PROBLEMS IN PARTICLE-FLUID MECHANICS

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ABSTRACT

The continuum equations describing the motion of a fluid containing small solid particles are discussed and stated. The examples considered fall into two categories: (1) when the fluid is incompressible and viscous, with simultaneous occurrence of particle-fluid momentum relaxation and fluid viscous diffusion; and (2) when the fluid can be considered as "inviscid" but compressible, with simultaneous occurrence of coupled particle-fluid momentum and thermal relaxations and fluid compressibility.

Under (1), the low Mach-number Rayleigh problem is studied. Many of the physical features of the non-linear steady (constant pressure) laminar boundary-layer problem are recovered from appropriate expansions from this exact solution. One obtains answers to questions about the modifications on the boundary layer growth and skin friction; particularly their transition from the "frozen" value near the leading edge, where the viscous layer is "thin" and the fluid viscous diffusion behaves as if in the absence of particles with the ordinary fluid kinematic viscosity, $\nu = \frac{\mu}{\rho}$, to the ultimate "equilibrium" value far downstream where the mixture then behaves as a single heavier fluid and viscous diffusion takes place with the "equilibrium" kinematic viscosity augmented by the particle density $\bar{\nu} = \frac{\mu}{\rho + \rho_p}$. The uncoupled thermal Rayleigh problem (small relative temperature differences) is directly inferred, and this answers questions about the modifications on the surface heat-transfer rates and particularly about the possibility of similarity with the velocity boundary layer. Similarity of the two boundary layers is

possible when, in addition to lateral diffusion effects being similar as indicated by Prandtl number unity, the streamwise relaxation processes must also be similar. The infinite flat plate oscillating in its own plane is studied, and appropriate expansions from the exact solutions point out how approximate treatment of periodic boundary layers in the absence of a mean flow may be made.

Under (2), the first-order small perturbation theory is discussed, leading from the equation for acoustic propagation to that for linearized supersonic flow. The two-dimensional steady case, or the Ackeret problem, is considered in detail. The Mach wave structure induced by a thin obstacle is deduced and shows a rapid damping of the disturbance along the "frozen" Mach wave (based on the sound speed of a gas in the absence of particles), both damping and diffusiveness along an intermediary Mach wave, and diffusiveness along the "equilibrium" Mach wave (based on the sound speed of an equilibrium mixture of gas and particles) and along which the bulk of the disturbance is carried to regions far from the obstacle. An exact form of the pressure coefficient is obtained for any surface shape (consistent with the linear theory), and involves a convolution integral of two Bessel functions with imaginary argument which is analytically evaluated. When the particle-fluid density ratio is small, the "frozen" and "equilibrium" Mach waves are very closely clustered together. A "boundary layer technique", based on the fact that changes across the Mach waves are rapid compared to changes along Mach waves, is then applied to obtain a simplified version of the linearized equation that describes Mach waves inclined toward the downstream direction only. While the Mach wave structure is consistent with the exact treatment, the pressure coefficient takes on the much simpler form of decreasing exponentials. The transition is, again, from the "frozen" value at the leading edge towards the "equilibrium" value in the downstream direction insofar as the surface shape permits.

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PART	TITLE	PAGE

LIST OF SYMBOLS

a	speed of sound
c_p, c_v	specific heats of gas
c_s	specific heat of solid material
C	airfoil chord length
C_p	pressure coefficient
$\mathcal{D}_{12}$	an effective diffusivity
e	internal energy per unit mass
$f(x)$	local surface inclination of airfoil
F_p	force per unit volume exerted by particles on fluid
h	static enthalpy per unit mass
I_ν	Bessel function with imaginary argument of order ν
k	thermal conductivity of fluid
m_p	particle mass
M	Mach number
n_p	number of particles per unit volume
p	fluid pressure
$p' = p - p_0$	fluid perturbation pressure
$P_r = \frac{c_p \mu}{k}$	fluid Prandtl number
$\dot{q}_k$	fluid heat flux vector
$\dot{q}_w$	surface heat transfer rate
Q_p	heat transfer rate per unit volume from particles to fluid
r_p	particle radius
R	universal gas constant

Re_x	local Reynolds number
Re_{λ_v}	Reynolds number based on particle velocity equilibration distance
s	entropy per unit mass
s_1, s_2	defined by equation (5.102)
$\tilde{s}_1, \tilde{s}_2$	defined by equation (6.36)
T	temperature
$T' = T - T_o$	perturbation temperature
v_i ; u, v, w	Cartesian velocity components
$(u', v', w') = (\frac{u}{u_o}, \frac{v}{u_o}, \frac{w}{u_o})$	non-dimensional Cartesian velocity components
x_i ; x, y, z	Cartesian coordinates
$(x', y', z') = (\frac{x}{\lambda_v}, \frac{y}{\lambda_v}, \frac{z}{\lambda_v})$	non-dimensional Cartesian coordinates
$Y(x')$	convolution integral defined by equation (5.108)
$\tilde{Y}(x')$	convolution integral defined by equation (6.39)
α_o	wedge angle
$\beta = \sqrt{M^2 - 1}$	
$\gamma = \frac{c_p}{c_v}$	specific heat ratio of gas
$\bar{\gamma}$	specific heat ratio of gas and particles in equilibrium
δ	boundary layer thickness
$\epsilon_1 = \frac{\beta_1}{\beta_o} - 1$	
$\epsilon_2 = \frac{\beta_2}{\beta_o} - 1$	
$\eta = \frac{y}{2\sqrt{\nu t}}$; $y\sqrt{\frac{\omega}{2\nu}}$	"frozen" boundary layer coordinate normal to surface
$\bar{\eta} = \frac{y}{2\sqrt{\bar{\nu} t}}$; $y\sqrt{\frac{\omega}{2\bar{\nu}}}$	"equilibrium" boundary layer coordinate normal to surface

θ	lag in shearing stress at the wall
$\theta_p(x')$	local particle streamline inclination at boundary
$\Theta = \frac{T - T_0}{T_w - T_0}$	non-dimensional temperature
$\kappa = \rho_p / \rho$	particle-fluid density ratio at reference state
λ	characteristic length associated with equilibration of particle
μ	fluid viscosity
ν	kinematic viscosity of fluid
$\bar{\nu} = \frac{\mu}{\rho + \rho_p}$	kinematic viscosity of fluid and particles in equilibrium
$(\xi, \eta) = \left(\frac{x' - \beta_0 y'}{\beta_0^2}, \frac{y'}{\beta_0}\right)$	one-sided characteristic coordinates
ρ	mass density
$\rho' = \rho - \rho_0$	perturbation mass density
$\sigma = \frac{\lambda_v}{\lambda_T}\left(1 + \kappa \frac{c_s}{c_p}\right)$	
τ	characteristic time associated with equilibration of particle
τ_{ij}	fluid viscous stress tensor
τ_w	surface shearing stress
ϕ	perturbation velocity potential
$\phi' = \frac{\phi}{\lambda_v u_0}$	non-dimensional perturbation velocity potential
$\Phi'(s, y')$	Laplace transform of $\phi'(x', y')$
$\Phi_s(x, y, z)$	steady part of perturbation velocity potential
Φ	fluid viscous dissipation function
Φ_p	rate of work done per unit volume by particles on fluid
$\chi = \frac{x - \beta_2 y}{\sqrt{4 \vartheta_{12} y / u_0}}$	similarity variable in far-field wave structure

$\psi = \tan^{-1}\left(\frac{\tau_v}{\omega^{-1}}\right)$	particle lag
ω	frequency
Ω	vorticity

Subscripts:

e	equilibrium
f	frozen
p	particle
s	solid; particle-fluid slip; also steady state
v	velocity ($\tau_v \rightarrow$ velocity equilibration time)
T	temperature ($\tau_T \rightarrow$ temperature equilibration time)
o	reference state; free stream

Parametric Groups:

$\frac{\tau_v}{t}, \frac{\lambda_v}{x}$	local velocity equilibration parameter
$\frac{\tau_v}{\omega^{-1}}, \frac{\lambda_v}{u_o \omega^{-1}}$	oscillatory velocity equilibration parameter
$\frac{\tau_T}{t}, \frac{\lambda_T}{x}$	local temperature equilibration parameter
$\kappa = \frac{\rho_p}{\rho}$	momentum interaction parameter
$\kappa \frac{c_s}{c_p}$	thermal interaction parameter

I. INTRODUCTION

Historically, the study of the motion of a fluid containing small, solid particles has taken on numerous aspects of interest by many scientific and engineering disciplines since the turn of the century. The interest in this subject within the discipline of fluid mechanics is, however, only recent. In his 1947 paper on sand ripples in the desert, the late Professor von Kármán[1] made an invitation as well as pointed out the challenge to workers in fluid mechanics for the elucidation of problems in this field.

One is referred to existing comprehensive reviews concerning the earlier works[2, 3, 4, 5], largely empirical, on such problems as atomization of liquids, fluidization, powder beds, and smoke, to mention a few; and on problems of raindrop impingement and icing on airfoils[6, 7, 8] that are mainly concerned with trajectories of a single droplet in an undisturbed aerodynamic flow field.

The recent aeronautical interest has been motivated by the use of high-energy solid rocket propellants which stimulated the study of rocket-nozzle exhaust gases containing finely distributed solid particles. Other aspects that may be of aeronautical interest, which are not entirely speculative, concern the aerodynamics of a dusty planetary atmosphere. Aside from the evidence of the presence of fine dust particles of micron size in the atmospheres of Mars and Venus[9], however, various other essential information concerning the order of the particle number density, the question as to their charge, and even the detailed gaseous atmospheric properties are not as yet available.

The various significant papers on the nozzle problem were recently reviewed by Hoglund[10]; these are largely concerned with tedious numerical procedures. Recent analytical formulations of the nozzle problem are given by Rannie[11] and by Marble[12]. The problem of a normal shock wave in a gas containing small solid particles is studied by Carrier[13]. The stability of laminar flow of an incompressible fluid containing particles is studied by Saffman[14] from the small disturbance point of view. Consideration of the turbulent boundary layer is made by Soo[15], however, unaccompanied by the necessary extensive experimental program that has lent success in the understanding of turbulent boundary layers in ordinary fluid mechanics.

Only in recent years has the study of the motion of a fluid containing small solid particles departed from association with individual problems, and efforts have been made for its introduction as a new borderline area in the general discipline of fluid mechanics[16-20] to which the description as "particle-fluid mechanics" may be appropriately assigned. In this context, the inter-particle distances are large compared to the sizes of the particles, whose presence in the fluid renders a fluid-like behavior on their own part. This will be discussed in more detail in Chapter II on the general conservation equations, and where a review of the more general approaches, discussions of our ranges of interest, and the implications of "incompressible" and "inviscid" flow in the present context are also given.

Several examples that are typical fluid-mechanical problems encountered in aeronautics are studied to which the conservation equations are specialized. As an example of the competing effects of fluid viscous diffusion and its inhibition by the force exerted on the fluid by the particle cloud, the subsequent motions of the fluid and the particle cloud after an infinite flat plate is impulsively set into motion in its own plane are studied in Chapter III. This is the corresponding incompressible Rayleigh problem in particle-fluid mechanics, and except for details, possesses nearly all the physical features of the non-linear (constant pressure) laminar boundary layer problem(19). The corresponding uncoupled heat transfer problem can be directly inferred. These answer important questions, with mathematical simplicity, such as the modifications of the surface shearing stress and surface heat transfer rate as a result of distortions of the fluid velocity and temperature profiles due to particle-fluid interactions, and also on boundary layer growth and particularly on the possibility of similarity. As a resemblance to a class of periodic boundary layers, the infinite flat plate oscillating in its own plane is studied in Chapter IV. These comprise the incompressible viscous flows studied and take their place as exact solutions, within the particular assumptions of the form of the particle-fluid interaction, in the discipline of particle-fluid mechanics.

As an example of the simultaneous occurrence of fluid compressibility and particle-fluid momentum and thermal interactions in regions of the flow field outside of fluid boundary layers subjected to small disturbances, linearized supersonic flow is considered in

Chapter V. The resulting Mach wave structure, common to flows of relaxing gases, is studied and the exact form of the pressure coefficient is derived. It is noticed that when the mass content of the particle cloud is small compared to that of the fluid in a unit volume, a physical approximation which is equivalent to the more formal "boundary layer technique" may be used to obtain a considerably simpler form for the pressure coefficient, while the flow field characteristics are consistent with the exact approach.

II. GENERAL CONSERVATION EQUATIONS FOR A GAS CONTAINING SMALL SOLID PARTICLES

1. General Discussion

The motion of a gas containing small solid particles is discussed in rather general terms in the pioneering work of Kiely[16], who stated the necessary equations of change and subsequently applied the techniques in the theory of irreversible processes for small departures from thermodynamic equilibrium to deduce the forms of the particle-fluid interaction "forces". However, Kiely does not make clear the existence of a dissipation arising as a result of the work done due to particle-fluid momentum interaction, and this essentially perpetrated into his subsequent calculations of entropy sources. More recently, and independently, Chu and Parlange[20] discussed the conservation equations,and reference is made to the theory of irreversible processes in deducing the particle-fluid interaction "forces". The interaction laws obtained are linear in the velocity difference between the two phases for momentum interaction and linear in the temperature difference for thermal interaction, which is the forms which Stokes' law takes. In this situation, the linear interaction "forces" are placed in the same footing as the Newtonian and Fourier linear relation between the fluxes of momentum and heat and the gradient of velocity and temperature, respectively, for small departures from thermodynamic equilibrium.

The conservation equations are also discussed by Van Deemter and Van Der Lann[16] and by Hinze[17]. These authors discussed only the conservation equations of mass, momentum, and kinetic energy and are incomplete in the sense that the thermal energy conservations are omitted from their discussion.

The general conservation equations obtained on the basis of a particle distribution function are given by Marble[19]. Prior to stating the result of his derivation, however, we first discuss some aspects of our range of interest in terms of the number, mass, and size of the particles as compared to that of the fluid medium in which they are immersed. This points to the way in which the conservation equations are obtained and the form which they take.

When the solid particles are metallic, the ratio of the mass density of the solids to the mass density of a gas at standard conditions ρ_s/ρ, is of the order of 10^3. Our interest falls in the range when the total mass content of the particles is of the same order as that of the total mass content of the gas in a unit volume of mixture, that is, $\rho_p/\rho = \mathcal{O}(1)$. In this situation, the ratio of the total volume occupied by the particles to that of the gas in the unit volume of mixture is then of the order of 10^{-3}. Hence, one now speaks of a quantity in terms of per unit volume of mixture synonymously as per unit volume of gas due to the negligible volume occupied by the solids. Furthermore, when $\rho_p/\rho = \mathcal{O}(1)$, if all the particles are of radius $r_p = 1$ micron, the number of particles present in a cubic millimeter is of the order of 10^5, and the inter-particle distance is about 10^{-2} mm; when $r_p = 10$ microns, then they are 10^3 and 10^{-1}, respectively. In this situation, one can certainly define a macroscopic "point" of the order of a fraction of a millimeter over which an average quantity of the particle cloud may be suitably defined. At the same time, the interparticle distance is sufficiently large compared to the sizes of the particles themselves so as to render particle-particle interaction* (if at all present) secondary when compared to the particle-fluid interaction. The latter interaction is a continuous one, since the mean free path of a gas at standard conditions is about 5×10^{-5} mm. Finally, the ratio of the mass of a single particle to that of a gas molecule is of the order of 10^{11} for $r_p = 1$ micron and 10^{14} for $r_p = 10$ microns.

From the above discussion, we can now proceed to discuss a distribution of fine particles, which we assume to be spheres of uniform radius, immersed in a perfect gas, and are sufficiently rare and are non-interacting. Within a macroscopic "point", in general, the individual particles may have different velocities and directions of motion and at different temperatures. However, due to the large number of particles present in a small volume, which we consider as a macroscopic "point", one can certainly define averaged local quantities on the basis of a particle distribution function and consider the particle cloud as a continuum. The conservation equations for the particle cloud were obtained by Marble[19] from an appropriate form of the Boltzmann equation for the distribution function of non-interacting particles by taking appropriate moments over the individual particle velocities, temperatures, and sizes. Here, we consider particles of a single family. When differences in particle sizes are present, however, we shall then assume that the differences are

* Particle-particle interaction is important when particles of extreme difference in sizes are present, and forms a separate discussion. In this paper, we only consider particles of the same size.

negligible to the extent that particle-particle interactions are negligible. Since the particles are non-interacting with themselves and are extremely massive compared with a gas molecule, the randomizing tendencies of the properties of the particles are absent. In the following, the appropriate conservation equations will be stated, adopting the notation of Marble[19].

2. Conservation Equations

Conservation of Mass. - In the absence of mass exchange between the two phases and the negligible volume occupied by the solid particles, the continuity equation for the gas phase is

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_j)}{\partial x_j} = 0 \qquad (2.1)$$

and for the particle cloud,

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial (\rho_p v_{p_j})}{\partial x_j} = 0 \qquad (2.2)$$

which is analogous to that for the gas.

Conservation of Momentum. - The momentum equation for the gas phase takes the usual Navier-Stokes form, but is augmented by the force per unit volume exerted on the gas by the particle cloud in that volume

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + F_{p_i} \qquad (2.3)$$

Here, τ_{ij} is the usual viscous stress tensor. The gas strain tensor, to which it is linearly related, refers to "smooth" derivatives of the gas velocity vector. In this case, the detailed disturbances on the gas velocity due to the passage of particles are neglected. The momentum equation for the particle cloud is

$$\rho_p \frac{\partial v_{p_i}}{\partial t} + \rho_p v_{p_j} \frac{\partial v_{p_i}}{\partial x_j} = \frac{\partial S_{ij}}{\partial x_j} + (-F_{p_i}) \qquad (2.4)$$

The particle "slip" stress tensor S_{ij} , analogous to the gas stress tensor, is the momentum flux due to deviations of the individual particle motions from the mean particle velocity v_{p_i} . This is negligible compared to the corresponding gas stress tensor τ_{ij} since the randomizing tendency of the particle velocity is absent or negligible if present. $(-F_{p_i})$ is the force per unit volume exerted on the particles in that volume by the gas.

<u>Conservation of Energy.</u> - The energy equation for the gas in the form of the First Law of thermodynamics for the gas internal energy per unit mass of the gas is

$$\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial x_j} + p \frac{\partial v_j}{\partial x_j} = \Phi - \frac{\partial \dot{q}_k}{\partial x_k} + \Phi_p + Q_p \qquad (2.5)$$

which is augmented by the rate in which work is being done per unit volume on the gas by the particles in that volume; $\Phi_p = (v_{p_i} - v_i) F_{p_i}$, and the rate in which heat is transferred per unit volume to the gas Q_p from the particles in that volume. Φ is the usual viscous dissipation and $\dot{q}_k$ the usual Fourier heat flux vector linearly related to the gas temperature gradient. By analogy with the discussion of the viscous stress tensor, the gas temperature gradient referred to is the "smooth" one. The energy equation for the particle cloud, in terms of the particle-cloud internal energy $e_p = c_s T_p$, is

$$\rho_p c_s \frac{\partial T_p}{\partial t} + \rho_p c_s v_{p_j} \frac{\partial T_p}{\partial x_j} = - \frac{\partial \mathcal{H}_{p_j}}{\partial x_j} + (-Q_p) \qquad (2.6)$$

$\mathcal{H}_{p_j}$ denotes the particle "slip" heat flux vector, which is due to deviations in the individual particle motion and temperature from the mean particle velocity v_{p_i} and temperature T_p respectively. By analogy with the particle "slip" stress tensor, this is negligible compared to the corresponding gas heat flux vector $\dot{q}_k$. $(-Q_p)$ is the rate in which heat is transferred per unit volume to the particle cloud in that volume from the gas.

<u>Equation of State.</u> - The equation of state for the gas at moderate temperatures and pressures is simply that of a perfect gas,

$$p = \rho R T \qquad (2.7)$$

The particle cloud, however, is not constrained by an equation of state due to its lack of volume and randomizing tendencies.

<u>Interaction Force and Heat Transfer.</u> - Since the particle cloud occupies negligible volume, the force exerted on the particle cloud due to the pressure gradient in the gas, as well as that due to virtual mass, is neglected. We assume that the interaction force is given by a linear relation in the relative velocity,

$$F_{p_i} = \frac{\rho_p}{\tau_v} (v_{p_i} - v_i) \qquad (2.8)$$

In particular, if we assume Stokes' law holds, then the velocity relaxation time is given by the relation

$$\tau_v = m_p / 6 \pi \mu r_p$$

which is order of the time required for the particle cloud and the gas to come into local velocity equilibrium. Similarly, we assume the rate at which heat is transferred to a unit volume of gas from the particle cloud in that volume depends on a linear relation in the relative temperature,

$$Q_p = \frac{\rho_p \frac{c_s}{c_p}}{\tau_T} (T_p - T) \tag{2.9}$$

For particles obeying the Stokes' law, the temperature relaxation time is then

$$\tau_T = \frac{3}{2} P_r \frac{c_s}{c_p} \tau_V$$

which is the order of the time required for the particle cloud and the gas to come into local temperature equilibrium. For metallic particles in a gaseous medium, τ_T and τ_V are of the same order.

We have tacitly assumed, since the interparticle distance in our range of interest is much larger than the particle size, that the total interaction of the particle cloud on the gas in a unit volume is the number of particles in that volume times the corresponding effect of a single particle.

Rubinow and Keller[21] showed that the transverse force on a sphere in shear flow is solely due to its spin; however, when the sphere originally has zero spin, it remains so thereafter. These apply to our range of interest in small departures from thermodynamic equilibrium where the particle-fluid interaction is describable by a linear relation in the relative velocity, and the transverse force will not be included in our subsequent considerations.

3. "Incompressible" Flow

In the classical sense, when one considers the special fluid that is called "incompressible", one restricts considerations to small relative temperature differences in the fluid, in which case the fluid density and properties variations are negligible. Simultaneously, the flow Mach number has to be much less than unity, in which case the heat source in the fluid due to viscous dissipation can be neglected. In particle-fluid mechanics, the additional restrictions are that the relative temperature difference and that the Mach number based on the relative velocity difference between the fluid and the particle cloud be small. The first in keeping the variations of fluid density and properties negligible, the second in keeping the heat source in the fluid due to the work done on the fluid by the force exerted from the particle cloud negligible. In this situation, the momentum and thermal considerations are uncoupled. Since the particle cloud is not subjected to an equation of state, the variation of particle cloud density, ρ_p , is then governed solely by mass conservation considerations. In this sense the particle-cloud continuity equation does not have an "incompressible" form.

For the "incompressible" problem, the fundamental equations reduce to the following form:

$$\frac{\partial v_j}{\partial x_j} = 0 \tag{2.10}$$

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{\rho} F_{p_i} \tag{2.11}$$

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial (\rho_p v_{p_j})}{\partial x_j} = 0 \tag{2.12}$$

$$\frac{\partial v_{p_i}}{\partial t} + v_{p_j} \frac{\partial v_{p_i}}{\partial x_j} = \frac{1}{\rho_p} \left(-F_{p_i}\right) \tag{2.13}$$

These equations are in the form given by Saffman(14), which forms the basis of his study of the "incompressible" laminar stability in particle-fluid mechanics.

Entirely uncoupled from the dynamical problem, the "incompressible" form, in the context discussed, of the energy equations are:

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p v_j \frac{\partial T}{\partial x_j} = -\frac{\partial P}{\partial t} - v_j \frac{\partial P}{\partial x_j} + k \frac{\partial^2 T}{\partial x_j^2} + \Phi + Q_p + \Phi_p \tag{2.14}$$

$$\rho_p c_s \frac{\partial T_p}{\partial t} + \rho_p c_s v_{p_j} \frac{\partial T_p}{\partial x_j} = \left(-Q_p\right) \tag{2.15}$$

In the consideration of the heat transfer problem, the dissipation terms are neglected; while in the thermometer problem, they are retained as the only sources contributing to temperature variations.

4. "Inviscid" Flow

In the study of flow fields of aeronautical interest in particle-fluid mechanics, i.e., those which result from the presence of an obstacle placed in the stream, the classification of certain regions in the flow field is similar to that in classical fluid mechanics for fluids with negligible friction, since here the only transport effects are those of the fluid phase. In other words, shearing stresses within the fluid itself are confined to certain thin boundary-layer regions in space and adjacent to solid boundaries when the flow Reynolds number, based on some characteristic length, is large. Outside such regions, the particle-fluid momentum interaction, which arises due to fluid viscosity, is more important compared to the viscous forces within the fluid itself. Similarly, for other transport effects, such as fluid heat conductivity and mass diffusivity, this is also true. Loosely speaking, then, outside of certain thin boundary-layer regions, we may consider the "inviscid" flow in particle-fluid mechanics.

The conservation equations for "inviscid" flow, in the context just discussed, is simply the corresponding Euler equations in particle-fluid mechanics. For the fluid, which can be compressible, the equations take the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_j)}{\partial x_j} = 0 \qquad (2.16)$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + F_{p_i} \qquad (2.17)$$

$$\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial x_j} + P \frac{\partial v_j}{\partial x_j} = Q_p + (v_{p_i} - v_i) F_{p_i} \qquad (2.18)$$

Alternative forms of the fluid energy equation may be written in terms of the gas static enthalpy per unit of its own mass $h = e + P/\rho$:

$$\rho \frac{\partial h}{\partial t} + \rho v_j \frac{\partial h}{\partial x_j} = \frac{\partial P}{\partial t} + v_j \frac{\partial P}{\partial x_j} + Q_p + (v_{p_i} - v_i) F_{p_i} \qquad (2.19)$$

and in terms of the gas entropy per unit mass:

$$\rho\frac{\partial s}{\partial t} + \rho v_j\frac{\partial s}{\partial x_j} = \frac{1}{T}\left(\rho\frac{\partial h}{\partial t} + \rho v_j\frac{\partial h}{\partial x_j} - \rho\frac{\partial p}{\partial t} - \rho v_j\frac{\partial p}{\partial x_j}\right)$$

$$= \frac{1}{T}\left(Q_p + (v_{p_i} - v_i)F_{p_i}\right) \tag{2.20}$$

The corresponding equations for the particle cloud are

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial(\rho_p v_{p_j})}{\partial x_j} = 0 \tag{2.21}$$

$$\rho_p\frac{\partial v_{p_i}}{\partial t} + \rho_p v_{p_j}\frac{\partial v_{p_i}}{\partial x_j} = (-F_{p_i}) \tag{2.22}$$

$$\rho_p c_s\frac{\partial T_p}{\partial t} + \rho_p c_s v_{p_j}\frac{\partial T_p}{\partial x_j} = (-Q_p) \tag{2.23}$$

where $c_s T_p = e_p = h_p$. In terms of the particle cloud entropy per unit of its own mass, the energy equation takes the form

$$\rho_p\frac{\partial s_p}{\partial t} + \rho_p v_{p_j}\frac{\partial s_p}{\partial x_j} = \frac{1}{T_p}\left(\rho_p\frac{\partial h_p}{\partial t} + \rho_p v_{p_j}\frac{\partial h_p}{\partial x_j}\right) \tag{2.24}$$

$$= \frac{1}{T_p}(-Q_p)$$

III. THE RAYLEIGH PROBLEM

1. General Discussion

In classical fluid mechanics, the most enlightening arguments concerning the existence and nature of boundary layers were derived from considerations of the diffusion of vorticity[22]. The equation satisfied by the vorticity appears in the same form as the heat diffusion equation, the analogy with which implies that vorticity cannot originate in an infinitesimal part of the fluid which is entirely enclosed by fluid without vorticity. Like heat, vorticity must be diffused into the interior of the fluid from the boundaries from which it must originate. Finally, the analogy implies that the total amount of vorticity which was originally produced at the boundaries must remain conserved in the fluid. The simplest example is the infinite flat plate started from rest impulsively, now commonly known as the Rayleigh problem, was first considered by Stokes[23], and its experimental verification is only recent[24]. In this case, concentrated vorticity is produced at the plate at $t = 0$ and after some time, t, the extent of its penetration into the interior of the fluid is of the order $\sqrt{\nu t}$ where ν is the kinematic viscosity of the fluid. It was suggested by Rayleigh[25] that when a moving stream of fluid with velocity u_0 reaches the leading edge of a plate, the situation is somewhat similar in that a concentrated vortex sheet is then produced. The time required for a fluid element to travel a distance x from the leading edge is approximately $t \sim x/u_0$ by which time the extent in which vorticity has diffused into the interior of the fluid is of the order of $\sqrt{\nu x/u_0}$

which is the order of the classical laminar boundary-layer thickness. The resemblance of the two problems, except for details, arises from the nature of high-Reynolds-number viscous flow where diffusion effects are in a direction normal to the plate only. Rayleigh inferred from the infinite flat plate impulsively started from rest the shear stress at the wall of a semi-infinite flat plate moving in a stream of small viscosity by applying the transformation $t = x/u_0$. Although approximate, this lead to a result for the shear stress at the wall differing only by a modest numerical factor from the more exact calculations of Blasius[22].

In classical fluid mechanics, the insight gained from the Rayleigh problem (for a recent review of a variety of Rayleigh's problem in fluid mechanics, see Stewartson[26]) in understanding the boundary layer concept renders the motivation of the present chapter rather obvious. However, in particle-fluid mechanics, the Rayleigh problem can only be considered as possessing features of a more restrictive number of physically important high-Reynolds-number viscous flow problems. For instance, one must exclude those problems where particle-boundary collisions and relatively large deviations of solid particle paths from fluid streamlines due to centrifugal acceleration become appreciably important. Hence, the Rayleigh problem in particle-fluid mechanics is then limited to an idealization of the constant-pressure laminar boundary layer or nearly parallel flows. For a general discussion of the implications of Prandtl's boundary layer concept in particle-fluid mechanics, one is referred to Marble[19].

2. The "Relaxation-Diffusion Equation"

We consider a special fluid that can be treated as "incompressible", as discussed in Chapter II, where the special case of the conservation equations for an "incompressible" fluid is stated. Here, we further reduce the equations to an appropriate form for the description of the Rayleigh problem. In the present problem, the mixture of fluid and particles is bounded by a flat plate, infinite in extent, at the $y = 0$ plane, as shown in Figure 1. We consider the motions of the plate to be in its own plane so that there is no displacement of fluid in the y-direction due to the plate motion. Neglecting viscous dissipation for our incompressible fluid, there is no expansion of the fluid near the plate, hence $v = v_p = 0$. Furthermore, since the plate extends to infinity in the $\pm x$ directions, there are no changes in the x-direction of any property. The streamlines of the fluid now coincide with that of the particles and are parallel to the plate. In this situation, the density of the particles ρ_p is fixed according to its initial value along streamlines which we take as constant throughout. This is essentially the main difference from the boundary layer problem considered by Marble[19] where the particle density is variable due to the variation of the particle velocity in the direction vertical to the plate. Now the quantities u and u_p are functions of y and t only. The momentum equations (2.11) and (2.13), with the force law given by equation (2.8), then appear in the form

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{K}{\tau_v}(u_p - u) \qquad (3.1)$$

$$\frac{\partial u_p}{\partial t} = -\frac{1}{\tau_v}(u_p - u) \tag{3.2}$$

where $\kappa = \rho_p / \rho$ is constant. The equations are rendered linear by virtue of the geometry of the problem and within the restrictions of an incompressible fluid. Within the assumption of the linear force law in the particle-fluid momentum interaction, these are the "exact" Navier-Stokes equations governing the problem in particle-fluid mechanics. Differentiating equation (3. 1) with respect to t and using equation (3. 2), we obtain the following "relaxation diffusion equation" for u (or for u_p):

$$\tau_v^* \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial t} - \bar{\nu} \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{3.3}$$

where $\tau_v^* = \tau_v / 1+\kappa$ is an effective velocity equilibration time, and $\bar{\nu} = \nu / 1+\kappa$ is the kinematic viscosity based on the density of the mixture and has the physical significance of being the "equilibrium" kinematic viscosity, and ν is the "frozen" kinematic viscosity. The fluid vorticity $\Omega = -\frac{\partial u}{\partial y}$ obeys the same equation. Equation (3. 3) appears in the same form as that governing elastic wave propagation in a visco-elastic solid for a three-parameter model with one elastic and two viscous elements derived by Morrison[(27)], and as that for the Rayleigh problem in a visco-elastic fluid considered by Tanner[(28)].*

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* Unfortunately, Tanner's exact solution is incorrect due to a mistake in making branch-cuts for the inversion of the LaPlace transformation.

3. Mechanical Energy Dissipation

In classical fluid mechanics, the dissipation of fluid kinetic energy arises due to the presence of viscosity which is ultimately degraded into heat. Due to the presence of particles, a new mechanism for the dissipation of kinetic or mechanical energy arises which is the result of the work done due to the particle-fluid interaction. This may be simply demonstrated in the following.

Let us consider the total kinetic energy of the mixture, which is the sum of the fluid and the particle kinetic energies per unit volume of space integrated over a unit width of space:

$$E_{kin} = \int_0^\infty \left(\rho \frac{u^2}{2} + \rho_p \frac{u_p^2}{2} \right) dy \tag{3.4}$$

where E_{kin} is the total kinetic energy of the mixture and $\rho \frac{u^2}{2}$, $\rho_p \frac{u_p^2}{2}$ are the fluid and particle kinetic energies per unit volume of space, respectively. Now the rate of increase of the total kinetic energy is then

$$\dot{E}_{kin} = \frac{\partial}{\partial t} \int_0^\infty \left(\rho \frac{u^2}{2} + \rho_p \frac{u_p^2}{2} \right) dy$$

$$= \int_0^\infty \left(\rho u \frac{\partial u}{\partial t} + \rho_p u_p \frac{\partial u_p}{\partial t} \right) dy \tag{3.5}$$

Using the momentum equations of the fluid and of the particles given in equations (3.1) and (3.2) respectively, then equation (3.5) becomes:

$$\dot{E}_{kin} = -u_0 \left(\mu \frac{\partial u}{\partial y} \right)_{y=0} - \left[\mu \int_0^\infty \left(\frac{\partial u}{\partial y} \right)^2 dy + \frac{\rho_p}{\tau_V} \int_0^\infty (u_p - u)^2 dy \right] \tag{3.6}$$

Since we are considering the plate suddenly set into motion at velocity u_o , then $u_o \tilde{\tau}_w$, where $\tilde{\tau}_w = -\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$, is the rate at which work is done on the fluid by the plate. Rewriting,

$$u_o \tilde{\tau}_w = \dot{E}_{kin} + \mu \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy + \frac{\rho_p}{\tau_v} \int_0^\infty (u_p - u)^2 \, dy \qquad (3.7)$$

(where $\mu > 0$, $\tau_v > 0$) . We see that not all of this work goes to increasing the kinetic energy of the mixture. The loss is given by the rate at which fluid kinetic energy is dissipated through viscosity, which is represented by the second term on the right where the integral is positive definite, and by the rate of particle and fluid kinetic energy dissipation due to particle-fluid momentum interaction, which is represented by the third term on the right where again the integral is positive definite. Hence, rate of increase of the total kinetic energy of the mixture is equal to the rate of work done on the fluid by the plate motion subtracted by that which is dissipated into heat. Of course, in the incompressible fluid which we consider, this heat is rapidly conducted away and gives rise to a negligible increase in temperature of the mixture.

4. The Rayleigh Problem

The Rayleigh problem in the classical sense considers the state of a viscous fluid at subsequent times after an infinite plate has been set into motion in its own plane. In our present context, the fluid contains small solid particles, and we consider the Rayleigh problem for such a mixture. Here, we encounter the competing effect between the two types of forces acting on the fluid. First is the fluid viscosity which, after the plate has been set into motion, transmits momentum in a normal direction, tending to accelerate adjacent layers of fluid increasingly outwards. On the other hand, there is the force acting on the fluid by the particles. Because of their inertia, the particles tend to remain still immediately after the plate has been set into motion, and this tends to hold back the fluid acceleration.

It is physically clear that our Rayleigh problem has two near limiting regimes of particle-fluid momentum interaction. This is measured by the ratio t/τ_V, where t is the time measured from the start of the plate motion. Initially, the fluid near the wall accelerates to the plate motion immediately. However, the particles in this region take a time of the order of τ_V before they are accelerated to the fluid velocity at the plate. So when $t/\tau_V \leq \mathcal{O}(1)$, the slip between the particles and the fluid is relatively large. At times much larger than the particle velocity equilibration time, τ_V, the particles themselves are very nearly following the fluid velocity. However, due to fluid viscosity effects, the fluid is being continuously accelerated by the momentum transmitted from the plate, and the particles are hence prevented from attaining the velocity of the fluid exactly.

In the range when $t/\tau_V \ll 1$ the particle-fluid interaction is in the "strong" interaction regime, and the particle motion is determined by the initial condition. In the range when $t/\tau_V \gg 1$ the particle-fluid interaction is in the "weak" interaction regime, and particles are very nearly following the fluid motion. In the transition regime when $t/\tau_V = \mathcal{O}(1)$, both the initial conditions and the local conditions are of importance.

Fortunately, the Rayleigh problem here is amenable to an exact solution for the entire range of t/τ_v. However, asymptotic forms of the solution can be deduced from appropriate expansions of the exact solution for the near limiting cases. This is particularly useful in that they suggest the form of expansion that can be made in closely related but nevertheless more complicated situations when an exact solution is not available. For instance, the laminar boundary layer is an example. When $t/\tau_v \ll 1$, the form of the expansion for the fluid velocity is recovered as

$$\frac{u}{u_o}(\eta, t/\tau_v) = \frac{u_{f_o}(\eta)}{u_o} + \frac{t}{\tau_v}\,\frac{u_{f_1}(\eta)}{u_o} + \vartheta\left(\left(\frac{t}{\tau_v}\right)^2\right)$$

where $\eta = y/2\sqrt{\nu t}$ is the similarity variable in the near "frozen" limit appropriately defined with ν as the "frozen" kinematic viscosity. The particles are very nearly stationary, and the interaction with the fluid, which has a tendency to hold it back, is everywhere "strong" when the fluid is in motion. Thus, the first-order connection term $\frac{u_{f_1}(\eta)}{u_o}$ is everywhere negative. $\frac{u_{f_o}(\eta)}{u_o}$ is, of course, the "frozen" solution of the classical Rayleigh problem. The effect on the shear stress is to prevent it from falling off like $1/\sqrt{t}$. The effect on the growth of the effective diffusion layer thickness is to thin down the thickness from the "frozen" parabolic behavior $\sqrt{\nu t}$.

