INFLUENCE OF RADIATIVE DISSIPATION ON THE SHOCK WAVE STRUCTURE

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ABSTRACT

The influence of radiation on a steady, one dimensional flow is considered. Only radiative heat transfer is taken into account; viscosity, heat conduction and mass diffusion are neglected. It is further assumed that the radiative heat transfer is adequately described by the quasi equilibrium theory relative to a grey gas.

Under these conditions, the velocity of the fluid satisfies an integral equation which has been investigated by various methods. It is shown that under certain conditions the influence of radiation alone is not sufficient to smooth out the shock profile and a discontinuity in velocity still appears; mass diffusion processes are dominant in these cases.

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LIST OF SYMBOLS

A	A constant
C	A constant measuring the ratio of radiative to hydrodynamic energy
C_p	Specific heat at constant pressure
E	A constant measuring the total energy flux at a location x
E_1, E_2	Integro exponential function of rank 1 or 2
$F = (v-v_1)(v-v_2)$	
$G = (v - v^2)^4$	
K	A constant analogous to C when the exponential approximation is used
M	A very large positive number
Q	Rate of gain of radiative energy
R	Gas constant
S	Energy flux due to radiation
T	Absolute temperature
f	Velocity of the fluid at zero pressure
h	Enthalpy of the fluid
p	Hydrodynamic pressure
u	Velocity of the fluid
$v = \frac{u}{f}$	Reduced velocity
$z = \frac{dv}{d\eta}$	Rate of change of the velocity
α	Absorption coefficient
γ	Ratio of specific heats

LIST OF SYMBOLS (CONTD.)

Γ	Momentum flux
σ	Stephan Boltzmann constant
η	Optical thickness
ρ	Density of the fluid

Superscripts

$*$	Isentropic speed of sound
$+$	Conditions at high velocities (toward $-\infty$)
$-$	Conditions a low velocities (toward $+\infty$)

Indices

v_1	Velocity at $-\infty$
v_2	Velocity at $+\infty$

INTRODUCTION

The interest in this work arose because of the investigation by John F. Clarke (Ref. 1) of "Radiation resisted shock waves". By including non linear effects in the study of wave propagation in a thermally radiating gas, Clarke found that steady disturbances, with continuous velocity profiles which have continuous first and second derivatives, could exist, but that in some cases discontinuities in the first derivative appeared. Heaslet and Baldwin, in a paper to be published shortly, showed that, in some range of the parameter measuring the ratio of radiative to hydrodynamic energy, discontinuities in the function itself had to be introduced. Both of these investigations were made by reducing the integral equation involved to a differential equation, through an approximation of the radiative energy integral.

The object of this paper is to investigate the influence of this approximation on the results which were predicted.

As will be shown continuous profiles are indeed obtained when the radiative energy flux is high compared to the hydrodynamic energy flux. However when the radiative energy flux becomes small, radiative dissipation is not large enough to prevent the steepening of the wave front, and a discontinuity in the velocity profile itself appears. Molecular transport processes can no longer be neglected. The description of a radiative shock including such processes becomes rather difficult although some progress has been made in this respect by S. C. Traugott (Ref. 2).

Moreover it is to be expected that as the wave strength increases and as the velocity at the tail of the wave becomes smaller than the isothermal speed of sound, the steepening of the velocity profile will increase. In this case the temperature reaches a peak which increases the smoothing effect of the radiation. Thus the character of the velocity profile can be expected to differ from that found for a weaker wave.

HYPOTHESES AND EQUATIONS

I. Following Clarke, it will be assumed that

1) The flow is steady and one dimensional
2) Viscosity, heat conduction and mass diffusion can be neglected
3) The radiation pressure is negligible compared to the hydrostatic pressure; and the contribution of radiation to the internal energy of the gas is negligible compared with the specific enthalpy
4) The gas is a calorically perfect gas (R and are constants).

II. With the above assumptions the integrated equations of mass, momentum and energy conservation can be written as follows

$$\rho u = \Gamma \tag{1}$$

$$p + \rho u^2 = \Gamma f \tag{2}$$

$$\rho u \left(h + \frac{u^2}{2} \right) = S + E_0 \tag{3}$$

where Γ, Γf and E_0 are constants measuring the mass, momentum and energy fluxes per unit area respectively and S is the energy flux per unit lost by radiation.

III. The determination of $Q = \frac{dS}{dx}$ under the assumption of a grey gas of infinite extent in quasi thermodynamic equilibrium has been derived by Clark (Ref. 1) (see also Chandrasekhar, Vincenti and Baldwin, Lighthill, Refs. 3, 4, 5). The energy flux due to radiation can be obtained by direct integration of the equation of radiative transfer (Appendix 1).

Introducing the optical thickness

$$\eta = \int_0^x \alpha(x')\,dx'$$

where α is the absorption coefficient of the assumed grey gas, we find

$$S = 2\sigma \left\{ \int_\eta^{+\infty} E_2(\eta' - \eta)\, T^4(\eta')\, d\eta' - \int_{-\infty}^{\eta} E_2(\eta - \eta')\, T^4(\eta')\, d\eta' \right\} \qquad (4)$$

where

$$E_2(x) = \int_0^1 e^{-\frac{x}{y}}\, dy = \int_1^{+\infty} e^{-xz}\, \frac{dz}{z^2}$$

IV. Since the gas is assumed calorically perfect

$$p = \rho R T$$

$$h = c_p T = \frac{\gamma}{\gamma - 1} R T = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$$

But according to equations 1 and 2

$$\frac{p}{\rho u} + u = f$$

Therefore

$$\frac{p}{\rho} = u(f - u) = RT \tag{5}$$

$$h = \frac{\gamma}{\gamma-1} u(f-u) \tag{6}$$

Introducing this value of h in equation 3

$$\Gamma\left[\frac{\gamma}{\gamma-1} u(f-u) + \frac{u^2}{2}\right] = S + E_o$$

or

$$u^2 - \frac{2\gamma}{\gamma+1} f u = -\frac{2(\gamma-1)}{\Gamma(\gamma+1)}(S + E_o) \tag{7}$$

As $\eta \to \pm\infty$ all physical quantities must remain bounded. The energy flux due to radiation S (Eq. 4) tends to zero and equations 1, 2 and 3 must be

$$\rho u = \Gamma$$

$$p + \rho u^2 = \Gamma f$$

$$\rho u\left(h + \frac{u^2}{2}\right) = E_o$$

Therefore the relations between the quantities far upstream (index 1) and far downstream (index 2) are identical with the normal shock relation (no molecular transport process, no radiation).

