BIANCHI TYPE I COSMOLOGICAL MODELS

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ABSTRACT

This thesis begins with a brief review of observations of cosmological interest and with a sketch of the "standard" spatially homogeneous and isotropic cosmological models of our Universe that are currently in vogue. Following this introduction we investigate in great detail anisotropic cosmologies and cosmological models of Bianchi Type I. Our primary goal is to understand the consequences of expansion anisotropies in the general relativistic, hot big-bang theory of cosmology.

We use the Einstein field equations with vanishing cosmological constant, and Maxwell's equations, to study the temporal evolution of anisotropic Bianchi Type I cosmologies. These cosmologies are spatially homogeneous, but anisotropic; and they have no rotation. We consider only cosmologies with the "flat", diagonal, Bianchi Type I metric $ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2$.

We begin by studying the general properties of Bianchi Type I cosmologies. Then we consider the stress-energy tensor for massless-particle gases (either degenerate or non-degenerate) which decouple from thermal equilibrium and become freely-propagating in our diagonal Bianchi Type I metric. We investigate the dynamical effects of anisotropic neutrino stresses, and we show how neutrino viscosity damps out most of the existing expansion anisotropies when neutrinos decouple.

Finally, we elucidate the structure and properties of the Einstein field equations for anisotropic Bianchi Type I cosmologies by deriving a large number of analytical and numerical solutions to these equations. Our stress-energy tensor consists, in general, of perfect-fluid matter with the barotropic equation of state $p_m = \gamma \rho_m$ $(0 \leq \gamma \leq 1)$, and a uniform comoving magnetic field, with energy-density ρ_b, aligned along the z-axis. We first consider the PERFECT-FLUID case where $\rho_b \equiv 0$. We find the general analytical solution (for all γ), and construct semi-realistic cosmological models of our Universe using this solution. Then we consider the PERFECT-FLUID-MAGNETIC case where $\rho_b \neq 0$. We derive several analytical solutions, find the behavior near the initial physical singularity for the remaining cases, and study those remaining cases by numerical integration of the field equations. We then consider semi-realistic PERFECT-FLUID-MAGNETIC cosmological models of our Universe.

In our semi-realistic cosmological models we study the possible effects of expansion anisotropies and of a uniform primordial magnetic field upon the following: (a) the type of initial physical singularity, (b) the thermal history and temporal evolution of our Universe, (c) primordial element formation, (d) the time when expansion anisotropies become small, and (e) the temperature isotropy of the observed 2.7 ^{o}K cosmic microwave radiation.

TABLE OF CONTENTS

I. INTRODUCTION

"... what's past is prologue; what to come,
In yours and my discharge".

William Shakespeare, The Tempest,
Act II, Scene i, Line 242.

I. A. THE PHILOSOPHY OF THIS THESIS

One of the "evolutionary" characteristics of the scientific method is the impetus given to theoretical investigations by new observational data and the similar stimulus provided the experimentalist by new theoretical formulations and predictions. In recent years the field of relativistic cosmology and astrophysics has been vigorously reawakened by this phenomenon. A host of startling observations has transformed relativity theory from a quiet, philosophical endeavor into a living, breathing science. This thesis presents some research into anisotropic cosmological models of our Universe --- research generated by and made possible by the current "revolution" in cosmology.

Why do we study anisotropic cosmological models? Isn't it true, as McCrea (1968) implies, that the rash of recent observational discoveries --- in particular, the apparent observation of the primeval fireball as 2.7 $^{\circ}$K cosmic microwave radiation ---

vindicates the "standard" isotropic, Lemaître models[1] of our Universe? No! All observations to date merely show that the general features of hot, big-bang cosmology are correct and that the "standard" Lemaître models are a reasonable representation of the recent evolution of our Universe. The data give us no secure handle on the early stages of cosmic evolution, which might well have been anisotropic. There are many good reason for considering anisotropic cosmological models of our Universe, as we do in this thesis:

a) The "standard" Lemaître models are unique and extraordinarily simple. To be able to interpret any observational tests of these models we must investigate the structure and consequences of other, less simple, cosmological models. For example, the recent observations of the degree of spatial isotropy of the 2.7 $^{\circ}$K cosmic microwave radiation (see below) are meaningless without an explicit set of anisotropic cosmological models with which to compare them.

b) In anisotropic cosmological models the observed 2.7 $^{\circ}$K cosmic microwave radiation can exhibit spectral distortions of its initially blackbody spectrum and can have non-zero polarization, whereas such phenomena cannot occur in the "standard" Lemaître models. Hence, the study of anisotropic cosmological models suggests new

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[1]The current "standard" models contain both "dust" and "radiation", and are more properly termed Lemaître models than Friedmann models.

observational possibilities for the experimentalist.

c) We stated above that the observational data to date tell us essentially nothing about the earliest stages of cosmic evolution. Several questions connected with the "initial conditions" of the Universe naturally arise: (1) Shall we adopt the philosophy of C. W. Misner (1968) of a chaotic origin for the Universe at an initial physical singularity? His idea is that we should assume completely random, "white-noise" initial conditions (i.e., chaos) and then try to demonstrate that the theory of general relativity predicts the subsequent evolution of the Universe into its present organized state. In this program the "standard" Lemaître models form only a subset of measure zero of the possible general relativistic cosmological models which satisfy the current observations. (2) In what type of initial physical singularity did the Universe originate? In the "standard" Lemaître models we have an initial POINT physical singularity and all regions of the models are spacelike to one another (i.e., they have no causal connection) in the earliest stages of cosmic evolution. This lack of initial physical interaction between regions leads one to be skeptical of the assumptions of perfect spatial isotropy and homogeneity which lead to the Lemaître models. Some classes of anisotropic cosmological models exhibit initial CIGAR physical singularities where regions arbitrarily far apart are always in causal contact along at least one spatial axis. (3) Was there, in fact, an initial physical singularity at the beginning of our Universe? A <u>physical</u> singularity appears when at least one of the world lines of the matter cannot be extended to arbitrary values of its affine parameter. It is well-known that all world lines of matter end at a physical singularity in the spatially homogeneous and isotropic, "standard" Lemaître cosmological models and in all cosmologies of Bianchi Type I and IX (see the references cited in Hawking and Ellis 1968). A <u>mathematical</u> singularity occurs when at least one timelike world line terminates; this does not imply that any of the world lines of matter terminates. By combining extremely powerful mathematical analyses with the results of observations of the 2.7 °K cosmic microwave radiation, Hawking and Ellis (1968) have shown, in the context of standard general relativity theory, that our Universe must have encountered a mathematical singularity in its past. It is not yet known whether our Universe originated in a physical singularity. The equation of Raychaudhuri (1955, 1957) --- relating the cosmic "expansion", "shear", and "rotation" to the material content of the Universe via the Einstein field equations --- indicates that cosmic rotation might determine whether the initial singularity was physical or mathematical. The fact that the presence of both <u>cosmic expansion and cosmic rotation necessarily implies the existence of cosmic shear</u> (<u>anisotropic expansion</u>) has been made very clear by Heckmann and Schücking (1962), as well as by Shepley (1965). In order to better understand the nature of the original singularity and the effects of cosmic rotation we would like to become intimately acquainted with the properties of anisotropic cosmologies.

d) There are some indications that several of the properties of the "standard" Lemaître models might be at variance with the present observational data: (1) The recent work of Wagoner, Fowler, and Hoyle (1967) shows that primordial helium (^4He) production in the primeval fireball of the Lemaître models yields 25% - 30% helium by mass. If the recently measured 8% reduction in the lifetime of the neutron is confirmed, this prediction is lowered to 23% - 28% helium by mass (see Tayler 1968, and the references cited therein). The only observations which seriously disagree with this prediction are observations of the atmospheres of some old Population II halo stars in our Galaxy (see Sargent and Searle 1966, and the references cited therein), and observations of the atmospheres of some blue horizontal-branch stars in globular clusters (Sargent 1967). These observations imply a primordial helium abundance less than 5% by mass, unless these stellar atmospheres are helium deficient due to some mechanism like gravitational segregation of the elements. Though this problem is still unresolved, the theoretical studies of Faulkner and Iben (1966), Faulkner (1967), and Iben and Faulkner (1968) on the evolution of Population II stars strongly suggest that though the stellar atmospheres are helium deficient the interiors are not. Should the primordial helium abundance actually turn out to be much less than 20% by mass, however, the simple Lemaître models would have to be abandoned. The studies of Hawking and Tayler (1966) and Thorne (1967) on cosmological models with anisotropic expansion show that primordial helium abundances below 20% by mass (and down to almost 0% by mass) can be obtained if spatial anisotropy is introduced <u>and</u> if large quantities of ionized hydrogen (H II) existed in our Universe at redshifts of less than 10^3. The primordial helium abundance can also be lowered appreciably by spatial inhomogeneities at the time of element production, by the rapid expansion rates in the Brans-Dicke theory of cosmology (Dicke 1968), and by a huge excess in our Universe of leptons over antileptons or antileptons over leptons (Wagoner, Fowler, and Hoyle 1967). A second possible conflict of observations with the "standard" Lemaître models is this: (2) The "standard" Lemaitre models do not admit large-scale, ordered, cosmic magnetic fields. For example, a uniform primordial magnetic field in our Universe would indicate a preferred direction (the axis of the field) and, hence, directional anisotropy. Galaxies are observed to have very strong magnetic fields, of the order of 10^{-6} gauss, which are "theoretically impossible" to generate during the lifetime of the Universe[2] (see Hoyle 1958), but which might be remnants of

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[2] Cameron (1967) suggests a method for generating these fields which might overcome the difficulties pointed out by Hoyle.

a primordial magnetic field. Thorne (1967) shows that if the magnetic fields were primordial, they must have produced large anisotropies in the early stages of cosmic evolution. Cosmic magnetic fields of the order of 10^{-8} gauss today have also been suggested by Peebles (1967) as a way to solve the "problem" of the energetics of galaxy formation, another headache of the Lemaitre models.

e) A final reason for investigating anisotropic cosmological models is curiosity. The intellectual satisfaction derived from studying the Einstein-Maxwell field equations and better understanding their properties is enough of a reason for an inquisitive man to tackle the project.

Armed with these fairly interesting reasons for considering anisotropic cosmologies, we must now decide which theory of cosmology to use. Many theories of cosmology have been put forward, but the histories of even the most successful theories have been chequered. Adequate accounts of theories of the past are contained in McVittie (1949), Bondi (1961), Dicke (1964, 1967, and the references cited therein), and North (1965). The only theories which currently demand consideration are: (a) the steady-state theory of Bondi and Gold (1948), (b) the scalar-tensor theory of Brans and Dicke (1961), (c) the standard tensor-geometric theory of general relativity of Einstein (1915), and (d) the theory of general relativity with a cosmological constant (Λ) explored most vigorously by Lemaître (1927).[3] Of these four theories, the first and third are to me the most aesthetically compelling, while the latter three are the more observationally well-verified. It appears very likely that recent observations --- especially those of the 2.7 °K cosmic microwave radiation, of the Hubble expansion and deceleration rates, and of the space volume counts of radio galaxies --- militate strongly against the steady-state theory. I base my rejection of this theory here upon these observations. I eliminate the operationally-beautiful Brans-Dicke scalar-tensor theory on two rather weak grounds: (1) In my opinion, the excellent pre-1967 agreement between the prediction of general relativity and the observations, concerning the perihelion shift of the orbit of the planet Mercury, is probably not fortuitous but actual; and (2) the introduction of a scalar field merely complicates the investigation of cosmology without, in my mind, having sufficient justification. Finally, I consider the introduction of the cosmological constant Λ rather ad hoc, and though observationally permitted, an inelegant appendage to the elegant geometrical picture of the simple Einstein theory of general relativity.

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[3] Lemaître, when asked "What is the greatest contribution that general relativity has made to intellectual thought?" is reported by S. Chandrasekhar to have replied, without hesitation, "The cosmical constant".

Hence, using the principle of simplicity and the observations of the cosmic microwave radiation as my principal justifications, I choose in this thesis to consider only the hot, big-bang, general relativistic theory of cosmology with vanishing cosmological constant Λ.

The format of the remainder of this thesis is, in outline, the following: We conclude Chapter I with a brief review of observations of cosmological interest and with a sketch of the currently fashionable "standard" Lemaître cosmological models, which are spatially homogeneous and isotropic. In Chapter II we present the results of our investigations into anisotropic cosmological models of Bianchi Type I. Bianchi Type I cosmologies are spatially homogeneous, but anisotropic; and they exhibit no rotation. In an appropriate coordinate system they have the diagonal, spatially-Euclidean metric

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2 \ . \qquad \text{(I.A.1)}$$

We study the Einstein-Maxwell field equations in this metric, with our stress-energy tensor $T^{\mu\nu}$ consisting, in general, of perfect-fluid matter with the equation of state

$$p_m = \gamma \rho_m \quad (0 \leq \gamma \leq 1) \ , \qquad \text{(I.A.2)}$$

and a uniform comoving magnetic field, with energy density ρ_b, aligned along the z-axis.

In particular, we first consider the PERFECT-FLUID case, wherein the magnetic field vanishes. The general analytical solution to the entire problem is obtained, and we use this solution to construct semi-realistic cosmological models of our Universe. Secondly, we consider the PERFECT-FLUID-MAGNETIC case, wherein the magnetic field does not vanish. We find several analytical solutions, investigate the behavior of the equations near the initial singularity, and "solve" the entire problem using numerical integration. Finally, we construct semi-realistic cosmological models of our Universe with a primordial magnetic field.

In these semi-realistic models which we have built we study the effects of departures from isotropy and of a uniform primordial magnetic field upon (a) the thermal history and temporal evolution of our Universe, (b) the character of the initial physical singularity, (c) primordial element formation, (d) the time when the expansion anisotropies effectively become small, and (e) the possible temperature anisotropy of the observed 2.7 $^{\circ}$K cosmic microwave radiation.

I. B. COSMOLOGICAL OBSERVATIONS

In his task of constructing believable cosmological models the cosmologist is absolutely constrained by the available observational data. In this thesis we shall constantly be referring to the current observational data of cosmology; hence it will be to our advantage to list the pertinent data here before we begin. This list will mainly be a simple paraphrasing of the more complete coverage provided by the works of Sandage (1961a), Dicke (1964, 1967), Zel'dovich (1965a), Schücking (1966), Kristian and Sach (1966), Zel'dovich (1966), Wagoner, Fowler, and Hoyle (1967), Chiu (1967), and Novikov and Zel'dovich (1967). The references cited in these reviews give excellent coverage. The observations most pertinent to cosmological model construction are:

1) GEOMETRICAL OBSERVATIONS

(a) The spatial homogeneity of our Universe is inferred from number counts of galaxies versus their apparent magnitudes, from number counts of radio galaxies versus the energy flux received from them, and from the observations of the isotropy of the spatial distribution of extra-galactic sources. On the small scale the Universe is populated with galaxies and clusters of galaxies, and is quite inhomogeneous. Averaging over dimensions greater than forty megaparsecs, however, reveals statistical homogeneity out to distances of about 2×10^9 light-years from us. The work of Holden (1966), Longair (1966), and Pooley and Ryle (1968) on number counts of radio sources indicates statistical homogeneity out to redshifts $z \approx 3$ or 4, where the luminosity function appears to have a cut-off (i.e., this appears to be the epoch where galaxy formation occurred). The extreme isotropy of the observed 2.7 °K cosmic microwave radiation (see below) implies a high degree of homogeneity out to redshifts of at least 9.

(b) The <u>spatial isotropy</u> of our Universe is deduced from the angular distribution of extra-galactic radio sources, from the spatial distribution of the redshifts of extra-galactic objects, and from the temperature distribution (over the sky) of the observed 2.7 °K cosmic microwave radiation. Holden (1966) and Hughes and Longair (1967) find that, out to redshifts $z \approx 3$ or 4, the distribution of extra-galactic radio sources is statistically isotropic over angular scales larger than $1/2^{\circ}$. According to Kristian and Sachs (1966), the anisotropy in the Hubble expansion rates of galaxies and radio galaxies is less than 30%. The measurements of the directional isotropy of the temperature of the 2.7 °K cosmic microwave radiation (Wilson and Penzias 1967; Partridge and Wilkinson 1967; Wilkinson and Partridge 1967; Conklin and Bracewell 1967a, b; Epstein 1967) show that the temperature anisotropy $|\Delta T/T|$, is probably less than 3% over the entire Northern sky, less than 0.15% (root mean square fluctuations) over angular distances greater than 10' at $+ 40.6^{\circ}$ declination, and less than 0.20 (0.10) % in the magnitude of the 12 (24)-hour harmonic of anisotropy around the celestial equator (actually at -8°

declination). These limits to the temperature anisotropy imply corresponding limits on the large-scale anisotropy of the Universe's expansion rate out to redshifts $z \gtrsim 9$.

2) KINEMATICAL OBSERVATIONS

(a) If our Universe began in a hot big-bang the ages of its constituent parts should be less than the age of the Universe itself. In general relativistic cosmology (with vanishing cosmological constant) the observed Hubble expansion rate of our Universe at present (see below) sets an upper limit of about 18×10^9 years to the age of our Universe. From Dicke's (1964) compilation of data --- the most recent and extensive that I could find --- we see that elliptical galaxies might be older than 16×10^9 years and that some globular clusters might be older than 25×10^9 years. These excessive ages are based upon the currently popular theory of stellar evolution and rather crude studies of galactic content and evolution, and they must be considered highly uncertain. For example, Sandage (1962) and Faulkner and Iben (1966) have shown that a high primordial helium abundance (in the vicinity of 30% by mass) can reduce the deduced ages of the oldest globular clusters to a value nearer 15×10^9 years. The most recent work by Iben and Faulkner (1968) indicates that globular cluster ages near 9×10^9 years follow from the assumption of $\sim 30\%$ by mass primordial helium. Hence, the age of our Universe probably lies in the range from 8×10^9 to 18×10^9

years, and my personal preference (considering all the data) is in the neighborhood of 10×10^9 years.

(b) The average Hubble expansion rate of our Universe is derived from the observed correlation between the redshift and the apparent magnitude of brightest members of clusters of galaxies. The excellence of this correlation has been taken as evidence that the Hubble expansion law applies out to distances of about 2×10^9 light-years (these distances are extrapolated from independent measures of distance on a much smaller scale). If we take R to be the distance scale factor and let a subscript zero (o) denote the present time, then from Sandage (1968, and the references cited therein) we have for the present average expansion rate:

$$H_o \equiv \left(\frac{1}{R} \frac{dR}{dt}\right)_o = \begin{Bmatrix} (75 \pm 25)\text{km sec}^{-1}\text{ Mpc}^{-1} \\ [(13 \pm 5) \times 10^9 \text{ years}]^{-1} \end{Bmatrix} . \qquad \text{(I.B.1)}$$

(c) The gravitational attraction of the material within our Universe, as described by classical general relativity theory, implies a deceleration of the cosmic expansion. We represent this deceleration by Sandage's deceleration parameter, q_o. The most reasonable observational limits upon this parameter are those of Sandage (1961a), who gives:

$$q_o \equiv -\left(\frac{1}{RH^2} \frac{d^2R}{dt^2}\right)_o = +1 \pm \left(\frac{1}{2}\right) . \qquad \text{(I.B.2)}$$

If the evolution of galaxies is taken into account this value is lowered (see Sandage 1961b).

(d) If the Hubble expansion of our Universe is not isotropic, there is cosmic shear. In our previous discussion of the spatial isotropy of our Universe (above) we presented the current observational limits on the isotropy of the cosmic microwave radiation. From this data we conclude that $|\Delta H/H|$ (a reasonable measure of expansion anisotropies) is probably less than 0.2% back to a redshift of at least 9 (see Thorne 1967; Jacobs 1968).

(e) There is cosmic rotation in our Universe if extra-galactic objects exhibit proper motions with respect to the local inertial reference frame of our Local Cluster of galaxies. According to Clemence (1957), Kristian and Sachs (1966), and Wayman (1966) the present limit to such proper motions is rather less than one second of arc per century. Impressive as this limit may seem it is actually extremely weak in a cosmological context: angular velocities of the order of the Hubble rate are consistent with it. This limit should improve considerably when the extra-galactic system of proper motions is firmly established in about five years (Waymann 1966) and when improved isotropy measurements of the 2.7 °K cosmic microwave radiation are available (Sciama 1967).

3) OBSERVATIONS OF MATTER AND RADIATION

(a) Oort (1958), Van den Bergh (1961), and Abell (1965) find that the present average mass-density of visible matter (essentially the luminous matter in stars and galaxies) in our Universe lies in the range from 3×10^{-31} gm cm^{-3} to 3×10^{-30} gm cm^{-3}. On scales larger than about forty megaparsecs the distribution of this matter is homogeneous, while on smaller scales it is extremely heterogeneous --- aggregating preferentially into stars, galaxies, and clusters of galaxies.

(b) The virial theorem applies only to equilibrium, bound systems. Applying this theorem to our Galaxy, which most likely is a bound, quasi-equilibrium configuration of stars, Oort (1965, and the references cited therein) finds that about 50% of the mass of our Galaxy must reside in "invisible" matter. It is quite possible that the virial theorem is not applicable to clusters of galaxies. If, however, the theorem is applied to clusters of galaxies, it is found (see Conference on the Instability of Systems of Galaxies 1961; Woolf 1967, and the references cited therein) that the amount of undetected material necessary to bind the clusters is enough to give our Universe an average mass-density of between 10^{-29} gm cm^{-3} and 10^{-28} gm cm^{-3}. Abell (1965), however, feels strongly that this figure should be below 10^{-29} gm cm^{-3}.

(c) The fossil radiation from the primeval fireball of our Universe has apparently been observed as the cosmic microwave radiation. This radiation was predicted by Dicke, Peebles, Roll, and Wilkinson (1965, and the references cited therein). Observations of this radiation in the wavelength range from decimeters to millimeters are consistent with a blackbody spectrum with a characteristic temperature of 2.7 ± 0.1 $^{\circ}$K (Penzias and Wilson 1965; Roll and Wilkinson 1966; Field and Hitchcock 1966a; Thaddeus and Clauser 1966; Howell and Shakeshaft 1966; Field and Hitchcock 1966b; Welch, Keachie, Thornton, and Wrixon 1967; Wilkinson 1967; Stokes, Partridge, and Wilkinson 1967; Ewing, Burke, and Staelin 1967; Penzias and Wilson 1967; Puzanov, Salomonovich, and Stankevich 1967). The equivalent mass-density of this radiation today is $(4.5 \pm 0.7) \times 10^{-34}$ gm cm^{-3}.

(d) The equivalent mass-density of starlight in the metagalaxy is, according to Felten (1966), about 2×10^{-35} gm cm^{-3}. This represents only a small perturbation to the total mass-density of radiation in our Universe (Schücking 1966).

(e) As a consequence of numerous observations, our estimate of the abundance of primordial helium (^{4}He) in our Universe appears to be stabilizing at about (25 ± 5)% by mass. Only the atmospheres (1) of some hot halo B stars of Population II and (2) of some blue horizontal-branch stars in globular clusters are "helium-deficient" compared to this value, and the present consensus is that this deficiency is indicative only of the surface evolution of these stars and not necessarily of a universally low primordial helium abundance (Sargent and Searle 1966; Greenstein and Münch 1966; Strom and Strom 1967; Sargent and Searle 1967; Sargent 1967; Cayrel 1968; Sargent and Searle 1968).

(f) The observed energy spectrum of cosmic-ray electrons is beautifully compatible with the spectrum of background x-rays and gamma rays observed above the Earth's atmosphere, if the latter are considered to be Compton recoil photons from collisions of the former with a universal 2.7 °K cosmic blackbody radiation (Felten and Morrison 1966; Felten, Gould, Stein, and Woolf 1966; Brecher and Morrison 1967; Silk 1968).

(g) At present the temperature and hydrogen content of intergalactic space are fairly well delineated (Field 1962; Field and Henry 1964; Gunn and Peterson 1965; Gould and Ramsay 1966; Bahcall and Salpeter 1966; Field, Solomon, and Wampler 1966; Weymann 1966; Koehler and Robinson 1966; Koehler 1966; Rees and Sciama 1967; Weymann 1967; Penzias and Scott 1968; Henry, Fritz, Meekins, Friedmann, and Byram 1968). Only two pictures are consistent with the observational data on absorption lines in quasar spectra, with the upper limit to the free-free (bremsstrahlung) x-ray background observed, and with the limits on distortions of the blackbody spectrum of the 2.7 °K cosmic microwave radiation: (1) The temperature of intergalactic space is about 0 °K and the present mass-density of H I (H_2) is below 5×10^{-36} (5×10^{-34}) gm cm^{-3}; or (2) From the present back to a redshift of at least 2 the intergalactic medium has consisted almost entirely of ionized hydrogen (H II), with a mass-density of less than or about 10^{-29} gm cm^{-3} in the temperature range 10^4 $^oK \lesssim T \lesssim 10^6$ ^oK with less than 1% admixture of H I and H_2. The picture which is currently popular is (2).

(h) In the "standard", hot, big-bang, Lemaître models of our Universe where neutrinos and gravitons are in thermal equilibrium near the initial physical singularity their present equivalent mass-density should be about the same as that of the primordial photon gas at present (Chiu 1967). The only observational statement that is possible at present is this, that the upper limit on q_o implies an upper limit of about 10^{-28} gm cm^{-3} for the equivalent mass-density of these massless particles today.

(i) Galactic magnetic fields of the order of 10^{-6} gauss appear to imply the existence of large-scale primordial intergalactic magnetic fields [see Hoyle 1958; note, however, that Cameron (1967) has suggested a method which may solve the problem of the origin of galactic magnetic fields without invoking cosmic fields.]. The observational upper limit on the magnitude of such a pervasive field is about 10^{-6} gauss also, which implies an equivalent mass-density of about 10^{-35} gm cm^{-3} (Schücking 1966; Thorne 1967). If the galactic fields were trapped from an intergalactic field during galaxy formation we would expect the present magnitude of the intergalactic magnetic field to be of the order of 10^{-8} gauss (or about 10^{-39} gm cm^{-3}). Presently popular (and reasonable) values for the intergalactic magnetic field lie in the range from zero to about 10^{-7} gauss.

I. C. THE "STANDARD" ISOTROPIC COSMOLOGICAL MODELS

The observational data of § I. B. suggest that cosmological models exhibiting perfect spatial homogeneity[1] and isotropy[2] should be good first-order approximations to the actual Universe. Within the context of standard, hot, big-bang, general relativity theory the assumption of perfect spatial homogeneity and isotropy has led to the currently fashionable "standard" models of our Universe. Here we shall content ourselves with a brief sketch of the current picture of these models. For an excellent review of the history of their development see Tolman (1934), North (1965), Harrison (1967), and the references cited in these works.

The Einstein field equations (with vanishing cosmological constant) are:[3]

$$G^{\mu}{}_{\nu} = 8\pi\, T^{\mu}{}_{\nu} \quad . \tag{I.C.1}$$

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[1] Homogeneity implies that every observer in our Universe sees the same general surroundings.

[2] Isotropy implies that all observations are independent of the direction in which an observer may look.

[3] Throughout this thesis the signature of the metric is -2, the summation convention is used with Greek indices running from 0 to 3, and geometrized units are used (wherein the speed of light, c, and the gravitational constant, G, are set equal to one).

The assumption of perfect spatial isotropy, which implies perfect spatial homogeneity, leads directly to the Robertson-Walker line element:

$$ds^2 = dt^2 - R^2(t) \quad [d\chi^2 + \begin{pmatrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{pmatrix} (d\theta^2 + \sin^2\theta \, d\phi^2)] \quad . \quad (I.C.2)$$

Here t is cosmic proper time and R is a distance scale factor picturesquely referred to by some as "the radius of the universe". In equation (I.C.2) we have ($\sin^2\chi$, χ^2, $\sinh^2\chi$) when the curvature of the space sections is positive ("closed"), zero ("flat"), and negative ("open"), respectively. In the "standard" models, the stress-energy-momentum tensor, $T^\mu{}_\nu$, is assumed to be that of a perfect fluid:

$$T^\mu{}_\nu = (\rho + p)\, u^\mu\, u_\nu - p\, \delta^\mu{}_\nu \quad , \qquad (I.C.3)$$

where ρ is the uniform mass-density and p is the isotropic pressure of the fluid. From equation (I.C.1) follow the Bianchi identities

$$G^\mu{}_{\nu;\,\mu} = 0 \quad , \qquad (I.C.4)$$

which in turn imply the conservation equations

$$T^\mu{}_{\nu;\,\mu} = 0 \quad . \qquad (I.C.5)$$

With the metric of equation (I.C.2) equations (I.C.1), (I.C.3), and (I.C.5) reduce to:

$$\left(\frac{dR}{dt}\right)^2 = \left(\frac{8\pi}{3}\right) \rho R^2 - \kappa \quad , \qquad \text{(I.C.6)}$$

$$\frac{d(\rho R^3)}{dt} + p \frac{d R^3}{dt} = 0 \quad , \qquad \text{(I.C.7)}$$

where κ equals (+ 1, 0, - 1) when the space sections are ("closed", "flat", "open"), respectively. Equations (I.C.6) and (I.C.7), together with an equation of state $p = p(\rho)$, completely determine the evolution of the "standard" cosmological models of our Universe.

In the "standard" models the material content is idealized to be presureless "dust" plus isotropic "radiation". Large-scale primordial magnetic fields are excluded, since they are inherently anisotropic (the field direction defines a preferred direction). Therefore, we have

$$\rho = \rho_d + \rho_r \qquad \text{(I.C.8)}$$

and the equation of state

$$p = p_d + p_r = p_r = \rho_r/3 \quad , \qquad \text{(I.C.9)}$$

where the subscripts d and r denote "dust" and "radiation", respectively. The "dust" represents all forms of matter with finite rest mass (intergalactic, primeval gas; clusters of galaxies; etc.), while the "radiation" is a mixture of isotropically distributed photons, neutrinos, and gravitons. The final idealization made in the "standard" models is that $(\rho_d R^3)$ is a constant for all time. This means that the total material mass-energy is conserved, and hence, that there is no interaction between the "dust" and the "radiation". This assumption of non-interaction is an excellent approximation[4] from the time when electron-positron pairs disappear at a temperature near 5×10^9 °K until the present. From the constancy of $(\rho_d R^3)$ and equations (I.C.7) and (I.C.9) we have:

$$\rho_d = \rho_{do} (R/R_o)^{-3} \quad , \quad \rho_r = \rho_{ro} (R/R_o)^{-4} \quad . \qquad \text{(I.C.10)}$$

Here the subscript zero (o) denotes the present value of a quantity.

---

[4]In the "standard" models neutrinos and gravitons have negligible interaction with matter after the temperature drops below 10^{10} °K (see Misner 1967, 1968). The enormous heat capacity of the photons relative to the "dust" insures that the "dust" temperature will be kept equal to the photon temperature until plasma recombination (T ~ 3000 °K), without any significant energy exchange occuring. (See Harrison 1968.) After plasma recombination there is essentially no energy exchange.

Combining equations (I.C.6), (I.C.8), and (I.C.10), we reduce the problem of the temporal evolution of the "standard" models to quadratures:

$$t = \int \{(8\pi/3)[\rho_{do}(R/R_o)^{-1} + \rho_{ro}(R/R_o)^{-2}] - \varkappa\, R_o^{-2}\}^{-1/2}\, d(R/R_o) \ . \tag{I.C.11}$$

The solutions to equation (I.C.11) are well-known (see Chernin 1965; Alpher, Gamow, and Herman 1967; Jacobs 1967; Cohen 1967; McIntosh 1968; Harrison 1968; and the earlier authors cited in these recent works). These solutions may be written as (see Chernin 1965):

$$(\text{"closed"}) \left\{ \begin{aligned} R &= \alpha\ (1 - \cos\eta) + \beta \sin\eta \\ t &= \alpha\ (\eta - \sin\eta) + \beta\ (1 - \cos\eta) \end{aligned} \right\} \ , \tag{I.C.12}$$

$$(\text{"flat"}) \left\{ \begin{aligned} R &= \tfrac{1}{2}\alpha\, \eta^2 + \beta\, \eta \\ t &= \tfrac{1}{6}\alpha\, \eta^3 + \tfrac{1}{2}\beta\, \eta^2 \end{aligned} \right\} \ , \tag{I.C.13}$$

$$(\text{"open"}) \left\{ \begin{aligned} R &= \alpha\ (\cosh\eta - 1) + \beta \sinh\eta \\ t &= \alpha\ (\sinh\eta - \eta) + \beta\ (\cosh\eta - 1) \end{aligned} \right\} \ , \tag{I.C.14}$$

We have set $dt = R\, d\eta$ (η is a dimensionless variable). The non-negative constants, α and β, are defined as:

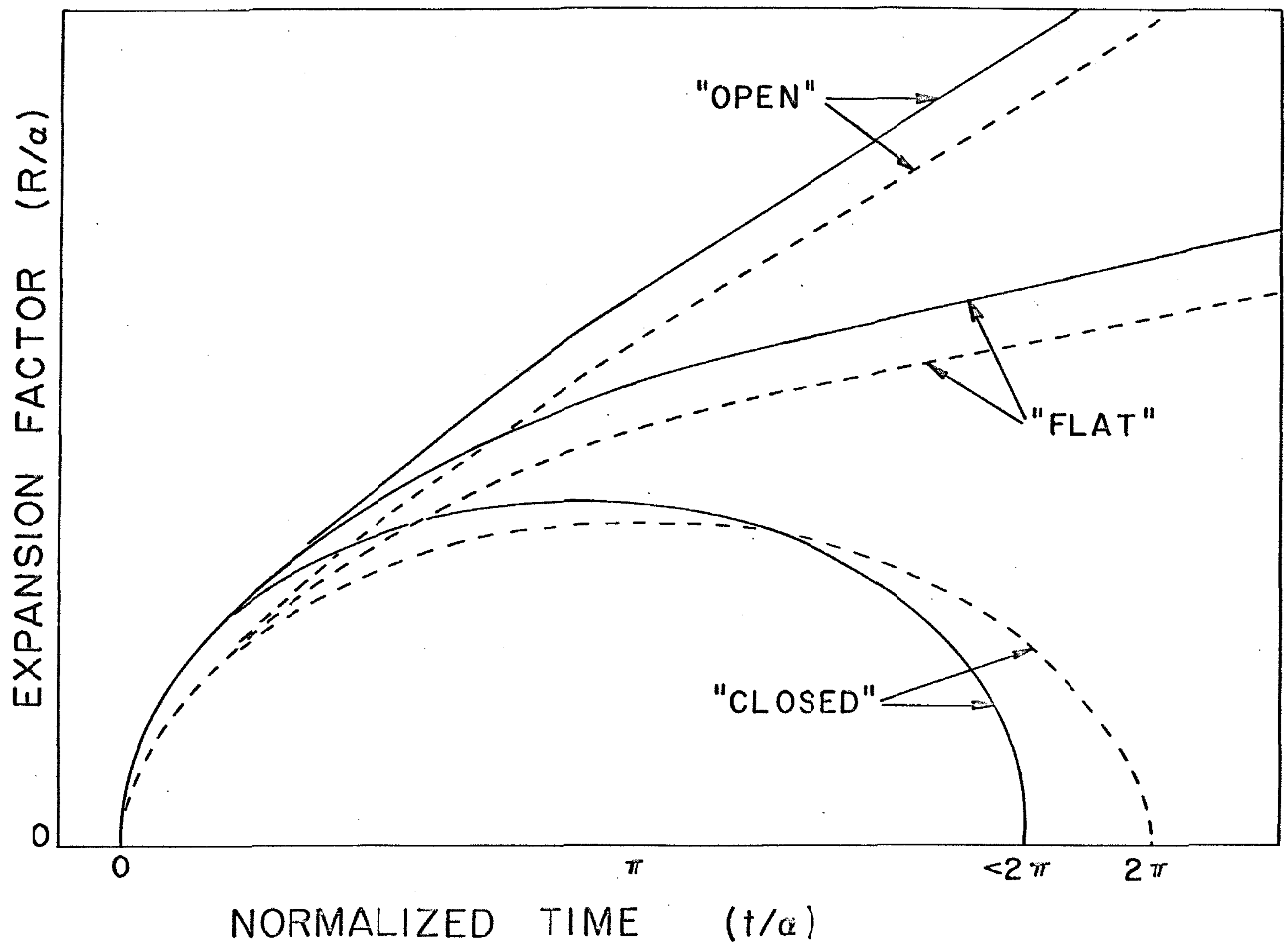
$$\alpha = (4\pi/3)\, \rho_{do}\, R_o^{\,3} \quad , \quad \beta^2 = (8\pi/3)\, \rho_{ro}\, R_o^{\,4} \ . \tag{I.C.15}$$

The analytical solutions, equations (I.C.12) through (I.C.14), are called Lemaître models since cosmological models containing both "dust" and non-interacting "radiation" were first studied by Lemaître (1927, 1930, 1931). When $\beta = 0$ these solutions are known as Friedmann models, since Friedmann (1922, 1924) first considered the homogeneous, isotropic models containing only "dust". Tolman (1934) was the first to study the models containing only "radiation"; hence, the solutions are called Tolman models when $\alpha = 0$. Equation (I.C.13) with $\alpha \neq 0$ and $\beta = 0$ is the well-known Einstein-de Sitter (1932) model.

The properties of the "standard" Lemaître models are now well-known. Their temporal evolution is depicted schematically in Figure 1. As $\eta \to 0$ they all behave like the "flat" model of equation (I.C.13). All of the models emerge from an initial physical singularity (where $R \to 0$ and $\rho \to \infty$) at $t = 0$. The "open" and "flat" models expand outward forever, but the "closed" models expand to a maximum radius and then recontract to a final physical singularity after a finite proper time. All of the physical characteristics of the "standard" Lemaître models may be derived, albeit tediously, from equations (I.C.12) through (I.C.14). An approximation procedure which greatly simplifies calculations is suggested by the observational data. From § I.B. we see that our Universe is probably characterized by $\rho_{ro}/\rho_{do} \lesssim 10^{-3}$ (i.e., $\beta/\alpha << 1$). This implies that approximate models can be constructed by beginning with a Tolman "radiation"

FIGURE 1

The temporal behavior (schematic) of the distance scale factor, R(t), in the "standard" Lemaître cosmological models of our Universe. All such models are spatially homogeneous and isotropic, and they are filled with a uniform perfect fluid consisting of "dust" ($\gamma = 0$) and/or "radiation" ($\gamma = 1/3$). The designation ("closed", "flat", "open") implies that the curvature of the space sections is (positive, zero, negative). The "flat" and "open" models expand forever after emerging from the initial physical singularity at $t = 0$, while the "closed" models encounter a second-and final-singularity after a finite proper time. The dashed lines represent Friedmann models, which contain only "dust". The solid lines represent general Lemaître models containing both "dust" and "radiation". Note how, for fixed α, the addition of "radiation" shortens the life-time of the "closed" model. The quantity α is a non-negative constant related to the mass-density of "dust" today [cf. equation (I.C.15)].


"OPEN"
"FLAT"
"CLOSED"
EXPANSION FACTOR (R/α)
0
0
π
<2π
2π
NORMALIZED TIME (t/α)

model at the initial singularity, joining it smoothly (with R and dR/dt continuous) to a later Friedmann "dust" model at the point where $\rho_r = \rho_d$, and continuing with the Friedmann model to the present time. This procedure is extremely accurate everywhere except near the join point, as demonstrated for the "flat" model in Figure 2.

To select the "standard" Lemaître model appropriate to our Universe we must resort to the observational data: (1) Equation (I.C.6) immediately tells us that:

$$\rho_o \gtreqless (3/8\pi)\, H_o^{\,2} \equiv \rho_{co} \quad \text{when } \kappa \gtreqless 0 \quad , \qquad \text{(I.C.16)}$$

and

$$q_o \gtreqless (1/2)[1 + 2(\rho_{ro}/\rho_{do})]\,[1 + (\rho_{ro}/\rho_{do})]^{-1} \equiv q_{co} \text{ when } \kappa \gtreqless 0, \qquad \text{(I.C.17)}$$

where H is the Hubble expansion rate defined by equation (I.B.1) and q is the deceleration parameter defined by equation (I.B.2). Here the subscript zero (o) denotes the present value of a quantity. The <u>critical</u> values of the total mass-density, ρ_{co}, and of the deceleration parameter, q_{co}, are the values for the "flat" model (where $\kappa = 0$). From the observations $H_o^{-1} \approx (13 \pm 5) \times 10^9$ years and, most probably, $(\rho_{ro}/\rho_{do}) << 1$. Hence, the critical values are:

$$\rho_{co} \approx 10^{-29}\ \text{gm cm}^{-3} \quad , \quad q_{co} \approx 1/2 \quad . \qquad \text{(I.C.18)}$$

The observed luminous matter and 2.7 $^{\circ}$K cosmic microwave radiation imply $\rho_o < \rho_{co}$, while the limits on undetected dark matter, ionized hydrogen (H II), neutrinos, and gravitons imply the distinct possibility that $\rho_o \gtrsim \rho_{co}$. According to Sandage (1961a), the value of q_o lies in the range $0 \lesssim q_o \lesssim 2$, with the most likely value being $q_o = 1 \pm 1/2$. Therefore, the tests implied by equations (I.C.16) and (I.C.17) are, unfortunately, inconclusive at present.

(2) From equations (I.C.12) through (I.C.15) and Figure 1 we see that the present age of our Lemaître models is:

$$t_o \begin{matrix} < \\ \approx \\ > \end{matrix} (2/3)\, H_o^{-1} \equiv t_{co} \quad \text{when} \quad \varkappa \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad . \qquad \text{(I.C.19)}$$

Since $H_o^{-1} = (13 \pm 5) \times 10^9$ years, the critical value is:

$$t_{co} = (8.7 \pm 3.3) \times 10^9 \text{ years} \quad . \qquad \text{(I.C.20)}$$

Because of the great uncertainty in the evolutionary ages of elliptical galaxies and globular clusters, this test [equation (I.C.19)] is also inconclusive.