In the opposite near-limiting case, when $t/\tau_v \gg 1$, the form of the expansion for the fluid velocity is

$$\frac{u}{u_o}(\bar{\eta}, t/\tau_v) = \frac{u_{e_o}(\bar{\eta})}{u_o} + \frac{\tau_v}{t}\,\frac{u_{e_1}(\bar{\eta})}{u_o} + \vartheta\left(\left(\frac{\tau_v}{t}\right)^2\right)$$

where $\bar{\eta} = y/2\sqrt{\bar{\nu}t}$ is the similarity variable in the near "equilibrium" limit appropriate defined with $\bar{\nu} = \nu/1+K$, the "equilibrium" kinematic viscosity. $\frac{u_{e_0}(\bar{\eta})}{u_0}$ is just the "equilibrium" classical Rayleigh solution. We are particularly interested in the shape of the fluid velocity profile near the plate which determines the shear stress at the wall. We have previously suggested that the fluid velocity is being continuously accelerated, due to viscosity, which transmits the momentum of the plate motion to the layers of fluid in the direction away from the plate. In other words, the diffusion of this momentum of plate motion accelerates the fluid locally. In this case, the diffusion takes place with $\bar{\nu}$, and the fluid behaves nearly like $\operatorname{erfc} \bar{\eta}$ according to the classical Rayleigh solution. However, in the immediate vicinity of the plate, when the particle-fluid momentum interaction force is the largest and the particles tend to drag the fluid back, in this situation $\frac{u_{e_1}(\bar{\eta})}{u_0}$ is expected to be negative, at least in the neighborhood near the plate. Hence, the shear stress at the plate approaches the final equilibrium value asymptotically from above. The effect on the effective diffusion-layer thickness is to approach the final parabolic equilibrium form $\sqrt{\bar{\nu}t}$ from below.

In this section, the application of the method of Laplace transformation to find the solution for the Rayleigh problem is presented. We fix the coordinate system in space, and the plate is suddenly set in motion. A simple transformation can be made leading to the solution with coordinates fixed on the plate with the fluid suddenly set into motion with velocity u_0 by taking $1 - \frac{u}{u_0}$ where $\frac{u}{u_0}$ is our solution. We will consider the problem in terms of nondimensional quantities denoted by a prime. Let $u' = \frac{u}{u_0}$ where u_0 is the velocity of the plate motion in the Rayleigh problem, $t' = t/\tau_V^*$, $y' = y/\sqrt{\nu\tau_V^*}$. Hence, equation (3.3) becomes

$$\frac{\partial}{\partial t'}\left(\frac{\partial u'}{\partial t'} - \frac{\partial^2 u'}{\partial y'^2}\right) + \left(\frac{\partial u'}{\partial t'} - \frac{1}{1+K}\frac{\partial^2 u'}{\partial y'^2}\right) = 0 \qquad (3.8)$$

where we have used the fact that $\bar{\nu}/\nu = 1/(1+K)$.

4.1. Fluid Velocity Profile. - We shall now consider obtaining the solution for u' from equation (3.8). Initially, the fluid and the particles are in equilibrium and are at rest, so that $u'(y', 0) = 0$, $\frac{\partial u'}{\partial t'}(y', 0) = 0$. The boundary conditions are $u'(0, t') = 0$ when $t' < 0$ and $u'(0, t') = 1$ when $t' \geq 0$. The fluid remains undisturbed far away from the plate, and $u'(y', t') = 0$ as $y' \to \infty$ or at least the disturbance far away should remain finite.

Denote $U(y', s)$ as the Laplace transformation of $u'(y', t')$ which is defined as

$$U(y', s) = \int_0^\infty e^{-st'} u'(y', t')\, dt' .$$

With the use of the initial conditions, the transformed differential equation for the fluid velocity then becomes

$$\frac{d^2 U}{dy'^2} - W^2(s)\, U = 0$$

where $W(s) = \sqrt{\frac{s(s+1)}{s + \frac{1}{1+K}}}$ and the boundary conditions become $U(0, s) = \frac{1}{s}$ and $U(\infty, s) = 0$. The appropriate solution is simply

$$U(y', s) = \frac{1}{s}\, e^{-W(s)\, y'}$$

The final solution in the (y', t') plane is obtained formally from the inversion integral

$$u'(y',t') = \frac{1}{2\pi i} \int_{L_1} U(y',s)\, e^{st'} ds$$

where L_1 is the Bromwich path parallel to the imaginary axis and to the right of all singularities of $U(y', s)$. For the details of the conditions assumed to be satisfied by $U(y', s)$ leading to the validity of the inversion integral and the differentiation and integration under the contour integral, one is referred to, for instance, Carslaw and Jaeger[29] or Churchill[30]. Our desired solution is then reduced to an appropriate evaluation of a contour integral in the complex s-plane. A brief discussion of obtaining the solution by contour integration is provided in Appendix III-A. Here, we will just state our final result, which appears in the form of real, definite integrals:

$$u'(y',t') = 1 - \frac{2}{\pi} \int_0^{\frac{1}{\sqrt{1+\kappa}}} e^{-\beta^2 t'} \sin\left(\beta \sqrt{\frac{1-\beta^2}{\frac{1}{1+\kappa}-\beta^2}}\, y'\right) \frac{d\beta}{\beta} - \frac{2}{\pi} \int_1^{\infty} e^{-\beta^2 t'} \sin\left(\beta \sqrt{\frac{\beta^2-1}{\beta^2-\frac{1}{1+\kappa}}}\, y'\right) \frac{d\beta}{\beta} \qquad (3.9)$$

Of course, when the particle concentration vanishes and κ is identically zero, we obtain

$$u'(y',t') = 1 - \frac{2}{\pi} \int_0^{\infty} e^{-\beta^2 t'} \sin \beta y' \, \frac{d\beta}{\beta}$$

$$= \operatorname{erfc}\left(y'/2\sqrt{t'} \right)$$

which is the solution to the classical Rayleigh problem, and the combination of the non-dimensional independent variables y', t' in the form $y'/2\sqrt{t'}$ becomes just $y/2\sqrt{\nu t}$ in the actual physical variables where the particle relaxation time, τ_V, does not appear, as indeed it should not.

The fluid velocity profile given by equation (3.9) is in an appropriate form for numerical evaluation if desired. It is also in a form appropriate for asymptotic approximation for $t/\tau_V >> 1$, and the discussion of applying Watson's lemma for such an approximation is provided in Appendix III-B. It is found that the fluid velocity profile can be expressed in the form:

$$u'(\bar{\eta},t') = u'_{e_0}(\bar{\eta}) + \frac{\tau_V}{t} u'_{e_1}(\bar{\eta}) + \mathcal{O}((\frac{\tau_V}{t})^2)$$

when $t/\tau_V >> 1$, which is the weak particle-fluid interaction regime. Here, $\bar{\eta} = y/2\sqrt{\bar{\nu} t}$, and is just the similarity variable appropriately based on the "equilibrium" kinematic viscosity $\bar{\nu}$. The zeroth order function is just the Rayleigh solution for the fully developed profile

$$u'_{e_0}(\bar{\eta}) = \operatorname{erfc} \bar{\eta} \tag{3.10}$$

The first order function is

$$u'_{e_1}(\bar{\eta}) = \frac{\kappa}{1+\kappa} \frac{1}{2\pi^{1/2}} \bar{\eta}(2\bar{\eta}^2-1) e^{-\bar{\eta}^2} \tag{3.11}$$

and $\frac{1+\kappa}{\kappa} u'_{e_1}(\bar{\eta})$ is a universal function of $\bar{\eta}$ independent of explicitly, and is shown in Figure 2.

When $t/\tau_V << 1$, it is expected that the approximation is of the form

$$u'(\bar{\eta},t') = u'_{f_0}(\eta) + \frac{t}{\tau_V} u'_{f_1}(\eta) + \mathcal{O}((\frac{t}{\tau_V})^2)$$

starting from the "frozen" limit as the zeroth order term, and

$\eta = y/2\sqrt{\nu t}$ is the similarity variable using the "frozen" kinematic viscosity, ν . The appropriate approximations found in Appendix III-C for $t/\tau_v << 1$ gives

$$u'_{f_0}(\eta) = \operatorname{erfc} \eta \tag{3.12}$$

which is the classical Rayleigh solution in the absence of particles, or the "frozen" solution. The first order term is

$$u'_{f_1}(\eta) = -\frac{\kappa}{2}\left(4\pi^{-1/2}\eta e^{-\eta^2} - 4\eta^2 \operatorname{erfc}\eta\right) \tag{3.13}$$

which is a negative quantity. We note that here, $\frac{1}{\kappa} u'_{f_1}(\eta)$ is a universal function of η , independent of κ and is shown in Figure 3.

4.2. Particle Velocity Profile. - The momentum equation for the nondimensional particle velocity $u'_p = u_p/u_0$ from equation (3.2) appears in the form:

$$(1+\kappa)\frac{\partial u'_p}{\partial t'} = -(u'_p - u) \tag{3.14}$$

and it is recalled that $t' = t/\tau_v^* = (1+\kappa)t/\tau_v$. Initially, the particles are at rest and in equilibrium with the fluid at rest, $u'_p(y',0) = 0$. The Laplace transformation of $u'_p(y',t')$ will be denoted as $U_p(y',s)$ which is, by definition,

$$U_p(y',s) = \int_0^\infty u_p(y',t')\, e^{-st'}\, dt'$$

With the use of the initial condition for u'_p the transformed momentum equation for $U_p(y',s)$ then appears in the form:

$$U_p(y',s) = \frac{1}{1+(1+\kappa)s}\,\frac{1}{s}\, e^{-W(s)\,y'}$$

where we have used the results of paragraph 4.1 and $W(s) = \sqrt{\frac{s(s+1)}{s+\frac{1}{1+\kappa}}}$

Again, formally, the inversion integral giving the solution in the (y', t') plane is

$$u_p'(y',t') = \frac{1}{2\pi i}\int_{L_1} U_p(y',s)\, e^{st'}\, ds$$

and L_1 is the Bromwich path. The equivalent path along which the consideration of the contour integral is made is the same as that for $u'(y',t')$ with the exception of an additional contribution of a pole of order one at $s = -1/1+\kappa$ in $U_p(y',s)$, which is found to be e^{-t/τ_v}.

The result in terms of real, definite integrals is then:

$$u_p'(y',t') = 1 - e^{-t'} - \frac{2}{\pi}\int_0^{\frac{1}{\sqrt{1+\kappa}}} e^{-\beta^2 t'} \sin\left(\beta\sqrt{\frac{1-\beta^2}{\frac{1}{1+\kappa}-\beta^2}}\, y'\right)\frac{d\beta}{\beta\left(1-(1+\kappa)\beta^2\right)}$$

$$-\frac{2}{\pi}\int_1^{\infty} e^{-\beta^2 t'} \sin\left(\beta\sqrt{\frac{\beta^2-1}{\beta^2-\frac{1}{1+\kappa}}}\, y'\right)\frac{d\beta}{\beta\left(1-(1+\kappa)\beta^2\right)} \qquad (3.15)$$

At the plate itself, the particle velocity responds to the fluid velocity $u'(0,t') = 1$ in the manner:

$$u_p'(0,t') = 1 - e^{-t'}$$

The corresponding asymptotic approximation for u_p' when $t/\tau_v \gg 1$, discussed in Appendix III-B, is written in terms of the slip velocity

$$u'(y',t') - u_p'(y',t') = e^{-t'} + \frac{\tau_v}{t}\, g_{e_1}(\bar{\eta}) + \mathcal{O}\left(\left(\frac{\tau_v}{t}\right)^2\right)$$

We have retained the term $e^{-t'}$, even for $t/\tau_v \ll 1$, since it is important near the plate and gives the correct value of the slip velocity at the plate:

$$u'(0,t') - u_p'(0,t) = e^{-t'}$$

The first-order slip velocity profile is given as

$$g_{e_1}(\bar{\eta}) = \frac{\pi^{-1/2}}{2} \; \bar{\eta} \, e^{-\bar{\eta}^2} \tag{3.16}$$

which is a universal function of $\bar{\eta}$ independent of κ, and is shown in Figure 4. One notes that the local acceleration of the zeroth-order equilibrium flow is proportional to

$$\frac{\partial u_{e_0}(\bar{\eta})}{\partial t} \propto \frac{u_0}{t} \frac{\pi^{-1/2}}{2} \; \bar{\eta} \, e^{-\bar{\eta}^2}$$

which implies that the first-order slip velocity, except for the term $e^{-t'}$ which is only important near the plate, is proportional to the local acceleration. This useful concept has been utilized by Marble(19) in an approximate treatment of the laminar boundary-layer problem. Here, however, it is recovered from the exact solution.

Initially, the situation is quite different, and the slip velocity is no longer small compared with the velocity of the plate motion, since the particles are very nearly standing still when $t/\tau_V \ll 1$. Instead of writing the particle velocity in terms of the slip velocity, we write directly

$$u_p'(\eta, t') = \frac{t}{\tau_V} \, u_{p_{f_1}}'(\eta) + \mathcal{O}\left(\left(\frac{t}{\tau_V}\right)^2\right)$$

From Appendix III-C, it was found that

$$u_{p_{f_1}}'(\eta) = \operatorname{erfc} \eta - \frac{1}{2}\left(4\pi^{-1/2} \eta e^{-\eta^2} - 4\eta^2 \operatorname{erfc} \eta\right) \tag{3.17}$$

This is shown in Figure 5. At the plate itself, $u'_{p_{f_1}}(0) = 1$ and $u'_p(0,t') = \frac{t}{\tau_V} + \vartheta((\frac{t}{\tau_V})^2)$, which is an incipient acceleration of the particles towards the velocity of the plate motion in which the fluid possesses at the plate itself. When $t/\tau_V \ll 1$, the particle motion depends on the initial condition, which is $u'_p(\eta, 0) = 0$.

4.3. <u>Vorticity Thickness. Shear Stress at the Plate.</u> - Recalling the results in paragraph 4.1, we can define the fluid vorticity in the transformed plane (y', s) as

$$\bar{\Omega}(y', s) = -\frac{dU}{dy'} = -\frac{1}{s} W(s) e^{-W(s) y'}$$

and the total content of vorticity is

$$\int_0^\infty \bar{\Omega}(y', s)\, dy' = \frac{1}{s} \int_0^\infty e^{-W(s) y'} d(W(s) y') = \frac{1}{s}$$

and, formally, in the (y', t') plane, it is simply unity. The initial amount of vorticity produced by the sudden motion of the plate is

$$\int_0^\infty \Omega'(y', t')\, dy' = -\int_0^\infty \frac{\partial u'}{\partial y'} dy' = -[u'(\infty) - u'(0)]$$
$$= 1$$

Hence, we conclude that vorticity is conserved, as we would expect. We can now define an "effective" vorticity thickness, as in the classical Rayleigh problem, in the form

$$\delta'(t') \sim \frac{1}{\Omega'(0,t')} \int_0^\infty \Omega' \, dy' = \frac{1}{\Omega'(0,t')} \tag{3.18}$$

where $\Omega'(0,t') = -\left(\frac{\partial u'}{\partial y}\right)_{y=0}$ and $\delta' = \delta / \sqrt{\nu \tau_v^*}$. Again, using the result of paragraph 4.1, we find that the "effective" vorticity thickness in actual physical coordinates is

$$\delta(t') \sim \frac{u_o}{\sqrt{\nu \tau_v^*}} \frac{\pi}{2} \left[\int_0^{\frac{1}{\sqrt{1+K}}} e^{-\beta^2 t'} \sqrt{\frac{1-\beta^2}{\frac{1}{1+K}-\beta^2}} \, d\beta + \int_1^\infty e^{-\beta^2 t'} \sqrt{\frac{\beta^2-1}{\beta^2-\frac{1}{1+K}}} \, d\beta \right]^{-1} \tag{3.19}$$

It is particularly interesting to observe the transition of the vorticity layer thickness from the "frozen" parabolic growth $\sqrt{\pi \nu t}$ to the final "equilibrium" parabolic growth $\sqrt{\pi \bar{\nu} t}$. When $t/\tau_v \gg 1$, the near-equilibrium asymptotic behavior is obtained from Appendix III-B:

$$\delta(t') \sim \sqrt{\pi \bar{\nu} t} \left[1 - \frac{1}{4} \frac{K}{1+K} \frac{\tau_v}{t} + \mathcal{O}\left(\left(\frac{\tau_v}{t}\right)^2 \right) \right] \tag{3.20}$$

which approaches the equilibrium layer thickness from below. Initially, the near-frozen asymptotic behavior is obtained from Appendix III-C:

$$\delta(t') \sim \sqrt{\pi \nu t} \left[1 - K \frac{t}{\tau_v} + \mathcal{O}\left(\left(\frac{t}{\tau_v}\right)^2 \right) \right] \tag{3.21}$$

in which thinning from the frozen layer thickness takes place. The discussion of the physical interpretation of the results is delayed to section 5 in connection with the laminar boundary layer.

The shear stress at the plate is defined as

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\frac{u_0}{\sqrt{\nu\tau_v^*}}\left(\frac{\partial u'}{\partial y'}\right)_{y'=0}$$

and we have

$$\frac{c_f}{2} = \frac{\tau_w}{\rho u_0^2} = -\sqrt{\frac{\nu}{u_0^2\tau_v^*}}\,\frac{2}{\pi}\left[\int_0^{\frac{1}{\sqrt{1+\kappa}}} e^{-\beta^2 t'}\sqrt{\frac{1-\beta^2}{\frac{1}{1+\kappa}-\beta^2}}\,d\beta + \int_1^{\infty} e^{-\beta^2 t'}\sqrt{\frac{\beta^2-1}{\beta^2-\frac{1}{1+\kappa}}}\,d\beta\right] \tag{3.22}$$

which is in a convenient form for numerical evaluation with t' as a parameter. One obtains similarly asymptotic approximations when $t/\tau_v \gg 1$ from Appendix III-B, the near-equilibrium behavior

$$\frac{\tau_w}{\rho u_0^2} = -\pi^{-1/2}\sqrt{\frac{\nu}{u_0^2 t}}\,\sqrt{1+\kappa}\left[1+\frac{1}{4}\frac{\kappa}{1+\kappa}\frac{\tau_v}{t} + \vartheta\left(\left(\frac{\tau_v}{t}\right)^2\right)\right] \tag{3.23}$$

The equilibrium value is $\frac{(\tau_w)_e}{(1+\kappa)\rho u_0^2} = \pi^{-1/2}\sqrt{\frac{\nu}{u_0^2 t}} = \pi^{-1/2}\sqrt{\frac{\nu/1+\kappa}{u_0^2 t}}$, which is the first term in the expansion. When $t/\tau_v \ll 1$, the near-frozen behavior of the shear stress on the plate is obtained from Appendix III-C,

$$\frac{\tau_w}{\rho u_0^2} = -\pi^{-1/2}\sqrt{\frac{\nu}{u_0^2 t}}\left[1+\kappa\frac{t}{\tau_v} + \vartheta\left(\left(\frac{t}{\tau_v}\right)^2\right)\right] \tag{3.24}$$

The frozen value in the absence of particles is just $\frac{(\tau_w)_f}{\rho u_0^2} = \pi^{-1/2}\sqrt{\frac{\nu}{u_0^2 t}}$. Again, the discussion of the physical interpretation of our results will be given in section 5 in connection with the laminar boundary layer.

5. Relation to the Constant Pressure Laminar Boundary Layer

We have discussed much of the formalisms involved in obtaining the solution of the Rayleigh problem in section 4. In classical fluid mechanics, many of the features of the laminar boundary layer are exhibited in the Rayleigh problem. The simplicity of the latter enables one to derive a physical picture easily for the high-Reynolds-number viscous flow problem. Here, we will follow a similar procedure and discuss the features of the laminar boundary layer in particle-fluid mechanics as derived from our Rayleigh problem.

Let us fix our coordinates on a sime-infinite flat plate with x measuring from the leading edge and y normal to the plate. Consider the flow to be steady and the oncoming stream, where the particles and fluid are in equilibrium, moves with velocity u_0 in the positive x- direction. The t in our unsteady Rayleigh problem will be replaced by x/u_0 for our steady boundary-layer coordinates. Now the observer is moving downstream of the plate with the velocity u_0. The resemblance in both cases, except for details, lies in the fact that viscous diffusion takes place in the normal y-direction only and in addition, the particle-fluid momentum interaction is t-like in the Rayleigh problem and x-like in the boundary layer problem.

Now,immediately when the fluid reaches the leading edge, it is stopped at the plate, but in an immediate outside layer, the fluid is still moving at velocity u_0 and a concentrated vortex sheet is set up. The tendency of viscosity is to smooth out the steepness in the velocity profile and diffusion takes place. Initially, the viscous diffusion takes place without knowing that the particles are present, since the viscous layer is very thin. Hence the zeroth order terms in the "frozen" limit do not contain any parameters indicating the presence of the particles. On the other hand, when the particles reach the leading edge of the plate, they slip completely and travel at u_0, not knowing the presence of the viscous layer. In the next instant, however, when the viscous layer has contained sufficient particles, the force as a result of the difference in velocity of the particles that are slipping and that of the fluid that is being slowed down by viscosity is to tend to accelerate the fluid and prevent its velocity profile from being smoothed by viscous diffusion and to decelerate the particles. The result is then to prevent the shear stress at the plate from decreasing like $1/\sqrt{x}$. Friction in this case is only partially effective in preventing the tendency of changes in the fluid velocity profile due to the particle-fluid momentum interaction near the wall. The shear stress at the plate behaves like

$$\frac{c_f}{2} = \frac{\tau_w}{\rho u_o^2} = \frac{\pi^{-1/2}}{\sqrt{Re_x}}\left[1 + \kappa\frac{x}{\lambda_v} + \vartheta\left(\left(\frac{x}{\lambda_v}\right)^2\right)\right] \tag{3.25}$$

when $x/\lambda_v \ll 1$. At first, the boundary layer tends to grow in the usual parabolic manner $\sqrt{\nu x}$ with the "frozen" kinematic viscosity, now knowing the particles are present. At some distance slightly downstream, the acceleration of the fluid caused by the particles then contributes to a thinning of the fluid stream-tube and decreases the boundary layer thickness:

$$\delta(x) \sim \sqrt{\frac{\pi\nu x}{u_o}}\left[1 - \kappa\frac{x}{\lambda_v} + \vartheta\left(\left(\frac{x}{\lambda_v}\right)^2\right)\right] \tag{3.26}$$

when $x/\lambda_v \ll 1$. Here, in the limiting case of the "frozen" situation, the shear stress is the familiar Rayleigh problem solution $\frac{(\tau_w)_f}{\rho u_o^2} = \pi^{-1/2}/\sqrt{Re_x}$ and the boundary thickness $\delta_f \sim \sqrt{\frac{\pi\nu x}{u_o}}$

When $x/\lambda_v \gg 1$, the particles and the fluid are very nearly moving together. In this regime, the thickening of the fluid boundary layer caused by viscous diffusion continuously decelerates the fluid in the boundary layer. Here, the controlling factor is the "equilibrium" diffusivity and the fluid tends to behave like the equilibrium profile in its stretching, and the dominant force is the shearing stress. However, because the fluid velocity is continuously changing due to viscosity, the particles never quite attain the actual fluid velocity. Particularly near the wall, where the particle-fluid interaction is relatively the largest, the fluid velocity will suffer from this interaction with the faster-moving particles by becoming fuller than the equilibrium profile. In this case, if the particles follow identically the fluid motion, the mixture behaves as a single fluid with an increased density $(1+\kappa)\rho$, and the "equilibrium" shear stress at the wall is simply $\frac{(\tau_w)_e}{(1+\kappa)\rho u_0^2} = \pi^{-1/2}\Big/\sqrt{(1+\kappa)R_{e_x}}$, or simply $(\tau_w)_e = \sqrt{1+\kappa}\;\pi^{-1/2}\Big/\sqrt{R_{e_x}}$. According to our reasoning, then the shear stress at the wall approaches its equilibrium value from above:

$$\frac{C_f}{2} = \frac{\tau_w}{\rho u_0^2} = \sqrt{1+\kappa}\,\frac{\pi^{-1/2}}{\sqrt{R_{e_x}}}\left[1+\frac{1}{4}\frac{\kappa}{1+\kappa}\frac{\lambda_v}{x} + \mathcal{O}\left(\left(\frac{\lambda_v}{x}\right)^2\right)\right] \qquad (3.27)$$

In the incompressible laminar boundary layer problem[19], $\pi^{-1/2}$ here is replaced by the familiar Blasius factor 0.332, and the factor 1/4 is replaced by 0.49 obtained through numerical integration. In the equilibrium limit, the boundary layer grows in the parabolic manner $\sqrt{\pi\bar{\nu}x/u_0}$ determined by the "equilibrium" kinematic diffusivity $\bar{\nu} = \nu/(1+\kappa)$. Since the boundary layer thickness is inversely proportional to the shear stress at the wall, it then approaches the final equilibrium value from below:

$$\delta(x) \sim \sqrt{\frac{\pi \bar{\nu} x}{u_0}} \left[1 - \frac{1}{4} \frac{\kappa}{1+\kappa} \frac{\lambda_v}{x} + \mathcal{O}\left(\left(\frac{\lambda_v}{x} \right)^2 \right) \right] \tag{3.28}$$

The shearing stress at the plate and the boundary layer growth are shown in Figures 6 and 7, respectively.

6. The Thermal Rayleigh Problem. Thermal Boundary Layer in Laminar Flow

In our incompressible problem, the momentum and energy equations are uncoupled. In this section, we consider the thermal Rayleigh problem for small temperature differences and the solutions are easily obtained by a generalization of the results of section 4. We are particularly interested in deriving from this the physical picture of the behavior of the thermal boundary layer in a laminar flow, and of special interest is the behavior of the surface heat transfer rate and the thermal boundary-layer thickness. This essentially has as its counterpart the Pohlhausen problem in boundary layer theory[31].

The energy equations can be similarly obtained for our problem as the momentum equations in section 4 from section 3 of Chapter II. The energy equation for the fluid is

$$\frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial y^2} + \frac{\kappa^{c_s/c_p}}{\tau_T} (T_p - T) \tag{3.29}$$

and for low Mach numbers we have neglected the fluid viscous dissipation and the work done on the fluid due to particle-fluid interactions. $\chi = k/\rho c_p$ is the thermal diffusivity of the fluid, k the thermal conductivity, and c_p the heat capacity per unit mass of the fluid , c_s the heat capacity per unit mass of the solid particles. τ_T is the temperature relaxation time of the particles, and $\tau_T = \frac{3}{2} Pr \frac{c_s}{c_p} \tau_V$ for particles obeying the Stokes law. Here, κ^{c_s}/c_p is the corresponding thermal equilibration parameter and is a measure of the relative temperature changes during the process of equilibration. The energy equation for the particles is

$$\frac{\partial T_p}{\partial t} = -\frac{1}{\tau_T}(T_p - T) \tag{3.30}$$

We can combine the two energy equations to form a single, but one order higher, differential equation in a similar manner as for the momentum equation of section 4. However, we now define the non-dimensional quantities as $y'' = y \Big/ \sqrt{\frac{\nu}{Pr}\tau_T^*}$, $t'' = t/\tau_T^*$, $\Theta = (T - T_0)/(T_w - T_0)$. It is noted that we can write the "frozen" thermal diffusivity as $\chi = \nu/Pr$ and the "equilibrium" thermal diffusivity as $\bar{\chi} = \frac{\bar{\nu}}{\bar{Pr}} = \frac{\bar{\nu}}{Pr}\left(\frac{1+\kappa}{1+\kappa^{c_s}/c_p}\right)$ and $\tau_T^* = \tau_T/(1+\kappa^{c_s}/c_p)$. Hence, the single differential equation appears in the form

$$\frac{\partial}{\partial t''}\left(\frac{\partial \Theta}{\partial t''} - \frac{\partial^2 \Theta}{\partial y''^2}\right) + \left(\frac{\partial \Theta}{\partial t''} - \frac{1}{1+\kappa\frac{c_s}{c_p}}\frac{\partial^2 \Theta}{\partial y''^2}\right) = 0 \tag{3.31}$$

Initially, the fluid and particles are in equilibrium at temperature T_0 so that $\Theta(y'', 0) = 0$ and $\frac{\partial \Theta}{\partial t''}(y'', 0) = 0$. The boundary conditions are that when $t'' < 0$, the plate is at T_0 so $\Theta(0, t'') = 0$. When $t'' > 0$, the plate is heated to temperature T_W so that $\Theta(0, t'') = 1$. Far from the wall the disturbances vanish so that $\Theta(\infty, t'') = 0$. It is implicit that the particles adjacent to the plate have a temperature slip as in our momentum problem where the particles have a velocity slip at the plate. The temperature rise of the particles next to the plate is then solely due to the heat received from the fluid surrounding the particles in that region.

In particle-fluid mechanics, the existence of the Crocco particular integral, which in the present problem is simply $\Theta = u'$, requires, in addition to $Pr = 1$, that $\tau_T/\tau_v = 1$ and that $\kappa/\kappa \frac{c_s}{c_p} = 1$. However, for particles obeying the Stokes' law when $\tau_T/\tau_v = \frac{3}{2} Pr \frac{c_s}{c_p}$, the conditions necessary for Crocco's integral cannot be all simultaneously satisfied. However, for gases and for metal particles, the conditions are very nearly satisfied.

Drawing upon the results of section 4, we can write the fluid temperature profile in the form:

$$\Theta(y'', t'') = 1 - \frac{2}{\pi}\int_0^{\frac{1}{\sqrt{1+\kappa c_s/c_p}}} e^{-\beta^2 t''} \sin\left(\beta\sqrt{\frac{1-\beta^2}{\frac{1}{1+\kappa c_s/c_p}-\beta^2}}\, y''\right)\frac{d\beta}{\beta} - \frac{2}{\pi}\int_1^{\infty} e^{-\beta^2 t''} \sin\left(\beta\sqrt{\frac{\beta^2-1}{\beta^2-\frac{1}{1+\kappa c_s/c_p}}}\, y''\right)\frac{d\beta}{\beta} \quad (3.32)$$

The surface heat transfer rate is

$$\dot{q}_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{k(T_w - T_0)}{\sqrt{\frac{\nu}{Pr}\tau_T^*}}\,\frac{2}{\pi}\left[\int_0^{\frac{1}{\sqrt{1+\kappa c_s/c_p}}} e^{-\beta^2 t''}\sqrt{\frac{1-\beta^2}{\frac{1}{1+\kappa c_s/c_p}-\beta^2}}\,d\beta + \int_1^{\infty} e^{-\beta^2 t''}\sqrt{\frac{\beta^2-1}{\beta^2-\frac{1}{1+\kappa c_s/c_p}}}\,d\beta\right] ; \quad (3.33)$$

In a similar fashion, the heat diffusion-layer thickness may be estimated from the expression:

$$\delta_T \sim \frac{k(T_w - T_0)}{\dot{q}_w} \quad (3.34)$$

Other quantities can also be as easily obtained. Here, we kept Pr , τ_T/τ_v , and $\kappa/\kappa \frac{c_s}{c_p}$ arbitrary but constant. It is then interesting to discuss the thermal boundary layer in a laminar flow using the information we have obtained from the thermal Rayleigh problem.

Both situations, except for details, are again similar in that heat diffusion takes place in the y-direction only, particle-fluid thermal interaction is t-like in the Rayleigh problem, and x-like in the boundary layer problem. We again replace t by x/u_0 and the observer of the course of events travels at velocity u_0 towards the downstream direction of the plate. The resemblance, again, is to a constant pressure boundary layer in low speed flow where the frictional heating due to fluid viscous dissipation and the work done on the fluid due to particle-fluid momentum interaction are neglected. The heat diffusion problem arises then only as the result of a temperature difference between the plate and the free stream, but $|T_w - T_0|/T_0 \ll 1$ in our problem. For the purpose of fixing our ideas, let us consider the plate is cooled to below the free stream temperature $(T_w < T_0)$ so that heat will flow from the fluid to the plate. When the stream of fluid reaches the leading edge of the plate, the fluid at the plate itself then is at temperature T_w, while the layer immediately next to it remains at the stream temperature, T_0. Immediately, then, the usual heat diffusion takes place with the "frozen" thermal diffusivity $\chi = k/\rho c_p$ not knowing the presence of the particles in the region very close to the leading edge $(x/\lambda_T \ll 1)$ since the boundary layer is very thin. On the other hand, the particles remain at the "hot" stream temperature T_0. Subsequently, the heat received by the fluid from the particles is to tend to steepen the fluid temperature profile and to prevent it being smoothed by the mechanism of heat diffusion when the boundary layer has become sufficiently thick so as to contain sufficient particles for interaction. Conduction in the fluid in this case is then only partially effective in preventing the changes in the fluid temperature profile due to particle-fluid interaction near the wall. Hence, in the region $x/\lambda_T \ll 1$ close to the leading edge, the local surface heat-transfer rate behaves like:

$$\dot{q}_w = \frac{k(T_w - T_o)}{\sqrt{\frac{\nu x}{u_o}}} \pi^{-1/2} P_r^{1/2} \left[1 + \kappa \frac{c_s}{c_p} \frac{x}{\lambda_T} + O\left(\left(\frac{x}{\lambda_T} \right)^2 \right) \right] \qquad (3.35)$$

where $\lambda_T = u_o \tau_T$. We can define a local Nusselt number as

$$\mathcal{N}u_x = \frac{x \dot{q}_w}{k(T_w - T_o)}$$

$$= \pi^{-1/2} P_r^{1/2} Re_x^{1/2} \left[1 + \kappa \frac{c_s}{c_p} \frac{x}{\lambda_T} + O\left(\left(\frac{x}{\lambda_T} \right)^2 \right) \right] \qquad (3.36)$$

The steepening effect of the particle-fluid interaction is, of course, determined by the thermal interaction parameter $\kappa c_s/c_p$ which is a measure of the extent of relative temperature changes during the process of interaction. The zeroth order term is the solution of the classical Rayleigh problem with the "frozen" thermal diffusivity $\chi = \nu/P_r$. The steepening effect of the particles on the fluid temperature profile is then to confine the extent of fluid heat diffusion closer to the plate itself, and the thermal boundary-layer thickness behaves like:

$$\delta_T(x) \sim \sqrt{\frac{\pi \nu x}{u_o}} P_r^{-1/2} \left[1 - \kappa \frac{c_s}{c_p} \frac{x}{\lambda_T} + O\left(\left(\frac{x}{\lambda_T} \right)^2 \right) \right] \qquad (3.37)$$

which is thinned from the "frozen" layer thickness $\sqrt{\frac{\pi \nu x}{u_o}} P_r^{-1/2}$. In the thermal boundary-layer problem, the scaling of the distance from the leading edge is by λ_T , the temperature equilibration length; in the momentum boundary layer, it is by λ_V, the velocity equilibration length. Also, a measure of the relative temperature changes in the former is by the thermal interaction parameter $\kappa \frac{c_s}{c_p}$; in the latter, the measure of the relative acceleration or deceleration gained by the fluid and particles is by the momentum interaction parameter κ. Hence, physically, the temperature and the velocity equilibration processes are not expected to have the possibility of being similar unless $\kappa = \kappa \, c_s/c_p$ and $\lambda_V = \lambda_T$. The ratio of the two boundary layer thicknesses is of the form which exhibits the dependence of the history of the interaction processes:

$$\frac{\delta_T}{\delta_V} \sim P_r^{-1/2}\left[1 + \left(\kappa \frac{x}{\lambda_V} - \kappa \frac{c_s}{c_p}\frac{x}{\lambda_T}\right) + \mathcal{O}\left(\left(\frac{x}{\lambda}\right)^2\right)\right] \tag{3.38}$$

When $\kappa = \kappa \, c_s/c_p$ and $\lambda_V = \lambda_T$, the ratio δ_T/δ_V reduces to a similar behavior as in the classical boundary-layer theory.

When $\frac{x}{\lambda_T} >> 1$, we have the opposite limiting case of nearly complete equilibrium. For the usual physical cases, $\frac{\lambda_T}{\lambda_V} = \mathcal{O}(1)$ and this implies $\frac{x}{\lambda_V} >> 1$ as well. In this case, then, one has the "equilibrium" thermal diffusivity $\bar{\chi} = \frac{\bar{\nu}}{\bar{P}_r} = \frac{\nu}{P_r}\frac{1}{1+\kappa c_s/c_p}$, and the equilibrium limit of the surface heat-transfer rate is

$$(\dot{q}_w)_e = \pi^{-1/2}\,\bar{P}_r^{\,1/2}\sqrt{\frac{u_0}{\bar{\nu}x}}\;k(T_w - T_0)$$

and the temperature boundary layer then grows parabolically in the manner $\sqrt{\pi\bar{\chi}x/u_0}$. The ratio of the two boundary layer thicknesses then behaves like $(\delta_T/\delta_V)_e \sim \bar{P}_r^{\,-1/2} = \sqrt{\bar{\nu}/\bar{\chi}}$, and $\bar{P}_r = \bar{\nu}/\bar{\chi}$ is the "equilibrium" Prandtl number. When $\bar{P}_r = 1$, the two layers in the equilibrium limit are then the same in thickness.

However, like the velocity boundary layer, here the particle temperature never actually attains the fluid temperature. The fluid temperature profile is stretched continuously through the process of heat diffusion which is the dominant mechanism of heat transfer in the near-equilibrium regime, with the thermal diffusivity $\overline{\chi}$. Here, the fluid temperature tends to behave very nearly like the local equilibrium profile. On the other hand, the particles are always hotter (here, we keep $T_W < T_o$ as discussed previously); hence, particularly near the plate, they tend to give rise to a fuller fluid-temperature profile near the wall. Thus the fluid temperature gradient near the wall exceeds the local equilibrium-temperature gradient. The local surface heat-transfer rate then approaches the final equilibrium limit from above:

$$\dot{q}_w = \frac{k(T_w - T_o)}{\sqrt{\frac{\nu x}{u_o}}} \pi^{-1/2} P_r^{1/2} \sqrt{1+\kappa\frac{c_s}{c_p}} \left[1 + \frac{1}{4}\frac{\kappa\frac{c_s}{c_p}}{1+\kappa\frac{c_s}{c_p}}\frac{\lambda_T}{x} + \mathcal{O}\left(\left(\frac{\lambda_T}{x}\right)^2\right)\right] \qquad (3.39)$$

and the local Nusselt number is

$$Nu_x = \pi^{-1/2} P_r^{1/2} \sqrt{1+\kappa\frac{c_s}{c_p}}\, Re_x^{1/2} \left[1 + \frac{1}{4}\frac{\kappa\frac{c_s}{c_p}}{1+\kappa\frac{c_s}{c_p}}\frac{\lambda_T}{x} + \mathcal{O}\left(\left(\frac{\lambda_T}{x}\right)^2\right)\right] \qquad (3.40)$$

In a corresponding manner, the temperature boundary-layer thickness approaches the equilibrium behavior from below:

$$\delta_T(x) \sim \sqrt{\frac{\pi \nu x}{u_o}}\; P_r^{-1/2} \frac{1}{\sqrt{1+\kappa\frac{c_s}{c_p}}} \left[1 - \frac{1}{4}\frac{\kappa\frac{c_s}{c_p}}{1+\kappa\frac{c_s}{c_p}}\frac{\lambda_T}{x} + \mathcal{O}\left(\left(\frac{\lambda_T}{x}\right)^2\right)\right] \qquad (3.41)$$

The comparison with the velocity boundary-layer behavior near the equilibrium limit gives the history depending ratio:

$$\frac{\delta_T}{\delta_V} \sim P_r^{-1/2}\sqrt{\frac{1+\kappa}{1+\kappa\frac{c_s}{c_p}}}\left[1+\frac{1}{4}\left(\frac{\lambda_V}{x}\frac{\kappa}{1+\kappa}-\frac{\lambda_T}{x}\frac{\kappa\frac{c_s}{c_p}}{1+\kappa\frac{c_s}{c_p}}\right)+\mathcal{O}\left(\left(\frac{\lambda}{x}\right)^2\right)\right] \quad (3.42)$$

When $\kappa=\kappa\frac{c_s}{c_p}$, $\lambda_V=\lambda_T$, then $\delta_T/\delta_V \sim P_r^{-1/2}$. The behavior of $\dot{q}_W(x)$ and of $\delta_T(x)$ is qualitatively like $\tau_W(x)$ and $\delta_V(x)$ as shown in Figures 6 and 7, respecitvely, except that the scale is modified by λ_T instead of λ_V.