The maximum of $\frac{u_2}{u_1}$ will consequently be $\frac{\gamma-1}{\gamma+1}$ (Corresponding to M_1 infinite) ; moreover

$$u_1 + u_2 = \frac{2\gamma}{\gamma+1} f$$

In order to simplify the equation a reduced velocity will now be introduced.

$$V = \frac{u}{f} = \frac{2\gamma}{\gamma+1} \frac{u}{u_1+u_2} \tag{8}$$

Let

$$V_1 = \frac{2\gamma}{\gamma+1} \frac{u_1}{u_1+u_2}$$

$$V_2 = \frac{2\gamma}{\gamma+1} \frac{u_2}{u_1+u_2}$$

$$2V^* = V_1 + V_2 = \frac{2\gamma}{\gamma+1}$$

Introducing V into equation 7 we find

$$(V - V_1)(V - V_2) = - \frac{2(\gamma-1)}{f^2 \Gamma(\gamma+1)} S$$

If $C = \frac{4f^6\sigma}{R^4} \frac{\gamma-1}{\gamma+1} > 0$ the equation of the problem becomes:

$$(V-V_1)(V-V_2) = C \int_{-\infty}^{+\infty} \mathrm{sgn}(\eta-\eta') E_2(|\eta-\eta'|)(V-V^2)^4 \, d\eta' \tag{9}$$

V. For the following study the function $v(\eta)$ will be assumed bounded.

In addition

$$v(-\infty) = v_1$$

$$v(+\infty) = v_2$$

$$v_1 > v_2$$

$$v_{1\,max} = 1$$

$$v_{2\,min} = \frac{\gamma-1}{\gamma+1} \quad (\sim 0.167)$$

$$v^* = \frac{v_1 + v_2}{2} = \frac{\gamma}{\gamma+1} \quad (\sim 0.583)$$

So that

$$\frac{\gamma}{\gamma+1} < v_1 \leq 1$$

$$\frac{\gamma-1}{\gamma+1} \leq v_2 < \frac{\gamma}{\gamma+1}$$

Although this is not a necessary assumption, it may be expected that $v_2 \leq v \leq v_1$. All approximate solutions (Clarke, Ref. 1 and Rosseland, Ref. 6) lead to that conclusion. Moreover no contradiction with this assumption will arise in the following study.

Since the main feature of a shock wave, with no molecular transport processes, is a discontinuity in the velocity profile, it is interesting to investigate whether the introduction of the radiative energy loss will prohibit such a discontinuity. The following section is therefore an investigation of the continuity of equation 9.

CONTINUITY

I. Continuity of the Function

Let

$$F(\eta)' = (\nu - \nu_1)(\nu - \nu_2)$$

$$G(\eta) = (\nu - \nu^2)^4$$

$$K(\eta - \eta') = \operatorname{sgn}(\eta - \eta')E_2(|\eta - \eta'|)$$

Equation 9 can be written

$$F(\eta) = C\int_{-\infty}^{+\infty} K(\eta - \eta')\, G(\eta')\, d\eta'$$

Since $K(\eta - \eta')$ is an integrable function

$$\lim_{\delta \to 0} \int_{-\infty}^{+\infty} [K(\eta - \eta' + \delta) - K(\eta - \eta')]\, d\eta' = 0$$

It will now be assumed that $\nu(\eta)$ is discontinuous for $\eta = \eta_1$. Since ν and therefore $G(\eta)$ are bounded

$$\lim_{\delta \to 0} |F(\eta_1 + \delta) - F(\eta_1 - \delta)| = \lim_{\delta \to 0} C\int_{-\infty}^{+\infty} [K(\eta_1 - \eta' + \delta) - K(\eta_1 - \eta' - \delta)]\, G(\eta')\, d\eta'$$

$$= C\, G_{max} \lim_{\delta \to 0} \int_{-\infty}^{+\infty} [K(\eta_1 - \eta' + \delta) - K(\eta_1 - \eta' - \delta)]\, d\eta' = 0$$

so that F is a continuous function even though ν is discontinuous. Therefore, using a superscript $+$ for quantities at $(\eta_1 - 0)$ and $-$ for $(\eta_1 + 0)$ we shall have

$$(V^+ - V_1)(V^+ - V_2) = (V^- - V_1)(V^- - V_2)$$

$$(V^+ - V^-)\left[(V^+ + V^-) - (V_1 + V_2)\right] = 0$$

This relation shows that V may be discontinuous and the values of V on both sides of the discontinuity are related by the following equation:

$$V^+ + V^- = 2V^* \tag{10}$$

For a decreasing solution any discontinuity will jump across V^*.

II. Continuity of the Derivative

If V is continuous, the derivative of equation 9 becomes:

$$2(V - V^*)\frac{dV}{d\eta} = C\left\{2G(\eta) + \int_{-\infty}^{\eta} \left.\frac{dE_2}{d\xi}\right|_{\eta-\eta'} G(\eta')\,d\eta' - \int_{\eta}^{+\infty} \left.\frac{dE_2}{d\xi}\right|_{\eta'-\eta} G(\eta')\,d\eta'\right\}$$

$$2(V - V^*)\frac{dV}{d\eta} = C\left\{\int_{-\infty}^{\eta} E_2(\eta-\eta')\frac{dG}{d\eta'}\,d\eta' - \int_{\eta}^{+\infty} E_2(\eta'-\eta)\frac{dG}{d\eta'}\,d\eta'\right\}$$

Assuming $\frac{dG}{dV} = 4(V-V^2)^3(1-2V)\,\frac{dV}{d\eta}$ is bounded, that is $\frac{dV}{d\eta}$ is bounded, the same procedure as in paragraph I shows that $(V-V^*)\,\frac{dV}{d\eta}$ is always continuous. Thus when V is continuous, its first derivative is always continuous, except possibly at $v = v^*$.