(3) Wagoner, Fowler, and Hoyle (1967, and the references cited therein) have studied primordial element production in the primeval fireball of the "standard" Lemaître models. With ρ_o in the observationally allowed range, 10^{-31} gm cm$^{-3} \lesssim \rho_o \lesssim 10^{-28}$ gm cm^{-3}, they find that the primordial helium abundance produced by the fireball should be 20% - 30% by mass (as long as the lepton number of

our Universe does not deviate significantly from zero). The present observations concerning primordial helium are still too crude to tell us the sign of $\varkappa$.

Therefore, current astronomical observations cannot tell us with any certainty whether our Universe is best described by the "closed", "flat", or "open" Lemaître models. If we still desire to make a choice, we can only fall back on such intangibles as whims of fashion, philosophical predilections, intuitive preferences, and/or a desire for ease of mathematical computation. In Chapter II I shall be investigating cosmological models of Bianchi Type I, in an effort to understand the properties of cosmic shear (anisotropic expansion). All Bianchi Type I cosmologies are generalizations of the "flat" (zero curvature) isotropic models, an example of which is the "standard" Lemaître model of equation (I.C.13). For this reason I choose to discuss here, in somewhat more detail than above, the properties of "flat" isotropic cosmologies.

From equation (I.C.2) we see that the "flat" isotropic metric can be written as:

$$ds^2 = dt^2 - R^2(t) \left(dx^2 + dy^2 + dz^2\right) , \qquad \text{(I.C.21)}$$

where (x, y, z) are Cartesian coordinates and where the scale factor, R, is now dimensionless. Equations (I.C.6) and (I.C.7) can now be written as:

$$\left[\frac{d(R/R_o)}{dt}\right]^2 = \left(\frac{8\pi}{3}\right) \rho \left(\frac{R}{R_o}\right)^2 \quad , \qquad \text{(I.C.22)}$$

$$\frac{d[\rho(R/R_o)^3]}{dt} + p \frac{d[(R/R_o)^3]}{dt} = 0 \quad , \qquad \text{(I.C.23)}$$

where the subscript zero (o) denotes the present value of a quantity. It is very useful to study, in addition to the Lemaître models, also models containing perfect-fluid matter that obeys the general equation of state:

$$p = \gamma \rho \quad (\gamma = \text{constant}, \ 0 \leq \gamma \leq 1)^5 \quad . \qquad \text{(I.C.24)}$$

Equations (I.C.23) and (I.C.24) then imply:

$$(\rho/\rho_o) = (R/R_o)^{-3(1 + \gamma)} \quad . \qquad \text{(I.C.25)}$$

The analytical solution to equation (I.C.22) becomes:

---

[5]The demand that the pressure be non-negative implies $\gamma \geq 0$, while $\gamma > 1$ is forbidden by causality (see Harrison, Thorne, Wakano, and Wheeler 1965, pp. 105-106). Zel'dovich (1961) argues that $1/3 < \gamma \leq 1$ is possible, but Harrison (1965) supports the popular consensus that only $0 \leq \gamma \leq 1/3$ occurs in Nature.

$$(R/R_o) = (t/t_o)^{2/[3(1+\gamma)]} , \qquad (I.C.26)$$

where the present age of a given model is $t_o \equiv [6\pi(1+\gamma)^2\rho_o]^{-1/2}$. The Hubble expansion rate, H, and the deceleration parameter, q, follow from equation (I.C.26):

$$H \equiv (1/R)\,(dR/dt) = 2/[3(1+\gamma)\,t] , \qquad (I.C.27)$$

$$q \equiv -\,(1/RH^2)\,(d^2R/dt^2) = (1+3\,\gamma)/2 . \qquad (I.C.28)$$

Note that $t_o = 2/[3\,H_o\,(1+\gamma)]$.

The Einstein-de Sitter (1932) "dust" model follows from equations (I.C.24) through (I.C.28) when $\gamma = 0$. If such a model is an adequate representation of our Universe, we have $p = 0$, $(\rho/\rho_o) = (R/R_o)^{-3}$, $(R/R_o) = (t/t_o)^{2/3}$, $q_o = 1/2$, $t_o = (2/3)\,H_o^{-1} \approx 9 \times 10^9$ years, and $\rho_o = \rho_{co} \approx 10^{-29}$ gm cm^{-3}.

The "flat" isotropic model can better describe our Universe if we assume that the material content is a mixture of several non-interacting perfect fluids. Let us take $\rho = \sum_i \rho_i$ and $p_i = \gamma_i\,\rho_i$, where the subscript i denotes the ith species of fluid. Then equation (I.C.23) implies:

$$(\rho_i/\rho_{io}) = (R/R_o)^{-3(1+\gamma_i)} , \qquad (I.C.29)$$

and equation (I.C.22) yields the temporal behavior:

$$(t/t_{co}) = (3/2) \int (R/R_o) \left[\Sigma_i (\rho_{io}/\rho_{co}) \, (R/R_o)^{3(1 - \gamma_i)} \right]^{-1/2} d(R/R_o). \tag{I.C.30}$$

Here $\rho_{co} \approx 10^{-29}$ gm cm^{-3} is the critical mass-density, and $t_{co} \equiv (6\pi \, \rho_{co})^{-1/2} \approx 9 \times 10^9$ years the critical age, of the "flat" model. Assuming that our fluid species are only "dust" ($\gamma = 0$) and "radiation" ($\gamma = 1/3$), we re-derive the "flat" Lemaître model of equation (I.C.13) in the form:

$$(t/t_{co}) = Q_o^{-2} \, \{[Q_o(R/R_o) - 2 \, S_o] \, [Q_o(R/R_o) + S_o]^{1/2} + 2 \, S_o^{3/2}\} \, . \tag{I.C.31}$$

Here we retain the generality of equation (I.C.30) by disregarding the observational data for the moment and by setting $Q_o \equiv \rho_{do}/\rho_{co}$ (this is the ratio of the mass-density of the "dust" to the critical mass-density today) and $S_o \equiv \rho_{ro}/\rho_{co}$ (this is the same ratio for "radiation" today). In the "flat" Lemaître model of equation (I.C.31) we can also find the total mass-density ρ, the total pressure p, the Hubble expansion rate H, and the deceleration parameter q:

$$\left.\begin{aligned}
(\rho/\rho_{co}) &\equiv (\rho_d + \rho_r)/\rho_{co} = (R/R_o)^{-4}\,[Q_o(R/R_o) + S_o] \quad , \\
(p/\rho_{co}) &\equiv (p_r/\rho_{co}) = (S_o/3)\,(R/R_o)^{-4} \quad , \\
(H\,t_{co}) &= (2/3)\,(R/R_o)^{-2}\,[Q_o(R/R_o) + S_o]^{1/2} = (2/3)\,(\rho/\rho_{co})^{1/2} , \\
q &= (1/2)\,[Q_o(R/R_o) + 2\,S_o]/[Q_o(R/R_o) + S_o] \quad .
\end{aligned}\right\} \text{(I.C.32)}$$

Present observational data indicate only that $\rho_{do} \gtrsim 10^{-30}$ gm cm^{-3} and $\rho_{ro} \gtrsim 5 \times 10^{-34}$ gm cm^{-3}. The "flat" Lemaître models specified by equations (I.C.31) and (I.C.32) are constrained by observations to lie between the following two extreme cases: (1) The observed luminous matter is augmented by undetected dark matter and ionized hydrogen (H II) so that $\rho_{do} \cong \rho_{co}$, and the "radiation" consists only of the observed 2.7 ^{o}K cosmic microwave radiation with $\rho_{ro} \cong 5 \times 10^{-34}$ gm cm^{-3}. This is the most widely used and popular model at present, since there are physical arguments (see Chiu 1967; Misner 1967, 1968) indicating that undetected neutrinos and gravitons should have a total mass-density of the order of ρ_{ro} today. (2) Only the detected luminous matter contributes to $\rho_{do} \approx 10^{-30}$ gm cm^{-3}, while the 2.7 ^{o}K cosmic microwave radiation is augmented by undetected neutrinos and gravitons so that $\rho_{ro} \cong \rho_{co}$. In Figure 2 we display this range of "flat" Lemaître models which is consistent with the available data.

An approximation to these "flat" Lemaître models, which is oftentimes mathematically convenient, is to smoothly (with R and dR/dt continuous) join an earlier pure "radiation" model ($\gamma = 1/3$) to a later pure "dust" model ($\gamma = 0$) at the point where $\rho_r = \rho_d$. In this case the scale factor goes as:

$$(R/R_o) = \left\{ \begin{matrix} (16\ S_o/9)^{1/4}\ (t/t_{co})^{1/2} \\ \\ [Q_o^{1/2}(t/t_{co}) + (1/4)(S_o/Q_o)^{3/2}]^{2/3} \end{matrix} \right\} \text{when } (t/t_{co}) \lesseqgtr (3/4) Q_o^{-2} S_o^{3/2}. \qquad \text{(I.C.33)}$$

All other quantities of interest may be easily computed from equation (I.C.33). The excellence of this approximation procedure is illustrated in Figure 2.

Figure 2

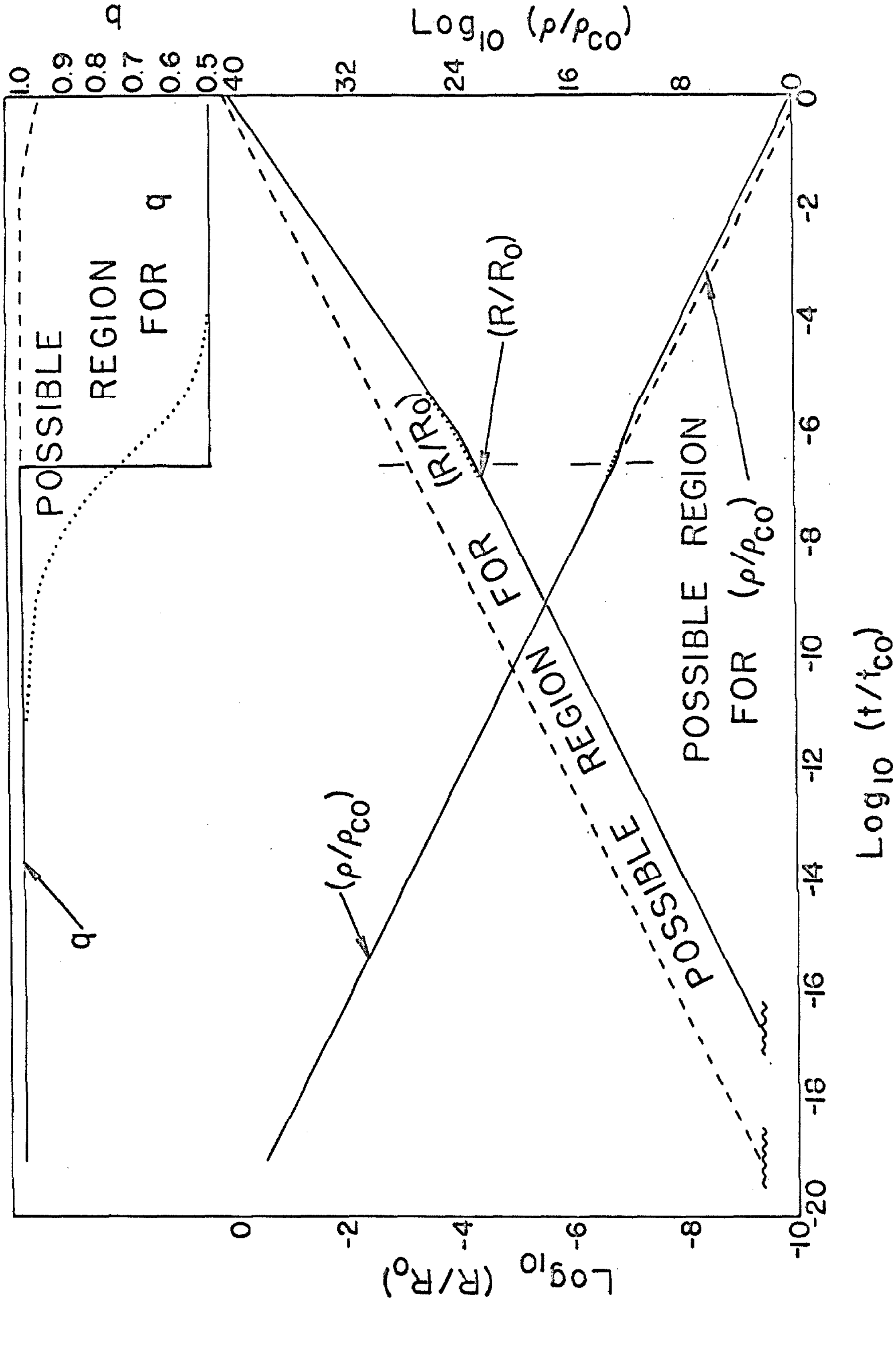
The explicit temporal evolution of all possible "flat" Lemaître cosmological models of our Universe which are consistent with present observations. We have plotted the normalized scale factor (R/R_o), the normalized total mass-density (ρ/ρ_{co}), and the deceleration parameter (q) versus normalized time (t/t_{co}). The subscript zero (o) refers to the present value of a quantity. The normalizing parameters are R_o, the present scale factor; $\rho_{co} \equiv (3/8\pi)\, H_o^2 = 10^{-29}$ gm cm^{-3}, the present critical mass-density necessary to close the Universe [where H_o is the present Hubble expansion rate, $H_o^{-1} = 13 \times 10^9$ years]; and $t_{co} \equiv (6\pi\ \rho_{co})^{-1/2} = 8.7 \times 10^9$ years, the present critical age of the model.

The <u>solid</u> lines represent the model in which an earlier "radiation" model is joined smoothly (R and H continuous) to a later "dust" model at the point where $\rho_r = \rho_d$; in this model we choose $\rho_{do} \equiv \rho_{co}$ and $\rho_{ro} = 5 \times 10^{-34}$ gm cm^{-3} (the observed 2.7 ^{o}K cosmic microwave radiation). The <u>solid vertical</u> lines at $(t/t_{co}) = 2.7 \times 10^{-7}$ mark the transition from the "radiation" phase to the "dust" phase.

The <u>dotted</u> lines represent the exact Lemaître model of Jacobs (1967) with the same present parameters as the <u>solid</u> line model. Note how closely the two models match (the dotted lines are coincident with the <u>solid</u> lines everywhere that the former are not visible).

The dashed lines represent the other limiting case (wherein the "radiation", and not the "dust", provides the critical mass-density) with present parameters $\rho_{ro} \equiv \rho_{co}$ and $\rho_{do} = 10^{-30}$ gm cm^{-3} (the present observed luminous mass-density). In this model the "hidden radiation" could be either neutrinos or gravitons.

All possible "flat" Lemaître models of our Universe must lie in regions indicated between the dotted and the dashed lines. Note that the appearance of electron-positron pairs at $(R/R_o) \approx 5 \times 10^{-10}$ marks the lower limit of validity of all these models.


q
Log10 (ρ/ρco)
1.0
0.9
0.8
0.7
0.6
0.5
40
32
24
16
8
0
POSSIBLE
REGION
FOR q
(R/R0)
(R/R0)
POSSIBLE REGION FOR (R/R0)
POSSIBLE REGION FOR (ρ/ρco)
(ρ/ρco)
(ρ/ρco)
q
0
-2
-4
-6
-8
-10
-20
-18
-16
-14
-12
-10
-8
-6
-4
-2
0
Log10 (t/tco)
Log10 (R/R0)

II. BIANCHI TYPE I COSMOLOGICAL MODELS

"I have a bit of FIAT in my soul,
And can myself create my little world".

Thomas Lovell Beddoes, Death's Jest Book,
Act V, Scene i, Line 38.

II. A. ANISOTROPY AND BIANCHI TYPE I COSMOLOGIES

In Chapter I we gave a brief sketch of the simplest cosmological models of our Universe within the context of standard general relativity theory: the "standard" Lemaître models. These models followed readily from our basic assumptions of perfect spatial homogeneity and isotropy. These two basic assumptions were indicated by the observational data, but they represent an extrapolation and idealization of that data. In Chapter I we also saw some reasons for abandoning the assumption of perfect spatial isotropy and for considering cosmologies with spatial anisotropy. Here, in Chapter II, we will implement an investigation of spatially homogeneous, but anisotropic, cosmological models of our Universe.

In this thesis our fundamental objectives are twofold: (a) to gain as complete and clear an understanding of anisotropic expansion within the context of standard general relativity theory as possible with a minimum of mathematical complications, and (b) to use this understanding to construct anisotropic cosmological models which adequately describe our Universe. Our models will be physically realistic only if they closely resemble the "standard" Lemaître models at the present stage of evolution of our Universe. This criterion and the desire for mathematical tractability (see below) delimit the class of cosmologies that we shall consider here.

In order to choose a suitable class of cosmologies to study, we proceed as follows:

a) We first demand that the cosmologies be spatially homogeneous. This maneuver is justifiable observationally due to the overall homogeneity of the luminous matter in our Universe and due to the spatial homogeneity implied by the extreme isotropy of the 2.7 ^{o}K cosmic microwave radiation. This demand is philosophically appealing since it implies that we do not occupy a unique position in our Universe. The most important justification for this limitation, however, is that it allows us to avoid the well-known mathematical difficulties inherent in inhomogeneous cosmologies.

b) Spatial homogeneity implies that space-time consists of a family of space-like hypersurfaces. These hypersurfaces define a "cosmic time". All "observers" within a given hypersurface see exactly the same surroundings. This means that the hypersurfaces must admit transitive groups of motions. From Saunders (1967) we see that this symmetry restriction implies that we need only consider three- and four-parameter isometry groups. These isometry groups consist essentially of two kinds. The four-parameter groups admitting no simply transitive subgroups have been studied extensively by Oszvath (1962), Doroshkevich (1965), Kantowski and Sachs (1966), and Thorne (1965, 1967). We shall not consider these cosmologies further here. Bianchi (1918) and Behr (see Estabrook, Wahlquist, and Behr 1968) have given equivalent classifications of the three-parameter simply transitive groups in three-space. There are nine algebraically inequivalent types of such groups, and they are designated Bianchi Type I through IX. The structure constants of these nine distinct Lie groups may be found in the works of Bianchi (1918), Taub (1951), and Petrov (1961). Hereafter we shall consider only these Bianchi Type cosmologies.

c) Within the context of the Bianchi classification, the generalizations of the "closed", "flat", and "open" Lemaître models are cosmologies of Bianchi Type IX, I, and V, respectively. In order to most closely mimic the present behavior of the Lemaître models we shall restrict ourselves to these three Bianchi types.

d) In § I.C. we saw that the observational data do not yet permit us to conclude whether our Universe is best described by the "closed", "flat", or "open" Lemaître model. Hence, Bianchi Type I, V, and IX cosmologies are all equally valid candidates for consideration. We finally choose Bianchi Type I as the basis of our investigation on the grounds of mathematical simplicity. This simplicity is best illustrated by the "flat" Lemaître solution of equation (I.C.13). In the investigation which follows we shall see that this choice is, indeed, a good one.

In the remainder of this section (§ II.A.) we will consider the general properties of Bianchi Type I cosmologies. In general, the Riemannian metric of space-time is:

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \quad . \qquad \text{(II.A.1)}$$

According to Taub (1951) and Heckmann and Schücking (1962), spatial homogeneity means that there is a three-parameter, simply transitive group of motions operating upon minimum invariant varieties which are three-dimensional spacelike hypersurfaces. Eisenhart (1926) and Taub (1951) show that these invariant hypersurfaces are geodesically parallel, and that they define a "cosmic time" via a hypersurface-orthogonal timelike congruence of geodesics. Then from the Appendix of Heckmann and Schücking (1962), we see that the metric can be put into the form:

$$ds^2 = dt^2 - g_{ij} \, (x^\mu) \, dx^i \, dx^j \quad . \qquad \text{(II.A.2)}$$

Here the g_{ij} are the coefficients of a positive definite quadratic form, since our convention is that the metric of equation (II.A.1) has signature -2. Taub (1951) and Heckmann and Schücking (1962) show that we can always make the decomposition:

$$g_{ij}\,(x^{\mu}) \;=\; \gamma_{ab}(t)\; e^{a}_{i}\,(x^{k})\; e^{b}_{j}\,(x^{k}) \qquad , \qquad \text{(II.A.3)}$$

where the $e^{c}_{\ell}\,(x^{k})$ are determined by the group of motions (isometries), and where the evolution of the $\gamma_{ab}(t)$ is governed by the Einstein field equations.

The isometry group of Bianchi Type I is the three-parameter group of translations in Euclidean three-space. It is an Abelian group, and hence, all of its structure constants are zero. Bianchi (1918) demonstrated that the invariant hypersurfaces then have zero curvature ("flat" space), and that we can always choose $e^{a}_{i} = \delta^{a}_{i}$ [see also Heckmann and Schücking 1962]. Therefore, the most general Bianchi Type I metric may be written as:

$$ds^{2} \;=\; dt^{2} \;-\; \gamma_{ij}(t)\; dx^{i}\, dx^{j} \qquad , \qquad \text{(II.A.4)}$$

where $\gamma_{ij}(t)$ is a symmetric, 3 × 3 matrix.

The Einstein field equations (with vanishing cosmological constant) are:

$$G_{\mu\nu} \;\equiv\; R_{\mu\nu} - (1/2)\, g_{\mu\nu}\, R = 8\pi\; T_{\mu\nu} \qquad , \qquad \text{(II.A.5)}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the contracted Riemann tensor, and $T_{\mu\nu}$ is the material stress-energy tensor. Equation (II.A.5) may be inverted to give the form:

$$R_{\mu\nu} = 8\pi\ [T_{\mu\nu} - (1/2)\ g_{\mu\nu}\ T] \quad . \qquad \text{(II.A.6)}$$

For the most general Bianchi Type I metric of equation (II.A.4), a full set of field equations is the following [see, e.g., equations (11-A-2.52) and (11-A-2.53) of Heckmann and Schücking 1962; also equations (99.10) through (99.12) of Landau and Lifshitz 1962]:

$$8\pi\ T_{00} = G_{00} = (1/8)[(\gamma^{ij}\ \dot{\gamma}_{ij})^2 - \gamma^{\ell m}\ \dot{\gamma}_{mk}\ \gamma^{kj}\ \dot{\gamma}_{jl}] \quad , \qquad \text{(II.A.7.a)}$$

$$8\pi\ T_{0j} = G_{0j} = 0 \quad , \qquad \text{(II.A.7.b)}$$

$$8\pi[T_{ij} - (1/2)g_{ij}\ T] = R_{ij} = -\ (1/2)[\ddot{\gamma}_{ij} - \dot{\gamma}_{im}\gamma^{mk}\dot{\gamma}_{kj} + (1/2)\dot{\gamma}_{ij}\gamma^{km}\dot{\gamma}_{km}] \ , \qquad \text{(II.A.7.c)}$$

where a dot renotes differentiation with respect to proper time, t. According to equation (II.A.7.b) there can be absolutely no energy flow relative to the particular coordinates [cf., equation (II.A.4)] of a Bianchi Type I cosmology.

The coordinate system of equation (II.A.4) is in part arbitrary. Consider a particular moment of time, t_o. On the hypersurface $t = t_o$ we can choose our space coordinates to be a set of Cartesian coordinates measuring proper distance, so that:

$$\gamma_{i'j'}\ (t_o) = \delta_{i'j'} \quad . \qquad \text{(II.A.8)}$$

We can then examine the time rate of change of the metric, $\dot{\gamma}_{i'j'}(t_o)$, in this coordinate system. By a pure rotation of space axes on the hypersurface $t = t_o$ we can bring $\dot{\gamma}_{i'j'}$ into the diagonal form[1]:

$$\dot{\gamma}_{i'j'}(t_o) = \dot{\gamma}_{i'i'}(t_o)\,\delta_{i'j'} \text{ (no sum)} \quad . \qquad \text{(II.A.9)}$$

A complete set of initial value data for Einstein's field equations (II.A.7) on this hypersurface will be T_{00}, $T_{i'j'}$, and $\dot{\gamma}_{i'i'}$. These data are constrained by the initial value equation (II.A.7.a), but are otherwise arbitrary. The evolution of the metric coefficients as one moves off the hypersurface $t = t_o$ is determined by the initial value data and by the dynamical equations (II.A.7.c.):

$$\ddot{\gamma}_{ij}(t_o) = -16\pi[T_{ij} + (1/2)\gamma_{ij}T] + \dot{\gamma}_{im}\gamma^{mk}\dot{\gamma}_{kj} - (1/2)\dot{\gamma}_{ij}\gamma^{km}\dot{\gamma}_{km} . \qquad \text{(II.A.10)}$$

In § II.D. we will be working with cosmological models filled with perfect fluid, for which the stress-energy tensor is:

$$T^{(PF)}_{\alpha\beta} = (\rho + p)\,u_\alpha u_\beta - p g_{\alpha\beta} \quad . \qquad \text{(II.A.11)}$$

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[1]This follows from the fact that $\gamma_{ij}(t)$, and hence $\dot{\gamma}_{ij}(t)$, are symmetric matrices. Symmetric matrices always have three orthogonal principal axes.

Here u_α is the covariant four-velocity of the perfect fluid in the coordinate system of equation (II.A.4), and ρ is the total density of mass-energy and p is the isotropic pressure of the fluid as measured in the proper reference frame of the fluid. For a perfect fluid, energy flow is absent[cf., equation (II.A.7.b)] if and only if the fluid is at rest in the coordinate system of equation (II.A.4):

$$u_j = u^j = 0, \quad u^0 = u_0 = 1 \quad . \tag{II.A.12}$$

Note that we have made the normalization, $u^\mu u_\mu = 1$, here. Therefore, for Bianchi Type I cosmologies filled with perfect fluid we must have:

$$T^{(PF)}_{00} = \rho(t) \quad , \quad T^{(PF)}_{0j} = 0 \quad , \quad T^{(PF)}_{ij} = p(t)\ \gamma_{ij} \quad . \tag{II.A.13}$$

For the perfect fluid of equation (II.A.13), equation (II.A.10) reads

$$\ddot{\gamma}_{ij}\ (t) = -\ 8\pi\ (\rho - p)\ \gamma_{ij} + \dot{\gamma}_{im}\gamma^{mk}\dot{\gamma}_{kj} - (1/2)\dot{\gamma}_{ij}\gamma^{km}\dot{\gamma}_{km} \quad . \tag{II.A.14}$$

It is easy to verify that the initial data of equations (II.A.8), (II.A.9), and (II.A.13) guarantee --- via equation (II.A.14) --- that $\dot{\gamma}_{ij}$ and γ_{ij} will remain diagonal for all time. Hence, <u>for any perfect-fluid cosmology of Bianchi Type I the metric can always be put into the diagonal form</u>

$$ds^2 = dt^2 - [A^2(t)\ dx^2 + B^2(t)\ dy^2 + C^2(t)\ dz^2] \quad , \tag{II.A.15}$$

and the stress-energy tensor takes the form

$$T^0{}_0 = \rho(t) \quad , \quad T^0{}_j = 0 \quad , \quad T^i{}_j = -\, p(t)\, \delta^i{}_j \quad . \qquad \text{(II.A.16)}$$

In § II.E. we will examine cosmologies of Bianchi Type I with a source-free cosmological magnetic field present, along with perfect fluid. Maxwell's equations for such a sourceless magnetic field and any associated electric fields read:

$$F^{\alpha\beta}{}_{;\beta} = {}^*F^{\alpha\beta}{}_{;\beta} = 0 \quad , \qquad \text{(II.A.17)}$$

where $F^{\alpha\beta}$ and ${}^*F^{\alpha\beta}$ are the electromagnetic field tensor and its dual. In the most general Bianchi Type I metric [equation (II.A.4)] these equations become expressions for the conservation of electric and magnetic flux through a given coordinate region:

$$\left.\begin{aligned} \gamma^{-1/2}\,(\gamma^{1/2}\,F^{\alpha\beta})_{,\beta} &= \gamma^{-1/2}(\gamma^{1/2}\,F^{\alpha 0})_{,0} = 0 \\ \gamma^{-1/2}\,(\gamma^{1/2}\,{}^*F^{\alpha\beta})_{,\beta} &= \gamma^{-1/2}(\gamma^{1/2}\,{}^*F^{\alpha 0})_{,0} = 0 \end{aligned}\right\} , \qquad \text{(II.A.18.a)}$$

or equivalently

$$\left.\begin{aligned} \gamma^{1/2}\,F^{j0} &= E^j = \text{constant throughout all spacetime} \\ \gamma^{1/2}\,{}^*F^{j0} &= B^j = \text{constant throughout all spacetime} \end{aligned}\right\} . \qquad \text{(II.A.18.b)}$$

Here γ is the determinant of $\gamma_{ij}(t)$. The stress-energy tensor for this electromagnetic field has components

$$4\pi\, T^{(EM)}_{00} = (\gamma_{jk}/2\gamma)[E^j E^k + B^j B^k] \quad , \qquad \text{(II.A.19.a)}$$

$$4\pi\, T^{(EM)}_{0j} = -\, \gamma^{-1/2}\, \epsilon_{0\ell mj}\, E^\ell B^m \quad , \qquad \text{(II.A.19.b)}$$

$$4\pi\, T^{(EM)}_{jk} = [-\, \gamma_{j\ell}\gamma_{km} + (1/2)\, \gamma_{jk}\gamma_{\ell m}]\, \gamma^{-1}\, [E^\ell E^m + B^\ell B^m] \quad , \qquad \text{(II.A.19.c)}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric symbol with $\epsilon_{0123} \equiv +1$. If both perfect fluid and electromagnetic field are present, then equation (II.A.7.b) implies that the sum of their energy flows must vanish:

$$T^{(EM)}_{0j} + T^{(PF)}_{0j} = (4\pi\gamma^{+1/2})^{-1}\epsilon_{0\ell mj}\, E^\ell B^m + (\rho + p)\, u_0 u_j = 0 \quad . \qquad \text{(II.A.20)}$$

This will generally be impossible unless the two energy flows vanish individually, since the time dependence of the two will generally be different. Vanishing of $T^{(PF)}_{0j}$ leads again to the fluid being at rest [cf., equation (II.A.12)] in the coordinate system of equation (II.A.4). Vanishing of $T^{(EM)}_{0j}$ forces the electric and magnetic fields --- via equation (II.A.19b) --- to be parallel[2]:

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[2]The energy flow in equation (II.A.19.b) is the Poynting vector, $T^{(EM)}_{0j}(t_o) \propto \underset{\sim}{E} \times \underset{\sim}{B}$. This vector cross-product vanishes only when either $\underset{\sim}{E}$ or $\underset{\sim}{B}$ vanishes or when $\underset{\sim}{E}$ is parallel to $\underset{\sim}{B}$.

$$E^i = hB^i \quad , \quad h = \text{constant}. \tag{II.A.21}$$

From equations (II.A.19) we can easily verify that the trace of the electromagnetic stress-energy tensor always vanishes identically, $T^{(EM)} \equiv 0$. Now the dynamical equations for $\gamma_{ij}(t)$ [cf., equation (II.A.10)] read:

$$\ddot{\gamma}_{ij}(t) = -8\pi(\rho - p)\gamma_{ij} + (4\gamma_{i\ell}\gamma_{jm} - 2\gamma_{ij}\gamma_{\ell m})\gamma^{-1}(1 + h^2)B^\ell B^m$$

$$+ \dot{\gamma}_{im}\gamma^{mk}\dot{\gamma}_{kj} - (1/2)\dot{\gamma}_{ij}\gamma^{km}\dot{\gamma}_{km} \quad . \tag{II.A.22}$$

The initial value equation (II.A.7.a) places no constraint upon either the magnitude or the direction of the magnetic field at time $t = t_o$. If the magnetic field [and hence, from equation (II.A.21), also the electric field] happens to lie along one of the coordinate axes [i.e., principal axes of $\dot{\gamma}_{ij}$; i.e., "principal shear directions" (see below)] at time t_o, then $\ddot{\gamma}_{ij}$ will be diagonal at time t_o, and equation (II.A.22) will guarantee that γ_{ij}, $\dot{\gamma}_{ij}$, and $\ddot{\gamma}_{ij}$ always remain diagonal. But, <u>if the magnetic field is not along a principal shear axis at time t_o, then $\ddot{\gamma}_{ij}(t_o)$ will not be diagonal, and it will be impossible to diagonalize the metric [as in equation (II.A.15)] for all time</u>.

We shall henceforth take the electric field to be zero for all time:

$$E^i = 0 \quad , \tag{II.A.23}$$

since no large-scale cosmic electric field could exist in the early, ionized stage of our Universe. For the sake of simplicity, we shall confine our attention in the remainder of this thesis to cosmic magnetic fields aligned along a shear axis. Hence, $\gamma_{ij}(t)$ will always be diagonal, and the metric will always assume the standard diagonal form of equation (II.A.15). No other investigation of anisotropic Bianchi Type I cosmologies to date has been more general than this.

As a consequence of our analysis (above), we have restricted ourselves to the consideration of the diagonal Bianchi Type I metric:

$$ds^2 = dt^2 - [A^2(t)\, dx^2 + B^2(t)\, dy^2 + C^2(t)\, dz^2] \quad . \tag{II.A.24}$$

In general, the streamlines of the motion of a cosmic fluid are characterized kinematically by their "expansion" (Θ), "shear" ($\sigma_{\mu\nu}$), and "rotation" ($\omega_{\mu\nu}$), defined by:

$$\left.\begin{aligned} \Theta &\equiv (1/3)\, u^{\mu}{}_{;\mu} \\ \sigma_{\mu\nu} &\equiv (1/2)\, (u_{\mu;\nu} + u_{\nu;\mu}) - \Theta\, (g_{\mu\nu} - u_{\mu}u_{\nu}) \\ \omega_{\mu\nu} &\equiv (1/2)\, (u_{\mu;\nu} - u_{\nu;\mu}) \end{aligned}\right\} . \tag{II.A.25}$$

From equations (II.A.12) and (II.A.24) we immediately find:

$$\omega_{\mu\nu} = 0 \quad , \tag{II.A.26.a}$$

and

$$\Theta = (1/3)\ [(\dot{A}/A) + (\dot{B}/B) + (\dot{C}/C)] \quad . \tag{II.A.26.b}$$

Hence, all cosmic "rotation" vanishes, and the cosmic "expansion" is just the average Hubble expansion rate of our cosmologies. The "shear" tensor, which represents the expansion anisotropies, is readily found to have only the mixed, physical, spatial components:

$$\left.\begin{aligned} \sigma^{(x)}{}_{(x)} &= (1/3)\ [2\ (\dot{A}/A) - (\dot{B}/B) - (\dot{C}/C)] \\ \sigma^{(y)}{}_{(y)} &= (1/3)\ [-\ (\dot{A}/A) + 2\ (\dot{B}/B) - (\dot{C}/C)] \\ \sigma^{(z)}{}_{(z)} &= (1/3)\ [-\ (\dot{A}/A) - (\dot{B}/B) + 2\ (\dot{C}/C)] \end{aligned}\right\} \quad . \tag{II.A.27}$$

Therefore, we see that our coordinate axes (x, y, z) are the principal axes of the fluid "shear" tensor.

In the diagonal Bianchi Type I metric of equation (II.A.24) the Einstein field equations

$$G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - (1/2)\ R\delta^{\mu}{}_{\nu} = 8\pi\ T^{\mu}{}_{\nu} \tag{II.A.28}$$

become:

$$\left.\begin{aligned} ab + ac + bc &= 8\pi\, T^0{}_0 \\ (\dot{b} + \dot{c}) + b^2 + c^2 + bc &= 8\pi\, T^1{}_1 \\ (\dot{a} + \dot{c}) + a^2 + c^2 + ac &= 8\pi\, T^2{}_2 \\ (\dot{a} + \dot{b}) + a^2 + b^2 + ab &= 8\pi\, T^3{}_3 \end{aligned}\right\} . \qquad \text{(II.A.29)}$$

Here we denote the Hubble expansion rates in the (x,y,z) directions by[3]

$$(a,\ b,\ c) \equiv (\dot{A}/A,\ \dot{B}/B,\ \dot{C}/C) \qquad , \qquad \text{(II.A.30)}$$

and $T^{\alpha}{}_{\beta}$ is the stress-energy tensor of the comoving material content of our cosmology. The Einstein tensor satisfies the contracted Bianchi identities

$$G^{\mu}{}_{\nu;\mu} = 0 \qquad . \qquad \text{(II.A.31)}$$

From equations (II.A.28) and (II.A.31) we readily see that only three of equations (II.A.29) are independent, if we guarantee ab initio that the material content of our cosmologies satisfies the equations

[3]This c is a variable, the Hubble expansion rate along the z-axis. Recall that we are using geometrized units, in which the speed of light is set equal to unity.

of motion (conservation equations):

$$T^{\mu}{}_{\nu;\mu} = 0 \quad . \qquad \text{(II.A.32)}$$

The dynamical evolution of our cosmologies is completely determined when we specify the components of $T^{\mu}{}_{\nu}$ as functions of A, B, and C, with the constraint of equation (II.A.32). In § II.C. below we shall examine some of the general properties of equations (II.A.29) in greater detail.

There are two interesting sub-cases of our Bianchi Type I cosmologies. The axially-symmetric case arises when we assume that the z-axis is a symmetry axis and set A = B for all time. Then we must have $T^{1}{}_{1} = T^{2}{}_{2}$, and the system of equations in equation (II.A.29) reduces to:

$$\left.\begin{aligned} a^2 + 2\,ac &= 8\pi\, T^{0}{}_{0} \\ (\mathring{a} + \mathring{c}) + a^2 + c^2 + ac &= 8\pi\, T^{1}{}_{1} \\ 2\,\mathring{a} + 3\,a^2 &= 8\pi\, T^{3}{}_{3} \end{aligned}\right\} . \qquad \text{(II.A.33)}$$

Note that only two of these equations are independent, if equation (II.A.32) is satisfied already by the stress-energy tensor. In § II.B. below we shall see that this sub-case has already been studied extensively. The second interesting sub-case is the isotropic limit. In this case we set $A = B = C \equiv R(t)$ for all time, and there is no preferred direction (i.e., no anisotropy). Thence, we must have $T^1{}_1 = T^2{}_2 = T^3{}_3$, and the dynamical evolution is governed by the Friedmann-like equation:

$$(\dot{R}/R)^2 = (8\pi/3)\ T^0{}_0 \quad , \qquad \text{(II.A.34)}$$

and by the conservation equation (II.A.32). Note that we have gotten back to the "flat" isotropic metric of equation (I.C.21), and that all of our considerations there concerning "flat" models are applicable here also.

II.B. PREVIOUS WORK ON BIANCHI TYPE I COSMOLOGIES

A great deal of research --- most of it very recent --- has been done on Bianchi Type I cosmologies. In order to have an overview of the context of our work here, I shall now give a synopsis of the previous work in this field. A brief statement on terminology is in order, however, before I begin.

The most general anisotropic Bianchi Type I metric is

$$ds^2 = dt^2 - g_{ij}(t)dx^i\, dx^j \quad . \tag{II.B.1}$$

When there is no a priori preferred direction this metric may always be written in the diagonal form

$$ds^2 = dt^2 - [A^2(t)\, dx^2 + B^2(t)dy^2 + C^2(t)dz^2] \quad . \tag{II.B.2}$$

When A = B for all time, we have the axisymmetric case

$$ds^2 = dt^2 - [A^2(t)\, (dx^2 + dy^2) + C^2(t)dz^2] \quad , \tag{II.B.3}$$

with the z-axis being the axis of symmetry. The limiting isotropic case occurs when $A = B = C \equiv R(t)$ for all time:

$$ds^2 = dt^2 - R^2(t)\, (dx^2 + dy^2 + dz^2) \quad , \tag{II.B.4}$$

and there is no longer any spatial anisotropy.

The material content of any cosmology is specified by its stress-energy tensor $T^\mu{}_\nu$. We shall only be concerned with diagonal stress-energy tensors in the comoving reference frame here. A PERFECT-FLUID is characterized by $T^0_{\ 0} = \rho$, $T^1_{\ 1} = T^2_{\ 2} = T^3_{\ 3} = -p$. The perfect fluid has a barotropic equation of state (see Zel'dovich 1961) if

$$p = \gamma \rho \qquad (0 \leq \gamma \leq 1) \quad . \tag{II.B.5}$$

The most important cases of equation (II.B.5) are called DUST ($\gamma = 0$), RADIATION ($\gamma = 1/3$), and ZEL'DOVICH ($\gamma = 1$). The stress-energy tensor for a uniform comoving magnetic field, of energy-density ρ_b, aligned along the z-axis is characterized by $T^0_{\ 0} = T^3_{\ 3} = \rho_b$, $T^1_{\ 1} = T^2_{\ 2} = -\rho_b$. We call this the MAGNETIC case. Finally, when $T^\mu_{\ \nu} \equiv 0$ we have the VACUUM case.

The previous investigations of Bianchi Type I cosmologies are the following:

1) THE ISOTROPIC CASE

The discussion and references cited in § I.C. above give adequate coverage of this case.

2) THE AXISYMMETRIC CASE

(a) Rosen (1964) gave the general solution for the PURE-MAGNETIC case.

(b) In a comparison of the Newtonian and Einstenian formulations of cosmology Zel'dovich (1964) gave the general solution for the DUST case, and he described the singularity behavior of this solution.

(c) Kompaneets and Chernov (1964) gave the general solution for the DUST case, and they discussed the singularity behavior of the RADIATION case. They also found the general solution when the material content consists solely of isentropic gas with the equation of state

$$\rho = Mn + Kn^{1 + \delta} \quad , \quad p = \delta Kn^{1 + \delta} \quad . \qquad \text{(II.B.6)}$$

Here M and K are constants, n is the number density of the gas particles, and δ is the "isentropic index" (a constant parameter).

(d) Zel'dovich (1965b) mentioned this metric in a qualitative discussion of universal primordial magnetic fields.

