$A = -\log(\sqrt{\bar{z}^2 + \tilde{r}^2} + \bar{z}) + \log 4 - \gamma - E_1(\overline{Re})$$

$$+ O(\nu, \frac{\tilde{r}}{\nu}) \qquad \text{(B-10)}$$

The asymptotical expansion of B can be obtained by direct integration

$$B = \frac{\tilde{x}^2}{2} \frac{1}{\tilde{r}^2} \left[\frac{\tilde{z} + \overline{Re}}{[(\tilde{z}+\overline{Re})^2 + \tilde{r}^2]^{\frac{1}{2}}} - \frac{\tilde{z} - \overline{Re}}{[(\tilde{z}-\overline{Re})^2 + \tilde{r}^2]^{\frac{1}{2}}} \right] \qquad \text{(B-11)}$$

For $\tilde{r} \to 0$ and $-\overline{Re} < \tilde{z} < \overline{Re}$, we have

$$B = \frac{\tilde{x}^2}{\tilde{r}^2} + O(\tilde{r}^2) \qquad \text{(B-12)}$$

For $\tilde{r} \to 0$ and $\bar{z} = \tilde{z} + \overline{Re} \to 0$, we can easily see

$$B = \frac{1}{2} \frac{\tilde{x}^2}{\tilde{r}^2} \left[\frac{\bar{z}}{\sqrt{\bar{z}^2 + \tilde{r}^2}} + 1 \right] + O(\tilde{r})$$

$$= \frac{1}{2} \frac{\tilde{x}^2}{\bar{z}^2 + \tilde{r}^2} \frac{1}{\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}}$$

$$= \frac{1}{2} \tilde{x} \frac{\partial}{\partial \tilde{x}} \log(\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}) + O(\tilde{r}) \qquad \text{(B-13)}$$

The asymptotical expansion of C can easily be obtained by the relation

$$C = \frac{B^{\tilde{y}}}{\tilde{x}} \qquad \text{(B-14)}$$

The easiest way for obtaining the asymptotical expansion for D is by use of series.

$$D = \int_{-\overline{Re}}^{\overline{Re}} \left[\frac{(\tilde{z}-\tilde{\zeta})\, e^{-\frac{1}{2}(\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}\, -\tilde{x})}}{[(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2]^{3/2}} - \frac{(\tilde{z}-\tilde{\zeta})}{[(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2]^{3/2}} \right.$$

$$\left. + \frac{1}{2} \frac{(\tilde{z}-\tilde{\zeta}) e^{-\frac{1}{2}(\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}\, - \tilde{x})}}{(\tilde{z} - \tilde{\zeta})^2 + \tilde{r}^2} \right] d\tilde{\zeta}$$

$$= \frac{1}{2} \tilde{x} \int_{-\overline{Re}}^{\overline{Re}} \frac{(\tilde{z}-\tilde{\zeta})}{[(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2]^{3/2}} \, d\tilde{\zeta}$$

$$+ \int_{-\overline{Re}}^{\overline{Re}} \frac{(\tilde{z} - \tilde{\zeta})}{[(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2]^{3/2}} \sum_{n=1}^{\infty} \left(\frac{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}{2} \right)^n \frac{1}{n!} \, d\tilde{\zeta}$$

$$+ \frac{1}{2} \int_{-\overline{Re}}^{\overline{Re}} \frac{(\tilde{z} - \tilde{\zeta})}{(\tilde{z}-\tilde{\zeta})^2+\tilde{\rho}^2} \sum_{n=0}^{\infty} \left(- \frac{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}}{2} \right)^n \frac{d\tilde{\zeta}}{n!}$$

$$+ O(\tilde{r} \log \tilde{r})$$

$$= \left[- \frac{1}{2} \frac{\tilde{x}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}} \right]_{-\overline{Re}}^{\overline{Re}} + \left[\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} \frac{[(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2]^{\frac{n}{2}}}{2^n\, (n+1)!} \right]_{-\overline{Re}}^{\overline{Re}}$$

$$+ \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{[(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2]^{\frac{n}{2}}}{n!\, 2^n} \right]_{-\overline{Re}}^{\overline{Re}} + O(\tilde{r} \log \tilde{r})$$

$$= \left[- \frac{1}{2} \frac{\tilde{x}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2}} \right]_{-\overline{Re}}^{\overline{Re}}$$

$$+\left[\frac{e^{-\frac{1}{2}\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}} - \frac{1}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}\right]_{-\overline{Re}}^{\overline{Re}} + O(\tilde{r}\log\tilde{r}) \quad \text{(B-15)}$$