APPENDIX III-A

Consideration of a Contour Integral

Here we will consider the inversion integral as obtained in paragraph 4.1:

$$u'(y',t') = \frac{1}{2\pi i}\int_{L_1} \frac{1}{s}\, e^{st' - W(s)\,y'}\, ds$$

where

$$W(s) = \sqrt{\frac{s(s+1)}{s+\frac{1}{1+k}}}$$

is a double-valued function, and a pole of order one at $s=0$ exists in the integrand. We can rewrite the function $W(s)$ in the form

$$W(s) = \sqrt{\frac{\rho_1\rho_3}{\rho_2}}\ \exp\left\{ i\ \frac{\theta_1-\theta_2+\theta_3}{2}\right\}$$

which has branch points located at $s=0$, $-1/_{1+k}$, and -1. Since $k>0$, the point $-1/_{1+k}$ lies between 0 and -1. It can be shown that $s=-\infty$ is a branch point also by taking the arbitrary point s_0 to circumscribe all three points 0, $-1/_{1+k}$, -1 according to the sense of the angles shown in Figure 8, and the value of $W(s)$ itself changes. The angles θ_1, θ_2, and θ_3 vary from 0 to 2π in the sense shown. The branch cuts for the purpose of evaluating the contour integral are: a cut connecting 0 and $-1/_{1+k}$, and a cut connecting -1 and $-\infty$, all on the negative real axis. It can easily be verified that $W(s)$ is then made single-valued by these branch cuts.

A remark concerning the sense of θ_3 is appropriate. The function $W(s)$ as $\theta_3 \to \pi^+$ above the cut will be denoted by $W_{\pi^+}(s_r)$, and as $\theta_3 \to \pi^-$ below the cut by $W_{\pi^-}(s_r)$. As we cross the cut, $W(s)$ on the lower side of the branch between -1 and $-\infty$ is obtained by analytic continuation, namely, $W_{\pi^-}(s_r) = -W_{\pi^+}(s_r)$.

Consider now the closed contour shown in Figure 9 where the large circular arc of radius $R > 1$ with center at the origin intersects AA' and is open at CC'. Since the integrand $\frac{1}{s} e^{st' - W(s)y'}$ is analytic in the region enclosed by the contour, then by Cauchy's theorem, the integral taken around the entire contour vanishes. Furthermore, on the circular arcs $\left| \frac{1}{s} e^{-W(s)y'} \right| < \left| \frac{1}{s} \right|$, the contribution of the integral on the arcs AB, BC, and $C'B'$, $B'A'$ vanishes $(t' > 0)$ as $R \to \infty$ (see Carslaw and Jaeger[29]). Hence, as $R \to \infty$ the Bromwich path AA' can be replaced by the equivalent paths $C'D'DC$ and $E'F'FE$ in the opposite sense shown in Figure 9.

Consider first the integral taken over path $C'D'DC$. The integral around the small circle at $s = -1$ of radius r_{-1} is of the order of r_{-1} and vanishes as $r_{-1} \to 0$. Then the integral over $C'D'DC$ is contributed from the sum of the integrals taken over DC and over $C'D'$ as $r_{-1} \to 0$ and is found to be

$$\frac{1}{2\pi i} \int_{C'D'DC} \frac{1}{s} e^{st' - W(s)y'} ds = -\frac{1}{\pi} \int_{1}^{\infty} e^{-\alpha t'} \sin\left(\sqrt{\frac{\alpha(\alpha-1)}{\alpha - \frac{1}{1+K}}}\, y' \right) \frac{d\alpha}{\alpha}$$

Consider now the integral taken over the path $E'F'FE$. The integral around the small circle at $s = -\frac{1}{1+K}$ of radius $r_{-\frac{1}{1+K}}$ is again of the order of $r_{-\frac{1}{1+K}}$ and vanishes as we take $r_{-\frac{1}{1+K}} \to 0$. The integral around the small circle at the origin gives a contribution of

$\frac{1}{2\pi i}(2\pi i) = 1$ as $r_0 \to 0$ from the pole of order one at $s = 0$. Finally, the sum of the integrals taken along FE and along $E'F'$, together with the contribution of the pole of order one at $s=0$, gives

$$\frac{1}{2\pi i}\int_{E'F'FE} \frac{1}{s} e^{st' - W(s) y'} ds = 1 - \frac{1}{\pi}\int_0^{\frac{1}{1+\kappa}} e^{-\alpha t'} \sin\left(\sqrt{\frac{\alpha(1-\alpha)}{\frac{1}{1+\kappa} - \alpha}}\, y'\right)\frac{d\alpha}{\alpha}$$

If we make the substitution $\alpha = \beta^2$ for the variable of integration, then finally, the desired solution may be written in the form as equation (3. 9) of paragraph 4. 1.

The vanishing of the particle concentration, i. e., $\kappa \to 0$, implies that the branch point $-1/_{1+\kappa}$ moves closer and closer towards -1, and as κ becomes identically zero the two points coincide and neutralize each other as branch points. The cut is then from 0 to $-\infty$ and we recover the classical Rayleigh solution.

APPENDIX III-B

Asymptotic Approximations for Large t/τ_v

Here, we will obtain approximate representations directly from the exact solution obtained in section 4 when t/τ_v is large for quantities such as the fluid velocity u', particle velocity u'_p, and the shearing stress at the wall τ_w. The exact solutions are given in terms of definite integrals of the form

$$\int_0^{\beta_1} e^{-\beta^2 t'} f(\beta; y')\, d\beta$$

in which t/τ_v enters as a (large) parameter. β is real and positive. When $t/\tau_v >> 1$, the main contribution to the value of the integral comes from the immediate vicinity of $\beta = 0$. However, here $f(\beta; y')$ involves y' as a parameter; the appropriate consideration will be discussed when the actual approximations are made in B. 1 and B. 2 . Here, it suffices to say that $f(\beta; y')$ is then expanded, in the manner appropriate, in the nieghborhood of $\beta = 0$ and the series is then integrated term by term with the upper limit taken to ∞. The change in the upper limit contributes only an exponentially small error to the value of the integral. The resulting approximation is in the form of an asymptotic series in inverse powers of (t/τ_v). This is essentially the spirit of Watson's lemma, and according to H. Jeffreys[32], its remarkable property is that it does not require specific conditions for its truthfulness provided that the result is, and in our case physically, meaningful.

B. 1 Fluid Velocity Profile

Let us consider u' as a function of t' and of

$$\bar{\eta} = \frac{y'}{2\sqrt{t'}}\sqrt{1+\kappa} = \frac{y}{2\sqrt{\bar{\nu}t}}$$

where $\bar{\nu} = \nu/_{1+\kappa}$. The similarity variable $\bar{\eta}$ is appropriately defined in terms of the "equilibrium" kinematic viscosity $\bar{\nu}$. The integral to be considered from paragraph 4. 1, equation (3. 9), is then of the form

$$\frac{2}{\pi}\int_0^{\frac{1}{\sqrt{1+\kappa}}} e^{-\beta^2 t'} \sin\left(\sqrt{\frac{1-\beta^2}{1-(1+\kappa)\beta^2}}\, 2\bar{\eta}\beta\sqrt{t'}\right)\frac{d\beta}{\beta} .$$

There are several ways to approximate the sine function, but all require specifying the order of $\bar{\eta}$. Since the effective diffusion region is of the order $y \sim \sqrt{\bar{\nu} t}$, it is then appropriate to take $\bar{\eta} = \mathcal{O}(1)$ in this case. Now for $t/\tau_\nu >> 1$, the main contribution comes from β near zero, but we now need to consider the order of the factor $(\beta\sqrt{t'})$ before approximating the sine function. When $\beta\sqrt{t'} >> 1$ the sine function oscillates rapidly but is nevertheless submerged by the overwhelmingly small exponential factor $e^{-\beta^2 t'}$. The sine function behaves smoothly when $\beta\sqrt{t'} << 1$, and a Taylor series expansion can be made. However, contributions to the value of the integral when $\beta\sqrt{t'} = \mathcal{O}(1)$ are also important. Hence we will approximate the sine function by taking $\beta\sqrt{t'} \leq \mathcal{O}(1)$, $\bar{\eta} \leq \mathcal{O}(1)$ and $\beta \to 0$:

$$\frac{1}{\beta} \sin\left(\sqrt{\frac{1-\beta^2}{1-(1+\kappa)\beta^2}}\, 2\bar{\eta}\,(\beta\sqrt{t'})\right) = \frac{1}{\beta} \sin\left(2\bar{\eta}\,(\beta\sqrt{t'})\left[1+\frac{\kappa}{2}\beta^2 + \mathcal{O}(\beta^4)\right]\right)$$

$$= \frac{1}{\beta}\sin(2\bar{\eta}(\beta\sqrt{t'})) + \kappa\bar{\eta}\,\beta(\beta\sqrt{t'})\cos(2\bar{\eta}(\beta\sqrt{t'})) + \mathcal{O}(\beta^3)$$

One then obtains the well-known error function from first term,

$$\frac{2}{\pi}\int_0^\infty e^{-\beta^2 t'} \sin(2\bar{\eta}(\beta\sqrt{t'}))\frac{d\beta}{\beta} = \operatorname{erf}\bar{\eta}$$

The second term is, from Gröbner and Hofreiter(33),

$$\kappa\bar{\eta}\,\frac{2}{\pi}\int_0^\infty \beta^2(\beta\sqrt{t'})e^{-\beta^2 t'}\cos(2\bar{\eta}(\beta\sqrt{t'}))\frac{d\beta}{\beta} = -\kappa\bar{\eta}\,\frac{\pi^{-1/2}}{4}\,e^{-\bar{\eta}^2}\,H_2\!\left(\bar{\eta}\sqrt{t'}, \frac{1}{t'}\right)$$

where the Hermite polynomial H_2 is

$$H_2(\bar{\eta}\sqrt{t'},\frac{1}{t'}) = e^{\bar{\eta}^2}\frac{d^2 e^{-\bar{\eta}^2}}{d(\bar{\eta}\sqrt{t'})^2}$$

$$= \frac{2}{t'}(2\bar{\eta}^2 - 1)$$

Finally, the fluid velocity profile when $t/\tau_v >> 1$ appears in the form

$$u'(\bar{\eta},t') = u'_{e_0}(\bar{\eta}) + \frac{\tau_v}{t}u'_{e_1}(\bar{\eta}) + \mathcal{O}((\frac{\tau_v}{t})^2)$$

where

$$u'_{e_0}(\bar{\eta}) = \text{erfc}\,\bar{\eta}$$

and

$$u'_{e_1}(\bar{\eta}) = -\frac{\kappa}{1+\kappa}\frac{\pi^{-1/2}}{2}\bar{\eta}(1-2\bar{\eta}^2)e^{-\bar{\eta}^2}$$

It is noted that $\frac{1+\kappa}{\kappa}u'_{e_1}(\bar{\eta})$ is a universal function of $\bar{\eta}$ and explicitly independent of κ.

B. 2 Particle Velocity Profile

The various aspects discussed in B. 1 concerning the approximation for u' also apply here, namely, that $\bar{\eta} \leq \mathcal{O}(1)$, $(\beta\sqrt{t'}) \leq \mathcal{O}(1)$, and $\beta \to 0$. The integral to be considered from paragraph 4. 2, equation (3. 15), is of the form:

$$\frac{2}{\pi}\int_0^{\frac{1}{\sqrt{1+\kappa}}} e^{-\beta^2 t'}\sin\left(\sqrt{\frac{1-\beta^2}{1-(1+\kappa)\beta^2}}\,2\bar{\eta}\,(\beta\sqrt{t'})\right)\frac{d\beta}{\beta(1-(1+\kappa)\beta^2)}$$

We expand

$$\frac{1}{\beta(1-(1+\kappa)\beta^2)} = \frac{1}{\beta}\left[1+(1+\kappa)\beta^2 + \mathcal{O}(\beta^3)\right]$$

and similarly expand the sine function as in B. 1. Hence,

$$\frac{1}{\beta(1-(1+K)\beta^2)}\sin\left(\sqrt{\frac{1-\beta^2}{1-(1+K)\beta^2}}\,2\bar{\eta}(\beta\sqrt{t'})\right) = \frac{1}{\beta}\sin(2\bar{\eta}(\beta\sqrt{t'})) + K\bar{\eta}\beta(\beta\sqrt{t'})\cos(2\bar{\eta}(\beta\sqrt{t'}))$$
$$+ (1+K)\beta\sin(2\bar{\eta}(\beta\sqrt{t'}))$$
$$+ \mathcal{O}(\beta^3)$$

It is recognized that the first two terms are those obtained for u' in B. 1 . The integral involving the third term is obtained from Gröbner and Hofreiter[33]:

$$(1+K)\frac{2}{\pi}\int_0^\infty e^{-\beta^2 t'}\beta\sin(2\bar{\eta}(\beta\sqrt{t'}))\,d\beta = \frac{\tau_v}{t}\bar{\eta}\,\Gamma(\tfrac{3}{2})\,\mathcal{F}(\tfrac{3}{2},\tfrac{3}{2};-\bar{\eta}^2)$$

where $\Gamma(\frac{3}{2}) = \frac{\pi^{-1/2}}{2}$. The confluent hypergeometric function $\mathcal{F}(\frac{3}{2},\frac{3}{2};-\bar{\eta}^2)$ in this case is recognized to be simply the representation for the exponential function[34]:

$$\mathcal{F}(\tfrac{3}{2},\tfrac{3}{2};-\bar{\eta}^2) = \sum_{n=0}^{\infty}\frac{(-\bar{\eta}^2)^n}{n!}$$
$$= e^{-\bar{\eta}^2}$$

Finally, we will write the particle velocity u_p' in terms of the slip velocity $(u'-u_p')$, combining the asymptotic forms of u' and of u' for $t/\tau_v >> 1$, obtaining

$$u'-u_p' = e^{-t'} + \frac{\tau_v}{t}\,g_{e_1}(\bar{\eta}) + \mathcal{O}((\tfrac{\tau_v}{t})^2)$$

where $g_{e_1}(\bar{\eta}) = \frac{\pi^{-1/2}}{2}\bar{\eta}\,e^{-\bar{\eta}^2}$ is a universal function of $\bar{\eta}$.

B. 3 Shear Stress at the Plate

The integral to be considered from paragraph 4. 3, equation (3. 22),is of the form:

$$\frac{2}{\pi}\int_0^{\frac{1}{\sqrt{1+K}}} e^{-\beta^2 t'} \sqrt{\frac{1-\beta^2}{1-(1+K)\beta^2}}\,\sqrt{1+K}\,d\beta$$

A simple and direct application of Watson's lemma can be made. The Taylor expansion about β of the function in the integrand is:

$$\sqrt{\frac{1-\beta^2}{1-(1+K)\beta^2}} = 1+\frac{K}{2}\beta + K\left(1+\frac{3}{4}K\right)\beta^3 + \mathcal{O}(\beta^4)$$

and the resulting integration by taking the upper limit to ∞ gives the asymptotic form for the shear stress from equation (3. 22) for $t/\tau_v >> 1$ as

$$\frac{c_f}{2} = \frac{\tau_w}{\rho u_0^2} = \pi^{-1/2}\sqrt{\frac{\nu}{u_0^2 t}}\,\sqrt{1+K}\left[1+\frac{1}{4}\frac{K}{1+K}\left(\frac{\tau_v}{t}\right)+\frac{3}{4}\frac{K}{1+K}\left(1+\frac{3}{4}K\right)\left(\frac{\tau_v}{t}\right)^2+\mathcal{O}\left(\left(\frac{\tau_v}{t}\right)^3\right)\right].$$

APPENDIX III-C

Asymptotic Approximations for Small t/τ_v

C. 1 Fluid Velocity Profile

When $t/\tau_v << 1$ we obtain the relevant approximations directly from the transformed quantities by keeping st' fixed and expanding in powers of t'. Here, as in Appendix III-B, we keep $y' \sim \sqrt{t'}$ fixed, or namely, $\eta = y'/2\sqrt{t'} \doteq \mathcal{O}(1)$.

Hence we may write the fluid velocity from paragraph 4. 1, in the transformed plane, as

$$U(\eta,s) = \frac{t'}{\zeta} \exp\left\{-2\eta \sqrt{\frac{\zeta(1+t'/\zeta)}{1+\frac{1}{1+\kappa}t'/\zeta}}\right\}$$

where $\zeta = st'$. Then for $t' \ll 1$, $\zeta \leq \mathcal{O}(1)$, $\eta \leq \mathcal{O}(1)$:

$$U(\eta,s) = \frac{t'}{\zeta}\, e^{-2\eta\sqrt{\zeta}\left[1+\frac{1}{2}\frac{\kappa}{1+\kappa}\frac{t'}{\zeta}+\mathcal{O}(t'^2)\right]}$$

$$= \frac{t'}{\zeta} e^{-2\eta\sqrt{\zeta}} - \frac{1}{2}\frac{\kappa}{1+\kappa} 2\eta \frac{t'^{3/2}}{\zeta^{3/2}} e^{-2\eta\sqrt{\zeta}} + \mathcal{O}(t'^2)$$

$$= \frac{1}{s} e^{-y'\sqrt{s}} - \frac{1}{2}\frac{\kappa}{1+\kappa}\frac{1}{s^{3/2}} e^{-y'\sqrt{s}} + \mathcal{O}(s^{-2})$$

The inverse transformation gives

$$u'(\eta,t') = u'_{f_0}(\eta) + \frac{t}{\tau_v} u'_{f_1}(\eta) + \mathcal{O}\left(\left(\frac{t}{\tau_v}\right)^2\right)$$

where $u'_{f_0}(\eta) = \operatorname{erfc}\eta$ is the classical Rayleigh solution with $\eta = y/2\sqrt{\nu t}$ and $u'_{f_1}(\eta) = -\frac{\kappa}{2}\left[4\pi^{-1/2}\eta e^{-\eta^2} - 4\eta^2 \operatorname{erfc}\eta\right]$ is a negative quantity. $\frac{1}{\kappa}u'_{f_1}(\eta)$ is a universal function of η .

C. 2 Particle Velocity Profile

From paragraph 4. 2, the transformed particle velocity profile may be approximated in a similar manner:

$$U_p(y',s) = \frac{t'}{\zeta}\left[1-(1+\kappa)\frac{t'}{\zeta} + \mathcal{O}(t'^2)\right] U(y',s)$$

$$= \frac{1}{s^2} e^{-y'\sqrt{s}} - \kappa y' \frac{1}{s^{5/2}} e^{-y'\sqrt{s}} + \mathcal{O}(s^{-3})$$

with $\eta = y'/2\sqrt{t'} \leq \mathcal{O}(1)$. The inverse transform then becomes

$$u'_p(\eta, t') = \frac{t}{\tau_v} u'_{p_{f_1}}(\eta) + \vartheta\left(\left(\frac{t}{\tau_v}\right)^2\right)$$

where

$$u'_{p_{f_1}}(\eta) = \operatorname{erfc} \eta - \frac{1}{2}\left[4\pi^{-1/2}\eta e^{-\eta^2} - 4\eta^2 \operatorname{erfc} \eta\right]$$

is a universal function of η, independent of κ.

C. 3 Shear Stress at the Plate

The function to be approximated becomes, from paragraph 4. 3,

$$\frac{1}{s} W(s) = \sqrt{\frac{s+1}{s\left(s + \frac{1}{1+\kappa}\right)}} = \frac{1}{s^{1/2}} + \frac{1}{2} \frac{\kappa}{1+\kappa} \frac{1}{s^{3/2}} + \vartheta\left(s^{-5/2}\right)$$

which gives

$$\frac{c_f}{2} = \frac{\tau_w}{\rho u_o^2} = \pi^{-1/2} \sqrt{\frac{\nu}{u_o^2 t}} \left[1 + \kappa \frac{t}{\tau_v} + \vartheta\left(\left(\frac{t}{\tau_v}\right)^2\right)\right].$$

IV. THE OSCILLATING INFINITE FLAT PLATE

1. General Discussion

Previously we have discussed impulsive motions of an infinite flat plate, or the Rayleigh problem, and brought out certain physical features of the laminar boundary layer. Here, we consider the infinite flat plate in periodic oscillations. This is the simplest problem which involves periodic boundary layers in the absence of a mean flow. The corresponding classical problem, being an exact solution of the Navier-Stokes equations, was first given by Stokes[23]. The periodic vorticity produced at the plate diffuses into the interior of the fluid and gives rise to a boundary layer whose thickness is of the order $\sqrt{\nu/\omega}$, where ω is the frequency of the plate motion. Hence, when the frequency of oscillation becomes larger, the viscous layer becomes confined closer to the plate. In particular, the amplitude of the fluid motion falls off in the interior of the fluid as $u_o e^{-\sqrt{\omega/2\nu}\, y}$, where u_o is the amplitude of plate motion. By the time that the vorticity generated at the plate at one instant is diffused to the outer layers of the fluid, the inner layers will have already responded to the vorticity generated at the plate at the next instant; consequently, the fluid at the layers far away lags behind the fluid in the layers near the plate. This lag is $\sqrt{\omega/2\nu}\, y$ with respect to the plate.

In particle-fluid mechanics, the questions that naturally arise are the modifications of the boundary layer thickness, fluid vorticity, and the shear stress at the plate due to the particle-fluid momentum interaction. Of course, the particle motions and their "slip" relative to the fluid are of interest also. The presence of the particle cloud on the viscous layer thickness is somewhat similar to the Rayleigh problem in that the inertia of the particles distorts the fluid profile through momentum interaction, and part of the momentum generated by the plate that would otherwise diffuse into the interior of the mixture now is absorbed by the particle cloud, thereby confining the periodic momentum produced at the wall in a region closer to the wall than in the absence of the particles. Since the particles do not follow the fluid motions precisely, it is then expected that the phase lag of the fluid layers with respect to the plate is augmented by the presence of the particle cloud. Of course, the distortion of the fluid velocity due to particle inertia gives rise to a relatively higher instantaneous shear stress at the plate.

The controlling parameter indicating the extent of particle-fluid equilibration is the fluid characteristic time, ω^{-1} , over the particle velocity equilibration time, τ_V . It is then physically clear that there are two near-limiting regimes characterized by $\omega^{-1}/\tau_V \ll 1$ which is the near-frozen regime, and by $\omega^{-1}/\tau_V \gg 1$ which is the near-equilibrium regime. This is in contrast to the Rayleigh problem, where the fluid characteristic time is the time t measured from the start of the impulsive plate motion. However, as in the Rayleigh problem, expansions about these two limiting regimes from the exact solution will provide suggestions for the approximate treatment of similar problems which are not amenable to an exact solution.

2. Fluid Velocity Profile

Let us now consider the problem of an infinite flat plate oscillating parallel to its own plane, with frequency ω, as shown in Figure 10, in a mixture of fluid and particles otherwise at rest. Our interest is in the induced motions as a result of the transmission of momentum produced at the plate into the interior of the mixture through the action of fluid viscosity and interaction between the fluid and the particle cloud. The differential equation, which is the exact Navier-Stokes equation in particle-fluid mechanics within the particular assumption of the particle-fluid interaction law, as derived in Chapter III, is repeated here for convenience:

$$\tau_v^* \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial t} - \bar{\nu} \frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad (4.1)$$

where $\tau_v^* = \tau_v / 1+K$ and $\bar{\nu} = \nu / 1+K$. We restrict our attention to the $y > 0$ plane. The motion of the plate is given as $u(0,t) = u_0 \cos \omega t = R\left(u_0 e^{i\omega t}\right)$, and $u \to 0$ as $y \to \infty$. For periodic oscillations, $u(y,t)$ is proportional to $R(e^{i\omega t})$ and a solution is given by the form

$$u(y,t) = R\left(u_0 e^{-\alpha y} e^{i\omega t}\right) \qquad (4.2)$$

Then equation (4.1) gives

$$\alpha^2 = X_1 + i Y_1 \qquad (4.3)$$

where we have denoted the real and positive quantities X_1 and Y_1 by:

$$X_1 = \frac{\omega}{\nu} \frac{(\omega \tau_v) K}{1 + (\omega \tau_v)^2} \quad ; \quad Y_1 = \frac{\omega}{\nu} \frac{(1+K) + (\omega \tau_v)^2}{1 + (\omega \tau_v)^2} \qquad (4.4)$$

Now the complex quantity α may be written as

$$\alpha = X_2 + i\,Y_2 \tag{4.5}$$

where X_2 and Y_2 are real quantities. The solution of equation (4.3) is then obtained by the elementary means of equating the real parts and imaginary parts of the same complex number:

$$X_1 = X_2^2 - Y_2^2 \quad ; \qquad Y_1 = 2 X_2 Y_2 \tag{4.6}$$

and it is found that

$$X_2 = \pm \frac{1}{\sqrt{2}} \left[X_1 \pm \sqrt{X_1^2 + Y_1^2} \right]^{1/2} \quad ; \quad Y_2 = \pm \frac{1}{\sqrt{2}} \left[-X_1 \pm \sqrt{X_1^2 + Y_2^2} \right]^{1/2} \tag{4.7}$$

Since $\sqrt{X_1^2 + Y_1^2} > X_1$ and both X_2 and Y_2 are real, we choose the positive sign under the root. For the signs outside the root, we choose the positive one for X_2 in the $y > 0$ plane, and this gives the solution which vanishes as $y \to \infty$ as physically required. Since $Y_1 = 2 X_2 Y_2 > 0$, this requires the same sign outside the root for Y_2 as that for X_2, namely, the positive one. Thus, finally,

$$\begin{aligned} X_2 &= + \sqrt{\frac{\omega}{2\nu}} \left\{ \frac{(\omega\tau_v)\kappa}{1+(\omega\tau_v)^2} + \sqrt{\frac{(1+\kappa)^2 + (\omega\tau_v)^2}{1+(\omega\tau_v)^2}} \right\}^{1/2} \\ Y_2 &= + \sqrt{\frac{\omega}{2\nu}} \left\{ - \frac{(\omega\tau_v)\kappa}{1+(\omega\tau_v)^2} + \sqrt{\frac{(1+\kappa)^2 + (\omega\tau_v)^2}{1+(\omega\tau_v)^2}} \right\}^{1/2} \end{aligned} \tag{4.8}$$

The solution for the fluid velocity profile is

$$\begin{aligned} u(y,t) &= R\left(u_0\, e^{-X_2 y}\, e^{i(\omega t - Y_2 y)} \right) \\ &= u_0\, e^{-X_2 y} \cos(\omega t - Y_2 y) \end{aligned} \tag{4.9}$$

As in the classical problem of Stokes[23], this result represents a transverse wave, one which propagates into the interior of the particle-fluid mixture from the plate in the direction perpendicular to the motion of the plate. The wave velocity in this case is ωY_2^{-1} and the wave length is $2\pi Y_2^{-1}$. The solution also exhibits the boundary layer property; the motion of the plate becomes insignificant when $X_2 y$ reaches some value. X_2^{-1} is that distance which the amplitude of u drops off by a factor of e and is known as the depth of penetration, of the order of the boundary layer thickness δ :

$$\delta \sim \sqrt{\frac{2\nu}{\omega}} \left\{ \frac{(\omega\tau_v)\kappa}{1+(\omega\tau_v)^2} + \sqrt{\frac{(1+\kappa)^2+(\omega\tau_v)^2}{1+(\omega\tau_v)^2}} \right\}^{-1/2} \tag{4.10}$$

Of course, when the particle concentration by mass vanishes and κ is identically zero, both X_2 and Y_2 become $\sqrt{\omega/2\nu}$ and we recover the classical solution

$$u = u_0 e^{-\sqrt{\frac{\omega}{2\nu}}\, y} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\, y\right)$$

with the boundary layer thickness $\delta \sim \sqrt{\frac{2\nu}{\omega}}$.

Although equation (4.9) for the fluid velocity profile is in a form suitable for numerical calculation for any value of the equilibration parameter ω^{-1}/τ_v, it is instructive to study the near-limiting cases. When $\omega^{-1}/\tau_v >> 1$, which corresponds to the "weak" particle-fluid interaction regime, or the near-equilibrium regime, one can expand X_2 and Y_2 as follows:

$$\begin{aligned} X_2 &= \sqrt{\frac{\omega}{2\nu}} \left[1 + \frac{1}{2}\frac{\kappa}{1+\kappa}\frac{\tau_v}{\omega^{-1}} + \mathcal{O}\left(\left(\frac{\tau_v}{\omega^{-1}}\right)^2\right) \right] \\ Y_2 &= \sqrt{\frac{\omega}{2\nu}} \left[1 - \frac{1}{2}\frac{\kappa}{1+\kappa}\frac{\tau_v}{\omega^{-1}} + \mathcal{O}\left(\left(\frac{\tau_v}{\omega^{-1}}\right)^2\right) \right] \end{aligned} \tag{4.12}$$

In this case, it is found that the fluid velocity profile can be expressed

$(\bar{\eta}, (\omega t-\bar{\eta}))$ and independent of κ. As in the Rayleigh problem, we note here that the local acceleration of the zeroth order equilibrium flow is proportional to $\frac{u_0}{\omega^{-1}} e^{-\bar{\eta}} \sin(\omega t - \bar{\eta})$ and we recover from the exact solution the useful concept that the first-order "slip" velocity is proportional to the local acceleration. This, again, is not surprising. The particle of mass m is subjected to an inertia force of order $m\, u_0/\omega^{-1}$, where ω^{-1} is the only characteristic inertia time. This must be balanced by the force exerted on the particle by the fluid, which is $m\, u_s/\tau_v$ and u_s is the "slip" velocity. Hence, $\frac{u_s}{u_0} \approx \frac{\tau_v}{\omega^{-1}}$. Thus, when $\frac{\omega^{-1}}{\tau_v} >> 1$, the "slip" velocity is small compared to the amplitude of plate motion. This is not unlike the similar feature recovered by the Rayleigh problem, which substantiates the approximation used by Marble[19] for the steady laminar boundary-layer problem. The similar feature here will then provide the suggestion concerning approximate considerations of those periodic boundary layers that are mathematically less accessible.

In the opposite situation, when $\frac{\omega^{-1}}{\tau_v} << 1$, the near-frozen regime, it is more convenient to write the particle velocity instead of the "slip" velocity as

$$u_p' = \frac{\omega^{-1}}{\tau_v} u'_{p_{f_1}}(\eta, (\omega t - \eta)) + \mathcal{O}\left(\left(\frac{\omega^{-1}}{\tau_v}\right)^2\right)$$

where

$$u'_{p_{f_1}}(\eta, (\omega t - \eta)) = e^{-\eta} \sin(\omega t - \eta) \tag{4.25}$$

and $\eta = y\sqrt{\frac{\omega}{2\nu}}$ is the similarity variable based on the frozen kinematic viscosity ν. The particle oscillations lag behind the fluid by $\pi/2$ in the first approximation. The amplitude of particle oscillation is simply proportional to ω^{-1}/τ_v and is nearly standing in the form

$$u'(\bar{\eta},(\omega t-\bar{\eta})) = u'_{e_0}(\bar{\eta},(\omega t-\bar{\eta})) + \frac{\tau_v}{\omega^{-1}} u'_{e_1}(\bar{\eta},(\omega t-\bar{\eta})) + \vartheta\left(\left(\frac{\tau_v}{\omega^{-1}}\right)^2\right)$$

when $\omega^{-1}/\tau_v >> 1$. Here $\bar{\eta} = y\sqrt{\frac{\omega}{2\bar{\nu}}}$ is the similarity variable based on the "equilibrium" kinematic viscosity $\bar{\nu}$. The zeroth order function is simply the Stokes solution[23] for the oscillating plate when the particle cloud and the fluid are moving together:

$$u'_{e_0}(\bar{\eta},(\omega t-\bar{\eta})) = e^{-\bar{\eta}} \cos(\omega t-\bar{\eta}) \tag{4. 13}$$

The first order function is

$$u'_{e_1}(\bar{\eta},(\omega t-\bar{\eta})) = -\frac{\kappa}{1+\kappa}\frac{1}{2}\bar{\eta}e^{-\bar{\eta}}\left[\sin(\omega t-\bar{\eta}) + \cos(\omega t-\bar{\eta})\right] \tag{4. 14}$$

As in the Rayleigh problem in Chapter III, $\frac{1+\kappa}{\kappa} u'_{e_1}$ is a universal function here of $\bar{\eta}$ for the amplitude and of $(\omega t-\bar{\eta})$ for the periodic part, and is independent of κ . The presence of the particle serves the purpose of augmenting the damping of the action of the plate, since the momentum generated by the plate that would otherwise diffuse into the interior of the mixture is now absorbed by particle inertia. Since the particle cloud lags behind, the fluid lag with respect to the plate is also augmented. This is more obviously seen when equation (4. 12) is directly substituted into equation (4. 9):

$$u' \cong e^{-\sqrt{\frac{\omega}{2\bar{\nu}}}\left(1+\frac{1}{2}\frac{\kappa}{1+\kappa}\frac{\tau_v}{\omega^{-1}}\right)y} \cos\left(\omega t - \sqrt{\frac{\omega}{2\bar{\nu}}}\,y + \sqrt{\frac{\omega}{2\bar{\nu}}}\,y\,\frac{1}{2}\frac{\kappa}{1+\kappa}\frac{\tau_v}{\omega^{-1}}\right) \tag{4. 15}$$

The boundary layer thickness is then

$$\delta \sim \sqrt{\frac{2\bar{\nu}}{\omega}}\left(1-\frac{1}{2}\frac{\kappa}{1+\kappa}\frac{\tau_v}{\omega^{-1}}\right) \tag{4. 16}$$

which approaches the equilibrium value $\sqrt{\frac{2\bar{\nu}}{\omega}}$ from below, similar to that discussed for the Rayleigh problem.

When $\omega^{-1}/\tau_v \ll 1$, which is the near-frozen regime, the time it takes the particle cloud to adjust itself to come into equilibrium with the local surrounding fluid is long compared with the time it takes the plate to go through one period of its motion. One can now expand X_2 and Y_2 as

$$X_2 = \sqrt{\frac{\omega}{2\nu}}\left[1 + \frac{K_v}{2}\frac{\omega^{-1}}{\tau_v} + \mathcal{O}\left(\left(\frac{\omega^{-1}}{\tau_v}\right)^2\right)\right]$$

$$Y_2 = \sqrt{\frac{\omega}{2\nu}}\left[1 - \frac{K_v}{2}\frac{\omega^{-1}}{\tau_v} + \mathcal{O}\left(\left(\frac{\omega^{-1}}{\tau_v}\right)^2\right)\right] \tag{4.17}$$

In this case, one expects that the fluid velocity can be expressed as

$$u'(\eta,(\omega t-\eta)) = u'_{f_0}(\eta,(\omega t-\eta)) + \frac{\omega^{-1}}{\tau_v}u'_{f_1}(\eta,(\omega t-\eta)) + \mathcal{O}\left(\left(\frac{\omega^{-1}}{\tau_v}\right)^2\right)$$

where $\eta = y\sqrt{\frac{\omega}{2\nu}}$ is the similarity variable based on ν and

$$u'_{f_0}(\eta,(\omega t-\eta)) = e^{-\eta}\cos(\omega t-\eta) \tag{4.18}$$

is the "frozen" Stokes' solution[23]. The first order function is

$$u'_{f_1}(\eta,(\omega t-\eta)) = -\frac{K_v}{2}\eta e^{-\eta}\left[\sin(\omega t-\eta)+\cos(\omega t-\eta)\right]. \tag{4.19}$$

$\frac{1}{K_v}u'_{f_1}$ is a universal function of $(\eta,(\omega t-\eta))$ independent of K_v. The boundary layer thickness, similar to the Rayleigh problem, now breaks away from its frozen value

$$\delta \sim \sqrt{\frac{2\nu}{\omega}}\left[1 - \frac{K_v}{2}\frac{\omega^{-1}}{\tau_v} + \mathcal{O}\left(\left(\frac{\omega^{-1}}{\tau_v}\right)^2\right)\right] \tag{4.20}$$

The transition from "frozen" to "equilibrium" is similar to the Rayleigh problem, except that the parameter here is ω^{-1}/τ_v instead of t/τ_v, and is shown in Figure 11.

3. Particle Velocity Profile

The momentum equation for the particle cloud is repeated here for convenience

$$\rho_p \frac{\partial u_p}{\partial t} = -\frac{\rho_p}{\tau_v}(u_p - u) \tag{4.21}$$

Since the fluid velocity is proportional to $R(e^{i\omega t})$, we set

$u_p = R(\tilde{u}_p(y)\, e^{i\omega t})$ and obtain

$$\begin{aligned} u_p &= R\left(u_o e^{-X_2 y} \cos\psi \; e^{i(\omega t - Y_2 y - \psi)}\right) \\ &= u_o e^{-X_2 y} \cos\psi \cos(\omega t - Y_2 y - \psi) \end{aligned} \tag{4.22}$$

It is noted that the particle lag $\psi = \tan^{-1}(\frac{\tau_v}{\omega^{-1}})$ is the same for all layers of y. This is what is physically expected, since the particles are non-interacting and respond only to the momentum exchange with the fluid along each streamline. It is of interest to consider the particle-fluid "slip" velocity, which is

$$u' - u_p' = e^{-X_2 y} \sin\psi \sin(\omega t - Y_2 y - \psi) \tag{4.23}$$

In the near-equilibrium regime, the approximation for the "slip" velocity when $\omega^{-1}/\tau_v \gg 1$ may be written in the form

$$u' - u_p' = \frac{\tau_v}{\omega^{-1}} g_{e_1}(\bar{\eta}, (\omega t - \bar{\eta})) + \mathcal{O}((\frac{\tau_v}{\omega^{-1}})^2)$$

where

$$g_{e_1}(\bar{\eta}, (\omega t - \bar{\eta})) = e^{-\bar{\eta}} \sin(\omega t - \bar{\eta}) \tag{4.24}$$

and $\bar{\eta} = y\sqrt{\frac{\omega}{2\bar{\nu}}}$ is the similarity variable based on the equilibrium kinematic viscosity $\bar{\nu}$. Here, g_{e_1} is a universal function of still when $\frac{\omega^{-1}}{\tau_v} << 1$, since in this case, the equilibration time is much longer than the fluid characteristic time ω^{-1} .