(e) Doroshkevich (1965) proved that all PERFECT-FLUID-MAGNETIC cases have an initial physical singularity and that all those with $\gamma < 1$ become isotropic as $t \to \infty$. He gave the general solutions for the DUST case and for the RADIATION case, and described their singularity behavior in detail. He presented the general solutions for the ZEL'DOVICH case and for the DUST-MAGNETIC case. Finally, he solved the ZEL'DOVICH-MAGNETIC case (leaving the time dependence in the form of an integral), and he investigated in detail the singularity behavior of the RADIATION-MAGNETIC case. In a second paper, Doroshkevich (1966) studied the evolution of density perturbations in the DUST, RADIATION, and ZEL'DOVICH solutions of his first paper.

(f) Hawking and Tayler (1966) mentioned some of the singularity properties of this metric in their brief discussion of the effects of expansion anisotropies upon primordial element formation in the big-bang fireball.

(g) Shikin (1966) solved the DUST-MAGNETIC case. In a later paper (Shikin 1967) he gave the general solution to the RADIATION case, and he described some of the properties of the RADIATION-MAGNETIC case.

(h) Saunders (1967) used the DUST solution in his numerical evaluation of the current luminosity distance-redshift and source count-redshift relations in this metric.

(i) Thorne (1967) gave the general solutions for the DUST case, for the RADIATION case, and for the DUST-MAGNETIC case. He found the singularity behavior of all of the PERFECT-FLUID-MAGNETIC cases. He constructed semi-realistic cosmological models of our Universe (with vanishing magnetic field) by smoothly joining an earlier RADIATION solution to a later DUST solution at the point where $\rho_r = \rho_d$. Finally, in these semi-realistic models he investigated the effects of anisotropic expansion upon primordial element formation and upon the isotropy of the observed 2.7 $^{\circ}$K cosmic microwave radiation.

(j) Ellis (1967) mentioned the DUST solution, both with and without the cosmological constant Λ.

(k) Tomita (1968) described the singularity behavior of the DUST case, and used the DUST solution in his numerical evaluation of the current apparent magnitude-redshift and number count-redshift relations in this metric.

(1) Stewart and Ellis (1968) have found the general solution for the PERFECT-FLUID cases in this metric.

3) THE GENERAL ANISOTROPIC CASE

(a) We will first consider the VACUUM case. The general solution was first given by Kasner (1921) in the form

$$ds^2 = dt^2 - [t^{2\ell}dx^2 + t^{2m}dy^2 + t^{2n}dz^2] \quad , \qquad \text{(II.B.7)}$$

with

$$\ell + m + n = \ell^2 + m^2 + n^2 = 1 \quad . \qquad \text{(II.B.8)}$$

It appears that this solution was independently rederived later by Narlikar and Karmarkar (1946), Taub (1951), and Lifshitz and Khalatnikov (1960). This VACUUM solution was given --- and attributed to Kasner --- by Petrov (1961), Dautcourt, Papapetrou, and Treder (1962), Lifshitz and Khalatnikov (1963a,b), Doroshkevich, Zel'dovich, and Novikov (1967), and Misner (1968). Lifshitz and Khalatnikov (1960, 1963a,b) used Kasner's solution in an investigation of the singularities of Einstein's field equations. They gave the useful parameterization

$$(\ell,m,n) = [-s,\ s(1 + s),\ 1 + s]/(1 + s + s^2) \quad , \qquad \text{(II.B.9)}$$

with the range of the parameter s being $0 \leq s \leq 1$. Doroshkevich et al. (1967) employed the solution in a study of the behavior of non-interacting particles (neutrinos and gravitons) in anisotropic cosmologies. Finally, Ellis and MacCallum (1968) gave the general VACUUM solutions in the case with a cosmological constant, and they attributed the $\Lambda = 0$ solution to Kasner.

(b) Raychaudhuri (1958) gave the general DUST solution in the general anisotropic metric of equation (II.B.1).

(c) The diagonal DUST solution, which is equivalent to Raychaudhuri's (1958) solution, was first found by Schücking and Heckmann (1958). [See also Heckmann and Schücking 1962.] They claimed that the DUST solutions with a cosmological constant (Λ) were also easy to obtain, but they didn't give these solutions nor did they give references to where these solutions might be found.

(d) Robinson (1961) independently found the general DUST solution. He described how it behaves near the initial physical singularity and how it becomes isotropic as $t \to \infty$.

(e) Rosen (1962) gave the general solution in the PURE-MAGNETIC case and also (equivalently) in the case with a uniform comoving electric field aligned along the z-axis.

(f) Zel'dovich (1965c) mentioned the DUST solution of Schücking and Heckmann (1958) while discussing the inapplicability of Newtonian cosmology near the physical singularity in anisotropic cosmologies.

(g) Grishchuk (1967) obtained the general anisotropic

Bianchi Type I metric of equation (II.B.1) and gave references to a few of its known solutions while analyzing a new criterion for spatial homogeneity in cosmology.

(h) Saunders (1967) derived the general DUST solutions in the case with a cosmological constant, and he attributed the $\Lambda = 0$ solution to Heckmann and Schücking (1962). He also analyzed the singularity behavior of these DUST solutions, and he made some brief comments on the primordial helium abundance, the 2.7 °K cosmic microwave radiation, and the DUST-plus-RADIATION case in Bianchi Type I cosmologies.

(i) Doroshkevich _et_ _al_. (1967) gave the general solution (leaving the time dependence in the form of an integral) for the case where the stress-energy tensor is characterized by $T^0{}_0 = -T^3{}_3 = \rho$, $T^1{}_1 = T^2{}_2 = 0$. [This stress-energy tensor corresponds to non-interacting, massless particles (gravitons, neutrinos, or photons) all moving in the z-direction.] They also mentioned a case where the metric coefficients exhibit damped sinusoidal oscillations (see below).

(j) Ellis and MacCallum (1968) derived the Friedmann-like equation and the metric-dependence of the mass-density ρ in the case of a PERFECT-FLUID with the barotropic equation of state [equation (II.B.5)]. They also gave the general DUST solutions for the cases with a cosmological constant ($\Lambda \gtreqless 0$).

(k) Misner (1967, 1968) has given the most complete analysis to date of the general anisotropic Bianchi Type I cosmologies. He referred to the solutions of Heckmann and Schücking (1962) and Thorne (1967). He considered the following subjects: the stress-energy tensor for collisionless particles, the temperature anisotropy of the photon gas, a Lagrangian for the anisotropy with the anisotropy "energy" being deriveable from a potential, and the effects of viscous stresses. Finally, he gave solutions for the following general cases (and constructed heuristic models from these solutions): anisotropic expansion with isotropic stresses, the viscous damping of anisotropy (using neutrino viscosity in an electron-positron gas), small anisotropy in a RADIATION dominated cosmology (this is where the metric coefficients exhibit damped sinusoidal oscillations), large anisotropy in general, and small anisotropy in a DUST dominated cosmology. He used his results to predict an upper limit to the temperature anisotropy of the 2.7 ^{o}K cosmic microwave radiation.

(1) Jacobs (1968, 1969) has also studied anisotropic Bianchi Type I cosmologies in great detail. In § II.D. below I shall give an expanded version of Jacobs' (1968) work, while in § II.E. the same will be done with Jacobs' (1969) work. The meat of this thesis is contained in these two papers.

In the first paper, Jacobs (1968) found the general solution of the PERFECT-FLUID case with the barotropic equation of state [equation (II.B.5)]. He used this solution to construct semi-realistic cosmological models of our Universe, and in these models he investigated in detail the effects of expansion anisotropy upon the following: the evolutionary history of our Universe, primordial element production, and the possible temperature anisotropy of the observed 2.7 $^{\circ}K$ cosmic microwave radiation.

The second paper (Jacobs 1969) dealt with the PERFECT-FLUID-MAGNETIC case in the diagonal metric of equation (II.B.2). Solutions were obtained for the PURE-MAGNETIC, ZEL'DOVICH-MAGNETIC, HARD-MAGNETIC (axially-symmetric), and DUST-MAGNETIC (axially-symmetric) subcases. The singularity behavior of all PERFECT-FLUID-MAGNETIC cases was found, and all unsolved subcases were "solved" by numerical integration. Finally, the effects of a uniform, comoving, primordial magnetic field upon primordial element formation and upon the isotropy of the 2.7 $^{\circ}K$ cosmic microwave radiation were briefly considered.

II.C. SOME USEFUL PRELIMINARY RESULTS IN BIANCHI TYPE I COSMOLOGIES

We begin our analysis of anisotropic cosmology by deriving several interesting preliminary results in Bianchi Type I cosmologies. These results are collected together in this section either because of their specialized interest or because of their general, and prefatory, nature. We will first consider some of the general properties of the field equations (II.A.29). Then we will derive the stress-energy tensor for <u>any</u> non-interacting gas of massless boson or fermion particles (including the degenerate cases) in the diagonal metric of equation (II.B.2), and we will use it to study oscillatory damping of anisotropy by non-interacting particles. This analysis supplements the work of Doroshkevich, Zel'dovich, and Novikov (1967) and Misner (1968). Finally, we shall present a more exact analysis of Misner's (1968) work on the damping of anisotropy by neutrino viscosity.

1) GENERAL PROPERTIES OF BIANCHI TYPE I COSMOLOGIES

Let us consider the diagonal metric of equation (II.B.2). We shall be using this metric almost exclusively later. In § II.A. we found that the comoving observer in this metric sees "expansion" and "shear", but no "rotation". If we construct a Hubble expansion matrix from equation (II.A.30):

$$H_{ij} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \qquad (i,j = 1,2,3), \qquad \text{(II.C.1)}$$

we see that the "expansion" is just the average of a,b, and c or

$$\Theta = (1/3)\ \text{Trace}(H_{ij}) = (a+b+c)/3 \quad . \qquad \text{(II.C.2)}$$

The "shear" is essentially the traceless part of H_{ij}. It has the mixed, physical components:

$$\sigma^{(i)}{}_{(j)} = H_{ij} - \Theta\delta_{ij} = (1/3) \begin{bmatrix} 2a-b-c & 0 & 0 \\ 0 & 2b-a-c & 0 \\ 0 & 0 & 2c-a-b \end{bmatrix} . \qquad \text{(II.C.3)}$$

With our diagonal metric, the Einstein field equations (II.A.28) lead to equations (II.A.29). We want to consider equations (II.A.29) in greater detail here.

Let us define a "volume" scale factor V by

$$V \equiv ABC \quad . \qquad \text{(II.C.4)}$$

It has the properties that:

$$(\dot{V}/V) = (a+b+c) = \text{Trace}(H_{ij}) \equiv 3\Theta \quad , \qquad \text{(II.C.5)}$$

and when $V \to 0$ we must encounter a physical singularity. By a rather straightforward manipulation of equations (II.A.29) and (II.C.5) we find:

$$\ddot{V} = 4\pi V\,[2T^0{}_0 + \mathrm{Trace}(T^\mu{}_\nu)] \quad . \qquad \text{(II.C.6)}$$

For any physically reasonable stress-energy tensor $T^\mu{}_\nu$ the right-hand side of equation (II.C.6) is non-negative[1]. Therefore, since scale factors are non-negative by definition, equation (II.C.6) implies an initial physical singularity ($V = 0$) for any expanding Bianchi Type I cosmology.

If the components of the stress-energy tensor, $T^\mu{}_\nu$, in our coordinate system [equation (II.B.2)] depend only upon V (as is the case for a perfect fluid with a barotropic equation of state), we may immediately find $V(t)$ by solving the non-linear, second-order differential equation (II.C.6). It is this property which will guide us to the general solution of the PERFECT-FLUID case in a later section. This property is also the keystone to our understanding of primordial element production in Bianchi Type I PERFECT-FLUID cosmologies, since the only dynamical quantity needed for this problem is $(\dot{V}/V)$.

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[1]For a perfect fluid at rest in our coordinate system we have $2T^0{}_0 + \mathrm{Trace}(T^\mu{}_\nu) = 3(\rho - p)$, so the causality condition $p \leq \rho$ guarantees that the right-hand side of equation (II.C.6) is non-negative. For an arbitrary electromagnetic field it follows from the positive-definiteness of the energy-density and the characteristic zero trace.

Let us now consider the consequences of specific symmetry properties of the components of the stress-energy tensor, $T^{\mu}{}_{\nu}$, in our coordinate system. In most of our later work we have $T^1{}_1 = T^2{}_2$. If this is the case, equations (II.A.29) immediately have the first integral:

$$(a - b)\, V = \text{constant} \quad . \tag{II.C.7}$$

If any two of $T^1{}_1$, $T^2{}_2$, and $T^3{}_3$ are equal, a similar first integral is obtained. When we have

$$T^1{}_1 = T^2{}_2 = T^3{}_3 \quad , \tag{II.C.8}$$

only two <u>independent</u> first integrals result since:

$$\left.\begin{aligned} (a - b)\, V &= c_1 = \text{constant} \quad , \\ (a - c)\, V &= c_2 = \text{another constant} \quad , \\ (b - c)\, V &= c_3 \equiv c_2 - c_1 \quad . \end{aligned}\right\} \tag{II.C.9}$$

Similarly, if $T^1{}_1 = - T^0{}_0$, we obtain the first integral

$$(a + c)\, V = \text{constant} \quad . \tag{II.C.10}$$

We can easily see that only two independent first integrals result if two or more of $T^1{}_1$, $T^2{}_2$, and $T^3{}_3$ are equal to $- T^0{}_0$.

The inverse situation is also true. Symmetry restrictions imposed upon the metric constrain the stress-energy tensor (and, in some cases, the equation of state). For example, if A = B for all time (the axisymmetric case) we must have $T^1{}_1 = T^2{}_2$. The most stringent limitation imposed by the metric occurs when A = B = C for all time (the isotropic case). Then we must have $T^1{}_1 = T^2{}_2 = T^3{}_3$. This limitation would guarantee the absence of large-scale magnetic fields.

Whenever equations (II.A.29) admit two first integrals of the form of equation (II.C.7) and/or equation (II.C.10), we find that only one more general condition is needed in order to immediately reduce the problem of equations (II.A.29) to quadratures. This final condition, which we will meet again in a later section, is that the quantity ($V^2\, T^0{}_0$) --- by virtue of the particular form of $T^0{}_0$ --- be a function of only one of the variables A, B, C, or V. Most of the general solutions which we obtain below follow from this property of equations (II.A.29).

In the diagonal Bianchi Type I metric of equation (II.B.2), the equations of motion (II.A.32) reduce to a single equation:

$$\left.\begin{aligned}
0 &= T^\mu{}_{\nu;\mu} \\
&= T^\mu{}_{\nu,\mu} + \Gamma^\mu{}_{\mu\lambda}\, T^\lambda{}_\nu - \Gamma^\lambda{}_{\mu\nu}\, T^\mu{}_\lambda \\
&= T^0{}_{0,0} + \Gamma^1{}_{10}\,(T^0{}_0 - T^1{}_1) + \Gamma^2{}_{20}\,(T^0{}_0 - T^2{}_2) + \Gamma^3{}_{30}\,(T^0{}_0 - T^3{}_3) \\
&= \dot{T}^0{}_0 + (\dot{A}/A)(T^0{}_0 - T^1{}_1) + (\dot{B}/B)(T^0{}_0 - T^2{}_2) + (\dot{C}/C)(T^0{}_0 - T^3{}_3)
\end{aligned}\right\} \quad .\text{(II.C.11)}$$

Using equations (II.A.30) and (II.C.5) we can write equation (II.C.11) in the convenient form:

$$(VT^0{}_0)^{\cdot} - (aT^1{}_1 + bT^2{}_2 + cT^3{}_3)\, V = 0 \quad . \qquad \text{(II.C.12)}$$

When our stress-energy tensor is that of a perfect fluid [see equation (I.C.3)], equation (II.C.12) reduces to the familiar form:

$$d(\rho V) + p\, dV = 0 \quad . \qquad \text{(II.C.13)}$$

Equation (II.C.12) is the final general property of our Bianchi Type I cosmologies that we shall consider here.

2) THE STRESS-ENERGY TENSOR FOR NON-INTERACTING PARTICLES AND OSCILLATORY DAMPING OF ANISOTROPY

In this sub-section we shall derive some new, and interesting, results in Bianchi Type I cosmologies. In the diagonal metric of equation (II.B.2) we will study the form of the stress-energy tensor for gases consisting of one type of boson or fermion. The particles of the gas will be massless, and we shall consider both the degenerate and non-degenerate cases. In particular, we shall investigate the evolution of the form of the stress-energy tensor for a gas of massless particles initially in thermal equilibrium, which then decouples from its surroundings (when its interaction cross-section with its surroundings becomes sufficiently small) to become freely-propagating and non-interacting. We shall also examine the effect of the resultant anisotropies in the stress-energy tensor on the anisotropic expansion of the cosmological model.

A thorough understanding of the "classical" statistical mechanics of boson and fermion gases is a necessary prerequisite to the study of statistical mechanics in general relativity theory, since we will be dealing with local --- as opposed to global --- gas properties. Such an understanding is provided by the works of Morse (1962), Kubo (1965), Chiu (1967), and the references cited therein. The transition to general relativistic statistical mechanics is adequately covered in the works of Tauber and Weinberg (1961), Chernikov (1962a,b,c, 1963), Bardeen (1965), Vignon (1966), and Lindquist (1966). Here we shall only consider the general relativistic case.

In the diagonal metric of equation (II.B.2) the metric tensor is:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -A^2(t) & 0 & 0 \\ 0 & 0 & -B^2(t) & 0 \\ 0 & 0 & 0 & -C^2(t) \end{pmatrix} . \qquad \text{(II.C.14)}$$

The contravariant four-momentum of a gas particle is

$$P^{\alpha} = (P^0, P^1, P^2, P^3) \quad , \qquad \text{(II.C.15)}$$

and it is related to the covariant four-momentum by

$$P_\alpha = g_{\alpha\beta}P^\beta = g_{\alpha\alpha}P^\alpha \text{ (no sum)} \quad . \tag{II.C.16}$$

The stress-energy tensor is defined as:

$$T^\mu{}_\nu = \iiint N(x^\alpha,\ P^\alpha)\ P^\mu P_\nu\ (-g)^{1/2}\ (P_0)^{-1}\ dP^1\ dP^2\ dP^3 \quad , \tag{II.C.17}$$

where $N(x^\alpha, P^\alpha)$ is the scalar distribution function in relativistic phase space, and g is the determinant of $g_{\mu\nu}$. The number-density of gas particles is given by:

$$n = \iiint N(x^\alpha,\ P^\alpha)\ (-{}^{(3)}g)^{1/2}\ dP^1\ dP^2\ dP^3 \quad , \tag{II.C.18}$$

where ${}^{(3)}g$ is the determinant of the space part of the metric tensor, g_{ij}. Note that equation (II.C.14) implies $(-g)^{1/2} = (-{}^{(3)}g)^{1/2} = V$ here.

We are considering only massless particles (photons, neutrinos, and gravitons) here. The generalization to massive particles is completely straightforward, but it eventually leads to complicated mathematical expressions and it is also not of great cosmological interest. For massless particles we know that

$$g_{\alpha\beta}P^\alpha P^\beta = (\text{mass})^2 = 0 \quad , \tag{II.C.19}$$

which implies

$$P^0 = [(AP^1)^2 + (BP^2)^2 + (CP^3)^2]^{1/2} \quad . \tag{II.C.20}$$

We know, however, that the contravariant physical components of the four-momentum are:

$$P^{(\alpha)} = | g_{\alpha\alpha} |^{1/2} P^{\alpha} \text{ (no sum)} \quad . \qquad \text{(II.C.21)}$$

Therefore, we can introduce spherical coordinates (P^0, θ, φ) into the four-momentum phase space, so that:

$$\left. \begin{aligned} P^{(1)} &= AP^1 = P^0 \sin\theta \cos\varphi \\ P^{(2)} &= BP^2 = P^0 \sin\theta \sin\varphi \\ P^{(3)} &= CP^3 = P^0 \cos\theta \end{aligned} \right\} . \qquad \text{(II.C.22)}$$

Then we see that equation (II.C.20) is satisfied identically. When we calculate the transformation Jacobian, we find that:

$$dP^1 \, dP^2 \, dP^3 = V^{-1} \, (P^0)^2 \, dP^0 \, d\Omega \quad , \qquad \text{(II.C.23)}$$

where $V \equiv ABC$ as before and the angular element is

$$d\Omega = \sin\theta \, d\theta \, d\varphi \quad . \qquad \text{(II.C.24)}$$

Let us first consider the case of thermal equilibrium. Here the characteristic interaction time of our massless particles is much shorter than the dynamical expansion time of our cosmological model. The scalar distribution function is given by[2]

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[2]Note that Boltzmann's constant, k, is equal to one in our geometrized units.

$$N_{\pm} = (\tilde{\mu} h^{-3}) \{ \exp[(P^0/T) - D] \pm 1\}^{-1} \quad , \qquad \text{(II.C.25)}$$

where the (+, -) sign denotes (fermions, bosons), respectively. Here $\tilde{\mu}$ is the multiplicity of spin states of our particles ($\tilde{\mu} = 1$ for neutrinos and for anti-neutrinos; $\tilde{\mu} = 2$ for photons and gravitons), h is Planck's constant, D is the degeneracy parameter of the gas, and T is the thermodynamic temperature of the gas. Chiu (1967) shows that D vanishes for massless bosons, and that D is a constant during the adiabatic expansion of a massless fermion gas (the case under consideration here). In Appendix A we show how equations (II.C.17), (II.C.18), (II.C.23), and (II.C.25) lead to the thermal equilibrium properties:

$$\left.\begin{aligned} n\,(\pm) &= (4\pi\tilde{\mu} h^{-3})\, T^3 f_1^{(\pm)}(D) \\ T^0{}_0\,(\pm) &= (4\pi\tilde{\mu} h^{-3})\, T^4 f_2^{(\pm)}(D) \\ T^1{}_1\,(\pm) = T^2{}_2\,(\pm) &= T^3{}_3\,(\pm) = -(1/3)\, T^0{}_0\,(\pm) \\ T^{\mu}{}_{\nu}\,(\pm) &= 0 \quad (\mu \neq \nu) \end{aligned}\right\} , \qquad \text{(II.C.26)}$$

where

$$f_{\ell}^{(\pm)}(D) \equiv \int_0^{\infty} x^{\ell+1} [\exp(x-D) \pm 1]^{-1}\, dx \quad . \qquad \text{(II.C.27)}$$

We see that the thermal equilibrium configuration of massless particles is a perfect fluid with the equation of state $p = \rho/3$.

Let us recall that $D \equiv 0$ for massless bosons. In Appendix B we investigate the properties of equation (II.C.27). From the results of Appendix B we find the following forms of equation (II.C.26) for massless bosons (photons or gravitons) and for non-degenerate ($D = 0$) massless fermions (neutrinos and anti-neutrinos):

PHOTONS OR GRAVITONS

$$\left.\begin{aligned} n &= [16\pi h^{-3}\zeta(3)]\, T^3 \\ \rho \equiv T^0{}_0 &= \tilde{a}\, T^4 \\ p \equiv -T^i{}_i \text{ (no sum)} &= \rho/3 = (\tilde{a}/3)\, T^4 \end{aligned}\right\}, \qquad \text{(II.C.28)}$$

NON-DEGENERATE NEUTRINOS OR ANTI-NEUTRINOS

$$\left.\begin{aligned} n &= (3/8)\, [16\pi h^{-3}\zeta(3)]\, T^3 \\ \rho &= (7/16)\, \tilde{a}\, T^4 \\ p &= \rho/3 = (7/48)\, \tilde{a}\, T^4 \end{aligned}\right\}. \qquad \text{(II.C.29)}$$

Here $\tilde{a} = (8\pi^5/15h^3)$ is the Stefan-Boltzmann constant, and $\zeta(\ell)$ is the Riemann zeta-function defined by the series:

$$\zeta(\ell) \equiv \sum_{k=1}^{\infty} k^{-\ell} \quad . \qquad \text{(II.C.30)}$$

The above results are fairly well-known (see Chiu 1967). We will now proceed into almost virgin territory and consider the effects upon equations (II.C.26) when the massless particles decouple from their surroundings. These particles then become non-interacting, and they experience adiabatic expansion while propagating freely in the metric of equation (II.B.2). Thorne (1967) briefly considered such a situation for photons during an analysis of the temperature isotropy of the observed 2.7 $^{\circ}$K cosmic microwave radiation. Doroshkevich et al. (1967) considered the effects of anisotropic expansion upon the stress-energy tensor of neutrinos and anti-neutrinos in such a situation in Bianchi Type I cosmologies. Finally, Misner (1968) has analyzed this situation in great detail, in a manner which is complementary to our present analysis.

Let us first assume that our gas of massless particles is in thermal equilibrium (see above). We idealize the decoupling of the gas from its surroundings by specifying that total decoupling occurs at the time t_o. In the remainder of this discussion the subscript zero (o) will denote the value of a quantity at the time of decoupling. For $t > t_o$ the gas is wholly non-interacting, and Liouville's theorem implies that $(N_{\pm})$ [see equation (II.C.25)] is a constant throughout all phase space and for all time. The degeneracy parameter, D, is also a constant since we are assuming adiabatic expansion. Therefore [cf. equation (II.C.25)], as the energy (P^0) of a particle is redshifted, the temperature which characterizes the distribution function in its neighborhood is also redshifted by the same amount:

$$(P^0/T) = \text{constant} \quad , \tag{II.C.31a}$$

or equivalently, the redshift factor for a particular particle is equal to

$$P^0/(P^0)_o \equiv \zeta = (T/T_o)_{\text{in neighborhood of particle}} \quad . \tag{II.C.31b}$$

The evolution of the form of the stress-energy tensor depends only upon the function ζ, which we now determine. From equation (II.C.20) we have:

$$(P^0)_o = [(AP^1)_o^2 + (BP^2)_o^2 + (CP^3)_o^2]^{1/2} \quad , \tag{II.C.32}$$

but we know that P_1, P_2, and P_3 are constants of the motion in our Bianchi Type I cosmologies. Therefore, we have:

$$(AP^1)_o = - (P_1/A)_o = - (P_1/A_o) = (A/A_o)\ (AP^1) \quad , \tag{II.C.33}$$

and similarly for $(BP^2)_o$ and $(CP^3)_o$. Using equations (II.C.22) and (II.C.32) we quickly find that:

$$\zeta(\theta,\varphi) = \left[\left(\frac{A \sin\theta \cos\varphi}{A_o}\right)^2 + \left(\frac{B \sin\theta \sin\varphi}{B_o}\right)^2 + \left(\frac{C \cos\theta}{C_o}\right)^2 \right]^{-1/2} , \tag{II.C.34}$$

where (θ, φ) characterize the direction of motion of the particle in whose neighborhood the redshift factor ζ is measured. From equation (II.C.34), we see that the characteristic temperature of our gas is direction-dependent. Thorne (1966) arrives at essentially the same conclusion by a somewhat different means. Note that, when

$A = B = C \equiv R$ for all time (the isotropic case), we have

$$T \propto R^{-1} \quad , \qquad \text{(II.C.35)}$$

the well-known standard behavior of the temperature.

Using equations (II.C.31b) and (II.C.34) and the method of Appendix A, we can calculate the components of the stress-energy tensor at a time $t > t_o$ after decoupling of the gas particles from each other and from their surroundings. The calculation is identical to that of Appendix A except that the angular integrations are altered:

$$n = n_o \left[(4\pi)^{-1} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \; \zeta^3(\theta,\varphi) \right] \quad , \qquad \text{(II.C.36)}$$

$$\left.\begin{aligned}
T^0{}_0 &= (T^0{}_0)_o \left[(4\pi)^{-1} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \; \zeta^4(\theta,\varphi) \right] \\
T^1{}_1 &= (T^1{}_1)_o \left[(3/4\pi) \int_0^{2\pi} \cos^2\varphi \, d\varphi \int_0^{\pi} \sin^3\theta \, d\theta \; \zeta^4(\theta,\varphi) \right] \\
T^2{}_2 &= (T^2{}_2)_o \left[(3/4\pi) \int_0^{2\pi} \sin^2\varphi \, d\varphi \int_0^{\pi} \sin^3\theta \, d\theta \; \zeta^4(\theta,\varphi) \right] \\
T^3{}_3 &= (T^3{}_3)_o \left[(3/4\pi) \int_0^{2\pi} d\varphi \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \; \zeta^4(\theta,\varphi) \right]
\end{aligned}\right\} , \qquad \text{(II.C.37)}$$

$$T^\mu{}_\nu = 0 \quad (\mu \neq \nu) \quad . \qquad \text{(II.C.38)}$$

The vanishing of all off-diagonal terms in the stress-energy tensor again follows from the fact that there is an odd function of angle in one of the angular integrals in each case [see equations (A.9) and (II.C.34)]. Note that in the isotropic case ($A = B = C$) we once again recover equations (II.C.26) and (II.C.27). Therefore, in the isotropic case, our gases behave exactly as they would in thermal equilibrium, even though the gas particles are completely decoupled from each other and from their surroundings.

Let us now simplify our notation by setting

$$\bar{\alpha} \equiv (A/A_o),\ \bar{\beta} \equiv (B/B_o),\ \bar{\chi} \equiv (C/C_o) \quad , \qquad \text{(II.C.39)}$$

and

$$n = n_o\, J_0,\quad T^{\alpha}{}_{\alpha} = (T^{\alpha}{}_{\alpha})_o\, J_{\alpha\alpha}\ (\text{no sum}) \quad . \qquad \text{(II.C.40)}$$

The J's are angular integrals over (θ,φ) resulting in functions of $(\bar{\alpha}, \bar{\beta}, \bar{\chi})$. Only J_0 can be evaluated in terms of elementary functions in the general case. This is accomplished as follows:

$$\begin{aligned} J_0 &= (4\pi)^{-1} \iint_{\Omega} d\Omega[(\bar{\alpha} \sin\theta \cos\varphi)^2 + (\bar{\beta} \sin\theta \sin\varphi)^2 + (\bar{\chi}\cos\theta)^2]^{-3/2} \quad , \\ &= (2\pi)^{-1} \int_0^{2\pi} d\varphi \int_0^1 dx\ [A(\varphi) + B(\varphi)\, x^2]^{-3/2} \end{aligned} \qquad \text{(II.C.41)}$$

where we have $A(\varphi) \equiv (\bar{\alpha}\cos\varphi)^2 + (\bar{\beta}\sin\varphi)^2$ and $B(\varphi) \equiv \bar{\chi}^2 - A(\varphi)$, and we have made the change of variable $x \equiv \cos\theta$. The integral over x

(dummy variable) is standard[3], giving us:

$$J_0 = (2\pi\bar{\chi})^{-1} \int_0^{2\pi} d\varphi \, [\bar{\alpha}^2 \cos^2 \varphi + \bar{\beta}^2 \sin^2 \varphi]^{-1} \quad . \qquad \text{(II.C.42)}$$

The integral over φ is also standard[4], and our final result is:

$$J_0 = |\bar{\alpha}\bar{\beta}\bar{\chi}|^{-1} \equiv (V_0/V) \quad , \qquad \text{(II.C.43)}$$

where V is the "volume" scale factor of equation (II.C.4). This result tells us the obvious fact that the number-density of particles, n, is inversely proportional to the volume factor V (conservation of particles).

We can readily see that the $J_{\alpha\alpha}$'s can be expressed in terms of elliptic functions, but such a reduction serves only to obscure their properties. The most useful form for the $J_{\alpha\alpha}$'s is an integral form. By using the known integrals [equations (57a) and (58c) on page 102 of Gröbner and Hofreiter 1950]:

$$\int_0^{\pi/2} dx \, (a^2 \cos^2 x + b^2 \sin^2 x)^{-2} = \pi \, (a^2 + b^2)/4 \, a^3 b^3 \ , \ ab > 0, \qquad \text{(II.C.44)}$$

and

---

[3] It is $\int dx(A + Bx^2)^{-3/2} = (x/A) \, (A + Bx^2)^{-1/2}$.

[4] It is $\int_0^{\pi/2} d\varphi \, (\bar{\alpha}^2 \cos^2 \varphi + \bar{\beta}^2 \sin^2 \varphi)^{-1} = (\pi/2 \, |\bar{\alpha}\bar{\beta}|)$.

$$\int_0^{\pi/2} dx(A\cos^2 x + B\sin^2 x)(a^2\cos^2 x + b^2\sin^2 x)^{-2} = \pi(Ba^2 + Ab^2)/4\,a^3b^3,\ ab>0, \tag{II.C.45}$$

we can straightforwardly reduce the $J_{\alpha\alpha}$'s to the beautiful form:

$$\left.\begin{aligned}
J_{00} &= \left(\frac{\bar\alpha^2 + \bar\beta^2}{2\bar\alpha^3\bar\beta^3}\right)\int_0^1 \frac{(1 + Ex^2)\,dx}{[(1 + Fx^2)(1 + Gx^2)]^{3/2}} \\
J_{11} &= \left(\frac{3}{2\bar\alpha^3\bar\beta}\right)\int_0^1 \frac{(1 - x^2)\,dx}{[(1 + Fx^2)^3(1 + Gx^2)]^{1/2}} \\
J_{22} &= \left(\frac{3}{2\bar\alpha\bar\beta^3}\right)\int_0^1 \frac{(1 - x^2)\,dx}{[(1 + Fx^2)(1 + Gx^2)^3]^{1/2}} \\
J_{33} &= \left[\frac{3(\bar\alpha^2 + \bar\beta^2)}{2\bar\alpha^3\bar\beta^3}\right]\int_0^1 \frac{x^2(1 + Ex^2)\,dx}{[(1 + Fx^2)(1 + Gx^2)]^{3/2}}
\end{aligned}\right\} . \tag{II.C.46}$$

Here we use the functions

$$E \equiv \left(\frac{2\bar\chi^2}{\bar\alpha^2 + \bar\beta^2}\right) - 1\ ,\quad F \equiv \left(\frac{\bar\chi}{\bar\alpha}\right)^2 - 1\ ,\quad G \equiv \left(\frac{\bar\chi}{\bar\beta}\right)^2 - 1\ . \tag{II.C.47}$$

Note that in the isotropic case $(A = B = C \equiv R)$ all of the $J_{\alpha\alpha}$'s are equal to $(R_o/R)^4$.

Equation (II.C.46) is especially useful for studying various asymptotic limits of the $J_{\alpha\alpha}$'s. For example, a violent relative contraction along the z-axis for $t > t_o$ implies that $\tilde{\chi} << \tilde{\alpha}$ or $\tilde{\beta}$. The integrals in equations (II.C.46) can then be expanded using the binomial expansion, and integrated term by term. In this limit we find:

$$J_{33} \approx 3\, J_{00} \quad , \quad J_{11} \approx J_{22} \approx 0 \quad . \qquad \text{(II.C.48)}$$

This means that

$$T^3{}_3 \approx - T^0{}_0 \quad , \quad T^1{}_1 \approx T^2{}_2 \approx 0 \quad , \qquad \text{(II.C.49)}$$

which is exactly the form of the stress-energy tensor considered by Doroshkevich et al. (1967) (see § II.B.) in the Bianchi Type I cosmology characterized by the diagonal metric of equation (II.B.2). They found the general solution for this case:

$$\left.\begin{aligned} t &= t_* \int \eta^{(k^2 - \frac{1}{4})} \exp(\eta)\, d\eta \\ (A,B,C) &= [A_* \, \eta^{(\frac{1}{2} + k)} , B_* \, \eta^{(\frac{1}{2} - k)} , C_* \, \eta^{(k^2 - \frac{1}{4})} \exp(\eta)] \\ 8\pi\, T^0{}_0 &= [t_* \, \eta^{(k^2 + \frac{1}{4})} \exp(\eta)]^{-2} \\ & - 1/2 \le k \le + 1/2 \quad , \quad 0 < \eta < \infty \end{aligned}\right\} . \text{(II.C.50)}$$

Here t_*, A_*, B_*, and C_* are constants, k is a parameter, and η is the independent variable. We note here that the integral for the time-dependence of equation (II.C.50) can be evaluated approximately in the following fashion:

$$
\left.\begin{aligned}
(t/t_*) &\equiv \int \eta^{(k^2 - \frac{1}{4})} \exp(\eta)\, d\eta \\
&= \sum_{\ell=0}^{\infty} (1/\ell\,!) \int \eta^{(k^2 - \frac{1}{4} + \ell)}\, d\eta \\
&= \eta^{(k^2 + \frac{3}{4})} \sum_{\ell=0}^{\infty} \left\{ \frac{\eta^{\ell}}{\ell![(\ell + 1) + (k^2 - \frac{1}{4})]} \right\} \\
&\approx \eta^{(k^2 - \frac{1}{4})} \sum_{\ell=0}^{\infty} \left[\frac{\eta^{(\ell + 1)}}{(\ell + 1)!} \right] \\
&\approx \eta^{(k^2 - \frac{1}{4})} [\exp(\eta) - 1]
\end{aligned}\right\} . \quad \text{(II.C.51)}
$$

Our approximation here consists in setting $[(k^2 - \frac{1}{4})/(\ell + 1)]$ equal to zero, and it is justified by the fact that $0 \leq k^2 \leq 1/4$.

The general solution of equation (II.C.50) shows that the enormous pressure $- T^3_{\ 3} = T^0_{\ 0}$ along the z-axis converts an initial rapid contraction along the z-axis into a very rapid expansion along the z-axis, while affecting the initial expansion in the x - and y - directions very little. This implies that the stress-energy tensor has the limiting form of equation (II.C.49) for only a short period of time, and that the solution of equation (II.C.50) is valid only during this short epoch. After this epoch, we find a violent expansion along the z-axis, but only a very mild expansion along the x- and y- axes. For a time we must once again consider the general behavior of the stress-energy tensor indicated in equations (II.C.46) and (II.C.47). Eventually, however, the violent z-expansion leads to another, different limiting form of the stress-energy tensor. We shall study this second limiting form, and its effect upon the metric, in detail below.

In the general case, equations (II.C.46) and (II.C.47) cannot be evaluated in terms of elementary functions. Such an evaluation can, however, be accomplished in the axisymmetric case (e.g., A = B). Then $\bar{\alpha} = \bar{\beta}$ for all time, and we have

$$E = F = G \equiv (\bar{X}/\bar{\alpha})^2 - 1 \quad . \tag{II.C.52}$$

When $\bar{X} < \bar{\alpha}$, we set $r \equiv |\bar{X}/\bar{\alpha}|$ and obtain:

$$J_{00} = [2\bar{\alpha}^4 r^2 (1-r^2)^{1/2}]^{-1} \left\{ (1-r^2)^{1/2} + r^2 \ln \left| \frac{(1+r)^{1/2} + (1-r)^{1/2}}{(1+r)^{1/2} - (1-r)^{1/2}} \right| \right\}$$

$$J_{33} = \left[\frac{3}{2\bar{\alpha}^4 r^2 (1-r^2)^{3/2}} \right] \left\{ (1-r^2)^{1/2} - r^2 \ln \left| \frac{(1+r)^{1/2} + (1-r)^{1/2}}{(1+r)^{1/2} - (1-r)^{1/2}} \right| \right\} .$$

$$J_{11} = J_{22} = (3\, J_{00} - J_{33})/2 \tag{II.C.53}$$

The behavior of equations (II.C.53) as a function of r is illustrated in Figure 3. When $r = 1$ we are at $t = t_o$, and the thermal equilibrium results of equations (II.C.26) and (II.C.27) hold. As $r \to 0$, the relative contraction along the z-axis leads to the stress-energy tensor of equation (II.C.49). When $\bar{\chi} > \bar{\alpha}$, we set $s \equiv |\bar{\alpha}/\bar{\chi}|$ and obtain:

$$J_{00} = (2\,\bar{\alpha}^4)^{-1} \left\{ s\ (1-s^2)^{-1/2} \arctan\left[\frac{(1-s^2)^{1/2}}{s} \right] + s^2 \right\}$$

$$J_{33} = \left[\frac{3\ s^2}{2\bar{\alpha}^4(1-s^2)} \right] \left\{ s\ (1-s^2)^{-1/2} \arctan\left[\frac{(1-s^2)^{1/2}}{s} \right] - s^2 \right\}$$

$$J_{11} = J_{22} = (3\ J_{00} - J_{33})/2 \qquad \text{(II.C.54)}$$

The behavior of equations (II.C.54) versus s is displayed in Figure 4. At $s = 1$ ($t = t_o$) we have thermal equilibrium. As $s \to o$ there is expansion along the z-axis relative to the x- and y- axes, and the pressure in the z-direction becomes less than that in the x- and y- directions.

For an extremely violent expansion along the z-axis relative to the x- and y- axes, equations (II.C.54) and (II.C.40) have the asymptotic form: $T^1{}_1 = T^2{}_2 \approx -(1/2)\ T^0{}_0$, $T^3{}_3 \approx 0$. Let us here consider a stress-energy tensor of this asymptotic form in the diagonal metric of equation (II.B.2). In Appendix C we show how the Einstein field equations (II.A.29) and the conservation equation (II.C.12)

FIGURE 3

The dependence of the stress-energy tensor on the anisotropic expansion parameter r for a gas of non-interacting massless particles in an axisymmetric (e.g., $A = B$) Bianchi Type I cosmology. Here we have contraction along the z-axis relative to the x- and y- axes. The gas decouples from thermal equilibrium at $t = t_o$ $(r = 1)$. The subsequent anisotropic expansion is parameterized by $r \equiv (CA_o/AC_o)$, and the changing stress-energy tensor is described by $T^{\alpha}{}_{\alpha} \equiv (T^{\alpha}{}_{\alpha})_o\, J_{\alpha\alpha}$ (no sum). As the anisotropic expansion increases $(r \to 0)$, the metric pumps energy into the $T^3{}_3$ component, until we reach the limit: $T^3{}_3 \approx - T^0{}_0$, $T^1{}_1 = T^2{}_2 \approx 0$. In this limiting state our gas consists of only two beams of particles, one flowing in the z- direction and the other (with identical properties) flowing in the minus z- direction.