III. Continuity of the Temperature

As will be shown later, an approximate solution of equation 9 in the case of an optically dense medium (Rosseland's approximation, Ref. 6) leads to the result that the temperature profile remains continuous when the velocity profile is discontinuous.

It is therefore interesting to note here that using the full equation the only possible jump in velocity satisfies

$$v^+ + v^- = 2v^* = \frac{2\gamma}{\gamma+1}$$

and since the temperature is proportional to $(v - v^2)$ a continuous temperature profile would require that

$$v^+(1-v^+) = v^-(1-v^-)$$

$$(v^+ - v^-)\left[1 - (v^+ + v^-)\right] = 0$$

The last relation cannot be satisfied if v is discontinuous since in general $\frac{2\gamma}{\gamma+1} \neq 0$.

BEHAVIOR AT INFINITY

I. Full Equation

With $G=(v-v^2)^4$ equation 9 can be rewritten as follows:

$$(v-v_1)(v-v_2) = C\left\{\int_{-\infty}^{\eta} E_2(\eta-\eta')\,G(\eta')\,d\eta' - \int_{\eta}^{+\infty} E_2(\eta'-\eta)\,G(\eta')\,d\eta'\right\}$$

or

$$(v-v_1)(v-v_2) = C\int_0^{\infty} E_2(x)\left[G(\eta-x) - G(\eta+x)\right]dx \qquad (11)$$

But

for x large $\qquad E_2(x) \sim \frac{e^{-x}}{x} + O\left(\frac{e^{-x}}{x^2}\right)$

for x small $\qquad E_2(x) \sim 1 - 1.577x + x\log x$

Therefore, provided the function G is bounded (which has been assumed throughout this paper), the main contribution to the integral appearing in equation 11 will come from small values of x .

Near $x=0$, if G is a continuous function of η , we can write

$$G(\eta-x) - G(\eta+x) = \left.\frac{dG}{dv}\right|_{\eta}\left\{v(\eta-x) - v(\eta+x)\right\}$$

$$= \left[4(v-v^2)^3(1-2v)\right]_{\eta}\left\{v(\eta-x) - v(\eta+x)\right\}$$

As $\eta \to \pm\infty$ it can be expected that v will tend continuously towards v_1 or v_2 and therefore the above approximation is reasonable.

Since $E_2(x)$ weights any function it multiplies most heavily near $x = 0$, the further assumption that the right hand side of equation 11 can be approximated by the following expression for $\eta \to \pm\infty$ will be made:

$$\int_0^\infty E_2(x) \left[G(\eta - x) - G(\eta + x) \right] dx \sim$$

$$\int_0^\infty E_2(x) \left[4(v - v^2)^3 (1 - 2v) \right]_\eta \left\{ v(\eta - x) - v(\eta + x) \right\} dx$$

Using this approximation for $v \to v_1$, $\eta \to -\infty$ and then assuming that:

$$v - v_1 \sim e^{-\lambda_1 \eta}$$

equation 11 can be written

$$(v_1 - v_2)\, e^{-\lambda_1 \eta} =$$

$$4(v_1 - v_1^2)(1 - 2v_1)\, C \int_0^\infty E_2(x) \left[e^{-\lambda_1(\eta - x)} - e^{-\lambda_1(\eta + x)} \right] dx$$

Let

$$A_1 = \frac{4C}{v_1 - v_2} (v_1 - v_1^2)^3 (1 - 2v_1) < 0$$

must then satisfy the following equation

$$1 = A_1 \int_0^\infty \left(e^{\lambda_1 x} - e^{-\lambda_1 x} \right) \left\{ \int_0^1 e^{-\frac{x}{y}} dy \right\} dx$$

$$1 = \frac{A_1}{\lambda_1} \left[-2 - \frac{1}{\lambda_1} \log \frac{1 - \lambda_1}{1 + \lambda_1} \right] \qquad (12)$$

The integral is convergent only for $|\lambda_1|<1$. We should note however that the condition $\lambda_1 > -1$ comes from the integration around v_2 $(\eta \to +\infty)$ where the approximation is not valid.

Moreover $\lambda_1 = 0$ is not a solution of equation 12 since for λ small the right hand side of the equation is equivalent to

$$\frac{2}{3} A_1 \lambda + O(\lambda^3)$$

For $\lambda_1 \neq 0$ the equation can be written

$$\lambda^2 + 2 A_1 \lambda = A_1 \log \frac{1+\lambda}{1-\lambda} \tag{13}$$

Figure 1 where

$$f_1 = \lambda^2 + 2 A_1 \lambda$$

and

$$f_2 = A_1 \log \frac{1+\lambda}{1-\lambda}$$

have been plotted, shows that equation 13 has two real roots

$$\lambda = 0$$

$$\lambda = -\alpha_1 \qquad 0 < \alpha_1 < 1$$

The second root is also a root of equation 12. Therefore when $\eta \to -\infty$

$$v - v_1 \sim e^{\alpha_1 \eta}$$

Similar steps can be taken to find an approximation

$$v - v_2 \sim e^{-\lambda_2 \eta}$$

when $\eta \to +\infty$.

In this case however, the parameter A_2 is not always positive since

$$A_2 = -\frac{4c}{v_1 - v_2} (v_2 - v_2^2)^3 (1 - 2v_2)$$

Therefore:

$$v_2 > \frac{1}{2} \qquad \lambda_2 = \alpha_2 \qquad 0 < \alpha_2 < 1$$

$$v_2 < \frac{1}{2} \qquad \lambda_2 = -\alpha_2 \qquad 0 < \alpha_2 < 1$$

In the case $v_2 > \frac{1}{2}$ then $(v - v_2) \sim e^{-\alpha_2 \eta}$ which is compatible with the assumption $v \to v_2$ as $\eta \to +\infty$.

