THE INFLUENCE OF RANDOM MEDIA
ON THE
PROPAGATION AND DEPOLARIZATION OF ELECTROMAGNETIC WAVES

Thesis by

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ABSTRACT

Electromagnetic wave propagation and depolarization in an inhomogeneous medium having random fluctuations in its permittivity are studied. The continuous space-time permittivity fluctuations are taken to be frozen-in, homogeneous, and isotropic. We find that the essential effect of the random permittivity is to destroy the time coherence and spatial orthogonality of the vector components of an electromagnetic wave penetrating the medium.

To study this problem, we develop a unique discrete model for the continuous random medium by dividing the volume occupied by the random inhomogeneities into independent elementary scattering volumes. Scattering by each of these elementary volumes is analyzed to obtain the complex amplitude and polarization of the single scattered field. Then multiple scattering among the many elementary volumes is used to estimate the composite values for scattering per unit length and depolarization per unit length of the medium. The manifestation of scattering in the medium is the generation of an incoherent or fluctuating electric wavefield and a coherent or average electric wavefield. It is shown that the total electric wavefield propagating in the medium satisfies an integral equation which is directly reducible to the classical equation for radiation transfer.

A novel result of this study is that only two phenomenological parameters are needed to describe the penetration of the wave into a plane-parallel medium, when a polarized plane wave is normally incident. These two parameters appear as diffusion constants in expressions for

the solution for the coherency and Stokes matrices. These solutions simply describe how wave energy is progressively converted from the initially coherent and polarized field to an incoherent and unpolarized field as the wave propagates. An initially polarized wave is gradually depolarized, yielding a completely unpolarized wave deep into the medium.

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I. INTRODUCTION

Electromagnetic wave propagation and scattering in random media have become increasingly important in recent years. In general, a random medium varying in space and time will cause the amplitude, phase, polarization, and direction of propagation of an electromagnetic field to fluctuate in a random manner. Such fluctuations in the wave parameters are important in many practical situations. For instance, random fluctuations may limit the coherent bandwidth of communication signals propagated through the medium, or cause signal fading beyond system margin capabilities. In other communication systems, scattering by a random medium may be utilized, as with over-the-horizon communication links. Remote sensing of geophysical and meteorological parameters is also an important application for the study of wave propagation in random media. Geophysicists and astronomers are interested in the use of wave fluctuations to remotely determine the dynamic and constituent characteristics of planetary atmospheres. And, subsurface exploration for hydrocarbons requires an understanding of statistical wave propagation characteristics in random media.

There are two general categories of random media: discrete random media and continuous random media. Both these categories are reviewed by Ishimaru (1). Discrete random media refers to a collection of discrete scatterers, such as rain, fog, molecules, or suspended particles in otherwise homogeneous media. Procedures for studying discrete random media can be divided into two principal steps. First, the scattering characteristics of a single scatterer is considered, and second, the characteristics of a wave scattered among many randomly distributed scatterers is considered. Historically, these steps have been undertaken using two distinct approaches; radiative transfer theory and multiple scattering theory. Schuster (2) initiated radiative transfer theory or "transport" theory in 1905 to study radiation in foggy atmospheres in order to explain absorption and emission lines in stellar spectra. This transfer theory is based upon the phenomenological occurrence of sources and sinks of wave energy or intensity, and its basic differential equation is equivalent to Boltzmann's equation in the kinetic theory of gases (3) and in neutron transport theory (4). In 1945, Chandrasekhar (5) developed a systematic approach to radiative transfer. His work has since become the foundation of modern radiative transfer theories. Alternatively, in multiple scattering theory, one begins with a wave equation, obtains solutions for a single particle, then rigorously introduces the interaction effects of many particles. Twersky (6,7) and Foldy (8) have provided some of the most useful multiple scattering theories.

In contrast, a "continuous random medium" is used to describe the medium whose constitutive parameters (permittivity, index of refraction, impedance, etc.) vary randomly yet continuously in space and time. Such media are the subject of this thesis. Primary examples are atmospheric turbulence, biological media and turbulent wakes and plumes of aircraft engines. Continuous random media have traditionally been treated using the Born approximation, wherein the effective field incident at any point in the medium is approximated with the free-space propagating incident field. However, much of the recent work in this area is based upon techniques employed by Tatarski (9), such as the method of smooth perturbations (Rytov's method). Other techniques and methods utilized for continuous random media are reviewed in a recent article by Lawrence and Strohbehn (10).

For continuous random media, a major shortcoming of much of the previous research is that polarization effects on electromagnetic waves have been neglected. The typical procedure is to begin with the vector wave equation, then invoke physical reasons to justify neglecting the change in polarization of the scattered field. The result is a scalar wave equation for each of the decoupled scalar components of the electromagnetic field. In general, no attempt is made to compute the magnitude of the neglected depolarized field. One notable exception is the quantitative discussion using an entropy argument given by Papas (11). Also, Chandrasekhar's work in discrete random media using transport theory involved polarization effects explicitly. In fact, Chandrasekhar noted that a scalar wave theory is never reliable and requires careful interpretation when compared with measured data.

The subject considered herein is electromagnetic wave propagation and scattering in a continuous random medium. The random variations in the medium's permittivity are assumed to arise from turbulence. This work is novel, however, in that discrete random media techniques are utilized by subdividing the continuous medium into discrete elementary scattering volumes. Propagation or scattering in the continuous medium is accounted for by multiple scattering among many elementary volumes. Also, the polarization of the waves in the random medium is consistently accounted for; no a priori assumption regarding the magnitude of the depolarized field is made.

The procedure is to first obtain the far-zone vector electric field from an elementary volume of the continuous random medium. Then rigorous multiple scattering among the elementary volumes using the Foldy-Twersky theory is used to find the average electric field propagating in the medium as well as the bulk effective refractive index of the random medium. The fluctuating part of the electric field is constructed from multiply scattered field contributions generated throughout the entire medium. Finally, by defining the total electric field as the sum of the average electric field and the fluctuating electric field, an integral equation for the total field is derived. This integral equation is then used to construct equations satisfied by the coherency and Stokes matrices. An approximate solution for the integral equation describing propagation of the coherence matrix of the field is obtained when a completely polarized plane wave is incident upon a slab of random medium. This approximate solution is then utilized to develop a solution which describes the diffusion of the coherency and Stokes matrices in the random medium.

An essential part of this thesis is to provide estimates of the polarization behavior of the electromagnetic field as it penetrates the medium.

II. GENERAL PROPERTIES OF ELECTROMAGNETIC WAVES IN RANDOM MEDIA

Consider a completely polarized and monochromatic plane wave initially traveling in the $\hat{z}$ direction. As this wave penetrates a random medium, its amplitude, phase, and polarization fluctuate randomly with time and position. If the frequency spectrum of the wave fluctuations is confined to a narrow bandwidth (quasi-monochromatic with mean frequency ω), then the electric field vector of the wave may be represented as

$$\underline{E}_0(\underline{r},t) = \mathrm{Re}[\underline{E}(\underline{r},t)\exp(-i\omega t)].$$

Throughout this study, the primary time dependence $\exp(-i\omega t)$ is assumed. In general, the field $\underline{E}$ is complex, and due to its narrow bandwidth, is a slowly varying function of time. In accord with the vector nature of the wave, the Cartesian components of $\underline{E}(\underline{r},t)$ are

$$E_x = U_x(\underline{r},t)\exp[i\phi_x(\underline{r},t)]$$

$$E_y = U_y(\underline{r},t)\exp[i\phi_y(\underline{r},t)]$$

$$E_z = 0 ,$$

where U_x, U_y, ϕ_x, and ϕ_y are also slowly varying functions of time.

2.1 Coherent and Incoherent Fields

Each of the scalar field components, E_x and E_y, is a random function of position and time and can be expressed as a sum of the average field, $\langle E \rangle$, and the fluctuating field E_f(1):

$$E(\underline{r},t) = \langle E(\underline{r},t)\rangle + E_f(\underline{r},t) \quad (2.1.1)$$

$$\langle E_f(\underline{r},t)\rangle = 0 \quad , \quad (2.1.2)$$

where E represents either of the field components E_x and E_y and the angle brackets denote the ensemble average discussed in Appendix A. The average field is also called the coherent field E_c and the fluctuating field is called the incoherent field. The energy density, J, of the total field is separated into coherent and incoherent parts also, yielding

$$J(\underline{r}) = J_c(\underline{r}) + J_i(\underline{r}) \quad (2.1.3)$$

$$J_c(\underline{r}) = |\langle E \rangle|^2 \quad (2.1.4)$$

$$J_i(\underline{r}) = \langle |E_f|^2 \rangle \; . \quad (2.1.5)$$

As the wave propagates in a random medium, its coherent intensity decreases due to scattering and absorption in the medium. The portion of the coherent wave which is scattered travels in all directions according to the scattering properties of each elementary scattering volume comprising the medium. These scattered waves represent the incoherent field. It is evident that as the wave propagates, wave energy is continuously transferred from the coherent field to the incoherent field by scattering. These scattered waves comprising the incoherent field also exchange energy but this incoherent energy cannot be converted into coherent energy. The total energy density of the wave, (2.1.3), decreases due to absorption only. If the medium is lossless (no absorption), the total energy density of the wave penetrating the medium is conserved except for backscattering.

The above description of propagation in random media applies generally to each scalar component of the vector electric field. However, when electromagnetic waves are multiply scattered, coupling between the orthogonal scalar components of the field occurs. This coupling causes a progressive depolarization of the electromagnetic wave propagating in a random medium. Because the depolarization is a random phenomenon affecting a polychromatic wave, the incoherent field is only partially polarized. This means that the energy is not only transferred from the coherent component of the field to an incoherent component but also from a polarized field to a partially polarized field. In the limit as the waves penetrate deep into the random medium, the total field becomes completely incoherent and unpolarized.

2.2 Coherency Matrix and Stokes Parameters

To describe the polarization and coherence of the polychromatic wavefield, Papas (12) has shown that the state of polarization can be specified by means of a matrix whose elements characterize the degree of coherency between the transverse components E_x and E_y of the wave. The elements of this coherency matrix $\underline{\underline{J}}$ are defined by

$$J_{pq} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A_p A_q^* dt \qquad (p,q = x,y) \quad . \tag{2.2.1}$$

Equation (2.2.1) represents a time averaging integration, where $A_p (p = x,y)$ represents either of the electric field components E_x or E_y.

By invoking Taylor's frozen-in turbulence hypothesis*, it is evident that the time average and ensemble average of the wavefield are identically the same quantity. Thus (2.2.1) is replaced with

$$J_{pq} = \langle A_p A_q^* \rangle \ . \tag{2.2.2}$$

Taylor's frozen-in hypothesis is discussed in detail in Appendix A. If A_p and A_q are independent waves, then $J_{pq} = 0$. Also, it is apparent that the coherency matrix

$$\underline{\underline{J}} = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix} \tag{2.2.3}$$

is hermitian, that is, $J_{xy} = J_{yx}^*$.

Throughout this paper, the electric field will be expressed as a column matrix with spatially orthogonal field components, i.e., E_x and E_y. Accordingly, we have

$$\underline{E}(\underline{r},t) = \begin{bmatrix} E_x(\underline{r},t) \\ E_y(\underline{r},t) \end{bmatrix} \ . \tag{2.2.4}$$

With this notation, the coherency matrix given in (2.2.3) may be formed as a product of the vector $\underline{E}$ with its hermitian conjugate $\underline{E}^\dagger$. However, in multiple scattering problems it is advantageous (as will be shown in Appendix E) to define the coherency matrix as the product of the electric field vector with its complex conjugate $\underline{E}^*$, yielding the column matrix

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* Taylor's frozen-in hypothesis simply states that any time variation of a quantity is a spatial variation translated with uniform velocity.

$$\underline{\underline{J}} = \underline{E} \times \underline{E}^* = \begin{bmatrix} J_{xx} \\ J_{xy} \\ J_{yx} \\ J_{yy} \end{bmatrix} . \qquad (2.2.5)$$

In view of the discussion at the beginning of this section, we see that the random fluctuations of the medium destroy the spatial orthogonality and time coherence between E_x and E_y as the wave propagates. J_{xy} and J_{yx} approach zero and J_{xx} and J_{yy} approach an equal-valued equilibrium. This situation is illustrated in Figure 2-1. These ideas suggest that the wave entropy is being steadily increased, behavior first suggested by Papas (11) in a recent paper. However, it should be pointed out that this phenomenon is strictly a consequence of the multiply scattered and polychromatic nature of the wavefield. For instance, a monochromatic wave, even after single scattering, is always in some state of general elliptical polarization, i.e., the end point of its electric field vector must periodically trace out an ellipse or one of its special forms, viz., a circle or a straight line. However, a polychromatic electromagnetic wave can be in any state of polarization ranging from elliptic to completely unpolarized. Mathematically, the polarization state of such polychromatic waves is described by the four Stokes parameters (5,12). The Stokes parameters may be derived directly from correlations involving the orthogonal field components, however, they are also simply related to components of the coherency matrix as follows:

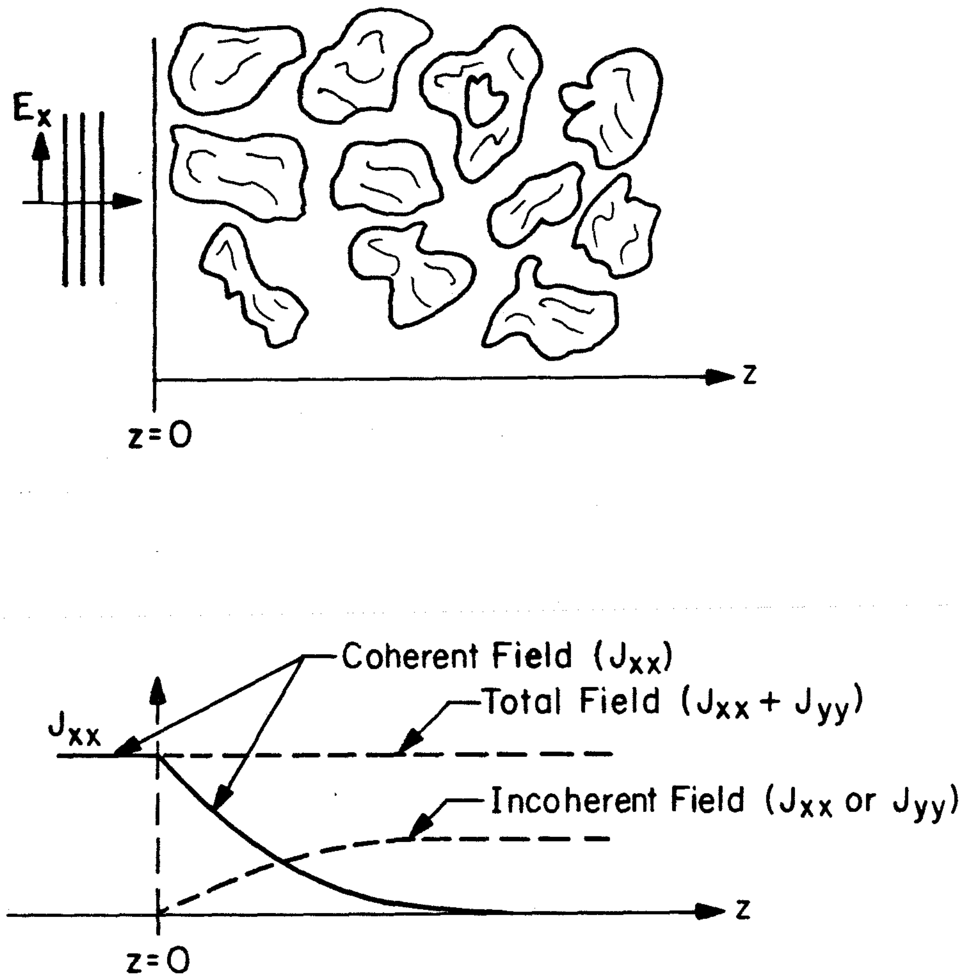

Figure 2-1. General behavior of a polarized plane wave incident upon a random medium ($z > 0$). The incident field is polarized along the $\hat{x}$-axis.

$$S_0 = J_{xx} + J_{yy} \; , \quad S_1 = J_{xx} - J_{yy}$$
$$S_2 = J_{xy} - J_{yx} \; , \quad S_3 = i(J_{yx} - J_{xy}). \qquad (2.2.6)$$

The parameters S_0, S_1, S_2, and S_3 are often collected into a column matrix and called the Stokes matrix. Only a simple transformation is required to convert the coherency matrix (2.2.5) into a Stokes matrix whose components are given by (2.2.6). It can be shown (12) that the polychromatic Stokes parameters satisfy the relation

$$S_0^2 \geq S_1^2 + S_2^2 + S_3^2 \; , \qquad (2.2.7)$$

where the equality sign holds only when the wave is completely polarized.

The degree of polarization P of the partially polarized wave is defined as the ratio of the energy in the polarized field to the total energy. In terms of the Stokes parameters, P is given by

$$P = [S_1^2 + S_2^2 + S_3^2]^{\frac{1}{2}}/S_0 \; . \qquad (2.2.8)$$

The range of P is $0 \leq P \leq 1$. For the two extreme values, P = 0 corresponds to a completely unpolarized wave and P = 1 corresponds to a completely polarized wave. When P is in the intermediate range of values $0 < P < 1$, the wave is partially polarized.

The degree of linearity L of the polarized part of the field is given by

$$L = [S_1^2 + S_2^2]^{\frac{1}{2}}/[S_1^2 + S_2^2 + S_3^2]^{\frac{1}{2}} \; . \qquad (2.2.9)$$

As with P, the degree of linearity L varies from zero to unity, $0 \leq L \leq 1$. When L = 1, the polarized part of the wave is linear.

When L = 0 the polarized part is circular. For $0 < L < 1$, the polarization is elliptical.

The degree of ellipticity E of the polarized part of the wave is given by

$$E = S_3/[S_1^2 + S_2^2 + S_3^2]^{1/2} , \qquad (2.2.10)$$

where the range of E is $-1 \leq E \leq 1$. When E = -1, the polarization is left-hand circular. When E = +1, the polarization is right-hand circular. E = 0 corresponds to linear polarization. For $0 < E < 1$ the polarization is left-hand elliptical and similarly, for $-1 < E < 0$ the polarization is right-hand elliptical.

III. SCATTERING BY AN ELEMENTARY VOLUME OF RANDOM MEDIUM

The physical model for studying propagation and scattering in the random medium is illustrated in Figure 3-1, where it is shown that the total volume occupied by the random inhomogeneities is divided into N identical elementary scattering volumes. Details of the electromagnetic scattering characteristics of the elementary scattering volume are given in this section.

3.1 Far-Field Scattering Amplitude

Consider a continuously varying random medium occupying an elementary volume V_s. The medium is characterized by the permittivity $\varepsilon(\underline{r},t)$, which can be complex, and the permeability μ_0 equal to that of free-space. The permittivity varies from point-to-point and time-to-time in a random manner and is described by a homogeneous, isotropic, stationary, and frozen-in random function of position and time. To represent the permittivity mathematically, one decomposes $\varepsilon(\underline{r},t)$ into its average value ε_0 and its fluctuating part:

$$\varepsilon(\underline{r},t) = \varepsilon_0(1 + \varepsilon_1(\underline{r},t)) \ , \qquad (3.1.1)$$

where ε_1 is the small fluctuation. The first two moments of ε_1 are

$$\langle\varepsilon_1(\underline{r},t)\rangle = 0 \qquad (3.1.2)$$

and

$$\langle\varepsilon_1(\underline{r},t)\varepsilon_1(\underline{r}+\underline{r}_d,t)\rangle = B_\varepsilon(r_d) \ . \qquad (3.1.3)$$

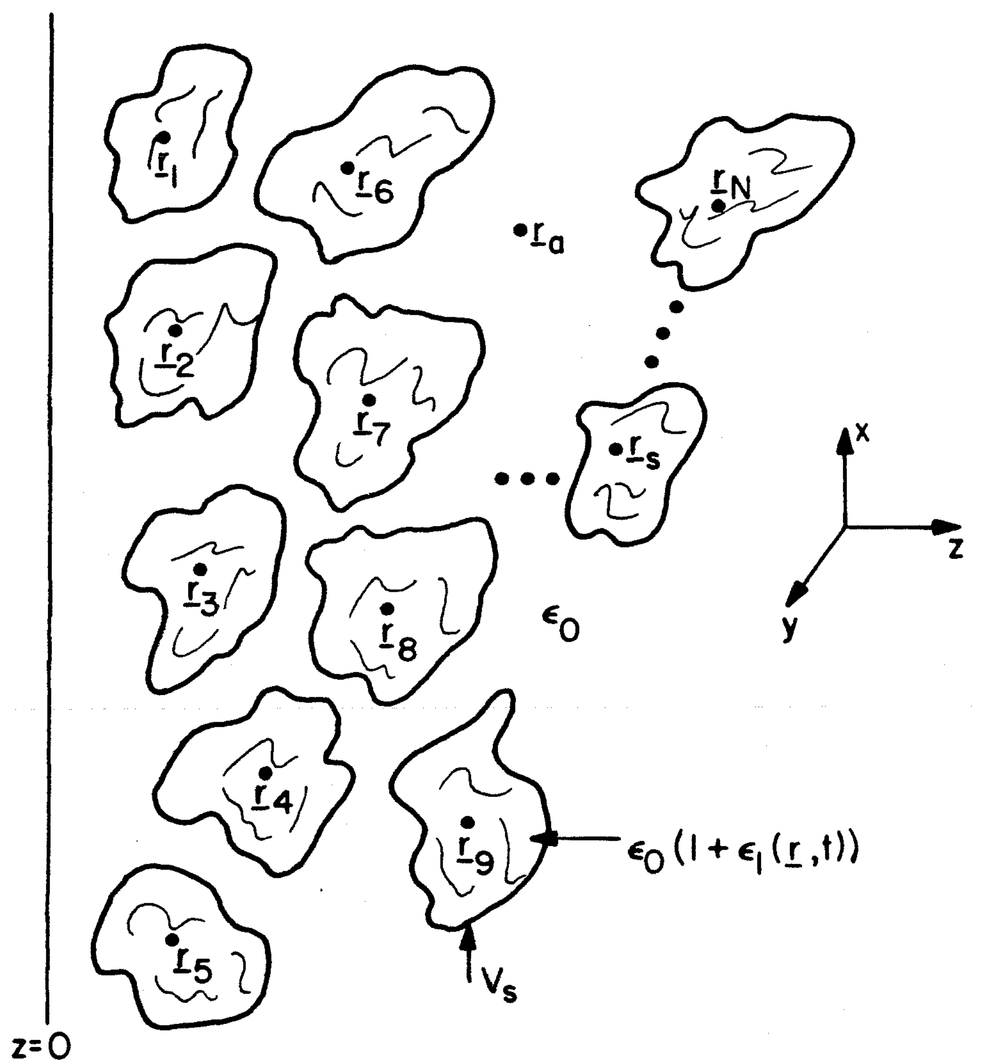

Figure 3-1. Illustration of the division of the continuous random medium into N elementary scattering volumes.

Throughout this paper, sharp brackets < > denote the ensemble, space, or time average, which is equivalent for a statistically stationary, homogeneous and frozen-in medium (see Appendix A). The space-correlation function $B_\varepsilon(r_d)$ is the fundamental measure of the random medium. Higher order moments of ε_1 are rarely discussed except to note that they are completely specified in terms of $B_\varepsilon(r_d)$ whenever ε_1 forms a gaussian random process. At this point in the problem development, the details of $B_\varepsilon(r_d)$ are unimportant; its assumed form will be given later. It should be emphasized, however, that any correlation function is only a model of the medium's irregularity structure and must be evaluated in terms of its ability to satisfactorily predict the results of experiments. An equivalent characterization of the medium is given by the spectral density of its irregularities $\Phi_\varepsilon(\kappa)$, which is the Fourier transform

$$\Phi_\varepsilon(\kappa) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} B_\varepsilon(r_d) \frac{\sin(\kappa r_d)}{\kappa} r_d \, dr_d \tag{3.1.4}$$

The spectrum $\Phi_\varepsilon(\kappa)$ represents the medium's ability to produce irregularities of a certain blob-size $\ell = 2\pi/\kappa$. Three frequently used permittivity spectra are discussed in Section 3.2.