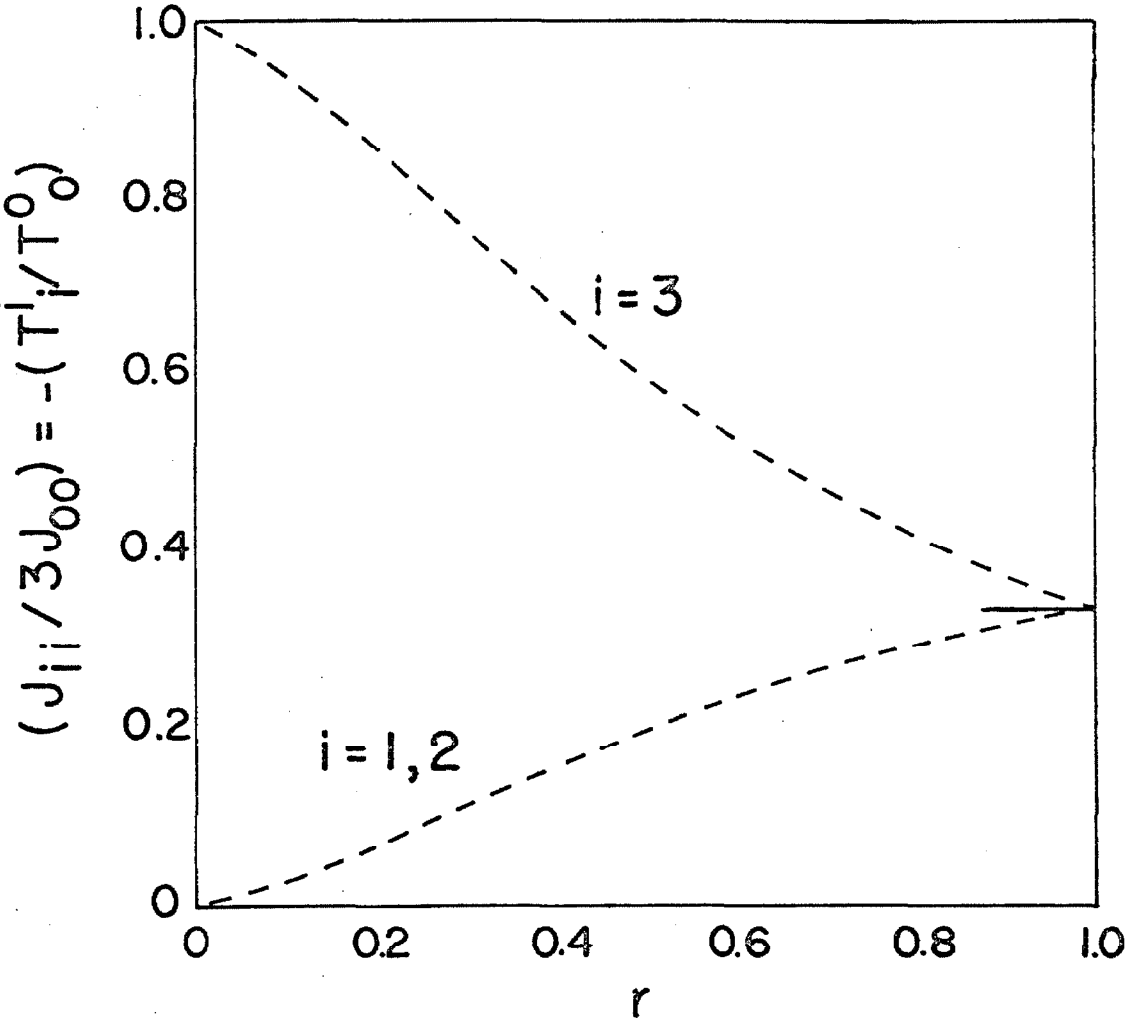
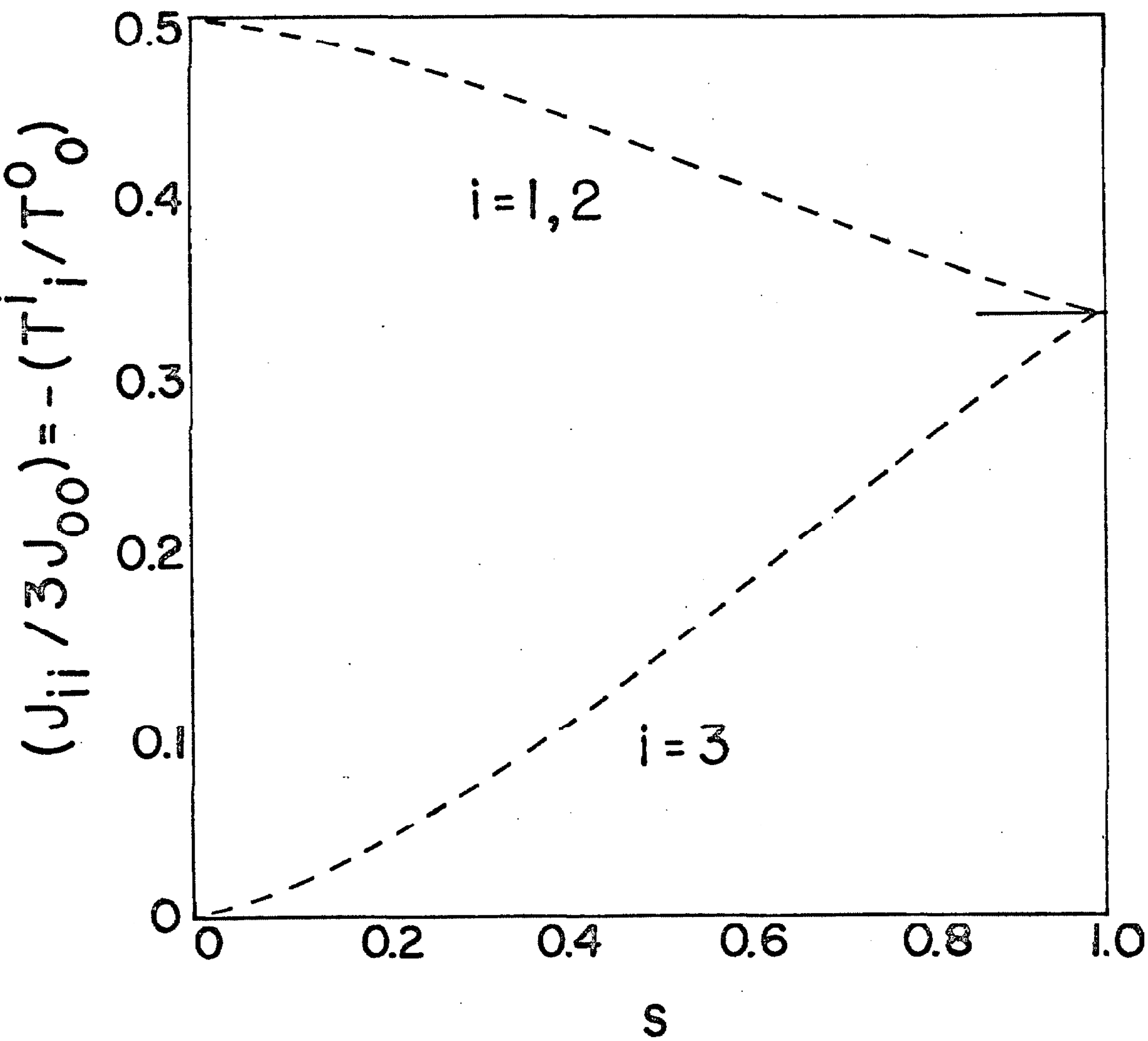

1.0
0.8
0.6
0.4
0.2
0
$(J_{ii}/3J_{00}) = -(T^i_i/T^0_0)$
i = 3
i = 1, 2
0
0.2
0.4
0.6
0.8
1.0
r

FIGURE 4

The dependence of the stress-energy tensor on the anisotropic expansion parameter s for a gas of non-interacting massless particles in an axisymmetric (e.g., A = B) Bianchi Type I cosmology. In this case, there is expansion along the z- axis relative to the x- and y- axes. The gas decouples from thermal equilibrium at $t = t_o$ $(s = 1)$. The subsequent anisotropic expansion is parameterized by $s \equiv (AC_o/CA_o)$, and $T^{\alpha}{}_{\alpha} \equiv (T^{\alpha}{}_{\alpha})_o \, J_{\alpha\alpha}$ (no sum). As the anisotropy becomes more extreme $(s \to 0)$, the metric depletes the energy in $T^3{}_3$ and distributes equal quantities of energy to $T^1{}_1$ and $T^2{}_2$. In the limiting case, essentially all of the gas energy resides in particles moving in directions that lie in the x-y plane, and we have: $T^1{}_1 = T^2{}_2 \approx -(1/2)\, T^0{}_0$, $T^3{}_3 \approx 0$.



$(J_{ii}/3J_{00}) = -(T^i_i/T^0_0)$
0.5
0.4
0.3
0.2
0.1
0
i = 1, 2
i = 3
0
0.2
0.4
0.6
0.8
1.0
s

may be solved in this case. The solution, which is new so far as I know, can be written in the form:

$$\left.\begin{aligned}
(A/A_*,\ B/B_*,\ C/C_*) &= \{f\ \exp[-(1-k)\eta],\ f\ \exp[-(1+k)\eta],\ \exp(\eta)\} \\
(t/t_*) &= \int_0^{\eta} f^2\ \exp(-\eta)\ d\eta \\
T^0{}_0 = -2\,T^1{}_1 &= -2\,T^2{}_2 = \rho \quad , \quad T^3{}_3 = 0 \\
8\pi\rho &= (t_*^{\ 2}\ f^3)^{-1}\ \exp(2\eta)
\end{aligned}\right\} . \qquad \text{(II.C.55)}$$

where

$$\left.\begin{aligned}
f &\equiv 4\varkappa\ (1+k^2)\ \exp[(1+k^2)^{1/2}\eta]\ \{1-\varkappa\ \exp[(1+k^2)^{1/2}\eta]\}^{-2} \\
\varkappa &\equiv [(2+k^2)^{1/2} - (1+k^2)^{1/2}]^2
\end{aligned}\right\} , \qquad \text{(II.C.56)}$$

and

$$-1 \leq k \leq +1 \quad , \quad 0 \leq \eta \leq (1+k^2)^{-1/2}\ \ln\ (1/\varkappa) \quad . \qquad \text{(II.C.57)}$$

Note that the axisymmetric case (e.g., $A = B$ for all time), the only case for which the assumption $T^1{}_1 = T^2{}_2$ is reasonable when the stress-energy is due to non-interacting particles, occurs when $k = 0$. In this case, the remaining integral in equation (II.C.55) is readily evaluated, and we find the behavior:

$$A = B \propto t^{2/3} \quad , \quad C \propto \text{(constant)} - \text{(another constant)}\ t^{-1/3} \ . \qquad \text{(II.C.58)}$$

This axisymmetric solution [equation (II.C.58)] was previously found by Doroshkevich et al. (1967). Equations (II.C.55) through (II.C.58) imply that the lack of pressure along the z-direction causes the initial rapid relative expansion in the z-direction to quickly slow down (approaching a state with no expansion in the z-direction), while the slower expansion in the x- and y- directions continues. As a result, the consition $T^3{}_3 << T^0{}_0$ is soon violated, and the solution of equations (II.C.55) through (II.C.58) soon loses its validity. Once again, we must consider the general behavior of the stress-energy tensor indicated in equations (II.C.46) and (II.C.47).

From the analysis above, we can see that the dynamical effects peculiar to non-interacting massless particles are the following: The VACUUM metric of Kasner (1921) induces a stress-energy tensor of the form of equation (II.C.49). The stress- energy then reacts upon the metric [see equation (II.C.50)] to generate a stress-energy tensor of the form:

$$T^1{}_1 \approx T^2{}_2 \approx - (1/2)\, T^0{}_0,\ T^3{}_3 \approx 0 \quad . \qquad \text{(II.C.59)}$$

This stress-energy [equation (II.C.59)] generates the metric behavior of equations (II.C.55) through (II.C.58), and leads again to a stress-energy tensor of the form of equation (II.C.49). Doroshkevich et al. (1967) were the first to demonstrate this oscillatory behavior of both the stress-energy and the anisotropy. They also showed that the anisotropy is damped by the expansion of the cosmology. Subsequently Misner (1967, 1968) found that there is oscillatory damping of the anisotropy in general Bianchi Type I cosmologies containing both RADIATION and non-interacting massless particles. When the anisotropy is "small" compared to the mass-energy of both the RADIATION and the non-interacting massless particles, Misner (1967, 1968) showed that {with $g_{ij}^{1/2} \equiv R(t) \exp[\beta_{ij}(t)]$}:

$$\left. \begin{aligned} R(t) &\propto t^{1/2} \\ \beta_{ij}(t) &\propto t^{-1/4} \sin[K^2 \ln (t/t_*)] \end{aligned} \right\}, \qquad \text{(II.C.60)}$$

where K and t_* are constants. Equation (II.C.60) explicitly shows the oscillatory damping of the anisotropy.

Here I summarize the above analysis. We began by briefly reviewing the statistical mechanics of a gas of massless particles in thermal equilibrium in general relativity theory. We considered both the non-degenerate and degenerate cases. Then we analyzed, in detail, the metric dependence of the stress-energy tensor for a non-interacting gas of massless particles in a Bianchi Type I cosmology. To closely mimic the actual physical situation, we let a gas in thermal equilibrium decouple completely from its surroundings at a time t_o, and we studied the subsequent evolution of the form of its stress-energy tensor. We mentioned the analysis of Doroshkevich et al. (1967) in one limiting case, derived a new general anisotropic solution to the field equations in another limiting case, and presented our own evaluation of the stress-energy tensor in the axisymmetric case (e.g., A = B). Finally, we studied the <u>dynamical</u> effects peculiar to non-interacting particles in anisotropic Bianchi Type I cosmologies. In particular, we considered the oscillatory damping of anisotropy (previously discussed by Doroshkevich <u>et al</u>. 1967 and Misner 1967, 1968). Our results here are directly applicable to the decoupling of gravitons, neutrinos and anti-neutrinos (at $T \approx 10^{10}$ °K), and photons (at $T \approx 3000$ °K). We did not consider gases of massive particles because they retain a thermal equilibrium stress-energy tensor until $T \lesssim 3000$ °K, and the effects of expansion anisotropies upon their stress-energy are extremely small after that.

3) VISCOUS DAMPING OF ANISOTROPY

In the previous subsection we studied the general metric dependence of the components of the stress-energy tensor for massless particles which decouple from their surroundings and become freely-propagating in Bianchi Type I cosmologies. When we study anisotropic cosmological models of our Universe in §§ II.D. and II.E., we will neglect all of the dynamical effects peculiar to non-interacting massless particles. In this subsection our objectives are threefold: (1) To summarize and critize the present state of knowledge concerning the dynamical effects of non-interacting massless particles in Bianchi Type I cosmological models. (2) To derive some new results which refine Misner's (1967, 1968) analysis of the viscous damping of anisotropy. (3) To justify, as far as is possible, our neglect of the peculiar dynamical effects of non-interacting massless particles in our anisotropic Bianchi Type I cosmological models in §§ II.D. and II.E.

At equation (II.C.6) we saw that all Bianchi Type I cosmologies encounter an initial physical singularity --- i.e., they have a hot, big-bang beginning. Near the singularity, Lifshitz and Khalatnikov (1960, 1963a,b) have shown that the dynamical effects of the anisotropy dominate those of the material content, and that Kasner's (1921) VACUUM solution [equations (II.B.7) and (II.B.8)] holds. Very near the singularity the temperature is high enough

($T >> 10^{12}$ °K) that all particles are in thermal equilibrium --- i.e., they behave like perfect fluids. The thermal equilibrium of all massive particles and photons is maintained by the strong and electromagnetic interactions until the photon temperature falls below 3000 °K. For $T \lesssim 3000$ °K the observed temperature isotropy of the 2.7 °K cosmic microwave radiation implies that the anisotropic stresses of all massive particles and photons are negligible. Gravitons and neutrinos, which participate only in weak interactions, are the only particle species which can decouple near the singularity (when $T \gtrsim 10^{10}$ °K) where large expansion anisotropies exist. Therefore, we must consider the possibility of large anisotropic stresses due to non-interacting gravitons and neutrinos.

Let us now summarize and critize all investigations to date of the dynamical effects of non-interacting gravitons and neutrinos in anisotropic Bianchi Type I cosmologies:

(a) The decoupling of gravitons has been mentioned by Doroshkevich *et al*. (1967) and Matzner (1967a), but no serious investigation of the anisotropic stresses of a free graviton gas in an anisotropic cosmology has yet been undertaken. The physical state of our Universe at the time that gravitons decoupled and the processes by which they decouple are both *very* uncertain at present. Due to this uncertainty, we choose here to simply assume that the dynamical effects of free gravitons are always negligible. Let us remember, however, the distinct possibility that free gravitons might dominate the dynamics in the early stages of anisotropic cosmologies. Let us also remark that, in analogy to the neutrino viscosity considered below, a graviton viscosity which eliminates most of the expansion anisotropies at the time of graviton decoupling is also a likely possibility!

(b) There is a general consensus of opinion on the properties of the decoupling of muon-neutrinos (ν_μ). The ν_μ decouple at approximately the same temperature that the $\mu^\pm$ pairs recombine ($T \approx 10^{12}$ °K). This decoupling of the ν_μ has been considered by Doroshkevich et al. (1967) and Misner (1968). The possibility of neutrino viscosity (see below) does not arise here, since the $\mu^\pm$ pairs necessary for ($\mu\ \nu_\mu$) weak interactions disappear rapidly due to recombination. Anisotropic stresses now begin to develop in the ν_μ gas due to the large expansion anisotropies near the initial singularity. The subsequent evolution of these anisotropic ν_μ stresses and of the Bianchi Type I cosmologies depends critically upon the processes which occur during the decoupling of the electron-neutrinos (ν_e) at $T \approx 10^{11}$ °K - 10^{10} °K. Doroshkevich et al. (1967), who neglect the possibility of ν_e viscosity, say that the anisotropic stresses induced in the neutrino gases have very important dynamical effects upon the subsequent evolution of these anisotropic cosmologies. However, Misner (1967, 1968), who first introduced the concept of ν_e viscosity, argues that the dynamical effects of anisotropic ν_μ stresses cannot manifest themselves before they are eliminated by the ν_e viscosity (see below).

(c) We see that the evolution of anisotropic Bianchi Type I cosmologies for $T \lesssim 10^{10}$ ^oK depends critically upon the physical processes occuring during ν_e decoupling. If there exists no reasonably efficient mechanism for damping out expansion anisotropies at $T \gtrsim 10^{10}$ ^oK, then we must accept the conclusion of Doroshkevich et al. (1967) that the anisotropic ν stresses will dominate the dynamics long after $T = 10^{10}$ ^oK. These anisotropic ν stresses (as we saw in the previous subsection) cause enormous oscillations of the expansion factors (A, B, C), and these oscillations are damped only by the general adiabatic expansion of these cosmologies. The two most important consequences of such oscillations are: (1) Primordial element formation at $T \approx 10^9$ ^oK - 10^8 ^oK is quite different from the results found in § II.D. and in the "standard" isotropic cosmological models. (2) The primordial neutrinos which exist today will probably have a highly anisotropic momentum distribution (i.e., they will probably exist in high-energy beams).

Misner (1967, 1968), however, has shown that a ν_e viscosity arises during the decoupling of the ν_e at $T \approx 10^{11}$ ^oK - 10^{10} ^oK. He argues that this viscosity will damp out most of the expansion anisotropy in this epoch, and that the anisotropic neutrino stresses which are generated after ν_e decoupling will have very small dynamical effects.

[Misner's work was subsequently extended to anisotropic Bianchi Type V and IX cosmologies by Matzner 1967b.] We will (below) derive some new results which refine Misner's analysis of ν_e viscosity, but first we must critize the analyses of Doroshkevich _et al_. (1967), Misner (1967, 1968), _and_ Matzner (1967b). The results of Doroshkevich _et al_. cannot be strictly correct, since they neglected the anisotropy-damping effects of ν_e viscosity. The investigation of Matzner is also incorrect, since he apparently assumed cosmologies containing only ν_e neutrinos during ν_e decoupling while neglecting the other particle species which exist at $T \geq 10^{10}$ °K. Misner _greatly overestimated_ the damping effects of ν_e viscosity in his approximate solution of the damping of anisotropy (see below). The true situation following ν_e decoupling must be the following: Expansion anisotropies have been substantially reduced by the ν_e viscosity, but enough expansion anisotropy probably remains to generate non-negligible anisotropic neutrino stresses.

In §§ II.D. and II.E. we will construct anisotropic Bianchi Type I cosmological models of our Universe in which we neglect _all_ of the possible dynamical effects of anisotropic neutrino stresses when $T \lesssim 10^{10}$ °K. Our conclusions above and our analysis below indicate that these models are a poor representation of the actual situation in the interval 10^9 °K $\lesssim T \lesssim 10^{10}$ °K, but that they are an adequate representation when $T \lesssim 10^9$ °K (where primordial element formation begins). In other words, the ν_e viscosity places limits upon the amount of expansion anisotropy with which our models may begin at $T \approx 10^{10}\ {}^oK$, and it reduces the dynamical effects of anisotropic neutrino stresses to the point where they can reasonably be neglected for $T \lesssim 10^9\ {}^oK$.

Let us now consider the damping of expansion anisotropies by ν_e viscosity in Bianchi Type I cosmologies. We begin by giving an abbreviated version of Misner's (1968) analysis. Misner began by assuming thermal equilibrium for $T \gtrsim 10^{11}\ {}^oK$, so that the diagonal Bianchi Type I metric of equation (II.B.2) holds. He made the change of variables:

$$(A, B, C) \equiv R(t) \exp(\beta_1, \beta_2, \beta_3) \qquad , \qquad \text{(II.C.61.a)}$$

with

$$\beta_1(t) + \beta_2(t) + \beta_3(t) = 0 \qquad . \qquad \text{(II.C.61.b)}$$

Here $R(t)$ represents the "mean expansion", and $\beta_i(t)$ the "anisotropic expansion", of these cosmologies. Then the Friedmann-like field equation (II.A.29)

$$ab + ac + bc = 8\pi T^0{}_0 \qquad , \qquad \text{(II.D.62)}$$

takes the form:

$$(\dot{R}/R)^2 = (8\pi/3)\ [T^0{}_0 + \rho_a] \qquad . \qquad \text{(II.C.63)}$$

Here we have defined the "anisotropy energy-density", ρ_a, which represents the "kinetic energy of shear" by:

$$\left.\begin{aligned} \rho_a &\equiv (16\pi)^{-1} \sum_{i=1}^{3} \dot{\beta}_i^{\,2} \\ &= (144\pi)^{-1}[(2a-b-c)^2 + (2b-a-c)^2 + (2c-a-b)^2] \\ &= (48\pi)^{-1} \; [(a-b)^2 + (a-c)^2 + (b-c)^2] \end{aligned}\right\} \quad . \qquad (II.C.64)$$

Proceeding in this fashion, Misner derives the remaining Einstein field equations and the conservation equation. When there is ν_e viscosity and $\rho_a >> T^0{}_0$ his equations are:

$$(\dot{R}/R)^2 = (8\pi/3)\, \rho_a \qquad , \quad (II.C.65.a)$$

$$(\dot{\beta}_i R^3)^{\cdot} = -16\pi\eta(t)\, (\dot{\beta}_i R^3) \qquad (i = 1,2,3) \quad , \quad (II.C.65.b)$$

$$(\rho_a R^6)^{\cdot} = -32\pi\eta(t)\, (\rho_a R^6) \qquad , \quad (II.C.65.c)$$

$$(\rho_\gamma R^4)^{\cdot} + (\rho_a R^6)^{\cdot}\, R^{-2} = 0 \qquad . \quad (II.C.65.d)$$

Here ρ_γ is the energy-density of all important thermalized particle species present (electron-neutrinos, $e^{\pm}$ pairs, and photons). The ν_e viscosity coefficient is $\eta(t)$, and the ν_e viscosity is due to small anisotropic ν_e stresses as the ν_e begin to decouple. Equation (II.C.65.d) shows that the viscous damping of anisotropy transfers energy from ρ_a to ρ_γ. Equation (II.C.65.d) is where Matzner assumed incorrectly that only the neutrinos experience viscous heating. In actuality, all of the thermalized particle species are heated by the viscous damping, but only the energy-density of the $\nu_e (\rho_\nu)$ contributes to the value of $\eta(t)$.

In equations (II.C.65), $\eta(t) = (4/15)\,\rho_\nu t_c$ is the coefficient of ν_e viscosity. Here t_c is the mean collision time for all important weak interaction (e ν_e) processes. From equation (II.C.29) and Matzner (1967b) we have:

$$\left.\begin{aligned} \eta &= (4/15)\,\rho_\nu t_c \\ &= (4/15)\,[(7/8)\,\tilde{a}T^4]\,[\sigma(1.202)\,45\tilde{a}T^3/2\pi^4]^{-1} \\ &\cong (0.84)\,G_W^{-2}T^{-1} \end{aligned}\right\} \quad . \quad \text{(II.C.66)}$$

Here we have set

$$t_c = (\sigma n)^{-1} \cong (G_W^2 T^2 n)^{-1} \qquad . \quad \text{(II.C.67)}$$

In equations (II.C.66) and (II.C.67) $\tilde{a}$ is the Stefan-Boltzmann constant; n is the number density of electrons and positrons, σ is the (V-A) weak interaction cross-section for all important (e ν_e) processes, and $G_W \approx 2 \times 10^{33}$ is the dimensionless weak coupling constant in our geometrized units. Equation (II.C.66) is valid only so long as the collision time is short compared with the expansion time of our cosmologies, $t_c < (\dot{R}/R)^{-1}$ [$\eta \equiv 0$ when $t_c(\dot{R}/R) \gtrsim 1$], and only so long as the $e^{\pm}$ pairs have not recombined. In equation (II.C.66) we have also assumed that the viscous heating is sufficiently slow so that $\rho_\nu \tilde{\propto} R^{-4}$ and $\rho_a \tilde{\propto} R^{-6}$.

In order to solve equations (II.C.65) and (II.C.66), Misner made the approximation:

$$\eta = \eta_o = \text{constant} \cong 4 \times 10^{-3}\ \text{sec}^{-1} \quad , \qquad \text{(II.C.68)}$$

where the value of η_o follows from equation (II.C.66) at the point where $t_c(\dot{R}/R) \approx 0.5$. Using the arbitrariness of coordinates in general relativity theory to place the initial singularity ($R = 0$) at $t = 0$, and to normalize to $R = 1$ as $t \rightarrow \infty$, the solution to equations (II.C.65) becomes:

$$R^3 = [1 - \exp(-t/\tau)] \quad , \text{(II.C.69.a)}$$

$$\beta_i = E_i \ ln\ [1 - \exp(-t/\tau)] \qquad (i = 1,2,3) \quad , \text{(II.C.69.b)}$$

$$\rho_a R^6 = (24\pi\tau^2)^{-1} \exp(-2t/\tau) \quad , \text{(II.C.69.c)}$$

$$\rho_\gamma R^4 = (\rho_\gamma R^4)_{R=0} + (16\pi\tau^2)^{-1}[1 - \exp(-t/\tau)]^{1/3}[3 + \exp(-t/\tau)] \ . \qquad \text{(II.C.69.d)}$$

Here $\tau \equiv (16\pi\eta_o)^{-1} \cong 4.8$ seconds is the viscous-damping time constant, and the E_i are constant parameters satisfying:

$$\sum_{i=1}^{3} E_i = 0 \ , \ \sum_{i=1}^{3} E_i^2 = 2/3 \qquad . \qquad \text{(II.C.70)}$$

This solution [equations (II.C.69) and (II.C.70)] is valid until $\rho_\gamma \gtrsim \rho_a$. Note that $\rho_\gamma = \rho_a$ when $t \cong \tau/8$, while $t_c(\dot{R}/R) < 1$ until $t \gtrsim \tau$.

When $\rho_\gamma > \rho_a$ and $t_c(\dot{R}/R) < 1$, Misner finds:

$$\left.\begin{aligned} R &\approx t^{1/2} \\ \rho_a R^6 &\approx \exp(-2t/\tau) \end{aligned}\right\} . \qquad \text{(II.C.71)}$$

Hence, ρ_a continues to decrease rapidly while the ν_e viscosity persists.

The ν_e are completely decoupled when $t_c(\dot{R}/R) \gtrsim 1$. Misner finds that this occurs when $t \approx \tau$ and $T \cong 2 \times 10^{10}$ °K. Non-negligible anisotropic neutrino stresses now begin to appear due to the expansion anisotropies which remain. As we previously mentioned in equation (II.C.60), Misner now finds the behavior:

$$\left.\begin{aligned} R &\approx t^{1/2} \\ \beta_i &\mathrel{\tilde{\propto}} t^{-1/4} \sin[K^2 \ln(t) + \text{constant}] \quad (i = 1,2,3) \\ \rho_a R^5 &\approx \text{constant} \end{aligned}\right\} , \text{(II.C.72)}$$

where K is a constant of order one. Hence, the anisotropic neutrino stresses cause small-amplitude damped oscillations of the expansion factors (A, B, C) about their average behavior. The effects of these oscillations become negligible soon after $T = 10^{10}$ °K.

Let us now carry out a somewhat more refined analysis of the effects of ν_e viscosity, and obtain some new results which give us a more realistic picture of the viscous damping of expansion anisotropies in Bianchi Type I cosmologies. We will use the form of the viscosity coefficient given by equation (II.C.66):

$$\eta \propto T^{-1} \propto R \quad . \qquad \text{(II.C.73)}$$

Although equation (II.C.73) is not strictly correct during ν_e decoupling, it is a much better approximation than η = constant, and it is strictly correct before decoupling becomes significant. Misner's approximation (η = constant) <u>greatly overestimates</u> the viscosity as $T \to \infty$ $(R \to 0)$. We shall write equation (II.C.73) as

$$\eta(t) = \eta_* R(t) \quad , \qquad \text{(II.C.74)}$$

where η_* is a constant [determined at the point where $t_c(\dot{R}/R) \cong 1$].

When $\eta(t) \neq 0$ and $\rho_a > T^0{}_0$, we must solve equations (II.C.65) and (II.C.74). From equations (II.C.65.c) and (II.C.74) we have:

$$(\dot{\rho}_a/\rho_a) + 6(\dot{R}/R) = -\,32\pi\eta_* R \quad . \qquad \text{(II.C.75)}$$

Using equation (II.C.65.a) in equation (II.C.75) gives:

$$(R^2\dot{R})^{\cdot} = -\,16\pi\eta_* R^3\dot{R} \quad . \qquad \text{(II.C.76)}$$

Now let R be our independent variable, so that:

$$(R^2\dot{R})^{\cdot} \equiv \dot{R}\ [d(R^2\dot{R})/dR] \qquad . \quad \text{(II.C.77)}$$

We can now immediately integrate equation (II.C.76) to find[5]:

$$16\pi\eta_* t = \ln\ [(1+R)/(1-R)] - 2\ \arctan\ (R) \qquad . \quad \text{(II.C.78)}$$

In equation (II.C.78) we have used the arbitrariness of coordinates in general relativity theory to place the initial singularity ($R = 0$) at $t = 0$, and to normalize to $R = 1$ as $t \to \infty$. Using equation (II.C.78) we can easily solve equations (II.C.75), (II.C.65.b), and (II.C.65.d), and we obtain:

$$\beta_i(t) = D_i \ln R(t) \qquad (i = 1,2,3), \quad \text{(II.C.79.a)}$$

$$\rho_a R^6 = 6\pi\eta_*^2\ (1-R^4)^2 \qquad , \quad \text{(II.C.79.b)}$$

$$\rho_\gamma R^4 = (\rho_\gamma R^4)_{R=0} + 8\pi\eta_*^2 R^2 (3-R^4) \qquad . \quad \text{(II.D.79.c)}$$

Here the D_i are constant parameters satisfying:

$$\sum_{i=1}^{3} D_i = 0 \quad , \quad \sum_{i=1}^{3} D_i^{\,2} = 6 \qquad . \quad \text{(II.C.80)}$$

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[5]We use the integral on page 21 of Gröbner and Hofreiter (1949):

$\int x^2(1-x^4)^{-1}dx = (1/4)\ \{\ln\ [(1+x)/(1-x]\ -\ 2\ \arctan\ (x)\}$.

Equations (II.C.79) and (II.C.80) represent our new, more realistic, results for the viscous damping of anisotropy when $\eta \neq 0$ and $\rho_a > \rho_\gamma$. From this solution, we find that $\rho_a \approx \rho_\gamma$ when $t_c(\dot{R}/R) \approx 1$. Therefore, expansion anisotropies are not negligible when the ν_e become completely decoupled. Misner's numerical solution when $\eta = 0$ [i.e., $t_c(\dot{R}/R) > 1$] and $\rho_a \gtrsim \rho_\nu$ [see Figure 2 of Misner 1968] now shows that anisotropic neutrino stresses are important in the interval $10^{10}\ {}^\circ K \gtrsim T \gtrsim 10^9\ {}^\circ K$. However, when $T \lesssim 10^9\ {}^\circ K$, we have $\rho_a < \rho_\nu$ and equations (II.C.72) hold. Hence, the expansion factors (A, B, C) exhibit small-amplitude damped oscillations about their average behavior, and the oscillations become negligible soon after $T \approx 10^9\ {}^\circ K$. We are, therefore, justified in neglecting the dynamical effects of anisotropic neutrino stresses when $T \lesssim 10^9\ {}^\circ K$.

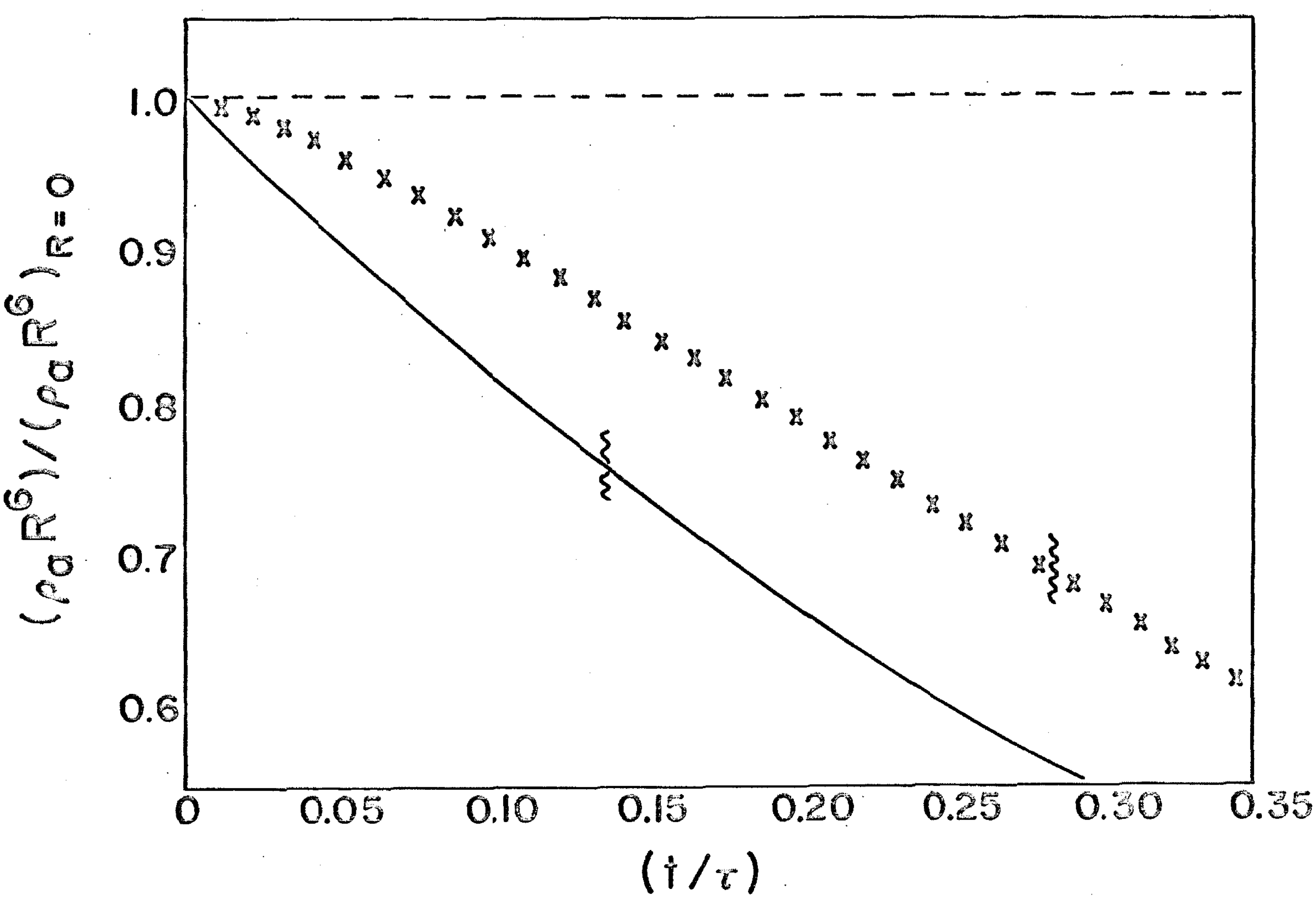
To compare equations (II.C.79) and (II.C.80) with the approximate solution of Misner when $\rho_a > \rho_\gamma$ [equations (II.C.69) and (II.C.70)], we must determine η_*. We find that $\eta \approx \eta_o$ when $t_c(\dot{R}/R) \approx 1$; but $R \approx 3/4$ when $t_c(\dot{R}/R) \approx 1$. Therefore, equation (II.C.74) implies

$$\eta_* \approx (4/3)\ \eta_o \qquad . \qquad \text{(II.C.81)}$$

We also note that $T \approx 2 \times 10^{10}\ {}^\circ K$ when $t_c(\dot{R}/R) \approx 1$. Comparing our solution with Misner's approximate solution, we find that $R(t)$ is practically the same in both cases. The behavior of $(\rho_a R^6)$ and (ρ_γ/ρ_a), however, is quite different in the two cases. In Figure 5

FIGURE 5

The viscous decay of the "anisotropy energy-density" (i.e., the magnitude of the shear) with time in Bianchi Type I cosmologies. Here $\tau \equiv (16\pi\eta_o)^{-1} \cong 4.8$ seconds is the characteristic viscous-damping time constant. When there is no neutrino viscosity ($\eta \equiv 0$), the dashed line shows that $\rho_a \propto R^{-6}$. When $\eta \neq 0$ there are two cases. The solid line is from Misner (1968), who made the approximation $\eta = \eta_o$ = constant. The starred line is the new result derived in this thesis, where we have used the more physical behavior $\eta(t) \propto R(t)$. The wiggly vertical lines indicate the point where $\rho_\gamma = \rho_a$ (see Figure 6). When $\rho_\gamma > \rho_a$ these viscous solutions are no longer valid.



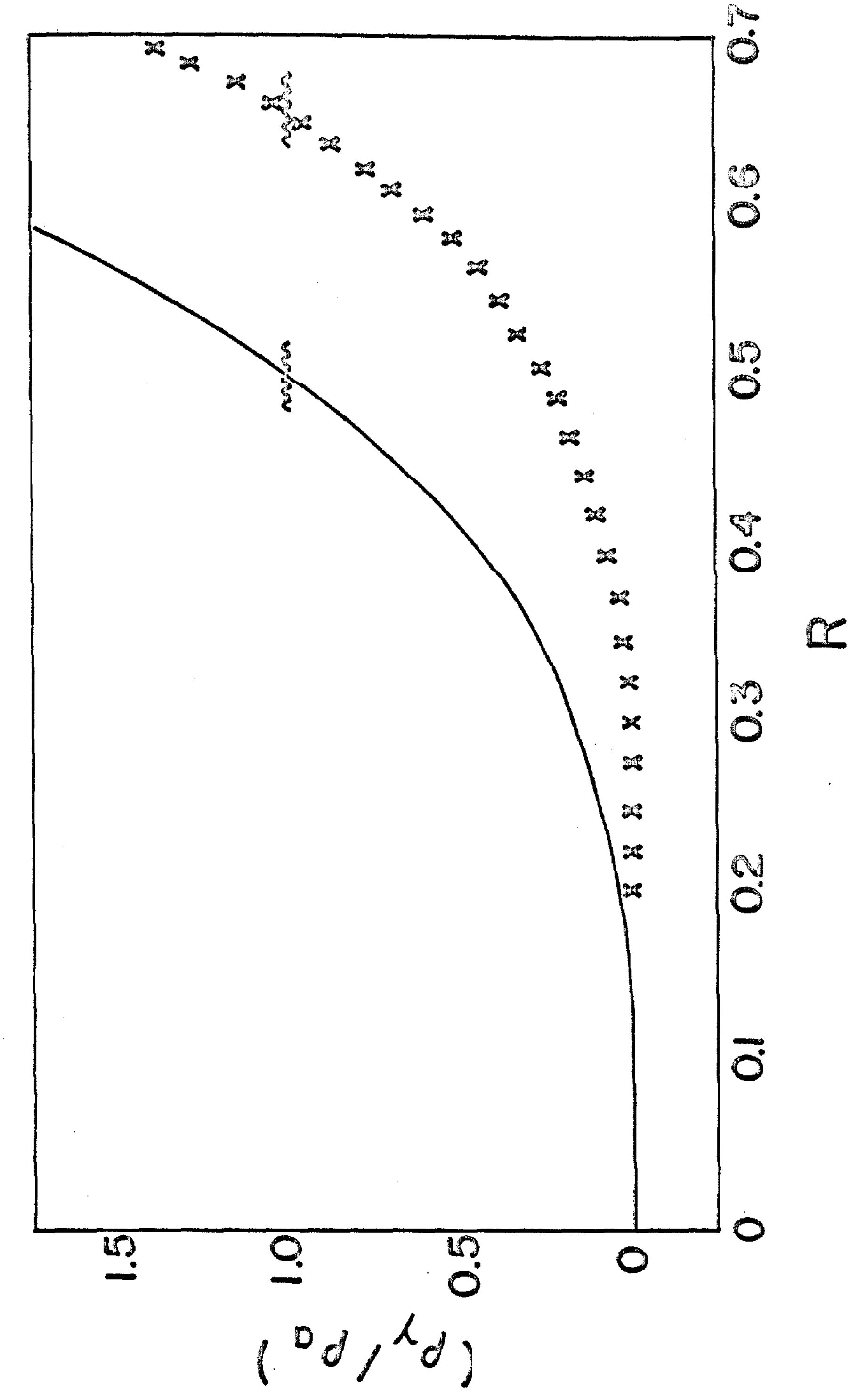
1.0
0.9
0.8
0.7
0.6
$(\rho_a R^6)/(\rho_a R^6)_{R=0}$
0
0.05
0.10
0.15
0.20
0.25
0.30
0.35
(t/τ)

we plot $(\rho_a R^6)/(\rho_a R^6)_{R=0}$ versus (t/τ), where $\tau \equiv (16\pi\eta_o)^{-1} \cong 4.8$ seconds, for the following three cases: (1) no ν_e viscosity $[\eta(t) = 0]$, (2) Misner's approximate solution $[\eta(t) = \eta_o = \text{constant}]$, and (3) our more realistic solution $[\eta(t) = \eta_* R(t) \approx (4/3)\cdot\eta_o R(t)]$. When $\eta(t) = 0$, we have the well-known behavior, $\rho_a \propto R^{-6}$. When $\eta \neq 0$, we see that ρ_a decays much faster than R^{-6}. In Misner's case (η = constant) we see that the effect of ν_e viscosity is greatly overestimated as we approach the singularity, while in our case here ($\eta \propto R$) we see the more physically reasonable behavior caused by $\eta \to 0$ as $R \to 0$. In Figure 6 we show the viscous heating of ρ_γ due to the damping of ρ_a. In this figure we have assumed the extreme behavior $(\rho_\gamma R^4) \to 0$ as $R \to 0$ in order to maximize the amount of anisotropy present near the singularity. When $(\rho_\gamma R^4)_{R=0} \neq 0$ we attain $\rho_\gamma = \rho_a$ for <u>smaller</u> values of R than those shown in Figure 6, and the amount of expansion anisotropy remaining after ν_e decoupling is <u>reduced</u>.