Thus for $\tilde{r} \to 0$ and $-\overline{Re} < \tilde{z} < \overline{Re}$, we have

$$D = \frac{e^{\frac{1}{2}(\overline{Re}-\tilde{z})}}{\overline{Re}-\tilde{z}} - \frac{1}{\overline{Re}-\tilde{z}} - \frac{e^{-\frac{1}{2}(\overline{Re}+\tilde{z})}}{\overline{Re}+\tilde{z}} + \frac{1}{\overline{Re}+\tilde{z}}$$

$$+ O(\tilde{r}\log\tilde{r}) \quad \text{(B-16)}$$

and for $\tilde{r} \to 0$, $\overline{z} = \tilde{z} + \overline{Re} \to 0$, we have

$$D = -\frac{1}{2}\tilde{x}\frac{\partial}{\partial \overline{z}} \log\left(\sqrt{\overline{z}^2+\tilde{r}^2} - \overline{z}\right)$$

$$+ \frac{e^{-\overline{Re}}}{2\overline{Re}} - \frac{1}{2\overline{Re}} + O(\overline{z}\log\overline{z}) \quad \text{(B-17)}$$

Thus the asymptotic evaluation of $\vec{g}_1(\tilde{x}_i)$ can be summarized as follows:

a) For $\tilde{r} \to 0$ and $-\overline{Re} < \tilde{z} < \overline{Re}$, we obtain

$$\vec{g}_1(\tilde{x}_i) = \vec{i}\,(\log\tilde{r} - \log 4 + \gamma) - \tilde{x}\,\tilde{\nabla}\log\tilde{r}$$

$$+ \frac{\vec{i}}{2}\left\{E_1\left(\frac{\overline{Re}+\tilde{z}}{2}\right) + E_1\left(\frac{\overline{Re}-\tilde{z}}{2}\right)\right\}$$

$$+ \vec{k}\left\{\frac{e^{-\frac{1}{2}(\overline{Re}-\tilde{z})}}{\overline{Re}-\tilde{z}} - \frac{1}{\overline{Re}-\tilde{z}} + \frac{1}{\overline{Re}+\tilde{z}} - \frac{e^{-\frac{1}{2}(\overline{Re}+\tilde{z})}}{\overline{Re}+\tilde{z}}\right\}$$

$$+ O(\tilde{r}\log\tilde{r}) \quad \text{(B-18)}$$

b) For $\tilde{r} \to 0$ and $\overline{z} = \tilde{z} + \overline{Re} \to 0$, we obtain

$$\vec{g}_1(\tilde{x}_i) = \frac{\vec{i}}{2}\Big\{\log\left(\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}\right) - \gamma + \log 4$$

$$+ E_1(\overline{Re})\Big\} - \tilde{x}\tilde{\nabla} \log\left(\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}\right)$$

$$+ \vec{k}\left(\frac{e^{-\overline{Re}}}{2\overline{Re}} - \frac{1}{2\overline{Re}}\right) + O(\bar{z} \log \bar{z}) \qquad \text{(B-19)}$$

It is worth noting that the same results can be obtained by differentiating the asymptotic expansion of A. This justifies the differentiation of the asymptotic expansion. Thus in evaluating of $\vec{g}_2^{\,1}(\tilde{x}_i)$ in Appendix C, we will not repeat the detail and will only discuss the corresponding essential term.