The medium's index of refraction $n = (\varepsilon/\varepsilon_0)^{\frac{1}{2}}$ can be expressed as $n(\underline{r},t) = 1 + n_1(\underline{r},t)$. For small ε_1, the fluctuation of the index of refraction is approximately given by

$$n_1(\underline{r},t) \approx \frac{1}{2} \varepsilon_1(\underline{r},t) \ . \tag{3.1.5}$$

Additional details about the treatment of random variables are discussed in Appendix A.

Consider a monochromatic ($\omega = ck$) plane electromagnetic wave incident upon the elementary volume V_s. Referring to Figure 3-2, the incident wave $\underline{E}^{inc} = \hat{e}_{s'} \exp(ik\hat{s}'\cdot\underline{r})$ is assumed to travel in the direction $\hat{s}'$ and is polarized along $\hat{e}_{s'}$ with unit amplitude. From the procedure outlined in Appendix B, one obtains the scattered field $\underline{E}^{sc}$ in the direction $\hat{s}$ at a distance r from the scattering volume in the far-zone:

$$\underline{E}^{sc}(r,\hat{s}) = \underline{f}(\hat{s},\hat{s}')\frac{e^{+ikr}}{r} = \hat{e}_s f(\hat{s},\hat{s}')\frac{e^{+ikr}}{r} \tag{3.1.6}$$

$$\underline{f}(\hat{s},\hat{s}') = \frac{k^2}{4\pi}\int_{V_s} \{-\hat{s} \times [\hat{s} \times \underline{E}(\underline{r}')]\}\varepsilon_1(\underline{r}')e^{-ik\hat{s}\cdot\underline{r}'}d\underline{r}' \ , \tag{3.1.7}$$

where $\underline{f}$ is the far-zone scattering amplitude and $\underline{E}(\underline{r}')$ is the total electric field at $\underline{r}'$. Since ε_1 is small and occupies only the elementary volume V_s, it is appropriate and applicable to replace $\underline{E}(\underline{r}')$ in the integral of (3.1.7) with the incident field $\underline{E}^{inc}(\underline{r}')$. This substitution is known as the Born approximation and yields the scattering amplitude

$$\underline{f}(\hat{s},\hat{s}') = \hat{e}_s \sin\chi \frac{k^2}{4\pi}\int_{V_s} \varepsilon_1(\underline{r}')e^{+ik(\hat{s}'-\hat{s})\cdot\underline{r}'}d\underline{r}' \ , \tag{3.1.8}$$

where χ is the angle between the polarization vector of the incident wave $\hat{e}_{s'}$ and the direction of observation $\hat{s}$, and $\hat{e}_s \sin\chi = -\hat{s} \times (\hat{s} \times \hat{e}_{s'})$ (see Figure 3-2).

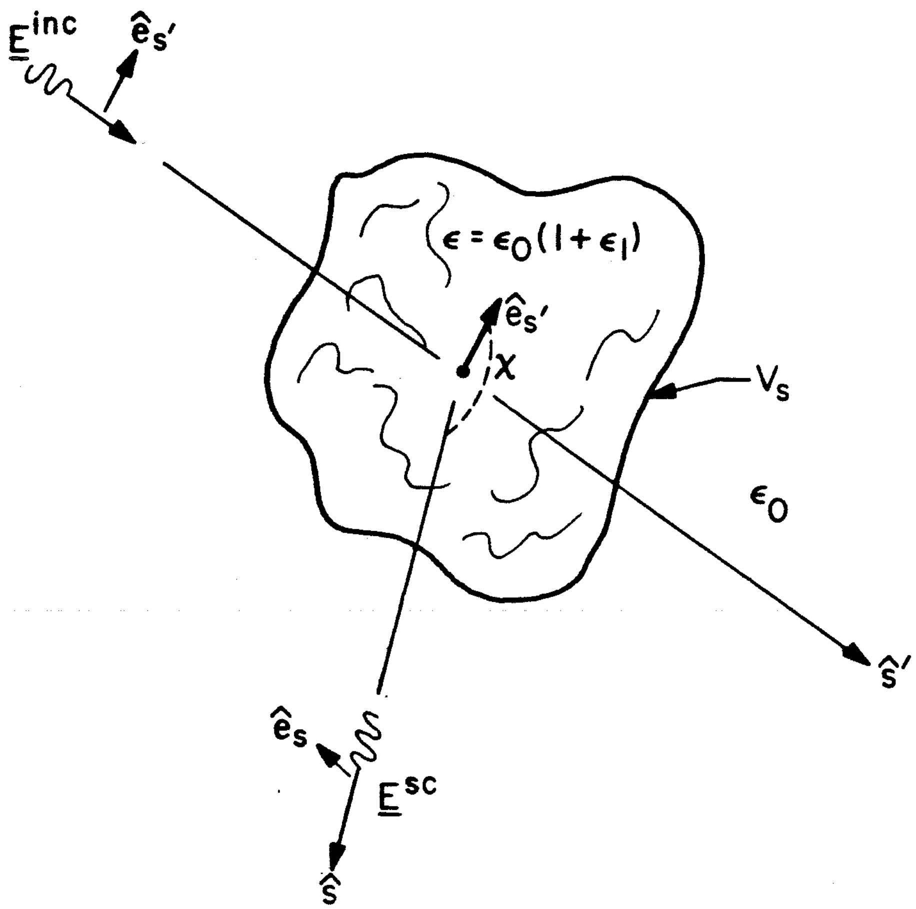

Figure 3-2. Elementary volume of random medium illuminated by a plane electromagnetic wave polarized along $\hat{e}_{s'}$. The incident wave is directed along $\hat{s}'$ and the scattered wave is observed along $\hat{s}$.

The vector scattering amplitude $\underline{f}$ can be expressed as a matrix $\underline{\underline{f}}$, whose components relate the incident and scattered electric field parallel and perpendicular to the scattering plane, the plane containing the unit vectors $\hat{s}$ and $\hat{s}'$. The matrix formulation simplifies treatment of scattering from a single volume, and is used throughout this study. Its geometry is illustrated in Figure 3-3. Accordingly, the far-zone scattered field becomes

$$\begin{bmatrix} E_{\parallel}^{sc} \\ E_{\perp}^{sc} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} E_{\parallel}^{inc} \\ E_{\perp}^{inc} \end{bmatrix} \frac{e^{+ikr}}{r} \tag{3.1.9}$$

$$\begin{bmatrix} f \end{bmatrix} = \frac{k^2}{4\pi} \int_{V_s} \varepsilon_1(\underline{r}')e^{+i\underline{k}_s \cdot \underline{r}'} d\underline{r}' \begin{bmatrix} \cos\underset{\sim}{\theta} & 0 \\ 0 & 1 \end{bmatrix} , \tag{3.1.10}$$

where $\underset{\sim}{\theta}$ is the angle between $\hat{s}$ and $\hat{s}'$ (see Figure 3-3), $\underline{k}_s = k(\hat{s}'-\hat{s})$, and $|\underline{k}_s| = 2k \sin(\underset{\sim}{\theta}/2)$. The angle $\underset{\sim}{\theta}$ is obtained by noting that $\chi=90°-\underset{\sim}{\theta}$ for the component of electric field parallel to the scattering plane ($E_{\parallel}$), and $\chi = 90°$ for the component perpendicular to the scattering plane ($E_{\perp}$). The $\cos(\underset{\sim}{\theta})$ factor in (3.1.10) represents the familiar scattering pattern of an electric dipole. For further discussion on the scattering amplitude matrix, see Van de Hulst (13). The elements of the scattering matrix depend only upon $\cos\underset{\sim}{\theta} = \hat{s}\cdot\hat{s}'$, and from the Helmholtz reciprocity theorem (13), it follows that f_{12} and f_{21} must vanish. This simplification results from the scattering symmetry about the axis $\hat{s}'$.

Although the selection of the reference axes in directions parallel and perpendicular to the scattering plane substantially simplifies treatment of scattering from a single particle, this selection is quite

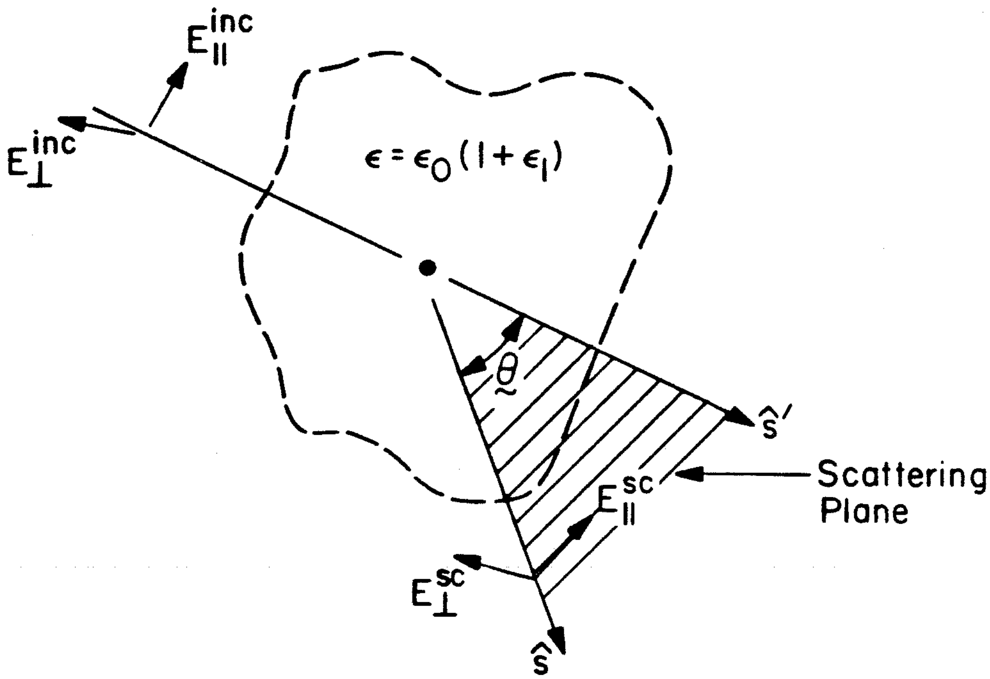

Figure 3-3. Scattering geometry for the far-field scattering matrix, referenced to the scattering plane.

impractical for multiple scattering. In this case, since the amplitudes or energy densities of several scattered waves must be added, a system of reference axes (for use in each scattering event) which are fixed and mutually parallel rather than varying with the changing orientations of the scattering plane is required. For a plane parallel medium with complete statistical homogeneity, it is customary to select the reference axes parallel and perpendicular to the plane through the normal to the medium boundary and the direction $\hat{s}$. With the normal to the plane parallel boundaries, $\hat{z}$, taken as the polar axis, the direction $\hat{s}$ is defined by the polar angle θ, or its cosine μ ($\mu = \cos\theta = \hat{s}\cdot\hat{z}$) and by the azimuth angle ϕ of the plane through the axis $\hat{z}$ and the direction $\hat{s}$.

The relationship (3.1.10) then has to be modified by: (a) a rotation of the reference axes around $\hat{s}'$ by the angle η to orient them to the scattering plane, and (b) a rotation through the angle $(\pi-\gamma)$ to reorient them after scattering. Figure 3-4 illustrates these rotations. If $\underline{\underline{R}}(\alpha)$ is defined as this clockwise rotation matrix,

$$\underline{\underline{R}}(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \qquad (3.1.11)$$

where α represents the rotation of the reference axes, then (3.1.10) is replaced with

$$\underline{\underline{A}}(\hat{s},\hat{s}') = \underline{\underline{R}}(\pi-\gamma)\underline{\underline{f}}(\cos\underset{\sim}{\theta})R(-\eta) \; . \qquad (3.1.12)$$

From the derivations of Chandrasekhar (5), the rotation matrix $\underline{\underline{R}}$ is obtained and $\underline{\underline{A}}$ may be written in the form

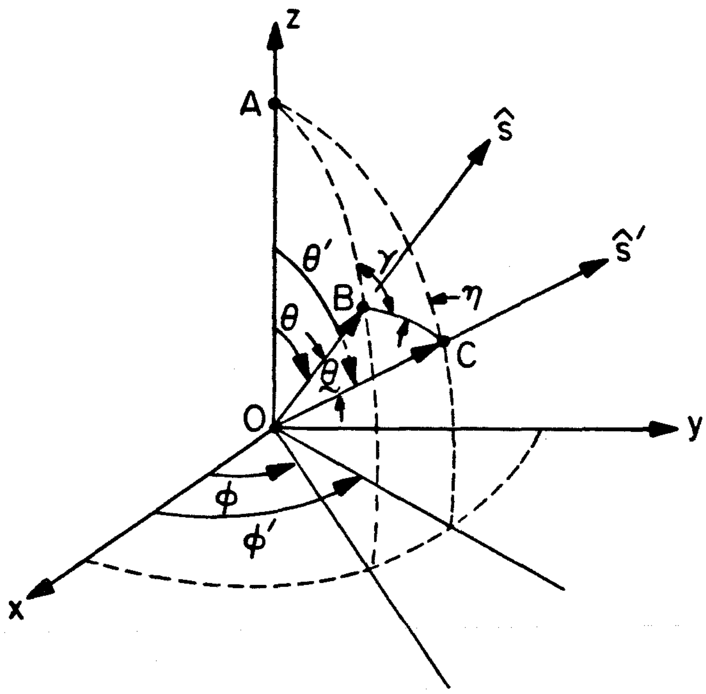

Figure 3-4. Reference planes for incident and scattered fields for use in multiple scattering analyses. The scattering plane is through the points OBC, The meridian planes OBA and OCA are used for multiple scattering.

$$\underline{\underline{A}}(\hat{s},\hat{s}') = \frac{k^2}{4\pi} \int_{V_s} \varepsilon_1(\underline{r}')e^{+i\underline{k}_s\cdot\underline{r}'} d\underline{r}' \begin{bmatrix} (\ell,\ell) & (r,\ell) \\ (\ell,r) & (r,r) \end{bmatrix} \quad (3.1.13)$$

where

$$\begin{aligned} (\ell,\ell) &= \sin\theta\sin\theta' + \cos\theta\cos\theta'\cos(\phi'-\phi) \\ (r,\ell) &= +\cos\theta\sin(\phi'-\phi) \\ (\ell,r) &= -\cos\theta'\sin(\phi'-\phi) \\ (r,r) &= \cos(\phi'-\phi) \ . \end{aligned} \quad (3.1.14)$$

The angle pair (θ',ϕ') specifies the incident field direction $\hat{s}'$; (θ,ϕ) determine the scattered field direction $\hat{s}$. The matrix elements given in (3.1.14) correspond to the electric dipole scatterer. To differentiate between the scattering elements considered here and those strictly for the electric dipole, the scattering amplitude given in (3.1.13) is referred to as the "augmented dipole" pattern of the random medium.

3.2 Differential Scattering Cross-Section

The ensemble average of the scattered field $\langle \underline{E}^{sc} \rangle$ is zero due to the vanishing first moment of ε_1 given in (3.1.2). However, the second statistical moment of the scattered field is nonzero; this moment represents the scattered power. The differential scattering cross-section per unit volume is defined as (see Appendix B)

$$\sigma(\hat{s},\hat{s}') = \frac{1}{V_s} |f(\hat{s},\hat{s}')|^2 \ , \quad (3.2.1)$$

where the units of $\sigma(\hat{s},\hat{s}')$ are inverse length per steradian. However, f is a random function because it depends upon ε_1 which is random; therefore, (3.2.1) is replaced by its ensemble average

$$\sigma(\hat{s},\hat{s}') = \frac{1}{V_s} \langle f(\hat{s},\hat{s}') f^*(\hat{s},\hat{s}') \rangle \quad . \tag{3.2.2}$$

Substitution of (3.1.8) into (3.2.2) gives an integral expression for $\sigma(\hat{s},\hat{s}')$:

$$\sigma(\hat{s},\hat{s}') = \frac{1}{V_s} \frac{k^4}{(4\pi)^2} \sin^2\chi \int_{V_s}\int_{V_s} \langle \varepsilon_1(\underline{r}_1)\varepsilon_1(\underline{r}_2)\rangle e^{+i\underline{k}_s\cdot(\underline{r}_1-\underline{r}_2)} d\underline{r}_1 d\underline{r}_2 \tag{3.2.3}$$

By defining a function $V(\underline{r})$ according to the equations,

$$V(\underline{r}) = 1 \qquad \underline{r} \quad \text{inside} \quad V_s$$

$$V(\underline{r}) = 0 \qquad \underline{r} \quad \text{outside} \quad V_s \quad ,$$

(3.2.3) can be rewritten as

$$\sigma(\hat{s},\hat{s}') = \frac{1}{V_s} \cdot \frac{k^4}{(4\pi)^2} \sin^2\chi \iint_{-\infty}^{+\infty} V(\underline{r}_1)V(\underline{r}_2)\langle \varepsilon_1(\underline{r}_1)\varepsilon_1(\underline{r}_2)\rangle e^{+i\underline{k}_s\cdot(\underline{r}_1-\underline{r}_2)} d\underline{r}_1 d\underline{r}_2 . \tag{3.2.4}$$

Next, change variables from $\underline{r}_1$ and $\underline{r}_2$ to $\underline{r}_c$ and $\underline{r}_d$ by means of the formulae

$$\underline{r}_c = \frac{1}{2}(\underline{r}_1 + \underline{r}_2)$$

$$\underline{r}_d = \underline{r}_1 - \underline{r}_2 \quad . \tag{3.2.5}$$

The integral expression for the differential scattering cross-section then becomes

$$\sigma(\hat{s},\hat{s}') = \frac{1}{V_s}\frac{k^4}{(4\pi)^2}\sin^2\chi\iint_{-\infty}^{+\infty} V(\underline{r}_c + \tfrac{1}{2}\underline{r}_d)V(\underline{r}_c - \tfrac{1}{2}\underline{r}_d)B_\varepsilon(r_d)e^{+i\underline{k}_s\cdot\underline{r}_d}d\underline{r}_d d\underline{r}_c \tag{3.2.6}$$

Depending upon the value of $\underline{r}_d$, the integrand in (3.2.6) is nonzero only inside the common volume specified by the overlap of the product $V(\underline{r}_c + \frac{1}{2}\underline{r}_d)\cdot V(\underline{r}_c - \frac{1}{2}\underline{r}_d)$. If this product is factored into the form $V(\underline{r}_c)\cdot D(\underline{r}_d)$, then (3.2.6) is reduced to

$$\sigma(\hat{s},\hat{s}') = 2\pi k^4 \sin^2\chi\ \overline{\Phi}_n(\underline{k}_s), \tag{3.2.7}$$

where

$$4\overline{\Phi}_n(\underline{k}_s) = \overline{\Phi}_\varepsilon(\underline{k}_s) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} D(\underline{r}_d)B_\varepsilon(r_d)e^{+i\underline{k}_s\cdot\underline{r}_d}d\underline{r}_d\ . \tag{3.2.8}$$

$D(\underline{r}_d)$ may be interpreted as a volume overlap function. Its significance is illustrated in Figure 3.5. $\overline{\Phi}_n(\underline{k}_s)$ is an averaged spectral density of the refractive index fluctuations, the average resulting from the overlap function $D(\underline{r}_d)$ due to the finite size scattering volume. Note that even if the random field $\varepsilon_1(\underline{r})$ is homogeneous and isotropic, its average spectral density still depends upon the direction of $\underline{k}_s$. If the scattering volume is infinite, then $D(\underline{r}_d) = 1$ and $\overline{\Phi}_n \equiv \Phi_n$. In this work, it is assumed that the scattering volume is sufficiently large so that $D(\underline{r}_d)$ is essentially constant over the interesting range of $B_\varepsilon(r_d)$ as illustrated in Figure 3-5. If ℓ denotes this range of $B_\varepsilon(r_d)$ and L denotes the characteristic size of V_s, then this criterion can be stated as

$$L \gg \ell\ . \tag{3.2.9}$$

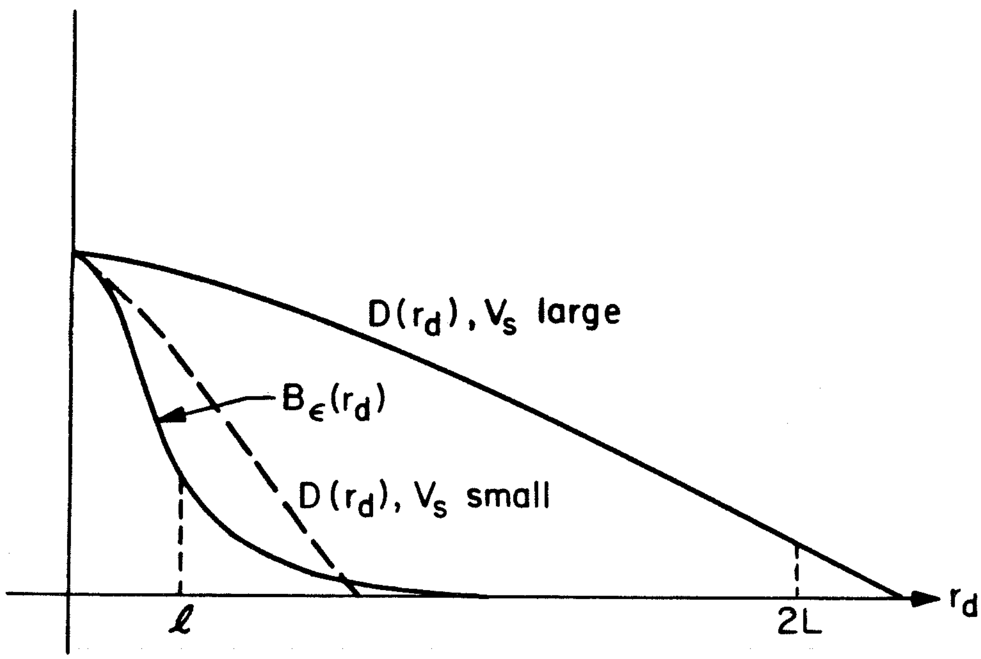

Figure 3-5. Approximate behavior of a "large" scattering volume overlap function compared to the correlation function of the random medium.

Strictly speaking, an infinite scattering volume is not allowed since the validity of the Born approximation would be severely limited in such cases. The equivalent scattering matrix for the differential scattering cross section is

$$\underline{\underline{\sigma}}(\hat{s},\hat{s}') = 2\pi k^4 \Phi_n(k_s) \begin{bmatrix} \cos^2\underset{\sim}{\theta} & 0 \\ 0 & 1 \end{bmatrix} \qquad (3.2.10)$$

$$\cos\underset{\sim}{\theta} = \hat{s}\cdot\hat{s}' \quad \text{and} \quad k_s = 2k\sin(\underset{\sim}{\theta}/2) \ .$$

The significance of using $\Phi_n(k_s)$ instead of $\overline{\Phi}_n(\underline{k}_s)$ in $\underline{\underline{\sigma}}(\hat{s},\hat{s}')$ is easily related to the scattering physics. First we note that the $\cos^2\underset{\sim}{\theta}$ factor is simply the electric dipole pattern of the scatterer. When $\Phi_n(k_s)$ is used, it is evident from (3.2.10) that scattering into the direction θ_s is produced by blobs or irregularities of characteristic size ℓ_s given by

$$\ell_s = \frac{2\pi}{k_s} = \lambda/\sin(\theta_s/2) \quad . \qquad (3.2.11)$$

However, when $\overline{\Phi}_n(\underline{k}_s)$ is used, scattering in direction θ_s is contributed to by a collection of blobs which have a narrow spectrum of sizes centered about ℓ_s. In an analogous fashion, the averaging process specified by $\overline{\Phi}_n$ smears the angular selectivity indicated by (3.2.11). That is, blobs of characteristic size ℓ_s produce a scattered wave only in the direction θ_s according to (3.2.11) if V_s were infinite, whereas a finite scattering volume produces a scattered beam in a narrow angular spread about the direction θ_s.