4. Shear Stress at the Plate

The frictional force exerted on a unit area of the plate is

$$\tau_W = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0}$$

Substituting equation (4. 4), and forming the coefficient of skin friction, we have

$$\frac{C_f}{2} = \frac{\tau_W}{\rho u_0^2} = R\left(\sqrt{X_2^2 + Y_2^2}\; e^{i(\omega t + \theta)} \right)$$

$$= \sqrt{\frac{\omega \nu}{u_0^2}} \left[\frac{(1+K)^2 + (\frac{\tau_v}{\omega^{-1}})^2}{1 + (\frac{\tau_v}{\omega^{-1}})^2} \right]^{\frac{1}{4}} \cos(\omega t + \theta) \tag{4. 26}$$

where $\theta = \tan^{-1}\left(\frac{Y_2}{X_2}\right)$. When the particles are absent and $K = 0$, θ becomes $\pi/4$ and we have

$$\frac{c_f}{2} = \sqrt{\frac{\omega \nu}{u_0^2}} \cos(\omega t + \frac{\pi}{4})$$

which is the classical expression for $c_f/2$, showing that the maximum of the shear stress occurs at intervals of 1/8 of a period behind the mean position of the plate. For the near equilibrium limit in which $\frac{\omega^{-1}}{\tau_v} >> 1$, then $\theta = \frac{\pi}{4} - \frac{1}{2}\frac{K}{1+K}\frac{\tau_v}{\omega^{-1}} + \vartheta((\frac{\tau_v}{\omega^{-1}})^2)$ and the additional lag $-\frac{1}{2}\frac{K}{1+K}\frac{\tau_v}{\omega^{-1}}$ due to the inertia of the particle cloud now renders the maximum of the shear stress to take place at intervals of $\left(\frac{1}{8} - \frac{1}{4\pi}\frac{K}{1+K}\frac{\tau_v}{\omega^{-1}}\right)$ of a period behind the mean position of the plate. Expanding the cosine function, we have

$$\frac{c_f}{2} = \frac{\tau_w}{\rho u_o^2} = \sqrt{\frac{\omega\nu}{u_o^2}}\sqrt{1+K_V}\left[\cos(\omega t+\frac{\pi}{4})+\frac{\tau_V}{\omega^{-1}}\frac{1}{2}\frac{K_V}{1+K_V}\sin(\omega t+\frac{\pi}{4})+\vartheta((\frac{\tau_V}{\omega^{-1}})^2)\right] \quad (4.27)$$

the first term of which corresponds to the equilibrium value with the modified density $(1+K_V)\rho$.

When $\frac{\omega^{-1}}{\tau_V} << 1$, $\theta = \frac{\pi}{4} - \frac{\omega^{-1}}{\tau_V}\frac{K_V}{2} + \vartheta((\frac{\omega^{-1}}{\tau_V})^2)$, and the maximum of the shear stress takes place at $\left(\frac{1}{8} - \frac{1}{4\pi}K_V\frac{\omega^{-1}}{\tau_V}\right)$ of a period behind the mean position of the plate, and

$$\frac{c_f}{2} = \frac{\tau_w}{\rho u_o^2} = \sqrt{\frac{\omega\nu}{u_o^2}}\left[\cos(\omega t+\frac{\pi}{4})+\frac{\omega^{-1}}{\tau_V}\frac{K_V}{2}\sin(\omega t+\frac{\pi}{4})+\vartheta((\frac{\omega^{-1}}{\tau_V})^2)\right] . \quad (4.28)$$

V. LINEARIZED SUPERSONIC FLOW

1. General Discussion

The pioneering work of Ackeret[35], von Kármán and N. B. Moore[36], and the lectures of Prandtl[37], von Kármán[38], and Busemann[39] at the Fifth Volta Congress for High Speed Aeronautics held in 1935 in Rome essentially established the foundations of the linearized theory of supersonic flow. One need only consult the two subsequent general lectures of von Kármán, the Tenth Wright Brothers Lectures of 1947[40], and the Fifth Guggenheim Memorial Lecture of 1958[41], to gain perception of the subsequent developments of the linearized theory of supersonic flow, which has now become an important branch of fluid mechanics. In considering supersonic flows in particle-fluid mechanics, it is then natural to inquire as to how the concepts of the linearized theory, which has seen such fruitful service in the classical fluid mechanics, may be extended to this new borderline area of fluid mechanics.

We have discussed in Chapter II the significance of "inviscid" but compressible flow. The appropriate conservation equations were also stated. These are the so-called Euler equations in particle-fluid mechanics, from which linearization is to be made leading to the formulation of the small perturbation theory. This is discussed in detail in section 2, where the corresponding Prandtl-Glauert equation in particle-fluid mechanics is obtained. This, in fact, has been derived by Marble[42] through the perturbation of a uniform stream familiar in aerodynamics, and applied to the two-dimensional steady flow over a wavy wall for both subsonic and supersonic free-stream velocities. On the other hand, Chu and Parlange[20] obtained the corresponding equation for acoustic propagation in particle-fluid mechanics in which the particles and gas are initially at rest and in thermal equilibrium. The latter is essentially the counterpart of Rayleigh's theory[43] of acoustics. Here, however, we begin from the equation of acoustics and invoke the well-known physical interpretation of the linearization process in the classical high-speed, thin-airfoil theory that, in the reference frame fixed on an observer moving with the airfoil, the flow is described by the equation of acoustic propagation. We obtain the fundamental equations for thin-airfoil theory in particle-fluid mechanics. These include the equation for non-uniform motion, the equation for harmonic motion, and the equation for steady motion in the reference frame fixed on the airfoil. The last is the corresponding Prandtl-Glauert equation, and the alternate derivation of this essentially connects the acoustics and aerodynamic concepts of small perturbation theory in particle-fluid mechanics. This idea is emphasized by Sears[44] in his discussion of the classical theory of small perturbations. We also discuss in section 2, in terms of the small perturbation theory, the role of entropy and the existence of the velocity potential for, respectively, the gas and particles. Some limiting cases are discussed, and finally, in order that the linearized theory can be of fruitful service, one must be aware of and hence discuss its limitations or shortcomings.

In section 3, the two-dimensional, steady supersonic flow, or the Ackeret problem, is discussed in detail. The consideration is made from the exact form of the Prandtl-Glauert equation derived in section 2. The wave structure, particularly the far-field behavior, is deduced. An exact form of the pressure coefficient is obtained from which the aerodynamic forces and moments could be obtained when the airfoil shape is specified.

The consideration of two-dimensional, linearized supersonic flow with the additional restriction and simplification of small particle-fluid density ratio ($\kappa \ll 1$) is given in Chapter VI.

2. Small Perturbation Theory

Prior to obtaining the equations for thin-airfoil theory, some preliminary discussions are desired which will enhance our later considerations. The first of these concerns the role of entropy production, if any, in the small perturbation theory and the question of its relevance in the problem of drag. The second of these demonstrates an important theorem due to Marble[42] and its application to the small perturbation theory, and concerns the possibility of expressing the perturbation velocities of the gas and of the particle cloud as gradients of their respective potential functions. The existence of the potential functions is apparently overlooked in Chu and Parlange's[20] consideration of the acoustic situation. We first quote the equations for the small perturbations about a gas and particle cloud at rest and in thermal equilibrium, denoting the stationary coordinate system as $(\tilde{x}_i, \tilde{t})$ or as $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$. The original undisturbed value of the gas pressure is p_o , gas density ρ_o , and the original temperature of the gas and particle cloud is T_o ; the undisturbed particle-cloud density is ρ_{p_o} . We denote the first order departures from equilibrium by $p' = p - p_o$, $\rho' = \rho - \rho_o$, $\rho_p' = \rho_p - \rho_{p_o}$, $T' = T - T_o$, and $T_p' = T_p - T_o$. Neglecting higher order terms in the perturbation quantities and their derivatives, then the Euler equations given in section 4 of Chapter II take the following form.

The continuity equation for the gas becomes

$$\frac{\partial \rho'}{\partial \tilde{t}} + \rho_o \frac{\partial v_i}{\partial \tilde{x}_i} = 0 \qquad (5.1)$$

and the momentum for the gas becomes

$$\rho_o \frac{\partial v_i}{\partial \tilde{t}} = - \frac{\partial p'}{\partial \tilde{x}_i} + \frac{\rho_{p_o}}{\tau_{v_o}} (v_{p_i} - v_i) \qquad (5.2)$$

with the use of the linear momentum interaction law. The energy equation for the gas in terms of the static enthalpy $h = c_p T$ is:

$$\rho_o c_p \frac{\partial T'}{\partial \tilde{t}} = \frac{\partial p'}{\partial \tilde{t}} + \frac{\rho_{p_o}}{\tau_{T_o}} c_s (T_p' - T) \qquad (5.3)$$

with the use of the linear thermal interaction law, and the equation of state

$$\frac{p'}{p_o} = \frac{\rho'}{\rho_o} + \frac{T'}{T_o} \qquad (5.4)$$

The corresponding small perturbation forms of the conservation equations for the particle cloud become:

the continuity equation is

$$\frac{\partial \rho_p'}{\partial \tilde{t}} + \rho_{p_o} \frac{\partial v_{p_i}}{\partial \tilde{x}_i} = 0 \qquad (5.5)$$

the particle-cloud momentum equation is

$$\rho_{p_0} \frac{\partial v_{p_i}}{\partial \bar{t}} = - \frac{\rho_{p_0}}{\tau_{v_0}} (v_{p_i} - v_i) \qquad (5.6)$$

and the energy equation is

$$\rho_{p_0} c_s \frac{\partial T_p'}{\partial \bar{t}} = - \frac{\rho_{p_0}}{\tau_{T_0}} c_s (T_p' - T) \qquad (5.7)$$

The small perturbation equations in their individual forms will facilitate our subsequent discussions.

2.1. <u>The Role of Entropy.</u> - It is particularly important to discuss the mechanism for net entropy production, if at all, in terms of the small perturbation theory. This will clarify the situation of whether one could attribute a rise in drag of an obstacle to mechanisms for entropy increase.

There are, in general, two sources for the net production of entropy when there are particle-fluid momentum and thermal interactions present. The first of these is the work done on the fluid as a result of the particle-fluid equilibration process, which contributes to the increase in the fluid entropy per unit volume, and is

$$\frac{1}{T} (v_{p_i} - v_i) F_{p_i}$$

as given by the second term on the right hand side of equation (2.20), and this also contributes to a net entropy increase of the mixture. Since $F_{p_i} = \frac{\rho_p}{\tau_v} (v_{p_i} - v_i)$ and $\tau_v > 0$, it is a positive-definite quantity. However, in the first-order small-perturbation theory, this dissipative work is absent, and it enters into a second order consideration only.

Now Q_p is the local heat transferred to a unit volume of fluid from the particle cloud in that volume, which contributes to an increase or decrease of the fluid entropy per unit volume depending on whether the fluid temperature is lower or higher than the local temperature of the particle cloud. This is given by the first term on the right side of equation (2.20), and normally, the heat transfer to the fluid is received by the fluid at its local temperature T ; the local increase or decrease of fluid entropy per unit volume is Q_p/T . Similarly, the local decrease or increase of the particle cloud entropy per unit volume is $-Q_p/T_p$; the transfer of heat to or from the particle cloud takes place at its local temperature T_p . In order to fix our ideas, suppose now $T > T_p$ and locally heat is transferred from the fluid to the particle cloud. The transference of heat from the fluid takes place at T , and is received by the particle cloud at a lower temperature, T_p . In this process, net entropy production results. However, in the small perturbation theory, the transfer process takes place at the same temperature T_o , and the mechanism for net entropy production is absent and enters only in a second order consideration.

Denoting the first-order heat transfer by Q_p' , the local entropy increase or decrease of the fluid obeys the equation

$$\rho_o \frac{\partial s'}{\partial \tilde{t}} = \frac{Q_p'}{T_o}$$

for the acoustic situation in a gas and particle cloud originally at rest, obtained from equation (2.20). Similarly, for the particle cloud:

$$\rho_{p_0} \frac{\partial s_p'}{\partial \hat{t}} = - \frac{Q_p'}{T_0}$$

obtained from equation (2. 24).

In the first-order small-perturbation theory to follow, therefore, there is no net production of entropy. Hence, when a symmetrical, thin obstacle or thin airfoil is moving at supersonic speeds in a particle-fluid mixture, the resistance it experiences other than particle-boundary collisions, which are of second order importance, is the analogous "wave drag" in ordinary gas dynamics. There is, of course, no drag rise due to mechanisms for a net entropy increase, since such mechanisms are absent in the first-order small-perturbation theory regardless of subsonic or supersonic velocities. The expectation of the appearance of a drag rise of a nonlifting, symmetrical obstacle moving at subsonic velocities, for instance, is solely attributed to the destruction, due to particle-fluid interaction, of the symmetry of pressure distribution between the fore and aft sections on the obstacle.

Precisely, in the absence of net entropy production in the streamwise direction in the first-order small-perturbation theory, the periodic solution for the flow over a wave-shaped wall is possible(42).

2. 2. Velocity Potentials - In this section, we examine the possibility, in terms of the first-order small-perturbation theory, of expressing both the gas velocity and particle cloud velocity as the gradient of their respective potential functions. We proceed, again, from the case when the particle cloud and the gas are initially at rest and in thermal equilibrium, i. e., we proceed from the acoustical

situation.

The dynamical equation of the gas, equation (5.2), may be re-written in the vector form as:

$$\frac{\partial \vec{v}}{\partial \tilde{t}} = -\frac{1}{\rho_0}\tilde{\nabla} p' + \frac{\kappa}{\tau_{v_0}}(\vec{v}_p - \vec{v}) \tag{5.8}$$

where $\tilde{\nabla}(\ )$ is the gradient operator in the stationary coordinate system $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ and we have denoted $\kappa = \rho_{p_0}/\rho_0$ which is the initial undisturbed mass ratio of the particle cloud to the gas in a unit volume. Similarly, the dynamical equation of the particle cloud, equation (5.6), may be rewritten in the vector form as:

$$\frac{\partial \vec{v}_p}{\partial \tilde{t}} = -\frac{1}{\tau_{v_0}}(\vec{v}_p - \vec{v}) \tag{5.9}$$

Let us denote the gas vorticity vector as $\vec{\Omega}$ which indicates the intrinsic rotation of an element of the gas,

$$\vec{\Omega} = \tilde{\nabla} \times \vec{v} \tag{5.10}$$

Similarly, we introduce the analogous vorticity vector $\vec{\Omega}_p$ as indicating the intrinsic rotation of an element of the particle cloud as

$$\vec{\Omega}_p = \tilde{\nabla} \times \vec{v}_p \tag{5.11}$$

where $\tilde{\nabla}\times(\ )$ is the rotation operator in the stationary coordinate system $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$. If we take the $\tilde{\nabla}\times(\ )$ of both sides of equations (5.9) and (5.10), respectively, and noting that $\tilde{\nabla}\times(\tilde{\nabla} p') = 0$ we then arrive at

$$\frac{\partial \vec{\Omega}}{\partial \tilde{t}} = \frac{\kappa}{\tau_{v_0}}(\vec{\Omega}_p - \vec{\Omega}) \tag{5.12}$$

and

$$\frac{\partial \vec{\Omega}_p}{\partial \tilde{t}} = -\frac{1}{\tau_{v_0}}(\vec{\Omega}_p - \vec{\Omega}) \tag{5.13}$$

respectively. With these equations, the behavior of vorticity in the motion due to small disturbances in particle-fluid mechanics can now be studied. Combining equations (5.12) and (5.13), we see that $\vec{\Omega} + \kappa \vec{\Omega}_p =$ constant along the path following an element of the mixture. If at some point $\vec{\Omega} = \vec{\Omega}_p = 0$, then $\vec{\Omega} = 0$ and $\vec{\Omega}_p = 0$ along such paths. Hence, when motions originate from a state of uniform rest, then $\vec{\Omega} = \vec{\Omega}_p = 0$ subsequently everywhere. The gas velocity $\vec{v}$ and the particle cloud velocity $\vec{v}_p$ can then be represented as the gradient of their respective potentials:

$$\vec{v} = \tilde{\nabla} \phi \tag{5.14}$$

and

$$\vec{v}_p = \tilde{\nabla} \phi_p \tag{5.15}$$

This demonstrates an important theorem due to Marble[42]: for the case of small perturbation theory, the first-order perturbation potentials ϕ and ϕ_p exist. In our subsequent considerations, the potential functions will be used throughout, and the representation of other properties of the gas and of the particle cloud can be simply related through equations (5.1) to (5.7).

2.3. <u>The Equation for Acoustic Propagation.</u> - In this section we state the equations for acoustic propagation, which are obtained from equations (5.1) through (5.7):

$$\left[\frac{\partial^2}{\partial \tilde{t}^2}\left(\frac{\partial^2}{\partial \tilde{t}^2} - a_0^2 \tilde{\nabla}^2\right) + \left(\frac{1+\kappa}{\tau_{V_0}} + \frac{1+\gamma\kappa\frac{c_s}{c_p}}{\tau_{T_0}}\right)\frac{\partial}{\partial \tilde{t}}\left(\frac{\partial^2}{\partial \tilde{t}^2} - a_1^2 \tilde{\nabla}^2\right) + \frac{(1+\kappa)\left(1+\gamma\kappa\frac{c_s}{c_p}\right)}{\tau_{V_0}\tau_{T_0}}\left(\frac{\partial^2}{\partial \tilde{t}^2} - a_2^2 \tilde{\nabla}^2\right)\right]\phi = 0 \tag{5.16}$$

The form of the wave operator was obtained by Chu and Parlange[20] for the acoustical situation. However, they appear to have overlooked the existence of the potential functions. Here, we write the acoustic equation in terms of the velocity potential ϕ of the gas. $\tilde{\nabla}^2$ is the Laplacian operator in the stationary coordinate system $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$. The acoustic propagation speed in the undisturbed state of the gas, as if no particles were present, is

$$a_0^2 = \gamma \frac{p_0}{\rho_0} \tag{5.17}$$

and is appropriately called the "frozen" speed of sound. The acoustic propagation speed when the particle cloud and the gas are in ultimate equilibrium, after momentum and thermal equilibration processes have subsided, is interpreted as

$$a_2^2 = \bar{\gamma} \frac{\bar{p}}{\bar{\rho}} \tag{5.18}$$

where $\bar{\gamma}$ is the "equilibrium" heat capacity ratio and is correspondingly defined as

$$\bar{\gamma} = \frac{\rho_{p_0} c_s + \rho_0 c_p}{\rho_{p_0} c_s + \rho_0 c_v} = \frac{1 + \kappa \frac{c_s}{c_p}}{1 + \gamma \kappa \frac{c_s}{c_p}} \gamma \tag{5.19}$$

The equilibrium density of the mixture is simply $\bar{\rho} = \rho_0 + \rho_{p_0} = \rho_0 (1 + \kappa)$; the pressure of the mixture is still the same gas pressure p_0 , since the particle cloud does not contribute to a partial pressure. We then have the interpretation of the "equilibrium" speed of sound:

$$a_2^2 = \frac{1}{1+\kappa} \frac{1 + \kappa \frac{c_s}{c_p}}{1 + \kappa \gamma \frac{c_s}{c_p}} a_0^2 \tag{5.20}$$

The propagation speed associated with an intermediary wave is

$$a_1^2 = \frac{\frac{1}{\tau_{v_0}} + \frac{1}{\tau_{T_0}}\left(1+\kappa\frac{c_s}{c_p}\right)}{\frac{1+\kappa}{\tau_{v_0}} + \frac{1+\gamma\kappa\frac{c_s}{c_p}}{\tau_{T_0}}}\, a_0^2 \tag{5.21}$$

Since $\kappa, \frac{c_s}{c_p}, \gamma, \tau_{v_0}, \tau_{T_0}$ all are positive quantities, it is then observed that

$$a_0 > a_1 > a_2$$

The domain of dependence is characterized by the highest order wave.

The equation for the perturbation pressure p' follows from the dynamical equation of the gas

$$\rho_0 \frac{\partial \phi}{\partial \tilde{t}} = - p' + \frac{\kappa}{\tau_{v_0}}(\phi_p - \phi) \tag{5.22}$$

which is coupled through ϕ_p with the dynamical equation of the particle cloud

$$\frac{\partial \phi_p}{\partial \tilde{t}} = - \frac{1}{\tau_{v_0}}(\phi_p - \phi) \tag{5.23}$$

The forms of equations (5.22) and (5.23) are purposely retained here to exhibit their physical significance. Equations (5.16), (5.22), and (5.23) play the same role here as Rayleigh's[43] acoustic propagation equations in ordinary aerodynamics. That is, they describe those flows that are generated by small perturbations external to fluid-boundary layers and in the absence of shock waves. As in the ordinary aerodynamics, if shock waves exist, our approximations are valid provided that the shocks are weak. We shall accordingly obtain, in the following sections, the fundamental equations of the linear theory.

2.4. The Equation for Unsteady, Thin-Airfoil Theory. - We use the acoustic propagation equations of section 2.3 as the starting point, together with the well-known physical interpretation of the linearization process in the classical high-speed, thin-airfoil theory. This states that, in the reference frame fixed on an observer moving with the airfoil, the flow is described by the acoustic propagation equations. We then derive the appropriate equations for the linear theory in particle-fluid mechanics.

The thin obstacle is considered as a source of small disturbances, or acoustic disturbances, and the resulting flow field is built up of superpositions of such small disturbances. The flow field in the absence of the obstacle is a steady, uniform, parallel stream of velocity u_0 along the positive x-direction. The reference frame fixed on the obstacle will be represented by (x, y, z, t). The Galilean transformation fixing the reference frame on the obstacle is related to the stationary frame $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ by:

$$x = \tilde{x} + u_0 \tilde{t}, \quad y = \tilde{y}, \quad z = \tilde{z}, \quad t = \tilde{t} \tag{5.24}$$

and

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{y}}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial \tilde{z}}, \quad \frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \tag{5.25}$$

Of course, the corresponding gradient and rotation operators are related as

$$\tilde{\nabla} = \nabla, \qquad \tilde{\nabla} \times (\ ) = \nabla \times (\ ), \tag{5.26}$$

invariant in the transformation. In the moving reference frame, equation (5.16) then becomes

$$\left\{\left(\frac{1}{u_0}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^2\left[(1-M_0^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2\frac{M_0}{a_0}\frac{\partial^2}{\partial x\partial t}-\frac{1}{a_0^2}\frac{\partial^2}{\partial t^2}\right]\right.$$

$$+\left(\frac{1}{\lambda_v}+\frac{1+\kappa\frac{c_s}{c_p}}{\lambda_T}\right)\left(\frac{1}{u_0}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left[(1-M_1^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2\frac{M_1}{a_1}\frac{\partial^2}{\partial x\partial t}-\frac{1}{a_1^2}\frac{\partial^2}{\partial t^2}\right]$$

$$\left.+\frac{1+\kappa\frac{c_s}{c_p}}{\lambda_v\lambda_T}\left[(1-M_2^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2\frac{M_2}{a_2}\frac{\partial^2}{\partial x\partial t}-\frac{1}{a_2^2}\frac{\partial^2}{\partial t^2}\right]\right\}\phi=0 \qquad (5.27)$$

where we have introduced $M_0 = u_0/a_0$ as the "frozen" Mach number, $M_1 = u_0/a_1$ as the "intermediary" Mach number, and $M_2 = u_0/a_2$ as the "equilibrium" Mach number. Since $a_0 > a_1 > a_2$, then $M_0 < M_1 < M_2$. Here, the "zone of action" is determined by characteristics of the highest order wave. We also introduced the velocity equilibration distance $\lambda_v = \tau_{v_0} u_0$ and the temperature equilibration distance $\lambda_T = \tau_{T_0} u_0$.

The corresponding equations for the perturbation pressure from equations (5.22) and (5.23) are:

$$\left(\frac{1}{u_0}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\phi = -\frac{p'}{\rho_0 u_0}+\frac{\kappa}{\lambda_v}(\phi_p-\phi) \qquad (5.28)$$

and

$$\left(\frac{1}{u_0}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\phi_p = -\frac{1}{\lambda_v}(\phi_p-\phi) \qquad (5.29)$$

respectively. Equations (5.27), (5.28), and (5.29) constitute the fundamental equations for an unsteady thin-airfoil theory in particle-fluid mechanics. The form of the individual wave operators in equation (5.27) is the familiar one in ordinary aerodynamics of unsteady airfoil theory[45].

If the obstacle exerts harmonic oscillations with frequency ω while in a uniform stream with steady constant velocity u_o, the perturbation potentials are then of the form:

$$\phi(x,y,z,t) = e^{i\omega t}\ \Phi_s(x,y,z) \tag{5.30}$$

$$\phi_p(x,y,z,t) = e^{i\omega t}\ \Phi_{p_s}(x,y,z) \tag{5.31}$$

and similarly, the perturbation pressure is then of the form:

$$p'(x,y,z,t) = e^{i\omega t}\ P_s(x,y,z) \tag{5.32}$$

Hence the fundamental equations for harmonic motions of thin airfoil theory become

$$\begin{aligned}\Bigg\{&\left(\frac{i\omega}{u_o}+\frac{\partial}{\partial x}\right)^2\left[(1-M_o^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2i\omega\frac{M_o}{a_o}\frac{\partial}{\partial x}+\frac{\omega^2}{a_o^2}\right]\\ &+\left(\frac{1}{\lambda_v}+\frac{1+\kappa\frac{c_s}{c_p}}{\lambda_T}\right)\left(\frac{i\omega}{u_o}+\frac{\partial}{\partial x}\right)\left[(1-M_1^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2i\omega\frac{M_1}{a_1}\frac{\partial}{\partial x}+\frac{\omega^2}{a_o^2}\right]\\ &+\frac{1+\kappa\frac{c_s}{c_p}}{\lambda_v\lambda_T}\left[(1-M_2^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2i\omega\frac{M_2}{a_2}\frac{\partial}{\partial x}+\frac{\omega^2}{a_2^2}\right]\Bigg\}\Phi_s=0\end{aligned} \tag{5.33}$$

$$\left(\frac{i\omega}{u_o}+\frac{\partial}{\partial x}\right)\Phi_s=-\frac{P_s}{\rho_o u_o}+\frac{\kappa}{\lambda_v}\left(\Phi_{p_s}-\Phi_s\right) \tag{5.34}$$

$$\left(\frac{i\omega}{u_o}+\frac{\partial}{\partial x}\right)\Phi_{p_s}=-\frac{1}{\lambda_v}\left(\Phi_{p_s}-\Phi_s\right) \tag{5.35}$$

Unlike equation (5.27), equation (5.33) is hyperbolic only when $M_o > 1$, in which case, the "zone of action" is again determined by the characteristics of the highest order wave and is just the "frozen" Mach conoid of semivertex angle given by $\tan^{-1}(1/\sqrt{M_0^2-1})$.

2.5. The Equation for Steady Thin-Airfoil Theory. - When the motion is steady in the reference frame fixed on the obstacle, we then obtain the fundamental equation for steady flow in the linear theory in particle-fluid mechanics:

$$\left\{\frac{\partial^2}{\partial x^2}\left[(1-M_0^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]+\left(\frac{1}{\lambda_v}+\frac{1+\kappa\frac{c_s}{c_p}}{\lambda_T}\right)\frac{\partial}{\partial x}\left[(1-M_1^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]+\frac{1+\kappa\frac{c_s}{c_p}}{\lambda_v\lambda_T}\left[(1-M_2^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\right\}\phi=0 \tag{5.36}$$

which has, as its counterpart in ordinary aerodynamics, the Prandtl-Glauert equation. Again, equation (5.36) is hyperbolic only when $M_0 > 1$ and the "zone of action" is determined by the characteristics of the highest order wave, which is the "frozen" Mach conoid of semi-vertex angle $\tan^{-1}(1/\sqrt{M_0^2-1})$.

The corresponding equations for the perturbation pressure p' become:

$$\frac{\partial \phi}{\partial x} = -\frac{p'}{\rho_0 u_0} + \frac{\kappa}{\lambda_v}(\phi_p - \phi) \tag{5.37}$$

and

$$\frac{\partial \phi_p}{\partial x} = -\frac{1}{\lambda_v}(\phi_p - \phi). \tag{5.38}$$

Equations (5.36), (5.37), and (5.38) are obtained by Marble[41] directly by applying the Prandtl-Glauert type of perturbation of a uniform, steady, parallel stream. In the present context, however, the alternate derivation essentially connects the acoustic and aerodynamic concepts of the small perturbation theory in particle-fluid mechanics. This idea is emphasized by Sears[44] in his discussion of the classical small perturbation theory.

As in the Prandtl-Glauert problem, our system of equations here holds for both subsonic and supersonic free-stream Mach numbers M_o . In section 3, we will discuss in detail application to two-dimensional, supersonic thin-airfoil theory, or the corresponding Ackeret[35] problem. This is essentially the simplest problem, and one wishes to follow the way of simplification, which leads to an understanding of the phenomena and from which subsequent extensions and improvements can be made.

It is necessary to remark that the equations for the perturbation pressure given in sections 2. 4 and 2. 5 need to be improved, as in the classical problems, when one discusses slender bodies of revolution, particularly the pressure in the vicinity of the body surface where the perturbation velocities are not of the same order of magnitude. However, the pressure relations given in sections 2. 4 and 2. 5 can be applied to planar systems.

2. 6. Some Limiting Cases. - There are two equilibration processes that enter into our consideration. They are exhibited in the linearized equation by the simultaneous occurrence of the velocity equilibration time τ_{V_0} and the temperature equilibration time τ_{T_0} . In general, for solid particles immersed in a gas, the ratio τ_{V_0} / τ_{T_0} is of the order unity, and this will be taken as the case in our detailed consideration of linearized supersonic flow in section 3. In this section, we consider three limiting situations in which the linearized equation reduces to the form familiar in the consideration

of relaxation processes or finite chemical reactions in a gas involving a single characteristic time.

Again, we proceed from the equation of acoustic propagation in the consideration of limiting cases, and the forms of the equations in a moving coordinate system follow after a Galilean transformation. To enhance our consideration, let us rewrite equation (5.16) in the form:

$$\left\{\frac{\partial^2}{\partial\tilde{t}^2}\left(\frac{\partial}{\partial\tilde{t}}+\frac{1+\kappa}{\tau_{V_0}}\right)\left(\frac{\partial}{\partial\tilde{t}}+\frac{1+\kappa\gamma\frac{c_s}{c_p}}{\tau_{T_0}}\right)-a_0^2\left(\frac{\partial}{\partial\tilde{t}}+\frac{1}{\tau_{V_0}}\right)\left(\frac{\partial}{\partial\tilde{t}}+\frac{1+\kappa\frac{c_s}{c_p}}{\tau_{T_0}}\right)\tilde{\nabla}^2\right\}\phi=0 \qquad (5.39)$$

When the actions of the two particle-fluid relaxation processes are entirely similar, that is, when the momentum and thermal interaction parameters are the same $\kappa=\kappa\frac{c_s}{c_p}$, and when the velocity and temperature equilibration times are the same $\tau_{V_0}=\tau_{T_0}$, the entire relaxation process is then accomplished by a single mechanism and is governed by a single equilibration time τ. Equation (5.39) then reduces to

$$\left\{\frac{\tau}{1+\kappa\gamma}\frac{\partial}{\partial\tilde{t}}\left(\frac{\partial^2}{\partial\tilde{t}^2}-a_0^2\tilde{\nabla}^2\right)+\left(\frac{\partial^2}{\partial\tilde{t}^2}-a_{2e}^2\tilde{\nabla}^2\right)\right\}\phi = 0 \qquad (5.40)$$

for wave motions. Where the "equilibrium" sound speed in this limiting case is

$$a_{2e}^2=\bar{\gamma}\frac{P_0}{\bar{\rho}}=\frac{a_0^2}{1+\kappa\bar{\gamma}} \qquad (5.41)$$

and the ratio of specific heats of the mixture when $\kappa=\kappa\frac{c_s}{c_p}$ is simply $\bar{\gamma}=\frac{1+\kappa}{1+\kappa\gamma}$; $\bar{\rho}=(1+\kappa)\rho_0$. The equations for the perturbation pressure are still given by the dynamical equations (5.22) and (5.33).

For the case when $\frac{\tau_{v_0}}{\tau_{T_0}} \to 0$, equation (5.39) then reduces to

$$\left\{ \frac{\tau_{v_0}}{1+\kappa} \frac{\partial}{\partial \tilde{t}} \left(\frac{\partial^2}{\partial \tilde{t}^2} - a_0^2 \tilde{\nabla}^2 \right) + \left(\frac{\partial^2}{\partial \tilde{t}^2} - a_{2_m}^2 \tilde{\nabla}^2 \right) \right\} \phi = 0 \qquad (5.42)$$

In this situation, while the particle-fluid momentum equilibration process is taking place, the particle cloud remains "thermally frozen" at the initial undisturbed temperature T_0. In the absence of the thermal equilibration process, the heat capacity ratio of the ultimate (momentum) equilibrium mixture is γ, that of the gas alone. However, since the particle cloud and the gas are in final equilibrium dynamically, the density of the equilibrium mixture is then $\bar{\rho} = (1+\kappa)\rho_0$. Hence we now have the interpretation of the "thermally frozen, dynamical equilibrium" acoustic-propagation speed:

$$a_{2_m}^2 = \frac{a_0^2}{1+\kappa} \qquad (5.43)$$

The equations for the perturbation pressure p' are again given by equations (5.22) and (5.23).

In the opposite limiting case, when $\frac{\tau_{v_0}}{\tau_{T_0}} \to \infty$, equation (5.39) reduces to

$$\left\{ \frac{\tau_{T_0}}{1+\kappa\gamma\frac{c_s}{c_p}} \frac{\partial}{\partial \tilde{t}} \left(\frac{\partial^2}{\partial \tilde{t}^2} - a_0^2 \tilde{\nabla}^2 \right) + \left(\frac{\partial^2}{\partial \tilde{t}^2} - a_{2_T}^2 \tilde{\nabla}^2 \right) \right\} \phi = 0 \qquad (5.44)$$

In this situation, while the particle-fluid thermal equilibration process is taking place, the particle cloud remains "dynamically frozen" at its initial zero velocity. In the absence of the momentum equilibration process, the density of the ultimate (thermal) equilibrium mixture is ρ_o , that of the gas alone, while the heat capacity ratio is $\bar{\gamma}$ as interpreted in equation (5.19). In this situation, we now have the interpretation of the opposite limiting case of "dynamically frozen, thermal equilibrium" acoustic-propagation speed:

$$a_{2T}^2 = \bar{\gamma}\, \frac{P_o}{\rho_o} = \frac{1+\kappa\frac{c_s}{c_p}}{1+\kappa\gamma\frac{c_s}{c_p}}\, a_o^2 \tag{5.45}$$

The equations for the perturbation pressure p' for this "dynamically frozen" case reduce to the simple form

$$\rho_o \frac{\partial \phi}{\partial t} = - p' \tag{5.46}$$

For small departures from the last two limiting cases, that is, when either of the ratios τ_{v_o}/τ_{T_o} or τ_{T_o}/τ_{v_o} is very small but not identically zero, one then has a "boundary layer" region similar to that discussed by Marble[19]. For instance, for the near "thermally-frozen" case, the "rapid" momentum equilibration process appears as a boundary layer zone imbedded in a "slow" thermal equilibration zone. Similarly, the opposite takes place in the near "dynamically-frozen" case.

The forms of the limiting cases exhibited in equations (5.40), (5.42), and (5.44) are identical to the acoustical propagation situation in a relaxing or reacting gas involving a single characteristic time. The equation in this form appears to have been first derived by Stokes[46] in his consideration of the effect of radiation of heat, using the linear Newtonian relation, on the propagation of sound. Stokes also gave the periodic solution to his equation. One is also referred to its discussion by Chu[47] and Moore and Gibson[48, 49]. An extensive discussion of more general forms of the relaxation wave equation involving a single characteristic time is given by Whitham[50].

The limiting forms of the equation for the perturbation potential in the moving coordinate system, or the airfoil coordinates, follows immediately. When $\kappa = \kappa \frac{c_s}{c_p}$ and $\tau_{v_0} = \tau_{T_0} = \tau$, we have, for unsteady motions in the airfoil coordinates, with $\lambda = u_0 \tau$:

$$\left\{ \lambda \left(\frac{1}{u_0}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \left[(1-M_0^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{2M_0}{a_0}\frac{\partial^2}{\partial x \partial t} - \frac{1}{a_0^2}\frac{\partial^2}{\partial t^2} \right] \right.$$

$$\left. + \left[(1-M_{2e}^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{2M_{2e}}{a_{2e}}\frac{\partial^2}{\partial x \partial t} - \frac{1}{a_{2e}^2}\frac{\partial^2}{\partial t^2} \right] \right\} \phi = 0 \qquad (5.47)$$

where we have defined the "equilibrium" Mach number as $M_{2e} = u_0 / a_{2e}$. The perturbation pressure is still given by equations (5.28) and (5.29). For harmonic motions in the airfoil coordinates with the perturbation potentials expressed in the forms given by equations (5.30) and (5.31), the equation for the potential $\Phi_s(x, y, z)$ becomes

$$\left\{ \lambda \left(\frac{i\omega}{u_0} + \frac{\partial}{\partial x} \right) \left[(1-M_0^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - 2i\omega\frac{M_0}{a_0}\frac{\partial}{\partial x} + \frac{\omega^2}{a_0^2} \right] \right.$$

$$\left. + \left[(1-M_{2e}^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - 2i\omega\frac{M_{2e}}{a_{2e}}\frac{\partial}{\partial x} + \frac{\omega^2}{a_{2e}^2} \right] \right\} \Phi_s = 0 \qquad (5.48)$$

with the perturbation pressure given by equations (5.34) and (5.35). For steady flow in airfoil coordinates, we have

$$\left\{ \lambda\frac{\partial}{\partial x}\left[(1-M_0^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] + \left[(1-M_{2e}^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \right\} \phi = 0 . \qquad (5.49)$$

For the "thermally-frozen" case, we have the following corresponding equation for unsteady motions in airfoil coordinates, with $\lambda_V = u_0 \tau_{V_0}$:

$$\left\{\lambda_V\left(\frac{1}{u_0}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left[(1-M_0^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{2M_0}{a_0}\frac{\partial^2}{\partial x\partial t}-\frac{1}{a_0^2}\frac{\partial^2}{\partial t^2}\right]\right.$$

$$\left.+\left[(1-M_{2m}^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{2M_{2m}}{a_{2m}}\frac{\partial^2}{\partial x\partial t}-\frac{1}{a_{2m}^2}\frac{\partial^2}{\partial t^2}\right]\right\}\phi = 0 \qquad (5.50)$$

where we have defined the "thermally-frozen, dynamical equilibrium" Mach number as $M_{2m} = u_0 / a_{2m}$. The perturbation pressure is again given by equations (5.28) and (5.29). For harmonic motions:

$$\left\{\lambda_V\left(\frac{i\omega}{u_0}+\frac{\partial}{\partial x}\right)\left[(1-M_0^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2i\omega\frac{M_0}{a_0}\frac{\partial}{\partial x}+\frac{\omega^2}{a_0^2}\right]\right.$$

$$\left.+\left[(1-M_{2m}^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2i\omega\frac{M_{2m}}{a_{2m}}\frac{\partial}{\partial x}+\frac{\omega^2}{a_{2m}^2}\right]\right\}\Phi_s = 0 \qquad (5.51)$$

with perturbation pressure given by equations (5.34) and (5.35). For steady motion in the airfoil coordinates:

$$\left\{\lambda_V\frac{\partial}{\partial x}\left[(1-M_0^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]+\left[(1-M_{2m}^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\right\}\phi = 0 \qquad (5.52)$$

with perturbation pressure given by equations (5.37) and (5.38).