But for $v_2 < \frac{1}{2}$

$$v - v_2 \sim e^{\alpha_2 \eta}$$

The following interpretations are possible in this case:

a) $v \to v_2$ as $\eta \to +\infty$ as the above relation seems to indicate. That means that the solution is double-valued (see Fig. 2); such a behavior will also appear as the result of the investigation of the solution near $v \simeq v^*$ but should be discarded on physical grounds.

b) Equation 13 written for λ_2, A_2 has imaginary roots with a positive real part. Further investigation is necessary in this case.

c) The assumptions made to obtain equation 13 break down. Physically this would mean that the contribution of radiation from points far upstream is not negligible compared with the contribution of neighboring points. This is certainly true in Clarke's approximation as will be shown below.

II. Approximate Solution

Following Clarke let us now replace $E_2(x)$ by $m e^{-bx}$. Then equation 11, when $K = Cm$, is written

$$(v-v_1)(v-v_2) = K \int_0^{\infty} e^{-bx} \left[G(\eta - x) - G(\eta + x) \right] dx \qquad \text{(11-a)}$$

It can readily be transformed into Clarke's differential equation:

$$\frac{d^2}{d\eta^2}\left[(v-v_1)(v-v_2)\right] - 4K(v-v^2)^3(1-2v)\frac{dv}{d\eta} - b^2(v-v_1)(v-v_2) = 0$$

When $\eta \to \pm\infty$, the differential equation shows that v behaves like:

$$(v-v_1) \sim e^{-\lambda_1 \eta} \qquad \eta \to -\infty$$

$$(v-v_2) \sim e^{-\lambda_2 \eta} \qquad \eta \to +\infty$$

where

$$\lambda_1 = -\frac{2K(v_1-v_1^2)^3(1-2v_1)}{v_1-v_2} \pm \sqrt{\frac{4K^2(v_1-v_1^2)^6(1-2v_1)^2}{(v_1-v_2)^2} + b^2}$$

$$\lambda_2 = \frac{2K(v_2-v_2^2)^3(1-2v_2)}{v_1-v_2} \pm \sqrt{\frac{4K^2(v_2-v_2^2)^6(1-2v_2)}{(v_1-v_2)^2} + b^2}$$

These results should also be found by using the same method as in the previous case. The integral equation can be written:

$$(V_1 - V_2)\, e^{-\lambda_1 z} =$$

$$4K(V_1 - V_1^2)^3 (1-2V_1) \left\{ e^{-b\eta} \int_{-\infty}^{\eta} e^{(b-\lambda)\eta'} d\eta' - e^{b\eta} \int_{\eta}^{+\infty} e^{-(b-\lambda)\eta'} d\eta' \right\}$$

The integrals converge only for $|\lambda| < b$ and the equation in λ becomes:

$$\lambda^2 + 2A_1\lambda - b^2 = 0 \qquad (14)$$

or $$\lambda^2 + 2A_1\lambda = b^2$$

compared with the result found previously

$$\lambda^2 + 2A_1\lambda = A_1 \log \frac{1+\lambda}{1-\lambda}$$

Equation 14 is identical with the equation for λ obtained by using the Clarke differential equation.

The results can be summarized as follows:

$$V \to V_1 \text{ as } z \to -\infty \qquad \begin{cases} \lambda_1 = -\alpha_1 & 0 < \alpha_1 < b \\ \lambda_1 = \alpha_1' & \alpha_1' > b \end{cases}$$

Since $V \to V_1$ (and not infinity) as $\eta \to -\infty$ on the root α_1 should be kept and this is compatible with the condition found from the integral equation .

$v \to v_2$ as $\eta \to +\infty$

$v_2 > \frac{1}{2}$ $\quad \begin{cases} \lambda_2 = \alpha_2 & 0 < \alpha_2 < b \\ \lambda_2 = -\alpha_2' & \alpha_2' > b \end{cases}$

Here again the root α_2 is to be kept and is compatible with the relation $|\lambda| < b$.

$v < \frac{1}{2}$ $\quad \begin{cases} \lambda_2 = -\alpha_{2'} & 0 < \alpha_{2'} < b \\ \lambda_2 = \alpha_{2'}' & \alpha_{2'}' > b \end{cases}$

If $v \to v_2$ for $\eta \to +\infty$ the root $\alpha_{2'}'$ should be kept: but here this is in contradiction with the requirement that $|\lambda| < b$.

This last case can be investigated a little further: Let $\eta \to +\infty$ and M be a very large positive number (of order $\sqrt{\eta}$). Since $v \to v_2$, as $\eta \to +\infty$ it can be assumed that

$$v = v_2 + \mathcal{E}(\eta) \quad \text{where} \quad \frac{\mathcal{E}(\eta)}{v_2} << 1 \quad \text{when} \quad \eta > M$$

Equation (11-a) can be written for $\eta \to +\infty$

$$(v_2 - v_1)\,\mathcal{E} = K \left\{ \int_{-\infty}^{M} e^{-b(\eta-\eta')} G(\eta')\,d\eta' + \int_{M}^{\eta} e^{-b(\eta-\eta')} \left[G_2 + 4(v_2 - v_2^2)^3 (1-2v_2)\,\mathcal{E}(\eta') \right] d\eta' \right.$$
$$\left. - \int_{\eta}^{+\infty} e^{-b(\eta'-\eta)} \left[G_2 + 4(v_2 - v_2^2)^3 (1-2v_2)\,\mathcal{E}(\eta') \right] d\eta' \right\}$$

or

$$(V_2 - V_1)\,\mathcal{E} = K \Big\{ \int_{-\infty}^{M} e^{-b(\eta-\eta')} [G(\eta') - G_2]\, d\eta' +$$

$$4(V_2 - V_2^2)^3 (1 - 2V_2) \Big[\int_{M}^{\eta} e^{-b(\eta-\eta')} \mathcal{E}(\eta')\, d\eta' - \int_{\eta}^{+\infty} e^{-b(\eta'-\eta)} \mathcal{E}(\eta')\, d\eta' \Big] \Big\} \qquad (15)$$

The first integral can be evaluated by the mean value theorem.