The total scattering cross-section σ_s (scattering loss per unit length) is obtained by integrating $\sigma(\hat{s},\hat{s}')$ in (3.2.10) or (3.2.7) over 4π steradians:

$$\sigma_s = \int_{4\pi} \sigma(\hat{s},\hat{s}')d\omega_s \ , \tag{3.2.12}$$

where $d\omega_s = \sin\theta d\theta d\phi$ is the differential solid angle. By taking $\hat{s}' = \hat{z}$, a change of variables in (3.2.12) from θ to $k_s = 2k\sin(\theta/2)$ yields

$$\sigma_s = 4\pi^2k^2 \int_0^{2k} \Phi_n(k_s)\sin^2\chi k_s dk_s d\phi \ . \tag{3.2.13}$$

From the scattering geometry, we have $\sin^2\chi = 1 - \sin^2\theta\cos^2\phi$. Thus,

$$\sigma_s = 4\pi^2k^2 \int_0^{2k} \Phi_n(k_s)[1 - \frac{1}{2}(\frac{k_s}{k})^2 + \frac{1}{8}(\frac{k_s}{k})^4]k_s dk_s \ . \tag{3.2.14}$$

The factor in the bracket of (3.2.14) accounts for the electromagnetic dipole pattern. For acoustic or scalar wave scattering, this factor is unity.

For multiple scattering, the differential scattering matrix (3.2.10) must be referenced to a fixed coordinate system just as discussed when the scattering amplitude matrix $\underline{\underline{A}}$ given in (3.1.13) was developed. This can be done by applying the appropriate rotation maxtrix to $\underline{\underline{\sigma}}(\hat{s},\hat{s}')$ or by simply taking the ensemble average of the product of $\underline{\underline{A}}$ with its complex conjugate. The latter procedure yields the following expression for the differential scattering cross-section for use in multiple scattering analyses:

$$\underline{\underline{S}}(\hat{s},\hat{s}') = \frac{1}{V_s} \langle \underline{\underline{A}} \times \underline{\underline{A}}^* \rangle = 2\pi k^4 \Phi_n(k_s)$$

$$\times \begin{bmatrix} (\ell,\ell)^2 & (\ell,\ell)(r,\ell) & (\ell,\ell)(r,\ell) & (r,\ell)^2 \\ (\ell,\ell)(\ell,r) & (\ell,\ell)(r,r) & (r,\ell)(\ell,r) & (r,\ell)(r,r) \\ (\ell,\ell)(\ell,r) & (\ell,r)(r,\ell) & (\ell,\ell)(r,r) & (r,\ell)(r,r) \\ (\ell,r)^2 & (\ell,r)(r,r) & (\ell,r)(r,r) & (r,r)^2 \end{bmatrix} \quad (3.2.15)$$

The matrix $\underline{\underline{S}}$ should be used in connection with the coherency matrix (2.2.5). The matrix elements enclosed within the bracket of (3.2.15) are formed from equations (3.1.14).

In the next paragraphs, three models for the correlation function $B_n(|\underline{r}_1-\underline{r}_2|)$ are discussed along with their respective spectrum $\Phi_n(\kappa)$. The three correlation functions and their spectra are

(1) Exponential:

$$B_n(r_d) = \langle n_1^2 \rangle e^{-r_d/\ell} \quad (3.2.16)$$

$$\Phi_n(\kappa) = \frac{\langle n_1^2 \rangle \ell^3}{[1+(\kappa\ell)^2]^2} \left(\frac{1}{\pi^2}\right) \quad (3.2.17)$$

(2) Gaussian:

$$B_n(r_d) = \langle n_1^2 \rangle e^{-r_d^2/\ell^2} \quad (3.2.18)$$

$$\Phi_n(\kappa) = \frac{\langle n_1^2 \rangle \ell^3}{8\pi\sqrt{\pi}} e^{-(\kappa\ell)^2/4} \quad (3.2.19)$$

(3) Kolmogorov:

$B_n(r_d)$ not specified

$$\Phi_n(\kappa) = 0.033C_n^2(\kappa^2 + \frac{1}{L_o^2})^{-11/6} e^{-\kappa^2/\kappa_m^2} \qquad (3.2.20)$$

In (3.2.20), $\kappa_m = 5.92/\ell$ and C_n^2 is the random medium structure constant. The exponential correlation function has been widely used to study over-the-horizon scatter propagation. Such random medium is described by two quantities: the variance $\langle n_1^2 \rangle$ and the correlation distance ℓ. This exponential model is possibly suggested by the physical theory of pressure fluctuations in the turbulent atmosphere. Use of this model however is largely based upon the ease of mathematical operation involving the exponential function. The gaussian model has been widely used in line-of-sight theories mainly because it can be integrated easily. The gaussian model has the same basic shortcomings as the exponential model in that it has no strong foundation in physical theories of turbulence and does not adequately explain the results of scattering experiments. Note that both the exponential and gaussian models have only one characteristic scale size ℓ describing the random medium. Typical angular distributions for the differential scattering cross-section using the gaussian spectrum of index of refraction fluctuations are given in Figure 3-6.

The Kolmogorov model is based upon the physical theory of turbulence. According to the theory, the turbulence eddies or random irregularities of the medium are characterized by two scale sizes: The outer scale length L_o and the inner scale length ℓ. The spectrum in (3.2.20)

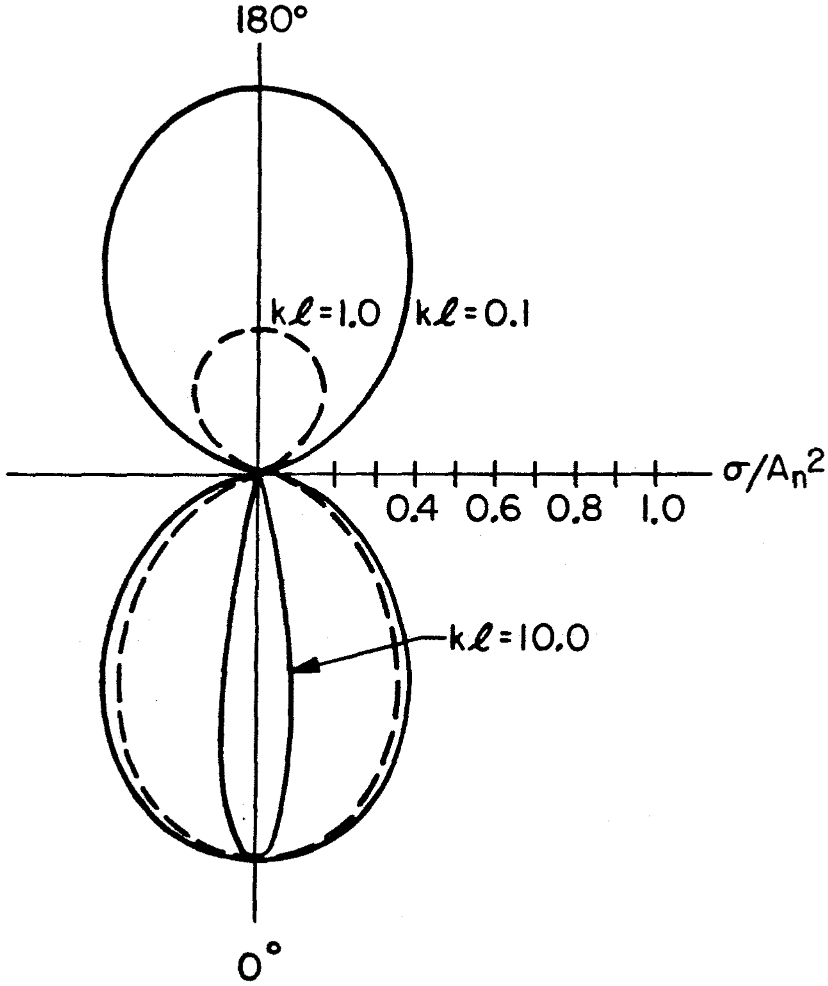

Figure 3-6. Differential scattering for polarization parallel to scattering plane and gaussian spectrum. $A_n^2 = \langle n_1^2 \rangle (k\ell)^4 / 4\sqrt{\pi}\,\ell$.

is sometimes called the Von Karman spectrum. The value of the structure constant in the atmosphere is on the order of $10^{-13}m^{-2/3}$ for strong turbulence and $10^{-17}m^{-2/3}$ for weak turbulence. In the atmosphere, ℓ is usually considered to be approximately 1 or 2 mm near the ground, increasing to about 1 cm at the tropopause. The outer scale length L_0 is generally taken to be about 100 meters.

IV. SCATTERING BY A COLLECTION OF ELEMENTARY VOLUMES

Section III provided the necessary details for scattering from an elementary volume of continuous random medium. In this section, integral equations which account for the multiple scattering interaction among many elementary scattering volumes are derived. Rigorous multiple scattering theory is reviewed in subsection 4.1 in order to place in evidence the scattering physics. This is followed by derivations for the average and fluctuating electric fields in subsections 4.2 and 4.3, respectively. In subsection 4.4, the basic integral equation for the total electric field $\underline{E}(\underline{r})$ is developed along with transport equations for the electric field and coherency matrix. Subsection 4.5 (along with Appendix C) summarizes a simple analytic connection between multiple scattering theory and radiative transfer theory. It is assumed that all elementary scattering volumes are statistically identical, in the far-zone of each other, and uniformly distributed through the total volume. In addition, the scattering volumes are uncorrelated and represented mathematically as point scatterers with all the scattering characteristics of the elementary volume.

4.1 Rigorous Multiple Scattering Theory

Rigorous multiple scattering theory is developed in this subsection to derive mathematical representations for the electric field. Consider a random collection of N elementary scattering volumes located at $\underline{r}_1, r_2$ --- $\underline{r}_N$ as illustrated in Figure 3-1. Each scattering volume is characterized by the same statistical moments. As discussed in Appendix A, a point scatterer with all the scattering characteristics of the augmented dipole represents the scattering volume. The electric field at points $\underline{r}_a$ in between the scatterers satisfies the free-space wave equation

$$\nabla \times \nabla \times \underline{E}(\underline{r}_a) - k^2 \underline{E}(\underline{r}_a) = 0 , \qquad (4.1.1)$$

where $k = \frac{2\pi}{\lambda}$ is the wavenumber of the medium surrounding the elementary scattering volumes. The total field at $\underline{r}_a$ is the sum of the incident field $\underline{E}^{inc}(\underline{r}_a)$ and the contributions $\underline{E}^{sc}(\underline{r}_a,\underline{r}_s)$ from all N scattering volumes:

$$\underline{E}(\underline{r}_a) = \underline{E}^{inc}(\underline{r}_a) + \sum_{s=1}^{N} \underline{E}^{sc}(\underline{r}_a,\underline{r}_s) . \qquad (4.1.2)$$

The scattered field $\underline{E}^{sc}$ is the wave at $\underline{r}_a$ scattered from the elementary volume at $\underline{r}_s$. This scattered wave can be expressed in terms of the incident field $\underline{\Psi}(\underline{r}_s)$ and the scattering characteristic of the scattering volume,

$$\underline{E}^{sc}(\underline{r}_a,\underline{r}_s) = \underline{\underline{U}}_s^a \, \underline{\Psi}(\underline{r}_s) . \qquad (4.1.3)$$

Note that, in general, $\underline{\underline{U}}_s^a \, \underline{\Psi}(\underline{r}_s)$ does not mean the product of $\underline{\underline{U}}_s^a$ and $\underline{\Psi}(\underline{r}_s)$, but rather a symbolic notation to indicate the field at $\underline{r}_a$ due the scattering from the volume at $\underline{r}_s$. The effective field $\underline{\Psi}(\underline{r}_s)$ is incident upon the scattering volume at $\underline{r}_s$. It consists of the incident field $\underline{E}^{inc}(\underline{r}_s)$ and the waves scattered from all scattering volumes except the one at $\underline{r}_s$. Thus, $\underline{\Psi}$ is given by

$$\underline{\Psi}(\underline{r}_s) = \underline{E}^{inc}(\underline{r}_s) + \sum_{\substack{t=1 \\ t \neq s}}^{N} \underline{E}^{sc}(\underline{r}_s,\underline{r}_t) . \qquad (4.1.4)$$

Equations (4.1.2), (4.1.3), and (4.1.4) are combined to form the fundamental pair of equations for scattering and propagation in the random medium. These are

$$\underline{E}(\underline{r}_a) = \underline{E}^{inc}(\underline{r}_a) + \sum_{s=1}^{N} \underline{\underline{U}}_s^a \, \underline{\Psi}(\underline{r}_s) \tag{4.1.5}$$

$$\underline{\Psi}(\underline{r}_s) = \underline{E}^{inc}(\underline{r}_s) + \sum_{\substack{t=1 \\ t\neq s}}^{N} \underline{\underline{U}}_t^s \, \underline{\Psi}(\underline{r}_t) \; . \tag{4.1.6}$$

In principle, $\underline{\Psi}$ can be eliminated from these two equations by substituting (4.1.6) into (4.1.5) and iterating (1). The iteration produces two groups of scattered waves. One group represents chains of successive scattering involving different scattering volumes. The second group represents successive scattering through the same scattering volume more than once. Twersky (6) included all the terms in the first group and neglected those in the second group. His theory yields the following series expansion for $\underline{E}(\underline{r}_a)$:

$$\underline{E}(\underline{r}_a) = \underline{E}_i^a + \sum_{s=1}^{N} \underline{\underline{U}}_s^a \, \underline{E}_i^s + \sum_{s=1}^{N} \sum_{\substack{t=1 \\ t\neq s}}^{N} \underline{\underline{U}}_s^a \, \underline{\underline{U}}_t^s \, \underline{E}_i{}^t$$

$$+ \sum_{s=1}^{N} \sum_{\substack{t=1 \\ t\neq s}}^{N} \sum_{\substack{m=1 \\ m\neq t \\ m\neq s}}^{N} \underline{\underline{U}}_s^a \, \underline{\underline{U}}_t^s \, \underline{\underline{U}}_m^t \, \underline{E}_i{}^m + \ldots , \tag{4.1.7}$$

where a short form notation has been used according to the equation $\underline{E}_i^a = \underline{E}^{inc}(\underline{r}_a)$. It is known that (4.1.7) takes care of almost all the multiple scattering and gives excellent results when backscattering is small compared to scattering in other directions. Furthermore, when N is large the difference between the exact scattering process (4.1.5) or (4.1.6) and the Twersky approximation (4.1.7) becomes very small. Even so, (4.1.7) offers little practical usefulness except in its interpretation of the physical scattering processes.

We note that when the incident wave at $\underline{r}_s$ can be approximated by a plane wave propagating the the directions $\hat{s}'$,

$$\underline{\Psi}(\underline{r}_s) \approx \hat{e}_{s'} e^{+ik\hat{s}'\cdot\underline{r}_s} \qquad (4.1.8)$$

and the distance between $\underline{r}_s$ and $\underline{r}_a$ is large, the scattering characteristic $\underline{\underline{U}}_s^a$ can be approximated by the far-zone scattering amplitude:

$$\underline{\underline{U}}_s^a \approx \underline{\underline{A}}(\hat{s},\hat{s}') \frac{e^{+ik|\underline{r}_a-\underline{r}_s|}}{|\underline{r}_a-\underline{r}_s|} , \qquad (4.1.9)$$

where $\hat{s}$ is the unit vector in the direction $\underline{r}_a-\underline{r}_s$. This far-zone approximation (discussed in Appendix B) will be used throughout this study.

It will now be shown that the Twersky multiple scattering theory can be used for deriving the basic integral equation satisfied by the coherent or average field. First, take the average of (4.1.7) with respect to the N scattering volumes $\langle E \rangle_N$ according to the definition given in Appendix A, equation (A.7). In the limit as $N\to\infty$, this averaging process yields

$$<\underline{E}(\underline{r}_a)>_N = \underline{E}_i^a + \int \underline{\underline{U}}_s^a \, \underline{E}_i^s \, \rho d\underline{r}_s + \iint \underline{\underline{U}}_s^a \, \underline{\underline{U}}_t^s \, \underline{E}_i^t \, \rho^2 d\underline{r}_s d\underline{r}_t$$

$$+ \iiint \underline{\underline{U}}_s^a \, \underline{\underline{U}}_t^s \, \underline{\underline{U}}_m^t \, \underline{E}_i^m \, \rho^3 d\underline{r}_s d\underline{r}_t d\underline{r}_m + \cdots \quad . \tag{4.1.10}$$

In (4.1.10), the integral expressions were obtained as in the following:

$$\sum_{s=1}^{N} <\underline{\underline{U}}_s^a \, \underline{E}_i^s> = \sum_{s=1}^{N} \int \underline{\underline{U}}_s^a \, \underline{E}_i^s \, \frac{\rho}{N} \, d\underline{r}_s = \int \underline{\underline{U}}_s^a \, \underline{E}_i^s \, \rho d\underline{r}_s$$

and (4.1.11)

$$\sum_{s=1}^{N} \sum_{\substack{t=1 \\ t\neq s}}^{N} <\underline{\underline{U}}_s^a \, \underline{\underline{U}}_t^s \, \underline{E}_i^t> = \frac{(N-1)\cdot N}{N^2} \iint \underline{\underline{U}}_s^a \, \underline{\underline{U}}_t^s \, \underline{E}_i^t \, \rho^2 d\underline{r}_s d\underline{r}_t \quad .$$

Finally, we observe that (4.1.10) is just an expanded form of the Foldy-Twersky integral equation (1):

$$<\underline{E}(\underline{r}_a)>_N = \underline{E}^{inc}(\underline{r}_a) + \int \underline{\underline{U}}_s^a \, <\underline{E}(\underline{r}_s)>_N \, \rho d\underline{r}_s \quad . \tag{4.1.12}$$

With the insertion of the far-zone expression (4.1.9) into (4.1.12), the integral equation for $<E>_N$ becomes

$$<\underline{E}>_N = \underline{E}^{inc} + \int \rho \, \underline{\underline{A}}(\hat{s},\hat{s}') \, <\underline{E}>_N \, \frac{e^{+ik|\underline{r}_a-\underline{r}_s|}}{|\underline{r}_a-\underline{r}_s|} \, d\underline{r}_s \quad . \tag{4.1.13}$$

This integral equation for $<E>_N$ was first obtained by Foldy (8), however, its physical significance was placed in evidence by Twersky (6).

4.2 Coherent Field

Let us consider a plane wave normally incident upon the random medium as shown in Figure 4-1. As derived on the preceding pages, the average field $\langle E \rangle_N$ satisfies the Foldy-Twersky integral equation (4.1.13). The average field in the medium travels in the same direction as the incident field, i.e., $\hat{s} = \hat{z}$. For a polarized monochromatic (ω=ck) infinite plane wave, the incident field is given by

$$\underline{E}^{inc}(z) = \begin{bmatrix} E_x(0) \\ E_y(0) \end{bmatrix} e^{+ikz} \quad . \tag{4.2.1}$$

It is clear that for a statistically homogeneous plane parallel medium as indicated in Figure 4-1, the average field varies only in the $\hat{z}$ direction. Thus we write

$$\langle \underline{E}(\underline{r}) \rangle_N = \langle \underline{E}(z) \rangle_N \; . \tag{4.2.2}$$

For this geometry, the integral equation for the average field becomes

$$\langle \underline{E}(z) \rangle_N = \underline{E}(0) e^{+ikz} + \int_0^z dz' \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \rho \underline{\underline{A}}(\hat{s},\hat{z}) \langle \underline{E}(z') \rangle_N \frac{e^{+ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \; . \tag{4.2.3}$$

The integrations with respect to x' and y' may be evaluated exactly (7). For the present purposes, we obtain the same result by using the method of stationary phase:

$$\iint G(x',y') e^{+ikg(x',y')} dx'dy' \sim \frac{2\pi i}{k} \frac{[G\, e^{+ikg}]}{[g_{xx} g_{yy} - g_{yx}^2]^{\frac{1}{2}}} \; , \tag{4.2.4}$$

where the brackets indicate that the functions g, G, and the second

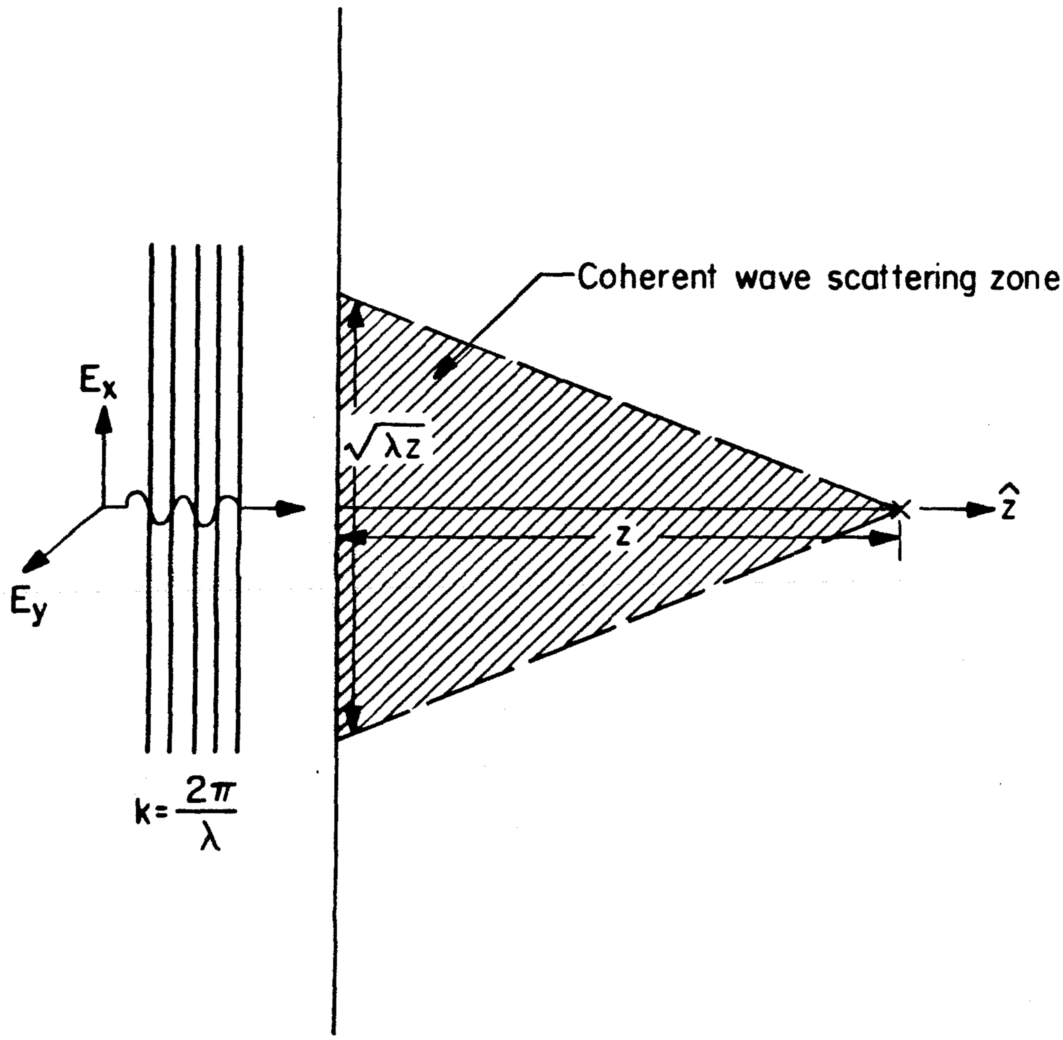

Figure 4-1. Fresnel zone scattering of the coherent wave in a statistically homogeneous plane parallel medium. See Van de Hulst (13).

derivatives of g (i.e., g_{xx}, g_{yy}, g_{yx}) are evaluated at the stationary points for which g_x and $g_y = 0$. Applying the results of (4.2.4) to the integral in (4.2.3) yields

$$<\underline{E}(z)>_N = E(0)e^{+ikz} + \int_0^z dz' \frac{2\pi i}{k} e^{+ik(z-z')} \rho \,\underline{\underline{A}}(\hat{z},\hat{z}) <\underline{E}(z')>_N$$
$$+ \int_z^\infty dz' \frac{2\pi i}{k} e^{+ik(z-z')} \rho \underline{\underline{A}}(-\hat{z},\hat{z}) <\underline{E}(z')>_N . \quad (4.2.5)$$

The second term on the right-hand side of (4.2.5) is the forward coherently scattered wave at z, whereas the third term is the backward coherently scattered wave at z. In many practical circumstances, as will be the case here, the backscattering amplitude is much less than the forward scattering amplitude, $\underline{\underline{A}}(-\hat{z},\hat{z}) <<< \underline{\underline{A}}(\hat{z},\hat{z})$; therefore, the backscattering integral is neglected and (4.2.5) becomes

$$<\underline{E}(z)>_N = \underline{E}(0)e^{+ikz} + \frac{2\pi i}{k} \rho \,\underline{\underline{A}}(\hat{z},\hat{z})e^{+ikz} \int_0^z e^{-ikz'} <\underline{E}(z')>_N dz' . \quad (4.2.6)$$

Finally, to obtain the coherent field we take the ensemble average with respect to the permittivity characteristics (according to (A.9)) of Appendix A) yielding

$$\underline{E}_c(z) = <<E>_N>_\varepsilon = \underline{E}(0)e^{+ikz} + \frac{2\pi i}{k} \rho <\underline{\underline{A}}(\hat{z},\hat{z})>_\varepsilon e^{+ikz} \int_0^z e^{-ikz'} \underline{E}_c(z')dz' .$$
$$(4.2.7)$$

In obtaining (4.2.7) it has been assumed that the statistics of $\underline{E}$ at the location z are independent of the random variable ε_1 at the same point z. The exact solution of (4.2.7) is

$$\underline{E}_c(z) = \underline{E}(0)e^{+i\underline{\underline{\beta}}z} \ , \tag{4.2.8}$$

$$\underline{\underline{\beta}} = k + \frac{2\pi}{k} \rho\langle\underline{\underline{A}}(\hat{z},\hat{z})\rangle \ . \tag{4.2.9}$$

Thus, the average field propagates in the medium with the effective propagation constant $\underline{\underline{\beta}}$, which in general is a matrix. This solution provides an equivalent matrix (tensor) refractive index for the medium. Such an equivalent refractive index describes a combination of several effects, including double refraction, polarization plane rotation, and linear and circular dichroism as suggested earlier by Van de Hulst (13). More specifically, however, for the augmented dipole scattering amplitude discussed in section III, $\underline{\underline{\beta}} = \beta\underline{\underline{I}}$, where $\underline{\underline{I}}$ is the identity matrix. Accordingly, none of the aforementioned effects associated with the matrix refractive index occur in the continuous random medium. In addition, Van de Hulst has shown that this forward traveling average field given by (4.2.8) is coherently scattered from a region known as "first few central Fresnel zones" with respect to the observation point z as illustrated in Figure 4-1.