APPENDIX C. ASYMPTOTIC EXPANSION OF $\vec{g}_2^1(\tilde{x}_i)$

$\vec{g}_2^1$ is defined in equation (6-56) and the asymptotic expansion of $\vec{g}_2^1(\tilde{x}_i)$ for $\tilde{r} \to 0$, $-\overline{Re} < \tilde{z} < \overline{Re}$ and for $\tilde{r} \to 0$, $\bar{z} = \tilde{z} + \overline{Re} \to 0$ can proceed in the same manner as for $\vec{g}_1$ in Appendix B. The details will not be repeated here. We only consider the evaluation of the following essential part

$$I(\tilde{z},\tilde{r}) = \int_{-\overline{Re}}^{\overline{Re}} \frac{f(\tilde{\zeta})e^{-\frac{1}{2}\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}}{\sqrt{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2}}\, d\tilde{\zeta} \tag{C-1}$$

a) For $\tilde{r} \to 0$, $-Re < \tilde{z} < \overline{Re}$, we can divide the integral into three parts as we did in Appendix B. Then we have

$$I = I_1 + I_2 + I_3 \tag{C-2}$$

and

$$I_1 = \int_{-\overline{Re}}^{\tilde{z}-\nu(\tilde{r})} \frac{f(\tilde{\zeta})e^{-\frac{1}{2}(\tilde{z}-\tilde{\zeta})}}{\tilde{z}-\tilde{\zeta}}\, d\tilde{\zeta} + O(\nu, \frac{\tilde{r}}{\nu}) \tag{C-3}$$

$$= f(z)E_1(\frac{\nu}{2}) - f(-\overline{Re})E_1(\frac{\overline{Re}+\tilde{z}}{2})$$

$$+ \int_{-\overline{Re}}^{z-\nu(\tilde{r})} E_1(\frac{\tilde{z}-\tilde{\zeta}}{2}) \frac{df}{d\tilde{\zeta}}\, d\tilde{\zeta} + O(\nu, \frac{\tilde{r}}{\nu}) \tag{C-4}$$

$$I_2 = \int_{\sinh^{-1}(-\frac{\nu}{\tilde{r}})}^{\sinh^{-1}(\frac{\nu}{\tilde{r}})} f(\tilde{z} + \tilde{r}\sinh\sigma)e^{-\frac{1}{2}\tilde{r}\sinh\sigma}\, d\sigma$$

$$= 2f(\tilde{z}) \log \frac{2\nu}{\tilde{r}} + O(\nu, \frac{\tilde{r}}{\nu}) \tag{C-5}$$

$$I_3 = \int_{\tilde{z}+\nu(\tilde{r})}^{\overline{Re}} \frac{f(\tilde{\zeta})\, e^{-\frac{1}{2}(\tilde{\zeta}-\tilde{z})}}{\tilde{\zeta}-\tilde{z}}\, d\tilde{\zeta} + O(\nu, \frac{\tilde{r}}{\nu})$$

$$= f(\tilde{z})\, E_1(\frac{\nu}{2}) - f(\overline{Re})\, E_1(\frac{\overline{Re}+\tilde{z}}{2}) + \int_{\tilde{z}+\nu}^{\overline{Re}} E_1\,(\frac{\tilde{\zeta}-\tilde{z}}{2})\,\frac{df}{d\tilde{\zeta}}\, d\tilde{\zeta}$$

$$+ O(\nu, \frac{\tilde{r}}{\nu}) \qquad \text{(C-6)}$$

Thus

$$I = -2f(\tilde{z})\ [\log \tilde{r} + \gamma - \log 4]$$

$$+ \int_{\tilde{z}}^{\overline{Re}} E_1\,(\frac{\tilde{\zeta}-\tilde{z}}{2})\,\frac{df}{d\tilde{\zeta}}\ d\tilde{\zeta} - \int_{-\overline{Re}}^{\tilde{z}} E_1(\frac{\tilde{z}-\tilde{\zeta}}{2})\,\frac{df}{d\tilde{\zeta}}\ d\tilde{\zeta}$$

$$-f(\overline{Re})\, E_1\,(\frac{\overline{Re}-\tilde{z}}{2}) - f(-\overline{Re})\, E_1(\frac{\overline{Re}+\tilde{z}}{2}) + O(\nu, \frac{\tilde{r}}{\nu}) \qquad \text{(C-7)}$$

The above evaluation is possible if $f(\overline{Re})$ and $f(-\overline{Re})$ are finite and the integrals in (C-7) are integrable.