Let $\mathcal{E} = \beta e^{-\lambda_2 \eta}$

Then:

$$\beta e^{-\lambda_2 \eta} = \frac{c}{b\beta (V_1 - V_2)} \Big[(V - V^2)^4 - (V_2 - V_2^2)^4 \Big]_{\text{mean value th.}} e^{-b(\eta - M)}$$

$$+ A_2 \Big[\frac{e^{-\lambda_2 \eta} - e^{-b(\eta - M) - \lambda_2 M}}{b - \lambda_2} - \frac{e^{-\lambda_2 \eta}}{b + \lambda_2} \Big]$$

Let

$$\frac{c}{b\beta} \, \frac{1}{V_1 - V_2} \Big[(V - V^2)^4 - (V_2 - V_2^2)^4 \Big] = B_2$$

Then

$$1 = \Big[B_2 - \frac{A_2 e^{-\lambda M}}{b - \lambda} \Big] e^{-(b-\lambda)\eta + bM} + A_2 \frac{2\lambda}{b^2 - \lambda^2} \qquad (16)$$

If $V_2 > \frac{1}{2}$, $A_2 > 0$ and choosing $\lambda < b$ the equation reduces (when $\eta \to +\infty$)to

$$\lambda^2 + 2A_2\lambda = b^2$$

If $V_2 < \frac{1}{2}$, $A_2 < 0$ we saw that the differential equation led to $\lambda > b$, Equation (16) cannot be satisfied since in this case as $\eta \to +\infty$, $e^{-\lambda M} \to 0$ and $e^{-(b-\lambda)\eta + bM} \longrightarrow \infty$; thus the dominant term of equation 15 becomes

$$\left[B_2 - \frac{A_2 e^{-\lambda M}}{b-\lambda} \right] e^{-(b-\lambda)\eta + bM}$$

Remembering that

$$B_2 \, e^{-(b-\lambda)\eta + bM} \sim \int_{-\infty}^{M} e^{-b(\eta-\eta')} [G - G_2] \, d\eta'$$

we can see that the radiation due to points far upstream which is measured by this integral is not negligible compared with near radiation as measured by $A_2 \frac{e^{-(b-\lambda)(\eta - M)}}{b-\lambda}$.

In fact, the differential equation shows that these two terms balance exactly (here the approximation of the upstream radiation involving B_2 is not valid). When such a balance is realized, the remaining equation for λ is the same as equation 14 and both integral and differential equation give the same value α_2', for λ .

Such a situation may also arise when the function $E_2(x)$ is kept. In that case, however, the integration of equation 16 would not lead to a term $\log \frac{1+\lambda}{1-\lambda}$ like equation 13. The restriction $|\lambda| < 1$, which arises for λ real because of the logarithmic term, will disappear, and a value of $|\lambda| > 1$ may be admissible. A computer may be useful in finding what the equation for λ would be in this case and if such an equation does indeed lead to a positive value of λ_2. A value of $\lambda_2 >>> 1$ corresponding to an extremely rapid exponential decay may explain the result found by Rosseland in the limit of $C \to \infty$, i.e., for $v < \frac{1}{2}$ the velocity profile is discontinuous and for η large $v \equiv v_2$. This approximation will be discussed later.

BEHAVIOR OF THE SOLUTION NEAR THE ISENTROPIC SPEED OF SOUND

If the solution and its first derivative are continuous equation 9 can be differentiated twice:

$$(v-v_1)(v-v_2) = C\int_0^\infty E_2(x)\left[G(y-x) - G(y+x)\right]dx \qquad (9)$$

$$2(v-v^*)\frac{dv}{d\eta} = C\int_0^\infty E_2(x)\left[G'(y-x) - G'(\eta+x)\right]dx \qquad (16)$$

$$2(v-v^*)\frac{d^2v}{d\eta^2} + 2\left(\frac{dv}{d\eta}\right)^2 = C\int_0^\infty E_2(x)\left[G''(y-x) - G''(y+x)\right]dx \qquad (17)$$

Let $z = \frac{dv}{d\eta}$. Equation 17 can be integrated by parts and we get:

$$2(v-v^*)z\frac{dz}{dv} + 2z^2 = 2C\left(\frac{dG}{dv}\right)_y z - C\int_0^\infty E_1(x)\left[G'(y-x) + G'(\eta+x)\right]dx \qquad (18)$$

Let $\varphi(v, C) = C\int_0^\infty E_1(x)\left[G'(y-x) + G'(y+x)\right]dx$

$$\varphi(v, C) = -C\int_0^\infty E_0(x)\left[G(y-x) - G(y+x)\right]dx$$

$$\varphi(v, C) = -C\int_0^\infty \frac{d^2E_2}{dx^2}\left[G(y-x) - G(\eta+x)\right]dx$$

But $$E_2(x) = \int_0^1 e^{-\frac{x}{y}}\, dy$$

$$\frac{d^2 E_2}{dx^2} = \int_0^1 e^{-\frac{x}{y}} \frac{dy}{y^2}$$

$$\frac{d^2 E_2}{dx^2} - E_2 = \int_0^1 e^{-\frac{x}{y}} \left(\frac{1}{y^2} - 1\right) dy \geq 0$$

But $E_1(x) > 0$ and if we assume a continuously decreasing solution and $V_2 > \frac{1}{2}$

$$\frac{dG}{dV} < 0 \qquad \frac{dV}{d\eta} < 0$$

$$G'(\eta - x) > 0 \qquad G'(\eta + x) > 0$$

So that

$$\varphi(V, c) = c \int_0^\infty E_1(x) \left[G'(\eta - x) + G'(\eta + x)\right] dx \geq 0$$

But $$(V - V_2)(V_1 - V) = -c \int_0^\infty E_2(x)\left[G(\eta - x) - G(\eta + x)\right] dx \geq 0$$

$$\varphi = -c \int_0^\infty \frac{d^2 E_2(x)}{dx^2} \left[G(\eta - x) - G(\eta + x)\right] dx$$

$$\frac{d^2 E_2}{dx^2} \geq E_2$$

So that $\varphi > 0$ and at most $\varphi = 0$ at $V = V_{1,2}$. Equation 18 can be written

$$-\frac{dz}{dv} = \frac{2z^2 - 8C(v-v^2)^3(1-2v)z + \varphi}{2(v-v^*)z} \tag{19}$$

At $v = v^*$

a)
$$2z^2 - 8C(v^* - v^{*2})^3(1-2v^*)z + \varphi(v^*; c) = 0$$

or
$$16C^2(v^* - v^{*2})^6(1-2v^*)^2 - 2\varphi(v^*; c) > 0$$

In this case $v(\eta)$ may cross the value v^* and a continuous profile may exist.