We note that $\underline{\underline{\beta}}$ is in general complex, even for a lossless medium, and so the average or coherent field is attenuated as it propagates in the medium. This attenuation is due to scattering as discussed in section II. The result given in (4.2.8) and (4.2.9) is also known from the Ewald-Oseen (see Born and Wolf (14)) extinction theorem, which established that the incident wave in the medium may be regarded as extinguished at any point and replaced by another wave with a different velocity (and generally different direction) of propagation.

The coherency matrix for the average field in the continuous random medium is

$$\underline{\underline{J}}_c(z) = \underline{\underline{J}}(0)e^{-\sigma_s z} , \quad (4.2.10)$$

where according to the forward scattering theorem (13), we have used

$$\sigma_s = \rho\sigma = \frac{4\pi}{k} \rho\langle \text{Im}A(\hat{z},\hat{z})\rangle . \quad (4.2.11)$$

As expected, $\underline{\underline{J}}_c$ shows that the average wave energy simply decays exponentially with propagation distance but its initial coherence, specified by $\underline{\underline{J}}(0)$ at z = 0, is unaltered.

It can be shown that $\underline{\underline{J}}_c(z)$ may be interpreted as the unscattered field arriving at z by the most direct paths. Let's consider the geometry illustrated in Figure 4-2. The cylindrical elementary volume indicated has a cross section area S and length Δ. The volume contains $\rho S\Delta$ scatterers, where ρ is the number density of scatterers. Each point scatterer scatters the energy $\sigma\underline{\underline{J}}/S$; therefore the total loss due to scattering is $\Delta\underline{\underline{J}} = \rho\Delta\sigma\underline{\underline{J}}$. The percentage of unscattered energy is

$$\frac{\underline{\underline{J}} - \Delta\underline{\underline{J}}}{\underline{\underline{J}}} = 1 - \rho\Delta\sigma \quad . \quad (4.2.12)$$

From (4.2.12), it is clear that as $\Delta \to 0$ the unscattered portion of $\underline{\underline{J}}$ decreases exponentially as given in (4.2.10).

4.3 Incoherent Field

Next, let us consider the fluctuating field at a point $\underline{r}$ in the

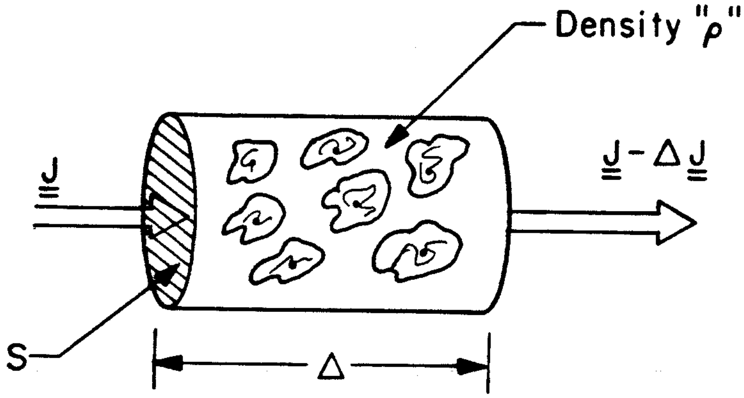

Figure 4-2. Geometrical interpretation of the coherent field energy as unscattered wavefield.

medium (see Figure 4-3). We know that this fluctuating field is composed of scattered waves originating from the many elementary scattering volumes making up the medium. The scattered fluctuating field is diffuse as opposed to specular like the average field. That is, at any point in the medium, the field can be decomposed into an angular distribution according to the equation

$$\underline{E}(\underline{r}) = \int_{4\pi} \underline{E}(\underline{r},\hat{s})d\omega \quad , \qquad (4.3.1)$$

where $\hat{s} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$,

and $d\omega = \sin\theta d\theta d\phi$. However, we choose to add one more dimension to the angular distribution of $\underline{E}(\underline{r})$ by expanding it in plane waves. Thus, (4.3.1) is generalized to become

$$\underline{E}(\underline{r}) = \int_{-\infty}^{+\infty} \underline{E}(\underline{\kappa})e^{+i\underline{\kappa}\cdot\underline{r}}\, d\underline{\kappa} \ , \qquad (4.3.2)$$

where $\underline{E}(\underline{\kappa})$ is simply the three-dimensional (3-D) Fourier transform of $\underline{E}(\underline{r})$. The transform quantity $\underline{E}(\underline{\kappa})$ is given analytically by

$$\underline{E}(\kappa) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \underline{E}(\underline{r})e^{-i\underline{\kappa}\cdot\underline{r}}\, d\underline{r} \ . \qquad (4.3.3)$$

Equations (4.3.2) and (4.3.3) form the complementary pair Fourier transform. The corresponding relationship to (4.3.1) may be seen by rewriting (4.3.2) as follows:

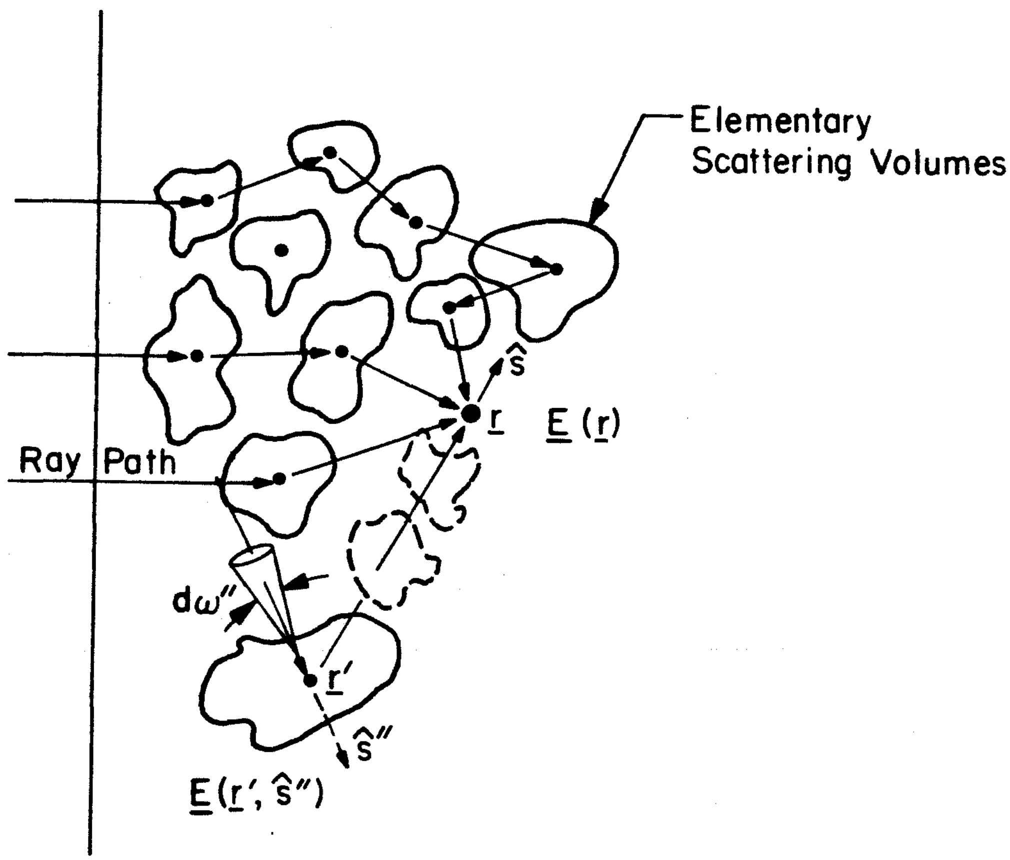

Figure 4-3. The physical construction of the incoherent field scattered from points $\underline{r}'$ and arriving at the point $\underline{r}$ via many ray paths.

$$\underline{E}(\underline{r}) = \int_{4\pi} \int_0^{\infty} \underline{E}(\underline{\kappa}) e^{+i\kappa\hat{s}\cdot\underline{r}} \kappa^2 d\kappa d\omega \ , \qquad (4.3.4)$$

where it is now clear that $\underline{E}(\underline{r},\hat{s})$ is given by

$$\underline{E}(\underline{r},\hat{s}) = \int_0^{\infty} \underline{E}(\underline{\kappa}) e^{+i\kappa\hat{s}\cdot\underline{r}} \kappa^2 d\kappa \ . \qquad (4.3.5)$$

It may first appear that (4.3.2) is an unnecessary complication of (4.3.1), but it will be shown that this plane wave expansion offers a significant generalization. Thus, the field at $\underline{r}$ is decomposed into components with generally different wavenumbers κ as well as directions $\hat{s}$.

Utilizing the angular expansion (4.3.1), one gets that the scattered field at $\underline{r}$, due to an elementary scattering volume dV' centered about $\underline{r}'$, illuminated by a narrow beam of radiation with angular spread $d\omega''$ in the direction $\hat{s}''$ is given by

$$d\ \underline{E}^{sc}(\underline{r},\underline{r}') = \frac{e^{+i\underline{\beta}|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \rho\ \underline{\underline{A}}(\hat{s}',\hat{s}'')\underline{E}(\underline{r}',\hat{s}'')d\omega''dV' \ . \qquad (4.3.6)$$

Figure 4-3 illustrates the construction of this contribution to the field at $\underline{r}$. Then, according to (4.3.2), when the incident field along $\hat{s}''$ has wavenumber κ'', $\underline{E}^{sc}(\underline{r},\underline{r}')$ becomes

$$d\ \underline{E}^{sc}(\underline{r},\underline{r}') = \frac{e^{+i\underline{\beta}|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \rho\ \underline{\underline{A}}(\hat{s}',\hat{s}'')\underline{E}(\underline{\kappa})e^{+i\underline{\kappa}''\cdot\underline{r}'}d\underline{\kappa}''dV' \ . \qquad (4.3.7)$$

The total scattered field at $\underline{r}$ due to all incident waves at $\underline{r}'$ is obtained by integrating $\underline{\kappa}''$ over all angles and wavenumbers:

$$\underline{E}^{sc}(\underline{r},\underline{r}') = \frac{e^{+i\underline{\beta}|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \int_{-\infty}^{+\infty} \rho\underline{\underline{A}}(\hat{s}',\hat{s}'')\underline{E}(\underline{\kappa}'')\, e^{+i\underline{\kappa}''\cdot\underline{r}'} d\underline{\kappa}'' dV' \quad . \qquad (4.3.8)$$

Noting that $\hat{s}$ is a unit vector in the direction $\underline{r}-\underline{r}'$, the total field at $\underline{r}$ is obtained by summing all the contributions along the variable direction $\hat{s}'$, from each elementary volumes dV'. Thus we have

$$\underline{E}_f(r) = \int_V \underline{E}^{sc}(\underline{r},\underline{r}')dV' \ ,$$

$$\underline{E}_f(\underline{r}) = \int_V \frac{e^{+i\underline{\beta}|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \int_{-\infty}^{+\infty} \rho\ \underline{\underline{A}}(\hat{s}',\hat{s}'')\underline{E}(\underline{\kappa}'')\, e^{+i\underline{\kappa}''\cdot\underline{r}'} d\underline{\kappa}'' dV' \quad . \qquad (4.3.9)$$

In allowing the integration in (4.3.9) to extend over all points $\underline{r}'$, we have assumed that the elementary scattering volumes are being replaced by point scatterers which scatter according to the augmented dipole pattern. This point scatterer is discussed in Apppendix A. Since the augmented dipole is represented in (4.3.9) by its far-field scattering amplitude $\underline{\underline{A}}$, the distance between scattering centers $\underline{r}'$ should also satisfy the far-zone conditions. To account for this effect, the number of augmented dipole point scatterers (ρ) per unit integration volume dV' is included in the expression for $\underline{E}_f$ as an averaging factor for the intervening space between the scatterers. Also, the propagation factor $\exp(i\underline{\beta}|\underline{r}-\underline{r}'|)$ is utilized instead of $\exp(ik|r-r'|)$ to account for scattering between $\underline{r}'$ and $\underline{r}$. In this fashion, (4.3.9) represents a two-space model wherein scattering from the elementary volume dV' is computed as if it were isolated in free-space, whereas propagation of the scattered waves away from dV' occur in a $\underline{\underline{\beta}}$-space medium. This $\underline{\underline{\beta}}$-space medium accounts for subsequent bulk scattering. Although the field at $\underline{r}'$ is expanded in general plane waves with both variable directions $\hat{s}''$ and wavenumber κ'', the scattering amplitude $\underline{\underline{A}}$ does not depend upon κ. Thus, there is no wavenumber (κ) conversion created by the scattering, only a redirection of energy.

4.4 Total Field and its Transport Equations

Combining the expression for the fluctuating field, (4.3.9), with the expression for the coherent field, (4.2.8), gives an integral equation for the total field

$$\underline{E}(\underline{r}) = \underline{E}(0)e^{+i\underline{\underline{\beta}}z} + \int_V \frac{e^{+i\underline{\underline{\beta}}|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \int_{-\infty}^{+\infty} \rho\, \underline{\underline{A}}(\hat{s}',\hat{s}'')\underline{E}(\underline{\kappa}'')e^{+i\underline{\kappa}''\cdot\underline{r}'} d\underline{\kappa}''dV' \quad . \tag{4.4.1}$$

The integral equation (4.4.1) has little practical usefulness, particularly because both the unknown quantity $\underline{E}(\underline{r})$ and its Fourier transform $\underline{E}(\underline{\kappa})$ appear in the integral equation. All is not lost however, forms of (4.4.1) are given in subsequent paragraphs, along with derivations of integral transport equations satisfied by the coherency matrix, which allows solutions for $\underline{E}(\underline{r})$ to be more easily found.

Transport Equation for $\underline{\underline{J}}$

Transport equations have found widespread use in physics and astronomy. These equations account for sources and sinks of field intensities along the propagation path. More generally, for electromagnetic fields the quantity satisfied by the transport equation is the coherency matrix or Stokes matrix discussed in section II. It is shown here that under certain simplifying conditions, a transport equation can be obtained from (4.4.1). Thus, the applicability of (4.4.1) and the resulting transport equation will be clearly placed in evidence.

To obtain the transport equation, first take the ensemble average of the product of $\underline{E}(\underline{r})$ with its complex conjugate:

$$<\underline{E}(\underline{r}) \times \underline{E}^*(\underline{r})> = <\underline{E}_c(\underline{r}) \times \underline{E}_c^*(\underline{r})> + <\underline{E}_f(\underline{r}) \times \underline{E}_f^*(\underline{r})> \qquad (4.4.2)$$

where

$$<\underline{E}_f \times \underline{E}_f^*> = < \int_V \rho \int_V \rho \iint_{-\infty}^{+\infty} \underline{\underline{B}}_1 \times \underline{\underline{B}}_2^* \cdot \underline{E}_1 \times \underline{E}_2^* e^{+i(\underline{\kappa}_1'' \cdot \underline{r}_1' - \underline{\kappa}_2'' \cdot \underline{r}_2')} d\underline{\kappa}_1'' d\underline{\kappa}_2'' dV_1' dV_2'$$

and

$$\underline{\underline{B}}_1 = e^{+i\underline{\underline{\beta}}|\underline{r}-\underline{r}'|} \underline{\underline{A}}(\hat{s}_1', \hat{s}_1'')/|\underline{r}-\underline{r}_1'|$$

$$\underline{\underline{B}}_2 = e^{+i\underline{\underline{\beta}}|\underline{r}-\underline{r}'|} \underline{\underline{A}}(\hat{s}_2', \hat{s}_2'')/|\underline{r}-\underline{r}_2'|$$

$$\underline{E}_1 = \underline{E}(\underline{\kappa}_1'')$$

$$\underline{E}_2 = \underline{E}(\underline{\kappa}_2'') \quad . \qquad (4.4.3)$$

No cross products between $\underline{E}_c$ and $\underline{E}_f$ appear in (4.2.2) since by definition the ensemble average of the fluctuating field is zero. In fact, it was shown in Appendix A that the amplitude of fluctuating field is Rayleigh distributed and the phase is uniformly distributed. The first simplification to (4.4.2) is that the scattering volumes (or equivalent point scatterers) are independent and uncorrelated. Thus, the integrand in (4.4.3) is a delta function correlated at $\underline{r}_1' = \underline{r}_2'$ and equation (4.4.3) becomes

$$\underline{\underline{J}}_i(\underline{r}) = \langle \underline{E}_f \times \underline{E}_f^* \rangle = \int_V \rho \iint_{-\infty}^{+\infty} \underline{\underline{B}} \times \underline{\underline{B}}^* \langle \underline{E}(\underline{\kappa}_1'') \times \underline{E}^*(\underline{\kappa}_2'') \rangle e^{+i(\underline{\kappa}_1''-\underline{\kappa}_2'')\cdot\underline{r}'} d\underline{\kappa}_1'' d\underline{\kappa}_2'' dV' \tag{4.4.4}$$

Mathematically, the independent and uncorrelated scattering volumes are represented by

$$\langle \underline{E}(\underline{\kappa}_1) \times \underline{E}^*(\kappa_2) e^{+i\underline{\kappa}_1\cdot\underline{r}_1 - i\underline{\kappa}_2\cdot\underline{r}_2} \rangle = \langle \underline{E}(\underline{\kappa}_1) \times \underline{E}^*(\underline{\kappa}_2) e^{+i(\underline{\kappa}_1-\underline{\kappa}_2)\cdot\underline{r}_1} \rangle \delta(\underline{r}_2-\underline{r}_1) \ . \tag{4.4.5}$$

Secondly, change variables from $\underline{\kappa}_1''$ and $\underline{\kappa}_2''$ to $\underline{\kappa}_c$ and $\underline{\kappa}_d$ according to

$$\underline{\kappa}_c = \frac{1}{2}(\underline{\kappa}_1'' + \underline{\kappa}_2'') \tag{4.4.6}$$

$$\underline{\kappa}_d = \underline{\kappa}_1'' - \underline{\kappa}_2'' \ . \tag{4.4.7}$$

With this change of variables, (4.4.4) becomes

$$\underline{\underline{J}}_i(r) = \int_V \rho \iint_{-\infty}^{+\infty} \underline{\underline{B}}(\hat{s}',\hat{s}_c) \times \underline{\underline{B}}^*(\hat{s}',\hat{s}_c)\cdot\underline{\underline{J}}(\underline{r}',\underline{\kappa}_c) d\underline{\kappa}_c |\underline{r}-\underline{r}'|^2 dr' d\omega' , \tag{4.4.8}$$

where the $\underline{\kappa}_d$ integration is represented by

$$\underline{\underline{J}}(\underline{r}',\underline{\kappa}_c) = \int_{-\infty}^{+\infty} \langle E(\underline{\kappa}_c + \frac{1}{2}\underline{\kappa}_d) \times \underline{E}^*(\underline{\kappa}_c - \frac{1}{2}\underline{\kappa}_d) \ e^{+i\underline{\kappa}_d \underline{r}'} d\underline{\kappa}_d \ , \tag{4.4.9}$$

and the scattering amplitude $\underline{\underline{A}}(\hat{s}';\hat{s}_c,\hat{s}_d)$ is assumed to depend only upon the average direction $\hat{s}_c = \frac{1}{2}(\hat{s}_1''+\hat{s}_2'')$. Also the elementary volume has been replaced with $dV' = |\underline{r}-\underline{r}'|^2 dr'\, d\omega'$. The integral equation for $\underline{\underline{J}}(\underline{r},\underline{\kappa}_c)$ is found by noting that $\underline{\underline{J}}(\underline{r})$ and $\underline{\underline{J}}_c(r)$ in (4.4.2) can be written in the forms

$$\underline{\underline{J}}(\underline{r}) = \langle \underline{E}(\underline{r}) \times \underline{E}^*(\underline{r}) \rangle = \int_{4\pi} \int_0^\infty \underline{\underline{J}}(\underline{r},\underline{\kappa}_c)\kappa_c^2 d\kappa_c d\omega' \tag{4.4.10}$$

$$\underline{\underline{J}}_c(r) = \langle \underline{E}_c(\underline{r}) \times \underline{E}_c^*(\underline{r}) \rangle = \int_{4\pi} \int_0^\infty \underline{\underline{J}}_c(\underline{r},\underline{\kappa}_c)\kappa_c^2 d\kappa_c d\omega' \ . \tag{4.4.11}$$

Thus, combining (4.4.10), (4.4.11), and (4.4.8) into (4.4.2) gives

$$\underline{\underline{J}}(\underline{r},\underline{\kappa}) = J_c(\underline{r},\underline{\kappa}) + \int_{r_b}^{r} \rho \int_{4\pi} \underline{\underline{B}} \times \underline{\underline{B}}^* \cdot \underline{\underline{J}}(\underline{r}',\underline{\kappa}')d\omega' |\underline{r}-\underline{r}'|^2 dr' \ . \tag{4.4.12}$$

The point r_b is located on the random medium boundary, originating the path integration in (4.4.12). Finally, we note that the random medium under consideration here is such that $\underline{\underline{\beta}} = \beta \underline{I}$, hence, the kernel of the integrand of (4.4.12) becomes

$$\rho \underline{\underline{B}}(\hat{s},\hat{s}') \times B^*(\hat{s},\hat{s}') = \frac{e^{-\sigma_s |\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|^2} \underline{\underline{S}}(\hat{s},\hat{s}') \ , \tag{4.4.13}$$

and the integral form of the radiative transfer equation results:

$$\underline{\underline{J}}(\underline{r},\underline{\kappa}) = \underline{\underline{J}}(0)e^{-\sigma_s z} + \int_{r_b}^{r} \int_{4\pi} \underline{\underline{S}}(\hat{s},\hat{s}')\cdot \underline{\underline{J}}(\underline{r}',\underline{\kappa}')d\omega' e^{-\sigma_s |\underline{r}-\underline{r}'|} dr' \tag{4.4.14}$$

where

$$\underline{\underline{S}}(\hat{s},\hat{s}') = \langle \rho \underline{\underline{A}}(\hat{s},\hat{s}') \times \underline{\underline{A}}^*(\hat{s},\hat{s}') \rangle \ . \qquad (4.4.15)$$

The differential equation form of the transport equation for $\underline{\underline{J}}(\underline{r},\underline{\kappa})$ is given by the following expression:

$$\hat{s}\cdot\nabla\underline{\underline{J}}(\underline{r},\underline{\kappa}) = -\sigma_s J(\underline{r},\underline{\kappa}) + \int_{4\pi} \underline{\underline{S}}(\hat{s},\hat{s}')\cdot\underline{\underline{J}}(\underline{r},\underline{\kappa}')d\omega' \ . \qquad (4.4.16)$$

Although (4.4.16) is a Boltzmann-type transport equation, the equation for the classical quantity $\underline{\underline{J}}(\underline{r},\hat{s})$ is obtained by integrating out the wavenumber dependence (κ) in (4.4.16) yielding

$$\hat{s}\cdot\nabla\underline{\underline{J}}(\underline{r},\hat{s}) = -\sigma_s\underline{\underline{J}}(\underline{r},\hat{s}) + \int_{4\pi} \underline{\underline{S}}(\hat{s},\hat{s}')\cdot\underline{\underline{J}}(\underline{r},\hat{s}')d\omega' \ . \qquad (4.4.17)$$

From either (4.4.16) or (4.4.17), it is easily shown that the Stokes matrix satisfies a Boltzmann-type transport equation also. Detailed discussion of the transport equation is given by Chandrasekhar (5) and Davison (3).