For the "dynamically-frozen" case, with $\lambda_T = u_0 \tau_{T_0}$, the equation for unsteady motions is:

$$\left\{\frac{\lambda_T}{1+K_v\frac{c_s}{c_p}}\left(\frac{1}{u_0}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left[(1-M_0^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{2M_0}{a_0}\frac{\partial^2}{\partial x\partial t}-\frac{1}{a_0^2}\frac{\partial^2}{\partial t^2}\right]\right.$$

$$\left.+\left[(1-M_{2T}^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{2M_{2T}}{a_{2T}}\frac{\partial^2}{\partial x\partial t}-\frac{1}{a_{2T}}\frac{\partial^2}{\partial t^2}\right]\right\}\phi = 0 \qquad (5.53)$$

with the "dynamically-frozen, thermal equilibrium" Mach number as $M_{2T} = u_0 / a_{2T}$. Here, the perturbation pressure is modified from equation (5.44) and is

$$\left(\frac{1}{u_0}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\phi = -\frac{p'}{\rho_0 u_0} \qquad (5.54)$$

For harmonic motions:

$$\left\{\frac{\lambda_T}{1+K_v\frac{c_s}{c_p}}\left(\frac{i\omega}{u_0}+\frac{\partial}{\partial x}\right)\left[(1-M_0^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2i\omega\frac{M_0}{a_0}\frac{\partial}{\partial x}+\frac{\omega^2}{a_0^2}\right]\right.$$

$$\left.+\left[(1-M_{2T}^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-2i\omega\frac{M_{2T}}{a_{2T}}\frac{\partial}{\partial x}+\frac{\omega^2}{a_{2T}}\right]\right\}\Phi_s = 0 \qquad (5.55)$$

with

$$\left(\frac{i\omega}{u_0}+\frac{\partial}{\partial x}\right)\Phi_s = -\frac{P_s}{\rho_0 u_0} \qquad (5.56)$$

where the perturbation pressure p' is expressed in equation (5.32) in terms of $P_s(x,y,z)$. For steady motions:

$$\left\{\frac{\lambda_T}{1+K_v\frac{c_s}{c_p}}\frac{\partial}{\partial x}\left[(1-M_0^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]+\left[(1-M_{2T}^2)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z}\right]\right\}\phi = 0 \qquad (5.57)$$

and

$$\frac{\partial \phi}{\partial x} = - \frac{p'}{\rho_o u_o} \tag{5.58}$$

The forms of the three limiting cases for steady motions in the airfoil coordinates appear identical to the corresponding linearized steady, small-perturbation theory in a relaxing or reacting gas with a single characteristic time. One is referred to the discussions by Moore and Gibson(48, 49), Vincenti(51), and Wegener and Cole(52). The linearized equation for unsteady motions in the airfoil coordinates involving a single characteristic relaxation time, however, does not appear to have been previously considered.

2.7. Limitations of the Linear Theory. - Similar to the situation in ordinary aerodynamics, the linear theory can only be of fruitful service when an understanding of its shortcomings and of the actual flow situation which it is intended to simulate is kept in mind. Many of the shortcomings of the linear theory in particle-fluid mechanics are, however, carried over from the similar situation in ordinary aerodynamics. Hence, we must exclude all problems in which viscosity and heat conduction within the fluid itself play an important role. It is further necessary that the magnitude of the induced velocities due to an obstacle in the flow field be small compared to the free stream velocity. We know from ordinary aerodynamics that the linear theory cannot describe situations close to the transonic and hypersonic flow regimes; in such cases, even the description of the flow due to small disturbances is of a nonlinear nature. When the flow is unsteady, the magnitude of the accelerations produced must be small compared to the ratio of the square of the free stream velocity to the characteristic length of the obstacle.

The relatively small magnitude required of the disturbance velocities familiar in ordinary aerodynamics then restricts our consideration of thin obstacles or airfoils with sharp leading edges and at small angles of inclination to the oncoming free stream. In fact, Ackeret[35] suggested that the tangent lines of the leading edge lie in the direction of the free stream. In this sense, the formulation of the linear theory in particle-fluid mechanics implies that the pressure distribution on an airfoil is primarily modified by the particle-fluid equilibration processes and that particle-boundary collisions are of much less importance. The latter will be discussed in detail in section 3.1 and is indeed a second order effect. On the other hand, in the flow in the vicinity of the stagnation point of a blunt-nosed body, where the linear theory does not apply, particle-boundary collisions are much more significant.

It is also well known in ordinary aerodynamics, in particular for supersonic flows, that the linear theory is not expected to describe the actual flow pattern in the far-field with any great accuracy, even for small disturbances, although it predicts the surface pressures fairly well within the limitations of the linear theory. In the linear theory, the inclinations of the individual Mach waves are approximated by a constant value; hence, the mechanism for overtaking of the Mach wave front by later Mach waves leading to the formation of shocks is precluded from the linear theory. If one is interested in a more realistic flow pattern far from the obstacle, the linear theory needs to be improved according to the spirit of "higher approximations" in ordinary aerodynamics discussed by Whitham[53] and by Lighthill[54].

3. Two-Dimensional Steady Supersonic Flow

Up to now we have discussed in rather general terms the linearized theory in particle-fluid mechanics. In this section, the two-dimensional form of equation (5. 36) will be applied in the specific discussion of steady, supersonic flow past a thin obstacle, which is the corresponding Ackeret[35] problem.

In the ordinary two-dimensional supersonic flow, it is well known that its simplicity lies on the fact that the pressure acting on an element of surface depends only on the local surface deflection itself. In this situation, the aerodynamic forces on a two-dimensional, thin obstacle are thereby easily obtained through a simple integration when the local slope is prescribed as a function of the streamwise distance. The fact that this simplicity no longer exists in particle-fluid mechanics is due to the particle-fluid equilibration processes which render the local pressure acting on an element of of surface to depend on its upstream history. In this situation, the aerodynamic forces then depend on the extent in which the equilibration processes take place over the surface of the obstacle. Hence, the natural non-dimensional parameters which arise as a consequence of the problem are the ratios of some characteristic length of the obstacle to the equilibration distances; C/λ_v and C/λ_T . We have already discussed in section 2.6 the opposite limiting cases of $\frac{\lambda_V}{\lambda_T} = \frac{\tau_{V_0}}{\tau_{T_0}} \to 0$ and $\frac{\lambda_V}{\lambda_T} = \frac{\tau_{V_0}}{\tau_{T_0}} \to \infty$, in which cases one recovers a form of the perturbation potential equation that is well known in the corresponding problem of a relaxing or reacting gas with a single characteristic time. For the consideration of small solid particles in a gas, we retain the ratio λ_V/λ_T to be of order unity. In fact, for particles obeying Stokes' law, $\frac{\lambda_V}{\lambda_T} = \frac{2}{3}\frac{c_p}{c_s} P_r^{-1}$. Hence the two equilibration processes in our problem take place over approximately the same streamwise distance.

When $\frac{c}{\lambda} \gg 1$, where λ is either λ_V or λ_T since we regard $\frac{\lambda_V}{\lambda_T} = \mathcal{O}(1)$, the equilibration processes are very nearly confined to the distance over the surface of the airfoil, and the particle cloud adjusts itself relatively quickly to the local environment, provided, of course, the shape of the surface of the airfoil is "slowly varying". When $\frac{c}{\lambda} \ll 1$, the disturbances induced by the airfoil make themselves felt, through the equilibration processes, far into the downstream regions of the wake. However, as far as the calculation of the surface pressures are concerned, one need only consider the changes in the flow field due to the presence of the obstacle in the region bounded by the "frozen" Mach lines, which determine the "zone of action", emanating from the leading and trailing edges of the obstacle. The upper and lower surfaces do not enter into each other's "zone of action" and hence may be considered independently. The situation is similar, for instance, for a finite wing of uniform cross section in the region away from the tip "frozen" Mach conoids where the flow is essentially of a two-dimensional nature.

It is well known in ordinary aerodynamics that the well developed theory of sound of Lord Rayleigh[43] is utilized to great advantage in deriving and interpreting the physical aspects of the linearized theory of supersonic thin airfoils. The situation in particle-fluid mechanics, however, does not have in its possession this advantageous counterpart, since the development of the acoustical situation is recent and rudimentary. Chu and Parlange[20] discussed the one-dimensional unsteady motion in which a piston is suddenly set into motion and is subsequently maintained in steady motion. This would correspond to the situation of a simple semi-infinite wedge in the steady, two-dimensional supersonic flow, where the streamwise distance plays the role of time. They showed that the disturbance, while it decays along the wave front defined by the "frozen" speed of sound a_0 , is ultimately propagated with the "equilibrium" sound speed a_2 . However, the pressure on the piston face for finite times is not given, and their consideration of the wave structure for "large" times does not yield a physically recognizable form. Soo's work[55], which did not proceed systematically from the fundamental equations, unfortunately contains misconceptions of a fundamental nature, such as writing the energy equation of the mixture as total enthalpy(in the form inconsistent with "small disturbances") equals constant for an unsteady flow.

The subsequent discussions of the linearized supersonic flow will be from the aerodynamic point of view. This can be regarded as simultaneously providing a complementary depiction of the acoustical situation in particle-fluid mechanics, reversing the order of the corresponding depiction in the classical fluid mechanics.

3.1. Boundary Conditions. Particle-Boundary Collisions. - The linearized equation obtained exhibits second order in the y-derivative, and the discussions on boundary conditions in the linearized theory in ordinary aerodynamics are carried over to particle-fluid mechanics. Namely, the solutions of the airfoil problem may be split into a thickness problem and a lifting problem(56), and the condition at the airfoil may be satisfied on the $y = 0$ plane. Hence, at $y = 0$, the condition is that the fluid is tangent to the surface along the airfoil. The condition far from the airfoil requires that the disturbance produced by the airfoil be at least bounded if it does not vanish. When the airfoil is present in a channel instead of being in a flow field of infinite extent, the same condition may also be applied provided that the reflection of Mach waves produced by the airfoil from the channel walls does not intersect the airfoil itself. The linearized equation exhibits fourth order in the x-derivative, and this requires that the disturbance quantity considered together with its first three x-derivatives be specified at $x = 0$ as initial conditions. When the particle cloud is in equilibrium with the fluid in the oncoming uniform stream, then initially all four initial conditions vanish.

Within the linear theory, collisions of the particles with the boundary have a particularly simple consideration. In the first place, all considerations are referred to the $y=0$ plane and are separable from the main problem of considering the particle-fluid interaction and its modification of the surface pressure. According to the linear theory, the surface pressure is proportional to the local deflection angle. On the other hand, the superimposable normal force per unit area due to particle-boundary collisions is a second order effect due to its Newtonian nature, and is proportional to the square of the angle between the particle streamline and the boundary at the point of impact:

$$\begin{aligned} F_N &\cong \rho_{p_0} u_0^2 \left(f(x') - \theta_p\right)^2 && \text{(Newtonian)} \\ &\cong 2\rho_{p_0} u_0^2 \left(f(x') - \theta_p\right)^2 && \text{(elastic)} \end{aligned} \tag{5.59}$$

with the usual approximation consistent with the linear theory. The geometry is shown in Figure 12, in which the surface inclination is highly exaggerated. The normal force coefficient is

$$\begin{aligned} C_{F_N} = \frac{F_N}{\frac{1}{2}\rho_0 u_0^2} &\cong 2k\left(f(x') - \theta_p\right)^2 && \text{(Newtonian)} \\ &\cong 4k\left(f(x') - \theta_p\right)^2 && \text{(elastic)} \end{aligned} \tag{5.60}$$

Here, θ_p is the local particle streamline inclination at the boundary and is determined by the particle-fluid interaction from the particle y-momentum equation in the $y=0$ plane:

$$\theta_p \cong \frac{v_p}{u_0} = \int_0^{x'} e^{-(x'-\xi)} f(\xi)\, d\xi \tag{5.61}$$

The particle streamline trajectory up to the point of collision is

$$y'_p = y'_{p_o} + \int_0^{x'} \theta_p(\xi)\, d\xi \tag{5.62}$$

where $y'_{p_o} = \frac{y_{p_o}}{\lambda_v}$ is the non-dimensional initial location of the particle. The point of collision $x'_c = \frac{x_c}{\lambda_v}$ is obtained implicitly from the intersection of the particle trajectory with the curve of the boundary:

$$y'_{p_o} + \int_0^{x'_c} \theta_p(\xi)\, d\xi = \int_0^{x'_c} f(\xi)\, d\xi \tag{5.63}$$

In the above, we have assumed the obstacle to be a pointed nosed one as required by the linear theory. The particle trajectory after collision with the boundary falls within the two limiting cases in which we obtained the normal force. In the Newtonian situation, the particle immediately follows the wall. This forms the initial condition for the subsequent particle motion subjected to the relaxation by the local boundary shape or fluid velocity at the boundary. In the opposite limiting case of an elastic collision, the initial condition is furnished by a reflection of the fictitious particle trajectory continued into the boundary surface at the point of collision. The limiting particle streamline, adjacent to a particle vacuum on one side in the aft portion of a curved boundary, is of interest in a second-order linear theory where the effect on the flow and the surface pressure due to particle vacuum near curved boundaries must be considered.

3.2. <u>Mach Wave Structure</u>. - The two-dimensional form of the equation for thin airfoil theory is obtained from equation (5.36):

$$\left\{\frac{\partial^2}{\partial x^2}\left[\beta_0^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right]+\left(\frac{1}{\lambda_V}+\frac{1+\kappa\frac{c_s}{c_p}}{\lambda_T}\right)\frac{\partial}{\partial x}\left[\beta_1^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right]+\frac{1+\kappa\frac{c_s}{c_p}}{\lambda_V\lambda_T}\left[\beta_2^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right]\right\}\phi=0 \quad (5.64)$$

where we have denoted

$$\beta_0^2 = M_0^2 - 1\,, \quad \beta_1^2 = M_1^2 - 1\,, \quad \beta_2^2 = M_2^2 - 1 \quad (5.65)$$

and are positive quantities for supersonic flow, and $\beta_2 > \beta_1 > \beta_0$. The geometry is shown in Figure 13.

The exact solutions for the flow over a wavy wall, that is, the periodic solutions, have been obtained by Marble(41). Here, we make some considerations of the airfoil problem, which is an initial value problem with vanishing disturbances at $x=0$ and conditions prescribed on $y=0$.

Let us introduce the following non-dimensional quantities:

$$x' = \frac{x}{\lambda_V}\,, \quad y' = \frac{y}{\lambda_V}\,, \quad \phi' = \frac{\phi}{\lambda_V u_0}\,, \quad \sigma = \frac{\lambda_V}{\lambda_T}\left(1+\kappa\frac{c_s}{c_p}\right) \quad (5.66)$$

so that

$$u' = \frac{u}{u_0} = \frac{\partial\phi'}{\partial x'}\,, \qquad v' = \frac{v}{u_0} = \frac{\partial\phi'}{\partial y'} \quad (5.67)$$

σ in our subsequent considerations is regarded as of order unity. Substituting the quantities in (5.66) into (5.64), we then have:

$$\left\{\frac{\partial^2}{\partial x'^2}\left[\beta_0^2\frac{\partial^2}{\partial x'^2}-\frac{\partial^2}{\partial y'^2}\right]+(1+\sigma)\frac{\partial}{\partial x'}\left[\beta_1^2\frac{\partial^2}{\partial x'^2}-\frac{\partial^2}{\partial y'^2}\right]+\sigma\left[\beta_2^2\frac{\partial^2}{\partial x'^2}-\frac{\partial^2}{\partial y'^2}\right]\right\}\phi' = 0 \quad (5.68)$$

The appropriate initial and boundary conditions are

$$\begin{aligned} &\phi' = \phi'_{x'} = \phi'_{x'x'} = \phi'_{x'x'x'} = 0\,, && x'=0 \quad y'>0 \\ &v' = \phi'_{y'} = f(x') && y'=0^+ \quad x'>0 \end{aligned} \quad (5.69)$$

Here, we consider the upper surface of the airfoil in the $y' > 0$ plane, which is independent of the lower surface, and the "frozen" Mach waves which determine the "zone of action" are inclined towards the downstream direction $x' - \beta_0 y' = \text{constant}$. $f(x')$ is the local inclination of the surface of the airfoil.

Denote $\Phi'(s, y')$ as the Laplace transformation of $\phi'(x', y')$ which is defined as

$$\Phi'(s, y') = \int_0^\infty e^{-sx'} \phi'(x', y')\, dx'$$

With the use of the initial conditions of vanishing disturbances, the transformed equation for the perturbation potential becomes

$$\frac{d\Phi'}{dy'^2} - S^2(s)\Phi' = 0 \tag{5.70}$$

and

$$\Phi'(s, y') = A(s)\, e^{-S_1(s) y'} + B(s)\, e^{-S_2(s) y'} \tag{5.71}$$

where it is found that $S_1(s)$ and $S_2(s)$ are the roots

$$\begin{matrix} S_1(s) \\ S_2(s) \end{matrix} = \pm \left[\frac{s^2 + (\frac{\beta_1}{\beta_0})^2 (1+\sigma) s + (\frac{\beta_2}{\beta_0})^2 \sigma}{s^2 + (1+\sigma) s + \sigma} \right]^{\frac{1}{2}} \beta_0 s \tag{5.72}$$

Now for large s we have $S_1(s) \sim +\beta_0 s$ and $S_2(s) \sim -\beta_0 s$. Hence the appropriate solution which gives Mach waves inclining towards the downstream direction only is represented by the first term in equation (5.71), which is essentially the "outgoing wave". The second term in equation (5.71) is deleted, since it represents Mach waves inclined towards the upstream direction or the "incoming wave" from infinity where no disturbances originate. The solution satisfying the condition $\Phi'_{y'}(s,0^+) = F(s)$ on $y' = 0^+$ is then

$$\Phi'(s,y') = -\frac{F(s)}{S_1(s)} e^{-S_1(s)y'} \tag{5.73}$$

where

$$F(s) = \int_0^\infty e^{-sx'} f(x')\,dx'$$

and the solution in the (x', y') plane is

$$\phi'(x',y') = \frac{1}{2\pi i}\int_{L_1} \frac{-F(s)}{S_1(s)} e^{sx' - S_1(s)y'} ds \tag{5.74}$$

where L_1 is the Bromwich path parallel to the imaginary axis and to the right of all singularities of $\frac{F(s)}{S_1(s)} e^{-S_1(s)y'}$. As far as the wave structure is concerned, the behavior can be deduced directly in the (s, y') plane. In what follows, the standard procedures in treating wave propagation problems will be followed[57].

It is convenient to discuss the behavior of the wave structure in terms of the induced normal velocity

$$V(s,y') = \Phi'_{y'}(s,y') = F(s)\, e^{-S_1(s)y'} \tag{5.75}$$

from which the relevant behavior may be deduced. It is desired to examine first,the behavior in the vicinity of the "frozen" Mach wave, which is the "wave front" and defines the zone of action; second, the location of the wave along which the main disturbance will ultimately be carried out, and the behavior in the vicinity of this wave. The behavior near the "frozen" Mach wave is obtained by expanding $S_1(s)$

in the exponential for large s :

$$-S_1(s) = -\beta_0 s - \beta_0 \frac{1+\sigma}{2}\left[\left(\frac{\beta_1}{\beta_0}\right)^2 - 1\right] + \mathcal{O}\left(\frac{1}{s}\right)$$

which gives the interpretation, keeping y' fixed:

$$v'(x',y') = f(x'-\beta_0 y')\, e^{-\frac{1+\sigma}{2}\left[\left(\frac{\beta_1}{\beta_0}\right)^2-1\right]\beta_0 y'}\left[1+\mathcal{O}(x'-\beta_0 y')\right] \qquad (5.76)$$

describing the exponential decay of signals along the "frozen" Mach wave. Referring to equation (5.68), the "frozen" wave operator characterized by β_0 is one order higher in the x'- derivative than the "intermediary" wave operator characterized by β_1 which in turn is one order higher than the "equilibrium" wave operator characterized by β_2. An examination of the damping factor in equation (5.76) shows that the decay of the highest order waves, $x' = \beta_0 y'$, is attributed by the presence of waves of one order lower, $x' = \beta_1 y'$. This is the familiar situation in wave motions in a relaxing gas involving only a single characteristic time[50]. The effect of the lowest order waves, $x' = \beta_2 y'$, on the behavior of the wave structure in the vicinity of the frozen Mach wave is contained in the factor $\mathcal{O}(x'-\beta_0 y')$ in equation (5.76) and is evidently negligible compared with the effect of the "intermediary" waves. One notes that the linearization process replaced the actual inclination of the "frozen" Mach waves $\tan^{-1}\left(\frac{1}{\beta_f}\right)$ by the constant value term $\tan^{-1}\left(\frac{1}{\beta_0}\right)$; hence, the mechanism for wave steepening due to the overtaking of the most forward wave by waves from the rear, which eventually leads to shock formation, is thereby ruled out in this approximation.

Since the signals are damped along the frozen Mach wave, it is obvious that the bulk of the disturbance is going to be propagated along waves other than the most forward wave, even though it actually determines the "zone of action". In this case, we again follow standard procedure[57] to study the behavior of the disturbance represented by equation (5.75) for large x'. However, for the study of wave motions, y'/x' is kept fixed in order that one follows along the various waves to study the behavior in their vicinity as $x' \to \infty$. We will consider a simple physical situation which enables us to show the relevant features, and the conclusions can be easily generalized to the general case. For instance, consider the disturbances generated by a simple semi-infinite wedge of half angle α_o, in which case

$$f(x') = \alpha_0 H(x')$$

where $H(x')$ is the unit Heaviside function. Furthermore, let us consider the behavior of $v'_{x'}(x', y')$ instead of $v'(x', y')$ itself. Thus the representation for $v'_{x'}(x', y')$ in terms of a contour integral is

$$v'_{x'}(x', y') = \frac{\alpha_o}{2\pi i} \int_{L_1} e^{sx' - S_1(s) y'} ds \tag{5.77}$$

The behavior of $v'_{x'}(x', y')$ for $x' \to \infty$ can now be deduced by an application of the saddle point method. Let us write $m = y'/x'$ in which we keep m fixed. The dominant contribution to the value of the integral in equation (5.77) comes from the vicinity of the saddle point, through which the contour is chosen to pass, when the exponential factor

$$e^{x'\{s - m S_1(s)\}} \tag{5.78}$$

is a maximum. The location of the saddle point $s = s_0$ is therefore obtained from

$$1 - m \frac{dS_1(s_0)}{ds} = 0 \tag{5.79}$$

and from which, in principle, one can solve for s_0 in terms of m, that is, $s_0 = s_0(m)$. This, in general, does not yield the asymptotic representation with the desired relevant simple physical features which we seek. Instead, we first seek to determine at what value of m does $v'_{x,1}(x',y')$ attain a maximum. In other words, we seek to locate the particular Mach wave in which the exponential factor in equation (5.78) is a maximum. This condition is simply

$$\frac{d}{dm}\left\{s_0 - m S_1(s_0)\right\} = 0 \tag{5.80}$$

Now since $\frac{dS_1(s_0)}{dm} = \frac{1}{m}\frac{ds_0}{dm}$ from equation (5.79), the above condition then implies that $S_1(s_0) = 0$. Upon examining equation (5.72) for $S_1(s)$, the value of s_0 for this case is $s_0 = 0$. Hence, equation (5.80) gives

$$m = \frac{1}{\frac{dS_1(0)}{dm}} = \frac{1}{\beta_2} \tag{5.81}$$

as the location of the maximum of the disturbance $v'_{x,1}(x',y')$, which is the equilibrium Mach wave, $x' = \beta_2 y'$. Hence, the disturbance is ultimately propagated along the lowest order waves. In this situation, we need only to study the wave structure for large x' in the vicinity of the "equilibrium" Mach wave, - that is, in the vicinity of the relevant saddle point $s_0 = 0$. Since $S_1(s)$ is analytic at $s=0$, we expand $S_1(s)$ about $s=0$ in a Taylor series of the form:

$$-m\, S_1(s) = -\alpha_1 s + \alpha_2 s^2 - \alpha_3 s^3 + \cdots \quad . \tag{5.82}$$

The contour is chosen to pass through $s=0$ and the Bromwich path now is simply along the imaginary axis and $s = i\, s_i$ along the path. In this case, equation (5.77) has the form

$$v'_{x'}(x',y') = \frac{d_0}{2\pi} \int_{-\infty}^{\infty} e^{i x' [(1-\alpha_1) s_i + \alpha_3 s_i^3 + \cdots]} \, e^{-x'[\alpha_2 s_i^2 - \cdots]} \, ds_i \tag{5.83}$$

where

$$\begin{aligned} \alpha_1 &= \beta_2 \frac{y'}{x'} \\ \alpha_2 &= \frac{1+\sigma}{\sigma} \frac{\beta_2}{2} \left[1 - \left(\frac{\beta_1}{\beta_2}\right)^2 \right] \frac{y'}{x'} \\ \alpha_3 &= \frac{\beta_2}{2\sigma} \left\{ \left[1 - \left(\frac{\beta_0}{\beta_2}\right)^2 \right] - \frac{(1+\sigma)^2}{\sigma} \left[1 - \left(\frac{\beta_1}{\beta_2}\right)^2 \right] \right\} \frac{y'}{x'} \end{aligned} \tag{5.84}$$

The dominant term in equation (5.83) is

$$v'_{x'}(x',y') = \frac{d_0}{\sqrt{2\pi}} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i x' (1-\alpha_1) s_i} \, e^{-\alpha_2 s_i^2 x'} \, ds_i \right\} \tag{5.85}$$

The interpretation of the integral in the bracket in equation (5.85) is the familiar one in Fourier transforms[58], and is simply

$$v_{x'}'(x',y') = \frac{d_0}{\sqrt{2\pi}}\,\frac{1}{\sqrt{2\alpha_2 x'}}\,e^{-\frac{(1-\alpha_1)^2 x'^2}{4\alpha_2 x'}}$$

$$= \frac{d_0}{\sqrt{2\pi}}\,\frac{1}{\sqrt{2\frac{1+\sigma}{\sigma}\frac{\beta_2}{2}\left[1-\left(\frac{\beta_1}{\beta_2}\right)^2\right]y'}}\,e^{-\frac{(x'-\beta_2 y')^2}{2\frac{1+\sigma}{\sigma}\beta_2\left[1-\left(\frac{\beta_1}{\beta_2}\right)^2\right]y'}} \tag{5.86}$$

The last expression is obtained upon substituting for α_1 and α_2 from equation (5.84). Hence, $v_{x'}'(x',y')$ is a Gaussian centered about the equilibrium Mach wave, $x' = \beta_2 y'$, and is interpreted as representing a diffusion-like behavior with the effective diffusivity given by

$$\mathcal{D}_{12} = \frac{1+\sigma}{\sigma}\,\frac{\beta_2}{2}\left[1-\left(\frac{\beta_1}{\beta_2}\right)^2\right](\lambda_V u_0)\,. \tag{5.87}$$

Since $\beta_2 > \beta_1$, then $\mathcal{D}_{12} > 0$. In the physical variables, the maximum of v_x decreases like

$$\frac{d_0 u_0}{\sqrt{4\pi\mathcal{D}_{12}\frac{y}{u_0}}}$$

and the extent in which the diffusion-like region spreads out from the equilibrium Mach wave is

$$\sqrt{4\pi\mathcal{D}_{12}\frac{y}{u_0}}\,.$$

Since the intermediary Mach wave extends to infinity like $x = \beta_1 y$, the growth of the diffusion-like region ahead of the equilibrium Mach wave never reaches $x = \beta_1 y$. The behavior of the smooth step itself is obtained from equation (5.86) through a simple integration:

$$v' = \frac{\alpha_0}{\sqrt{\pi}} \int_{-\infty}^{\chi} e^{-\bar{\chi}^2} d\bar{\chi} = \frac{\alpha_0}{2} (1 + \operatorname{erf} \chi) \qquad (5.88)$$

where

$$\chi = \frac{x - \beta_2 y}{\sqrt{4 \mathcal{D}_{12} \frac{y}{u_0}}} .$$

In the integration, use is made of the fact that the disturbances "far ahead" have become negligible through decay far upstream of the equilibrium Mach wave.

An examination of the effective diffusivity $\mathcal{D}_{12}$ shows that the diffusion of the lowest order waves $x' = \beta_2 y'$ is effected by the presence of waves of one order higher: $x' = \beta_1 y'$. This is also the familiar situation in wave motions in a relaxing gas with only a single characteristic time(50). The effect of the highest order waves $x' = \beta_0 y'$ on the behavior of the wave structure in the vicinity of the equilibrium Mach wave is contained in the neglected higher order terms in equation (5.83), and they will be examined later.

The saddle point s_0 for the intermediary Mach wave, for deducing the wave structure in its vicinity, is not easily obtainable. However, from heuristic arguments, one expects a simultaneous damping effected by the lower order wave $x' = \beta_2 y'$ with the decay factor containing $\left[\left(\frac{\beta_2}{\beta_1}\right)^2 - 1 \right]$, and diffusion effected by the higher order wave $x' = \beta_0 y'$ with the effective diffusivity containing $\left[1 - \left(\frac{\beta_0}{\beta_1}\right)^2 \right]$. Following the approximate treatment suggested by Whitham(50), one obtains

$$v'_{x'}(x',y') = \frac{\alpha_0}{\sqrt{2\pi}} \frac{1}{\sqrt{2\frac{1}{1+\sigma}\frac{\beta_1}{2}\left[1-\left(\frac{\beta_0}{\beta_1}\right)^2\right]y'}} e^{-\frac{(x'-\beta_1 y')^2}{2\frac{1}{1+\sigma}\beta_1\left[1-\left(\frac{\beta_0}{\beta_1}\right)^2\right]y'}} \cdot e^{-\frac{\sigma}{1+\sigma}\frac{\beta_1}{2}\left[\left(\frac{\beta_2}{\beta_1}\right)^2-1\right]y'}$$

for the structure in the vicinity of the intermediary Mach wave, $x' = \beta_1 y'$.

Thus far we have only considered the wave structure in the far field due to a smooth step in the normal velocity. When we consider disturbances generated by the presence of a thin, finite obstacle placed in the flow field, a characteristic length then enters. This is some characteristic length of the obstacle C . Some conclusions of a rather general nature can be drawn from the consideration of the far field behavior due to a flat-plate airfoil of finite length C at a small negative angle of attack α_0 . In this case, then

$$f(x') = \alpha_0 \left[H(x') - H(x'-C')\right]$$

where $C' = C/\lambda_v$, and

$$v'_{x'}(x',y') = \frac{\alpha_0}{2\pi i}\int_{L_1}\left\{1-e^{-C's}\right\}e^{sx'-S_1(s)y'}\,ds \,. \qquad (5.89)$$

The far field behavior is deduced in a similar manner

$$v'_{x'}(x',y') = \frac{\alpha_0\lambda_v}{\sqrt{4\pi\mathcal{D}_{12}\frac{y}{u_0}}}\left\{e^{-\frac{(x-\beta_2 y)^2}{4\mathcal{D}_{12}y/u_0}} - e^{-\frac{(x-C-\beta_2 y)^2}{4\mathcal{D}_{12}y/u_0}}\right\} \qquad (5.90)$$

which is the superposition of two Gaussians, one centered about the equilibrium Mach wave emanating from the leading edge of the airfoil, and the other centered about the equilibrium Mach wave emanating from the trailing edge. The behavior of the normal velocity is similarly obtained:

$$v' = \frac{\alpha_0}{2}\left[\operatorname{erf}\chi - \operatorname{erf}\chi_c\right] \tag{5.91}$$

where

$$\chi = \frac{(x-\beta_2 y)}{\sqrt{4\mathcal{D}_{12}\frac{y}{u_0}}} \quad \text{and} \quad \chi_c = \frac{(x-C-\beta_2 y)}{\sqrt{4\mathcal{D}_{12}\frac{y}{u_0}}}$$

which is the superposition of two smooth steps whose maximum slopes are centered about the leading and trailing edge equilibrium Mach waves, respectively. The situation gives a physical picture which is rather simple. The controlling factor is the ratio of the plate length C to the extent of the diffusion-like region, $\sqrt{4\pi\mathcal{D}_{12}\frac{y}{u_0}}$ centered about the equilibrium Mach waves from the leading and trailing edges, respectively. First, suppose the far field behavior is such that $C/\sqrt{4\pi\mathcal{D}_{12}\frac{y}{u_0}} \gg 1$; the extent of the diffusion-like region is very close about the equilibrium Mach wave compared to the length of the plate. In this situation, the activity of particle-fluid interactions takes place relatively rapidly in the regions about $x=\beta_2 y$ and about $x-C=\beta_2 y$, and in the region between the leading and trailing edge equilibrium Mach waves the flow is very nearly in equilibrium. In fact, the diffusion-like region is so thin that one could write approximately

$$v' = \alpha_0\left[H(x'-\beta_2 y') - H(x'-C'-\beta_2 y')\right]$$

as if the disturbances are concentrated at the equilibrium Mach waves. This situation is illustrated in the left of Figure 14. More generally, when $C/\sqrt{4\pi \mathcal{D}_{12} \frac{y}{u_0}} \gg 1$, we can write

$$v' = f(x' - \beta_2 y')$$

provided, however, that the shape of the airfoil surface is "slowly varying".

In the opposite limiting case, when $C/\sqrt{4\pi \mathcal{D}_{12} \frac{y}{u_0}} \ll 1$, the diffusion-like regions will have merged to form a single diffusion-like region, and the far field behavior then spreads out like

$$\sqrt{4\pi \mathcal{D}_{12} \frac{y}{u_0}}$$

centered about an equilibrium Mach wave emanating from the center of the plate. The amplitude of the disturbance decreases like

$$v' \sim \frac{\lambda v}{\sqrt{4\pi \mathcal{D}_{12} \frac{y}{u_0}}}$$

The situation is illustrated in the right of Figure 14, which is the ultimate far field behavior.

We have shown, as far as the far field behavior is concerned, by example of a flat-plate airfoil at an angle of attack, that the disturbances are propagated along the equilibrium Mach waves. Surrounding them is a diffusion-like structure with the effective diffusivity $\mathcal{D}_{12}$. We could interpret the far field behavior in terms of, for instance, a momentum source $\rho_0 u_0 v \big]_{y=0}$, which is ultimately diffused about and propagated along equilibrium Mach lines. Hence, when the airfoil has a variable surface, we may think of the far field behavior as the superposition of such diffusion-like sources that lie in the region $0 < x < C$ at $y = 0$ where the source strength per unit length is $\rho_0 u_0 v\Big]_{y=0} = \rho_0 u_0 f(x)$ and is zero elsewhere. By analogy with the conduction of heat[59], we may write, heuristically,

$$\rho_0 u_0 v = \frac{\rho_0 u_0}{\sqrt{\pi}} \int_{-\frac{x-\beta_2 y}{2\sqrt{\mathcal{D}_{12}\frac{y}{u_0}}}}^{0} f\left(x-\beta_2 y+2r\sqrt{\mathcal{D}_{12}\frac{y}{u_0}}\right) e^{-r^2}\, dr + \frac{\rho_0 u_0}{\sqrt{\pi}} \int_{0}^{\frac{c-(x-\beta_2 y)}{2\sqrt{\mathcal{D}_{12}\frac{y}{u_0}}}} f\left(x-\beta_2 y+2r\sqrt{\mathcal{D}_{12}\frac{y}{u_0}}\right) e^{-r^2}\, dr \qquad (5.92)$$

for the far field behavior. When $f(x) = \text{constant}$, as for the flat-plate airfoil at an angle of attack, this reduces to equation (5.91). We conclude also that, although the disturbances are no longer concentrated but are spread out about the equilibrium Mach waves far from the airfoil, by analogy with the conduction of heat, the totality of the disturbances is nevertheless preserved.

So far, we have only considered the ultimate far field behavior by retaining the dominant term in the asymptotic approximation, and where the effect on the ultimate structure that surrounds the equilibrium Mach wave is a diffusion-like behavior contributed by the presence of the intermediary Mach wave. Now the frozen Mach wave, which is two orders higher than the equilibrium Mach wave in the fundamental equation (5.64), is expected at first to exert a dispersion-like behavior on the structure about the equilibrium Mach wave. However, the presence of the intermediary Mach wave gives rise to the damping factor $e^{-x'\alpha_2 s_i^2}$ in equation (5.83) in our examination of the asymptotic behavior, and essentially submerges the oscillatory tendency contributed by the frozen Mach wave, whose effect is imbedded in the factor $e^{i x' \alpha_3 s_i^3}$. The overwhelming effect of the exponential damping in equation (5.83) renders the result that not even local approximate oscillations are exhibited. In fact, the result is monotonic. We can obtain estimates of the effect of the frozen Mach wave on the ultimate wave behavior by considering higher-order terms in the asymptotic approximation. The formal aspects of the above discussion are as follows.

We may write the exponential, using equation (5.82), in the form

$$e^{-x' m S_1(s)} \cong e^{-x'\alpha_1 s + x'\alpha_2 s^2} \left\{ 1 - x'\alpha_3 s^3 + \cdots \right\} .$$

In our consideration of the simple semi-infinite wedge of half angle α_0, for instance, we already have obtained the interpretation of what corresponds to the first term. The second term has the formal interpretation, except for a multiplicative factor $(x'\alpha_3)$, as being the third x'- derivative of the already known first term in the (x', y') plane, since the initial conditions in x' vanish. This is similar for other higher order terms. In this case,

$$v'_{x'}(x', y') = \frac{\alpha_0 \lambda v}{\sqrt{4\pi \mathcal{D}_{12} \frac{y}{u_0}}} e^{-\chi^2} \cdot \left\{ 1 - \frac{1+\sigma}{\sigma} \frac{\lambda v}{\sqrt{\mathcal{D}_{12} \frac{y}{u_0}}} \left(\frac{\mathcal{D}_{02}}{\mathcal{D}_{12}} - 1 \right) (3\chi - \chi^3) + \mathcal{O}\left(\frac{1}{y'} \right) \right\} \tag{5.93}$$

where

$$\chi = \frac{x - \beta_2 y}{\sqrt{4 \mathcal{D}_{12} \frac{y}{u_0}}}$$

and we have denoted

$$\vartheta_{02} = \frac{1}{1+\sigma}\frac{\beta_2}{2}\left[1-\left(\frac{\beta_0}{\beta_2}\right)^2\right](\lambda_v u_0) . \qquad (5.94)$$

Hence the term of order $(y')^{1/2}$ gives rise to a skewness in the Gaussian. The behavior of v' itself is then

$$v' = \frac{\alpha_0}{2}(1+\operatorname{erf}\chi) - \frac{\alpha_0}{2}\frac{1+\sigma}{\sigma}\frac{\lambda_v}{\sqrt{\pi\vartheta_{12}\frac{y}{u_0}}}\left(\frac{\vartheta_{02}}{\vartheta_{12}}-1\right)(2\chi^2-1)e^{-\chi^2} + O\left(\frac{1}{y'}\right) \qquad (5.95)$$

which shows the approach to the ultimate form of the wave structure. The situation is similar for the airfoil of finite length, and the higher order effect of the frozen Mach wave is in the term with the coefficient $\lambda_v / \sqrt{\pi\vartheta_{12}\frac{y}{u_0}}$, which ultimately becomes unimportant. The dominant effect is the diffusive effect exerted by the presence of the intermediary Mach wave.