We can note that when $E_2(x)$ is replaced by me^{-bx} Clarke found that

$$\varphi(v; c) = b^2(v_1 - v)(v - v_2)$$

so that this case occurs for

$$8C > \frac{\sqrt{2}\, b(v_1 - v_2)}{(v^* - v^{*2})^3(2v^* - 1)} = 8c'$$

and that a continuous profile exists with a finite slope if $C > \frac{3}{2\sqrt{2}}c'$ and an infinite slope for $c' < c < \frac{3}{2\sqrt{2}}c'$

b) $16C^2(v^* - v^{*2})^6(1-2v^*)^2 - 2\varphi(v^*; c) < 0$

which certainly occurs for $C \simeq 0$ since, for all C, if the

solution is continuous at v^*, $\mathcal{G}(v^*; c) > (v_1 - v^*)(v^* - v_2) \neq 0$
In this case $\frac{dz}{dv}$ becomes infinite.

It can easily be seen from equation 19 that in this case z cannot remain finite and $z(v - v^*)$ must remain constant across v^* (with the constant non zero).

Therefore as v crosses v^*, $z = \frac{dv}{dy}$ must change sign which corresponds to a locally double-valued solution (see Fig. 2).

This is always true in Clarke's approximation for $c < c'$ since $\mathcal{G}(v, c)$ is independent of c and of the sign of $(v_2 - \frac{1}{2})$.

When the full equation is used it has not yet been shown that $\mathcal{G}(v, c)$ is positive for $v_2 < \frac{1}{2}$ and c small. Therefore, the above conclusion is not necessarily true in this case. However for $v_2 > \frac{1}{2}$ and c sufficiently small, only a locally double-valued solution would be possible, and this is unacceptable. Therefore, the solution must be discontinuous.

Using the same method as in Chapter II, it can easily be found that, if v is discontinuous at η_1, the following relation must hold

$$\left[(v - v^*)\frac{dv}{d\eta}\right]_{\eta_1 - 0}^{\eta_1 + 0} = \left[G\right]_{\eta_1 - 0}^{\eta_1 + 0}$$

or

$$(v^+ - v^*)\frac{dv^+}{d\eta} - (v^- - v^*)\frac{dv^-}{d\eta} = (v^+ - v^{+2})^4 - (v^- - v^{-2})^4$$

Since $\quad v^{+} + v^{-} = v^{*}$

$$v^{+}\frac{dv^{-}}{d\eta} - v^{-}\frac{dv^{+}}{d\eta} = (v^{+} - v^{+2})^{4} - (v^{-} - v^{-2})^{4}$$

The two last relations, which give the conditions across the discontinuity, will determine the values $v^{+}(\eta_1)$ and $v^{-}(\eta_1)$ in a $\left(z = \frac{dv}{d\eta}, v\right)$ plot starting at the two saddle points $(v = v_1, z = 0)$ and $(v = v_2, z = 0)$ when, by Clarke's approximation, the equation of the problem is reduced to a differential equation.

ROSSELAND'S APPROXIMATION

OPTICALLY THICK GAS

The Rosseland approximation is equivalent to taking the first term of a Taylor expansion of the function T^4 which appears in the radiative energy integral:

$$S = 2\sigma \left\{ \int_{\eta}^{+\infty} E_2(\eta'-\eta)\, T^4\, d\eta' - \int_{-\infty}^{\eta} E_2(\eta-\eta')\, T^4\, d\eta' \right.$$

where $\eta = \int_0^{x} \alpha(x')\, dx'$

If a non dimensional length $\bar{x} = \frac{x}{L}$, and a non dimensional absorption coefficient $\bar{\alpha} = \frac{\alpha}{\alpha_0}$ are introduced, an optical "length" ζ_0 can be obtained by the following relation

$$\eta = \zeta_0 \bar{\zeta} = \int_0^{\bar{x}} \alpha_0 L\, \bar{\alpha}(\bar{x}')\, d\bar{x}' \qquad \text{or} \qquad \zeta_0 = \alpha_0 L$$

An optically dense medium is such that $\alpha_0 >> 1$; that is $\zeta_0 \to \infty$.

In terms of the bar variables, the energy flux becomes:

$$S = 2\sigma T_0^4 \left\{ \int_{\bar{\zeta}}^{+\infty} \zeta_0 E_2\left[\zeta_0(\bar{\zeta}'-\bar{\zeta})\right] \bar{T}^4 d\bar{\zeta}' - \int_{-\infty}^{\bar{\zeta}} \zeta_0 E_2\left[\zeta_0(\bar{\zeta}-\bar{\zeta}')\right] \bar{T}^4 d\zeta' \right\}$$

When $\zeta_0 \to \infty$, the terms

$$\zeta_0 E_2\left[\zeta_0 |\bar{\zeta}'-\bar{\zeta}|\right] = \int_0^1 \zeta_0\, e^{-\frac{\zeta_0 |\bar{\zeta}'-\bar{\zeta}|}{\mu}}\, d\mu$$

give the main contribution for $\bar{\xi}' = \bar{\xi}$.

In that case it is then possible to use a Taylor expansion of $T^4(\eta) = \theta(\eta)$.