Transport Equation for $\underline{E}$

As shown in the previous paragraphs, the integral equation for the total field $\underline{E}$ leads directly to the transport equations for the coherency matrix or Stokes matrix. In the next few paragraphs, a transport equation of the same mathematical form is derived for the electric field at a point $\underline{r}$ and in the general direction $\hat{s}$, i.e., $\underline{E}(\underline{r},\hat{s})$. This equation offers an advantage over the classical transport equations for electromagnetic fields in that its dimensionality is only one-half the classical form for $\underline{\underline{J}}$. We begin this derivation by noting that $\underline{E}(\underline{r})$ is the total field at $\underline{r}$ and the integral equation (4.4.1) relates this total field and its plane wave spectrum. Thus, we wish to decompose the total field on the left-hand side of (4.4.1) into terms of its specific value $\underline{E}(\underline{r},\hat{s})$ along the variable direction $\hat{s}$. Now, according to the derivation, the coherent field travels only in the forward direction and is scattered from a region loosely known as the "first few Fresnel zones" with respect to the observation point. Similarly, we assume that the scattered fluctuating field at point $\underline{r}$ and in the direction $\hat{s}$ is scattered from Fresnel zone regions with respect to $\underline{r}$ but aligned along the direction $\hat{s}$. This condition is illustrated in Figure 4-4. The integrations with respect to x_s and y_s can be approximated using the method of stationary phase just as performed for the coherent field yielding

$$\underline{E}(\underline{r},\hat{s}) = \underline{E}_c(\underline{r},\hat{s}) + \frac{2\pi i}{k}\int_{r_b}^{r} \rho\, e^{+i\underline{\underline{\beta}}|\underline{r}-\underline{r}'|}\int_{4\pi}\underline{\underline{A}}(\hat{s},\hat{s}')\cdot\underline{E}(\underline{r}',\hat{s}')\,d\omega' dr' \tag{4.4.18}$$

In analogy with (4.4.14), we note that (4.4.18) is simply the integral form of

$$\hat{s}\cdot\nabla\underline{E}(\underline{r},\hat{s}) = i\underline{\underline{\beta}}\cdot\underline{E}(\underline{r},\hat{s}) + \frac{2\pi i}{k}\int_{4\pi} \rho\,\underline{\underline{A}}(\hat{s},\hat{s}')\cdot\underline{E}(\underline{r},\hat{s}')\,d\omega' \;. \tag{4.4.19}$$

In deriving (4.4.18), $\underline{\underline{\beta}}$ that would appear in the denominator of the integral term has been replaced with the free-space wavenumber k. This

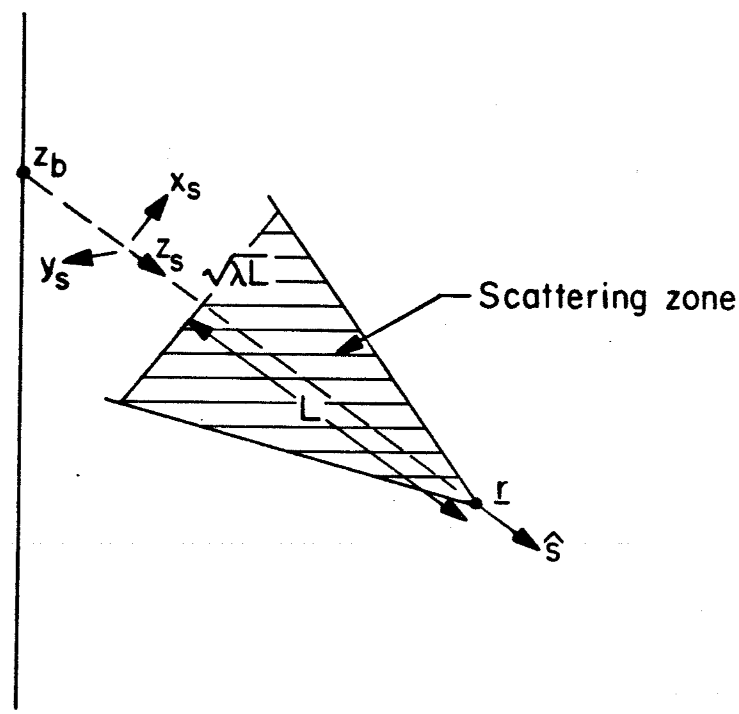

Figure 4-4. Illustration of scattering zone for fluctuating electric field arriving at $\underline{r}$ along ray path direction $\hat{s}$.

valid whenever the effective refractive index of the random medium is near unity, which is certainly the case of interest here. We coin the title "Coherent Transfer Equation" to describe (4.4.18) and (4.4.19). The terms in these transport equations can be associated with a conservation of electric field in each elementary volume of the medium just as done with $\underline{\underline{J}}$ in the classical version. There are, however, two distinct advantages of (4.4.19) compared to (4.4.17). First, media for which $\underline{\underline{\beta}}$ cannot be written as $\underline{\underline{\beta}} = \beta\underline{\underline{I}}$ can be studied directly using (4.4.19). Secondly, the dimensionality of (4.4.19) is only one-half the dimensionality of (4.4.17). That is, the scattering amplitude matrix $\underline{\underline{A}}$ is 2X2 whereas the differential cross section matrix $\underline{\underline{S}}$ is 4X4. The first term on the right-hand side of (4.4.19) has been studied by Van de Hulst (13). The second term represents the fluctuating electric field and is a new development.

It remains to be shown that (4.4.18) or (4.4.19) reduces to the transport equations for $\underline{\underline{J}}$. To show this, we first point out that the quantity $(2\pi i\rho\underline{\underline{A}}/k)$ is the electric field scattering amplitude per unit length of random medium. To form the transport equation for $\underline{\underline{J}}$, we make the same assumptions as used in deriving (4.4.14), namely that the scattering volumes (or scattering per unit volume) are uncorrelated. In the present case, however, it is the scattering per unit length which is computed as the statistically independent quantity. Thus, we take the product of $\underline{E}(\underline{r},\hat{s})$ with its complex conjugate and replace $(2\pi i\rho A/k)$ with $\langle\rho\underline{\underline{A}} \times \underline{\underline{A}}\rangle$. The integral form of the transport equation for $\underline{\underline{J}}(\underline{r},\hat{s})$ results directly.

4.5 Connection between Multiple Scattering Theory and Transport Theory

A quantity often computed in multiple scattering theory is the correlation of the total electric wavefield at two different space points. Mathematically, this quantity is given by

$$\langle \underline{\underline{\Gamma}}(\underline{r}_i,\underline{r}_2) = \langle \underline{E}(\underline{r}_1) \times \underline{E}^*(\underline{r}_2)\rangle \ . \tag{4.5.1}$$

Note that when $\underline{r}_1 = \underline{r}_2$, the correlation matrix $\underline{\underline{\Gamma}}$ reduces to the coherency matrix (4.4.2). In this subsection, it is shown that the correlation matrix is simply related to the conserved transport quantity $\underline{\underline{J}}(\underline{r}, \underline{\kappa})$ described by (4.4.16). Most of the details of this derivation are given in Appendix C for scalar waves. However, the relevant results for the vector electromagnetic field follows in the next few paragraphs.

From Appendix C, equation (C.11), we generalize to get the following result for electromagnetic fields:

$$\underline{\underline{\Gamma}}(\underline{r},\underline{r}_d) = \int_{-\infty}^{+\infty} \underline{\underline{J}}(\underline{r},\underline{\kappa})e^{+i\underline{\kappa}\cdot r_d}d\underline{\kappa} \tag{4.5.2}$$

where

$$\underline{r}_d = \underline{r}_1 - \underline{r}_2$$

$$\underline{r} = \frac{1}{2}(\underline{r}_1 + \underline{r}_2) \ .$$

Since $\underline{\kappa}$ has both varying magnitude and direction, i.e., $\underline{\kappa} = \kappa\hat{s}$, it is clear that (4.5.2) can be rewritten into the form

$$\underline{\underline{\Gamma}}(\underline{r},\underline{r}_d) = \int_{4\pi} \int_0^{\infty} \underline{\underline{J}}(\underline{r},\kappa\hat{s})e^{+i\kappa\hat{s}\cdot\underline{r}_d}\kappa^2 d\kappa d\omega \ .$$

In deriving (4.5.2), it was not necessary to assume a specific form for the correlation of the waves traveling along different directions in the medium. Thus $\underline{\underline{J}}(\underline{r},\underline{\kappa})$ is simply given by the Fourier transform relation (4.4.9). However, in deriving the transport equation for $\underline{\underline{J}}(\underline{r},\underline{\kappa})$, it was assumed that scattering from different elementary volumes was independent, viz. (4.4.4) and (4.4.5). This is important because in the classical derivation of the transport equation, "independent waves" are taken to mean uncorrelated in both direction and position as given by

$$<\underline{E}(\underline{r}_1,\hat{s}_1) \times \underline{E}^*(\underline{r}_2,\hat{s}_2)> = <\underline{E}(\underline{r}_1,\hat{s}_1) \times \underline{E}^*(\underline{r}_1,\hat{s}_1)>\delta(\underline{r}_2-\underline{r}_1)\delta(\hat{s}_2-\hat{s}_1) \ . \quad (4.5.4)$$

Clearly in our derivation, we have only assumed the following:

$$<\underline{E}(\underline{r}_1,\hat{s}_1) \times \underline{E}^*(\underline{r}_2,\hat{s}_2)> = <\underline{E}(\underline{r}_1,\hat{s}_1) \times \underline{E}^*(\underline{r}_1,\hat{s}_2)>\delta(\underline{r}_2-\underline{r}_1) \ . \quad (4.5.5)$$

As shown in Appendix C, assuming the correlation given in (4.5.4) is equivalent to specifying that the wavefield $\underline{E}(\underline{r})$ is homogeneous. And, for such a homogeneous wavefield the coherency matrix cannot depend explicitly upon the space variable $\underline{r}$, i.e., $J(\underline{r},\underline{\kappa}) \equiv J(\underline{\kappa})$. Thus, this inconsistency in classical transport theory is uncovered.

Concluding this section, we note that many researchers (1,9) have used the following relations between $\underline{\underline{\Gamma}}(\underline{r},\underline{r}_d)$ and $\underline{\underline{J}}(\underline{r},\hat{s})$:

$$\underline{\underline{\Gamma}}(\underline{r},\underline{r}_d) = \int_{4\pi} \underline{\underline{J}}(\underline{r},\hat{s})e^{+i\beta_r\hat{s}\cdot\underline{r}_d}d\omega \ . \quad (4.5.6)$$

It is pointed out in Appendix C that (4.5.6) is only valid when $\underline{\underline{J}}(r,\underline{\kappa})$ takes the form

$$\underline{\underline{J}}(\underline{r},\underline{\kappa}) = \frac{\delta(\kappa-\beta_r)}{\kappa^2}\,\underline{\underline{J}}(\underline{r},\hat{s}) \;, \qquad (4.5.7)$$

and that this condition applies only when the incident field is an infinite plane wave. Thus for all other cases, the general formula (4.5.2) must be used. Equation (4.5.7) simply describes an angularly diffuse field with a single wavenumber β_r.

The significance of the relationship between the correlation matrix $\underline{\underline{\Gamma}}$ and the specific coherency matrix $\underline{\underline{J}}$ is better understood with a few examples. First, note that the quantity $\underline{\underline{J}}(\underline{r},\underline{\kappa})$ represents the angular and wavenumber distribution of the second-order field incident at the point $\underline{r}$. If this field is an infinite plane wave, then $J(\underline{r},\underline{\kappa})$ is a delta function and the correlation distance specified by $\underline{\underline{\Gamma}}$ is infinite. The other extreme is when $J(\underline{r},\underline{\kappa})$ does not depend upon $\underline{\kappa}$, as with completely diffuse multiply scattered fields. In this case, $\underline{\underline{\Gamma}}$ becomes a delta function and the field has a zero correlation distance. Obviously, there are many intermediate cases of physical interest such as spherical waves and finite cross-section plane waves. All these cases can be effectively treated with the results given above.

V. THE PLANE - PARALLEL PROBLEM

Wave propagation in a random medium bounded by parallel planes represents many practical situations. For example, planetary atmospheres, oceans, and subsurface layers can be often approximated as plane-parallel media.

5.1 Normally Incident Plane Wave

As discussed in Sections II and IV, components of the coherency matrix consist of a coherent part $\underline{\underline{J}}_c$ and an incoherent part $\underline{\underline{J}}_i$. More specifically, it was shown that the sum of these two parts $\underline{\underline{J}}(\underline{r},\underline{\kappa})$ satisfies the following integral equation:

$$\underline{\underline{J}}(\underline{r},\underline{\kappa}) = \underline{\underline{J}}_c(\underline{r},\underline{\kappa}) + \int_{r_b}^{r}\int_{4\pi} \underline{\underline{S}}(\hat{s}',\hat{s}'')\cdot\underline{\underline{J}}(\underline{r}',\hat{s}'')d\omega'' e^{-\sigma_s|\underline{r}-\underline{r}'|} dr' \quad . \tag{5.1.1}$$

Now, let us consider a plane wave normally incident on a plane parallel slab of thickness L (see Figure 5-1). For an infinite or uniform plane wave it can be easily shown that $\underline{\underline{J}}_c$ is given by

$$\underline{\underline{J}}_c(\underline{r},\underline{\kappa}) = \underline{\underline{J}}(0)e^{-\sigma_s \hat{z}\cdot\underline{r}}\delta(\underline{\kappa}-\beta_r\hat{z}) \quad , \tag{5.1.2}$$

where β_r is the real part of the medium's effective coherent refractive index given in (4.2.1). The delta function in (5.1.2) is defined by

$$\int \delta(\underline{\kappa}-\beta_r\hat{z})d\underline{\kappa} = \int_{4\pi}\int_0^\infty \frac{\delta(\kappa-\beta_r)}{\kappa^2}\delta(\mu-\mu')\delta(\phi-\phi')\kappa^2 d\kappa d\mu d\phi = 1 \quad , \tag{5.1.3}$$

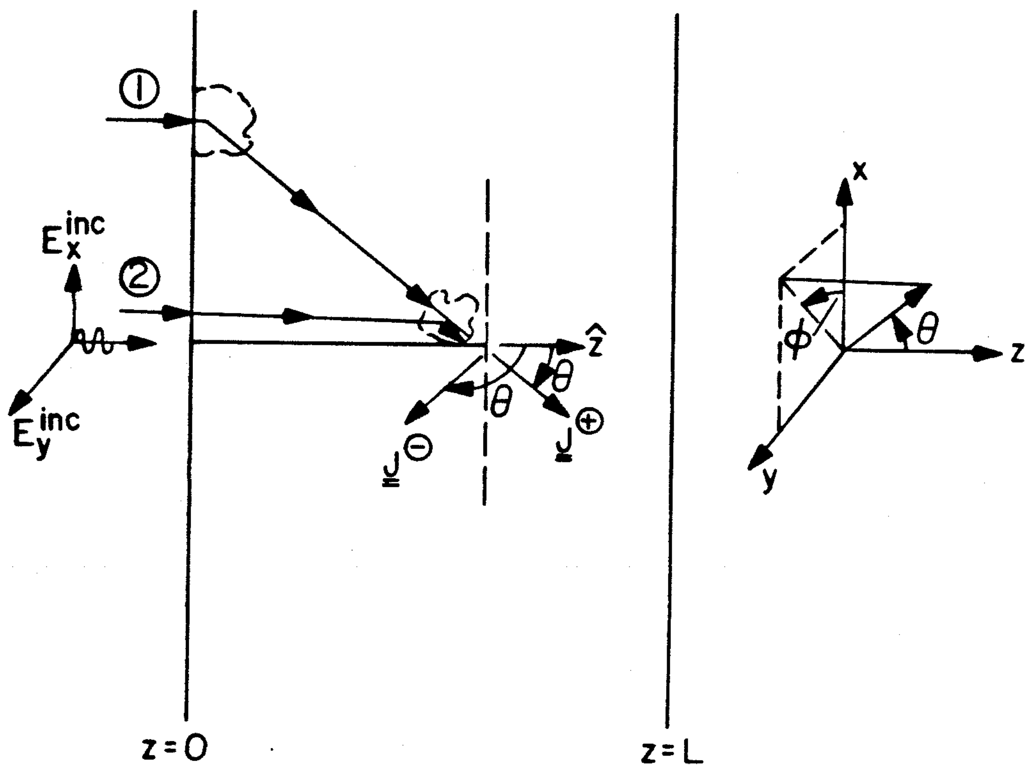

Figure 5-1. Plane-parallel slab with normally incident infinite plane wave.

where $\mu = \cos\theta$. We note that because of symmetry, the incoherent field depends only upon the two space variables z and μ; there is no ϕ dependence. To eliminate the ϕ dependence, we integrate (5.1.1) with respect to ϕ over 2π and obtain

$$\underline{\underline{J}}(z,\mu;\kappa) = \underline{\underline{J}}_c(z,\mu;\kappa)+2\pi \int_{z_b}^{z} \int_{-1}^{+1} \underline{\underline{S}}(\mu,\mu')\cdot\underline{\underline{J}}(z',\mu';\kappa)d\mu' e^{-\sigma_s \frac{|z-z'|}{\mu}} \frac{dz'}{\mu} \tag{5.1.4}$$

where

$$\underline{\underline{S}}(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} d\phi \, \frac{1}{2\pi}\int_0^{\pi} d\phi' \underline{\underline{S}}(\hat{s},\hat{s}') \tag{5.1.5}$$

and

$$\underline{\underline{J}}_c(z,\mu;\kappa) = \frac{1}{2\pi\kappa^2} \underline{\underline{J}}(0) e^{-\sigma_s z} \delta(\kappa-\beta_r)\delta(\mu-1) \ . \tag{5.1.6}$$

The matrix $\underline{\underline{S}}(\mu,\mu')$ is discussed in Appendix D. From the results given in this appendix, equation (D-13), the integral equations for J_{xx} and J_{yy} may be decoupled from the equations for J_{xy} and J_{yx}. Accordingly, the component equations for $\underline{\underline{J}}$ given in (5.1.4) become

$$J_{xx} = J_{xx,c} + 2\pi\int_{z_b}^{z}\int_{-1}^{+1}[S_{11}J_{xx} + S_{14}J_{yy}]d\mu' e^{-\sigma_s \frac{|z-z'|}{\mu}} \frac{dz'}{\mu} \tag{5.1.7}$$

$$J_{yy} = J_{yy,c} + 2\pi\int_{z_b}^{z}\int_{-1}^{+1}[S_{41}J_{xx} + S_{44}J_{yy}]d\mu' e^{-\sigma_s \frac{|z-z'|}{\mu}} \frac{dz'}{\mu} \tag{5.1.8}$$

$$J_{xy} = J_{xy,c} + 2\pi\int_{z_b}^{z}\int_{-1}^{+1}[S_{22}J_{xy} + S_{23}J_{yx}]d\mu' e^{-\sigma_s \frac{|z-z'|}{\mu}} \frac{dz'}{\mu} \tag{5.1.9}$$

$$J_{yx} = J_{jx,c} + 2\pi\int_{z_b}^{z}\int_{-1}^{+1}[S_{23}J_{xy} + S_{22}J_{yx}]d\mu' e^{-\sigma_s \frac{|z-z'|}{\mu}} \frac{dz'}{\mu} \ . \tag{5.1.10}$$

These four integral equations (5.1-7)-(5.1.10) constitute the basic mathematical formulations for studying normally incident plane wave propagation in a plane-parallel medium.

Since $\underline{\underline{J}}$ can be decomposed into coherent and incoherent parts, substitution of $\underline{\underline{J}} = \underline{\underline{J}}_c + \underline{\underline{J}}_i$ into (5.1.4) yields an integral equation for incoherent field $\underline{\underline{J}}_i$:

$$\underline{\underline{J}}_i(z,\mu;\kappa) = \underline{\underline{J}}_{ci}(z,\mu;\kappa) + 2\pi\int_{z_b}^{z} \int_{-1}^{+1} \underline{\underline{S}}(\mu,\mu'')\cdot\underline{\underline{J}}_i(z',\mu';\kappa)d\mu''\cdot e^{-\sigma_s\frac{|z-z'|}{\mu}} \frac{dz'}{\mu} , \tag{5.1.11}$$

where

$$\underline{\underline{J}}_{ci}(z,\mu;\kappa) = \frac{\delta(\kappa-\beta_r)}{\kappa^2} S(\mu,1)\cdot\underline{\underline{J}}(o)\int_{z_b}^{z} e^{-\sigma_s z'-\sigma_s\frac{|z-z'|}{\mu}} \frac{dz'}{\mu} . \tag{5.1.12}$$

5.2 First-Order Multiple Scattering Theory

In the first-order multiple scattering approximation, we assume that the total field illuminating each elementary volume is approximately equal to the known coherent field. Thus we obtain the first-order solution

$$\underline{\underline{J}}_i(z,\mu;\kappa) \approx \underline{\underline{J}}_{ci}(z,\mu;\kappa) \quad . \tag{5.2.1}$$

This first-order solution is applicable whenever the incoherent field is considerably small compared to the coherent field, i.e., $\underline{\underline{J}}_{ci} << \underline{\underline{J}}_c$, a condition which is satisfied for two important situations:

(1) Propagation for small distances into the random medium.