As a result of the analysis of this subsection, I have come to the following conclusions: (a) The dynamical effects of non-interacting neutrinos (i.e., anisotropic neutrino stresses and the damping of expansion anisotropies by neutrino viscosity) <u>must</u> be taken into account in any realistic anisotropic cosmological model of Bianchi Type I. (b) The true situation following neutrino decoupling is intermediate between that envisioned by Doroshkevich <u>et al</u>. (1967) and that envisioned by Misner (1967, 1968). (c) Neutrino viscosity

FIGURE 6

The viscous heating of the thermalized matter content (ρ_γ) during the viscous damping of the expansion anisotropy (ρ_a) in Bianchi Type I cosmologies. Here R(t) is the average distance scale factor (the "mean radius"). The solid curve is from Misner (1968), who uses the approximation $\eta = \eta_o$=constant. Using the more physically realistic behavior, $\eta(t) \propto R(t)$, we have derived in this thesis the new result represented by the starred curve. The wiggly horizontal lines mark the point where $\rho_\gamma = \rho_a$. When $\rho_\gamma > \rho_a$ these viscous solutions are no longer valid. Note that we have set $(\rho_\gamma R^4)_{R=0} \equiv 0$ in this figure so that we maximize ρ_a as $R \to 0$; when $(\rho_\gamma R^4)_{R=0} \neq 0$ we reach $\rho_\gamma = \rho_a$ at smaller values of R(and t) than those shown here.



R
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
(ρ_γ / ρ_a)
0
0.5
1.0
1.5

at T $\approx 10^{10}$ °K reduces expansion anisotropies to an essentially negligible level by the time that primordial element formation begins at T $\approx 10^9$ °K. (d) The "standard" isotropic cosmological models are an adequate representation of our Universe for T $\lesssim 10^9$ °K. (e) Anisotropic cosmological models of Bianchi Type I which totally neglect the effects of non-interacting neutrinos and which are constrained only by the available present-day observations (i.e., those models considered in §§ II.D. and II.E.) are grossly inadequate representations of the early stages of evolution of our Universe. The solutions and models presented in §§ II.D. and II.E. are useful and justifiable only because of the mathematical insight they provide us of possible anisotropic solutions to the Einstein field equations. The Einstein field equations are so complicated that any new analytical solutions which we can find --- and we derive many new solutions in §§ II.D. and II.E. --- are extremely useful in elucidating the structure and consequences of these equations.

II. D. THE PERFECT-FLUID MODELS

The primary purpose of this thesis is to gain an understanding of the effects of anisotropic expansion in cosmological models of our Universe. To facilitate this program we have restricted ourselves to spatially homogeneous cosmologies of Bianchi Type I (see § II.A.) with the diagonal metric

$$ds^2 = dt^2 - [A^2(t)\, dx^2 + B^2(t)\, dy^2 + C^2(t)\, dz^2] \quad . \qquad \text{(II.D.1)}$$

We saw in § II.A. that Einstein's field equations constrain a perfect fluid to be comoving in this coordinate system. In § II.C. we demonstrated why we will neglect the dynamical effects peculiar to non-interacting massless particles here, once the photon temperature drops below $T \approx 10^{10}\ {}^{\circ}K$ in our models. We now begin our investigation of semi-realistic[1] Bianchi Type I cosmological models of our Universe by considering the PERFECT-FLUID case. This work has been published, in condensed form, in Jacobs (1968).

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[1]Semi-realistic models are those which represent the gross features of the evolution of our Universe, while neglecting such fine-structure features as the development of density perturbations into galaxies, etc.

1) THE GOVERNING EQUATIONS

Let us consider cosmologies containing only a perfect fluid, characterized by the stress-energy tensor:

$$T^{\mu}{}_{\nu} = (\rho + p)\, u^{\mu}u_{\nu} - p\, \delta^{\mu}{}_{\nu} \quad . \qquad \text{(II.D.2)}$$

Here ρ is the uniform mass-density and p is the isotropic pressure of our comoving perfect fluid. In the diagonal Bianchi Type I metric of equation (II.D.1), the Einstein field equations (II.A.29) now become:

$$ab + ac + bc = +8\pi\rho \quad , \qquad \text{(II.D.3.a)}$$

$$(\dot{b} + \dot{c}) + b^2 + c^2 + bc = -8\pi p \quad , \qquad \text{(II.D.3.b)}$$

$$(\dot{a} + \dot{c}) + a^2 + c^2 + ac = -8\pi p \quad , \qquad \text{(II.D.3.c)}$$

$$(\dot{a} + \dot{b}) + a^2 + b^2 + ab = -8\pi p \quad , \qquad \text{(II.D.3.d)}$$

where our notation is the same as that of equations (II.A.29) and (II.A.30). As we remarked at equation (II.C.13), the conservation equation (II.A.32) for a perfect fluid becomes:

$$d(\rho V) + p\, dV = 0 \quad . \qquad \text{(II.D.4)}$$

Here we will study only perfect fluids with the barotropic equation of state

$$p = \gamma\rho \quad (0 \leq \gamma \leq 1) \quad . \qquad \text{(II.D.5)}$$

Combining equations (II.D.4) and (II.D.5) we have:

$$d\rho/[\rho(1 + \gamma)] = -\, dV/V \qquad , \qquad \text{(II.D.6.a)}$$

or

$$\rho \propto V^{-(1+\gamma)} \qquad . \qquad \text{(II.D.6.b)}$$

Our remarks at equations (II.C.6), (II.C.8), and (II.C.9) show that the general solution to equations (II.D.3) is now straightforward.

In order to clarify the structure of the system of equations (II.D.3), (II.D.5), and (II.D.6), let us make the change of variables

$$[A(t),\ B(t),\ C(t)] \equiv R(t)\ \exp[\alpha(t),\ \beta(t),\ \chi(t)] \qquad \text{(II.D.7.a)}$$

with

$$\alpha(t) + \beta(t) + \chi(t) = 0 \qquad . \qquad \text{(II.D.7.b)}$$

We will call (A, B, C) the "expansion functions", R the "mean radius", and (α, β, χ) the "anistropy functions". Equations (II.D.7) imply

$$V \equiv ABC = R^3 \qquad . \qquad \text{(II.D.8)}$$

Finally, we define the two independent anisotropy functions "perpendicular to" and "in" the x-y plane by:

$$(\eta, \sigma) \equiv (\alpha + \beta,\ \alpha - \beta) \qquad . \qquad \text{(II.D.9)}$$

It is now straightforward to see that the total system of equations governing the PERFECT-FLUID case becomes[2]:

$$3(\dot{R}/R)^2 - [3\dot{\eta}^2 + \dot{\sigma}^2)/4] = 8\pi\rho_* \,(R/R_*)^{-3(1+\gamma)} \qquad , \qquad \text{(II.D.10.a)}$$

$$\ddot{\eta} + 3(\dot{R}/R)\dot{\eta} = 0 \qquad , \qquad \text{(II.D.10.b)}$$

$$\ddot{\sigma} + 3(\dot{R}/R)\dot{\sigma} = 0 \qquad . \qquad \text{(II.D.10.c)}$$

Here the subscript star (*) denotes the value of a quantity at some fixed proper time.

2) THE GENERAL SOLUTION

Equations (II.D.10.b) and (II.D.10.c) immediately imply

$$(\dot{\eta}, \dot{\sigma})R^3 = (\dot{\eta}_*, \dot{\sigma}_*)R_*^3 = \text{constants} \qquad , \qquad \text{(II.D.11)}$$

and

$$(\eta, \sigma) = \int (\dot{\eta}_*, \dot{\sigma}_*)\,(R/R_*)^{-3}\,dt \qquad . \qquad \text{(II.D.12)}$$

Substituting equation (II.D.11) into equation (II.D.10.a) reduces the entire problem to quadratures:

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[2]This approach was developed independently by Misner (1967, 1968), in a slightly different context.

$$t = \int \{(8\pi\rho_*/3)\ (R/R_*)^{-(1+3\gamma)} + [(3\dot{\eta}_*^2 + \dot{\sigma}_*^2)/12]\ (R/R_*)^{-4}\}^{-\frac{1}{2}} d(R/R_*) . \tag{II.D.13}$$

Using equation (II.D.13) we can write equation (II.D.12) in the form:

$$(\eta,\sigma) = (\dot{\eta}_*,\dot{\sigma}_*)\int (R/R_*)^{-3}\{(8\pi\rho_*/3)(R/R_*)^{-(1+3\gamma)} + [(3\dot{\eta}_*^2 + \dot{\sigma}_*^2)/12]\ (R/R_*)^{-4}\}^{-\frac{1}{2}} d(R/R_*) \quad . \tag{II.D.14}$$

We now make some notational simplifications. Let

$$x \equiv t/\tau \equiv \text{ normalized time} \quad , \tag{II.D.15.a}$$

where

$$\tau \equiv (1 + \gamma)^{-1}\ (6\pi\rho_*)^{-\frac{1}{2}} \equiv \text{ time scale} \quad ; \tag{II.D.15.b}$$

and let

$$y \equiv [(R/R_*)^{3(1-\gamma)} + \Omega^2]^{\frac{1}{2}} \quad , \tag{II.D.15.c}$$

where

$$\Omega^2 \equiv 3[(1+\gamma)\tau/4]^2\ (3\dot{\eta}_*^2 + \dot{\sigma}_*^2) \equiv \text{ anisotropy parameter} \quad . \tag{II.D.15.d}$$

Note that Ω lies in the range $0 \leq |\Omega| < \infty$. Now our general solution of equations (II.D.13) and (II.D.14) becomes:

$$x = \begin{cases} [(1+\gamma)/(1-\gamma)] \int (y^2 - \Omega^2)^{[\gamma/(1-\gamma)]} dy & (0 \leq \gamma < 1) \\ (1+\Omega^2)^{-1/2} \, (R/R_*)^3 + \text{constant} & (\gamma = 1) \end{cases}, \quad \text{(II.D.16.a)}$$

$$\left(\frac{\eta}{\dot{\eta}_*}, \frac{\sigma}{\dot{\sigma}_*}\right) = \begin{cases} \left(\frac{\tau}{2\Omega}\right)\left(\frac{1+\gamma}{1-\gamma}\right) \ell n \left|\frac{y-\Omega}{y+\Omega}\right| + \text{constant} & (0 \leq \gamma < 1) \\ 3\tau(1+\Omega^2)^{-1/2} \, \ell n \, (R/R_*) + \text{constant} & (\gamma = 1) \end{cases}. \quad \text{(II.D.16.b)}$$

We note that there are two independent "anisotropy parameters" ($\dot{\eta}_*$ and $\dot{\sigma}_*$) in our general solution to the PERFECT-FLUID case. When $0 \leq \gamma < 1$, it is much more convenient to replace $\dot{\eta}_*$ and $\dot{\sigma}_*$ by two new anisotropy parameters, ϵ and ψ, defined by:

$$\epsilon \equiv -2\Omega \qquad (0 \leq |\epsilon| < \infty) \quad , \qquad \text{(II.D.17.a)}$$

$$\tau(\dot{\alpha}_*, \dot{\beta}_*, \dot{\chi}_*) \equiv \mp \left[\frac{2|\epsilon|}{3(1+\gamma)} \right] \sin(\psi, \ \psi + \frac{2\pi}{3}, \ \psi + \frac{4\pi}{3})$$

$$(0 \leq \psi < \frac{2\psi}{3}) \qquad . \quad \text{(II.D.17.b)}$$

Here the ($\mp$) sign is for $\epsilon \gtrless 0$, respectively. Note that equations (II.D.17) are consistent with equations (II.D.7.b), (II.D.9), and (II.D.15.d), as indeed they must be. The ψ - parameterization has the interesting and useful properties:

$$\sin(\psi) + \sin(\psi + \tfrac{2\pi}{3}) + \sin(\psi + \tfrac{4\pi}{3}) = 0$$

$$\sin^2(\psi) + \sin^2(\psi + \tfrac{2\pi}{3}) + \sin^2(\psi + \tfrac{4\pi}{3}) = 3/2$$

$$\sin(\psi)\sin(\psi + \tfrac{2\pi}{3}) + \sin(\psi)\sin(\psi + \tfrac{4\pi}{3}) + \sin(\psi + \tfrac{2\pi}{3})\sin(\psi + \tfrac{4\pi}{3}) = -3/4 \qquad .(\text{II.D.18})$$

Finally, we want to express our general solution in terms of the original variables. The expansion functions (A, B, C) are found by combining equations (II.D.7.a), (II.D.9), (II.D.16.b), and (II.D.17). The result is:

$$\frac{(A,B,C)}{R} \propto \left\{ \begin{array}{ll} \left\{\dfrac{[4(R/R_*)^{3(1-\gamma)} + \epsilon^2]^{1/2} + |\epsilon|}{[4(R/R_*)^{3(1-\gamma)} + \epsilon^2]^{1/2} - |\epsilon|}\right\}^{\pm \frac{2Z}{3(1-\gamma)}} & (0 \leq \gamma < 1) \\ (R/R_*)^{\mp 2|\epsilon| Z (4+\epsilon^2)^{-1/2}} & (\gamma = 1) \end{array} \right\}, \qquad (\text{II.D.19})$$

where the upper (lower) sign is for $\epsilon > (<)\ 0$, respectively, where

$$Z \equiv \sin(\psi,\ \psi + \tfrac{2\pi}{3},\ \psi + \tfrac{4\pi}{3}) \qquad , \qquad (\text{II.D.20})$$

and where R is given as a function of time, t, by equation (II.D.13) or equivalently (II.D.16.a). We obtain the Hubble expansion rates (a,b,c) from equations (II.A.30), (II.D.16.a), and (II.D.19):

$$(a,b,c) = \left\{ \frac{[4(R/R_*)^{3(1-\gamma)} + \epsilon^2]^{1/2} \mp 2|\epsilon|Z}{3(1+\gamma)\tau \ (R/R_*)^3} \right\} \quad (0 \leq \gamma \leq 1). \qquad \text{(II.D.21)}$$

From equations (II.D.6.b) and (II.D.8) we have the behavior of the total mass-density:

$$(\rho/\rho_*) = (R/R_*)^{-3(1+\gamma)} \quad . \qquad \text{(II.D.22)}$$

We have only to perform the remaining integral in equation (II.D.16.a) in order to find $R(t)$, the dependence of the mean radius upon proper time. We will examine this integral in the next subsection.

Let us consider here some of the properties of our general solution to the PERFECT FLUID case. Equation (II.D.22) shows that we always have an initial physical singularity ($R \to 0$). Equations (II.D.15.c) and (II.D.16.a) indicate that $R(t)$ is a monotonically increasing function of t. Equations (II.D.19) and (II.D.21) show that we have a continuous family of solutions for $0 \leq \gamma < 1$, with the following properties: (a) The initial singularity is of the PANCAKE type (i.e., $A \to 0$, B and $C \to$ constants, as $R \to 0$), when $\psi = \pi/6$ ($\epsilon > 0$) and $\psi = \pi/2$ ($\epsilon < 0$), while it is of the CIGAR type (i.e., $A \to \infty$, B and $C \to 0$, as $R \to 0$), for all other allowed values of ψ. (b) Near the initial singularity, the anisotropy dominates the dynamics and the metric asymptotically approaches that of Kasner's (1921) VACUUM solution [see equations (II.B.7) and (II.B.8)].

(c) As R (and t) $\to \infty$, the solutions become isotropic and asymptotically approach the completely isotropic ($\epsilon \equiv 0$) solutions, which have

$$A \equiv B \equiv C \equiv R \propto t^{2/[3(1+\gamma)]} \quad . \qquad \text{(II.D.23)}$$

(d) The solutions are axisymmetric (e.g., A = B for all t) when $\psi = \pi/6$ ($\epsilon > 0$) and $\psi = \pi/2$ ($\epsilon < 0$). Hence, axisymmetric cases can have PANCAKE or CIGAR singularities, while non-axisymmetric cases always have CIGAR singularities.

When $\gamma = 1$, we already have the complete explicit solution [i.e., no unevaluated integral in equation (II.D.16.a)]. This solution is quite different from the $0 \leq \gamma < 1$ cases. We shall examine this case, which we term the ZEL'DOVICH solution, in much greater detail a little later. Here we will only mention this, that the ZEL'DOVICH solution does not isotropize as $R \to \infty$ ($t \to \infty$), but remains highly anisotropic even in this limit.

3) SOME PARTICULAR EXAMPLES

In general, the remaining integral in equation (II.D.16.a) for $0 \leq \gamma < 1$ cannot be performed explicitly (i.e., in terms of elementary functions). This integral form is, however, admirably suited to evaluation by numerical integration. Here we will examine those cases in which I have been able to carry out the integration analytically. Fortunately, some of these cases happen to be the cases of greatest physical interest.

The DUST solution is characterized by $\gamma = 0$ (i.e., $p_d = 0$, where the subscript d denotes DUST). Now the integral of equation (II.D.16.a) yields directly:

$$x + \text{constant} = y \equiv [(R/R_*)^3 + \Omega^2]^{1/2} \quad . \qquad \text{(II.D.24)}$$

By a simple translation along the time axis, we can express equation (II.D.24) in the form:

$$(R/R_*) = [x_d\ (x_d + |\epsilon_d|)]^{1/3} \quad , \qquad \text{(II.D.25.a)}$$

where

$$\left.\begin{array}{l} x_d \equiv (t + t_d)/\tau_d = \text{normalized time} \\ \tau_d \equiv (6\pi\rho_{d*})^{-1/2} = \text{time scale} \end{array}\right\} \quad . \qquad \text{(II.D.25.b)}$$

Here t_d is a trivial constant of integration. The expansion functions (A, B, C) of equation (II.D.19) become:

$$(A, B, C) \propto x_d^{1/3(1\mp 2Z)}(x_d + |\epsilon_d|)^{1/3(1\pm 2Z)} \quad , \qquad \text{(II.D.26)}$$

while the Hubble expansion rates (a, b, c) take the form:

$$(a, b, c) = [2x_d + |\epsilon_d|(1 \mp 2Z)]/[3\tau_d x_d(x_d + |\epsilon_d|)] \quad . \qquad \text{(II.D.27)}$$

In equations (II.D.26) and (II.D.27) the upper (lower) sign is for $\epsilon_d > (<) 0$, and Z is given by equation (II.D.20). Finally, the total mass-density is:

$$(\rho_d/\rho_{d*}) = [x_d(x_d + |\epsilon_d|)]^{-1} \quad . \qquad \text{(II.D.28)}$$

The initial physical singularity occurs at $x_d = 0$. This general DUST solution [equations (II.D.25.b) through (II.C.28)] was found previously by Raychaudhuri (1958), Schücking and Heckmann (1958) [see also Heckmann and Schücking (1962)], and Robinson (1961). This solution has also been given and discussed recently by Saunders (1967), Ellis and MacCallum (1968), and Misner (1968). The axisymmetric case (A = B for all time) has been given and discussed by Zel'dovich (1964, 1965b), Kompaneets and Chernov (1964), Doroshkevich (1965, 1966), Thorne (1967), Ellis (1967), and Stewart and Ellis (1968). When $\epsilon_d = 0$ we recover the standard, isotropic Einstein-de Sitter (1932) solution with

$$A = B = C \equiv R(t) \propto x_d^{2/3} \quad . \qquad \text{(II.D.29)}$$

The RADIATION solution is characterized by $\gamma = 1/3$ (i.e., $p_r = \rho_r/3$, where the subscript r denotes RADIATION). This cosmology is filled with either massless particles or ultra-relativistic massive particles, all with isotropic velocity distributions. The analytical solution may be written in two different --- but equally useful and equivalent --- forms: (1) The integral of equation (II.D.16.a) is straightforward, yielding:

$$x_r = \left(\frac{1}{2}\right) \left[\left(\frac{R}{R_*}\right) F - \left(\frac{\epsilon_r^2}{2}\right) \ln \left| \frac{2(R/R_*) + F}{|\epsilon_r|} \right| \right] , \quad \text{(II.D.30)}$$

where

$$\left. \begin{aligned} F &\equiv [4(R/R_*)^2 + \epsilon_r^2]^{1/2} \\ x_r &\equiv (t + t_r)/\tau_r = \text{normalized time} \\ \tau_r &\equiv [3/(32\pi\rho_{r*})]^{1/2} = \text{time scale} \end{aligned} \right\} . \quad \text{(II.D.31)}$$

Here t_r is a constant of integration. From equations (II.D.19) through (II.D.22) we easily find the expansion functions (A, B, C), the Hubble expansion rates (a, b, c), and the total mass-density (ρ_r):

$$\left. \begin{aligned} (A, B, C)/R &\propto [(F + |\epsilon_r|)/(F - |\epsilon_r|)]^{\pm Z} \\ (a, b, c) &= (F \mp 2|\epsilon_r|Z)/[4\tau_r(R/R_*)^3] \\ \rho_r/\rho_{r*} &\equiv 3p_r/\rho_{r*} = (R/R_*)^{-4} \end{aligned} \right\} . \quad \text{(II.D.32)}$$

Here the upper (lower) sign is for $\epsilon_r > (<) 0$, and Z is given by equation (II.D.20). Note that R(t) is now defined implicitly by equations (II.D.30) and (II.D.31). The initial physical singularity occurs at $x_r = 0$. This general RADIATION solution [equations (II.D.30) through (II.D.32)] is a new result. The axisymmetric case was previously given in essentially this form by Thorne (1967) and Stewart and Ellis (1968), while Doroshkevich (1965, 1966) and Shikin (1966) previously gave equivalent axisymmetric solutions in a completely different form (see below). When $\epsilon_r = 0$ we recover the standard, isotropic Tolman (1934) solution with

$$A = B = C \equiv R(t) \propto x_r^{1/2} \quad . \qquad \text{(II.D.33)}$$

(2) A second --- completely equivalent --- form of the RADIATION solution consists of two parts. We obtain the $\epsilon_r > 0$ part of our RADIATION solution by using the arbitrariness of coordinates in general relativity theory to set

$$\epsilon_r^2 \equiv +4 \quad , \quad (R/R_*) \equiv [2\xi/(\xi^2 - 1)] \quad . \qquad \text{(II.D.34)}$$

Note that the range of ξ is $1 \leq \xi < \infty$. Equations (II.D.30) through (II.D.32) now take the form:

$$\left.\begin{aligned} x_r &= 2\left\{\left[\frac{\xi(\xi^2+1)}{(\xi^2-1)^2}\right] - \left(\frac{1}{2}\right) \ln\left|\frac{\xi+1}{\xi-1}\right|\right\} \\ (A,B,C) &\propto (\xi^2-1)^{-1}\,\xi^{1+2Z} \\ (a,b,c) &= (2\tau_r)^{-1}\left(\frac{\xi^2-1}{2\xi}\right)^3\left[\left(\frac{\xi^2+1}{\xi^2-1}\right) - 2Z\right] \\ \rho_r/\rho_{r*} &\equiv 3p_r/\rho_{r*} = [(\xi^2-1)/(2\xi)]^4 \end{aligned}\right\} . \qquad \text{(II.D.35)}$$

This solution emerges from the initial physical singularity at $\xi = \infty$ with highly anisotropic expansion rates, but becomes isotropic as $\xi \to 1$. The initial singularity is of the CIGAR (PANCAKE) type when $\psi \neq \pi/6$ ($\psi = \pi/6$). The axisymmetric case, which occurs when $\psi = \pi/6$ or $\pi/2$, was previously given in this form by Doroshkevich (1965, 1966). We note that this representation of the RADIATION solution is peculiar in that it cannot represent the limiting isotropic case. This peculiarity follows directly from equation (II.D.34) where we have stipulated $\epsilon_r \neq 0$. We obtain the $\epsilon_r < 0$ case of our RADIATION solution when we set

$$\epsilon_r^{\,2} \equiv +4 \quad , \quad (R/R_*) \equiv [2\xi/(1 - \xi^2)] \quad . \qquad \text{(II.D.36)}$$

The range of ξ is now $0 \leq \xi \leq 1$. Equations (II.D.30) through (II.D.32) become:

$$\left.\begin{aligned} x_r &= 2\left\{\left[\frac{\xi(\xi^2 + 1)}{(1 - \xi^2)^2}\right] - \left(\frac{1}{2}\right) \ln \left|\frac{1 + \xi}{1 - \xi}\right|\right\} \\ (A,B,C) &\propto (1 - \xi^2)^{-1}\, \xi^{1 + 2Z} \\ (a,b,c) &= (2\tau_r)^{-1} \left(\frac{1 - \xi^2}{2\xi}\right)^3 \left[\left(\frac{1 + \xi^2}{1 - \xi^2}\right) - 2Z\right] \\ \rho_r/\rho_{r*} &= 3p_r/\rho_{r*} = [(1 - \xi^2)/(2\xi)]^4 \end{aligned}\right\} . \qquad \text{(II.D.37)}$$

The initial physical singularity occurs at $\xi = 0$, and the expansion rates approach isotropy as $\xi \to 1$. The singularity is of the CIGAR (PANCAKE) type when $\psi \neq \pi/2$ ($\psi = \pi/2$). When $\psi = \pi/6$ or $\pi/2$, we have the axisymmetric case which was previously given in this form by Doroshkevich (1965, 1966). Again we can see why this representation cannot describe the limiting isotropic cases with $\epsilon_r \equiv 0$.

I have been able to find two infinite sequences of explicit solutions to the integral of equation (II.D.16.a), when $1/3 < \gamma < 1$. I call these the HARD solutions. These solutions are characterized by $\rho_h/3 < p_h = \gamma\rho_h < \rho_h$, where the subscript h denotes HARD. The solutions split into the following cases: (1) One infinite sequence of solutions results for

$$\gamma/(1 - \gamma) \equiv n = \text{integer} \qquad (1 \leq n < \infty) \quad . \qquad \text{(II.D.38.a)}$$

We then have the following sequence of γ's:

$$\gamma \equiv n/(n + 1) = 1/2,\ 2/3,\ 3/4,\ 4/5,\ \ldots \quad . \qquad \text{(II.D.38.b)}$$

From page 18 of Gröbner and Hofreiter (1949), we see that the integral of equation (II.D.16.a) yields:

$$x_h = \left\{ [(2n + 1)/4] \left[4(R/R_*)^{3/(n+1)} + \epsilon_h^2 \right]^{1/2} \sum_{\nu=0}^{n} (-\epsilon_h^2/4)^\nu \frac{n!(n-\nu+1/2)!}{(n-\nu)!(n+1/2)!} (R/R_*)^{3(n-\nu)/(n+1)} \right\}, \qquad \text{(II.D.39)}$$

where

$$\left. \begin{aligned} x_h &\equiv (t + t_h)/\tau_h = \text{normalized time} \\ \tau_h &\equiv [(n+1)/(2n + 1)]\,(6\pi\rho_{h*})^{-1/2} = \text{time scale} \end{aligned} \right\} . \qquad \text{(II.D.40)}$$

Equations (II.D.19) through (II.D.22) then become:

$$\left.\begin{aligned}
\frac{(A,B,C)}{R} &\propto \left\{\frac{[4(R/R_*)^{3/(n+1)} + \epsilon_h^2]^{1/2} + |\epsilon_h|}{[4(R/R_*)^{3/(n+1)} + \epsilon_h^2]^{1/2} - |\epsilon_h|}\right\}^{\pm 2(n+1)Z/3} \\
(a,b,c) &= \left[\frac{(n+1)}{3(2n+1)\tau_h(R/R_*)^3}\right] \left\{[4(R/R_*)^{3/(n+1)} + \epsilon_h^2]^{1/2} \mp 2|\epsilon_h|Z\right\} \\
\rho_h/\rho_{h*} &= (n+1)p_h/n\rho_{h*} = (R/R_*)^{-3(2n+1)/(n+1)}
\end{aligned}\right\} \quad .(\text{II.D.41})$$

Here the upper (lower) sign is for $\epsilon_h > (<) \; 0$, and Z is given by equation (II.D.20). We note that R(t) is defined implicitly by the finite series in equation (II.D.39). This first infinite sequence of HARD solutions emerges from the initial physical singularity (at R = 0) with highly anisotropic expansion rates, but as $x_h \to \infty$ the solutions become isotropic with $(A, B, C) \approx R(t) \underset{\sim}{\propto} x_h^{[2(n+1)/3(2n+1)]}$. This entire infinite sequence of HARD solutions is new. (2) A second infinite sequence of HARD solutions appears for

$$\gamma/(1 + \gamma) \equiv m + 1/2 = \text{half-integer} \qquad (0 \leq m < \infty) . (\text{II.D.42.a})$$

This gives the following sequence of γ's:

$$\gamma \equiv (2m + 1)/(2m + 3) = 1/3,\ 3/5,\ 5/7,\ 7/9,\ \ldots \qquad (\text{II.D.42.b})$$

Note that we are retaining the RADIATION case ($\gamma = 1/3$) here as a check on our computations. From pages 36 - 38 of Gröbner and Hofreiter (1949) we see that the integral in equation (II.D.16.a) yields:

$$x_h = 2[(2m + 1)!!/m!] \Bigg[\Bigg[\Bigg\{ (1/4)Y\ (4Y^2 + \epsilon_h^2)^{1/2} \sum_{\nu=0}^{m} \left(\frac{-\epsilon_h^2}{8}\right)^{\nu} \left[\frac{(m-\nu)!}{(2m-2\nu+1)!!} \right] Y^{2(m-\nu)} \Bigg\} + \Bigg\{ \left(\frac{-\epsilon_h^2}{8}\right)^{m+1} \ln \left|(1/2)\ (4Y^2 + \epsilon_h^2)^{1/2} + Y\right| \Bigg\} \Bigg]\Bigg] \quad , \text{(II.D.43)}$$

where

$$\left.\begin{aligned} Y &\equiv (R/R_*)^{[3/(2m+3)]} \\ x_h &\equiv (t + t_h)/\tau_h = \text{normalized time} \\ \tau_h &\equiv [(2m + 3)/\ 4(m + 1)]\ (6\pi\rho_{h*})^{-1/2} \end{aligned}\right\} . \quad \text{(II.D.44)}$$

Here t_h is a constant of integration. When $m = 0$ ($\gamma = 1/3$), equation (II.D.43) immediately reduces to our previous RADIATION solution of equation (II.D.30). In general here, equations (II.D.19) through (II.D.22) take the form:

$$\left.\begin{aligned}
\frac{(A,B,C)}{R} &\propto \left[\frac{(4Y^2+\epsilon_h^2)^{1/2}+|\epsilon_h|}{(4Y^2+\epsilon_h^2)^{1/2}-|\epsilon_h|}\right]^{\pm[(2m+3)Z/3]} \\
(a,b,c) &= \left[\frac{(2m+3)}{12(m+1)\tau_h(R/R_*)^3}\right]\ [(4Y^2+\epsilon_h^2)^{1/2} \mp 2|\epsilon_h|Z] \\
\rho_h/\rho_{h*} &= \quad (2m+3)p_h/[(2m+1)\rho_{h*}] = (R/R_*)^{-[12(m+1)/(2m+3)]}
\end{aligned}\right\}. \quad (II.D.45)$$

Here the upper (lower) sign is for $\epsilon_h > (<)\ 0$, and Z is given by equation (II.D.20). The initial singularity occurs at $R = 0$, and as $x_h \to \infty$ these solutions isotropize to $(A, B, C) \approx R(t) \underset{\sim}{\propto} x_h^{[(2m+3)/6(m+1)]}$. This entire second infinite sequence of HARD solutions $(0 \leq m < \infty)$ is new.

Zel'dovich (1961) (see also Harrison 1965) has discussed the possibility of matter with the equation of state $p_z = \rho_z$. This is the "hardest" equation of state permitted by causality (see pages 105-106 of Harrison et al. 1965). Here we will say that we have the ZEL'DOVICH case when $\gamma = 1$, with the subscript z denoting ZEL'DOVICH. We have already found the complete ZEL'DOVICH solution in equations (II.D.16) above. To simplify our result we make the change of parameter

$$\delta \equiv -\ |\epsilon_z|\ (4+\epsilon_z^2)^{-1/2} \qquad (0 \leq |\delta| < 1) \quad , \qquad (II.D.46)$$

where δ is now our new first anisotropy parameter. Equation (II.D.16.a) then becomes

$$(R/R_*) = x_z^{1/3} \quad , \qquad \text{(II.D.47.a)}$$

where

$$\left.\begin{aligned} x_z &\equiv (t + t_z)/\tau_z = \text{normalized time} \\ \tau_z &\equiv [(1 - \delta^2)/(24\pi\rho_{z*})]^{1/2} = \text{time scale} \end{aligned}\right\} . \qquad \text{(II.D.47.b)}$$

Equations (II.D.19) through (II.D.22) --- in our general ZEL'DOVICH solution --- now take the form:

$$\left.\begin{aligned} (A,B,C) &\propto x_z^{(1/3)(1\pm 2|\delta|Z)} \\ (a,b,c) &= (1 \pm 2|\delta|Z)/(3\tau_z x_z) \\ \rho_z/\rho_{z*} &= p_z/\rho_{z*} = x_z^{-2} \end{aligned}\right\} . \qquad \text{(II.D.48)}$$

Here the upper (lower) sign is for $\delta > (<) \ 0$, and Z is given by equation (II.D.20). The initial physical singularity occurs at $x_z = 0$. The expansion rates are always highly anisotropic, even as $x_z \to \infty$. The axisymmetric case of the ZEL'DOVICH solution occurs when $\psi = \pi/6$ or $\pi/2$ (for all δ). In the axisymmetric case we return to the well-known behavior (see Doroshkevich 1965, 1966):

$$\left.\begin{array}{c} A = B \propto x_z^{\,v} \quad , \quad C \propto x_z^{\,(1-2v)} \\ (1/3 \leq v < 2/3) \end{array}\right\} . \qquad \text{(II.D.49)}$$

When $\delta \equiv 0$ (all ψ) we return to the standard, isotropic result:

$$A = B = C \equiv R(t) \propto x_z^{\,1/3} \quad . \qquad \text{(II.D.50)}$$

The general ZEL'DOVICH solution of equations (II.D.47) and (II.D.48) exhibits initial singularities of the CIGAR, POINT (i.e., A and B and $C \to 0$ as $R \to 0$), and BARREL (i.e., A = constant, B and $C \to 0$ as $R \to 0$) types, but it has <u>no PANCAKE singularities</u>. In Table 1 we display the possible types of initial singularities in the ZEL'DOVICH solution and the ranges of δ and ψ within which each type is found. This general ZEL'DOVICH solution is entirely new.

The final case which we will consider here is the DUST-PLUS-RADIATION case. These anisotropic cosmologies contain a non-interacting mixture of DUST ($\gamma = 0$) and RADIATION ($\gamma = 1/3$). They are generalizations of the similar isotropic cosmologies of equations (I.C.32), which have been previously considered by Lemaître (1927, 1930, 1931), Chernin (1965), Alpher, Gamow, and Herman (1967), Jacobs (1967), Cohen (1967), McIntoch (1968), Harrison (1968), and the earlier authors cited in these works.

TABLE 1

Types of Singularity in the ZEL'DOVICH Solution.[a,b]

	CIGAR[c]	POINT[d]	BARREL[e]
$3^{-1/2} < \delta < 1$	$0 \leq \psi < \frac{2\pi}{3} - \psi_o$ $\frac{\pi}{3} + \psi_o < \psi < \frac{2\pi}{3}$	$\frac{2\pi}{3} - \psi_o < \psi < \frac{\pi}{3} + \psi_o$	$\psi = \frac{2\pi}{3} - \psi_o$ $\psi = \frac{\pi}{3} + \psi_o$
$\frac{1}{2} < \delta \leq 3^{-1/2}$	$\psi_o - \frac{\pi}{3} < \psi < \frac{2\pi}{3} - \psi_o$	$0 \leq \psi < \psi_o - \frac{\pi}{3}$ $\frac{2\pi}{3} - \psi_o < \psi < \frac{2\pi}{3}$	$\psi = \psi_o - \frac{\pi}{3}$ $\psi = \frac{2\pi}{3} - \psi_o$
$\lvert\delta\rvert = 1/2$		All ψ except $\psi = \pi/6$ $(\delta > 0)$ $\psi = \pi/2$ $(\delta < 0)$	$\psi = \pi/6$ $(\delta > 0)$ $\psi = \pi/2$ $(\delta < 0)$
$0 \leq \lvert\delta\rvert < 1/2$		All ψ	
$-(3^{-1/2}) < \delta < -\frac{1}{2}$	$\psi_o < \psi < \pi - \psi_o$	$0 \leq \psi < \psi_o$ $\pi - \psi_o < \psi < \frac{2\pi}{3}$	$\psi = \psi_o$ $\psi = \pi - \psi_o$
$\delta = -(3^{-1/2})$	$\frac{\pi}{3} < \psi < \frac{2\pi}{3}$	$0 < \psi < \frac{\pi}{3}$	$\psi = 0$ $\psi = \frac{\pi}{3}$
$-1 < \delta < -(3^{-1/2})$	$0 \leq \psi < \frac{\pi}{3} - \psi_o$ $\psi_o < \psi < \frac{2\pi}{3}$	$\frac{\pi}{3} - \psi_o < \psi < \psi_o$	$\psi = \frac{\pi}{3} - \psi_o$ $\psi = \psi_o$

[a]The effective range of ψ is $0 \leq \psi < 2\pi/3$.

[b]We define ψ_o by $\psi_o = \arcsin(1/2\lvert\delta\rvert)$, and it has the range $\pi/6 < \psi_o < \pi/2$.

[c]As we approach the singularity $A \to \infty$, B and $C \to 0$.

[d]As we approach the singularity A, B, and C all $\to 0$.

[e]As we approach the singularity A = constant, B and $C \to 0$.