Introducing it in the expression of S we find:

$$S = 4\sigma\theta_0 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{1}{\xi_0^{2n+1}} \frac{d^{2n+1}\bar{\theta}}{d\bar{\xi}^{2n+1}} \int_0^{\infty} E_2(x)\, x^{2n+1} dx$$

As $\xi_0 \to \infty$

$$S \sim 4\sigma \frac{d\bar{\theta}}{d\eta}$$

Therefore equation 9 becomes

$$(v - v_1)(v - v_2) = C \frac{d}{d\eta}\left[(v - v^2)^4\right] \qquad (20)$$

which is Rosseland's approximation.

Equation 20 can be simply written

$$4C(v - v^2)^3 (1 - 2v) \frac{dv}{d\eta} = (v - v_1)(v - v_2 \qquad (21)$$

when $v_2 > \frac{1}{2}$ it has a continuously decreasing solution from $v_1\ (\eta \to -\infty)$ to $v_2\ (\eta \to +\infty)$.

But when $v_2 < \frac{1}{2}$, if $v \to v_2$ as $\eta \to +\infty$, equation 21 shows that $v \equiv v_2$ and therefore the velocity must be discontinuous. Equation 20 then shows that the discontinuity must be such that $(v - v^2)^4$ is continuous or

$$\left(v^{+}-v^{+2}\right)^{4}=\left(v^{-}-v^{-2}\right)^{4}$$

That is $v^{+}=1-v_2$ and the temperature is continuous.

However it has been shown in the original exact formulation that the temperature cannot be continuous if the velocity is discontinuous. This apparent contradiction can be explained by the fact that in the true solution the discontinuity is followed by a rapid change in velocity over a very small distance; the total change appears as the discontinuity in the Rosseland approximation.

Rosseland's approximation is not uniformly valid in the case $v_2 < \frac{1}{2}$. The expansion which has been used is only valid outside a region of order ζ_0^{-1} around the point where the discontinuity appears. Outside the region Rosseland's approximation remains valid and leads to solutions which behave like

$$\begin{cases} v - v_1 \simeq e^{-\lambda_1 \eta} & \lambda_1 \sim \dfrac{1}{\zeta_0} \ll 1 \\ v - v_2 = 0 & \end{cases}$$

This can be compared with the results obtained by Clarke using a different approximation. For $v_2 < \frac{1}{2}$:

$$v - v_1 \sim e^{-\lambda_1 \eta} \qquad \lambda_1 \sim \frac{1}{K} \ll 1$$

$$v - v_2 \sim e^{-\lambda_2 \eta} \qquad \lambda_2 \sim K \gg 1$$

since K (or C in the full equation) and ζ_0 must be of the same order of magnitude (see Eq. 21 where $\eta = \zeta_0 \bar{\zeta}$).

According to the results found in Chapter II, within the region where the Rosseland approximation is not valid a discontinuity of magnitude $v^+ - v^-$ such that $v^+ + v^- = 2v^*$ must exist, and the velocity profile must decay very rapidly from the value v^- to v_2.

SUMMARY

The introduction of viscosity, heat conduction or molecular diffusion, which are properties depending on derivatives, into the equations of a supersonic one dimensional flow, remove the discontinuity of the velocity profile; it has been shown in this paper that this is not necessarily true when radiation, which is an integral property, is introduced.

When the ratio of radiative to dynamic energy (measured by the constant C) is small enough, and if the shock is weak $\left(\frac{V_2}{V_1} \sim 1\right)$, the velocity profile has a symmetric discontinuity around the isentropic speed of sound. The velocity reaches the values V_1 and V_2 at minus and plus infinity respectively exponentially, the decay towards V_2 being more rapid than the decay away from V_1 $(\alpha_2 > \alpha_1)$. The Rosseland approximation, for large values of C , shows that continuous profiles will occur when $V_2 > \frac{1}{2}$; it could be expected that this result remains true when the full equation is used.

However, when V_2 becomes smaller than 1/2 (downstream speed lower than the isothermal speed of sound), Rosseland's approximation indicates that the velocity profile has a discontinuity for large values of C . It could then be expected that for strong shocks $(V_2 << V_1)$, however high the radiative energy, it will not be sufficient to remove the discontinuity in the velocity profile. More work still remains to be done in order to obtain more information about this case.

APPENDIX 1

The equation of radiative transfer, in the case of a one dimensional problem can be written

$$\mu \frac{dI_\nu}{dx} = \alpha_\nu J_\nu - \alpha_\nu I_\nu$$

where:

$\mu = \cos\theta$ is the direction along which the radiation is observed with reference to the direction x .

I_ν is the intensity of the radiation

J_ν is the source term

α_ν is the absorption coefficient of the radiating gas.

We consider here a grey gas: α_ν is independent of the frequency of radiation ν . Moreover in thermodynamic equilibrium

$$J_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}\left(e^{\frac{h\nu}{kT}} - 1\right)^{-1}$$

Let $\quad d\eta = \alpha\, dx$

$$\mu \frac{dI_\nu}{d\eta} = J_\nu - I_\nu$$

$$I_\nu = e^{-\frac{\eta}{\mu}} \int_{-\infty}^{\eta} \frac{J_\nu e^{\frac{\eta'}{\mu}}}{\mu} d\eta' \qquad \text{for } \mu > 0$$

$$I_\nu = e^{-\frac{\eta}{\mu}} \int_{+\infty}^{\eta} \frac{J_\nu e^{\frac{\eta'}{\mu}}}{\mu} d\eta' \qquad \text{for } \mu < 0$$

Integrating over ν :

$$\int_0^\infty J_\nu \, d\nu = \frac{\sigma}{\pi} T^4$$

$$I_\nu = \frac{\sigma}{\pi} \int_{-\infty}^{\eta} T^4 \frac{e^{\frac{\eta'-\eta}{\mu}}}{\mu} d\eta' \qquad \mu > 0$$

$$I_\nu = -\frac{\sigma}{\pi} \int_{\eta}^{+\infty} T^4 \frac{e^{\frac{\eta'-\eta}{\mu}}}{\mu} d\eta' \qquad \mu < 0$$