(2) For waves confined to a narrow angular region as with microwave and optical propagation in the atmosphere.

From Figure 5-1, it is clear that $\underline{\underline{J}}_i^{\oplus}$ for $0 \leq \theta < \pi/2$ $(1 \geq \mu > 0)$ represents scattered contributions from the range 0 to z whereas $\underline{\underline{J}}_1^{\ominus}$ for $\frac{\pi}{2} < \theta \leq \pi$ $(0 > \mu \geq -1)$ represents contributions from the range z to L. From these definitions of $\underline{\underline{J}}_i^{\oplus}$ and $\underline{\underline{J}}_i^{\ominus}$, we get the following first-order solutions from (5.2.1)

$$\underline{\underline{J}}_i^{\oplus}(z,\mu;\kappa) = \frac{\delta(\kappa-\beta_r)}{\kappa^2}\frac{\underline{\underline{S}}(\mu,1)}{\sigma_s}\cdot\underline{\underline{J}}(0)G^{\oplus}(z,\mu) \qquad 0 < \mu \leq 1 \quad , \qquad (5.2.2)$$

$$\underline{\underline{J}}_i^{\ominus}(z,\mu;\kappa) = \frac{\delta(\kappa-\beta_r)}{\kappa^2}\frac{\underline{\underline{S}}(\mu,1)}{\sigma_s}\cdot\underline{\underline{J}}(0)G^{\ominus}(z,\mu) \qquad -1 \leq \mu < 0 \ . \qquad (5.2.3)$$

Accordingly, the $\oplus$ sign represents forward waves and the $\ominus$ sign represents backward waves. The propagation factors $G^{\oplus}$ and $G^{\ominus}$ are given by

$$G^{\oplus}(z,\mu) = \frac{e^{-\sigma_s z} - e^{-\sigma_s/\mu}}{(1-\mu)} \qquad (5.2.4)$$

$$G^{\ominus}(z,\mu) = \frac{e^{-\sigma_s z} - e^{-\sigma_s(z-L)/\mu-\sigma_s L}}{(1-\mu)} \qquad (5.2.5)$$

A simple physical description of the two terms comprising either (5.2.4) or (5.2.5) can be given. Let's consider the forward waves and $G^{\oplus}$. First note that the integral term in (5.1.4), often called path radiance, represents a summation of the scattered waves all along a ray path in the direction μ from the boundary z_b to a depth z. Referring to Figure 5-1, we see that scattering from points near the boundary at z = 0 (ray path ①) propagates a distance z/μ in reaching the point z producing the term exp $(-\sigma_s z/\mu)$. However, scattering from points near the observation point as with ray path ② propagates a distance **z** producing the term $\exp(-\sigma_s z)$. Thus the combined ray path at z is a weighted average of these two extreme paths, this average being given by $G^{\oplus}$ in (5.2.4). Finally, we know that the measurable quantity $\underline{\underline{J}}_i(z,\mu)$ is obtained from (5.2.2) and (5.2.3) by integrating with respect to κ:

$$\underline{\underline{J}}_i^{\oplus}(z,\mu) = \frac{\underline{\underline{S}}(\mu,1)}{\sigma_s} \cdot \underline{\underline{J}}(0) G^{\oplus}(z,\mu) \qquad (5.2.6)$$

$$\underline{\underline{J}}_i^{\ominus}(z,\mu) = \frac{\underline{\underline{S}}(\mu,1)}{\sigma_s} \cdot \underline{\underline{J}}(0) G^{\ominus}(z,\mu). \qquad (5.2.7)$$

The forward incoherent field $\underline{\underline{J}}_1^{\oplus}$ is zero at z = 0, increases with z, and reaches a maximum at

$$\sigma_s z = \frac{\mu \ln(\mu)}{\mu - 1} \quad . \qquad (5.2.8)$$

Along the line-of-sight $\mu = 1$, the forward incoherent field is given by

$$\underline{\underline{J}}_i^{\oplus}(z,\mu=1) = \underline{\underline{S}}(1,1) \cdot J(0) z e^{-\sigma_s z} \quad , \qquad (5.2.9)$$

and reaches a maximum at $\sigma_s z = 1$. The final example of first-order multiple scattering is incoherent reflection from an infinite half-space ($L \to \infty$). From (5.2.7) we have

$$\underline{\underline{J}}^{\ominus}(z=0,\mu) = \frac{\underline{\underline{S}}(\mu,1)}{\sigma_s} \cdot \underline{\underline{J}}(0) \frac{1}{1-\mu} \quad . \qquad (5.2.10)$$

This field is equal to the incoherent field anywhere in the region $z < 0$ and is known as the law of diffuse reflection (5).

To conclude this section on first-order multiple scattering, we provide a specific example of $\underline{\underline{J}}_i^{\oplus}$ and $\underline{\underline{J}}_i^{\ominus}$. These results are given for the gaussian spectrum of refractive index fluctuations (derived in Appendix D):

$$\underline{\underline{S}}(\mu,1) = \frac{A_n^2}{2} e^{-\frac{k^2\ell^2}{2}(1-\mu)} \begin{bmatrix} \mu^2 & 0 & 0 & \mu^2 \\ 0 & \mu & -\mu & 0 \\ 0 & -\mu & \mu & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} , \qquad (5.2.11)$$

where $A_n^2 = \langle n_1^2 \rangle (k\ell)^4 / 4\sqrt{\pi}\,\ell$.

Now, consider a linearly plane polarized wave incident upon the random medium, i.e., $J_{xx}(0) \neq 0$ and $J_{yy}(0) = 0$. In this case, we get

$$J_{xx}^{\oplus}(z,\mu) = J_{xx}(0)\delta(\mu-1)e^{-\sigma_s z} + J_{xx}(0)S_{11}(\mu,1)G^{\oplus}(z,\mu)$$

$$J_{yy}^{\oplus}(z,\mu) = J_{xx}(0)S_{41}(\mu,1)G^{\oplus}(z,\mu)$$

$$J_{xx}^{\ominus}(z,\mu) = J_{xx}(0)S_{11}(\mu,1)G^{\ominus}(z,\mu)$$

$$J_{yy}^{\ominus}(z,\mu) = J_{xx}(0)S_{41}(\mu,1)G^{\ominus}(z,\mu) \quad . \qquad (5.2.12)$$

Clearly, the x and y components of the scattered field are independent since $J_{xy}(z) = J_{yx}(z) = 0$. However, the scattered field is partially polarized, the degree of partial polarization, P^{sc}, given by

$$P^{sc} = \left| \frac{J_{xx} - J_{yy}}{J_{xx} + J_{yy}} \right| = \frac{1-\mu^2}{1+\mu^2} \quad . \qquad (5.2.13)$$

Thus the scattered field is unpolarized along the line-of-sight ($\mu = \pm 1$) but completely plane polarized at right angles to the line-of-sight ($\mu = 0$).

5.3 Diffusion of the Coherency Matrix and Stokes Parameters

In this section, we develop solutions which describe diffusion of the coherency matrix $\underline{\underline{J}}(z)$ and the Stokes parameters. This solution is derived for a semi-infinite half-space by cascading elemental layers of random medium for which the scattering per unit length and depolarization per unit length have been estimated using the first-order multiple scattering theory. Construction of this model is illustrated in Figure 5-2.

We begin by integrating the first-order multiple scattering solution for $\underline{\underline{J}}_i^{\oplus}(z,\mu)$, (5.2.6), with respect to μ to obtain the total forward incoherent field at the output of a layer of thickness L. This yields

$$\underline{\underline{J}}_i^{\oplus}(L) = \frac{1}{\sigma_s} \int_0^1 \underline{\underline{S}}(\mu,1) \cdot \underline{\underline{J}}(0) G^{\oplus}(L,\mu) d\mu \quad . \tag{5.3.1}$$

For a single isolated layer, the forward field at L, $\underline{\underline{J}}_i^{\oplus}(L)$, is identically equal to the total incoherent field $\underline{\underline{J}}_i(L)$ since there is no back scattering in the region $z > L$. When there are many layers present as indicated in Figure 5-2, equation (5.3.1) is still a valid approximation for the total field when scattering in the backward hemisphere is negligible compared to forward scattering. Furthermore, we note that the propagation factor $G^{\oplus}(L,\mu)$ in (5.3.1) varies from $\sigma_s L \exp(-\sigma_s L)$ to $\exp(-\sigma_s L)$ as μ varies from 1 to 0. In developing this approximation, we consider a strongly forward peaked scattering function $\underline{\underline{S}}(\mu,1)$ and let

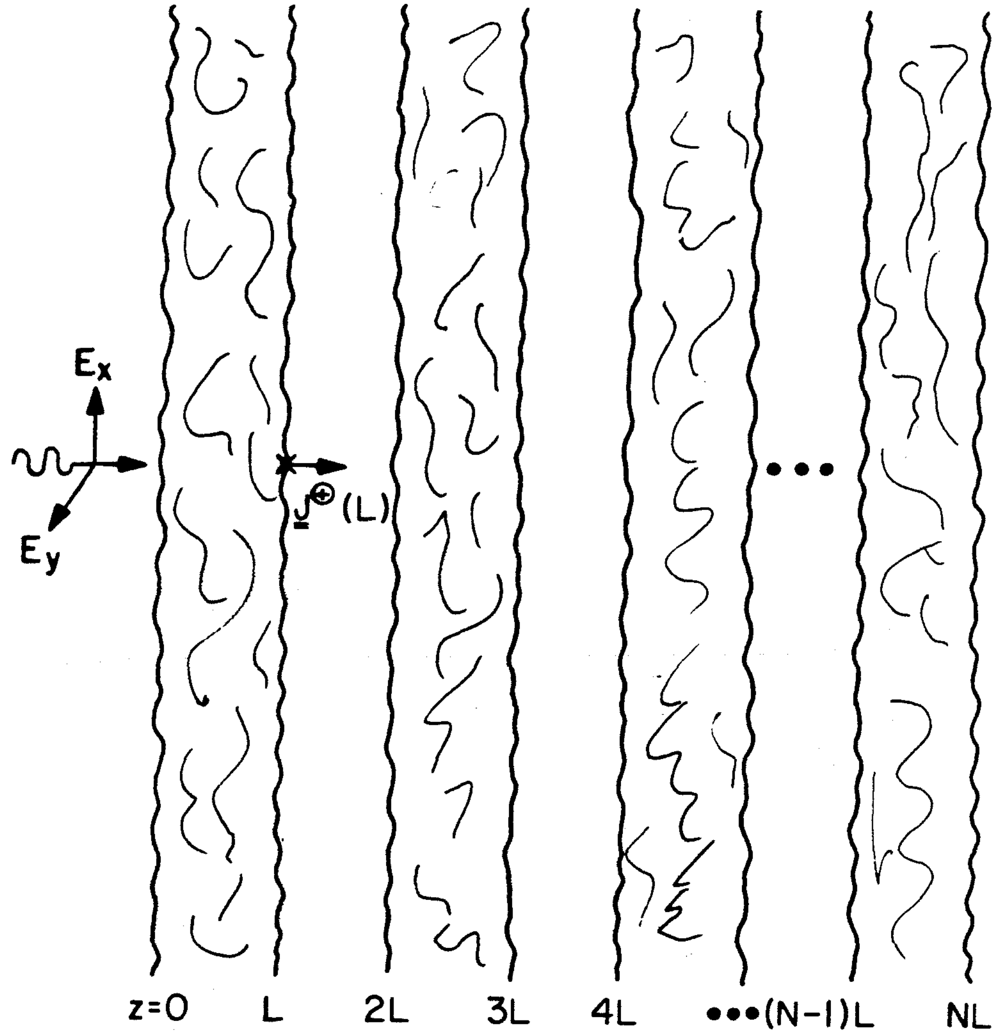

Figure 5-2. Successive layer model for the total field diffused into a random medium.

the factor $\overset{\oplus}{G}$ be $\sigma_s L \exp(-\sigma_s L)$, since most contributions to the integral (5.3.1) come from the neighborhood of $\mu = 1$. With this approximation we get

$$\underline{\underline{J}}_i(L) = \sigma_s L e^{-\sigma_s L} \underline{\underline{q}} \cdot \underline{\underline{J}}(0) \quad , \tag{5.3.2}$$

where

$$\underline{\underline{q}} = \frac{1}{\sigma_s} \int_0^1 \underline{\underline{S}}(\mu, 1) d\mu \quad . \tag{5.3.3}$$

The total field at $z = L$ is simply the sum of coherent field J_c and the incoherent field J_i. Thus, we have

$$\underline{\underline{J}}(L) = \underline{\underline{J}}(0) e^{-\sigma_s L} + \sigma_s L e^{-\sigma_s L} \underline{\underline{q}} \cdot \underline{\underline{J}}(0). \tag{5.3.4}$$

For an elemental layer of thickness $L \cong \Delta$, such that $\sigma_s \Delta \ll 1$, the total field becomes

$$\underline{\underline{J}}(\Delta) = (1 - \sigma_s \Delta) \underline{\underline{J}}(0) + \sigma_s \Delta \underline{\underline{q}} \cdot \underline{\underline{J}}(0) \quad , \tag{5.3.5}$$

where only terms of order $\sigma_s \Delta$ have been retained. In the limit as the geometric thickness of the layer approaches zero, the elements of (5.3.5) reduce to the following set of coupled ordinary differential equations:

$$\frac{d}{dz} J_{xx}(z) = -\sigma_s (1 - q_{11}) J_{xx}(z) + \sigma_s q_{14} J_{yy}(z) \tag{5.3.6}$$

$$\frac{d}{dz} J_{yy}(z) = -\sigma_s (1 - q_{44}) J_{yy}(z) + \sigma_s q_{41} J_{xx}(z) \tag{5.3.7}$$

$$\frac{d}{dz} J_{xy}(z) = -\sigma_s(1-q_{22})J_{xy}(z) + \sigma_s q_{23} J_{yx}(z) \tag{5.3.8}$$

$$\frac{d}{dz} J_{yx}(z) = -\sigma_s(1-q_{22})J_{yx}(z) + \sigma_s q_{23} J_{xy}(z) \quad . \tag{5.3.9}$$

Since backscattering has been neglected, the boundary conditions at $z = 0$ are simply given by the incident coherent field,

$$\underline{\underline{J}}(0) = \underline{\underline{J}}^{inc}(0) \quad . \tag{5.3.10}$$

For the gaussian spectrum of index fluctuations (see Appendix D), the solution (5.3.6)-(5.3.10) is easily obtained and given by

$$\begin{aligned} J_{xx}(z) &= \frac{1}{2} e^{-\sigma_s(1-q_1-q_4)z}[1 + e^{-\sigma_s(q_1+q_4)z}]J_{xx}(0) \\ &+ \frac{1}{2} e^{-\sigma_s(1-q_1-q_4)z}[1 - e^{-\sigma_s(q_1+q_4)z}]J_{yy}(0) \end{aligned} \tag{5.3.11}$$

$$\begin{aligned} J_{yy}(z) &= \frac{1}{2} e^{-\sigma_s(1-q_1-q_4)z}[1 + e^{-\sigma_s(q_1+q_4)z}]J_{yy}(0) \\ &+ \frac{1}{2} e^{-\sigma_s(1-q_1-q_4)z}[1 - e^{-\sigma_s(q_1+q_4)z}]J_{xx}(0) \end{aligned} \tag{5.3.12}$$

$$\begin{aligned} J_{xy}(z) &= \frac{1}{2} e^{-\sigma_s(1-2q_2)z}[1 + e^{-2\sigma_s q_2 z}]J_{xy}(0) \\ &- \frac{1}{2} e^{-\sigma_s(1-2q_2)z}[1 - e^{-2\sigma_s q_2 z}]J_{yx}(0) \end{aligned} \tag{5.3.13}$$

$$J_{yx}(z) = \frac{1}{2} e^{-\sigma_s(1-2q_2)z}[1 + e^{-2\sigma_s q_2 z}]J_{yx}(0)$$

$$- \frac{1}{2} e^{-\sigma_s(1-2q_2)z}[1 - e^{-2\sigma_s q_2 z}]J_{xy}(0) \qquad (5.3.14)$$

where we have $q_1 = q_{11} = q_{14}$, $q_4 = q_{41} = q_{44}$, and $q_2 = q_{22} = -q_{23}$. These dimensionless parameters are given by

$$q_1 = \frac{A_n^{\ 2}}{2\sigma_s} \int_0^1 \mu^2 e^{-\frac{k^2\ell^2}{2}(1-\mu)} d\mu \qquad (5.3.15)$$

$$q_2 = \frac{A_n^{\ 2}}{2\sigma_s} \int_0^1 \mu e^{-\frac{k^2\ell^2}{2}(1-\mu)} d\mu \qquad (5.3.16)$$

$$q_4 = \frac{A_n^{\ 2}}{2\sigma_s} \int_0^1 e^{-\frac{k^2\ell^2}{2}(1-\mu)} d\mu \quad . \qquad (5.3.17)$$

When the wave incident upon an elementary scattering volume is polarized along the x axis, $\sigma_s q_1$ equals the co-polarized scattering loss in the forward hemisphere and $\sigma_s q_4$ is the cross-polarized loss in the forward hemisphere. When backscattering is negligible, as assumed when developing the diffusion equations (5.3.11)-(5.3.14), we have the sum $q_1 + q_4$ approximately equal one, i.e., $q_1 + q_4 \simeq 1$. This is an affirmation of conservative scattering in the random medium. From this conservation condition, it is also evident that $2q_2 \leq 1$ $(1 + \mu^2 \geq 2\mu)$. Thus, the final form for the J_{xx} and J_{yy} elements of the coherency matrix is

$$J_{xx}(z) = \frac{1}{2}[1 + e^{-\sigma_s z}]J_{xx}(0) + \frac{1}{2}[1 - e^{-\sigma_s z}]J_{yy}(0) \qquad (5.3.18)$$

$$J_{yy}(z) = \frac{1}{2}[1 + e^{-\sigma_s z}]J_{yy}(0) + \frac{1}{2}[1 - e^{-\sigma_s z}]J_{xx}(0) \qquad (5.3.19)$$

The expressions for $J_{xy}(z)$ and $J_{yx}(z)$, (5.3.13) and (5.3.14), are unchanged. Equations (5.3.18) and (5.3.19) along with (5.3.13) and (5.3.14) constitute the diffusion approximation for the coherency matrix. These equations express the degree of coherence of the field as it penetrates the random medium. We see that a completely coherent and polarized wave at z = 0 becomes completely incoherent and unpolarized as $z \to \infty$, viz., $J_{xx} = J_{yy}$ and $J_{xy} = J_{yx} = 0$. Recently Papas (11) proposed a similar set of equations for the diffusion of the coherency matrix; his formulation was largely based upon a quantitative view of the phenomenon, whereas the results given herein have proceeded directly from scattering theory.

From (2.2.6), we define the components of the Stokes matrix in terms of the elements of the coherency matrix according to the relations

$$S_o = J_{xx} + J_{yy} \qquad (5.3.20)$$

$$S_1 = J_{xx} - J_{yy} \qquad (5.3.21)$$

$$S_2 = J_{xy} - J_{yx} \qquad (5.3.22)$$

$$S_3 = i(J_{yx} - J_{xy}) \quad . \qquad (5.3.23)$$

Utilizing the results for $\underline{\underline{J}}(z)$ from above we get

$$S_0(z) = S_0(0) \quad (5.3.24)$$

$$S_1(z) = S_1(0)e^{-\sigma_s z} \quad (5.3.25)$$

$$S_2(z) = S_2(0)e^{-\sigma_s z} \quad (5.3.26)$$

$$S_3(z) = S_3(0)e^{-\sigma_s (1-2q_2)z} . \quad (5.3.27)$$

Recall that for conservative scattering, we always have $(1-2q_2) \geq 0$. Thus, $S_3(z)$ is also an exponentially decreasing function of z. The polarization characteristics of the field are easily computed from the Stokes parameters yielding

$$P(z) = e^{-\sigma_s z}[L^2(0) + E^2(0)e^{+4\sigma_s q_2 z}]^{\frac{1}{2}} \quad (5.3.28)$$

$$E(z) = E(0)/[E^2(0) + L^2(0)e^{-4\sigma_s q_2 z}]^{\frac{1}{2}} \quad (5.3.29)$$

$$L(z) = L(0)/[L^2(0) + E^2(0)e^{+4\sigma_s q_2 z}]^{\frac{1}{2}} \quad (5.3.30)$$

where in section II, P(z), E(z) and L(z) were defined as the degree of polarization, degree of elliptic polarization, and the degree of linear polarization, respectively.

5.4 Examples

Next, let's consider examples of the polarization properties described by (5.3.28) through (5.3.30). If the incident wave is linearly polarized, then $L(0) \neq 0$ and $E(0) = 0$. As the linearly polarized wave penetrates the random medium, its coherent and polarized part decreases, with the factor $\exp(-\sigma_s z)$, but remains linearly polarized. The incoherent part of the total field increases as the wave penetrates the random medium and is unpolarized. From (5.3.28)-(5.3.30) we get

$$P(z) = L(0)e^{-\sigma_s z} \tag{5.4.1}$$

$$E(z) = 0 \tag{5.4.2}$$

$$L(z) = 1 \quad . \tag{5.4.3}$$

If the incident field is circularly polarized, the behavior of the field is similar to that given above for the linearly polarized field yielding

$$P(z) = e^{-\sigma_s (1-2q_2) z} \tag{5.4.4}$$

$$E(z) = \pm 1 \tag{5.4.5}$$

$$L(z) = 0 \quad . \tag{5.4.6}$$

The more interesting case occurs for a general elliptically polarized wave incident upon the random medium, one for which $L(0) \neq 0$ and $E(0) \neq 0$. For the elliptically polarized wave, we get

$$P(z) \to |E(0)|e^{-\sigma_s (1-2q_2)z} \tag{5.4.7}$$

$$E(z) \to \pm 1 \tag{5.4.8}$$

$$L(z) \to \frac{L(0)}{|E(0)|} e^{-2\sigma_s q_2 z} \tag{5.4.9}$$

as $z \to \infty$. In this general case, the eccentricity of the polarization ellipse approaches zero as the wave penetrates the medium. Deep into the medium ($\sigma_s z >>> 1$), the ellipse turns into a circle and eventually this circularly polarized part disappears completely.

VI. CONCLUSIONS

As stated in the introduction, our purpose has been to study the influence of a continuous random medium on the propagation of electromagnetic waves, with particular emphasis on polarization effects. This, the final section, will be devoted to a summary and evaluation of the general results of our investigation.

The essential effect of the random medium is to destroy the time coherence and spatial orthogonality of the vector components of an electromagnetic field, components which might otherwise exist independently. This destruction process is intepreted as producing an average or coherent electric wavefield and a fluctuating (noisy) or incoherent electric wavefield in the medium. The average wavefield, E_c, retains all the coherence properties of the incident wave, but decays exponentially (due to scattering) with propagation distance in the medium; its behavior is described by an effective bulk refractive index for the random medium, given by the wavenumber $\underline{\underline{\beta}}$ in equation (4.2.9). In general, this effective refractive index is a matrix and describes several coherent wave effects, including double refraction, polarization plane rotation, and linear and circular dichroism. The fluctuating electric wavefield, $\underline{E}_f$, has an average value of zero (Rayleigh distributed amplitude and uniformly distributed phase) and is given by the integral expression (4.3.9); however, its amplitude increases due to scattering as the wave penetrates the medium. Together, the average and fluctuating electric wavefields form an integral equation, (4.4.1), satisfied by the total field $\underline{E} = \underline{E}_c + \underline{E}_f$.