In the DUST-PLUS-RADIATION case we consider a perfect-fluid stress-energy tensor with:

$$\rho = \rho_d + \rho_r \quad , \quad p = p_r = \rho_r/3 \quad , \qquad \text{(II.D.51)}$$

where the subscripts d and r denote DUST and RADIATION, respectively. Since the two material constituents are non-interacting, equations (II.D.4) and (II.D.8) imply:

$$\rho_d/\rho_{d*} = (R/R_*)^{-3} \quad , \quad \rho_r/\rho_{r*} = (R/R_*)^{-4} \quad . \qquad \text{(II.D.52)}$$

Equations (II.D.3), (II.D.7), and (II.D.10) readily reduce this problem to quadratures:

$$\left.\begin{aligned} x &= \frac{3}{2} \int \frac{(R/R_*)^2 \, d(R/R_*)}{\{(R/R_*)^2[(R/R_*) + S_*] + (\epsilon^2/4)\}^{1/2}} \\ (\alpha,\beta,\chi) &= \mp |\epsilon| Z \int \frac{(R/R_*)^{-1} \, d(R/R_*)}{\{(R/R_*)^2[(R/R_*) + S_*] + (\epsilon^2/4)\}^{1/2}} \end{aligned}\right\}, \text{(II.D.53.a)}$$

where

$$\left.\begin{aligned} x &\equiv (t + \bar{t})/\tau_d = \text{normalized time} \\ \tau_d &\equiv (6\pi\rho_{d*})^{-1/2} = \text{time scale} \\ S_* &\equiv \rho_{r*}/\rho_{d*} = \text{mixture parameter} \end{aligned}\right\} . \qquad \text{(II.D.53.b)}$$

Here $\bar{t}$ is a constant of integration, we have the upper (lower) sign for $\epsilon > (<) \, 0$, and Z is given by equation (II.D.20). Note that our two independent anisotropy parameters are again ϵ $(0 \leq |\epsilon| < \infty)$ and ψ $(0 \leq \psi < 2\pi/3)$. The two integrals in equation (II.D.53.a) can be evaluated analytically in terms of elliptic functions (see page 17 of Abramowitz and Stegun 1965; also pages 60 - 61 and 75 ff. of Gröbner and Hofreiter 1949). When this is done, the complete DUST-PLUS-RADIATION solution takes on the complicated form:

$$(A,B,C) \propto R \exp(\alpha,\ \beta,\ \chi) \qquad ,\text{(II.D.54.a)}$$

$$\begin{aligned} x = {} & \{(R/R_*)^2[(R/R_*) + S_*] + (\epsilon^2/4)\}^{1/2} \\ & - \mu S_* \left\{ \left(\frac{2}{\mu^2}\right) \left[E(\Phi,k) + \left(\frac{1 + \cos\Phi}{\sin\Phi}\right)(1 - k^2 \sin^2\Phi)^{1/2} \right] \right. \\ & \left. - (r + s\cot\Theta)\, F(\Phi,k) \right\} \end{aligned} \qquad ,\text{(II.D.54.b)}$$

$$\begin{aligned} (\alpha,\beta,\chi) = {} & \mp |\epsilon|\, Z\, [\mu/(s\tan\Theta - r)] \left\{ F(\Phi,k) \right. \\ & \left. - (1 - \zeta)^{-1} \left[\Pi\left(\Phi, \frac{\zeta^2}{1-\zeta^2}, k\right) - \zeta\, D_4\left(\Phi, \frac{\zeta^2}{1-\zeta^2}, k\right) \right] \right\} , \end{aligned} \qquad \text{(II.D.54.c)}$$

$$(a,b,c) = \left[\frac{\{4(R/R_*)^2[(R/R_*) + S_*] + \epsilon^2\}^{1/2} \mp 2|\epsilon| Z}{3\tau_d (R/R_*)^3} \right] \qquad ,\text{(II.D.54.d)}$$

$$\rho/\rho_{d*} = (R/R_*)^{-4}[(R/R_*) + S_*] \quad , \quad p/\rho_{d*} = (S_*/3)(R/R_*)^{-4} \quad , \tag{II.D.54.e}$$

where

$$\left.\begin{aligned}
\mu &\equiv [(\sin 2\Theta)/s]^{1/2} \quad ; \quad k \equiv |\sin\Theta| \quad ; \\
\Phi &\equiv \arccos\left\{\frac{[(R/R_*)-r]-s\cot\Theta}{[(R/R_*)-r]+s\tan\Theta}\right\} \quad , \quad 0 \le \Phi \le \pi \quad ; \\
\Theta &\equiv (1/2)\arctan\left[\frac{(\Psi_+ - \Psi_-)}{3^{1/2}(\Psi_+ + \Psi_-)}\right] \quad , \quad 0 \le \Theta < \pi/2 \quad ; \\
r &\equiv -\left[\left(\frac{\Psi_+ + \Psi_-}{2}\right) + \left(\frac{S_*}{3}\right)\right] \quad , \quad s \equiv 3^{1/2}\left(\frac{\Psi_+ - \Psi_-}{2}\right) \quad ; \\
\zeta &\equiv \left(\frac{s\tan\Theta - r}{s\cot\Theta + r}\right) \quad ; \\
\Psi_\pm &\equiv \left\{-\left[\left(\frac{\epsilon^2}{8}\right) + \left(\frac{S_*^{\,3}}{27}\right)\right] \pm \left(\frac{|\epsilon|}{2}\right)\left[\left(\frac{\epsilon^2}{16}\right) + \left(\frac{S_*^{\,3}}{27}\right)\right]^{1/2}\right\}^{1/3} .
\end{aligned}\right\} \tag{II.D.55}$$

In equations (II.D.54) and (II.D.55) the upper (lower) sign is for $\epsilon > (<) 0$, and Z is given by equation (II.D.20). This DUST-PLUS-RADIATION solution is entirely new. It emerges from an initial physical singularity at $R = 0$ looking "exactly" like the RADIATION solution of equations (II.D.30) through (II.D.32). When $R/R_* \approx S_*$ it passes into the form of the DUST solution of equations (II.D.25) through (II.D.28). Finally, it isotropizes to the standard, isotropic Einstein-de Sitter (1932) DUST solution as $x \to \infty$. The axisymmetric case (e.g., $A = B$ for all time) occurs when $\psi = \pi/6$ and $\pi/2$; this axisymmetric solution is also a new result. When $\epsilon = 0$ (for all ψ) we recover the standard, "flat", isotropic Lemaître results of equations (I.C.32) for the DUST-PLUS-RADIATION case.

3) THE CONSTRUCTION OF SEMI-REALISTIC MODELS

We will now use the exact solutions which we have obtained above to construct semi-realistic, anisotropic, cosmological models of our Universe. In these models we desire only to represent the overall features of the possible evolution of our Universe, and to study the effects of anisotropic expansions, especially in the early stages of our Universe's evolution. We will proceed in three stages.

First, let us consider the material content of our models. Our models will contain only dust and radiation. Let us denote the present value of a quantity by the subscript zero (o). We assume that the "dust" today has the present critical mass-density necessary to give us the "flat" space sections of Bianchi Type I cosmologies:

$$\rho_{do} = 10^{-29} \text{ gm cm}^{-3} \quad . \qquad \text{(II.D.56)}$$

The radiation consists solely of the observed 2.7 $^{\circ}$K cosmic microwave radiation (photons) with

$$\rho_{ro} = 4.5 \times 10^{-34} \text{ gm cm}^{-3} \quad . \qquad \text{(II.D.57)}$$

(To include the energy-density in neutrinos and gravitons would probably change this by a factor of about 2.) We will idealize the dust and radiation as mutually non-interacting (see footnote 4 of § I. C. for justification). This implies that our models will be valid only after the relativistic electron-positron gas has recombined at a temperature $T \approx 10^{10}\ {}^{o}K$. We will neglect any effects due to unobserved gravitons and neutrinos. Although Doroshkevich et al. (1967) have shown that non-interacting (i.e., non-thermalized) gravitons and neutrinos are driven into extremely energetic beams near the initial physical singularity of an anisotropic cosmology, they have neglected the possibility of viscous damping which was subsequently pointed out by Misner (1967, 1968). In § II. C. above we illustrated and improved Misner's (1967, 1968) argument, that neutrino viscosity in the relativistic electron-positron gas (at $T \gtrsim 10^{10}\ {}^{o}K$) will probably largely eliminate those dynamical effects peculiar to non-interacting neutrinos by the time that our models here are really useful ($T \lesssim 10^{9}\ {}^{o}K$). We will assume hereafter that the peculiar dynamical effects of non-interacting neutrinos and gravitons can be neglected in our models. Finally, we will not consider perfect fluids with $p > \rho/3$ here, because we expect to encounter such matter --- if ever --- only when $\rho > \rho_{baryon} \approx 10^{14}$ gm cm^{-3}, and this occurs long before the pairs recombine at $T \approx 10^{10}\ {}^{o}K$.

Secondly, we must consider how we will construct our models.

We could use the "analytical" DUST-PLUS-RADIATION solution of equations (II.D.54) and (II.D.55), which is the exact mathematical solution to our problem. It is easy to see, however, that this DUST-PLUS-RADIATION solution is extremely cumbersome to use in practice. Instead, we shall use the same approximation procedure that we used in § I. C. and in Figure 2: We will join an earlier exact RADIATION solution [equations (II.D.30) through (II.D.32)] smoothly [i.e., with expansion functions (A, B, C) and Hubble expansion rates (a, b, c) continuous] to a later exact DUST solution [equations (II.D.25) through (II.D.28)], at the point where $\rho_d = \rho_r$. This join point, which marks the transition from the radiation-dominated phase to the dust-dominated phase, occurs at the time t_J where

$$(R_J/R_o) = S_o \equiv \rho_{ro}/\rho_{do} \approx 5 \times 10^{-5} \quad . \qquad \text{(II.D.58)}$$

Here the subscript J denotes the JOIN point. The excellence of this approximation procedure is of the same degree as that illustrated for the isotropic case in Figure 2: The approximation deviates from the exact solution only very near the join point, and then only very slightly. The algebra involved in carrying out this approximation procedure is quite straightforward, and the resulting semi-realistic models are as follows: (i) For $t > t_J$ we are in the dust-dominated phase. Equations (II.D.25) through (II.D.28) hold here, with equation (II.D.26) normalized to:

$$(A,B,C) = x_d^{(1/3)(1\mp 2Z)}(x_d + |\epsilon_d|)^{(1/3)(1\pm 2Z)} \quad , \qquad \text{(II.D.59)}$$

and with Z given by equation (II.D.20). The anisotropy parameters lie in the full ranges:

$$0 \leq |\epsilon_d| < \infty \quad , \quad 0 \leq \psi < 2\pi/3 \qquad , \qquad \text{(II.D.60.a)}$$

and we set:

$$\rho_{do} = 10^{-29} \text{ gm cm}^{-3} \quad , \quad \tau_d \equiv (6\pi\rho_{do})^{-1/2} = 9 \times 10^9 \text{ years.} \quad \text{(II.D.60.b)}$$

Finally, the constant of integration, t_d, turns out to be given by

$$t_d = (\tau_d/8) \left\{ (4S_o^{\ 3} + \epsilon_d^2)^{1/2} - 4|\epsilon_d| + (3/2)S_o^{-3/2}\epsilon_d^2 \ \ln \left| [2S_o^{\ 3/2} + (4S_o^{\ 3} + \epsilon_d^2)^{1/2}]/|\epsilon_d| \right| \right\}, \quad \text{(II.D.61.a)}$$

where

$$S_o \equiv \rho_{ro}/\rho_{do} = 4.5 \times 10^{-5} \qquad . \quad \text{(II.D.61.b)}$$

(ii) For $t < t_J$ we are in the radiation-dominated phase. Equations (II.D.30) through (II.D.32) hold here, in the form:

$$t/\tau_r = (1/2)[\bar{R}\ F - (\epsilon_r^2/2)\ \ln |\ (2\bar{R} + F)/\ |\epsilon_r|\ |\ ] \qquad , \quad \text{(II.D.62.a)}$$

$$(A/A_J, B/B_J, C/C_J) = \bar{R} \left\{ \left(\frac{F+|\epsilon_r|}{F-|\epsilon_r|} \right) \left[\frac{(4+\epsilon_r^2)^{1/2} - |\epsilon_r|}{(4+\epsilon_r^2)^{1/2} + |\epsilon_r|} \right] \right\}^{\pm Z} \qquad , \quad \text{(II.D.62.b)}$$

$$(a,b,c) = [F \mp 2\,|\epsilon_r|\,Z]/(4\tau_r\,\bar{R}^3) \quad , \qquad \text{(II.D.62.c)}$$

$$\rho_r/\rho_{rJ} = (\bar{R})^{-4} \quad , \qquad \text{(II.D.62.d)}$$

where

$$\left.\begin{array}{l} F \equiv (4\,\bar{R}^2 + \epsilon_r^2)^{1/2} \\ \bar{R} \equiv (R/R_J) \end{array}\right\} . \qquad \text{(II.D.62.e)}$$

Here the anisotropy parameter ψ is exactly the same as that appearing in equation (II.D.60.a), while we have

$$|\epsilon_r| = S_o^{-3/2}\,|\epsilon_d| \quad . \qquad \text{(II.D.63)}$$

We determine the remaining parameters and constants in equations (II.D.62) as follows:

$$\left.\begin{array}{c} S_o \equiv \rho_{ro}/\rho_{do} = 4.5 \times 10^{-5} \\ \rho_{ro} = 4.5 \times 10^{-34}\ \text{gm cm}^{-3} \ , \ \tau_r \equiv (3/4)S_o^{3/2}\tau_d = 2000\ \text{years} \\ (R_J/R_o) \equiv S_o \ , \ \rho_{rJ} \equiv \rho_{do}S_o^{-3} = 10^{-16}\ \text{gm cm}^{-3} \\ (A_J,B_J,C_J) \equiv S_o \left[\dfrac{(\epsilon_d^2 + 4S_o^3)^{1/2} + |\epsilon_d|}{(\epsilon_d^2 + 4S_o^3)^{1/2} - |\epsilon_d|}\right]^{\pm 2Z/3} \end{array}\right\} . \qquad \text{(II.D.64)}$$

(iii) Finally, the time of the transition (join time) is given by:

$$t_J = (3\tau_d/8)\left[(\epsilon_d^2 + 4S_o^{\ 3})^{1/2} - \left(\frac{\epsilon_d^2}{2S_o^{\ 3/2}}\right) \ \ell n \left|\frac{2S_o^{\ 3/2} + (\epsilon_d^2 + 4S_o^{\ 3})^{1/2}}{|\epsilon_d|}\right|\right] \quad . \text{(II.D.65)}$$

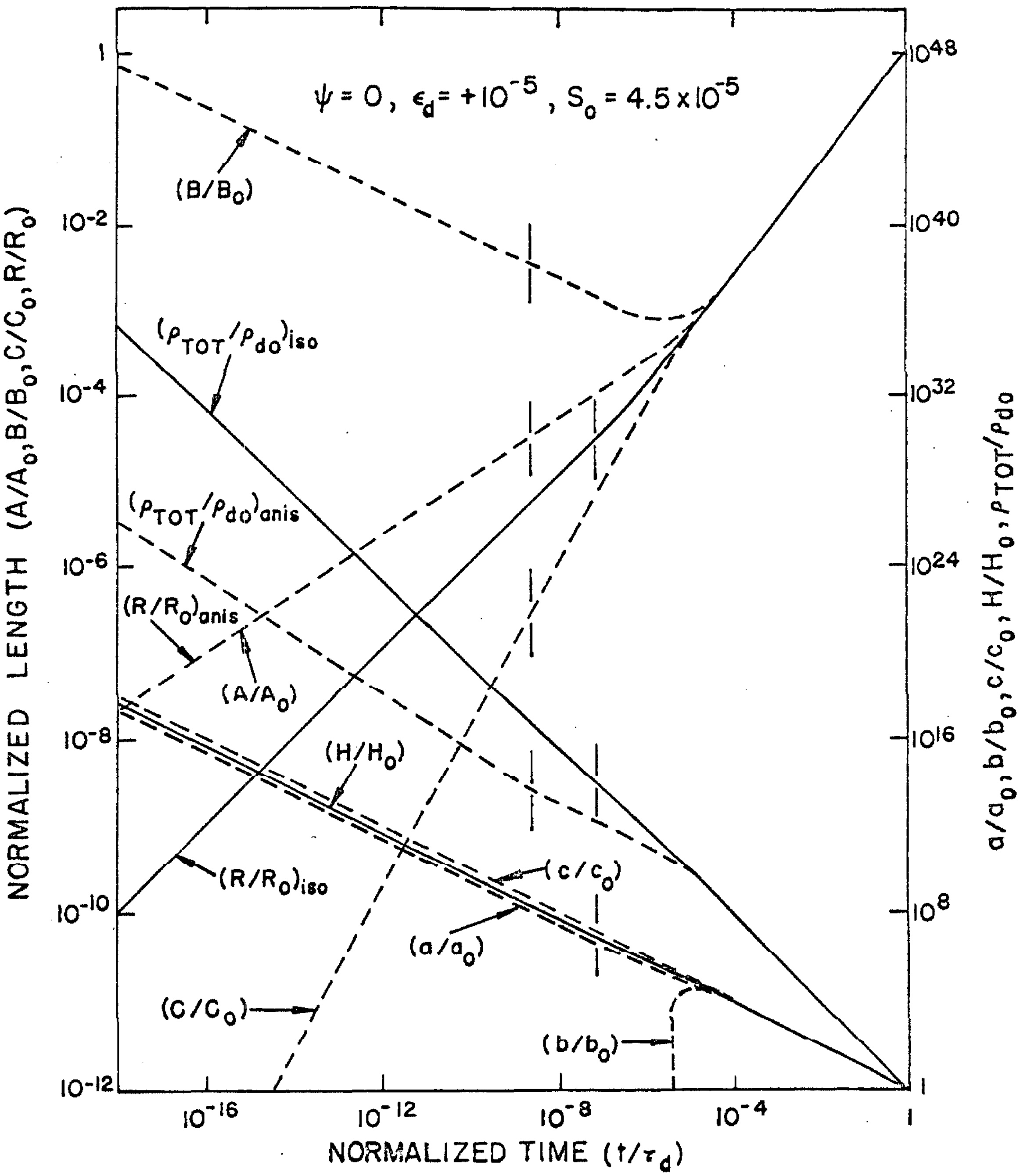
Thirdly, we want to see what our anisotropic, approximate, DUST-PLUS-RADIATION cosmological models look like and how they behave. When we examine the temperature anisotropy of the 2.7 °K cosmic microwave radiation below, we will find that, for two opposite extreme assumptions about the material content of intergalactic space, the recent observational data limit the permissible range of $|\epsilon_d|$ to

$$0 \leq |\epsilon_d| \lesssim \left\{\begin{matrix} 10^{-4} \text{ for H II} \\ 10^{-7} \text{ for H I} \end{matrix}\right\} \quad . \qquad \text{(II.D.66)}$$

Here H II signifies that the "dust" has consisted almost entirely of ionized intergalactic hydrogen since a redshift of at least $z \equiv (R_o/R) - 1 \approx 9$; while H I means that the ionized hydrogen from the primeval fireball recombined (i.e., neutralized) when the photon temperature dropped below about 3000 °K and the entire intergalactic "dust" content of our Universe has remained neutral since then. We will also find (below) that the observations of the 2.7 °K cosmic microwave radiation place no restrictions upon the range of ψ. Based upon these observational limitations on the anisotropy parameters (ϵ_d and ψ), I have explicitly evaluated equations (II.D.59) through (II.D.65) for a representative series of allowed anisotropic models. In Figure 7 we compare our anisotropic model with $\psi = 0$ and $\epsilon_d = +10^{-5}$

FIGURE 7

The semi-realistic, anisotropic DUST-PLUS-RADIATION cosmological model of our Universe with $\psi = 0$ and $\epsilon_d = +10^{-5}$ (dashed lines) compared with the "flat", isotropic DUST-PLUS-RADIATION model of Jacobs (1967) (solid lines). We show the "expansion functions" (A/A_o, B/B_o, C/C_o), the "mean radius" (R/R_o), the normalized Hubble expansion rates (a/a_o, b/b_o, c/c_o, and H/H_o), and the normalized total mass-density (ρ_{TOT}/ρ_{do}), all as functions of normalized time (t/τ_d). The constants which appear are: $H_o^{-1} = 13 \times 10^9$ years, $\rho_{do} = 10^{-29}$ gm cm^{-3}, $\tau_d \equiv (6\pi\rho_{do})^{-1/2} = 9 \times 10^9$ years, and the "mixture parameter" $S_o \equiv \rho_{ro}/\rho_{do} = 4.5 \times 10^{-5}$. The relativistic electron-positron pairs recombine at $R/R_o \approx 10^{-10}$; primordial element formation takes place in the range $R/R_o \approx 10^{-9} - 10^{-8}$; the anisotropic model enters the DUST phase (at $t = t_J$) at the left-hand set of vertical bars, while the isotropic models enters at the right-hand set ($t \approx 2000$ years); expansion anisotropies become small for $t/\tau_d \gtrsim 10^{-5}$. Note that the anisotropic model encounters a CIGAR type initial singularity at $t \to 0$.


ψ = 0, ε_d = +10^-5, S_0 = 4.5 x 10^-5
(B/B0)
(ρTOT/ρdo)iso
(ρTOT/ρdo)anis
(R/R0)anis
(A/A0)
(H/H0)
(c/c0)
(a/a0)
(R/R0)iso
(C/C0)
(b/b0)
NORMALIZED LENGTH (A/A0, B/B0, C/C0, R/R0)
a/a0, b/b0, c/c0, H/H0, ρTOT/ρdo
NORMALIZED TIME (t/τd)
1
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^48
10^40
10^32
10^24
10^16
10^8
1
10^-16
10^-12
10^-8
10^-4
1

to the corresponding isotropic DUST-PLUS-RADIATION model of Jacobs (1967). In Figure 8 we do the same for the anisotropic model with $\psi = \pi/2$ and $\epsilon_d = - 10^{-10}$. These two explicit cases demonstrate all of the essential features of our semi-realistic anisotropic DUST-PLUS-RADIATION models. From these two figures we note the following properties: (a) The period in which the expansion is appreciably anisotropic moves towards smaller values of t as $|\epsilon_d|$ decreases. (b) The point, at which $T = 10^{10}$ °K, moves rapidly towards smaller values of t as $|\epsilon_d|$ increases. (c) The number-density of baryons (i.e., protons) is much lower at any given time in the anisotropic case than in the isotropic case, during the period of primordial element formation (when $T \approx 10^9$ °K). Put differently, the average rate of expansion out of the initial singularity is much greater in the anisotropic case. This fact greatly affects primordial element production.

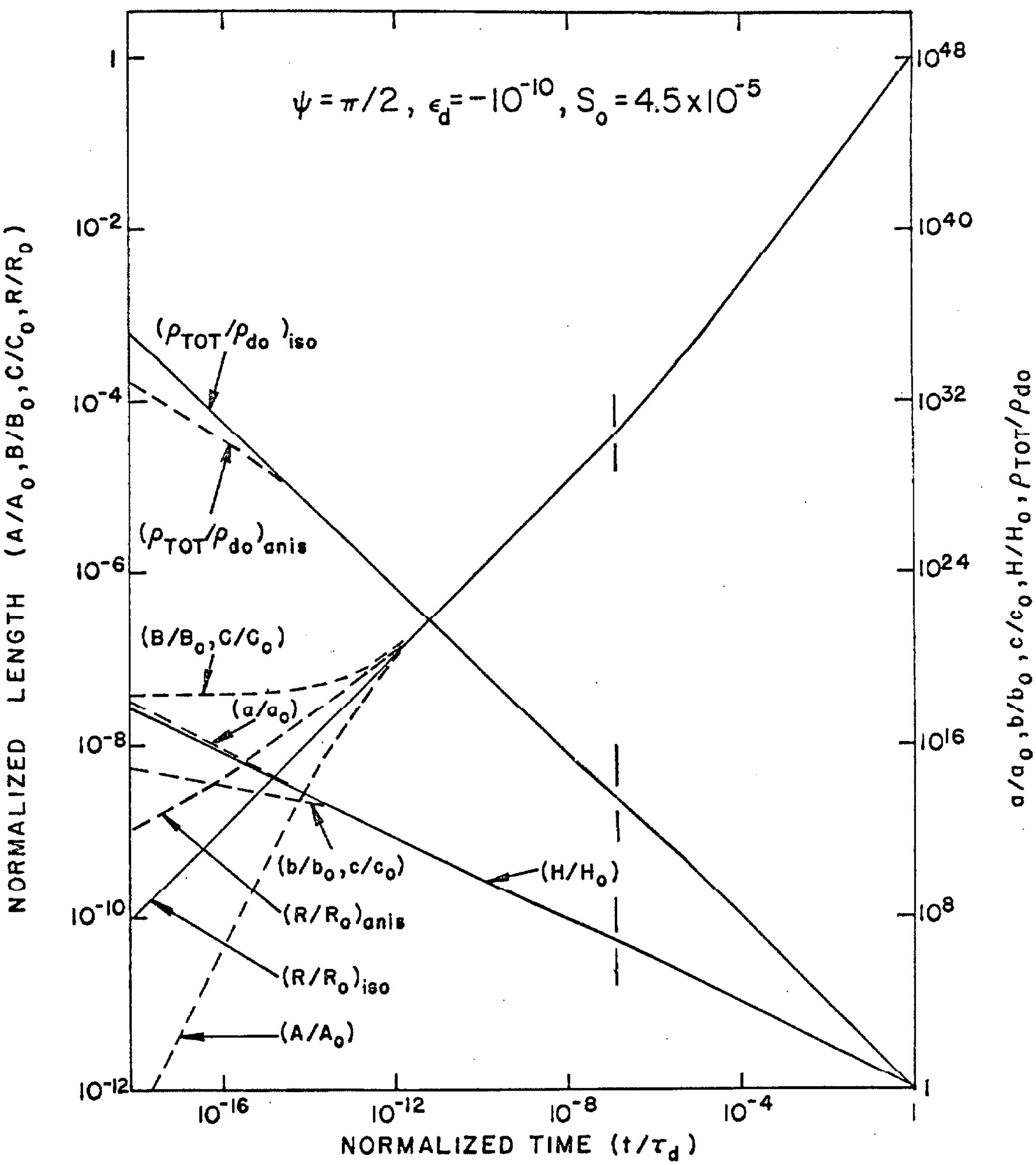
5) APPLICATIONS TO THE REAL UNIVERSE

Let us now apply the anisotropic cosmological models which we have constructed and illustrated above. We will list and analyze the important physical processes which can take place:

(a) We have neglected certain processes which take place during the interval between the initial physical singularity and the point (at $T \approx 10^{10}$ °K) when our models become valid. In general, the anisotropy dominates the dynamics during this period, and the

FIGURE 8

Comparing the semi-realistic, anisotropic DUST-PLUS-RADIATION cosmological model of our Universe with $\psi = \pi/2$ and $\epsilon_d = -10^{-10}$ (<u>dashed</u> lines) to the "flat", isotropic DUST-PLUS-RADIATION model of Jacobs (1967) (<u>solid</u> lines). We show the same quantities as in Figure 7, and the normalizing constants are the same as in Figure 7. Expansion anisotropies become small (at $t/\tau_d \approx 10^{-14}$) during the RADIATION phase, so that the transition to the DUST phase occurs at $t \approx 2000$ years for both the isotropic and the anisotropic cases (vertical bars). As in Figure 7, we have recombination of the relativistic electron-positron pairs at $R/R_o \approx 10^{-10}$ and primordial element formation near $R/R_o \approx 10^{-9} - 10^{-8}$. Note that the anisotropic model here is axisymmetric ($B = C$ for all time), and that it encounters a PANCAKE type initial singularity as $t \to 0$.



$\psi=\pi/2,\ \epsilon_d=-10^{-10},\ S_0=4.5\times10^{-5}$
NORMALIZED LENGTH $(A/A_0, B/B_0, C/C_0, R/R_0)$
$a/a_0, b/b_0, c/c_0, H/H_0, \rho_{TOT}/\rho_{do}$
NORMALIZED TIME (t/τ_d)
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{48}
10^{40}
10^{32}
10^{24}
10^{16}
10^{8}
1
10^{-16}
10^{-12}
10^{-8}
10^{-4}
1
$(\rho_{TOT}/\rho_{do})_{iso}$
$(\rho_{TOT}/\rho_{do})_{anis}$
$(B/B_0, C/C_0)$
(a/a_0)
$(b/b_0, c/c_0)$
(H/H_0)
$(R/R_0)_{anis}$
$(R/R_0)_{iso}$
(A/A_0)

Bianchi Type I metric which holds is Kasner's (1921) VACUUM solution. From equations (II.B.7) through (II.B.9) we see that there is only one PANCAKE type singularity, while in general we find CIGAR type singularities. As long as all elementary particle species remain thermalized, the behavior of the metric cannot affect the local thermodynamical behavior of the species, except for the dilution-effect caused by the expansion of proper volume elements. The gases expand adiabatically, except when the temperature drops below the mass of a given species of a massive particle-antiparticle gas. Then, the recombination of that relativistic particle-antiparticle gas causes heating of the material content. The final heating is due to the recombination of relativistic electron-positron pairs at $T \approx 10^{10}$ °K. Only gravitons and neutrinos can decouple from their surroundings when $T > 10^{10}$ °K. In § II.C. we showed how electron-neutrino viscosity in the relativistic electron-positron pair gas largely damps out the dynamical effects peculiar to non-interacting neutrinos before $T \approx 10^{10}$ °K (note, however, our qualifying remarks there). Our ignorance of the possible graviton content of the Universe, and of the processes in which gravitons are formed in its early stages leads us to make the simplifying assumption, that we can neglect <u>any</u> effects due to gravitons. Therefore, we assume that our simple models become valid at $T \approx 10^{10}$ °K. The work of Doroshkevich <u>et al</u>. (1967) and Misner (1967, 1968) indicates that this assumption is reasonable, though not rigorously justifiable.

(b) The first question to be answered in our anisotropic models is this, "When do expansion anisotropies become small?". Let us define the mean-square expansion anisotropy by:

$$\left(\frac{\Delta H}{H}\right)^2 \equiv 3 \left[\frac{(a-b)^2 + (a-c)^2 + (b-c)^2}{(a+b+c)^2}\right] \quad , \quad \text{(II.D.67.a)}$$

where the average Hubble expansion rate is:

$$H \equiv (a+b+c)/3 \quad . \quad \text{(II.D.67.b)}$$

The largest expansion anisotropy occurs in Kasner's (1921) VACUUM solution (near the initial singularity) where [see equations (II.B.7) and (II.B.8)]:

$$\left(\frac{\Delta H}{H}\right)^2 = 3 \left[\frac{(\ell-m)^2 + (\ell-n)^2 + (m-n)^2}{(\ell+m+n)^2}\right] = 6 \quad . \quad \text{(II.D.68)}$$

Therefore, we must have:

$$0 \leq |\Delta H/H| \leq 6^{1/2} \quad . \quad \text{(II.D.69)}$$

For $t < t_J$, we are in the RADIATION phase, where equations (II.D.18), (II.D.62.c), and (II.D.62.e) tell us that:

$$(\Delta H/H)^2 = 6\,[1 + (2\bar{R}/|\epsilon_r|^2]^{-1} \quad . \quad \text{(II.D.70)}$$

From equations (II.D.61.b) and (II.D.63), we see that expansion anisotropies become small [i.e., $(\Delta H/H) < 1$] during the RADIATION phase, only if $|\epsilon_d| \lesssim 3 \times 10^{-7}$; and equation (II.D.62.a) shows that in this case the time when the anisotropy becomes small is:

$$t \lesssim (3.8 \times 10^{6})\, \epsilon_d^2\, \tau_d = (3.4 \times 10^{+16})\, \epsilon_d^2 \text{ years.} \qquad \text{(II.D.71)}$$

For $t > t_J$, we are in the DUST phase, where equations (II.D.18), (II.D.27), and (II.D.67.a) tell us that:

$$(\Delta H/H)^2 = 6\, [1 + (2x_d/|\epsilon_d|)]^{-2} \quad . \qquad \text{(II.D.72)}$$

Equations (II.D.25.a), (II.D.64), and (II.D.72) imply that $(\Delta H/H) > 1$ during the DUST phase only if $|\epsilon_d| \gtrsim 3 \times 10^{-7}$; and in this case equation (II.D.72) tells us that the time when anisotropies become small is:

$$t \approx |\epsilon_d|\tau_d \approx (9 \times 10^{9})\, |\epsilon_d| \text{ years} \quad . \qquad \text{(II.D.73)}$$

Therefore, equation (II.D.66) indicates that expansion anisotropies are always small during the DUST phase for case H I, while in case H II the expansion anisotropies can be large in both the RADIATION and DUST phases.

(c) Next, we want to consider the effects of expansion anisotropy upon primordial element formation. Primordial element formation takes place at temperatures in the range $T \approx 10^9\ {}^oK - 10^8\ {}^oK$; this fact follows directly from nuclear physics. In our semi-realistic, anisotropic models --- as in the standard isotropic models --- this temperature range is always encountered in the RADIATION phase. To calculate the final relative abundances of the primordial elements formed, we need to know the average Hubble expansion rate, $(\dot{R}/R)$, as a function of the total mass-energy, ρ_{TOT} [cf., equation (B.18) and the associated discussion in Thorne 1967]. Such a relation, together with the equations:

$$\rho_d/\rho_{do} = (R/R_o)^{-3} \quad , \quad \rho_{TOT}/\rho_{ro} = (R/R_o)^{-4} \quad , \qquad \text{(II.D.74)}$$

tells us how the number-density of baryons varies with time during element production. Using equations (II.D.25) through (II.D.28) and (II.D.59) through (II.D.65) of our anisotropic DUST-PLUS-RADIATION models --- and especially equation (II.D.62.a) --- it is straightforward to find:

$$\left(\frac{\dot{R}}{R}\right) = (1.35 \times 10^7)\, \rho_{TOT}^{3/4} \left[\epsilon_d^2 + \frac{(3.1 \times 10^{-21})}{\rho_{TOT}^{1/2}} \right]^{1/2} \sec^{-1} , \quad \text{(II.D.75)}$$

where ρ_{TOT} is measured in gm cm^{-3}. This is <u>exactly the same</u> as equation (B.18) of Thorne (1967)! Therefore, all of the results on primordial element formation in axisymmetric models, as calculated by R. V. Wagoner and reported by Thorne (1967), apply directly to our more general anisotropic models here. In particular, we have the result (see Figure 1b of Thorne 1967) that the primordial helium abundance is considerably <u>below</u> ~ 30% by mass only if $|\epsilon_d| \gtrsim 3 \times 10^{-7}$.

(d) The next major event in the evolution of our anisotropic models is the decoupling of the primeval photons (and the attendant recombination of the ionized hydrogen) when the temperature drops below $T \approx 3000\ ^oK$. This process takes place in the DUST phase. Today we see these primeval photons as the $2.7\ ^oK$ cosmic microwave radiation. We shall find (below) that the "average" characteristic temperature of these photons goes as

$$T_{ave} \propto R^{-1} \quad , \qquad \text{(II.D.76)}$$

while they are non-interacting or in thermal equilibrium. Then using equations (II.D.25.a) and (II.D.76) we find that the time of decoupling is:

$$t_D/\tau_d = (T_o/T_D)^{3/2} \left[\{1 + [\epsilon_d^2/4(T_o/T_D)^3]\}^{1/2} - (|\epsilon_d|/2) \right] \quad , \qquad \text{(II.D.77)}$$

where the subscript D denotes the values of a quantity at the time of decoupling. We have already specified

$$T_o \equiv 2.7\ ^oK \quad , \quad T_D = 3000\ ^oK \quad . \qquad \text{(II.D.78)}$$

We obtain a reasonable approximation to t_D by taking

$$t_D \gtrapprox (T_o/T_D)^{3/2}\ \tau_d \cong 2.4 \times 10^5 \text{ years} \quad . \qquad \text{(II.D.79)}$$

(e) We now consider a final application of our anisotropic models to the real Universe: The isotropy of the cosmic microwave radiation. We will primarily study how the observed temperature isotropy of the radiation restricts the range of values of the anisotropy parameters (ϵ_d and ψ) which characterize our anisotropic models here.

First, we want to elucidate equation (II.D.76) above. In equation (II.C.35) we remarked that, in the isotropic cosmological models (where $A = B = C \equiv R(t)$ for all time), we have the temperature behavior:

$$T \propto R^{-1} \quad . \qquad \text{(II.D.80)}$$

That equation (II.D.80) also holds in our anisotropic models, when we have thermal equilibrium, follows directly from the first of equations (II.C.26). When a species of massless particles decouples from its surroundings, its characteristic temperature depends upon direction in our anisotropic models [see equation (II.C.34)]. The temperature of the particles along the (x, y, z) axes goes as:

$$(T_A, T_B, T_C) \propto (A^{-1}, B^{-1}, C^{-1}) \quad . \qquad \text{(II.D.81)}$$

If we define the "average" characteristic temperature T_{ave} by

$$T_{ave} \equiv (T_A \, T_B \, T_C)^{1/3} \qquad \text{(II.D.82)}$$

--- as does Misner (1968) --- we find with Misner that

$$T_{ave} \propto R^{-1} \quad . \qquad \text{(II.D.83)}$$

This result follows directly from equations (II.D.7), (II.D.9), (II.D.16.b), (II.D.17.b), and (II.D.18).

Next, we consider the temperature anisotropy of the 2.7 °K cosmic microwave radiation. Applying Liouville's theorem (see, e.g., Appendix B of Thorne 1966; also Thorne 1967) to the propagation of non-interacting photons in the diagonal Bianchi Type I metric of equation (II.B.2) gives the temperature distribution as a function of the observation direction. In spherical coordinates [see the derivation at equations (II.C.31) through (II.C.34)] the result is:

$$\frac{T_o(\theta,\varphi)}{T_s} = \left[\left(\frac{A_o \sin\theta \cos\varphi}{A_s}\right)^2 + \left(\frac{B_o \sin\theta \sin\varphi}{B_s}\right)^2 + \left(\frac{C_o \cos\theta}{C_s}\right)^2 \right]^{-1/2} .$$

(II.D.84)

Here the subscripts o and s denote, respectively, the value of a quantity "today (t_o)" and "at the time of the last scattering of the microwave photons by matter (at t_s)". Following Thorne (1967), I define the effective time of the last scattering by:

$$\int_{t_s}^{t_o} \lambda^{-1}(t)dt \equiv \text{optical depth} = 1 \quad , \qquad \text{(II.D.85)}$$

where $\lambda(t)$ is the photon mean free path at time t. If our Universe has been filled with ionized hydrogen for redshifts $z \lesssim 9$ (case H II), we have Thomson scattering by the free electrons and

$$\lambda(t) = [(2.67 \times 10^{-18}) / \rho_d(t)] \text{ years} \qquad , \qquad \text{(II.D.86)}$$

where ρ_d is in gm cm^{-3}. Since $t_s >> t_J$, we use the DUST phase equations, and readily obtain:

$$\int_{t_s}^{t_o} \lambda^{-1}(t)dt = [\rho_{do}\tau_d/(2.67 \times 10^{-18})] \int_{x_s}^{1} [x(x + |\epsilon_d|)]^{-1}dx$$

$$= (3.37 \times 10^{-2})|\epsilon_d|^{-1} \, ln \left\{ \left|\frac{x_s + |\epsilon_d|}{x_s}\right| - ln \left|\frac{1 + |\epsilon_d|}{1}\right| \right\} ,$$

(II.D.87)

where $t_o \equiv \tau_d$ and $x_s \equiv t_s/\tau_d$. We will see later that it is consistent to assume $x_s >> |\epsilon_d|$, and to expand the logarithms in series. We then find:

$$x_s \equiv t_s/\tau_d \cong 3.3 \times 10^{-2} \qquad , \qquad \text{(II.D.88.a)}$$

or

$$\left.\begin{aligned} t_s \text{ (H II)} &\cong 3.0 \times 10^8 \text{ years} \\ (R_o/R_s)\text{(H II)} &\cong 9.8 \\ z_s \equiv (\text{redshift})_s &\equiv [(R_o/R_s)(\text{H II})] - 1 \cong 8.8 \end{aligned}\right\} . \qquad \text{(II.D.88.b)}$$

If the ionized hydrogen recombined when the photon temperature dropped below about 3000 $^{\circ}$K and was never reionized thereafter (case H I), we find [see equations (II.D.77) through (II.D.79)]:

$$t_s \text{ (H I)} \equiv (T_o/T_s)^{3/2} \tau_d \approx 2.4 \times 10^5 \text{ years} \quad . \qquad \text{(II.D.89)}$$

To see what limits the observed isotropy of the 2.7 $^{\circ}$K cosmic microwave radiation places upon our anisotropy parameters (ϵ_d and ψ), we write:

$$T_A \equiv T_o(\pi/2,0) \quad , \quad T_B \equiv T_o(\pi/2,\pi/2) \quad , \quad T_C \equiv T_o(0,\varphi) \; . \qquad \text{(II.D.90)}$$

Then we define the present mean-square temperature anisotropy as:

$$\left(\frac{\Delta T}{T}\right)^2_o \equiv 3 \left[\frac{(T_A - T_B)^2 + (T_A - T_C)^2 + (T_B - T_C)^2}{(T_A + T_B + T_C)^2}\right] \; . \qquad \text{(II.D.91)}$$

Substituting equations (II.D.59), (II.D.84), and (II.D.90) into equation (II.D.91), and retaining only the lowest-order term, gives us the following limits --- for all ψ --- on $|\epsilon_d|$:

$$0 \leq |\epsilon_d| \lesssim (3/2)^{1/2} \, (t_s/\tau_d) \, (\Delta T/T)_o \quad . \qquad \text{(II.D.92)}$$

Therefore, although the range of allowed values of $|\epsilon_d|$ is limited by the observed temperature anisotropy and by the assumed DUST content of our anisotropic models (case H I and H II), there are no restrictions on ψ (to first-order). Recently, Partridge and Wilkins… found that the magnitude of the twelve-hour harmonic of the tempe… ture anisotropy around the celestial equator is:

$$(\Delta T/T)_0 \lesssim (1.6 \pm 0.7) \times 10^{-3} \quad . \qquad \text{(II.D.93)}$$

It should be noted that equation (II.D.91) defines a measure of temperature anisotropy over the entire sky, while all precise observations to date have been performed only over one great circle and over small portions of other great circles on the celestial sphere. Also note that our anisotropic models generate no twenty-four hour harmonics of temperature anisotropy; any observed twenty-four hour harmonics will reflect either the earth's motion relative to the local comoving frame of the cosmic microwave radiation or types of expansion anisotropy more complicated than those considered in this thesis. Equations (II.D.92) and (II.D.93) imply that:

$$0 \leq |\epsilon_d| \lesssim \left\{ \begin{array}{c} (6.5 \pm 3.0) \times 10^{-5} \\ \\ (5.3 \pm 2.3) \times 10^{-8} \end{array} \right\} \text{for case} \begin{pmatrix} \text{H II} \\ \\ \text{H I} \end{pmatrix} . \qquad \text{(II.D.94)}$$

Precise observations of the temperature anisotropy have been carried out by Partridge and Wilkinson (1967) around the celestial equator at declination - 8°, and by Conklin and Bracewell (1967a,b) and Epstein (1967) over short regions of right ascension at about + 40° declination. Only the results of Partridge and Wilkinson are useful in equation (II.D.92). Less precise measurements at selected points over the entire Northern sky have been performed by Wilson and Penzias (1967), and they provide the much weaker limits

$$0 \leq |\epsilon_d| \lesssim \begin{Bmatrix} 1.2 \times 10^{-3} \\ \\ 10^{-6} \end{Bmatrix} \text{for case} \begin{pmatrix} \text{H II} \\ \\ \text{H I} \end{pmatrix} . \quad \text{(II.D.95)}$$

If our Universe is approximately axisymmetric [e.g., $A(t) \cong B(t)$] during the DUST phase, there is a 3% probability that the celestial pole is so close to the axis of symmetry that only the weak limits of equation (II.D.95) and not the strong limits of equation (II.D.94) apply to $|\epsilon_d|$. It is easy to see that there is a pressing need for precise experimental investigations of temperature anisotropy along <u>several</u> complete great circles on the celestial sphere.