Integrating over the solid angle and taking the radiation flux in the x direction only:

$$S = -\frac{\sigma}{\pi} 2\pi \left\{ \int_{-\infty}^{\eta} T^4 d\eta' \int_0^1 \frac{e^{\frac{\eta'-\eta}{\mu}}}{\mu} \mu \, d\mu - \int_{\eta}^{+\infty} T^4 d\eta' \int_{-1}^{0} \frac{e^{\frac{\eta'-\eta}{\mu}}}{\mu} \mu \, d\mu \right\}$$

$$S = 2\sigma \left\{ \int_{\eta}^{+\infty} E_2 (\eta'-\eta) T^4 d\eta' - \int_{-\infty}^{\eta} E_2 (\eta-\eta') T^4 d\eta' \right\}$$

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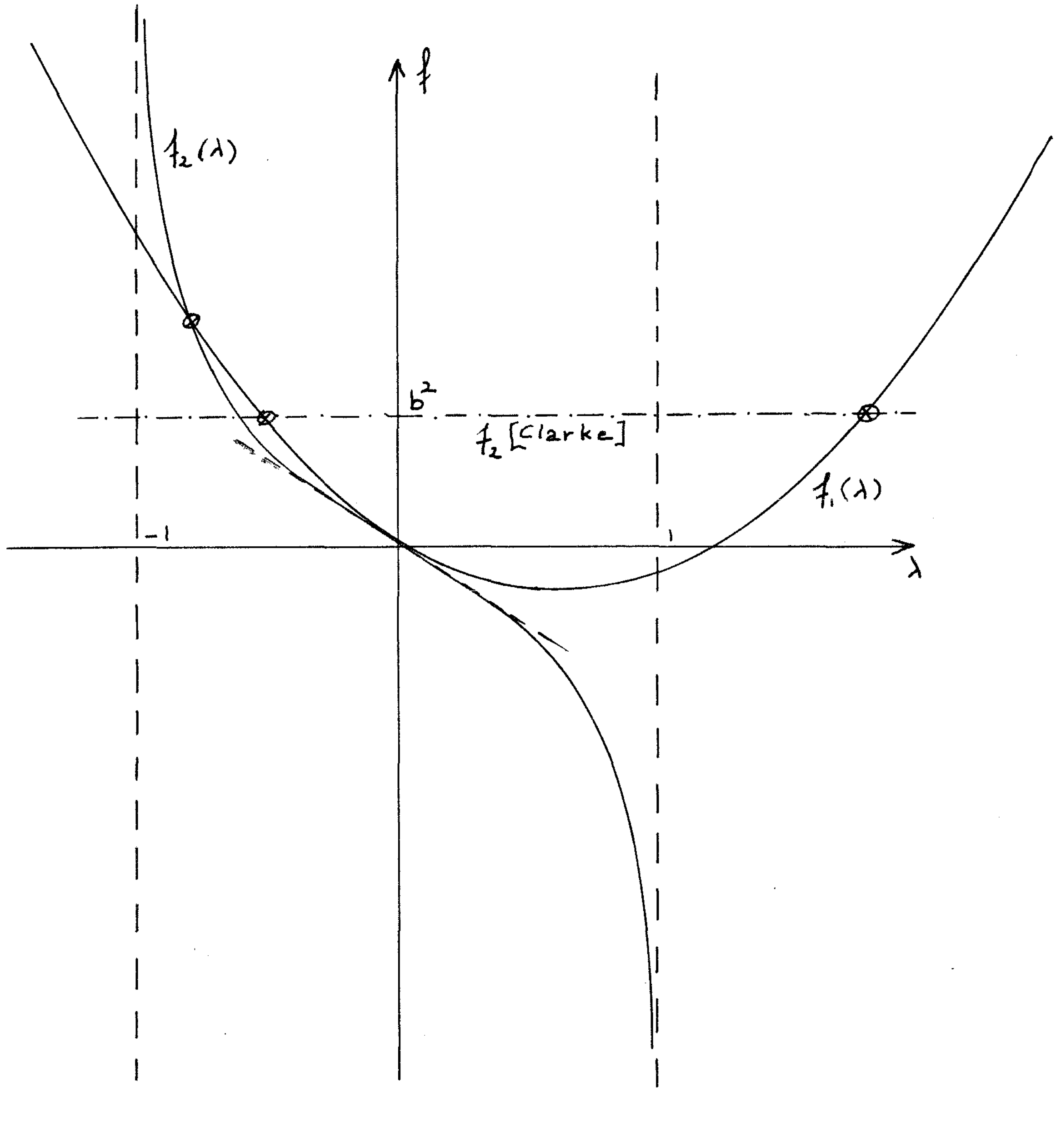

Fig. 1. Solution of equation 13

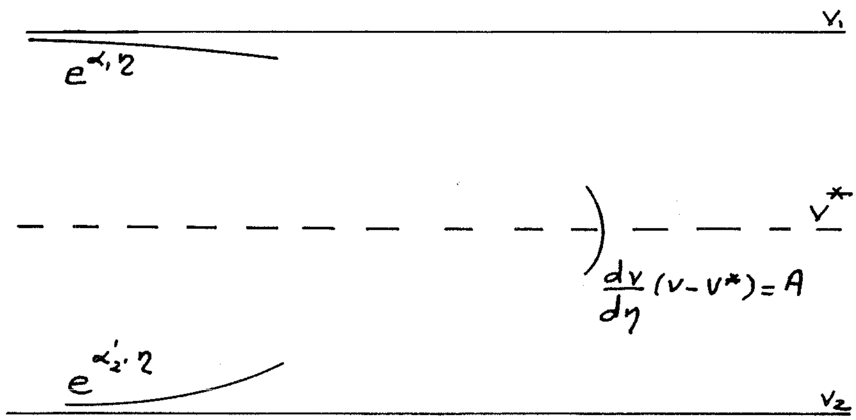

Fig. 2. Behavior of the solution at infinity for $v_2 < \frac{1}{2}$ and near the isentropic speed of sound when $\left.\frac{dS}{d\eta}\right|_{v=v^*} \neq 0$