For the continuous random medium considered, the effective refractive index is such that $\underline{\underline{\beta}} = \beta\underline{\underline{I}}$, where $\underline{\underline{I}}$ is the identity matrix. This special condition allows the derivation of Boltzmann-type transport equations, (4.4.14) or (5.1.4), for the coherency matrix. These transport equations follow directly from the integral equation (4.4.1) for the total electric field. For a plane parallel medium, the transport equation for the coherency matrix is solved in the first-order multiple scattering approximation given by (5.2.1). Results for a linearly plane polarized wave incident upon the medium, (5.2.12), show that the scattered incoherent wavefield along the line-of-sight is unpolarized, whereas the scattered incoherent wavefield perpendicular to the line-of-sight is completely plane polarized.

By cascading many elemental layers of random medium, for which the scattering per unit length and depolarization per unit length have been obtained using the first-order multiple scattering approximation (neglecting backscattering), the elements of the coherency matrix are found to satisfy a simple set of equations, (5.3.13), (5.3.14), (5.3.18), and (5.3.19), which are characterized by two phenomenological parameters σ_s and q_2. The parameter σ_s is the scattering loss per unit length of the medium, but may be interpreted as a depolarization parameter also, since the scattered waves are depolarized. Given by (3.2.14), σ_s varies directly with $(k\ell)^4\langle n_1^2\rangle/\ell$, where k is the free-space wavenumber, ℓ the characteristic length of the random inhomogeneities in the medium, and $\langle n_1^2\rangle$ the mean-square fluctuation of the medium's refractive index. The parameter q_2--given by (5.3.16)--also varies directly with $(k\ell)^4\langle n_1^2\rangle/\ell$, but is interpreted to be a coherence factor, appearing in the product $\sigma_s q_2$ in the equations for J_{xy} and J_{yx}. The phenomenological parameters are analogous to diffusion constants, which describe the diffusive transmission of the wavefield in the medium.

In general, this solution shows that as the wave penetrates the medium, the state of polarization of the polarized part becomes circular. Recalling that any generally elliptically polarized wave is the superposition of a circularly polarized wave (of the same sense) and a linearly polarized wave (along the major axis of the ellipse), one can reason that the coherence of the linearly polarized component decays more rapidly than the circularly polarized component, as indicated by (5.3.28). Thus, the medium tends to rub the edges off the ellipse causing its eccentricity to approach zero, eventually yielding a circularly polarized wave.

APPENDIX A

RANDOM FUNCTIONS AND STATISTICAL AVERAGES

In studying wave propagation in a random medium, one must treat quantities that are random functions of both position and time. As discussed in Sections III and IV, the total volume occupied by the continuous random medium is divided up into N independent elementary scattering volumes. Therefore, one must not only statistically characterize quantities which depend upon the random medium within each elementary volume, but also quantities which depend upon all N scattering volumes. These statistical characterizations denote finding ensemble averages of the quantities which are functions of a random variable. Thus, it will be helpful to examine the unique method required for determining these averages.

Let us consider a random function F which depends on all the N elementary scattering volumes comprising the medium. The ensemble average of F is given in terms of a probability density function $W(\underline{1}, \underline{2}, \underline{3},---N)$:

$$\langle F\rangle = \iint\cdots\int F\cdot W(\underline{1},\underline{2},\ldots\underline{s},\ldots\underline{N})\, d\underline{1}d\underline{2}\ldots d\underline{s}\ldots d\underline{N} \quad . \qquad (A.1)$$

The distribution function W is normalized such that $\int\cdots\int W d\underline{1}\ldots d\underline{N} = 1$, when integrated over the full domain of variables; $Wd\underline{1}\ldots d\underline{N}$ is the probability of finding the scattering medium in a certain configuration in the "volume" between $(\underline{1},\underline{2},\ldots,\underline{N})$ and $(\underline{1},\underline{2},\ldots\underline{N}) + d\underline{1}\ d\underline{2}\ldots dN$. Thus, in (A.1) $\underline{s}$ designates all the random characteristics of the medium including its location $\underline{r}_s$; therefore, we may write

$$d\underline{s} = d\underline{r}_s d\varepsilon \quad , \tag{A.2}$$

where $d\underline{r}_s$ designates the volume integral $dx_s\ dy_s\ dz_s$ and $d\varepsilon$ represents the random characteristics of the random permittivity. We assume that the number of elementary scattering volumes in a unit volume of space is low and that the separation between scattering volumes is large enough to satisfy the far-zone scattering conditions discussed in Appendix B. In this case, the finite size of the scattering volume can be neglected and the location and characteristics of each scattering volume are independent of the locations and characteristics of other scattering volumes. Thus, the elementary scattering volumes are considered as point scatterers with all the scattering characteristics of the random medium, and located precisely at $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$,---$\underline{r}_s$---$\underline{r}_N$. Under this assumption, we have

$$W(\underline{1},\underline{2},\ldots\underline{s},\ldots\underline{N}) = w(\underline{1})w(\underline{2})w(\underline{3})\ldots w(\underline{s})\ldots w(\underline{N}) \quad . \tag{A.3}$$

By assuming that all elementary scattering volumes have the same statistical characterization $w(\underline{s}) = w(\underline{r}_s,\varepsilon)$, the $d\varepsilon$ integration in (A.1) can be performed yielding

$$\langle F\rangle = \int\cdots\int \langle F\rangle_\varepsilon w(\underline{r}_1)w(\underline{r}_2)\ldots w(\underline{r}_s)\ldots w(\underline{r}_N)d\underline{r}_1\ldots d\underline{r}_N \quad , \tag{A.4}$$

where $\langle F\rangle_\varepsilon$ represents the average of F over the random configurations of the medium's permittivity.

Next, we note that $w(\underline{r}_s)\cdot d\underline{r}_s$ is interpreted as the probability of finding a random medium within the incremental volume $d\underline{r}_s$:

$$w(\underline{r}_s)\cdot d\underline{r}_s = \frac{\text{number of random medium scattering volumes in } d\underline{r}_s}{\text{total number of scattering volumes in the space V}}$$

$$= \rho(\underline{r}_s)d\underline{r}_s/N \quad , \qquad \text{(A.5)}$$

where $\rho(\underline{r}_s)$ is the "number density" defined by the number of random medium scatterers per unit volume. When the number density is uniform throughout all the space, we have $\rho = N/V$ and $w(\underline{r}_s) = 1/V$. With the substitution of (A.5) into (A.4), the ensemble average becomes

$$\langle F\rangle = \int\cdots\int \langle F\rangle_\varepsilon \frac{\rho(\underline{r}_1)\rho(\underline{r}_2)\ldots\rho(\underline{r}_N)}{N^2} d\underline{r}_1 d\underline{r}_2\ldots d\underline{r}_N \quad . \qquad \text{(A.6)}$$

However, it is apparent the average with respect to the permittivity depends only upon the location of the elementary volumes; accordingly, we can write $\langle F\rangle_\varepsilon = \langle F(\underline{r}_s)\rangle_\varepsilon$ and all the integrals in (A.6) can be performed except the one over $\underline{r}_s$. This leads to the final result

$$\langle F\rangle = \int \frac{\rho(\underline{r}_s)}{N} \langle F(\underline{r}_s)\rangle_\varepsilon d\underline{r}_s \quad . \qquad \text{(A.7)}$$

We note also that since the characteristics of the permittivity are independent of the location of scattering volume, the probability function in (A.3) can be written as

$$w(\underline{s}) = w(\underline{r}_s,\varepsilon) = \frac{\rho(\underline{r}_s)p(\varepsilon)}{N} \qquad \text{(A.8)}$$

Utilizing (A.8), an equivalent expression for the ensemble average of F is obtained as follows:

$$\langle F \rangle = \int p(\varepsilon) \langle F(\varepsilon) \rangle_N d\varepsilon \quad , \tag{A.9}$$

where $\langle F \rangle_N$ represents the average with respect to the N scattering volumes and $p(\varepsilon)$ is the probability density function describing the medium's permittivity.

Next, let us consider the random permittivity of the medium in some detail. The random variable $\varepsilon(\underline{r})$ discussed in Section III is a random function of position, which in this case is limited to being inside an elementary scattering volume. We call ε a random field and characterize it by ensemble averages of various functions of the random variable. In general, the statistics of the random field will also vary from point-to-point. Two statistical quantities particularly useful in characterizing the random field are its average value $\langle \varepsilon \rangle$ and correlation function $\langle \varepsilon(\underline{r}_1)\varepsilon(\underline{r}_2) \rangle$. The correlation function is symbolized by $B_\varepsilon(\underline{r}_1,\underline{r}_2)$. In many cases, the correlation function depends only upon the difference of the arguments, i.e., $B_\varepsilon(\underline{r}_1,\underline{r}_2) = B_\varepsilon(\underline{r}_1-\underline{r}_2)$, and the average value of the random field is independent of position. Such cases describe a homogeneous random field. Furthermore, if the correlation function of a homogeneous field depends only on the magnitude of the difference of the arguments, i.e., $B_\varepsilon(\underline{r}_1-\underline{r}_2) = B_\varepsilon(|\underline{r}_1-\underline{r}_2|)$, the field is isotropic as well as homogeneous. It is often useful to compute the spectral density of the homogeneous random field, defined as the Fourier transform of the correlation function. The spectral density is given by

$$\Phi_\varepsilon(\underline{\kappa}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} B_\varepsilon(\underline{r}) e^{+i\underline{\kappa}\cdot\underline{r}} d\underline{r} \quad , \tag{A.10}$$

where the integration is over all 3-D space. For the isotropic and homogeneous random field, the spectral density depends only on the magnitude of $\underline{\kappa}$. The complete transformation is specified by including the integral expression

$$B\ (\underline{r}) = \int_{-\infty}^{+\infty} \Phi_\varepsilon(\underline{\kappa}) e^{\mp i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa} \quad . \tag{A.11}$$

Up to this point, we have not discussed the time variations of the medium. In this paragraph, the relevant result for a slowly time-varying frozen-in permittivity are given. By "slowly time-varying," we mean that the time variation of the random permittivity is caused only by simple translational motion at uniform velocity $\underline{V}$. This assumption is known as Taylor's frozen-in hypothesis. Mathematically, we can write Taylor's frozen-in hypothesis as

$$\varepsilon(\underline{r},t) = \varepsilon(r-\underline{V}t,0) \ . \tag{A.12}$$

The correlation function becomes

$$\begin{aligned} B\ (\underline{r},\tau) &= \langle \varepsilon(\underline{r}_1-\underline{V}t_1,0)\cdot\varepsilon(\underline{r}_2-Vt_2,0) \rangle \\ &= B_\varepsilon(\underline{r}-V\tau) \ , \end{aligned} \tag{A.13}$$

where $\tau = t_1-t_2$. Comparing (A.13) with (A.11), we see that the spectral density for the frozen-in time-varying medium is

$$\Phi_\varepsilon(\underline{\kappa},\tau) = \Phi_\varepsilon(\underline{\kappa})e^{+i\underline{\kappa}\cdot\underline{V}\tau} \ , \tag{A.14}$$

In addition, it is evident that time averages and ensemble averages in the frozen-in medium are equivalent.

Finally, let us consider the probability distributions of the amplitude A and phase ϕ of a random field composed of many components, such as the fluctuating electric field at a point in the random medium comprised or waves scattered from many different elementary volumes. This fluctuating field can be written as

$$E = Ae^{i\phi} = X + iY \tag{A.15}$$

or

$$E = \sum_{n=1}^{N} A_n e^{i\phi_n} = \sum_{n=1}^{N} (X_n + Y_n) \quad . \tag{A.16}$$

From the "central limit theorem," we note that the probability distribution of a sum of N independent random variables approaches the normal distribution as $N\to\infty$, regardless of the probability distribution of each random variable. Assuming that each X_n and Y_n scattered from an elementary volume is an independent random variable, then we conclude that X and Y are normally distributed. We also assume that the phase of each of the elementary components is uniformly distributed over 2π, and the resulting sum (A.16) has no preferred phase. These conditions would only apply to the fluctuating field discussed in Section 4.3. It follows that the amplitude A and phase ϕ of the fluctuating field are independent.

The probability density function for A and ϕ can thus be written as

$$p(A,\phi) = p(A)p(\phi) \quad , \tag{A.17}$$

where

$$p(\phi) = \frac{1}{2\pi} \quad , \quad 0 < \phi < 2\pi \ . \tag{A.18}$$

Using (A.17) and (A.18), we can easily show that $\langle X\rangle_N = \langle A\cos\phi\rangle_N = 0$ and $\langle Y\rangle_N = \langle A\sin\phi\rangle_N = 0$; therefore, the average value of the composite field is zero, i.e., $\langle E\rangle_N = 0$.

The probability density function p(A) can be easily computed also. Since X and Y are normally distributed and independent (also uncorrelated), their joint probability density function p(x,y) is given by

$$p(x,y) = p(x)p(y) = (\frac{2\pi}{\sigma^2})e^{-(x^2+y^2)/2\sigma^2} \tag{A.19}$$

Noting that

$$p(x,y)dxdy = p(A)p(\phi)dAd\phi \tag{A.20}$$

and for the 2-dimension functions

$$dxdy = AdAd\phi \quad , \tag{A.21}$$

we get

$$p(A) = \int_0^{2\pi} p(A,\phi)d\phi = \frac{A}{\sigma^2}\, e^{-A^2/2\sigma^2} \quad . \tag{A.22}$$

The amplitude distribution given in (A.22) is known as the Rayleigh distribution. Furthermore, we note that $\langle X^2 \rangle_N = \langle Y^2 \rangle_N = \sigma^2$.

To summarize the results given in this appendix, we point out that (A.7) or (A.9) gives the mathematical expression for the ensemble average of a quantity F which depends upon all N elementary scattering volumes. Also, (A.18) and (A.22) show that the phase of the fluctuating electric field in a random medium is uniformly distributed between zero and 2π, and the amplitude is Rayleigh distributed.

APPENDIX B

INTEGRAL REPRESENTATIONS FOR THE SCATTERED FIELD

If a plane wave is incident upon a volume of space V_s with constitutive parameters different from the surrounding medium, a scattered field will be generated. The incident and scattered fields both obey Maxwell's equations. Let $\varepsilon(\underline{r})$ represent the spatially dependent permittivity descriptive of all space:

$$\varepsilon(\underline{r}) = \varepsilon_o \cdot \varepsilon_r(\underline{r}) \quad \text{inside } V_s \tag{B.1}$$

$$\varepsilon(\underline{r}) = \varepsilon_o \quad \text{outise } V_s \tag{B.2}$$

First, consider the Maxwell equations:

$$\nabla X \underline{E} = i\omega\mu_o \underline{H} \tag{B.3}$$

$$\nabla X \underline{H} = i\omega\varepsilon(\underline{r})\underline{E} \quad , \tag{B.4}$$

where the field time dependence $e^{-i\omega t}$ has been suppressed. Here it is assumed that the permeability μ_o is constant inside and outside the scattering volume. From Maxwell's equations, it is easy to obtain the vector Helmholtz equation for $\underline{E}$ in the form

$$\nabla X \nabla X \underline{E}(\underline{r}) - \omega^2\mu_o\varepsilon_o\underline{E}(\underline{r}) = \omega^2\mu_o\varepsilon_o(\varepsilon_r(\underline{r}) - 1)\underline{E}(\underline{r}) \quad . \tag{B.5}$$

The right-hand side of (B.5) is nonzero only inside the scattering volume V_s. If we define this right-hand side in terms of an equivalent current density inside the scattering volume $J(\underline{r})$, then equation (B.5) takes the form

$$\nabla X \nabla X \underline{E} - k^2\underline{E} = i\omega\mu_o\underline{J} \tag{B.6}$$

where

$$\underline{J}(\underline{r}) = -i\omega\varepsilon_o(\varepsilon_r(r) -1)\underline{E}(\underline{r}) \qquad \text{(B.7)}$$

and $k^2 = \omega^2\mu_o\varepsilon_o = 2\pi/\lambda$. The homogeneous solution of (B.6) is just the primary or incident field $\underline{E}^{inc}(\underline{r})$ which exists in the absence of the scatterer whereas the particular solution is the scattered field $\underline{E}^{sc}(\underline{r})$ generated by the scatterer. Of course, the total field $\underline{E}(\underline{r}) = \underline{E}^{inc}(\underline{r}) + \underline{E}^{sc}(r)$ also satisfies (B.6). Since $\underline{E}$ is linearly related to $\underline{J}$, it can be shown (12) that:

$$\underline{E}(\underline{r}) = \underline{E}^{inc}(\underline{r}) + i\omega\mu_o \int_{V_s} \underline{\underline{\Gamma}}(\underline{r},\underline{r}')\underline{J}(\underline{r}')dv'' , \qquad \text{(B.8)}$$

i.e.,

$$\underline{E}^{sc}(\underline{r}) = i\omega\mu_o \int_{V_s} \underline{\underline{\Gamma}}(\underline{r},\underline{r}')\underline{J}(\underline{r}')dv' , \qquad \text{(B.9)}$$

where

$$\underline{\underline{\Gamma}}(\underline{r},\underline{r}') = (\underline{\underline{U}} + \frac{1}{2}\nabla\nabla)\frac{e^{+ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \qquad \text{(B.10)}$$

$\underline{\underline{U}}$ is the unit dyadic, and $\nabla\nabla$ is the double-gradient dyadic which in a cartesian coordinate system is expressed by

$$\underline{\underline{U}} = \sum_{m=1}^{3}\sum_{n=1}^{3} \hat{e}_m\hat{e}_n\delta_{nm} \qquad \text{(B.11)}$$

$$\nabla\nabla = \sum_{m=1}^{3}\sum_{n=1}^{3} \hat{e}_m\hat{e}_n \frac{\partial}{\partial X_m}\frac{\partial}{\partial X_n} \qquad \text{(B.12)}$$

where $x_i (i=1,2,3)$ are the cartesian coordinates, $\hat{e}_i (i=1,2,3)$ are the unit base vectors, and the symbol δ_{mn} is the Kronecker delta, which is 1 for m = n and 0 for m ≠ n. From (B.9) and (B.7), the scattered field becomes

$$\underline{E}^{sc}(\underline{r}) = k^2 \int_{V_s} \{\varepsilon_r(\underline{r}')-1\} \, \underline{\underline{\Gamma}}(\underline{r},\underline{r}')\underline{E}(\underline{r}')dv' \tag{B.13}$$

In the far-zone, defined by r >> r' and kr >> 1, one can show (12) that

$$\underline{\underline{\Gamma}}(\underline{r},\underline{r}') = (\underline{\underline{U}} - \hat{s}\hat{s})e^{-ik\hat{s}\cdot\underline{r}'} \frac{e^{+ikr}}{4\pi r} , \tag{B.14}$$

and (B.13) becomes

$$\underline{E}^{sc}(r,\hat{s}) = \underline{f}(\hat{s},\hat{s}') \frac{e^{+ikr}}{r} , \tag{B.15}$$

where $\hat{s}$ and $\hat{s}'$ are the directions of the scattered and incident waves, respectively, and $\underline{f}$ is the far-zone scattering amplitude defined by

$$\underline{f}(\hat{s},\hat{s}') = \frac{k^2}{4\pi} \int_{V_s} \{-\hat{s} \times [\hat{s} \times \underline{E}(\underline{r}')]\}\{\varepsilon_r(\underline{r}')-1\}e^{-ik\hat{s}\cdot\underline{r}'}dv' \quad . \tag{B.16}$$

Note that $[\hat{s}\hat{s} - \underline{\underline{U}}]\cdot\underline{E}$ has been rewritten as $\hat{s} \times (\hat{s} \times \underline{E})$, which clearly shows that this is the component of $\underline{E}$ perpendicular to $\hat{s}$.

The scattered power flux density $E^{sc} = F^{sc}\hat{s}$ is used to define the differential scattering cross section of the volume. The power flux density vectors for time harmonic waves are

$$\underline{F}^{sc} = \frac{1}{2}[\underline{E}^{sc} \times \underline{H}^{sc}] = \frac{|\underline{E}^{sc}|^2}{2\eta_o} \tag{B.17}$$

$$\underline{F}^{inc} = \frac{1}{2}[\underline{E}^{inc} \times \underline{H}^{inc}] = \frac{|\underline{E}^{inc}|^2}{2\eta_o} \tag{B.18}$$

where $\eta_o = (\mu_o \varepsilon_o)^{\frac{1}{2}}$ is the characteristic impedance of the medium. The differential scattering cross section is defined as follows:

$$\sigma(\hat{s},\hat{s}') = \lim_{r\to\infty} [r^2 F^{sc}/F^{inc}] = |f(\hat{s},\hat{s}')|^2 \quad . \tag{B.19}$$

$\sigma(\hat{s},\hat{s}')$ has the dimension of area per solid angle.

The total observed scattered power at all angles is called the scattering cross section σ_s and is given by

$$\sigma_s = \int_{4\pi} \sigma(\hat{s},\hat{s}')d\omega = \int_{4\pi} |f(\hat{s},\hat{s}')|^2 d\omega \tag{B.20}$$

where $d\omega = \sin\theta d\theta d\phi$ is the differential solid angle. σ_s has the dimension of area.

APPENDIX C

A RELATIONSHIP BETWEEN MULTIPLE SCATTERING THEORY AND TRANSPORT THEORY

In general, there are two approaches to the study of wave propagation in random media: analytical or scattering theory and transport theory. In analytic theory, fundamental equations for statistical quantities describing the wavefield are derived starting with Maxwell's equations and the resulting wave equation. Perhaps the simplest of the analytic theories is the Born approximation, wherein the interaction among scattered waves is neglected. Another well developed analytic technique which is known to be superior to the Born approximation is the method of smooth perturbations or Rytov's theory. In classical transport theory, as was discussed in section IV of the main text, the propagation of field intensities (coherency matrix or Stokes matrix for an electromagnetic wave) in the random medium is investigated using a transport equation based upon conservation of energy. Transport theory has its beginnings in nuclear physics where studies of neutron transport were done in the early 1900's. In many important circumstances, both analytic theory and transport theory can be applied to the same problem; therefore, it is important to establish and understand the relationship between the two theories. Such a relationship is developed in this appendix.

Let $E(\underline{r})$ be random scalar field, with a monochromatic time-dependence $\exp(-i\omega t)$. Results for the electromagnetic field may be easily obtained by generalization. The quantity most often sought in the analytic theory is the second-order correlation function of the wavefield, which is given by

$$\Gamma(\underline{r}_1,\underline{r}_2) = \langle E(\underline{r}_1)E^*(\underline{r}_2)\rangle \qquad (C.1)$$

In (C.1) the asterisk denotes the complex conjugate and the sharp brackets denote an ensemble average. In transport theory the quantity sought is the specific intensity $J(\underline{r},\hat{s})$ of the wavefield (radiance) in a given direction $\hat{s}$ at the point in space $\underline{r}$. The quantity $J(\underline{r},\hat{s})$ is given by

$$J(\underline{r},\hat{s}) = \langle E(\underline{r},\hat{s})E^*(\underline{r},\hat{s})\rangle \qquad (C.2)$$

The derivations in this Appendix provide a relationship between $\Gamma(\underline{r}_1,\underline{r}_2)$ and $J(\underline{r},\hat{s})$. Note that in measurements, the quantity $\Gamma(\underline{r}_1,\underline{r}_2)$ would be determined by two omni-directional voltage receivers located at the points $\underline{r}_1$ and $\underline{r}_2$, whereas $J(\underline{r},\hat{s})$ would be determined by a very narrow-beam intensity receiver pointed in the direction $\hat{s}$ and located at $\underline{r}$. The energy density $I(\underline{r})$ of the wavefield at a point $\underline{r}$ can be obtained from either (C.1) or (C.2):

$$I(\underline{r}) = \Gamma(\underline{r},\underline{r}) = \langle E(\underline{r})E^*(\underline{r})\rangle \qquad (C.3)$$

and

$$I(\underline{r}) = \int_{4\pi} J(\underline{r},\hat{s})d\omega \quad . \qquad (C.4)$$

In (C.4) $d\omega = \sin\theta d\theta d\phi$ is the unit solid angle.