II. E. THE PERFECT-FLUID-MAGNETIC MODELS

It is presently known that large-scale magnetic fields exist in at least some galaxies in our Universe. The large-scale magnetic field in our own Galaxy has a magnitude of the order of 10^{-6} gauss. As we mentioned in § I.A., Hoyle (1958) has argued that such galactic magnetic fields do not have enough time to appear if the age of our Universe is about 10^{10} years, and that they therefore imply the existence of metagalactic magnetic fields. Cameron (1967), on the other hand, has suggested a mechanism by which the observed galactic fields may be generated, without invoking large-scale cosmic magnetic fields. Whether Cameron's mechanism will really work is not settled at this time. If neither Cameron's nor any other reasonable mechanism will work, then we are forced to consider large-scale cosmic magnetic fields, with magnitudes less than the observational limit of about 10^{-7} gauss, in our anisotropic Bianchi Type I cosmologies. Such fields have previously been considered in Bianchi Type I cosmologies by Rosen (1962, 1964), Zel'dovich (1965b), Doroshkevich (1965), Shikin (1966, 1967), Thorne (1967), and Jacobs (1968, 1969). In this section we shall present a more expanded and detailed version of the work of Jacobs (1969).

In § II. A. we showed that the most general Bianchi Type I metric [equation (II.A.4)] cannot be put into the diagonal form of equation (II.A.15) for all time, when a cosmic magnetic field is present.

For the sake of simplicity, however, we shall assume here that the magnetic field lies along a principal axis of shear, and hence, that our metric can always be written in the diagonal form [equation (II.A.15)]. We have also chosen to assume, on reasonable physical grounds, that the cosmic electric field vanishes [see the discussion leading to equation (II.A.23)], but we will mention here that our cosmological solutions (below) take exactly the same form if we substitute a cosmic electric field in place of the cosmic magnetic field. Let us now proceed to investigate anisotropic Bianchi Type I cosmological models containing both perfect fluid and a cosmic magnetic field.

1) THE GOVERNING EQUATIONS

In the diagonal metric of equation (II.A.15) we have the Einstein field equations (II.A.29). Our cosmologies contain both a PERFECT-FLUID with the barotropic equation of state

$$p_m = \gamma \rho_m \qquad (0 \leq \gamma \leq 1) \qquad , \qquad \text{(II.E.1)}$$

(as in § II.D.) and a uniform comoving MAGNETIC field, of energy-density ρ_b, aligned along the z-axis. With the same notation as in equations (II.A. 29) and (II.A.30), our field equations take the form:

$$ab + ac + bc = +8\pi(\rho_m + \rho_b) \quad , \qquad \text{(II.E.2.a)}$$

$$(\dot{b} + \dot{c}) + b^2 + c^2 + bc = -8\pi(p_m + \rho_b) \quad , \qquad \text{(II.E.2.b)}$$

$$(\dot{a} + \dot{c}) + a^2 + c^2 + ac = -8\pi(p_m + \rho_b) \quad , \qquad \text{(II.E.2.c)}$$

$$(\dot{a} + \dot{b}) + a^2 + b^2 + ab = -8\pi(p_m - \rho_b) \quad . \qquad \text{(II.E.2.d)}$$

The subscripts m and b denote, respectively, the "matter" (PERFECT-FLUID) and the MAGNETIC field. We will call the cosmologies we study here the PERFECT-FLUID-MAGNETIC cosmologies.

The conservation of magnetic flux [see equation (II.A.18.b)] implies

$$\rho_b \propto (AB)^{-2} \quad . \tag{II.E.3}$$

Note that equation (II.E.3) identically satisfies the conservation equation (II.A.32) for the magnetic field, $T^{(EM)\mu}{}_{\nu;\mu} = 0$, which we have seen [equation (II.C.12)] has the form:

$$(VT^{(EM)0}{}_0)^{\cdot} - (aT^{(EM)1}{}_1 + bT^{(EM)2}{}_2 + cT^{(EM)3}{}_3)\, V = 0 \quad . \tag{II.E.4}$$

Hence, the perfect fluid must have exactly the same properties that we found in § II.D., where we had:

$$\rho_m \propto V^{-(1+\gamma)} \quad , \quad . \tag{II.E.5}$$

Since equations (II.E.3) and (II.E.5) guarantee that $(T^{(EM)\mu}{}_{\nu} + T^{(PF)\mu}{}_{\nu})_{;\mu} = 0$, the Bianchi identities imply that only three of equations (II.E.2) can be independent. Let us write equations (II.E.3) and (II.E.5) in the form:

$$\rho_m = (\mu/8\pi)\, V^{-(1+\gamma)} \quad , \quad \rho_b = (\beta/8\pi)\, (AB)^{-2} \quad , \tag{II.E.6}$$

where μ and β are non-negative constants. The field equations

(II.E.2), of which only three are independent, now take the final form:

$$ab + ac + bc = + \mu V^{-(1+\gamma)} + \beta(AB)^{-2} \quad , \quad \text{(II.E.7.a)}$$

$$(\dot{b} + \dot{c}) + b^2 + c^2 + bc = - \gamma\mu V^{-(1+\gamma)} - \beta(AB)^{-2} \quad , \quad \text{(II.E.7.b)}$$

$$(\dot{a} + \dot{c}) + a^2 + c^2 + ac = - \gamma\mu V^{-(1+\gamma)} - \beta(AB)^{-2} \quad , \quad \text{(II.E.7.c)}$$

$$(\dot{a} + \dot{b}) + a^2 + b^2 + ab = - \gamma\mu V^{-(1+\gamma)} + \beta(AB)^{-2} \quad . \quad \text{(II.E.7.d)}$$

Subtracting equation (II.E.7.b) from equation (II.E.7.c) immediately gives us the first integral:

$$(a - b) V = \text{constant} \quad . \quad \text{(II.E.8)}$$

This implies that we have an axisymmetric cosmology (i.e., $A = B$ for all time) if the Hubble expansion rates a and b are equal at any given moment of time. The general equation (II.C.6) here takes the form:

$$\ddot{V} = (3/2)(1 - \gamma) \mu V^{-\gamma} + \beta(C^2/V) \geq 0 \quad , \quad \text{(II.E.9)}$$

and immediately implies that any expanding, Bianchi Type I, PERFECT-FLUID-MAGNETIC cosmology must begin at an initial physical singularity.

2) SOME PARTICULAR SOLUTIONS

Equations (II.E.7) are exceptionally difficult to solve analytically, except in a few special cases. Analytical solutions have been found in four cases. I have given these solutions the following names: (1) the PURE-MAGNETIC solution ($\mu = 0$, $\beta \neq 0$), (2) the ZEL'DOVICH-MAGNETIC solution ($\mu \neq 0$ and $\beta \neq 0$, $\gamma = 1$), (3) an axisymmetric ($A = B$) HARD-MAGNETIC solution ($\mu \neq 0$ and $\beta \neq 0$, $1/3 < \gamma < 1$), and (4) the axisymmetric ($A = B$) DUST-MAGNETIC solution ($\mu \neq 0$ and $\beta \neq 0$, $\gamma = 0$). The HARD-MAGNETIC solution is new. The ZEL'DOVICH-MAGNETIC solution was found in the general case by Jacobs (1969) and it is new; Doroshkevich (1965) discovered this solution independently in the axisymmetric case (leaving the time dependence in the form of an integral). The PURE-MAGNETIC solution was first found by Rosen (1962, 1964), and was rediscovered for the general case by Jacobs (1969) and for the case of axial symmetry ($A = B$) by Shikin (1966). The axisymmetric ($A = B$) DUST-MAGNETIC solution was found independently by Doroshkevich (1965), Shikin (1966), and Thorne (1967). We will now present the derivations of these four exact solutions to equations (II.E.7).

(a) The PURE-MAGNETIC and ZEL'DOVICH-MAGNETIC solutions are obtained simultaneously by the following procedure. When $\gamma = 1$ we can easily manipulate equations (II.E.7) to obtain the equivalent complete set of equations:

$$(a + c)\, V = \bar{c}_1 = \text{constant} \qquad ,(\text{II.E.10.a})$$

$$(b + c)\, V = \bar{c}_2 = \text{another constant}, (\text{II.E.10.b})$$

$$(ab + ac + bc)\, V^2 = \mu + \beta C^2 \qquad .(\text{II.E.10.c})$$

Then from equations (II.E.10) we readily obtain:

$$(A'/A)\ V = c_1 \mp (\xi^2 - C^2)^{1/2} \quad , \qquad \text{(II.E.11.a)}$$

$$(B'/B)\ V = c_2 \mp (\xi^2 - C^2)^{1/2} \quad , \qquad \text{(II.E.11.b)}$$

$$(C'/C)\ V = \pm (\xi^2 - C^2)^{1/2} \quad , \qquad \text{(II.E.11.c)}$$

where

$$\left.\begin{aligned} (c_1, c_2) &= \beta^{-1/2} (\bar{c}_1, \bar{c}_2) \\ \xi^2 &= c_1 c_2 - (\mu/\beta) > 0 \end{aligned}\right\} , \qquad \text{(II.E.12)}$$

and where a prime (') denotes differentiation with respect to the normalized time:

$$\tau \equiv \beta^{1/2}\, t \quad . \qquad \text{(II.E.13)}$$

We now notice that

$$V' \equiv (a + b + c)\ V = (c_1 + c_2) \mp (\xi^2 - C^2)^{1/2} \quad . \qquad \text{(II.E.14)}$$

From equations (II.E.11) and (II.E.14) we see that we always have $C \leq |\xi|$. Therefore, let us make the change of variable

$$X \equiv C/|\xi| \quad (0 \leq X \leq 1) \quad . \qquad \text{(II.E.15)}$$

Dividing equation (II.E.11.c) by equation (II.E.14), integrating, and noting that

$$\left[\frac{X}{1 + (1 - X^2)^{1/2}}\right] \equiv \left[\frac{1 - (1 - X^2)^{1/2}}{X}\right] \qquad , \text{(II.E.16)}$$

we immediately find:

$$\left(\frac{V}{V_*}\right) = \left(\frac{1 + \mathfrak{S}^2}{2\mathfrak{S}}\right) \mathfrak{S}^{\pm\sigma} \qquad , \text{(II.E.17)}$$

where

$$\left.\begin{aligned} \mathfrak{S} &\equiv X^{-1}\,[1 - (1 - X^2)^{1/2}] \quad (0 \leq \mathfrak{S} \leq 1) \\ \sigma &\equiv (c_1 + c_2)/|\xi| \end{aligned}\right\} . \text{(II.E.18)}$$

Here the +(-) sign in equation (II.E.17) is for $C' > (<) \; 0$, respectively, and the subscript $*$ denotes the value of a quantity at $X = 1$. Now we can go back and solve equations (II.E.11.a), (II.E.11.b), and (II.E.14) to obtain, finally, the complete solution:

$$\tau = \left(\frac{A_* B_*}{2}\right) \left[\frac{\mathfrak{S}^{\pm(\sigma-1)}}{(\sigma-1)} + \frac{\mathfrak{S}^{\pm(\sigma+1)}}{(\sigma+1)}\right] \qquad , \text{(II.E.19.a)}$$

$$(A/A_*,\ B/B_*,\ V/V_*) = \left(\frac{1+\mathfrak{S}^2}{2\mathfrak{S}}\right) \mathfrak{S}^{\pm(k_1,\ k_2,\ \sigma)} \qquad , \text{(II.E.19.b)}$$

$$C/|\xi| = 2\mathfrak{S}/(1 + \mathfrak{S}^2) \qquad . \text{(II.E.19.c)}$$

Here $V_* \equiv A_* B_* |\xi|$, the +(-) sign is for $C' > (<) \; 0$, and we have made the notational simplification

$$(k_1,\ k_2) \equiv |\xi|^{-1}\ (c_1,\ c_2) \quad . \qquad \text{(II.E.20)}$$

Our two independent anisotropy parameters here are k_1 and k_2. In our discussion of the possible types of initial singularities (below), we will find it much more convenient to use two different anisotropy parameters ($|\xi|$ and χ) defined by [see equation (II.E.12)]:

$$(k_1,\ k_2) \equiv |\xi|^{-1}[\xi^2 + (\mu/\beta)]^{1/2}(\cosh\chi - \sinh\chi,\ \cosh\chi + \sinh\chi). \qquad \text{(II.E.21)}$$

The ranges of $|\xi|$ and χ are

$$\left.\begin{array}{l} 0 < |\xi| < \infty \\ 0 \leq \chi < \infty \end{array}\right\} . \qquad \text{(II.E.22)}$$

Note that when $\chi \to -\chi$ we interchange k_1 and k_2. Equations (II.E.18) through (II.E.22) represent the four-parameter (μ, β, $|\xi|$, χ) family of ZEL'DOVICH-MAGNETIC solutions; these solutions are new. When $\mu = 0$ they reduce to the PURE-MAGNETIC solutions (see Rosen 1962, 1964).

These two classes of solution emerge from an initial physical singularity at $\tau = 0$ with highly anisotropic expansion rates. Near the singularity the solutions asymptotically approach Kasner's (1921) VACUUM solution, where the anisotropy dominates the dynamics. These solutions always remain highly anisotropic, even as $\tau \to \infty$. The expansion factor C first increases, attains a maximum value of $|\xi|$, and then decreases towards zero as $\tau \to \infty$. Figures 9 and 10 illustrate the temporal evolution of these two classes of solution. In these figures we have normalized to $\xi^2 = 1$, and we have used the arbitrariness of coordinates to set A = B = C = V = 1 at $\tau = \sigma/(\sigma^2 - 1)$ (where C reaches its maximum value).

The ZEL'DOVICH-MAGNETIC solution exhibits the following types of initial singularity:

(i) POINT

axisymmetric ($A \equiv B$ and C all $\to 0$ as $\tau \to 0$) when $\chi = 0$,

anisotropic (A, B, and C $\to 0$ as $\tau \to 0$) when $0 < \chi < \chi_o$;

(ii) BARREL

symmetric ($A \to$ constant, $B \approx C \to 0$ as $\tau \to 0$) when $\chi = \chi_o(\mu/\beta = \xi^2)$,

anisotropic ($A \to$ constant, B and $C \to 0$ as $\tau \to 0$) when $\chi=\chi_o(\mu/B \neq \xi^2)$;

(iii)CIGAR

symmetric ($A \to \infty$, $B \approx C \to 0$ as $\tau \to 0$) when $\chi = \ln 2 - \chi_o(0 < \mu/\beta < \xi^2)$,

anisotropic ($A \to \infty$, B and $C \to 0$ as $\tau \to 0$) when $\chi > \chi_o$ (in general).

Here we have used $\chi_o \equiv (1/2) \ln \left|\frac{\xi^2+(\mu/\beta)}{\xi^2}\right|$. The PURE-MAGNETIC solution has the following types of singularity:

(i) PANCAKE

axisymmetric ($A \equiv B \to$ constant, $C \to 0$ as $\tau \to 0$) only when $\chi = 0$;

FIGURE 9

Types of behavior of the expansion factors (A, B, C) in the ZEL'DOVICH-MAGNETIC solution. (Schematic) Notice that the expansion factor C, along the direction of the magnetic field, is bounded in magnitude (solid line). Cases with POINT singularities (dot-dash lines) occur for $0 \leq \chi < \chi_o$; those with BARREL singularities (dotted lines) occur at $\chi = \chi_o$; and those with CIGAR singularities (dashed lines) occur for $\chi > \chi_o$. Here $\chi_o = (\frac{1}{2})\ln\left|\frac{\xi^2+(\mu/\beta)}{\xi^2}\right|$.

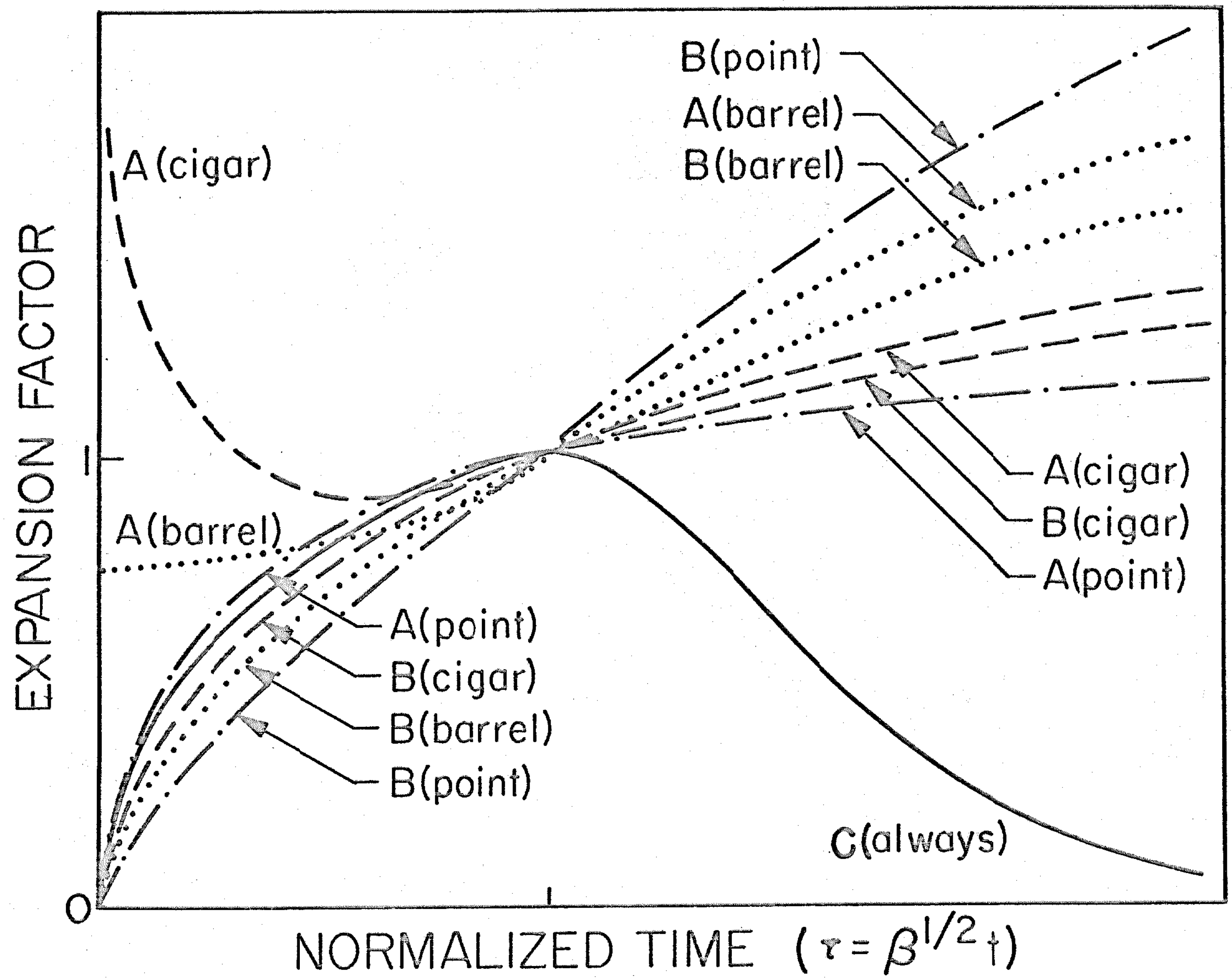
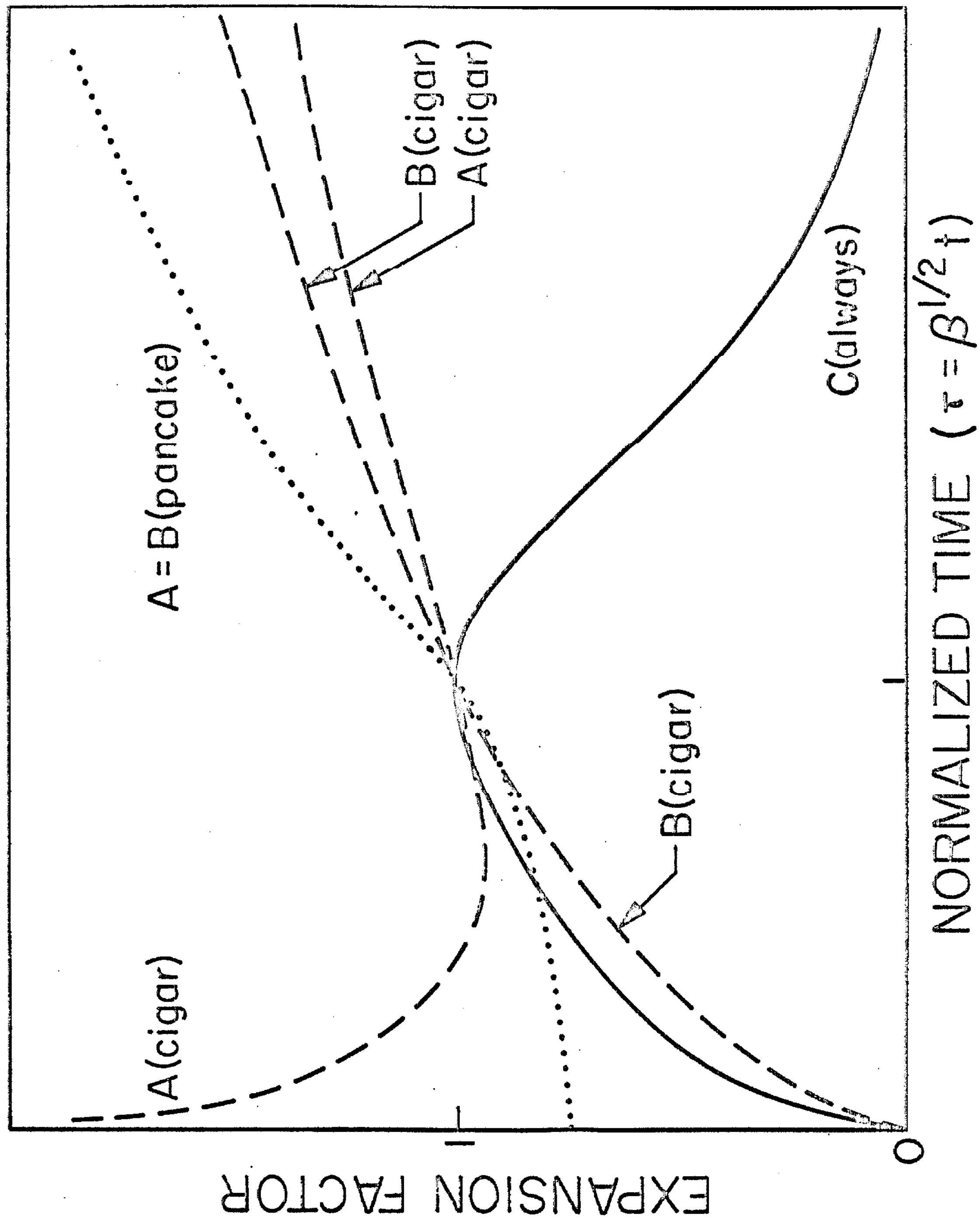

B(point)
A(barrel)
B(barrel)
A(cigar)
A(cigar)
B(cigar)
A(point)
A(barrel)
A(point)
B(cigar)
B(barrel)
B(point)
C(always)
EXPANSION FACTOR
1
0
NORMALIZED TIME ($\tau = \beta^{1/2} t$)

FIGURE 10

Types of behavior in the PURE-MAGNETIC solution. (Schematic) This solution follows from the ZEL'DOVICH-MAGNETIC solution when $\mu = 0$. Again C is bounded (solid line). This solution begins in (a) PANCAKE singularities for $\chi = 0$ (dotted lines; axial symmetry), and (b) CIGAR singularities for $\chi > 0$ (dashed lines). Note that $\chi_o \equiv 0$ for the PURE-MAGNETIC case.



EXPANSION FACTOR
NORMALIZED TIME ($\tau = \beta^{1/2} t$)
0
1
1
A(cigar)
A = B(pancake)
B(cigar)
B(cigar)
A(cigar)
C(always)

(ii) CIGAR

symmetric ($A \to \infty$, $B \approx C \to 0$ as $\tau \to 0$) when $\chi = \ln 2$,

anisotropic ($A \to \infty$, B and $C \to 0$ as $\tau \to 0$) when $\chi > 0$ (in general).

Note that the x- and y- axes are equivalent in all of these solutions (i.e., if in any given solution we interchange x and y, and A and B, we get another solution). Also note that there is only one PANCAKE singularity, that it is axisymmetric (A = B for all time), and that it occurs only in the PURE-MAGNETIC solution.

The axisymmetric (i.e., A = B) ZEL'DOVICH-MAGNETIC solution (found previously by Doroshkevich 1965) occurs when $\chi = 0$ (for all ξ). In this case, the complete solution is given by equations (II.E.19) with

$$k_1 = k_2 = \sigma/2 \equiv |\xi|^{-1} [\xi^2 + (\mu/\beta)]^{1/2} \quad . \qquad \text{(II.E.23)}$$

This solution displays only POINT type initial singularities. The axisymmetric (i.e., A = B) PURE-MAGNETIC solution (found previously by Rosen 1962, 1964) occurs when $\mu = 0$ and $\chi = 0$ (for all ξ). In this case, we have the simple result that

$$k_1 = k_2 = \sigma/2 \equiv 1 \quad . \qquad \text{(II.E.24)}$$

The complete solution follows from equations (II.E.19) as:

$$\tau = (A_*^2/6)\ (3\mathfrak{S}^{\pm 1} + \mathfrak{S}^{\pm 3}) \quad , \quad \text{(II.E.25.a)}$$

$$(A/A_*,\ V/V_*) = [(1 + \mathfrak{S}^2)/2\mathfrak{S}]\ \mathfrak{S}^{\pm(1,2)} \quad , \quad \text{(II.E.25.b)}$$

$$(C/|\xi|) = 2\mathfrak{S}/(1 + \mathfrak{S}^2) \quad , \quad \text{(II.E.25.c)}$$

where the +(-) sign is for $C' > (<) 0$. This solution displays only a PANCAKE type initial singularity.

(b) The analytical form of the axisymmetric (A = B) HARD-MAGNETIC solution was discovered by examining certain numerical solutions of equations (II.E.7). This exact solution (note: it is not the general solution) is essentially identical to one of the approximate solutions near the initial singularity given by Thorne (1967). We can write this HARD-MAGNETIC solution as:

$$A = B = x^{1/2} \quad , \quad C = x^{(1-\gamma)/(1+\gamma)} \quad , \quad \text{(II.E.26.a)}$$

$$a = b = (2\tau x)^{-1} \quad , \quad c = (\tau x)^{-1}\ (1-\gamma)/(1+\gamma) \quad , \quad \text{(II.E.26.b)}$$

$$\rho_m = (3-\gamma)[16\pi\tau^2(1+\gamma)^2]^{-1}x^{-2} \quad , \quad \text{(II.E.26.c)}$$

$$\rho_b = (1-\gamma)(3\gamma-1)[32\pi\tau^2(1+\gamma)^2]^{-1}x^{-2} \quad , \quad \text{(II.E.26.d)}$$

where

$$x \equiv (t/\tau) + \text{constant} \quad , \quad \text{(II.E.27)}$$

with τ being a constant. Equation (II.E.26.d) implies that γ can only lie in the range $1/3 < \gamma < 1$. This solution always begins in an axisymmetric POINT singularity at $x = 0$. It is always highly anisotropic (even as $x \to \infty$), since $(c/a) = 2(1 - \gamma)/(1 + \gamma) \neq 1$. We note that the ratio of the energy-density in the magnetic field (ρ_b) to that in the perfect fluid (ρ_m) is <u>a constant for all time</u>:

$$(\rho_b/\rho_m) = [(1-\gamma)\ (3\gamma-1)]/[2(3-\gamma)] = \text{constant}. \qquad \text{(II.E.28)}$$

Note carefully that this solution exists only for $1/3 < \gamma < 1$. This exact axisymmetric HARD-MAGNETIC solution is new.

(c) The axisymmetric ($A = B$) DUST-MAGNETIC solution is the remaining known exact solution of equations (II.E.7). It was discovered independently by Doroshkevich (1965), Shikin (1966), and Thorne (1967). Let us now derive this solution, which is characterized by $A \equiv B$ and $\gamma = 0$.

In the axisymmetric case, with $\gamma = 0$, the field equations (II.E.7) become:

$$a^2 + 2\,ac = +\mu V^{-1} + \beta A^{-4}\ , \qquad \text{(II.E.29.a)}$$

$$(\dot{a} + \dot{c}) + a^2 + c^2 + ac = -\beta A^{-4}\ , \qquad \text{(II.E.29.b)}$$

$$2\,\dot{a} + 3\,a^2 = +\beta A^{-4}\ , \qquad \text{(II.E.29.c)}$$

where

$$V \equiv A^2 C \ . \qquad \text{(II.E.30)}$$

Note that only two of equations (II.E.29) are independent. Using the relation

$$(A^3 a^2)^{\cdot} \equiv A^3 a \; (2\dot{a} + 3a^2) \qquad , \qquad \text{(II.E.31)}$$

we find that equation (II.E.29.c) can be immediately integrated to give:

$$\mu^{1/2} t = (2/3^{1/2}) \; (A + 6\delta) \; (A - 3\delta)^{1/2} \qquad . \qquad \text{(II.E.32)}$$

Here we have used $\delta \equiv (\beta/\mu) > 0$. Using equation (II.E.32) in equation (II.E.29.a) leads at once to the first-order linear differential equation:

$$2A \; (A - 3\delta) \; (dC/dA) + (A - 6\delta) \; C - 3A^2 = 0 \qquad . \qquad \text{(II.E.33)}$$

The general solution of equation (II.E.33) is:

$$C = A + 12\delta - 72\delta^2 A^{-1} + \epsilon A^{-1} (A - 3\delta)^{1/2} \qquad , \qquad \text{(II.E.34)}$$

where ϵ is our sole anisotropy parameter, with the allowed range $0 \leq |\epsilon| < \infty$. The complete axisymmetric DUST-MAGNETIC solution is given by equations (II.E.32), (II.E.34), and

$$\left.\begin{aligned} \rho_m &= (\mu/8\pi) \; (A^2 C)^{-1} \\ \rho_b &= (\beta/8\pi) \;\; A^{-4} \end{aligned}\right\} \; . \qquad \text{(II.E.35)}$$

Note that $\delta \equiv (\beta/\mu)$ is the ratio of the energy-density of the magnetic field to that of the perfect fluid when $A = C = 1$.

The initial singularity is always of the PANCAKE type, and it occurs when C = 0. The behavior of the solution near the singularity is:

$$\left.\begin{aligned} A \equiv B &\propto 1 + (\text{constant})\ t \\ C &\propto t \end{aligned}\right\} . \qquad \text{(II.E.36)}$$

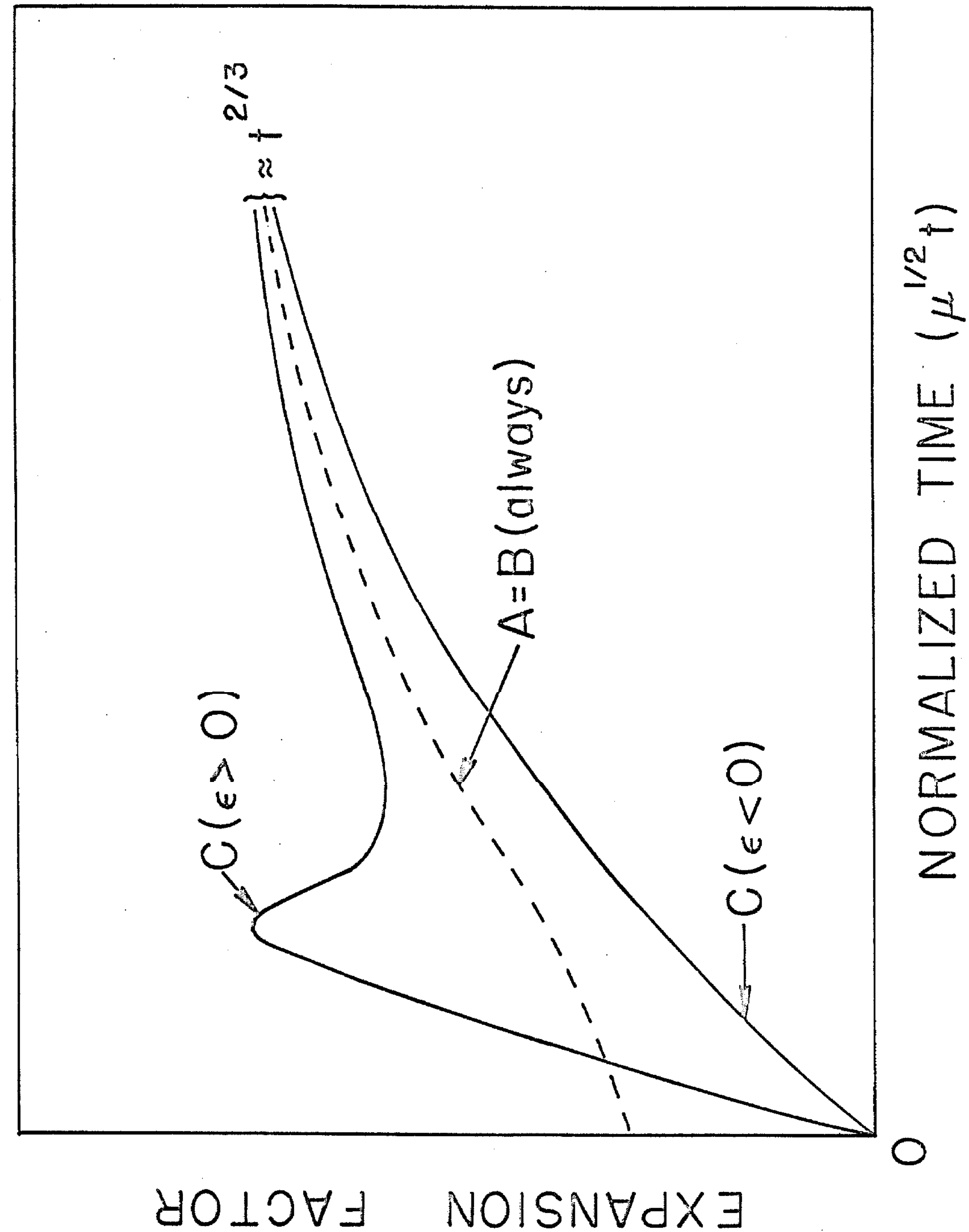
The solution emerges from the singularity with highly anisotropic expansion rates, but becomes asymptotically isotropic (with $A \equiv B \approx C \mathrel{\tilde{\propto}} t^{2/3}$) as $t \to \infty$. The two qualitatively different types of temporal evolution of this solution are illustrated in Figure 11. Note, especially, how the magnetic field causes the expansion factor C (along the field direction) to go to zero at the initial singularity. In fact, the non-magnetic ($\beta = 0$) CIGAR type singularity is converted into a PANCAKE singularity (with $C \to 0$) by the magnetic field.

3) ALL SINGULARITY SOLUTIONS

We mentioned above that equations (II.E.7) are extremely difficult to solve analytically. A case which is of great practical interest is the RADIATION-MAGNETIC case ($\mu \neq 0$, $\beta \neq 0$, $\gamma = 1/3$), but this case has not yet been solved analytically. In Appendix D we show how the axisymmetric (i.e., A = B) RADIATION-MAGNETIC case leads to a highly-nonlinear first-order Abel differential equation. Shikin (1967) independently arrived at this differential equation in his analysis of this case. It is this equation (D.9) which presently frustrates our efforts to obtain the analytical solution to the axisymmetric

FIGURE 11

The types of temporal behavior in the axisymmetric (i.e., A = B) DUST-MAGNETIC solution. (Schematic) This solution is defined by $\mu \neq 0$, $\beta \neq 0$, $\gamma = 0$. The initial singularity is always of the z-PANCAKE type (i.e., $A = B \to$ constant and $C \to 0$ at the singularity). The solution always isotropizes to the behavior $A = B \approx C \mathrel{\tilde{\propto}} t^{2/3}$ as $t \to \infty$. When $\epsilon < 0$ (dashed line and bottom solid line) the magnetic field affects the dynamics only slightly, whereas when $\epsilon > 0$ (dashed line and top solid line) the magnetic field dramatically converts a potential z-CIGAR singularity (where $C \to \infty$ at the singularity) into a z-PANCAKE singularity (where $C \to 0$).


EXPANSION FACTOR
NORMALIZED TIME (μ^1/2 t)
0
C(ε>0)
A=B (always)
C(ε<0)
≈ t^2/3

RADIATION-MAGNETIC case.

If we want to proceed further in our analysis of Bianchi Type I cosmologies containing a uniform magnetic field, we must turn to approximate methods. We have previously seen the general solutions to the ZEL'DOVICH-MAGNETIC and the PURE-MAGNETIC cases. Hence, the only remaining cases are those characterized by $\mu \neq 0$, $\beta \neq 0$, and $0 \leq \gamma < 1$: the PERFECT-FLUID-MAGNETIC cases. Only two particular analytical solutions have been found for these cases; they are our axisymmetric HARD-MAGNETIC case ($1/3 < \gamma < 1$) and the axisymmetric DUST-MAGNETIC case ($\gamma = 0$). The remaining cases must be studied by numerical integration of equations (II.E.7). Such numerical integration is "computationally stable" only if we integrate <u>out of</u> the initial singularity; hence we need to know the form of the solutions near the singularity. To find these "singularity solutions" we manipulate equations (II.E.7) into the form:

$$0 = (\dot{a} - \dot{b}) + (a - b)(a + b + c) \qquad ,\text{(II.E.37.a)}$$

$$(1 + \gamma)\,\mu V^{-(1+\gamma)} = 2ab + (a + b)(c - a - b) - (\dot{a} + \dot{b}) \qquad ,\text{(II.E.37.b)}$$

$$(1 + \gamma)\,\beta(AB)^{-2} = (\gamma - 1)\,ab + (a + b)(\gamma c + a + b) + (\dot{a} + \dot{b}),\text{(II.E.37.c)}$$

where a dot ($\cdot$) denotes differentiation with respect to proper time t. Let us place the initial singularity at $t = 0$, and let us set

$$(A,\ B,\ C) = t^{(\ell,m,n)} \quad . \qquad \text{(II.E.38)}$$

Then equation (II.E.37.a) becomes:

$$(\ell - m)\ (\ell + m + n - 1) = 0 \quad . \qquad \text{(II.E.39)}$$

Equation (II.E.39) implies three distinct possibilities:

$$\left.\begin{array}{lll} (1) & \ell = m\ , & \ell + m + n \neq 1 \\ (2) & \ell = m\ , & \ell + m + n = 1 \\ (3) & \ell \neq m\ , & \ell + m + n = 1 \end{array}\right\} . \qquad \text{(II.E.40)}$$

Since Thorne (1967) has given all of the axisymmetric ($\ell = m$) singularity solutions [see equations (A6) through (A11) of Thorne 1967], we need only find the non-axisymmetric ($\ell \neq m$) singularity solutions here. Equations (II.E.37) now take the form:

$$\ell + m + n = 1 \quad , \qquad \text{(II.E.41.a)}$$

$$(1+\gamma)\mu t^{-(1+\gamma)} = [1 - (\ell^2 + m^2 + n^2)]t^{-2} \quad , \qquad \text{(II.E.41.b)}$$

$$(1+\gamma)\beta t^{-2(1-n)} = (1/2)\ (\gamma-1)\ [1-(\ell^2+m^2+n^2)]t^{-2} \quad . \qquad \text{(II.E.41.c)}$$

Since $0 \leq \gamma < 1$ equations (II.E.41) imply that near the singularity:

$$\ell + m + n = \ell^2 + m^2 + n^2 = 1 \quad . \qquad \text{(II.E.42)}$$

Therefore, near the initial singularity, all anisotropic ($\ell \neq m$) PERFECT-FLUID-MAGNETIC solutions behave like the anisotropy-dominated Kasner (1921) VACUUM solution. From equations (II.E.41) we find that, near the initial singularity, the value of μ is arbitrary (undetermined), whereas

$$\beta = \begin{cases} 0 & , \text{ for } n \leq 0 \\ \text{arbitrary} & , \text{ for } n > 0 \end{cases} \quad . \tag{II.E.43}$$

When $\beta = 0$ we have the PERFECT-FLUID case, which we solved completely in § II.D.; hence, only the $\beta \neq 0$ ($n > 0$) singularity solutions remain to be considered. Equation (II.E.43) shows that a uniform magnetic field along the z-axis always causes the expansion factor in the z-direction (C) to vanish at the singularity. We can satisfy equation (II.E.42) with the following convenient parameterization:

$$(\ell, m, n) = (1/3)\,[1 + 2 \sin(\psi, \psi + 2\pi/3, \psi + 4\pi/3)] \,, \tag{II.E.44}$$

with $n > 0$ in the range $\pi/2 < \psi < 11\pi/6$. We now see that the only types of non-axisymmetric ($\ell \neq m$) singularities are:

(i) anisotropic y-CIGARS (A and C $\to 0$, B $\to \infty$ as $t \to 0$) for $\pi/2<\psi<7\pi/6$,

(ii) a symmetric y-CIGAR ($A = C \to 0$, $B \to \infty$ as $t \to 0$) at $\psi = 5\pi/6$,

(iii) anisotropic x-CIGARS ($A \to \infty$, B and C $\to 0$ as $t \to 0$) for $7\pi/6<\psi<11\pi/6$,

(iv) a symmetric x-CIGAR ($A \to \infty$, $B = C \to 0$ as $t \to 0$) at $\psi = 3\pi/2$.

Note that only CIGAR singularities appear. Also note that at $\psi = 7\pi/6$ an axisymmetric ($\ell = m$) z-PANCAKE ($A = B \to$ constant, $C \to 0$ as $t \to 0$) appeared for $\beta \neq 0$ and $0 \leq \gamma < 1$, but we have not listed it here because it is already given explicitly in equations (A11) of Thorne (1967).