3.3. Pressure Coefficient. - In this section, we are concerned primarily with obtaining the operational solution for the pressure coefficient

$$C_p = \frac{[p']_{y=0}}{\rho_0 u_0^2/2}$$

from which subsequent calculations of aerodynamic forces, if desired, can be made. Let us denote $\overline{C}_p(s)$ as the Laplace transformation of $C_p(x')$ and

$$\overline{C}_p(s) = \int_0^\infty e^{-sx'} C_p(x')\,dx'$$

in which the desired solution is obtained from

$$C_p(x') = \frac{1}{2\pi i}\int_{L_1} \overline{C}_p(s)\, e^{sx'}\,ds . \qquad (5.96)$$

Applying the transformation to the dynamical equation of the gas, equation (5.37) gives for the pressure coefficient:

$$\frac{1}{2}\bar{C}_p(s) = -s\Phi'(s,0) + K\left[\Phi'_p(s,0) - \Phi'(s,0)\right] \tag{5.97}$$

where $\Phi(s,0)$ is the Laplace transformation of $\phi'(x',0)$, and from equation (5.73),

$$\Phi'(s,0) = -\frac{F(s)}{S_1(s)} \tag{5.98}$$

where $F(s)$ is the Laplace transformation of $f(x')$ and $S_1(s)$ is defined by equation (5.72). The dynamical equation for the particle cloud, equation (5.38), gives

$$\Phi'_p(s,0) = \frac{1}{1+s}\Phi'(s,0) \tag{5.99}$$

Combining equations (5.97) through (5.99), we then have for the pressure coefficient

$$\frac{1}{2}\bar{C}_p(s) = \frac{sF(s)}{S_1(s)} + K\frac{sF(s)}{(1+s)\cdot S_1(s)} \tag{5.100}$$

Recalling the definition of $S_1(s)$ given by equation (5.72), and letting $s_1+s_2=(1+\sigma)\left(\frac{\beta_1}{\beta_0}\right)^2$ and $s_1s_2=\sigma\left(\frac{\beta_2}{\beta_0}\right)^2$, we can write

$$S_1(s) = \sqrt{\frac{(s+s_1)(s+s_2)}{(s+1)(s+\sigma)}}\;\beta_0 s \tag{5.101}$$

where

$$\begin{matrix}s_1\\ s_2\end{matrix} = \frac{1+\sigma}{2}\left(\frac{\beta_1}{\beta_0}\right)^2\left[1\pm\sqrt{1-\frac{4\sigma}{(1+\sigma)^2}\left(\frac{\beta_2}{\beta_1}\right)^2\left(\frac{\beta_0}{\beta_1}\right)^2}\,\right] \tag{5.102}$$

are real quantities.

Let us first discuss the situation for a simple wedge in which $F(s) = \frac{\alpha_0}{s}$, where α_0 is the half angle. The general case is then related to that of the simple wedge through Duhamel's formula, as will be shown subsequently. Equation (5.100), specialized for a wedge, appears in the form:

$$\frac{\beta_0}{2\alpha_0}\left[\overline{C}_p(s)\right]_{WEDGE} = s\, g_1(s)\, g_2(s) + (1+K_1+\sigma)\, g_1(s) g_2(s) + (1+K_1)\sigma \frac{1}{s}\, g_1(s) g_2(s) \qquad (5.103)$$

where we have denoted

$$g_1(s) = \frac{1}{\sqrt{(s+1)(s+s_1)}} \supset e^{-\frac{s_1+1}{2}x'} I_0\left(\frac{s_1-1}{2}x'\right) = G_1(x') \qquad (5.104)$$

$$g_2(s) = \frac{1}{\sqrt{(s+\sigma)(s+s_2)}} \supset e^{-\frac{s_2+\sigma}{2}x'} I_0\left(\frac{s_2-\sigma}{2}x'\right) = G_2(x') \qquad (5.105)$$

and I_0 is the Bessel function with imaginary argument of order zero[60]. The symbol $\supset$ implies the corresponding interpretation of the Laplace transform of the function, which is obtained from known results[61]. The interpretation of the pressure coefficient for the wedge is simply

$$\frac{\beta_0}{2\alpha_0}\left[C_p(x')\right]_{WEDGE} = \frac{dY(x')}{dx'} + (1+K_1+\sigma)Y(x') + (1+K_1)\sigma\int_0^{x'} Y(\xi)\,d\xi \qquad (5.106)$$

where

$$\frac{dY(x')}{dx'} = e^{-\frac{s_2+\sigma}{2}x'} I_0\left(\frac{s_2-\sigma}{2}x'\right) - \frac{s_1+1}{2} e^{-\frac{s_2+\sigma}{2}x'}\int_0^{x'} e^{\frac{s_2+\sigma-s_1-1}{2}\xi} I_0\left(\frac{s_1-1}{2}\xi\right) I_0\left(\frac{s_2-\sigma}{2}(x'-\xi)\right)d\xi$$

$$+ \frac{s_1-1}{2} e^{-\frac{s_2+\sigma}{2}x'}\int_0^{x'} e^{\frac{s_2+\sigma-s_1-1}{2}\xi} I_1\left(\frac{s_1-1}{2}\xi\right) I_0\left(\frac{s_2-\sigma}{2}(x'-\xi)\right)d\xi \qquad (5.107)$$

and

$$Y(x') = e^{-\frac{s_2+\sigma}{2}x'} \int_0^{x'} e^{\frac{s_2+\sigma-s_1-1}{2}\zeta} I_0\left(\frac{s_1-1}{2}\zeta\right) I_0\left(\frac{s_2-\sigma}{2}(x'-\zeta)\right) d\zeta \,. \qquad (5.108)$$

The representation of the pressure coefficient given in the above form is amenable to study of the forms for $x' \to 0^+$ and $x' \to \infty$. When $x' \to 0^+$, the value is simply unity since $I_0(0) = 1$, which corresponds to a "frozen" jump from the free-stream value of zero. When $x' \to \infty$, the first two terms in equation (5.106) have decayed to zero, and the asymptotic value of the pressure coefficient comes from the last term in the form of

$$(1+K)\sigma \int_0^{\infty} e^{-\frac{s_2+\sigma}{2}\xi} \left[\int_0^{\xi} e^{\frac{s_2+\sigma-s_1-1}{2}\zeta} I_0\left(\frac{s_1-1}{2}\zeta\right) I_0\left(\frac{s_2-\sigma}{2}(x'-\zeta)\right) d\zeta \right] d\xi \qquad (5.109)$$

and the interpretation of which is simply $(1+K)\sigma$ times the Laplace transform of the convolution integral in the bracket from the ξ- plane to the $\left(\frac{s_2+\sigma}{2}\right)$ - plane. This is then

$$(1+K)\sigma \int_0^{\infty} e^{-\frac{s_2+\sigma}{2}\xi} \left[e^{-\frac{s_2+\sigma-s_1-1}{2}\xi} I_0\left(\frac{s_1-1}{2}\xi\right) \right] d\xi \int_0^{\infty} e^{-\frac{s_2+\sigma}{2}\xi} \left[I_0\left(\frac{s_2-\sigma}{2}\xi\right) \right] d\xi \qquad (5.110)$$

which, from known results[61], gives

$$\frac{\beta_0}{2\alpha_0}\left[C_p(\infty)\right]_{\text{WEDGE}} = (1+K)\sigma \frac{1}{\sqrt{s_2}\sqrt{\sigma}\sqrt{s_1}\sqrt{1}} = (1+K)\frac{\beta_0}{\beta_2} \qquad (5.111)$$

using the fact that $s_1 s_2 = \sigma\left(\frac{\beta_2}{\beta_0}\right)^2$. Now from ordinary aerodynamics, it is well known that the pressure coefficient is simply

$\left[C_p\right]_{WEDGE} = \frac{2\alpha_o}{\beta_o}$. The pressure coefficient far downstream when the particle cloud and the gas are in ultimate equilibrium is

$$\frac{1}{2}\left[C_p\right]_{WEDGE,\,e} = \frac{(P')_{y=0,\,e}}{(1+\kappa)\rho_o u_o^2/2} \tag{5.112}$$

and corresponds to our asymptotic value, equation (5.111), since we normalized the pressure by $\rho_o u_o^2/2$ instead of $(1+\kappa)\,\rho_o u_o^2/2$.

Returning to equation (5.100) for the general shape of the boundary, the pressure coefficient may be written in the form:

$$\frac{\beta_o}{2}\overline{C}_p(s) = sF(s)\left\{\frac{\beta_o}{2\alpha_o}\left[\overline{C}_p(s)\right]_{WEDGE}\right\} \tag{5.113}$$

which may be interpreted simply, according to the Duhamel formula:

$$\frac{\beta_o}{2}C_p(x') = \frac{d}{dx'}\int_0^{x'} f(x'-\zeta)\left\{\frac{\beta_o}{2\alpha_o}\left[C_p(\zeta)\right]_{WEDGE}\right\}d\zeta\,. \tag{5.114}$$

This may also be written in the form

$$\frac{\beta_o}{2}C_p(x') = f(x') + \int_0^{x'} f(x'-\zeta)\left\{\frac{\beta_o}{2\alpha_o}\frac{d}{d\zeta}\left[C_p(\zeta)\right]_{WEDGE}\right\}d\zeta \tag{5.115}$$

since $\frac{\beta_o}{2\alpha_o}\left[C_p(0)\right]_{WEDGE} = 1$. A third form may be written as

$$\frac{\beta_o}{2}C_p(x') = f(0)\left\{\frac{\beta_o}{2\alpha_o}\left[C_p(x')\right]_{WEDGE}\right\} + \int_0^{x'} f'(x'-\zeta)\left\{\frac{\beta_o}{2\alpha_o}\left[C_p(\zeta)\right]_{WEDGE}\right\}d\zeta \tag{5.116}$$

In this situation, the pressure coefficient can be determined for arbitrary boundary shapes from that for the wedge. The pressure coefficient for the wedge given by equation (5.106) is determined when the integral $Y(x')$, as defined in equation (5.108), is evaluated. This is discussed in Appendix V-A.

A numerical example is considered, and we have taken $M_0 = 1.414$, $\gamma = 1.40$, $\kappa = 0.25$, $c_s/c_p = 1.10$, and $\lambda_v/\lambda_T = 0.819$. The calculated parameters are then $\beta_2/\beta_0 = 1.30984$, $\beta_1/\beta_0 = 1.15444$, and $\beta_0 = 1$; $s_1 = 1.61523$, $s_2 = 1.10917$. The function $Y(x')$ is more conveniently evaluated by directly calculating the integrals given in equation (5.119) through the use of a computer. The result of such a calculation, which will be used throughout, is shown as the solid line in Figure 15. For comparison purposes only, a three-term approximation with Y_1, Y_2, and Y_3, given by equations (5.121) through (5.123), is also shown in Figure 15 with the Bessel functions obtained from tables[60].

The first and third terms in the pressure coefficient given in equation (5.106) for the simple wedge are $\frac{d}{dx'} Y(x')$ and $\int_0^{x'} Y(\zeta)\, d\zeta$ respectively, and are more conveniently obtained by performing such numerical operations on $Y(x')$ directly. All three terms occurring in the pressure coefficient, equation (5.106), are shown as solid lines in Figure 16. They are subtracted by their respective "frozen" counterparts ($\kappa = 0$, with $\sigma = 1$)

$$\left(\frac{dY}{dx'}\right)_f = (1 - x')\, e^{-x'}$$

$$\Big((1+\kappa+\sigma)\, Y\Big)_f = 2\, x'\, e^{-x'}$$

$$\left((1+\kappa)\sigma \int_0^{x'} Y(\zeta)\, d\zeta\right)_f = 1 - (1+x')\, e^{-x'}$$

for purposes of facilitating comparison with the results of Chapter

VI. The sum of all three "frozen" counterparts gives $\left(\frac{\beta_o}{2\alpha_o} C_p\right)_f$, which is, of course, unity. The final form of the pressure coefficient for a simple wedge is shown as the solid line in Figure 17.

The perturbation pressure coefficient $\frac{\beta_o}{2\alpha_o} C_p$ at the leading edge of the wedge is attributed to the abrupt turning of the gas, and jumps from the zero free-stream value to the "frozen" value of unity. Since the turning of the gas is accomplished abruptly and is hence much shorter than the particle velocity and temperature equilibration distances, the particles there possess its free-stream velocity and temperature. Subsequently, the relaxations take place: the compressed gas is being accelerated and cooled by the particle cloud, which decreases the pressure coefficient. Both the gas and particle cloud then come to final equilibrium corresponding to the wedge flow of a single heavier gas with $\left(\frac{\beta_o}{2\alpha_o} C_p\right)_e = (1+\kappa) \frac{\beta_o}{\beta_2}$.

APPENDIX V-A

Consideration of a Convolution Integral of Two Bessel Functions with Imaginary Argument

The integral that needs to be evaluated in the exact form of the pressure coefficient for a wedge, given by equation (5.106), is

$$Y(x') = \int_0^{x'} e^{-a_1'\xi} I_0(b_1'\xi)\, e^{-a_2'(x'-\xi)} I_0(b_2'(x'-\xi))\,d\xi \tag{5.117}$$

where we have denoted $a_1' = \frac{s_1+1}{2}$, $a_2' = \frac{s_2+\sigma}{2}$, $b_1' = \frac{s_1-1}{2}$, $b_2' = \frac{s_2-\sigma}{2}$ for simplicity. The form of the integral $Y(x')$ does not appear to have been previously considered in the discussions of integrals of Bessel functions[62]. The procedure to be followed here is the following[60]: we replace the two Bessel functions with imaginary argument by their respective representation in terms of Poisson's integral,

$$I_\nu(z) = \frac{\frac{1}{2}\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \left(e^{z\cos\theta} - e^{-z\cos\theta}\right)\sin^{2\nu}\theta\,d\theta. \tag{5.118}$$

Change the order of integrations and carry out the integration involving x' first, giving

$$Y(x') = \frac{1}{(2\pi)^2}\int_0^\pi\int_0^\pi \left[\frac{e^{(-a_1'+a_2'+b_1'\cos\theta-b_2'\cos\bar\theta)x'}-1}{(-a_1'+a_2'+b_1'\cos\theta-b_2'\cos\bar\theta)} + \frac{e^{(-a_1'+a_2'-b_1'\cos\theta-b_2'\cos\bar\theta)x'}-1}{(-a_1'+a_2'-b_1'\cos\theta-b_2'\cos\bar\theta)}\right] e^{(-a_2'+b_2'\cos\bar\theta)x'}\,d\theta\,d\bar\theta$$

$$+\frac{1}{(2\pi)^2}\int_0^\pi\int_0^\pi\left[\frac{e^{(-a_1'+a_2'+b_1'\cos\theta+b_2'\cos\bar\theta)x'}-1}{(-a_1'+a_2'+b_1'\cos\theta+b_2'\cos\bar\theta)}\right.$$

$$\left.+\frac{e^{(-a_1'+a_2'-b_1'\cos\theta+b_2'\cos\bar\theta)x'}-1}{(-a_1'+a_2'-b_1'\cos\theta+b_2'\cos\bar\theta)}\right]e^{(-a_2'-b_2'\cos\bar\theta)x'}\,d\theta\,d\bar\theta. \quad (5.119)$$

The function involving the exponential in the bracketed terms is of the form

$$\frac{1}{\alpha}(e^{\alpha x'}-1)=\frac{1}{\alpha}\left(\alpha x'+\frac{(\alpha x')^2}{2!}+\frac{(\alpha x')^3}{3!}+\cdots\right)$$

$$=\sum_{m=1}^{\infty}\alpha^{m-1}\frac{(x')^m}{m!}$$

and the radius of convergence of the series is unlimited. Our integral can now be written

$$Y(x')=\frac{e^{-a_2'x'}}{(2\pi)^2}\sum_{m=1}^{\infty}\frac{(x')^m}{m!}\int_0^\pi\int_0^\pi\left\{\left[(-a_1'+a_2'+b_1'\cos\theta-b_2'\cos\bar\theta)^{m-1}\right.\right.$$

$$\left.+(-a_1'+a_2'-b_1'\cos\theta-b_2'\cos\bar\theta)^{m-1}\right]e^{(b_2'\cos\bar\theta)x'}$$

$$+\left[(-a_1'+a_2'+b_1'\cos\theta+b_2'\cos\bar\theta)^{m-1}\right.$$

$$\left.\left.+(-a_1'+a_2'-b_1'\cos\theta+b_2'\cos\bar\theta)^{m-1}\right]e^{(-b_2'\cos\bar\theta)x'}\right\}d\theta\,d\bar\theta. \quad (5.120)$$

The form of the integral is

$$Y(x') = \sum_{m=1}^{\infty} Y_m(x') \tag{5.121}$$

When $m = 1$,

$$Y_1(x') = x' I_0(b_2 x') \, e^{-a'_2 x'} \tag{5.122}$$

is simply obtained through use of the definition given by equation (5.118). For terms higher than $m = 1$, the following representation of Bessel functions with imaginary argument of integral order is required:

$$I_n(z) = \frac{(-1)^n}{\pi} \int_0^{\pi} e^{-z\cos\theta} \cos n\theta \, d\theta$$

$$= \frac{1}{\pi} \int_0^{\pi} e^{z\cos\theta} \cos n\theta \, d\theta$$

where n is an integer. A few higher terms are given in the following:

$$Y_2(x') = \frac{x'^2}{2!} \left\{ (-a'_1 + a'_2) I_0(b'_2 x') - b'_2 I_1(b'_2 x') \right\} e^{-a'_2 x'} \tag{5.123}$$

$$Y_3(x') = \frac{x'^3}{3!} \left\{ \left[(-a'_1 + a'_2)^2 + b'^2_2 \right] I_0(b'_2 x') - \left[2 b'_2 (-a'_1 + a'_2) \right] I_1(b'_2 x') \right.$$

$$\left. + \, b'^2_2 \, I_2(b'_2 x') \right\} e^{-a_2 x'} \tag{5.124}$$

$$Y_4(x') = \frac{x'^4}{4!}\Big\{ \left[(-a_1'+a_2')^3 + \frac{3}{2}(-a_1'+a_2')(b_1'^2 - b_2'^2) \right] I_0(b_2' x')$$

$$- \left[3 b_2' (-a_1'+a_2')^2 + \frac{3}{2} b_1'^2 b_2' + \frac{3}{4} b_2'^3 \right] I_1(b_2' x')$$

$$+ \left[\frac{3}{2}(-a_1'+a_2') b_2'^2 \right] I_2(b_2' x') - \left[\frac{1}{4} b_3'^3 \right] I_3(b_2' x') \Big\} e^{-a_2' x'} \qquad (5.125)$$

Generalizations to convolution integrals when the Bessel functions are of higher order can be made easily according to procedures similar to those described here. Here, it is sufficient to consider $Y(x')$; its derivative and integral that occur in the pressure coefficient can be more conveniently obtained numerically from $Y(x')$ itself.

VI. LINEARIZED SUPERSONIC FLOW FOR SMALL PARTICLE-FLUID DENSITY RATIO ($\kappa << 1$)

1. General Discussion

In this chapter, we consider the steady supersonic flow in terms of a "dilute" particle cloud. In this context, it is meant that the particle-cloud density, $\rho_{p_o} = n_o m_p$, where n_o is the number of particles per unit volume of space and m_p is the mass of a single particle, is small relative to the fluid density ρ_o . In this situation, it means that either the individual particles are "light weighted" or that the number density of the particles is small, or both. However, we always regard that there are sufficient numbers of particles present in the unit volume so that our consideration of the particle cloud as a continuum is still meaningful.

The ratio of the equilibration times τ_{v_o} / τ_{T_o} and the ratio of the heat capacities c_s / c_p are considered as of order unity in this study, as is expected in the actual situation.

In this context, the inclinations of the frozen Mach wave $\tan^{-1}(\frac{1}{\beta_o})$, the intermediary Mach wave $\tan^{-1}(\frac{1}{\beta_1})$, and the equilibrium Mach wave $\tan^{-1}(\frac{1}{\beta_2})$, are not significantly different from one another. In fact, when $\kappa = \frac{\rho_{p_o}}{\rho_o} << 1$, it is found that

$$\epsilon_1 \equiv \frac{\beta_1}{\beta_o} - 1 \cong \kappa \frac{1+\beta_o^2}{2\beta_o} \frac{1+(\gamma-1)\frac{c_s}{c_p}\frac{\lambda_v}{\lambda_T}}{1+\frac{\lambda_v}{\lambda_T}} \tag{6.1}$$

and

$$\epsilon_2 \equiv \frac{\beta_2}{\beta_0} - 1 \cong \kappa \frac{1+\beta_0^2}{2\beta_0} \left[1 + (\gamma - 1)\frac{c_s}{c_p}\right] \tag{6.2}$$

where use is made of the definitions of the sound speeds given in Chapter V, from which the expansions for $\kappa \ll 1$ are made. For the range of the free-stream Mach number M_0 consistent with the linearized theory, the factor $(1+\beta_0^2)/2\beta_0$ is of order unity. Hence $\epsilon_1 = \mathcal{O}(\kappa)$ and $\epsilon_2 = \mathcal{O}(\kappa)$. Of course, $\frac{\epsilon_1}{\epsilon_2} = \mathcal{O}(1)$.

The physical consequence of this situation is that the Mach waves emanating from the same point, bounded by the frozen Mach wave and the equilibrium Mach wave, are very closely clustered together. If this is a point disturbance, then the particle-fluid equilibration processes essentially take place in this small angular region. This physical situation will be explored to our advantage in a further simplification of the linearized supersonic flow involving a "dilute" particle cloud.

However, one most naturally is inclined to apply higher order corrections to a zeroth order approximation in which κ is identically zero beginning with

$$\beta_0^2 \phi_{xx} - \phi_{yy} = 0$$

On the other hand, one recognizes that in this approximation the mutual effect of the waves of various orders is lost. Hence, the scheme of this type of perturbation is then obviously a singular one; it does not provide us with the smooth transition from the frozen to the final equilibrium state.

In deriving a simplified but uniformly valid differential equation from the full equation for the "dilute" particle cloud situation, we shall begin first from a physically intuitive approach. Subsequently, we shall show that by properly applying the "boundary layer" concept(63, 64) in a transformed coordinate system, the resulting zeroth order equation is the same as that obtained from physical arguments. However, the formal perturbation scheme enables us to demonstrate how systematic higher corrections could be obtained.

2. The Simplified Linearized Equation for Two-Dimensional Steady Supersonic Flow.

Within the limitations placed on the linearized theory that the free-stream Mach number must be sufficiently larger than unity and the disturbances must be sufficiently small, the flow is everywhere supersonic. In the two-dimensional case or in the three-dimensional case away from the tip frozen Mach conoids, the upper surface of a thin airfoil is then independent of the lower surface. For the purpose of fixing our ideas, consider the upper surface of an airfoil in a stream of infinite extent. The disturbances produced by the airfoil are carried along Mach waves inclined towards the downstream direction, i.e., waves of the $x = \beta_i y$ family. Since disturbances are absent in the stream at infinity, in the consideration of the upper surface we need only to focus our attention on waves of a single family. This principle is well known in the ordinary supersonic aerodynamics in which the single function $\phi(x-\beta_0 y)$ is retained from the general solution for the upper surface, while $\phi(x+\beta_0 y)$ for the lower surface. It certainly suggests a similar isolation of Mach waves of a single family in our present problem, i.e., the family of downstream-running Mach waves. In fact, in Chapter V, this principle is already applied to the general solution in the Laplace transformed plane. However, here we take advantage of the "dilute" particle-cloud situation and apply this principle to simplify the full linearized equation (5.68).

2.1. Physical Derivation. - We essentially follow the spirit of Whitham's[50] approximate treatment of wave motions. The point of departure here is that instead of focusing our attention on the immediate vicinity of an individual downstream-running Mach wave, we make use of the physical fact that the various waves of one family are very closely clustered together, and hence to a good approximation the focusing of our attention on the downstream-running Mach waves holds for the entire structure enclosed in between the frozen and the equilibrium Mach waves emanating from the same point. The simplified equation which emerges from this approximation then enables one to obtain the pressure coefficient essential in thin airfoil theory. These ideas are interpreted more formally in the following.

Let us rewrite the full non-dimensional form of the linearized equation (5.68) in the form which exhibits the wave operators corresponding to the two families $x = \pm \beta_i y$ of waves:

$$\left\{\frac{\partial^2}{\partial x'^2}\left(\beta_0\frac{\partial}{\partial x'}-\frac{\partial}{\partial y'}\right)\left(\beta_0\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}\right) + (1+\sigma)\frac{\partial}{\partial x'}\left(\beta_1\frac{\partial}{\partial x'}-\frac{\partial}{\partial y'}\right)\left(\beta_1\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}\right)\right.$$

$$\left. + \sigma\left(\beta_2\frac{\partial}{\partial x'}-\frac{\partial}{\partial y'}\right)\left(\beta_2\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}\right)\right\}\phi' = 0 . \qquad (6.3)$$

Consider now the upper surface of the airfoil $(y' > 0)$, and the relevant wave operators that are to be retained are

$$\left(\beta_i\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}\right)$$

which describe waves inclined toward the downstream direction. We now make use of the principle[50] that for waves with Mach angles given by $\tan^{-1}\left(\frac{1}{\beta_i}\right)$, then

$$\beta_i\frac{\partial}{\partial x'} \approx -\frac{\partial}{\partial y'} .$$

In addition, we make use of the fact that when $K << 1$, then $\left(\frac{\beta_1}{\beta_0}-1\right) << 1$ and $\left(\frac{\beta_2}{\beta_0}-1\right) << 1$. Hence

$$-\frac{\partial}{\partial y'} \approx \beta_0\frac{\partial}{\partial x'} \approx \beta_1\frac{\partial}{\partial x'} \approx \beta_2\frac{\partial}{\partial x'} .$$

These approximations are to be made on the wave operators

$$\left(\beta_i\frac{\partial}{\partial x'}-\frac{\partial}{\partial y'}\right)$$

that are approximately insensitive as far as the description of waves that are inclined toward the downstream direction is concerned. The resulting equation, after integrating once with respect to x' and discarding the integration constant (function of y') for wave motions, then has the form:

$$\left\{\frac{\partial^2}{\partial x'^2}\left(\beta_0\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}\right) + (1+\sigma)\frac{\partial}{\partial x'}\left(\beta_1\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}\right) + \sigma\left(\beta_2\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}\right)\right\}\phi' = 0 \quad (6.4)$$

which describes Mach waves inclined towards the downstream direction.

2.2. Derivation by a "Boundary Layer Technique". - Implicit in the previous intuitive derivation is that the changes across the downstream-running Mach waves, that is, changes with respect to the variable $x'-\beta_0 y'$, are rapid compared with changes in the other independent variable, say y'. This, in fact, is the spirit of the "boundary layer" concept suggested by Moore and Gibson[48, 49] in their consideration of the non-equilibrium, linearized supersonic flow involving only a single characteristic time. The fact that changes with respect to y' are "slowly varying" is indicated by the behavior along the frozen Mach wave expressed by the exponential decaying factor from our exact considerations in equation (5.76):

$$e^{-\frac{1+\sigma}{2}\left[\left(\frac{\beta_1}{\beta_0}\right)^2-1\right]\beta_0 y'}$$

which, for $|\kappa| << 1$, is approximately

$$e^{-(1+\sigma)\epsilon_1\beta_0 y'} \quad (6.5)$$

where ϵ_1 is defined by equation (6.1). The "boundary layer" is more readily demonstrated after a transformation of the full linearized equation (5.68) from the physical plane (x', y') to the characteristic variable defined by

$$\beta_o^2 \xi = x' - \beta_o y'$$

and (6.6)

$$\beta_o \eta = y'$$

is made. With the transformation rules

$$\frac{\partial}{\partial x'} = \frac{1}{\beta_o^2}\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial y'} = \frac{1}{\beta_o}\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) \tag{6.7}$$

Then equation (5.68) takes the form:

$$\left\{\left(\frac{\partial^2}{\partial \xi^2} + (1+\sigma)\beta_o^2\frac{\partial}{\partial \xi} + \sigma\beta_o^4\right)\left(2\frac{\partial^2}{\partial\xi\partial\eta} - \frac{\partial^2}{\partial\eta^2}\right)\right.$$
$$\left. + \left((1+\sigma)\beta_o^2\frac{\partial}{\partial\xi} + \sigma\beta_o^4\right)\left(\epsilon_1 2\left(1+\frac{\epsilon_2}{\epsilon_1}\right) + \epsilon_1^2\left(1+\frac{\epsilon_2^2}{\epsilon_1^2}\right)\right)\frac{\partial^2}{\partial\xi^2}\right\}\phi' = 0 \tag{6.8}$$

where $\epsilon_1 << 1$ when $\kappa << 1$ and $\frac{\epsilon_2}{\epsilon_1} = \mathcal{O}(1)$. It is clear that, as in our previous discussion, the perturbation scheme using ϵ_1 identically zero as a starting point is a singular one. If we let

$$\bar{\xi} = \epsilon_1^a \xi \qquad \text{and} \qquad \bar{\eta} = \epsilon_1^b \eta \tag{6.9}$$

in order that the coordinates be stretched or contracted, we then determine a and b such that the resulting "boundary layer" equation becomes a uniformly valid one.(63, 64) It is found that this is obtained when $a = 0$ and $b = 1$:

$$\frac{\partial}{\partial\bar{\xi}}\left[\frac{\partial^3}{\partial\bar{\eta}\partial\bar{\xi}^3} + (1+\sigma)\beta_o^2\frac{\partial}{\partial\bar{\xi}}\left(\frac{\partial}{\partial\bar{\xi}} + \frac{\partial}{\partial\bar{\eta}}\right) + \sigma\beta_o^4\left(\frac{\epsilon_2}{\epsilon_1}\frac{\partial}{\partial\bar{\xi}} + \frac{\partial}{\partial\bar{\eta}}\right)\right]\phi'$$
$$= \frac{\epsilon_1}{2}\left[\frac{\partial^4}{\partial\bar{\eta}^2\partial\bar{\xi}^2} + (1+\sigma)\beta_o^4\frac{\partial}{\partial\bar{\xi}}\left(\frac{\partial^2}{\partial\bar{\eta}^2} - \frac{\partial^2}{\partial\bar{\xi}^2}\right) + \sigma\beta_o^4\left(\frac{\partial^2}{\partial\bar{\eta}^2} - \frac{\epsilon_2^2}{\epsilon_1^2}\frac{\partial^2}{\partial\bar{\xi}^2}\right)\right]\phi' . \tag{6.10}$$

Hence a systematic approximation for ϕ' can be made by expansion in ascending powers of the small parameter ϵ_1 :

$$\phi' = \phi^{(0)} + \epsilon_1 \phi^{(1)} + \epsilon_1^2 \phi^{(2)} + \cdots \quad (6.11)$$

Substituting equation (6.11) into equation (6.10) gives, except for a function of $\bar{\eta}$ which is discarded for wave motions, for the zeroth order equation:

$$\left\{ \frac{\partial^3}{\partial\bar{\eta}\partial\bar{\xi}^2} + (1+\sigma)\beta_0^2 \frac{\partial}{\partial\bar{\xi}}\left(\frac{\partial}{\partial\bar{\xi}} + \frac{\partial}{\partial\bar{\eta}}\right) + \sigma\beta_0^4\left(\frac{\epsilon_2}{\epsilon_1}\frac{\partial}{\partial\bar{\xi}} + \frac{\partial}{\partial\bar{\eta}}\right)\right\} \phi^{(0)} = 0 . \quad (6.12)$$

If we transform equation (6.12) back to the variables (x', y') , we obtain identically equation (6.4). This essentially demonstrates the equivalence of the more formal procedure with the earlier physically intuitive derivation. However, the more formal scheme enables one to obtain higher order corrections, if desired, from the non-homogeneous differential equations of the form

$$\frac{\partial}{\partial\bar{\xi}}\left[\frac{\partial^3}{\partial\bar{\eta}\partial\bar{\xi}^2} + (1+\sigma)\beta_0 \frac{\partial}{\partial\bar{\xi}}\left(\frac{\partial}{\partial\bar{\xi}} + \frac{\partial}{\partial\bar{\eta}}\right) + \sigma\beta_0^4\left(\frac{\epsilon_2}{\epsilon_1}\frac{\partial}{\partial\bar{\xi}} + \frac{\partial}{\partial\bar{\eta}}\right)\right] \phi^{(n)}$$

$$= \frac{1}{2}\left[\frac{\partial^4}{\partial\bar{\eta}^2\partial\bar{\xi}^2} + (1+\sigma)\beta_0^2 \frac{\partial}{\partial\bar{\xi}}\left(\frac{\partial^2}{\partial\bar{\eta}^2} - \frac{\partial^2}{\partial\bar{\xi}^2}\right) + \sigma\beta_0^4\left(\frac{\partial^2}{\partial\bar{\eta}^2} - \frac{\epsilon_2^2}{\epsilon_1^2}\frac{\partial^2}{\partial\bar{\xi}^2}\right)\right] \phi^{(n-1)} \quad (6.13)$$

where $n = 1, 2, \ldots$ indicates the order of the approximation. The appropriate condition at $\bar{\eta} = 0$, i.e., $y' = 0$, is to be satisfield by the zeroth order $(\phi_{y'}^{(0)})$ approximation, and all higher corrections then satisfy homogeneous conditions $\phi_{y'}^{(n-1)} = 0$ at $y' = 0$. In what follows, we only consider the zeroth order approximation exhibited by the equation given in the form of equation (6.4).

3. Mach Wave Structure

We now proceed to consider the wave structure as obtained from the simplified linear equation, and the problem is otherwise similar to that in section 3.2 of Chapter V. We drop the superscript on $\phi^{(0)}$, and it is understood that we are considering the zeroth order approximation. Equation (6.4) is repeated here for convenience:

$$\left\{ \frac{\partial^2}{\partial x'^2}\left(\beta_0 \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\right) + (1+\sigma)\frac{\partial}{\partial x'}\left(\beta_1 \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\right) + \sigma\left(\beta_2 \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\right) \right\} \phi' = 0 \qquad (6.14)$$

and the initial and boundary conditions are

$$\begin{aligned} &\phi' = \phi'_{x'} = \phi'_{x'x'} = 0, && x' = 0 \quad y' > 0 \\ &v' = \phi'_{y'} = f(x'), && y' = 0^+ \quad x' > 0 \end{aligned} \qquad (6.15)$$

and we consider the $y' > 0$ plane for the upper surface of the airfoil. We again apply the Laplace transformation and define

$$\Phi'(s,y') = \int_0^\infty e^{-sx'} \phi'(x',y')\, dx'$$

The transformed equation for the perturbation potential becomes

$$\frac{d\Phi'}{dy'} + S(s)\Phi' = 0 \qquad (6.16)$$

and

$$\Phi'(s,y') = A(s)\, e^{-S(s)y'} \qquad (6.17)$$

where

$$S(s) = \frac{s^2 + \frac{\beta_1}{\beta_0}(1+\sigma)s + \frac{\beta_2}{\beta_0}\sigma}{s^2 + (1+\sigma)s + \sigma}\ \beta_0 s \qquad (6.18)$$

For large s we then have $S(s) \sim +\beta_0 s$. Hence the solution expressed by equation (6. 17) describes waves inclined towards the downstream direction. The boundary condition on $y' = 0^+$ is then $\Phi'_{y'}(s, 0^+) = F(s)$ in the transformed plane and

$$F(s) = \int_0^\infty e^{-sx'} f(x')\,dx'$$

Hence, the appropriate solution is then

$$\Phi'(s, y') = -\frac{F(s)}{S(s)}\, e^{-S(s)y'} \tag{6.19}$$

and is interpreted as

$$\phi'(x', y') = \frac{1}{2\pi i}\int_{L_1} \frac{-F(s)}{S(s)}\, e^{sx' - S(s)y'}\, ds \tag{6.20}$$

in the (x', y') plane, where L_1 is the Bromwich path parallel to the imaginary axis and to the right of all singularities of $\frac{F(s)}{S(s)} e^{-S(s)y'}$.

It is again convenient to study the wave structure in terms of the normal velocity

$$V(s, y') = \Phi'_{y'}(s, y') = F(s)\, e^{-S(s)y'} . \tag{6.21}$$

Prior to deducing the asymptotic behavior of the wave structure, we first obtain the operational solution for $v'(x', y')$ from equation (6. 21). It can be shown that $S(s)$, as defined in equation (6. 18), can be rewritten in the form

$$S(s) = (1+\sigma)\epsilon_1\beta_0 + s\beta_0 - \Lambda_1 \frac{\beta_0\epsilon_1}{s+\sigma} - \Lambda_2 \frac{\beta_0\epsilon_1}{s+1} \tag{6.22}$$

with the use of partial fractions, where

$$\Lambda_1 = \frac{\sigma^2}{1-\sigma}\left[\frac{\epsilon_2}{\epsilon_1} - (1+\sigma)\right] \quad \text{and} \quad \Lambda_2 = \frac{1}{1-\sigma}\left[(1+\sigma) - \frac{\epsilon_2}{\epsilon_1}\sigma\right] \tag{6.23}$$

are both positive quantities. We now rewrite equation (6. 21) in the form:

$$V(s,y') = e^{-(1+\sigma)\epsilon_1\beta_0 y'} e^{-s\beta_0 y'} F(s)\left[1 + W(s,y')\right] \tag{6.24}$$

where

$$W(s,y') = \left(e^{\Lambda_1 \frac{\beta_0\epsilon_1}{s+\sigma} y'} - 1\right) + \left(e^{\Lambda_2 \frac{\beta_0\epsilon_1}{s+1} y'} - 1\right) + \left(e^{\Lambda_1 \frac{\beta_0\epsilon_1}{s+\sigma} y'} - 1\right)\left(e^{\Lambda_2 \frac{\beta_0\epsilon_1}{s+1} y'} - 1\right). \tag{6.25}$$

Again, let the symbol $\supset$ denote the corresponding interpretation of the Laplace transform of a function, and from known results(61),

$$W(s,y') \supset w(x',y') = \int_0^{x'} e^{-\sigma\zeta} \sqrt{\frac{\Lambda_1\beta_0\epsilon_1 y'}{\zeta}}\, I_1\left(2\sqrt{\Lambda_1\beta_0\epsilon_1 y'\zeta}\right) e^{-(x'-\zeta)} \sqrt{\frac{\Lambda_2\beta_0\epsilon_1 y'}{x'-\zeta}}\, I_1\left(2\sqrt{\Lambda_2\beta_0\epsilon_1 y'(x'-\zeta)}\right) d\zeta$$

$$+ e^{-\sigma x'} \sqrt{\frac{\Lambda_1\beta_0\epsilon_1 y'}{x'}}\, I_1\left(2\sqrt{\Lambda_1\beta_0\epsilon_1 y' x'}\right)$$

$$+ e^{-x'} \sqrt{\frac{\Lambda_2\beta_0\epsilon_1 y'}{x'}}\, I_1\left(2\sqrt{\Lambda_2\beta_0\epsilon_1 y' x'}\right) \tag{6.26}$$

where, in the first term, use is made of the convolution theorem, and I_1 is the Bessel function with imaginary argument of first order(60).