b) For $\tilde{r} \to 0$ and $\bar{z} = \overline{Re} + \tilde{z} \to 0$, we can divide the integration into two parts.

$$I = I_4 + I_5 \qquad \text{(C-8)}$$

and

$$I_4 = \int_0^{\bar{z}+\nu(\tilde{r})} \frac{f(\tilde{\zeta})e^{-\frac{1}{2}\sqrt{(\bar{z}-\tilde{\zeta})^2+\tilde{r}^2}}}{\sqrt{(\bar{z}-\tilde{\zeta})^2+\tilde{r}^2}}\, d\tilde{\zeta}$$

$$= f(\bar{z}) \log \frac{2\nu}{\tilde{r}} - f(\bar{z}) \log (\sqrt{\frac{\bar{z}^2}{\tilde{r}^2}+1} - \frac{\bar{z}}{\tilde{r}})$$

$$+O\,(\frac{\tilde{r}}{\nu}, \nu) \qquad \text{(C-9)}$$

$$I_5 = \int_{\bar{z}+\nu(\tilde{r})}^{2\overline{Re}} \frac{f(\zeta)e^{-\frac{1}{2}(\zeta-\bar{z})}}{\zeta-\bar{z}} \, d\bar{z} + O\left[\nu(\tilde{r}), \frac{\tilde{r}}{\nu(\tilde{r})}\right]$$

$$= - f(2\overline{Re})\, E_1\left(\frac{2\overline{Re}-\bar{z}}{2}\right) + f(\bar{z})\, E_1\left(\frac{\nu}{2}\right)$$

$$+ \int_{\bar{z}+\nu(\tilde{r})}^{2\overline{Re}} E_1\left(\frac{\zeta-\bar{z}}{2}\right) \frac{df}{d\zeta}\, d\zeta + O\left(\nu, \frac{\tilde{r}}{\nu}\right)$$

$$= -f(\bar{z})\left[\log\left(\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}\right) + \gamma - \log 4\right] + f(2\overline{Re})E_1(\overline{Re})$$

$$+ \int_0^{2\overline{Re}} E_1\left(\frac{\zeta}{2}\right) \frac{df}{d\zeta}\, d\zeta + O\left(\frac{\tilde{r}}{\nu}, \bar{z}\log\bar{z}\right) \qquad \text{(C-10)}$$

The above evaluation holds for any $f(\bar{z})$ such that $f(0)$ and $f(2\overline{Re})$ are finite and the last integral in the above equation exists. For $\vec{g}_2^1(\tilde{x}_i)$, $f(\tilde{z}) = f_1(\tilde{z})$ defined in (6-41a) which satisfies all the requirements in the above evaluation. Thus, by the above results, the asymptotic expansion can easily be summarized as follows.

1) For $\tilde{r} \to 0$, $-\overline{Re} < \tilde{z} < \overline{Re}$, we have

$$\vec{g}_2^1(\tilde{x}_i) \sim f(\tilde{z})\left\{\vec{i}\left[\log\tilde{r} + \gamma - \log 4\right] - \tilde{x}\tilde{\nabla}\log\tilde{r}\right\}$$

$$+ \vec{i}\, k_1(\tilde{z}) + \vec{k}\, k_2(\tilde{z}) + O(\tilde{r}\log\tilde{r}) \qquad \text{(C-11)}$$

where $k_1(\tilde{z})$ and $k_2(\tilde{z})$ are functions of $\tilde{z}$ only and are finite for $-\overline{Re} < \tilde{z} < \overline{Re}$.

2) For $\tilde{r} \to 0$, $\bar{z} = \overline{Re} + \tilde{z} \to 0$, we have

$$\vec{g}_2^1 (\tilde{x}_i) \sim f(\tilde{z}) \left\{ \frac{\vec{i}}{2} \left[\log \left(\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}\right) + \gamma - \log 4\right] \right.$$

$$\left. - \frac{\tilde{x}}{2} \tilde{\nabla} \log \left(\sqrt{\bar{z}^2 + \tilde{r}^2} - \bar{z}\right) \right\} + \vec{i}\, C_2 + \vec{k}\, D_2 \qquad \text{(C-12)}$$

where C_2 and D_2 are finite constants. The detailed evaluation will not be repeated here.