Now, at any point in space, $E(\underline{r})$ can be decomposed into an equivalent set of plane wave according to the equation (see (4.3.2) of the main text)

$$E(r) = \int_{-\infty}^{+\infty} E(\underline{\kappa}) e^{+i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa} \tag{C.5}$$

In (C.5) $E(\underline{\kappa})$ is simply the three-dimensional Fourier tra-sform of $E(\underline{r})$. For completeness, we include the following inverse transform relationship:

$$E(\underline{\kappa}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} E(\underline{r}) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{r} \quad . \tag{C.6}$$

The integrations in (C.5) and (C.6) involve the whole three dimensional space represented by the domain of the field quantities. In this representation, the correlation function Γ becomes

$$\Gamma(\underline{r}_1,\underline{r}_2) = \iint_{-\infty}^{+\infty} \langle E(\underline{\kappa}_1) E^*(\kappa_2) \rangle e^{+i(\underline{\kappa}_1\cdot\underline{r}_1 - \underline{\kappa}_2\cdot\underline{r}_2)} d\underline{\kappa}_1 d\underline{\kappa}_2 \tag{C.7}$$

Next, let us change variables from $(\underline{r}_1,\underline{r}_2)$ to $(\underline{r},\underline{r}_d)$ and $(\underline{\kappa}_1,\underline{\kappa}_2)$ to $(\underline{\kappa},\underline{\kappa}_d)$ by means of the formulas

$$\underline{r} = \frac{1}{2}(\underline{r}_1 + \underline{r}_2) \quad , \quad \underline{r}_d = \underline{r}_1 - \underline{r}_2$$

$$\underline{\kappa} = \frac{1}{2}(\underline{\kappa}_1 + \underline{\kappa}_2) \quad , \quad \underline{\kappa}_d = \underline{\kappa}_1 - \underline{\kappa}_2 \tag{C.8}$$

In terms of these new variables, the correlation function given by (C.7) becomes $\Gamma(\underline{r},\underline{r}_d)$ given by

$$\Gamma(\underline{r},\underline{r}_d) = \iint_{-\infty}^{+\infty} \langle E(\underline{\kappa} + \tfrac{1}{2}\underline{\kappa}_d) E^*(\underline{\kappa} - \tfrac{1}{2}\underline{\kappa}_d) \rangle \; e^{+i(\underline{\kappa}_d\cdot\underline{r} + \underline{\kappa}\cdot\underline{r}_d)} d\underline{\kappa} d\underline{\kappa}_d \tag{C.9}$$

Defining a new function $J(\underline{r},\underline{\kappa})$ according to the equation

$$J(\underline{r},\underline{\kappa}) = \int_{-\infty}^{+\infty} \langle E(\underline{\kappa} + \frac{1}{2}\underline{\kappa}_d)E^*(\underline{\kappa} - \frac{1}{2}\underline{\kappa}_d)\rangle e^{+i\underline{\kappa}_d \cdot \underline{r}} d\underline{\kappa}_d \qquad (C.10)$$

produces the final general expression for the correlation function:

$$\Gamma(\underline{r},\underline{r}_d) = \int_{-\infty}^{+\infty} J(\underline{r},\underline{\kappa}) e^{+i\underline{\kappa}_d \cdot \underline{r}_d} d\underline{\kappa} \qquad (C.11)$$

From (C.11) it can be identified that the energy density given by (C.3) is obtained by setting $\underline{r}_d = 0$. Thus, we have

$$I(\underline{r}) = \Gamma(\underline{r},\underline{r}_d{=}0) = \int_{\infty}^{\infty} J(\underline{r},\underline{\kappa}) d\underline{\kappa} \qquad (C.12)$$

Furthermore, when it is noted that $\underline{\kappa}$ has both magnitude and direction, i.e., $\underline{\kappa} = \kappa\hat{s}$ the vector wavenumber integration ($d\underline{\kappa}$) in (C.12) can be broken up into an angular integration ($d\omega$) and wavenumber integration ($d\kappa$) yielding

$$I(\underline{r}) = \int_{4\pi} \int_{0}^{\infty} J(\underline{r},\kappa\hat{s})\kappa^2 d\kappa d\omega \ . \qquad (C.13)$$

Comparing (C.13) and (C.14), we see that the specific intensity of the field $J(\underline{r},\hat{s})$ is simply related to the plane wave expansion field according to the equation

$$J(\underline{r},\hat{s}) = \int_{0}^{\infty} J(\underline{r},\kappa\hat{s})\kappa^2 d\kappa \qquad (C.14)$$

It should be noted that $J(\underline{r},\hat{s})$ is a directly measurable quantity whereas $J(\underline{r},\underline{\kappa})$ is not.

So far we have not entirely reached our goal of providing a relationship between $J(\underline{r},\hat{s})$ and $\Gamma(\underline{r},\underline{r}_d)$. However, we have shown that $J(\underline{r},\underline{\kappa})$ is a much more fundamental and general quantity than $J(\underline{r},\hat{s})$. In fact, in Section IV of the main text, it was shown that $J(\underline{r},\underline{\kappa})$ satisfies a transport equation just like $J(\underline{r},\hat{s})$. Thus, it appears that in mathematical solutions of transport phenomena, the quantity sought should be $J(\underline{r},\underline{\kappa})$ rather than $J(\underline{r},\hat{s})$. A few examples will help clarify the meanings of (C.11) and (C.14). These examples will also underline the applicability of classical transport theory as previously discussed in Section IV.

EXAMPLE A. HOMOGENEOUS WAVEFIELD

If the plane waves with different wavenumbers are independent, then we have

$$<E(\underline{\kappa} + \frac{1}{s}\underline{\kappa}_d)E^*(\underline{\kappa} - \frac{1}{2}\underline{\kappa}_d)< \; = \; <E(\underline{\kappa})E^*(\underline{\kappa})>\delta(\underline{\kappa}_d) \tag{C.15}$$

where $\delta(\underline{\kappa}_d)$ is the Dirac delta function which is zero except at $\underline{\kappa}_d = 0$. From (C.10) we have

$$J(r,\underline{\kappa}) = J(\underline{\kappa}) = <E(\underline{\kappa})E^*(\underline{\kappa})> \tag{C.16}$$

which is independent of the position vector $\underline{r}$. This result yields the following expression for the correlation function

$$\Gamma(\underline{r}_d) = \int_{\infty}^{\infty} J(\underline{\kappa})e^{+i\underline{\kappa}\cdot\underline{r}_d}d\underline{\kappa} \tag{C.17}$$

Thus, the second-order correlation function depends only upon the difference between the position vector $\underline{r}_1$ and $\underline{r}_2$. If we take $<E(\underline{r})> = 0$, then the above equation (C.17) is precisely a statement of statistical homogeneity of the wavefield $E(\underline{r})$. The significance of this example is that (C.16) is the often-used condition specified in transport theory for adding "independent" waves. From this example, it is clear that such a condition restricts the specific intensity to a form independent of position. Collet, Foley, and Wolf (15) reached a similar conclusion about transport theory, but came short of providing an alternative formulation as done here with $J(\underline{r},\underline{\kappa})$. From (C.17) the energy density of the wavefield can be found:

$$I(\underline{r}) = I(0) = \int_{-\infty}^{+\infty} J(\underline{\kappa})d\underline{\kappa} \quad . \qquad \text{(C.18)}$$

EXAMPLE B. UNCORRELATED WAVEFIELD

A useful and alternative expression of the correlation of the plane waves (which may have some physical significance) is that the correlation depends only on the wavenumber difference, i.e.,

$$<E(\underline{\kappa} + \frac{1}{2}\underline{\kappa}_d)E^*(\underline{\kappa} - \frac{1}{2}\underline{\kappa}_d)> = \frac{1}{(2\pi)^3} F(\underline{\kappa}_d) \quad . \qquad \text{(C.19)}$$

Thus, the specific intensity $J(\underline{r},\hat{s})$ or $J(\underline{r},\underline{\kappa})$ is independent of the observation direction;

$$J(\underline{r},\underline{\kappa}) = J(\underline{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} F(\underline{\kappa}_d)e^{+i\underline{\kappa}_d \cdot \underline{r}_d} d\underline{\kappa}_d \qquad \text{(C.20)}$$

The correlation function Γ is then simply related to $J(\underline{r})$ according to the equation

$$\Gamma(\underline{r},\underline{r}_d) = (2\pi)^3 J(\underline{r}_c)\delta(\underline{r}_d) \qquad (C.21)$$

and the energy density becomes

$$I(\underline{r}) = \int_{-\infty}^{+\infty} F(\underline{\kappa}_d) e^{+i\underline{\kappa}_d\cdot\underline{r}} d\underline{\kappa}_d \qquad (C.22)$$

From (C.21) we have the fact that the wavefield is completely uncorrelated for $\underline{r}_d = (\underline{r}_1 - \underline{r}_2) \neq 0$.

In some problems, the plane wave components of the wavefield may vary only with the observation angle $\hat{s}$, i.e., $J(\underline{r},\underline{\kappa}) = J(\underline{r},\hat{s})$, where β is fixed constant for all $\hat{s}$. For instance, if a plane wave is incident upon the random medium along the direction $\hat{s}'$, we have

$$E^{inc}(\underline{r}) = E(0)e^{+i\beta\hat{s}'\cdot\underline{r}} \qquad (C.23)$$

$$E^{inc}(\underline{\kappa}) = E(0)\delta(\underline{\kappa}-\beta\hat{s}')$$

$$= E(0)\,\frac{\delta(\kappa-\beta)}{\kappa^2}\,\delta(\hat{s}-\hat{s}') \qquad (C.24)$$

Since there is no wavenumber (κ) conversion in the medium (see Section IV), we know that the field everywhere includes the factor $\delta(\kappa-\beta)$.

$$J(\underline{r},\underline{\kappa}) = \frac{\delta(\kappa-\beta)}{\kappa^2}\,J(\underline{r},\hat{s}) \qquad (C.25)$$

Substitution of (C.25) into (C.11) yields

$$\Gamma(\underline{r},\underline{r}_d) = \int_{4\pi} J(\underline{r},\hat{s})e^{+i\beta\hat{s}\cdot\underline{r}_d}d\omega \qquad (C.26)$$

Equation (C.26) represents the result initially sought. However, this relationship was found to be applicable only when the incident wave upon the medium is a plane wave. The relationship in (C.26) has been postulated previously by Tartarski (9), Ishimaru (1) and others. However, in these previous papers, the limitation imposed by (C.25) was not given. Finally, we conclude this section by noting that for a plane parallel medium with a normally incident plane wave ($\hat{s}' = \hat{e}_z$) the expression for Γ in (C.26) reduces to

$$\Gamma(\underline{r},\underline{r}_d) = 2\pi \int_0^{\pi} J(\underline{r},\theta)J_o(\beta r_d \sin\theta)\sin\theta d\theta \qquad (C.27)$$

where $\underline{r}_d = \underline{r}_d\hat{e}_x$, $\cos\theta = \hat{s}\cdot\hat{e}_z$, and $J_o(x)$ is the zeroth order Bessel function.

APPENDIX D

THE SCATTERING MATRICES $\underline{\underline{A}}$ AND $\underline{\underline{S}}$

In Section III of the text, scattering by an elementary volume of random medium was discussed. The scattering amplitude matrix $\underline{\underline{A}}(\hat{s},\hat{s}')$ was given along with the differential scattering cross-section matrix $\underline{\underline{S}}(\hat{s},\hat{s}')$. In this appendix, more general details about these scattering matrices, along with some simplifications for the plane parallel medium, are derived. For convenience, let us first specify the meaning of the scattering matrices: the scattering amplitude matrix $\underline{\underline{A}}$ relates the scattered field in the far-zone to the incident field; this relationship can be written as

$$\underline{E}^{sc}(\hat{s}) = \underline{\underline{A}}(\hat{s},\hat{s}')\ \underline{E}^{inc}(\hat{s}') \quad . \tag{D.1}$$

The factor $\exp(ikr)/r$ has been suppressed in (D.1). In this appendix, we are primarily concerned with scattering from a single particle. In the plane parallel problem, $\underline{E}^{sc}$ and $\underline{E}^{inc}$ are column matrices with the elements (E_x^{sc}, E_y^{sc}) and (E_x^{inc}, E_y^{inc}), respectively, and $\underline{\underline{A}}$ is the 2x2 matrix given by

$$\underline{\underline{A}}(\hat{s},\hat{s}') = \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \quad . \tag{D.2}$$

From the Helmholtz reciprocity theorem it follows that

$$\underline{E}^{inc}(-\hat{s}) = \underline{\underline{A}}'(-\hat{s}',-\hat{s})\ \underline{E}^{inc}(-\hat{s}') \tag{D.3}$$

and

$$\underline{\underline{A}}'(\hat{s}',\hat{s}) = \begin{bmatrix} A_{xx} & -A_{yx} \\ -A_{xy} & A_{yy} \end{bmatrix} . \tag{D.4}$$

Now, according to the definitions (2.2.5) and (3.2.15) of the main text text, the coherency matrix of the scattered wave $\underline{\underline{J}}^{sc}$ is related to the coherency matrix of the incident wave $\underline{\underline{J}}^{inc}$ through the differential scattering cross-section matrix $\underline{\underline{S}}$; therefore, we have

$$\underline{\underline{J}}^{sc}(\hat{s}) = \underline{\underline{S}}(\hat{s},\hat{s}')\cdot\underline{\underline{J}}^{inc}(\hat{s}')$$

where

$$\underline{\underline{S}}(\hat{s},\hat{s}') = \underline{\underline{A}}(\hat{s},\hat{s}') \times \underline{\underline{A}}^{*}(\hat{s},\hat{s}') . \tag{D.5}$$

For the random medium described in Section III, the matrix $\underline{\underline{S}}$ takes the form

$$\underline{\underline{S}}(\hat{s},\hat{s}') = 2\pi\kappa^4\Phi_n(\kappa_s)\cdot\begin{bmatrix} (\ell,\ell)^2 & (\ell,\ell)(r,\ell) & (\ell,\ell)(r,\ell) & (r,\ell)^2 \\ (\ell,\ell)(\ell,r) & (\ell,\ell)(r,r) & (r,\ell)(\ell,r) & (r,\ell)(r,r) \\ (\ell,\ell)(\ell,r) & (\ell,r)(r,\ell) & (\ell,\ell)(r,r) & (r,\ell)(r,r) \\ (\ell,r)^2 & (\ell,r)(r,r) & (\ell,r)(r,r) & (r,r)^2 \end{bmatrix}, \tag{D.6}$$

where the elements of the matrix contained within the brackets in (D.6) may be computed from (3.1.14) of the main text. These elements are

$$(\ell,\ell)^2 = \frac{1}{2}[2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2] +$$

$$+ 2\mu\mu'(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos(\phi'-\phi) + \frac{1}{2}\mu^2\mu'^2\cos 2(\phi'-\phi)$$

$$(r,\ell)^2 = \frac{1}{2}\mu^2[1 - \cos 2(\phi'-\phi)]$$

$$(\ell,r)^2 = \frac{1}{2}\mu'^2[1 - \cos 2(\phi'-\phi)]$$

$$(r,r)^2 = \frac{1}{2}[1 + \cos 2(\phi'-\phi)]$$

$$(\ell,\ell)(r,\ell) = \mu(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\sin(\phi'-\phi) + \frac{1}{2}\mu^2\mu'\sin(\phi'-\phi)$$

$$(\ell,\ell)(\ell,r) = -\mu'(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\sin(\phi'-\phi) - \frac{1}{2}\mu\mu'^2\sin(\phi'-\phi)$$

$$(\ell,r)(r,r) = -\frac{1}{2}\mu'\sin 2(\phi'-\phi)$$

$$(r,\ell)(r,r) = \frac{1}{2}\mu\sin 2(\phi'-\phi)$$

$$(\ell,\ell)(r,r) = (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos(\phi'-\phi) + \frac{1}{2}\mu\mu'(1 + \cos 2(\phi'-\phi))$$

$$(r,\ell)(\ell,r) = -\frac{1}{2}\mu\mu'(1 - \cos 2(\phi'-\phi)) \qquad \text{(D.7)}$$

The coherency matrix in (D.5) is a four-element column matrix. An alternative representation is the 2x2 square matrix given in (2.2.3). Such a representation is obtained by taking the product of $\underline{E}^{sc}$ with its hermitian conjugate yielding

$$\underline{\underline{J}}^{sc}(\hat{s}) = \underline{\underline{A}}\cdot\underline{\underline{J}}^{inc}\cdot\underline{\underline{A}}^{+} \qquad \text{(D.8)}$$

The representation given in (D.8) is inconvenient, however, since the matrix describing the scatterer does not appear in a linear fashion.

This prevents many simple operations from being performed on $\underline{\underline{J}}^{sc}$.

We are interested in the plane-parallel medium, wherein field quantities depend only upon the depth into the medium (z) and polar angle $\theta (= \cos^{-1}\hat{s}\cdot\hat{z})$. As shown in Section V, more specifically equation (5.1.5), the appropriate scattering matrix for the plane-parallel medium is obtained by integrating out the ϕ and ϕ' dependence in $\underline{\underline{S}}(\hat{s},\hat{s}') = \underline{\underline{S}}(\theta,\phi;\theta';\phi')$. Thus, the applicable version of (D.5) is given by

$$\underline{\underline{S}}(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} d\phi' \, \frac{1}{2\pi}\int_0^{2\pi} d\phi \; \underline{\underline{S}}(\theta,\phi;\theta',\phi') \quad . \tag{D.9}$$

If $[S_{ik}](i,k = 1,4)$ are the elements of this scattering matrix $\underline{\underline{S}}$, then we have, for example, the first row (i = 1, k = 1,4)

$$S_{11}(\mu,\mu') = \frac{k^4}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \Phi_n(k_s)(\ell,\ell)^2 d\phi' d\phi \tag{D.10a}$$

$$S_{12}(\mu,\mu') = S_{13} = \frac{k^4}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \Phi_n(k_s)(\ell,\ell)(r,\ell) d\phi' d\phi \tag{D.10b}$$

$$S_{14}(\mu,\mu') = \frac{k^4}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \Phi_n(k_s)(r,\ell)^2 d\phi' d\phi \quad , \tag{D.10c}$$

Where $\Phi_n(k_s)$ is the spectral density of refractive index fluctuations. Fortunately, several important results can be drawn from expressions like (D.10) without explicit evaluation of the integrals. First, let us restrict ourselves to considering only the three refractive index spectral densities given in Section III; exponential, gaussian, and the physically-based Kolmogorov spectrums. This restriction is very weak.

For these three spectrums we note that $\Phi_n(k_s)$ is really a function of the square of k_s. That is, $\Phi_n(k_s) = F(k_s^2)$. From the definition of $\underline{k}_s$ given in (3.1.10) we have

$$\underline{k}_s = k(\hat{s}'-\hat{s}) = k[(\sin\theta'\cos\phi' - \sin\theta\cos\phi)\hat{x} + (\sin\theta'\sin\phi' - \sin\theta\sin\phi)\hat{y} + (\cos\theta' - \cos\theta)\hat{z}] \quad . \tag{D.11}$$

Using (D.11), it is easy to show that

$$k_s^2 = 2k^2[1 - \cos\theta'\cos\theta - \sin\theta'\sin\theta\cos(\phi'-\phi)] \quad . \tag{D.12}$$

Hence, $\Phi_n(k_s)$ is always an even function of ϕ or ϕ'. With $\Phi_n(k_s)$ even, integrals of the elements of $\underline{\underline{S}}(\hat{s},\hat{s}')$ are zero whenever the terms in the brackets of (D.7) are odd. Similarly, the integrals of $\underline{\underline{S}}(\hat{s},\hat{s}')$ are nonzero whenever the bracketed terms are even. From this simple symmetry, the matrix elements of $\underline{\underline{S}}(\mu,\mu')$ can be reduced to

$$S(\mu,\mu') = \begin{bmatrix} S_{11} & 0 & 0 & S_{14} \\ 0 & S_{22} & S_{23} & 0 \\ 0 & S_{23} & S_{22} & 0 \\ S_{41} & 0 & 0 & S_{44} \end{bmatrix} \quad . \tag{D.13}$$

For the gaussian correlation function, the nonzero elements of (D.13) can be readily computed. From (3.2.19) and (D.12) we have

$$\Phi_n(k_s) = \frac{\langle n_1^2\rangle \ell^3}{8\pi\sqrt{\pi}} e^{-\frac{k^2\ell^2}{2}[1-\cos\theta'\cos\theta-\sin\theta'\sin\theta\cos(\phi'-\phi)]} \tag{D.14}$$

Substitution of (D.14) into the appropriate integrals of (D.9) yields

$$S_{11}(\mu,\mu') = Q(\theta,\theta') \Big\{ [\sin^2\theta\sin^2\theta' + \frac{1}{2}\cos^2\theta\cos^2\theta']I_o(W) + 2\sin\theta\sin\theta'\cos\theta\cos\theta' I_1(W) + \frac{1}{2}\cos^2\theta\cos^2\theta' I_2(W) \Big\} \quad (D.15)$$

$$S_{14}(\mu,\mu') = Q(\theta,\theta') \frac{\cos^2\theta}{2} [I_o(W)-I_2(W)] \quad (D.16)$$

$$S_{41}(\mu,\mu') = Q(\theta,\theta') \frac{\cos^2\theta'}{2} [I_o(W)-I_2(W)] \quad (D.17)$$

$$S_{44}(\mu,\mu') = Q(\theta,\theta') \frac{1}{2} [I_o(W)+I_2(W)] \quad (D.18)$$

$$S_{22}(\mu,\mu') = Q(\theta,\theta') \frac{1}{2} \Big\{ \cos\theta\cos\theta' [I_o(W)+I_2(W)] + 2\sin\theta'\sin\theta I_1(W) \Big\} \quad (D.19)$$

$$S_{23}(\mu,\mu') = -Q(\theta,\theta') \frac{1}{2} \cos\theta\cos\theta' [I_o(W)-I_2(W)] \quad (D.20)$$

where

$$W = \frac{k^2\ell^2}{2} \sin\theta\sin\theta'$$

$$Q(\theta,\theta') = \frac{\langle n_1^2\rangle k^4\ell^3}{4\sqrt{\pi}} e^{-\frac{k^2\ell^2}{2}(1-\cos\theta\cos\theta')} \quad (D.21)$$

where $I_o(W)$, $I_1(W)$ and $I_2(W)$ are the modified Bessel functions of the zero , first, and second orders.

LOW FREQUENCY LIMIT: $k\ell \ll 1$ $(W \to 0)$

$$S_{11}(\mu,\mu') = \frac{\langle n_1^2\rangle k^4 \ell^3}{4\sqrt{\pi}} [\sin^2\theta \sin^2\theta' + \frac{1}{2} \cos^2\theta \cos^2\theta'] \qquad (D.22)$$

$$S_{14}(\mu,\mu') = \frac{\langle n_1^2\rangle k^4 \ell^3}{8\sqrt{\pi}} \cos^2\theta \qquad (D.23)$$

$$S_{41}(\mu,\mu') = \frac{\langle n_1^2\rangle k^4 \ell^3}{8\sqrt{\pi}} \cos^2\theta' \qquad (D.24)$$

$$S_{44}(\mu,\mu') = \frac{\langle n_1^2\rangle k^4 \ell^3}{8\sqrt{\pi}} \qquad (D.25)$$

$$S_{22}(\mu,\mu') = \frac{\langle n_1^2\rangle k^4 \ell^3}{8\sqrt{\pi}} \cos\theta \cos\theta' \qquad (D.26)$$

$$S_{23}(\mu,\mu') = -\frac{\langle n_1^2\rangle k^4 \ell^3}{8\sqrt{\pi}} \cos\theta \cos\theta' \qquad (D.27)$$

Equations (D.22) through (D.27) are the familiar matrix elements for Rayleigh scattering. These have been thoroughly studied for discrete random media by Chandrasekhar (5).

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