Now, with the use of the convolution theorem, we can write

$$F(s)W(s,y') \supset \int_0^{x'} f(\zeta)\, w(x'-\zeta, y')\, d\zeta \tag{6.27}$$

and the factor $e^{-s\beta_0 y'}$ in equation (6.24) lends itself to the "shift rule", and our operational solution for $v'(x',y')$ is then

$$V(s,y') \Rightarrow v'(x',y') = e^{-(1+\sigma)\epsilon_1\beta_0 y'} \int_0^{x'-\beta_0 y'} f(\zeta)\, w(x'-\beta_0 y'-\zeta, y')\, d\zeta$$

$$+ e^{-(1+\sigma)\epsilon_1\beta_0 y'} f(x'-\beta_0 y') \qquad (6.28)$$

where $w(x', y')$ is defined by equation (6.26). Along the frozen Mach wave, $x' = \beta_0 y'$, we recover the decay of signals:

$$e^{-(1+\sigma)\epsilon_1\beta_0 y'} f(0) .$$

We can interpret equation (6.28) in a similar manner as in the ordinary linearized supersonic flow as the result of the superposition of a sequence of small disturbances located on the $y' = 0^+$ plane, i.e., along the x'- axis. The distribution of sources is represented by $f(x')$. In supersonic flow, the induced velocity at any point (x', y') in the flow field is then determined by the superposition of those disturbances that are situated upstream from the point $\zeta = x' - \beta_0 y'$ in accordance with the "zone of action" determined by the frozen Mach wave, as indicated by the upper limit of the integral in equation (6.28).

The behavior of the ultimate wave structure is more readily deduced by considering $V(s, y')$ in equation (6.21). The standard procedure, which will not be repeated here, is identical to that described in section 3.2 of Chapter V, except that the discussion now refers to the function $S(s)$ defined by equation (6.18) instead of $S_1(s)$. Instead, we will show the consistency of the wave structure which emerges from the approximated equation with that from the full linearized equation of section 3. 2 of Chapter V. It is found here that the relevant saddle point is again $s_0 = 0$ and that the disturbances are ultimately propagated along the lowest order waves, i. e., the equilibrium Mach waves $x' - \beta_2 y' = \text{constant}$. $S(s)$ is analtyic at the saddle point $s_0 = 0$ and its Taylor expansion about $s = 0$ is then

$$-m\,S(s) = -\tilde{a}_1 s + \tilde{a}_2 s^2 - \tilde{a}_3 s^3 + \cdots \qquad (6.29)$$

where $m = y'/x'$ and

$$\begin{aligned} \tilde{a}_1 &= \beta_2 \frac{y'}{x'} \\ \tilde{a}_2 &= \frac{1+\sigma}{\sigma}\beta_2\left[1 - \frac{\beta_1}{\beta_2}\right]\frac{y'}{x'} \\ \tilde{a}_3 &= \frac{\beta_2}{\sigma}\left\{\left[1 - \frac{\beta_0}{\beta_2}\right] - \frac{(1+\sigma)^2}{\sigma}\left[1 - \frac{\beta_1}{\beta_2}\right]\right\}\frac{y'}{x'} \end{aligned} \qquad (6.30)$$

A comparison with equation (5. 84) of the corresponding coefficients in the Taylor series expansion for $S_1(s)$ in section 3. 2 of Chapter V shows the consistency. For instance, when $(1 - \frac{\beta_1}{\beta_2}) << 1$ as in our consideration of the "dilute" particle cloud $(\kappa << 1)$, the coefficient a_2 in equation (5. 84) reduces to that in equation (6. 30) in the zeroth order. The situation is similar for a_3, and other higher coefficients in the expansion. The situation is also similar for the structure about the intermediary Mach wave. The consistency is demonstrated, and our subsequent discussions from this point on in section 3. 2 of Chapter V holds for the "dilute" particle-cloud approximation as well. The advantage of the approximate treatment starting from a simpler differential equation, however, will be exhibited in our derivation of the pressure coefficient which is essential in airfoil theory.

4. Pressure Coefficient

In this section, the pressure coefficient

$$C_p = \frac{[p']_{y=0}}{\rho_0 u_0^2/2}$$

will be deduced for a thin airfoil in supersonic flow according to the "dilute" particle-cloud approximation. Again, we are concerned with obtaining its operational solution. For convenience, some of the general formulas given in section 3. 2 of Chapter V will be repeated here. The dynamical equation of the gas in the Laplace transformed plane $(s, 0)$ is

$$\frac{1}{2}\bar{C}_p(s) = -s\,\bar{\Phi}'(s,0) + \kappa\left[\bar{\Phi}_p'(s,0) - \bar{\Phi}'(s,0)\right] . \tag{6.31}$$

From equation (6. 19), we have

$$\bar{\Phi}'(s,0) = -\frac{F(s)}{S(s)} . \tag{6.32}$$

The dynamical equation of the particle cloud then becomes

$$\bar{\Phi}_p'(s,0) = \frac{\bar{\Phi}'(s,0)}{s+1} \tag{6.33}$$

Hence the equation for the pressure coefficient becomes

$$\frac{1}{2}\bar{C}_p(s) = \frac{sF(s)}{S(s)} + \kappa\frac{sF(s)}{(s+1)S(s)} \tag{6.34}$$

after inserting equations (6. 32) and (6. 33) into (6. 31). Recalling the definition of $S(s)$ defined in equation (6. 18), let $\tilde{s}_1 + \tilde{s}_2 = (1+\sigma)\frac{\beta_1}{\beta_0}$ $\tilde{s}_1\tilde{s}_2 = \sigma\frac{\beta_2}{\beta_0}$, then we can write

$$S(s) = \frac{(s+\tilde{s}_1)(s+\tilde{s}_2)}{(s+1)(s+\sigma)}\beta_0 s \tag{6.35}$$

where

$$\begin{matrix}\tilde{s}_1 \\ \tilde{s}_2\end{matrix} = \frac{1+\sigma}{2}\,\frac{\beta_1}{\beta_0}\left[1 \pm \sqrt{1 - \frac{4\sigma}{(1+\sigma)^2}\,\frac{\beta_2}{\beta_1}\,\frac{\beta_0}{\beta_1}}\,\right] \tag{6.36}$$

are real quantities. They differ from the results of our exact treatment, equation (5.102) through the absence of the square power attached to the ratios $\frac{\beta_1}{\beta_0}$, $\frac{\beta_2}{\beta_1}$. Again, we first discuss the simple wedge; the more general case can be expressed through Duhamel's formula. Now, equation (6.34) specialized for a wedge, in which $F(s) = \frac{\alpha_0}{s}$, becomes

$$\frac{\beta_0}{2\alpha_0}\left[\bar{C}_p(s)\right]_{WEDGE} = s\,h_1(s)h_2(s) + (1+\kappa+\sigma)h_1(s)h_2(s) + (1+\kappa)\sigma\frac{1}{s}\,h_1(s)h_2(s) \tag{6.37}$$

where

$$h_1(s) = \frac{1}{s+\tilde{s}_1} \supset e^{-\tilde{s}_1 x'} \equiv H_1(x')\ ;\quad h_2(s) = \frac{1}{s+\tilde{s}_2} \supset e^{-\tilde{s}_2 x'} \equiv H_2(x') \tag{6.38}$$

$$h_1(s)h_2(s) \supset \tilde{Y}(x') = \int_0^{x'} H_1(\zeta)H_2(x'-\zeta)\,d\zeta = \frac{e^{-\tilde{s}_1 x'} - e^{-\tilde{s}_2 x'}}{\tilde{s}_2 - \tilde{s}_1} \tag{6.39}$$

from known results[(61)]. The interpretation of the pressure coefficient then becomes

$$\frac{\beta_0}{2\alpha_0}\left[C_p(x')\right]_{WEDGE} = \frac{d}{dx'}\tilde{Y}(x') + (1+\kappa+\sigma)\tilde{Y}(x') + (1+\kappa)\sigma\int_0^{x'}\tilde{Y}(\xi)\,d\xi\,. \tag{6.40}$$

Comparison with the form of the exact pressure coefficient in equation (5.106) shows the correspondence of the terms, and this can be seen more readily by writing the approximate relation in the form

$$\frac{\beta_0}{2\alpha_0}\left[C_p(x')\right]_{WEDGE} = \left\{ e^{-\tilde{s}_2 x'} - \tilde{s}_1 e^{-\tilde{s}_2 x'} \cdot \frac{e^{(\tilde{s}_2-\tilde{s}_1)x'}-1}{\tilde{s}_2-\tilde{s}_1} \right\}$$
$$+ (1+\kappa+\sigma)\left\{ e^{-\tilde{s}_2 x'} \cdot \frac{e^{(\tilde{s}_2-\tilde{s}_1)x'}-1}{\tilde{s}_2-\tilde{s}_1} \right\}$$
$$+ (1+\kappa)\sigma \left\{ \frac{1}{\tilde{s}_1\tilde{s}_2} + \frac{\tilde{s}_1 e^{-\tilde{s}_2 x'} - \tilde{s}_2 e^{-\tilde{s}_1 x'}}{\tilde{s}_1\tilde{s}_2(\tilde{s}_2-\tilde{s}_1)} \right\} \quad . \tag{6.41}$$

When $x' \to 0^+$, we again have the "frozen" jump from the free-stream zero value to unity. When $x' \to \infty$, the exponentials die out and the surviving term is $(1+\kappa)\sigma\frac{1}{\tilde{s}_1\tilde{s}_2} = (1+\kappa)\frac{\beta_0}{\beta_2}$, which is the equilibrium value. The following form is more convenient for numerical calculations:

$$\frac{\beta_0}{2\alpha_0}\left[C_p(x')\right]_{WEDGE} = \left[1 - \frac{1+\kappa+\sigma-\tilde{s}_1}{(\tilde{s}_2-\tilde{s}_1)} + \frac{(1+\kappa)\sigma}{\tilde{s}_1\tilde{s}_2}\frac{\tilde{s}_1}{(\tilde{s}_2-\tilde{s}_1)}\right]e^{-\tilde{s}_2 x'} + \left[\frac{1+\kappa+\sigma-\tilde{s}_1}{(\tilde{s}_2-\tilde{s}_1)} - \frac{(1+\kappa)\sigma\,\tilde{s}_2}{\tilde{s}_1\tilde{s}_2(\tilde{s}_2-\tilde{s}_1)}\right]e^{-\tilde{s}_1 x'} + \frac{(1+\kappa)\sigma}{\tilde{s}_1\tilde{s}_2} \quad . \tag{6.42}$$

For the general boundary shape, it can be shown that the pressure coefficient can be written in the form

$$\frac{\beta_0}{2}\bar{C}_p(s) = sF(s)\left\{ \frac{\beta_0}{2\alpha_0}\left[\bar{C}_p(s)\right]_{WEDGE} \right\} \tag{6.43}$$

which is interpreted, as in the previous chapter, as

$$\frac{\beta_0}{2}C_p(x') = \frac{d}{dx'}\int_0^{x'} f(x'-\zeta)\left\{\frac{\beta_0}{2\alpha_0}\left[C_p(\zeta)\right]_{WEDGE}\right\}d\zeta \tag{6.44}$$

Other general forms are given at the end of section 3.3 of Chapter V.

For purposes of comparison with the exact form of the pressure coefficient in Chapter V, a similar numerical example is considered here: $M_0 = 1.414$, $\gamma = 1.40$, $\kappa = 0.25$, $c_s/c_p = 1.10$, and $\lambda_v / \lambda_T = 0.819$. The calculated parameters are again $\beta_2 / \beta_0 = 1.30984$, $\beta_1 / \beta_0 = 1.15444$, and $\beta_0 = 1$; with modified $\breve{\xi}_1 = 1.33666$, $\breve{\xi}_2 = 1.02329$. The function $\tilde{Y}(x')$ of our approximate treatment given by equation (6.39) is shown as a dashed curve in Figure 16, compared with the corresponding function from the exact treatment of Chapter V. The remaining two functions occurring in the pressure coefficient, $\frac{d\tilde{Y}(x')}{dx'}$ and $\int_0^{x'} \tilde{Y}(\xi)d\xi$, are similarly shown as dashed curves in Figure 16. Although the approximate treatment of the present chapter requires $\kappa \ll 1$, the comparisons shown in Figure 16 for the individual functions when $\kappa = 0.25$ is rather encouraging. However, the net result of the sum of all three terms, which gives the pressure coefficient shown in Figure 17, shows a difference of approximately 10% from the exact treatment within the range of variation of the pressure coefficient itself. This corresponds approximately an error of order κ^2 as is expected.

The results for the pressure coefficient on a simple wedge of this chapter is used to illustrate the effect of the equilibration parameter C/λ_v (here $\lambda_v/\lambda_T = \vartheta(1)$). The pressure distribution on a double-wedge airfoil is shown in Figure 18. When $\frac{C}{\lambda_v} \ll 1$, the pressure distribution very nearly corresponds to the frozen value on the front part of the airfoil. When $\frac{C}{\lambda_v} \gg 1$, equilibrium is reached rapidly. In any case, the abrupt change in body shape at mid-chord ($\frac{x}{C} = 0.5$) gives rise to an immediate expansion of the gas, which is accomplished in a distance much smaller than the particle equilibration distance, and the jump in pressure coefficient is then a frozen one of two units from whatever its previous value prior to the abrupt surface change. Subsequent equilibration then takes place in the aft portion of the airfoil, the extent of which is governed by the value of the parameter $\frac{c}{\lambda_v}$. Hence, even when $\frac{c}{\lambda_v} \gg 1$, the pressure coefficient is very nearly the equilibrium value only when the surface shape is "slowly varying".

VII. CONCLUDING REMARKS

Some problems of aeronautical interest in the motion of a fluid containing small solid particles have been studied. The examples considered fall into two categories: (1) incompressible viscous flow, with simultaneous occurrence of particle-fluid momentum relaxation and fluid viscous diffusion; (2) inviscid compressible flow, with simultaneous occurrence of coupled particle-fluid momentum and thermal relaxations and fluid compressibility.

Under the first category, the incompressible Rayleigh problem is studied. As in the classical fluid mechanics, the Rayleigh problem furnishes the physical insight for the more complicated problem of the laminar boundary layer. Thus, the relaxation-diffusion equation derived exhibits the manner in which the viscous diffusion process transits from "frozen" with the usual (frozen) diffusivity ν near $t/\tau_v << 1$ when the viscous layer is so thin that it contains negligible amounts of particles for interaction to "equilibrium" for $t/\tau_v >> 1$ with the diffusivity $\bar{\nu}$ referred to the density of the mixture $(1+\kappa)\rho$ as if the particle cloud and fluid were acting as a single heavier fluid. An exact solution is obtained, and the asymptotic behavior deduced for the two limiting regimes qualitatively confirms the expansion procedures used for the non-linear laminar boundary-layer problem where an exact solution is not available. The distortion of the fluid velocity due to particle-fluid interaction, particularly near the wall, gives rise to a new shear law as well as boundary layer growth, both indicating the transition from the "frozen" to the "equilibrium" regimes. The corresponding uncoupled (low Mach number) incompressible thermal Rayleigh problem is directly inferred. Just as Prandtl number unity indicates similarity of viscous and thermal diffusive effects across streamlines, then $\frac{\lambda_V}{\lambda_T}$ and $\frac{K'}{K \frac{c_s}{c_p}}$ being unity indicates similarity of the relaxation histories in the streamwise direction. Only when these are simultaneously satisfied can the velocity and temperature boundary layers be entirely similar.

The infinite flat plate oscillating in its own plane is also studied and periodic solutions of the relaxation-diffusion equation obtained. Here, the parameter indicating the extent of relaxation is $u_0 \omega^{-1} / \lambda_V$, where $u_0 \omega^{-1}$ is the wave length of the plate motion. The amplitude and phase lag of the particle-fluid slip-velocity is the same for all layers in the interior of the fluid, since the particles are non-interacting and respond to fluid motions only along their individual streamlines. The extent of the amplitude and lag is relatively large near the "frozen" regime when $u_0 \omega^{-1} / \lambda_V << 1$ and the particles are nearly standing still, and transits to the "equilibrium" regime when $u_0 \omega^{-1} / \lambda_V >> 1$ and the particles very nearly follow the fluid motion.

Under the second category, inviscid compressible flow, the first-order small perturbation theory is studied. Starting from the acoustical situation in a compressible fluid and particle cloud originally at rest and in thermal equilibrium, the relaxation-wave equations in airfoil coordinates are derived through a Galilean transformation, and essentially connect the acoustical and aerodynamical points of view in the small perturbation theory for a fluid containing small solid particles. It is shown that there is no net entropy generation due to particle-fluid interaction and that velocity potentials can be defined for both the fluid and the particle cloud in the first order theory. Two-dimensional steady supersonic flow is studied in detail; the wave structure deduced shows a rapid damping of disturbances along the "frozen" Mach wave which is the wave front, both damping and diffusiveness along an intermediate Mach wave, and finally diffusiveness along the "equilibrium" Mach wave which carries the bulk of the disturbance to regions far from the airfoil. An exact form of the pressure coefficient is obtained for arbitrary surface shape. The simple wedge, for instance, shows the transition from the "frozen" pressure jump at the leading edge to ultimate equilibrium far downstream. When the body is finite and of length C, then C/λ_v indicates the extent to which equilibration is possible over the surface of the body. In the study of linearized supersonic flow, λ_v/λ_T is taken as of order unity, which is the case when the fluid is a gas.

For two-dimensional supersonic flow or for planar problems of a quasi-two-dimensional nature, a simplification is afforded when $\kappa \ll 1$. In this situation, the various Mach waves from a single point are very closely clustered together, and physical approximations can be applied to the full linearized equation to retain those wave operators that describe Mach waves inclined to the downstream direction only. A more formal "boundary layer technique" is applied and the resulting zeroth order equation is the same as that obtained from physical arguments. The Mach wave structure is consistent with the exact treatment. The pressure coefficient, however, takes on a much simpler form.

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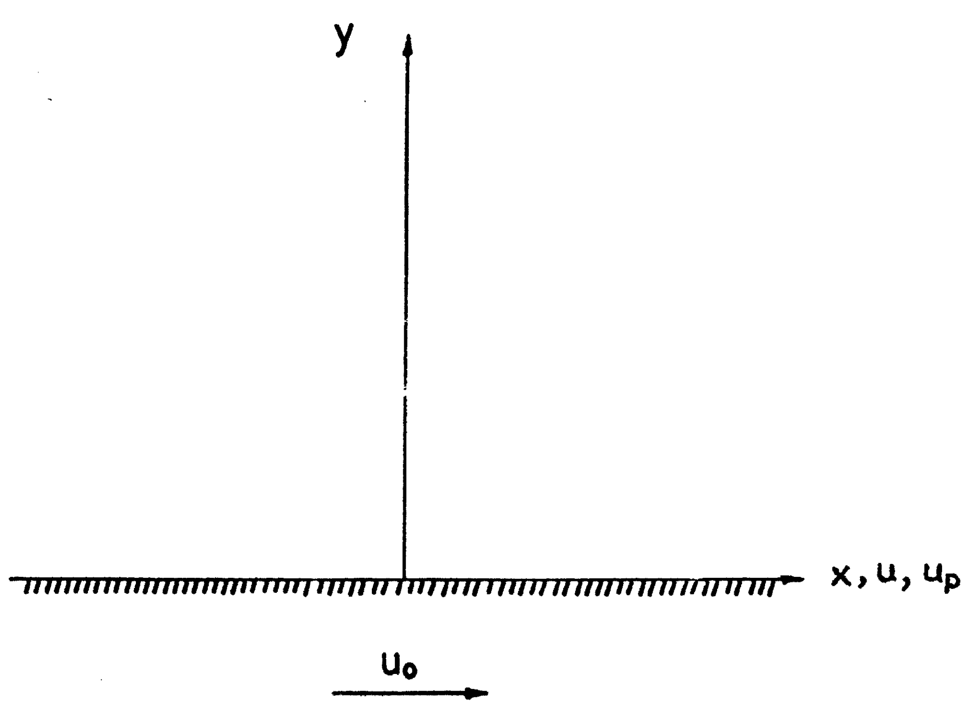


Figure 1. The Rayleigh problem.

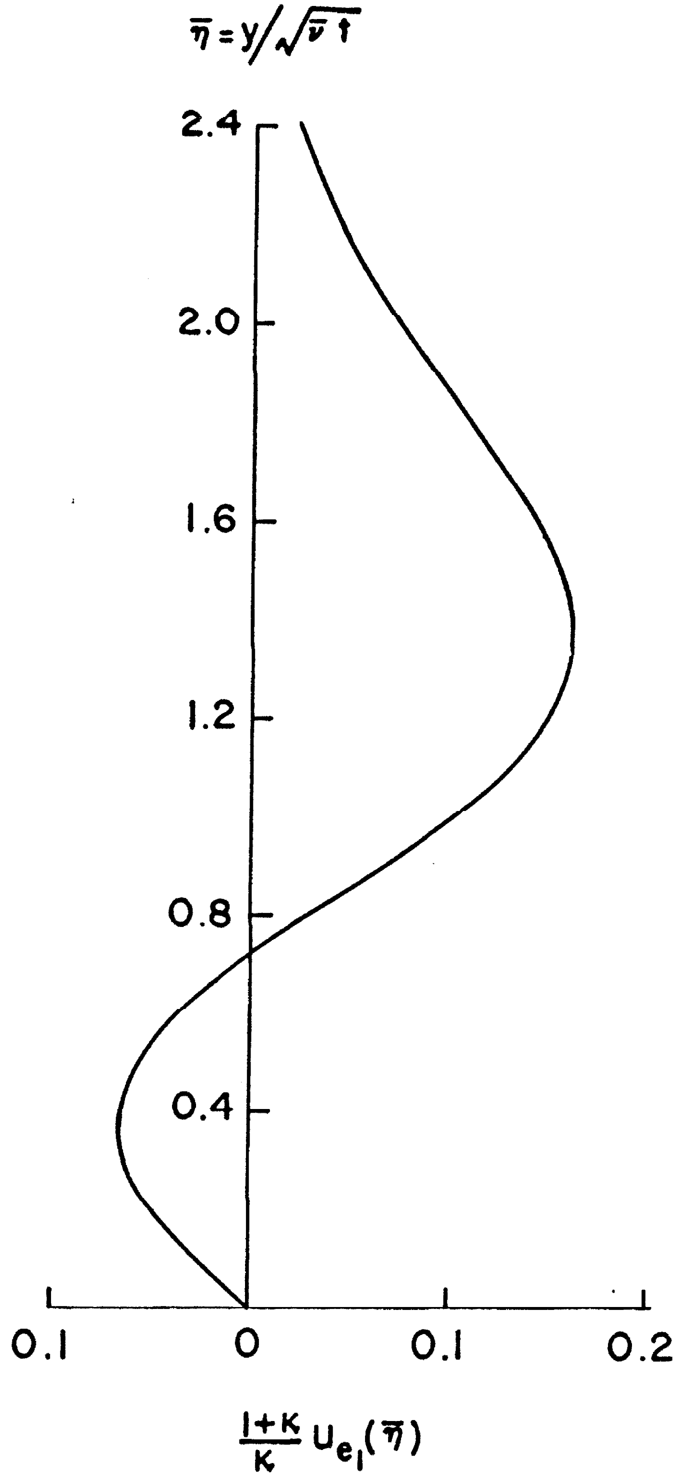


Figure 2. Coefficient of first-order expansion term for fluid velocity: t/τ_v large.

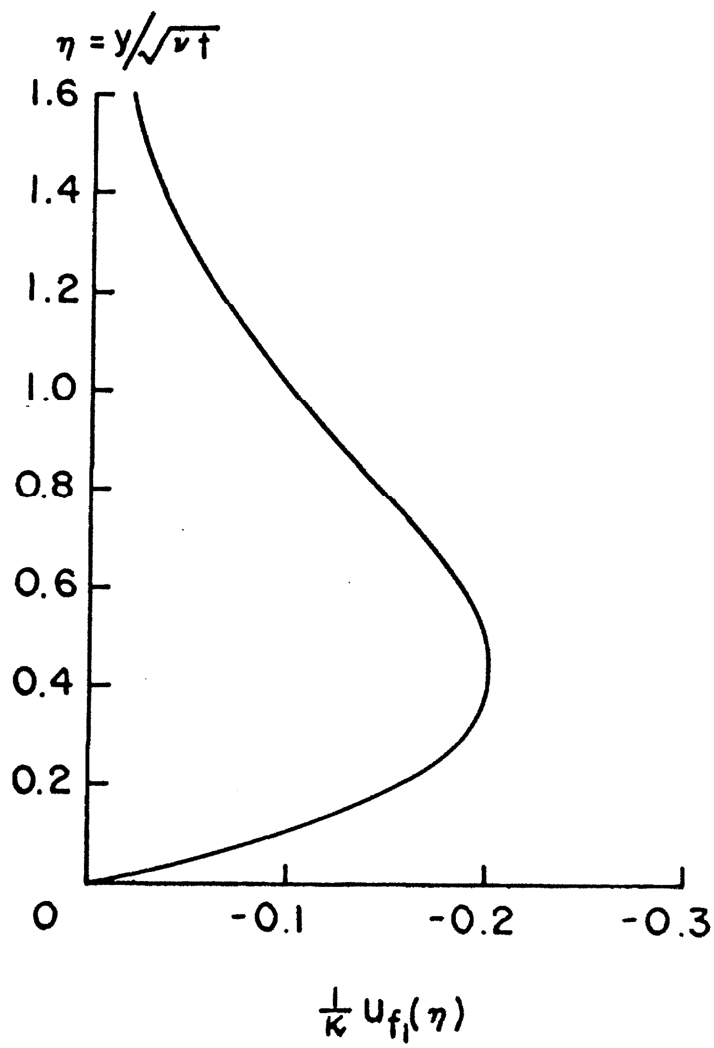


Figure 3. Coefficient of first-order expansion term for fluid velocity: t/τ_v small.

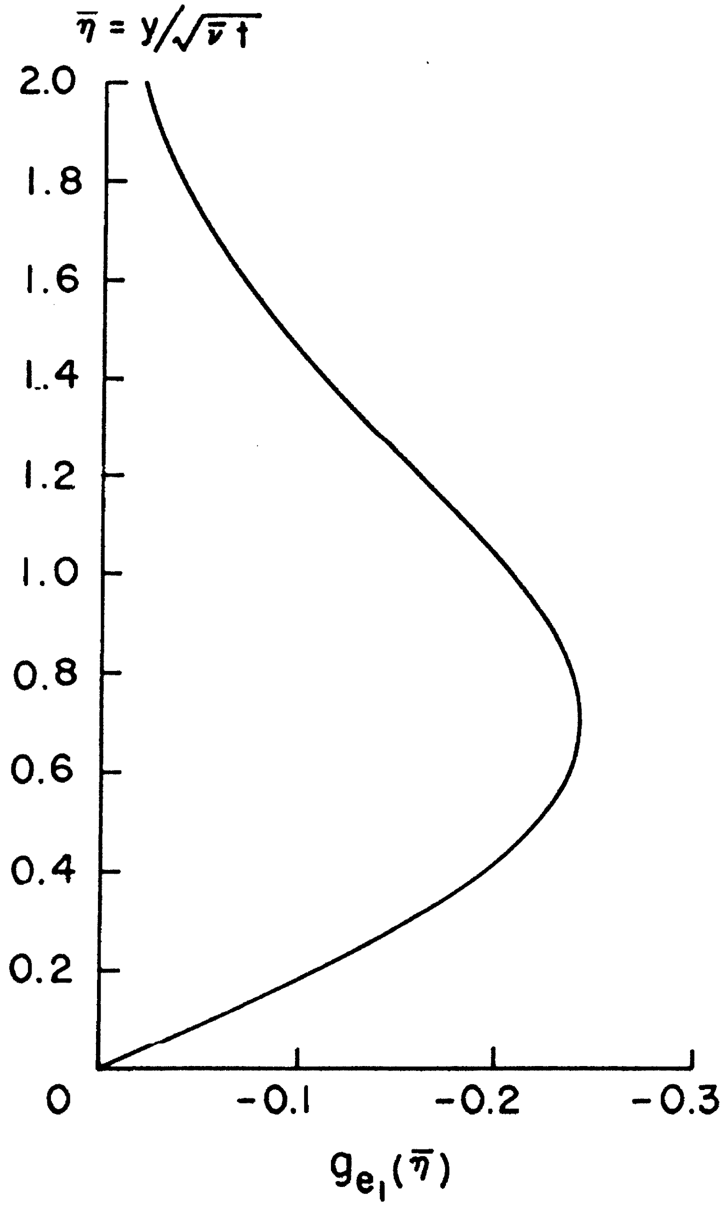


Figure 4. Coefficient of first-order expansion term for particle-fluid slip velocity: t/τ_v large.

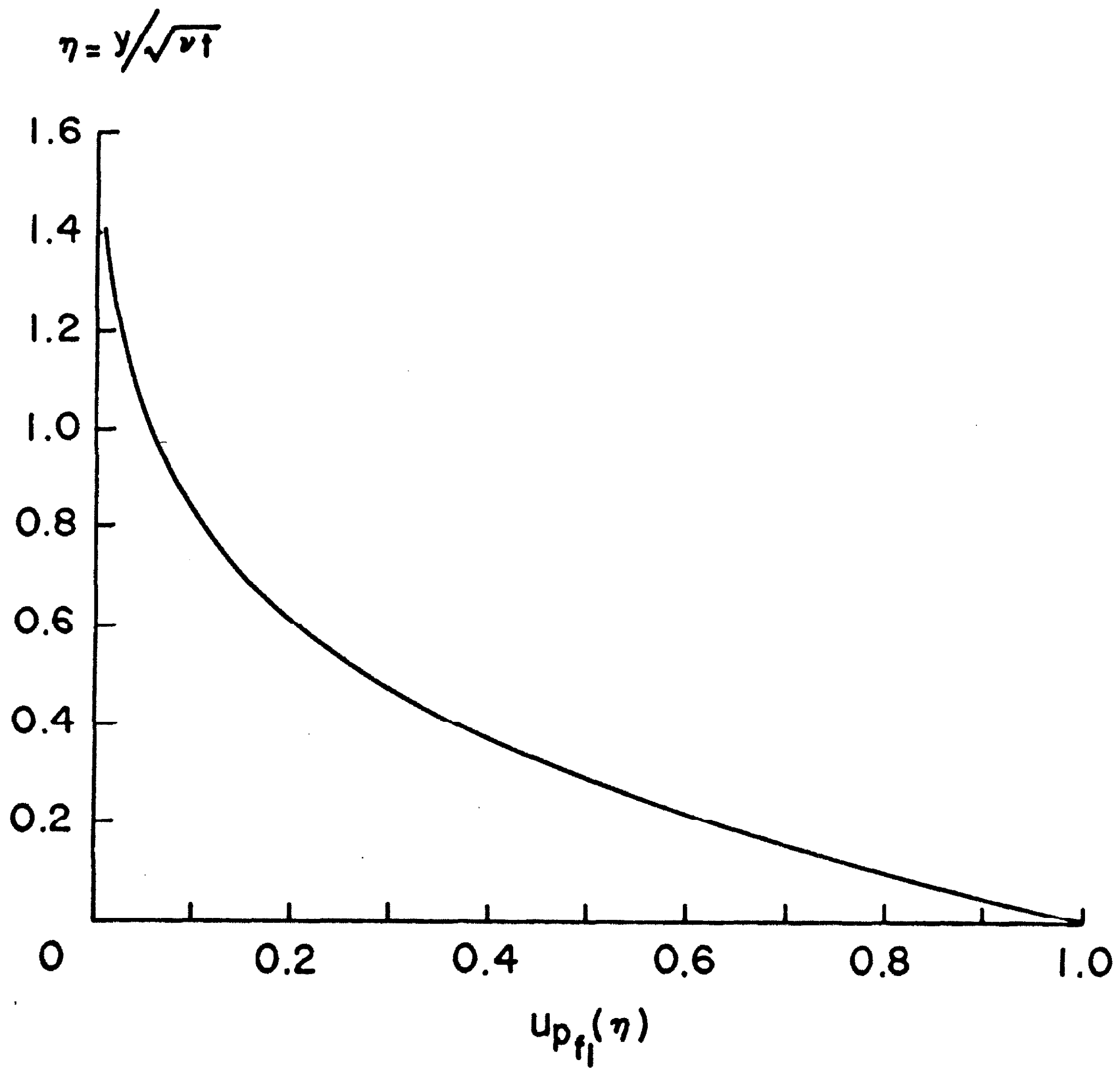


Figure 5. Coefficient of first-order expansion term for particle velocity: t/τ_v small.

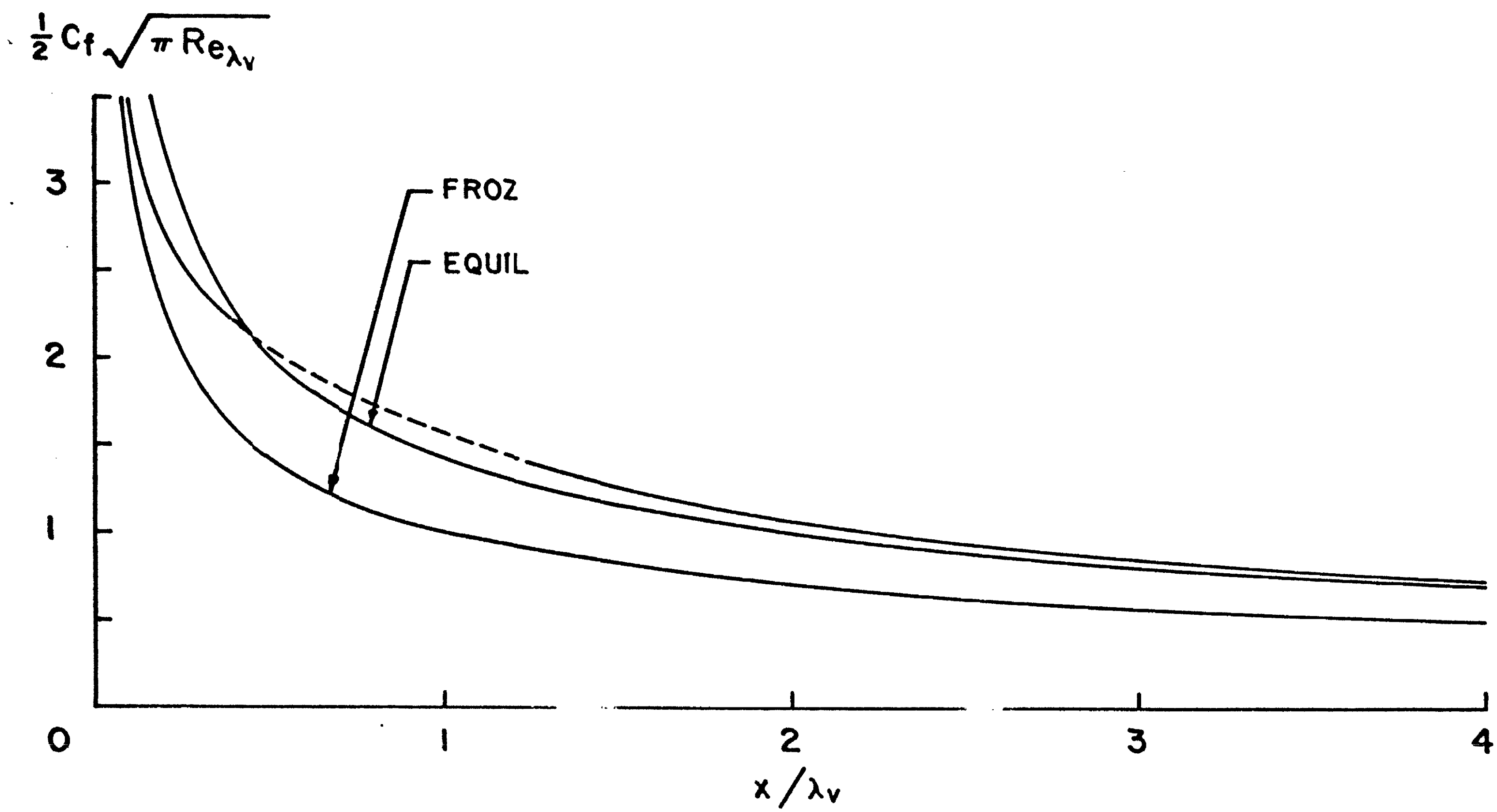


Figure 6. Rayleigh shear stress on a plate. $\kappa = 1$.

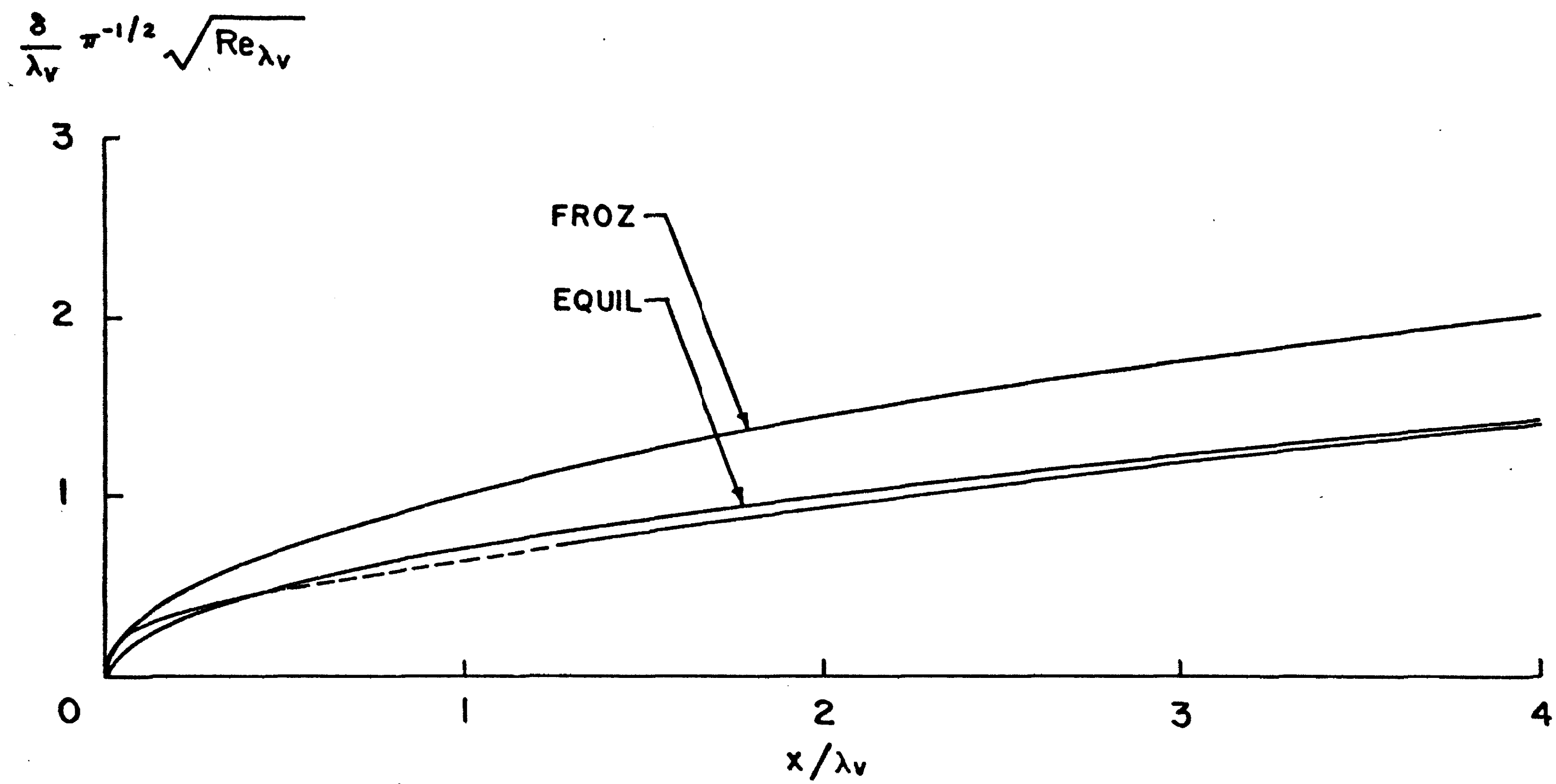


Figure 7. Rayleigh boundary layer thickness on a flat plate. $\kappa = 1$.

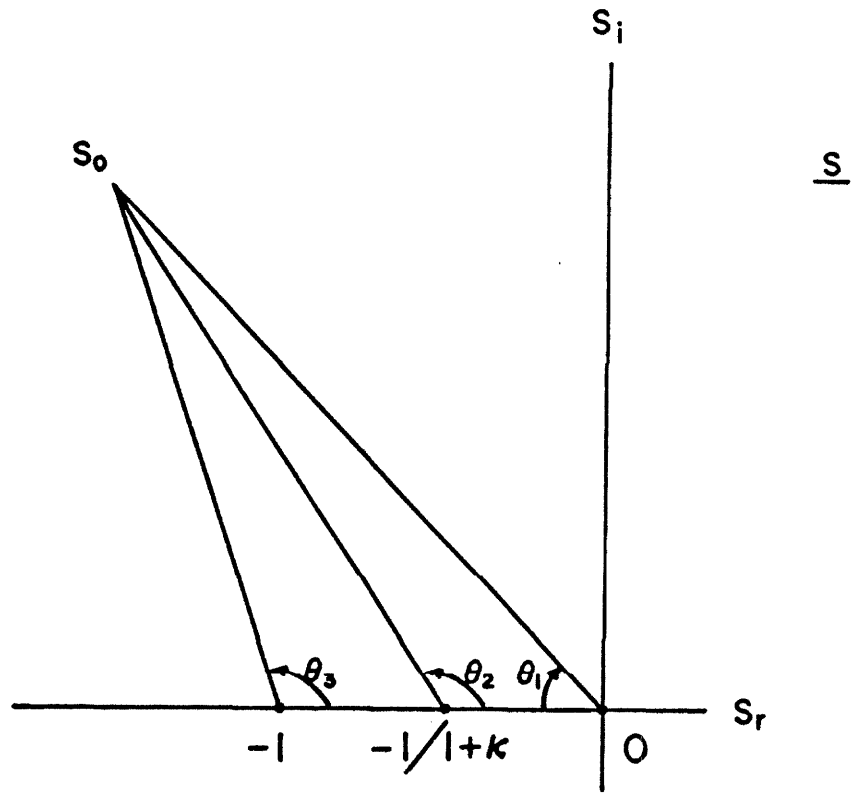


Figure 8.

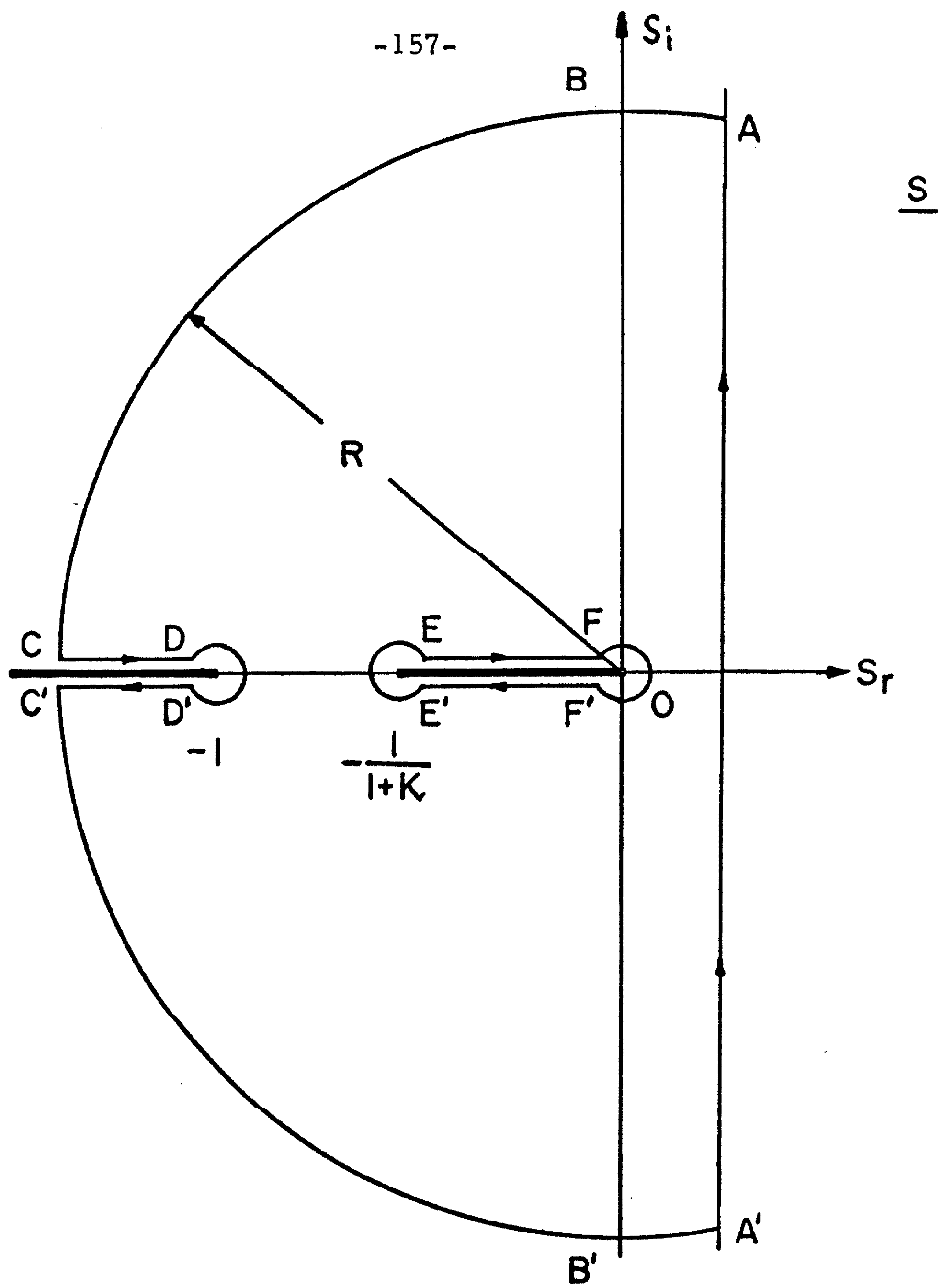


Figure 9.

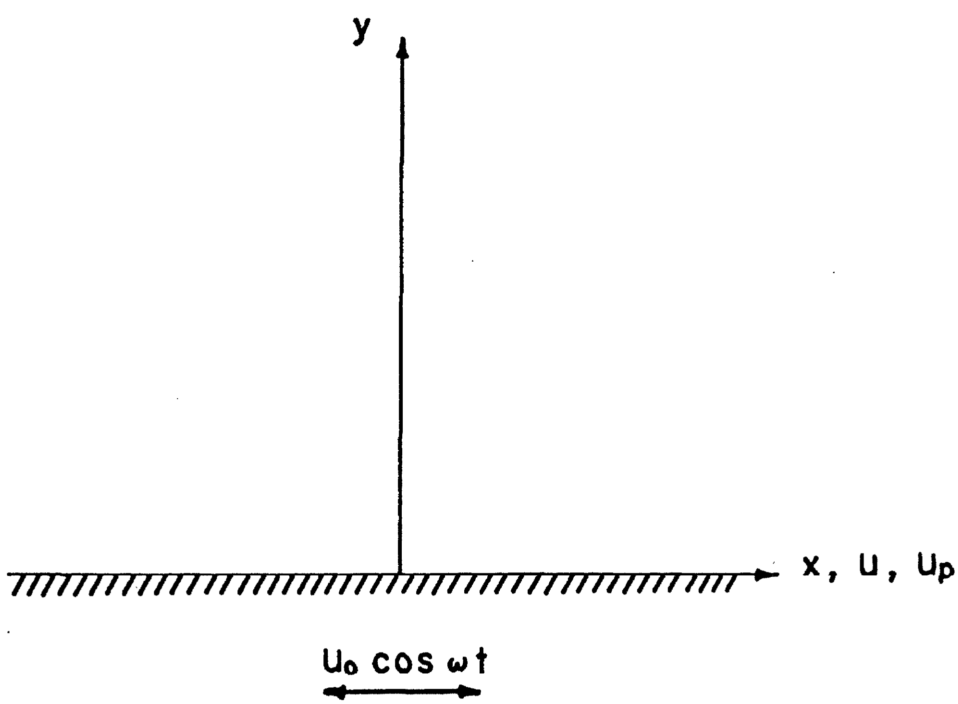


Figure 10. The oscillating infinite flat plate.

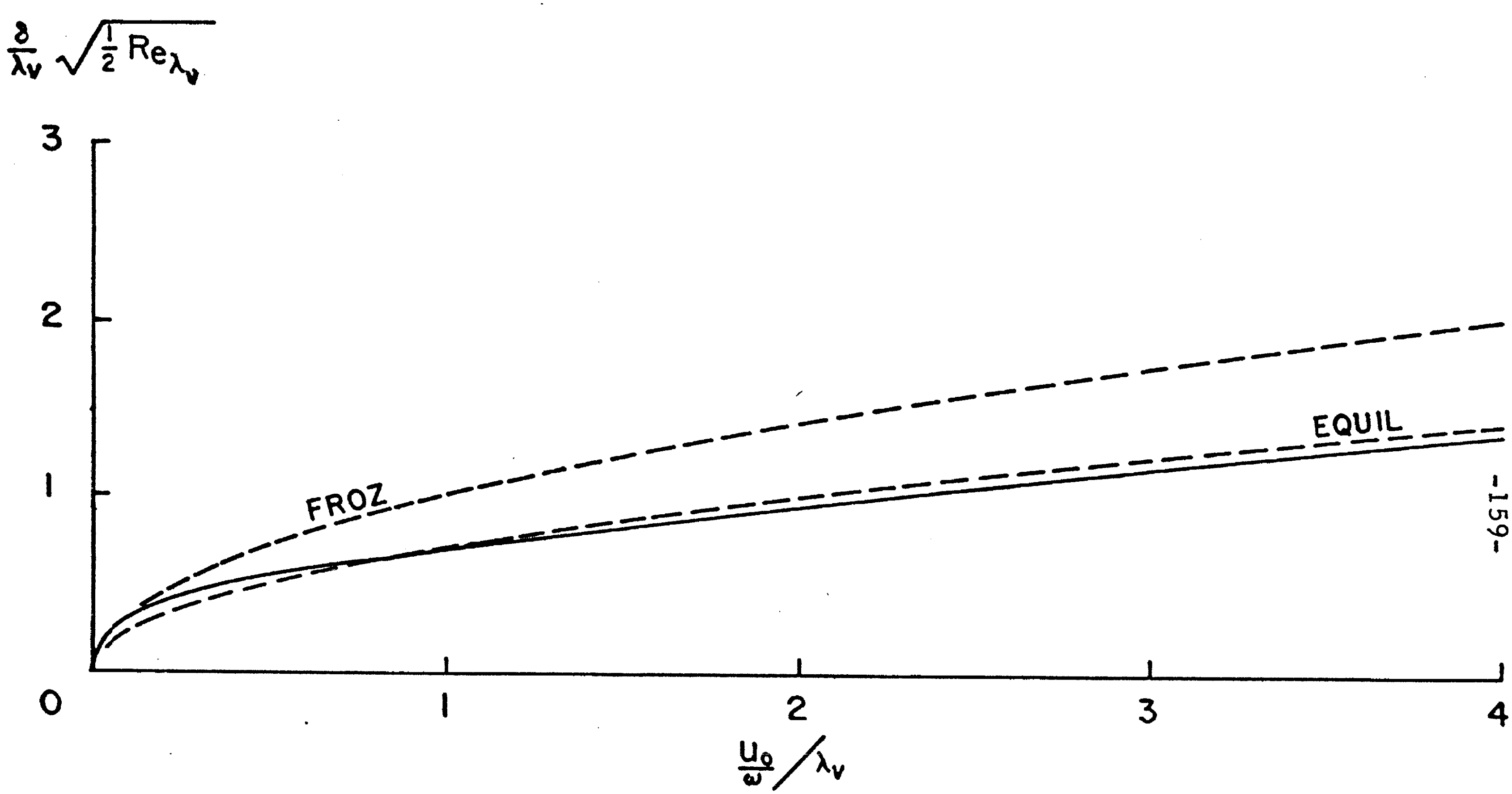


Figure 11. Boundary layer thickness for oscillating infinite flat plate.

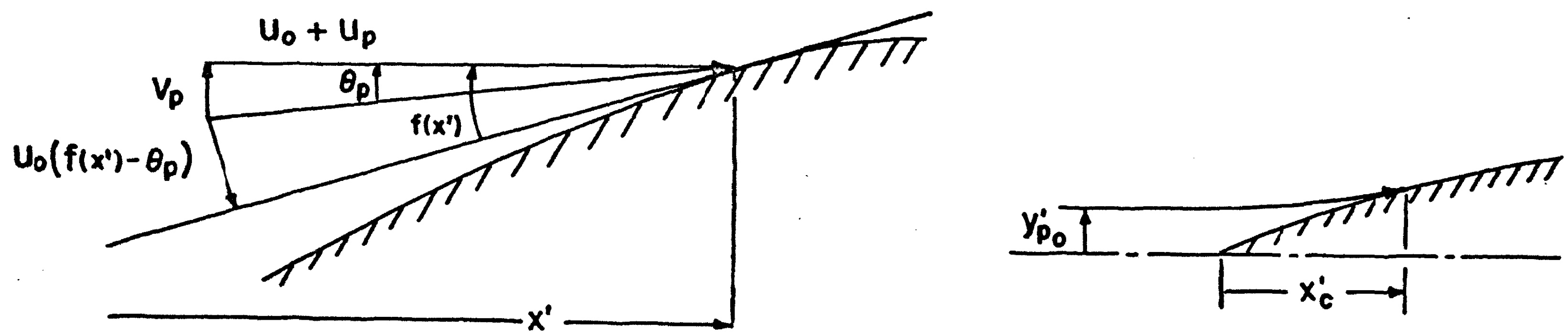


Figure 12. Particle-boundary collision in linearized theory, schematic.

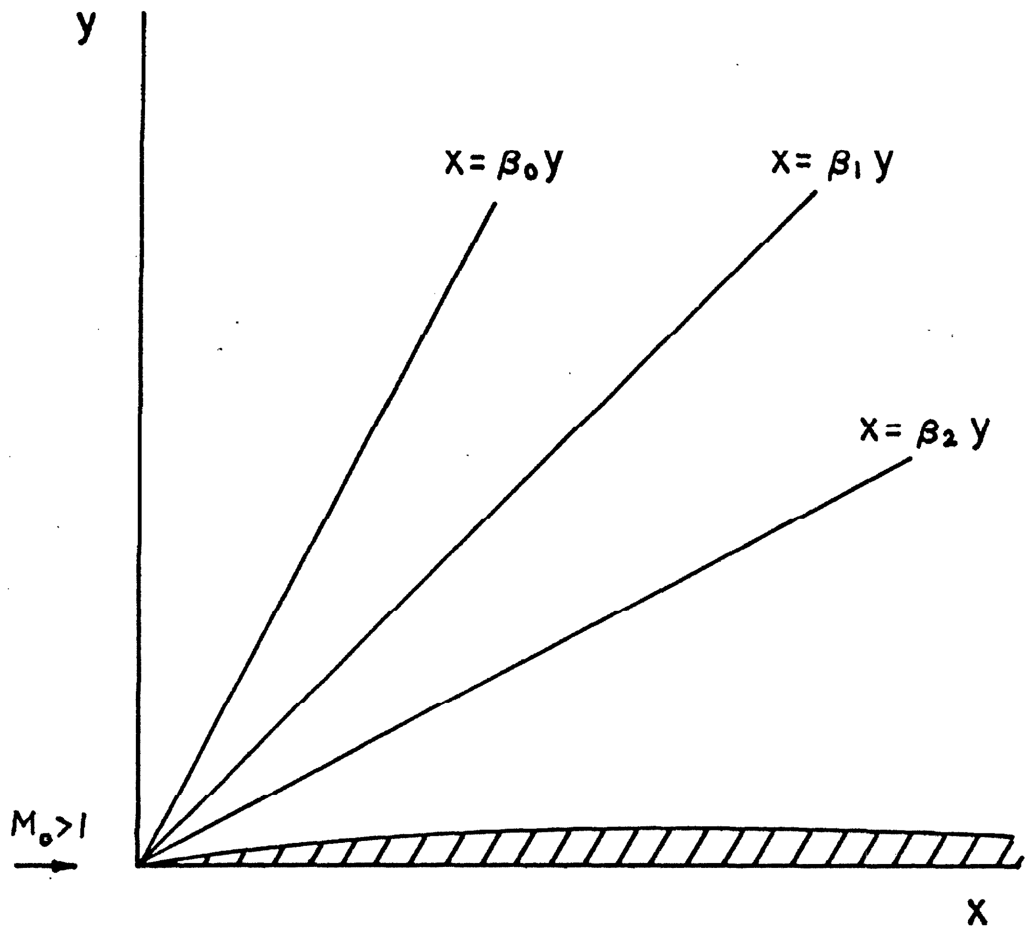


Figure 13. Linearized supersonic flow, schematic.

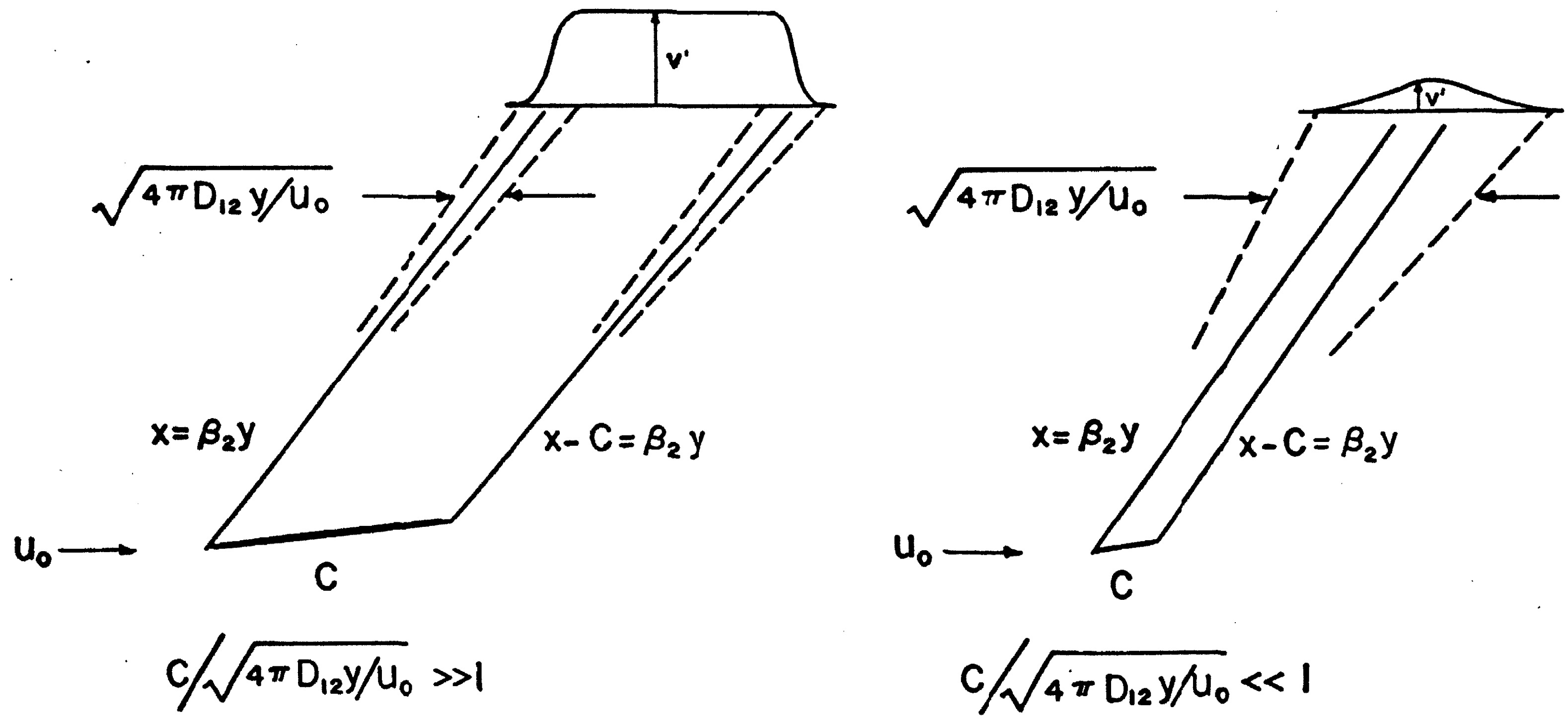
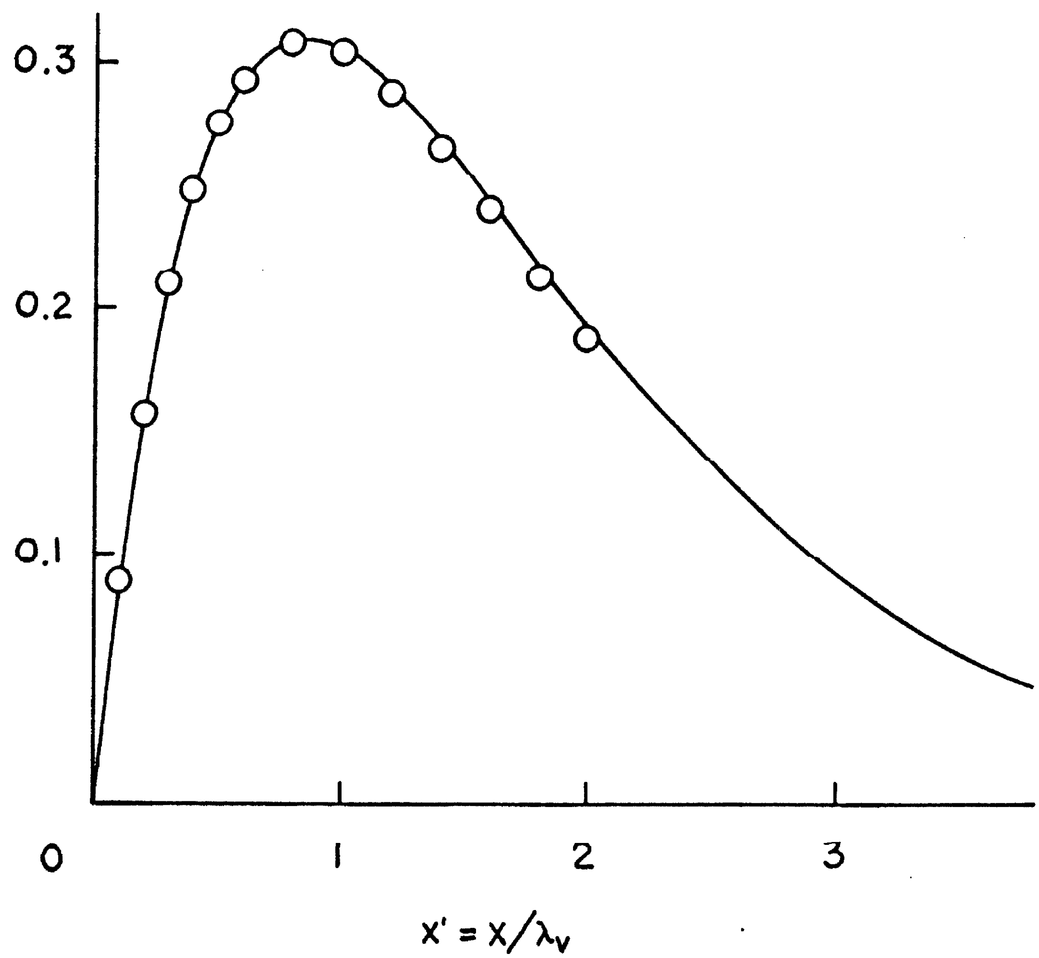


Figure 14. Far-field wave structure for finite flat-plate airfoil at an angle of attack according to the linearized theory, schematic.

$$Y(x') = \int_0^{x'} e^{-\frac{s_1+1}{2}\zeta} I_0\left(\frac{s_1-1}{2}\zeta\right) e^{-\frac{s_2+\sigma}{2}(x'-\zeta)} I_0\left(\frac{s_2-\sigma}{2}(x'-\zeta)\right) d\zeta$$



O INDICATES CALCULATED FROM SERIES (3 TERMS)

Figure 15. An integral occurring in the pressure coefficient from exact treatment of linearized theory. $M_0=1.414$, $\gamma=1.4$, $\kappa=0.25$, $c_s/c_p=1.1$, $\lambda_v/\lambda_T=0.819$.

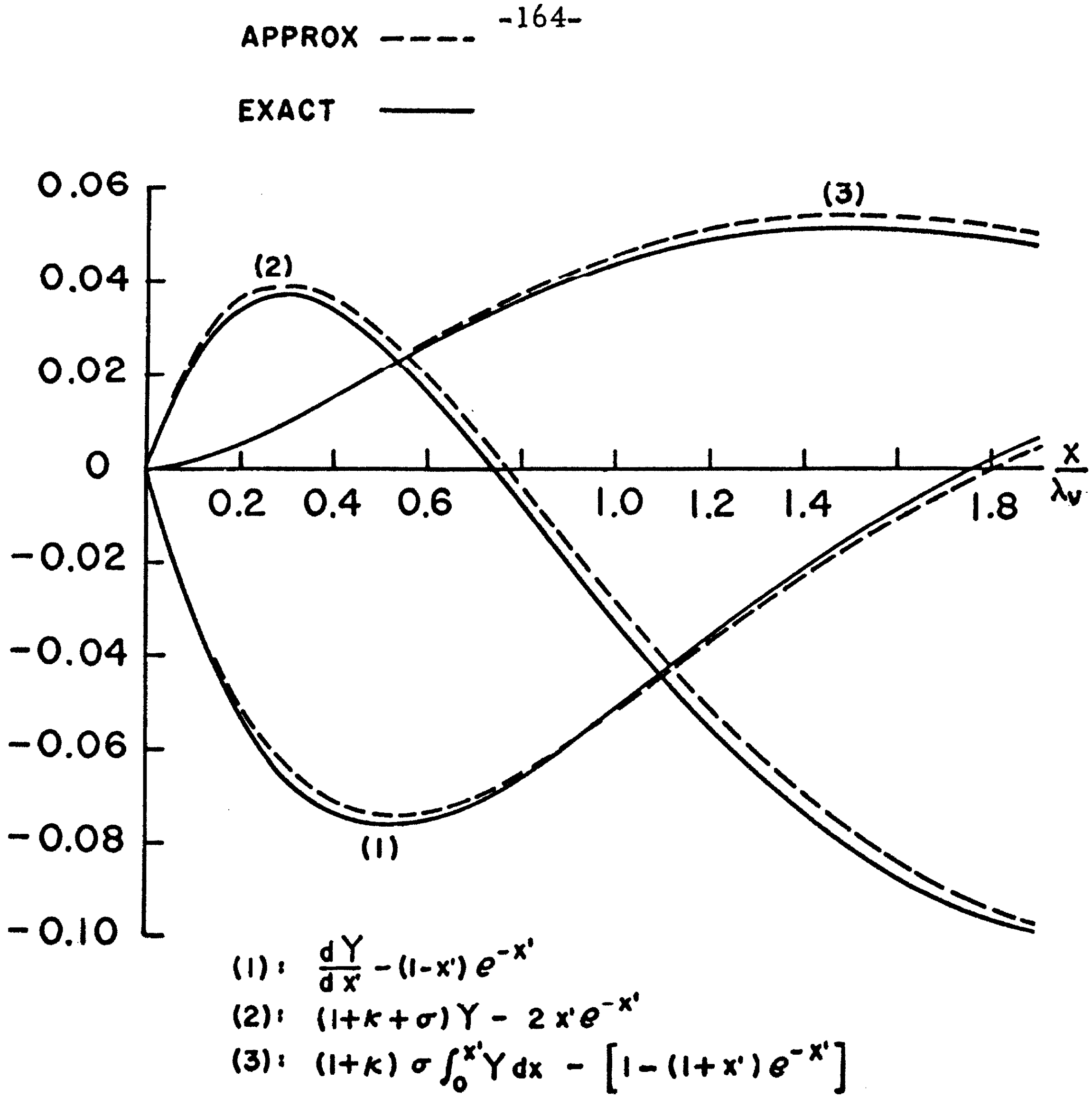


Figure 16. Integrals occurring in the pressure coefficient from exact and approximate treatments of linearized theory. M_o=1.414, γ=1.4, κ=0.25, c_s/c_p=1.1, λ_v/λ_T=0.819.

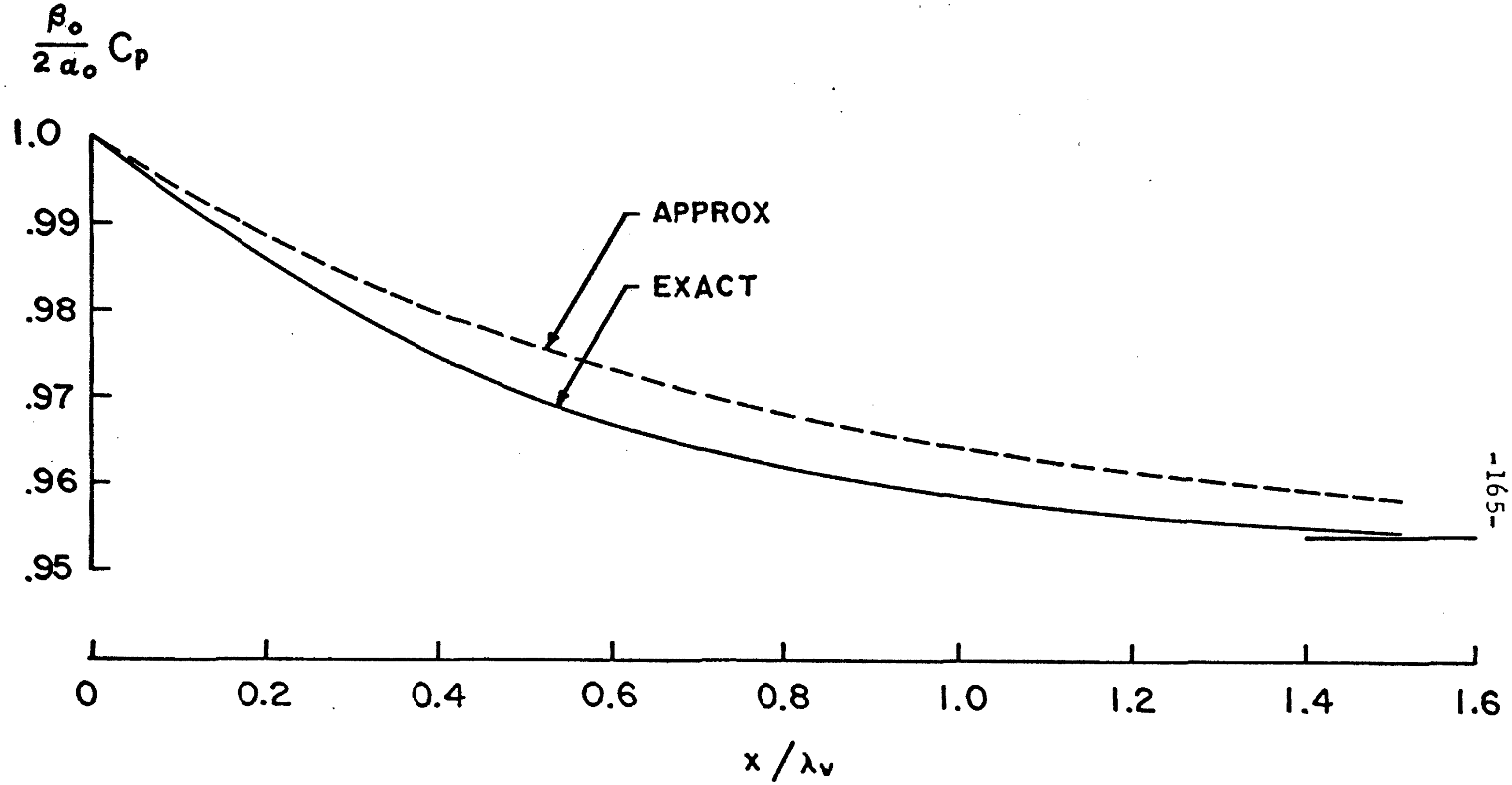


Figure 17. Pressure coefficient for a simple-wedge according to exact and approximate treatments of linearized theory. M_o=1.414, γ=1.4, κ=0.25, c_s/c_p=1.1, λ_v/λ_T=0.819.

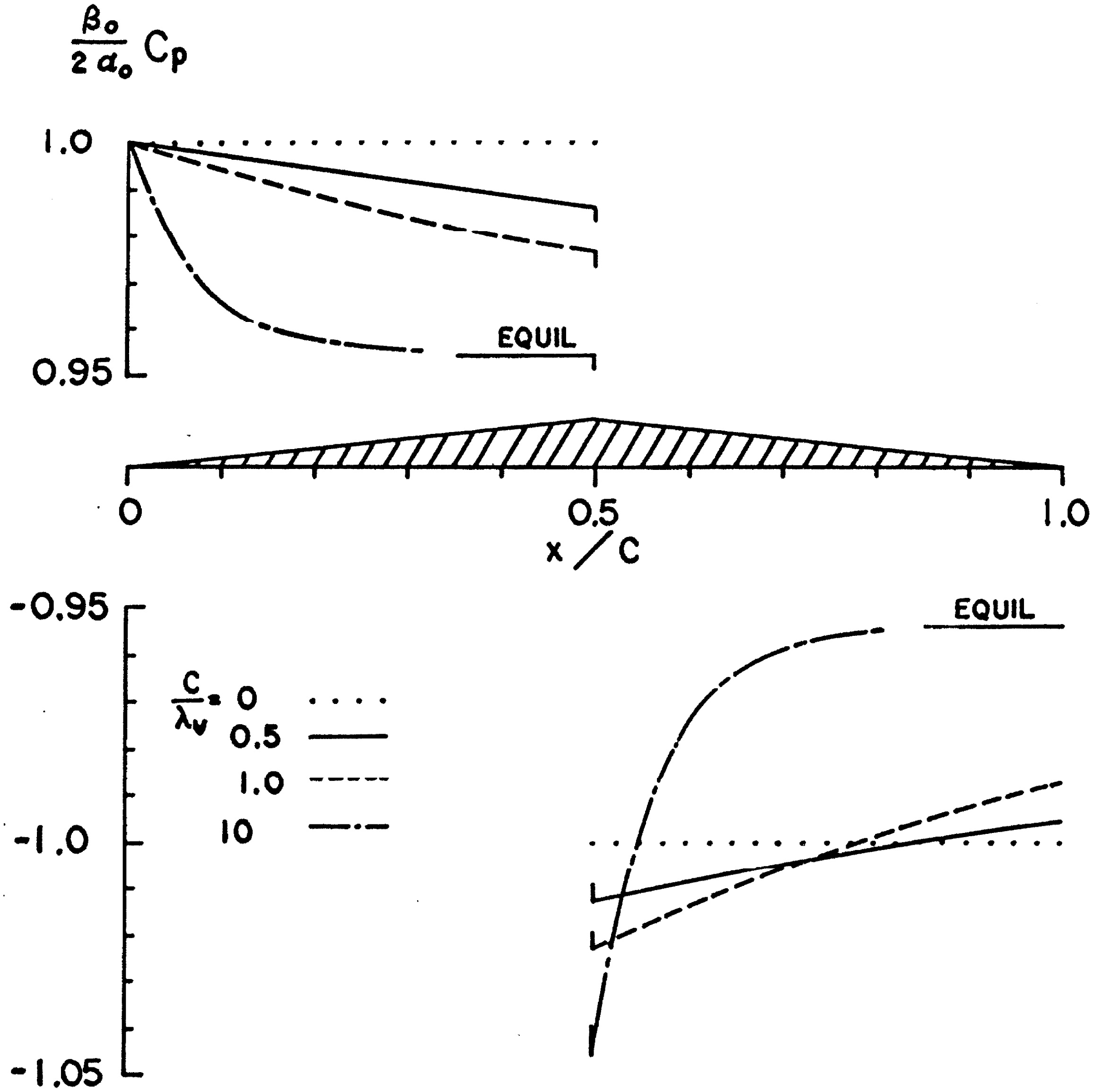


Figure 18. Pressure coefficient for a double-wedge airfoil from approximate treatment of linearized theory. $M_0=1.414$, $\gamma=1.4$, $\kappa=0.25$, $c_s/c_p=1.1$, $\lambda_v/\lambda_T=0.819$.