In order to see more clearly where we now stand with respect to the solutions of equations (II.E.7), I have prepared Table 1. Table 1 lists all possible solutions to equations (II.E.7) for all possible combinations of μ, β, and γ. I have included the non-magnetic ($\beta = 0$) PERFECT-FLUID solutions from § II.D. for completeness. In Table 2 all known analytical solutions to equations (II.E.7) are shown, and some references to where they may be found in the literature are given. Those cases which must still be solved by numerical integration (see below) are denoted "Numerical", and they include equations (A6) through (A11) of Thorne (1967) and the $\ell \neq m$ singularity solutions found above. From Table 1 we see that the only cases which need be studied numerically are the PERFECT-FLUID-MAGNETIC cases with the following behavior near the initial singularity (see also Thorne 1967):

(a) isotropic POINT singularities ($A = B = C \to 0$ as $t \to 0$)

$$A = B = C = t^{2/[3(1+\gamma)]} \quad , \qquad \text{(II.E.45.a)}$$

$$\rho_m = [6\pi(1+\gamma)^2]^{-1} t^{-2} \quad , \qquad \text{(II.E.45.b)}$$

TABLE 2

A CLASSIFICATION OF SOLUTIONS IN BIANCHI TYPE I COSMOLOGIES WITH MATTER AND MAGNETIC FIELD[a,b]

NAME	RANGE OF γ	TYPE OF SOLUTION	TYPES OF SINGULARITY BEHAVIOR[c,d]	PERTINENT REFERENCES
PURE MAGNETIC		Analytical	Axisymmetric PANCAKE Non-axisymmetric CIGARS	Rosen (1962, 1964), Shikin (1966), and Jacobs (1969) (§ IIIa, $\mu = 0$).
ZEL'DOVICH	$\gamma = 1$	Analytical	POINTS, BARRELS, and CIGARS	Jacobs (1968), § IIIe.
ZEL'DOVICH-MAGNETIC				Doroshkevich (1965); Jacobs (1969), § IIIa.
HARD-MAGNETIC	$\frac{1}{3} < \gamma < 1$	Analytical Particular Solution	Axisymmetric POINTS	Jacobs (1969), eqs. (16); Thorne (1967), eqs. (A7).
"ISOTROPIC"-HARD-MAGNETIC		Numerical	Isotropic POINTS	Thorne (1967), eqs. (A6); Jacobs (1969), eqs. (25).
PERFECT-FLUID	$0 \leq \gamma < 1$	Analytical	PANCAKES and CIGARS	Jacobs (1968), § III.
PERFECT-FLUID-MAGNETIC		Analytical for $\gamma = 0$ and Axisymmetry. DUST-MAGNETIC	Axisymmetric PANCAKES Non-axisymmetric CIGARS	Doroshkevich (1965), Shikin (1966), Thorne (1967), and Jacobs (1969) [eqs.(17)].
		Numerical		Jacobs (1969), eqs.(27); Thorne (1967), eqs.(A11).

[a] All of the solutions presented in Jacobs (1968, 1969) appear here.

[b] These cosmologies contain perfect-fluid matter with the equation of state, $p_m = \gamma \rho_m$ $(0 \leq \gamma \leq 1)$, and/or a uniform, comoving magnetic field of energy-density ρ_b aligned along the z-axis.

[c] Axisymmetric implies A = B, Non-axisymmetric implies $A \neq B$, and Isotropic implies A = B = C.

[d] The singularity is $\begin{pmatrix} \text{POINT} \\ \text{BARREL} \\ \text{PANCAKE} \\ \text{CIGAR} \end{pmatrix}$ when $\begin{pmatrix} A,\ B, \text{ and } C \to 0 \\ A \to \text{constant},\ B \text{ and } C \to 0 \\ A \text{ and } B \to \text{constant},\ C \to 0 \\ A \to \infty,\ B \text{ and } C \to 0 \end{pmatrix}$ as we approach the singularity.

$$\rho_b = (\beta/8\pi)\ t^{-8/[3(1+\gamma)]} \quad , \qquad \text{(II.E.45.c)}$$

$$\beta = \text{non-negative constant} \quad , \quad 1/3 < \gamma < 1 \quad ; \qquad \text{(II.E.45.d)}$$

(b) <u>axisymmetric PANCAKE singularities</u> ($A = B \rightarrow$ constant, $C \rightarrow 0$ as $t \rightarrow 0$)

$$A = B = 1 + \alpha t^{(1-\gamma)} \quad , \quad C = t \quad , \qquad \text{(II.E.46.a)}$$

$$\rho_m = (\alpha/4\pi)\ (1-\gamma)\ t^{-(1+\gamma)} \quad , \qquad \text{(II.E.46.b)}$$

$$\rho_b = (\beta/8\pi)\ [1 - 4\alpha t^{(1-\gamma)}] \quad , \qquad \text{(II.E.46.c)}$$

$$\alpha \text{ and } \beta = \text{non-negative constants},\ 0 < \gamma < 1 \quad ; \qquad \text{(II.E.46.d)}$$

(c) <u>anisotropic y-CIGAR singularities</u> (A and $C \rightarrow 0$, $B \rightarrow \infty$ as $t \rightarrow 0$)

$$(A,\ B,\ C) = t^{(l,m,n)} \quad , \qquad \text{(II.E.47.a)}$$

$$(l,m,n) = (1/3)\ [1 + 2\sin(\psi, \psi + 2\pi/3, \psi + 4\pi/3)]\ (\pi/2 < \psi < 7\pi/6) \quad ,$$

(II.E.47.b)

$$\rho_m = (\mu/8\pi)\ t^{-(1+\gamma)} \quad , \qquad \text{(II.E.47.c)}$$

$$\rho_b = (\beta/8\pi)\ t^{-(4/3)[1-\sin(\psi+4\pi/3)]} \quad , \qquad \text{(II.E.47.d)}$$

$$\mu \text{ and } \beta = \text{non-negative constants},\ 0 \leq \gamma < 1 \quad . \qquad \text{(II.E.47.e)}$$

Note that the symmetric y-CIGAR singularity is a particular case of equations (II.E.47). Also note that the anisotropic x-CIGAR singularity solutions are equivalent to equations (II.E.47), since the x- and y-axes are equivalent and interchangeable in all of the PERFECT-FLUID-MAGNETIC cases.

4) THE NUMERICAL INTEGRATIONS

To actually perform the numerical integration of equations (II.E.7) we must initialize with the singularity solutions of equations (II.E.45) through (II.E.47), and then we can compute the temporal evolution with the following convenient form of equations (II.E.7):

$$V'/V = \{(a^2 + b^2 + c^2) + 2\,[V^{-(1+\gamma)} + \delta(C/V)^2]\}^{1/2} \quad , \quad \text{(II.E.48.a)}$$

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = (1/2)\,(1-\gamma)\,V^{-(1+\gamma)} - \begin{pmatrix} a \\ b \\ c \end{pmatrix} (V'/V) \begin{pmatrix} + \\ + \\ - \end{pmatrix} \delta(C/V)^2 \quad . \quad \text{(II.E.48.b)}$$

Here a prime (') denotes differentiation with respect to the normalized time $\tau \equiv \mu^{\frac{1}{2}}t$, and we have

$$\left.\begin{aligned} V &\equiv ABC \\ \delta &\equiv (\beta/\mu) > 0 \end{aligned}\right\} . \qquad \text{(II.E.49)}$$

Trial runs of the computer program using equations (II.E.48) and (II.E.49) accurately reproduced the following: (a) the known isotropic solutions with $\beta = 0$, (b) the anisotropic PERFECT-FLUID solutions of Jacobs (1968) with $\beta = 0$, (c) the axisymmetric HARD-MAGNETIC solution of equations (II.E.26) with $\beta \neq 0$, and (d) the axisymmetric DUST-MAGNETIC solution of equations (II.E.32) through (II.E.35) with $\beta \neq 0$. Then computer studies of the PERFECT-FLUID-MAGNETIC cases characterized by the singularity solutions of equations (II.E.45) through (II.E.47) were carried out, with the following results.

The typical temporal behavior of solutions beginning in the <u>isotropic POINT singularities</u> ($1/3 < \gamma < 1$) of equations (II.E.45) is displayed in Figure 12. These solutions always remain axisymmetric (i.e., A = B). The magnetic field accelerates the transverse (A = B) expansions, while it decelerates the longitudinal (C) expansion. These solutions become highly anisotropic as time increases, and asymptotically (as $\tau \to \infty$) they approach the corresponding (same γ and δ) axisymmetric HARD-MAGNETIC solution of equations (II.E.26). The magnetic field (ρ_b) has negligible effect on the dynamics as $\tau \to 0$, but its influence becomes comparable to that of the perfect fluid (ρ_m) as τ increases. The effects of the magnetic field become readily apparent at a time which <u>decreases</u> as δ <u>increases</u>.

Some examples of solutions beginning in the <u>axisymmetric PANCAKE singularities</u> ($0 < \gamma < 1$) of equations (II.E.46) are shown in Figure 13. Their behavior is qualitatively the same as that of the axisymmetric DUST-MAGNETIC solution of equations (II.E.32) through (II.E.35) and Figure 11. They always remain axisymmetric (i.e., A = B) while the magnetic field accelerates their transverse (A = B)

FIGURE 12

The typical behavior of the PERFECT-FLUID-MAGNETIC numerical solutions beginning in isotropic (A = B = C) POINT singularities. (Schematic) These are "HARD-MAGNETIC" solutions ($1/3 < \gamma < 1$). They behave as $A = B = C \approx \tau^{2/[3(1+\gamma)]}$ near the initial singularity, but the magnetic field converts them into the corresponding (same γ) axisymmetric HARD-MAGNETIC solution of equation (II.E.26) as $\tau \to \infty$ [$A = B \approx \tau^{\frac{1}{2}}$ and $C \approx \tau^{(1-\gamma)/(1+\gamma)}$ for large τ]. They are highly anisotropic as $\tau \to \infty$! The parameter δ used in the figure measures the amount of magnetic field present [see equation (II.E.49)].

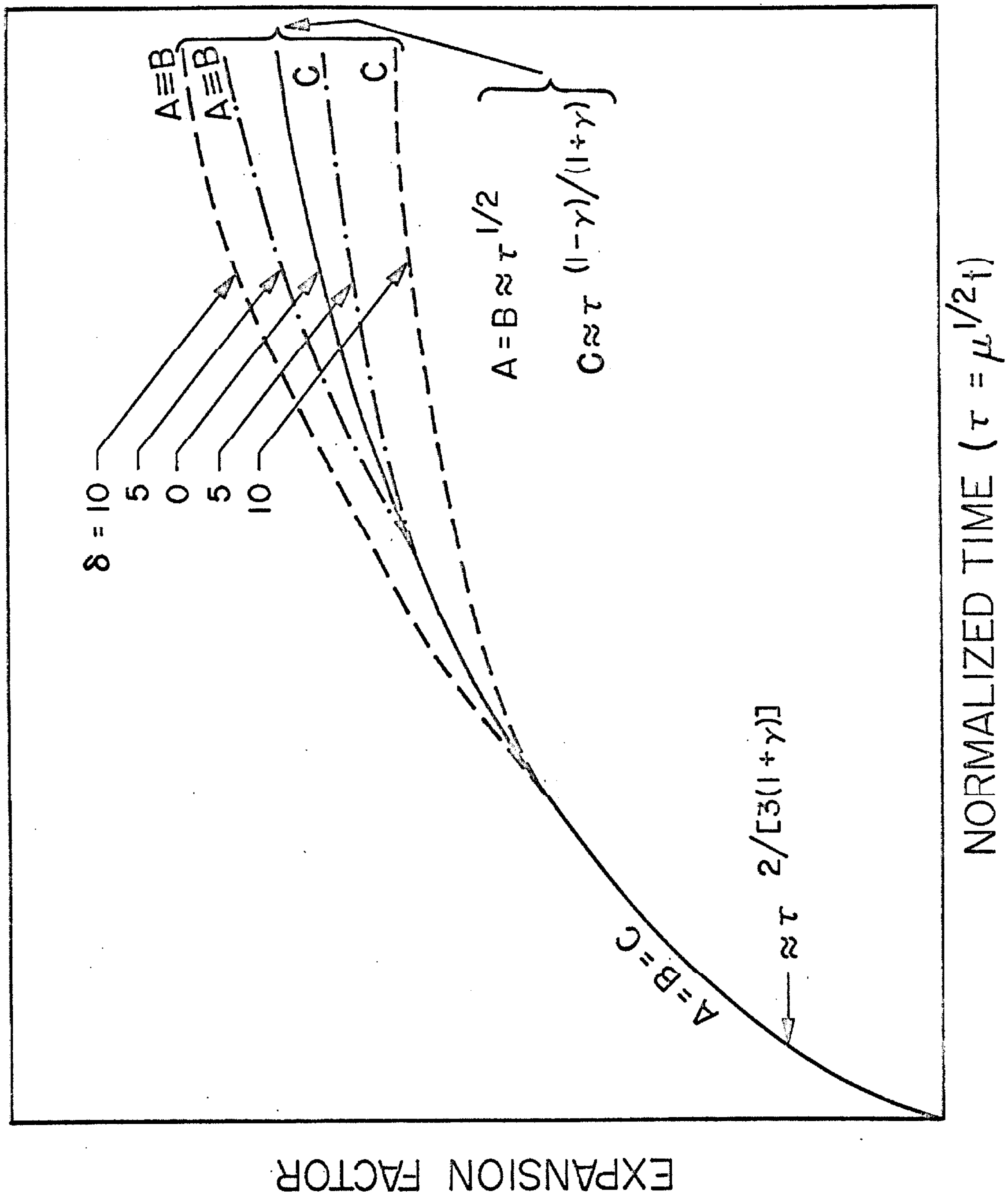
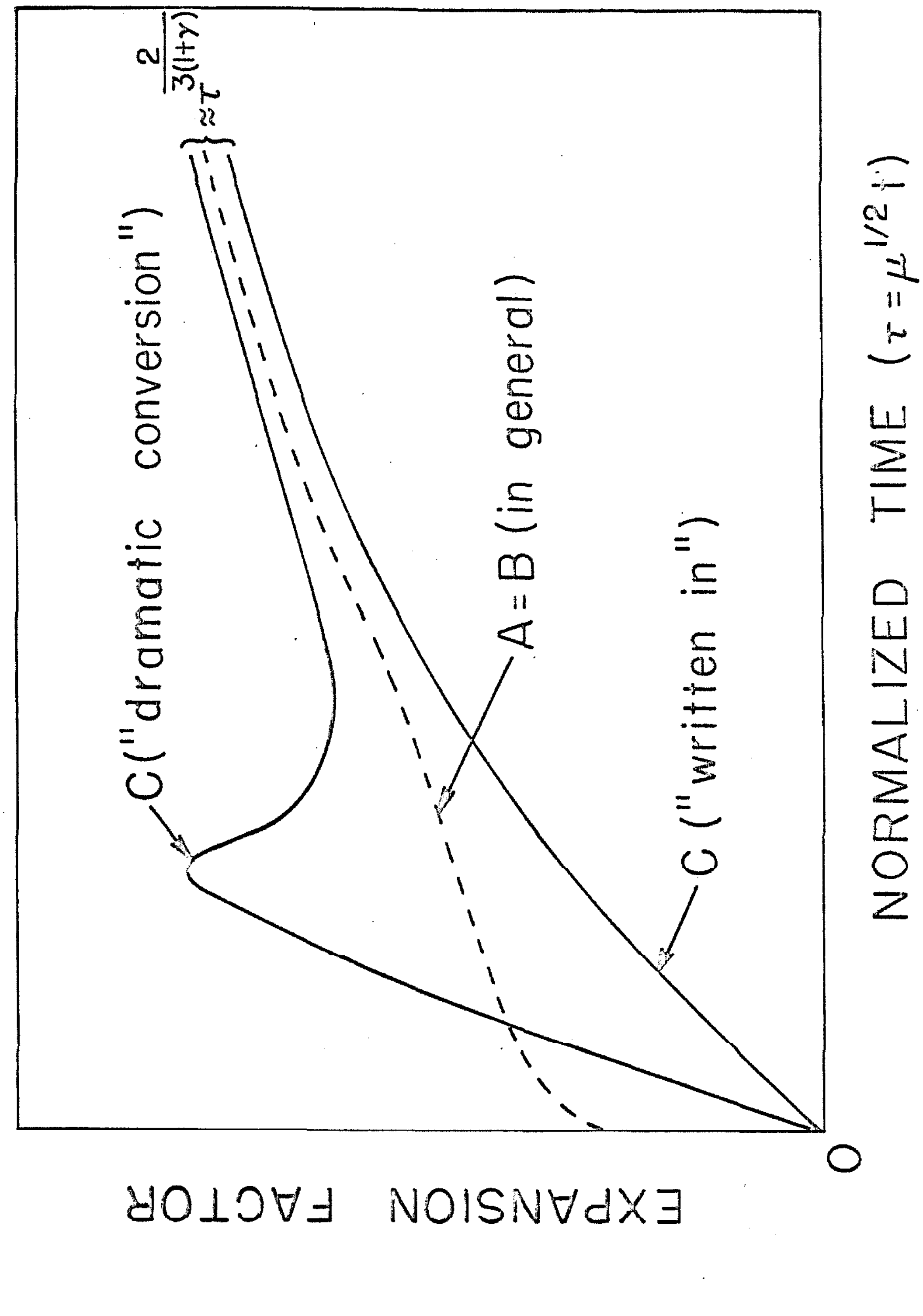

EXPANSION FACTOR
NORMALIZED TIME (τ = μ^1/2 t)
δ = 10
5
0
5
10
A≡B
A≡B
C
C
A = B ≈ τ^1/2
C ≈ τ^(1−γ)/(1+γ)
A = B = C
≈ τ^2/[3(1+γ)]

FIGURE 13

Representative types of temporal behavior for PERFECT-FLUID-MAGNETIC numerical solutions beginning in axisymmetric (i.e., A = B) PANCAKE singularities ($0 < \gamma < 1$). (Schematic) The expansion rates are highly anisotropic near the initial singularity, but they approach the isotropic behavior $A = B \approx C \tilde{\propto} \tau^{2/[3(1+\gamma)]}$ as $\tau \to \infty$. These numerical solutions have the same qualitative behavior as the axisymmetric DUST-MAGNETIC solution ($\gamma = 0$) shown in Figure 11. The term "written in" implies that the magnetic field does not affect the qualitative behavior of the solution as compared to the corresponding (same γ) PERFECT-FLUID ($\beta = 0$) solution of Jacobs (1968). The term "dramatic conversion" means that a potential z-CIGAR singularity is transformed into a z-PANCAKE singularity by the action of the magnetic field.



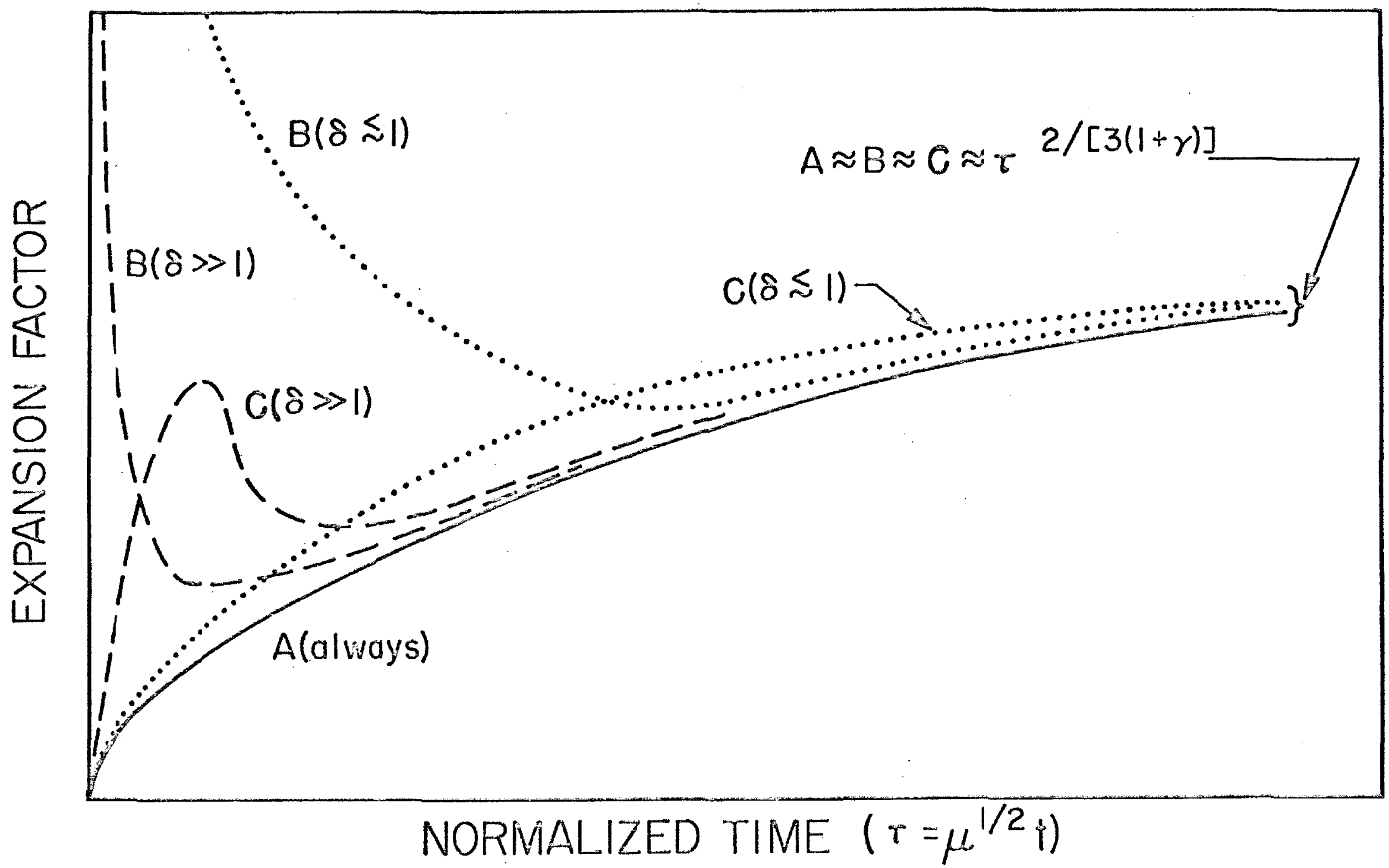
C("dramatic conversion")
A = B (in general)
C ("written in")
$\approx \tau^{\frac{2}{3(1+\gamma)}}$
0
NORMALIZED TIME ($\tau = \mu^{1/2} t$)
EXPANSION FACTOR

expansions and decelerates their longitudinal (C) expansion. They emerge from the initial singularity with highly anisotropic expansion rates, but asymptotically approach the corresponding (same γ) isotropic PERFECT-FLUID ($\beta = 0$) solution as $\tau \to \infty$. The magnetic field affects the evolution noticeably only at intermediate times, and has negligible effect as $\tau \to 0$ and as $\tau \to \infty$. The region in which the magnetic field appreciably affects the dynamics moves to <u>smaller</u> values of τ as δ <u>increases</u>.

Finally, some solutions which emerge from the <u>anisotropic y-CIGAR singularities</u> ($0 \leq \gamma < 1$) of equations (II.E.47) are shown in Figure 14. These solutions are never axisymmetric. They emerge from the initial singularity with highly anisotropic expansion rates, but as $\tau \to \infty$ they asymptotically isotropize to the corresponding (same γ) isotropic PERFECT-FLUID ($\beta = 0$) solution. The magnetic field is effective only for intermediate τ, where it accelerates the A and B expansions and decelerates the C expansion. The magnetic field has negligible effect on the dynamics as $\tau \to 0$ and as $\tau \to \infty$. The region in which the magnetic field is effective moves to <u>smaller</u> τ as δ <u>increases</u>. Qualitatively there are two types of temporal behavior: (1) Those anisotropic PERFECT-FLUID ($\beta = 0$) solutions of Jacobs (1968) which emerge from x- or y-CIGAR singularities are qualitatively unchanged by the magnetic field; we say that the magnetic field is simply "written in". (2) By contrast, the anisotropic

FIGURE 14

The typical behavior of the PERFECT-FLUID-MAGNETIC numerical solutions which begin at anisotropic CIGAR singularities. They exist for $0 \leq \gamma < 1$. (Schematic) The initial singularity is always x- or y-CIGAR, never z-CIGAR. As $\tau \to 0$ the behavior is $(A,B,C) \approx \tau^{(\ell,m,n)}$ where $(\ell,m,n) = (1/3)\ [1 + 2 \sin(\psi,\ \psi + 2\pi/3,\ \psi + 4\pi/3)]$ and $\pi/2 < \psi < 11\pi/6$. Isotropization of the expansion occurs as $\tau \to \infty$, with $A \approx B \approx C \approx \tau^{2/[3(1+\gamma)]}$. There are two types of behavior: (1) the magnetic field is essentially "written in" --- i.e., it does not affect the dynamics appreciably -- (dotted lines), and (2) the magnetic field dominates the dynamics for a short period of time (dashed lines).


EXPANSION FACTOR
NORMALIZED TIME ($\tau = \mu^{1/2} t$)
$B(\delta \lesssim 1)$
$B(\delta \gg 1)$
$C(\delta \gg 1)$
$C(\delta \lesssim 1)$
A(always)
$A \approx B \approx C \approx \tau^{2/[3(1+\gamma)]}$

PERFECT-FLUID ($\beta = 0$) solutions of Jacobs (1968) which emerge from a z-CIGAR singularity are dramatically converted into PERFECT-FLUID-MAGNETIC solutions with x- or y-CIGAR singularities, since the magnetic field always "pulls" C to zero at the singularity.

When we compare the singularity behavior [equations (II.E.45) through (II.E.47)] of these PERFECT-FLUID-MAGNETIC numerical solutions to that of the PERFECT-FLUID solutions ($0 \leq \gamma < 1$, $\beta = 0$) of Jacobs (1968), we see that we have made the normalization $|\epsilon| \equiv 1$ here. This normalization of the anisotropy parameter, ϵ, merely means that the normalized time, $\tau = \mu^{1/2}t$, appearing in our numerical integrations [see equations (II.E.48)] is scaled so that the expansion anisotropies due solely to ϵ become small for $\tau \gtrsim 1$. We may interpret the effects of the parameter δ as follows: If the singularity behavior of equations (II.E.45) through (II.E.47) persisted until $\tau = 1$, we would have $\delta \equiv \rho_b/\rho_m$. Therefore, δ represents the projected ratio of the energy-density in the magnetic field to that in the perfect fluid. For $\delta >> 1$ the magnetic field begins to affect the dynamics at $\tau << 1$, and only the anisotropy due to ϵ remains by the time that $\tau \approx 1$. For $\delta << 1$ the magnetic field can become effective only at $\tau >> 1$, and large anisotropies persist until $\tau >> 1$ due to the magnetic field.

5) THE CONSTRUCTION OF SEMI-REALISTIC MODELS

In order to construct semi-realistic anisotropic Bianchi Type I cosmological models of our Universe, we need to know at least the analytical forms of the axisymmetric (i.e., A = B) DUST-MAGNETIC and the axisymmetric RADIATION-MAGNETIC solutions. These two solutions could then be combined, by the approximation procedure of § II.D., to form approximate DUST-PLUS-RADIATION-MAGNETIC cosmological models. Unfortunately, the RADIATION-MAGNETIC solution has not been found yet. Appendix D illustrates how extraordinarily complicated the solution of even the simple axisymmetric RADIATION-MAGNETIC case is. Until the RADIATION-MAGNETIC case is solved, we cannot construct semi-realistic cosmological models as we did in § II.D.. We need not succumb to total despair however. Our numerical integrations of the field equations (II.E.7) and our knowledge of the existing analytical solutions in both the PERFECT-FLUID ($\beta = 0$) and PERFECT-FLUID-MAGNETIC ($\beta \neq 0$) cases provide enough information for a qualitative (and in some cases quantitative) analysis of many important physical properties of the DUST-PLUS-RADIATION-MAGNETIC cosmological models of our Universe. In the next subsection we shall extract as much information as we can about such models.

6) APPLICATIONS TO THE REAL UNIVERSE

Let us first consider the initial physical singularity of our heuristic DUST-PLUS-RADIATION-MAGNETIC semi-realistic cosmological models. In direct analogy to § II.D., we will say that our models are described by a RADIATION-MAGNETIC solution near the singularity. The <u>type</u> of initial singularity is of great importance. Thorne (1967) suggests that a primordial magnetic field probably could never exceed a field strength of $|\underset{\sim}{B}|_{critical} \approx 4.4 \times 10^{13}$ gauss. Otherwise, the field would have been quantized near the initial singularity, and a large-scale field would probably not have emerged from the initial quantum phase. This conjecture is supported by the work of O'Connell (1968) who shows that the anomalous magnetic moment of the electron implies the spontaneous creation of electron-positron pairs in a magnetic field with $|\underset{\sim}{B}| \gtrsim 4\pi\ (137)\ |\underset{\sim}{B}|_{critical} \approx 7.6 \times 10^{16}$ gauss. However, recent work by H-Y. Chiu and his colleauges (paper in press) suggests that electron-positron pair creation <u>might not</u> occur in an arbitrarily strong magnetic field. The problem of the existence of a large-scale magnetic field after an initial quantum phase must be considered unresolved at present. Let us suppose that Chiu <u>et al</u>. are wrong, and O'Connell right. Then if a primordial magnetic field originated with field strengths well above 10^{17} gauss, its energy would most probably have simply augmented the existing relativistic electron-positron gas, and no large-scale magnetic field would have emerged from the quantum phase. This argument implies that only z-PANCAKE type initial singularities might be compatible with a large-scale cosmic magnetic field today, in Bianchi Type I cosmologies ($|\underset{\sim}{B}| \to \infty$ for all other types of singularities). From Table 1 we see that z-PANCAKE singularities occur only for the <u>axisymmetric</u> (i.e., $A = B$) PERFECT-FLUID-MAGNETIC cosmologies with the singularity behavior of equations (II.E.46). Therefore, we might expect our models to have the following behavior in the RADIATION-MAGNETIC phase near the initial singularity:

$$A \equiv B \approx 1 + \alpha\, t^{2/3} \quad , \quad C \approx t \quad , \qquad \text{(II.E.50.a)}$$

$$\rho_m \approx (\alpha/6\pi)\, t^{-4/3} \quad , \qquad \text{(II.E.50.b)}$$

$$\rho_b \approx (\beta/8\pi)\, [1 - 4\alpha t^{2/3}] \quad , \qquad \text{(II.E.50.c)}$$

where α is a non-negative constant and where

$$\beta \leq \beta_{critical} \cong 3 \times 10^{+6}\ \text{gm cm}^{-3} \quad . \qquad \text{(II.E.51)}$$

Although the above arguments are rather weak, they suggest a serious need for an analytical solution to the axisymmetric RADIATION-MAGNETIC case.

From our numerical results in Figure 13 we see that A (= B) is a monotonically increasing function of time (t) for the axisymmetric RADIATION-MAGNETIC case. Hence, ρ_b is monotonically decreasing with t, and after the initial singularity we need no longer be concerned that $|\tilde{B}|$ might exceed $|\tilde{B}|_{critical}$. Equations (II.E.50) show that the anisotropy dominates the dynamics near the singularity [where the solution asymptotically approaches Kasner's (1921) axisymmetric VACUUM solution]. We note that $(\rho_m/\rho_b) \rightarrow \infty$ at the singularity.

Let us now leap-frog to the present state of our "models", and work back towards the singularity. Let the subscript zero (o) denote the present value of a quantity. Recalling (see § I.B.) that observations set an approximate upper limit of about 10^{-7} gauss to any present-day intergalactic magnetic field, let us write the present field strength as:

$$|\underset{\sim}{B}|_o \equiv \eta \times 10^{-9} \text{ gauss} \quad (0 \leq \eta \lesssim 10^2) \quad , \qquad \text{(II.E.52)}$$

where η is a non-negative constant. If the observed galactic magnetic fields ($\approx 10^{-6}$ gauss) were captured during galaxy formation from a cosmic magnetic field, the most reasonable value for η is $\eta \approx 10$. Peebles (1967) remarks that such a primordial magnetic field might also solve the "problem" of the energetics of galaxy formation. In any case, from equation (II.E.52) we know that:

$$\rho_{bo} \equiv |\underset{\sim}{B}|^2/8\pi = (4 \times 10^{-41})\ \eta^2 \text{ gm cm}^{-3} \quad . \qquad \text{(II.E.53)}$$

The value $(\rho_{bo})_{max} \approx 4 \times 10^{-37}$ gm cm^{-3} (for $\eta \approx 10^2$) is negligible compared to the critical energy-density ($\approx 10^{-29}$ gm cm^{-3}) necessary to have the "flat" space sections of a Bianchi Type I cosmology today. Hence, a possible large-scale primordial magnetic field can have no noticeable effect upon the dynamical evolution of our Universe today.

Let us now consider the possible relationship between a uniform primordial magnetic field and the observed isotropy of the 2.7 ^{o}K cosmic microwave radiation. Referring back to our analysis in § II.D. we recall that the observed upper limit to the twelve-hour harmonic of temperature anisotropy of the radiation (Partridge and Wilkinson 1967) was:

$$(\Delta T/T)_o \lesssim (1.6 \pm 0.7) \times 10^{-3} \quad . \qquad \text{(II.E.54)}$$

In § II.D. we considered two extreme cases: (1) the entire matter content of our Universe has been ionized hydrogen for redshifts $z \lesssim 9$ (case H II), and (2) the ionized hydrogen from the primordial fireball recombined and neutralized when the photon temperature dropped below about 3000 ^oK, and it was never reionized (case H I). Equation (II.E.54) implies that the expansion of our Universe has been <u>very nearly isotropic</u> since a time [see equations (II.D.88) and (II.D.89)]:

$$t_s \begin{pmatrix} \text{H II} \\ \\ \text{H I} \end{pmatrix} \approx \begin{Bmatrix} 3.0 \times 10^{+8} \\ \\ 2.4 \times 10^{+5} \end{Bmatrix} \text{ years after the singularity.} \qquad \text{(II.E.55)}$$

In an essentially isotropic cosmology we have the behavior:

$$\rho_m \tilde{\propto} (R/R_o)^{-3} \quad , \quad \rho_b \tilde{\propto} (R/R_o)^{-4} \quad , \qquad \text{(II.E.56)}$$

where R, the scale factor for proper lengths, goes as

$$R \tilde{\propto} t^{2/3} \quad , \qquad \text{(II.E.57)}$$

in the DUST phase of our Universe. Therefore, we easily find that:

$$(\rho_b/\rho_m)_s \approx (\rho_{bo}/\rho_{mo})\ [t_s/(\ 9 \times 10^9 \text{ years})]^{-2/3}$$

$$\approx \begin{Bmatrix} (3.9 \times 10^{-11})\ \eta^2 \\ (4.5 \times 10^{-9}\ )\ \eta^2 \end{Bmatrix} \lesssim \begin{Bmatrix} 4 \times 10^{-7} \\ 5 \times 10^{-5} \end{Bmatrix} \text{ for case } \begin{pmatrix} \text{H II} \\ \text{H I} \end{pmatrix} . \tag{II.E.58}$$

From equation (II.E.58) we see this, that the observed isotropy of the 2.7 $^{\circ}$K cosmic microwave radiation is consistent with the negligible dynamical effect of a possible cosmic magnetic field during the time that the photons of the radiation were freely-propagating ($t \gtrsim t_s$).

Going back even further towards the initial singularity, we must consider the time when expansion anisotropies become large. In our PERFECT-FLUID models of § II.D. we had only to consider two fairly distinct phases of the evolution of our Universe: (1) the earlier RADIATION phase ($\gamma = 1/3$), and (2) the later DUST phase ($\gamma = 0$). When we include primordial magnetic fields in our discussion, however, we introduce the possibility of a MAGNETIC phase where the energy-density of the magnetic field (ρ_b) dominates that of either the "radiation" (ρ_r) or the "dust" (ρ_d). The situation is not as complicated as it might be though. From equations (II.D.73), (II.D.94), and (II.E.58) we see that <u>the primordial magnetic field has negligible effect upon the expansion during the DUST phase</u> since $\eta \lesssim 10^2$, and the time when expansion anisotropies become large in the DUST phase is still determined by $|\epsilon_d|$ via equation (II.D.73). Therefore, a possible

MAGNETIC phase can only occur somewhere in the RADIATION phase. Large expansion anisotropies inevitably arise during the RADIATION phase if either η or $|\epsilon_d| \neq 0$. These anisotropies may be attributed to either ϵ (the anisotropy parameter), as in equations (II.D.71) for the PERFECT-FLUID models of § II.D., or to the magnetic field (η). When the anisotropies are due to ϵ we may find the time when the anisotropies become appreciable from equation (II.D.71), but when they are due to η we cannot make any firm quantitative statements since we do not know the analytical RADIATION-MAGNETIC solution.

Finally, we consider primordial element formation in our "models". Primordial element formation takes place in the RADIATION phase of our Universe at a temperature of about 10^9 ^oK. It is important that we know how the number-density of baryons varies with time at that epoch. This variation is governed by the quantity $[(dV/dt)/3V]$ where $V \equiv ABC$ is the volume scale factor. Since the RADIATION-MAGNETIC case had not yet been solved analytically, we must resort to either numerical integration or to approximate methods to calculate the primordial element production in any particular PERFECT-FLUID-MAGNETIC model of our Universe. However, we can make the following qualitative statements: (a) if the magnetic field is merely "written into" (see Figure 13) the dynamics of the RADIATION phase element production will be qualitatively similar to the results of § II.D., while (b) if the magnetic field dramatically converts a potential z-CIGAR singularity into a z-PANCAKE singularity (see Figure 13) we will encounter a MAGNETIC phase during a small part of the RADIATION phase and element production may be substantially different from the PERFECT-FLUID result of § II.D. In the latter case an approximate method which may prove useful is the following: Use the analytical RADIATION solution of § II.D. until the magnetic field dominates the "radiation" ($\rho_b >> \rho_r$). Then match smoothly [with (A, B, C) and (a, b, c) continuous] to the PURE-MAGNETIC solution of equations (II.E.19). Continue with the PURE-MAGNETIC solution until the "radiation" once again dominates the magnetic field ($\rho_r >> \rho_b$) --- as we have seen it must (above). Then return smoothly to the RADIATION solution. This approximate procedure, while conceptually easy, is rather difficult to carry out in practice. The best program would be to solve the RADIATION-MAGNETIC case analytically, and to obtain [(dV/dt)/3V] directly. Then the computation of primordial element formation would be straightforward.

In summary, we can see that our present knowledge of the anisotropic Bianchi Type I PERFECT-FLUID-MAGNETIC cosmological models is sketchy at best. We have a few analytical solutions when there is a magnetic field present, but we are in desperate need of the analytical DUST-MAGNETIC and RADIATION-MAGNETIC solutions. Without these solutions, or extensive numerical studies, our PERFECT-FLUID-MAGNETIC models can be little better than heuristic.

III. CONCLUSION

"The waters into which I am stepping
have not yet been crossed by any man".
("L'acqua ch'io prendo giammai no si corse".)

Dante Alighieri

III. A. A SUMMARY OF THE THESIS

Let us here summarize what we have done and what we have learned in this thesis.

In Chapter I we gave some useful background information on relativistic cosmology: In § I.A. we presented our reasons for considering anisotropic cosmological models of our Universe, and we decided to use the general relativistic, hot big-bang theory of cosmology (with vanishing cosmological constant) as the mathematical framework for our analysis of anisotropic cosmologies. In § I. B. we gave an extensive list of the observational data pertinent to cosmology. In § I. C. we briefly reviewed the "standard" isotropic Lemaître cosmological models of our Universe which are currently in vogue, and we considered the "flat" (zero curvature) isotropic Lemaître models in some detail (as a prelude to Chapter II).

In Chapter II we implemented our analysis of anisotropic cosmologies. Our primary purpose in this thesis was to understand the effects of expansion anisotropies in general relativistic cosmological models, and Chapter II was where we accomplished this goal.

To simplify the mathematical analysis, while still retaining the adequate representation of the current stage of evolution of our Universe provided by the "standard" isotropic cosmological models, we limited ourselves to the consideration of Bianchi Type I cosmologies in § II.A.. Bianchi Type I cosmologies are spatially homogeneous, but anisotropic; and they exhibit no rotation. In § II.A. we exhibited the most general Bianchi Type I metric, and we considered the stress-energy tensor for perfect-fluid matter and/or sourceless electromagnetic fields in this metric. We found that perfect-fluid matter is comoving in a diagonal metric in the most general case, while the electric and magnetic fields are parallel and uniform in a non-diagonal metric in the most general case. To further simplify the mathematics, we chose to consider only the diagonal Bianchi Type I metric:

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2 \quad . \qquad \text{(III.A.1)}$$

In § II.B. we listed the authors and the content of all of the previous work which has been done on Bianchi Type I cosmologies. This list indicated the areas of possible research which remained to be undertaken in these cosmologies.

The most important physical properties of Bianchi Type I cosmologies and cosmological models were investigated in § II.C.. In subsection II.C.1) we explicitly displayed the kinematical properties ("expansion", "shear", and "rotation") of the diagonal metric of equation (III.A.1). We also proved that all Bianchi Type I cosmologies encounter an initial physical singularity, and we investigated the important symmetry properties of the Einstein field equations for Bianchi Type I cosmologies.

In subsection II.C.2) we considered gases of massless particles (both degenerate and non-degenerate) which decouple from thermal equilibrium and become freely-propagating (i.e., non-interacting) in the metric of equation (III.A.1). We briefly reviewed the general relativistic statistical mechanics of massless-particle gases in thermal equilibrium, and we used the solution to the Liouville equation for non-interacting massless particles to study the metric dependence [i.e., the dependence upon (A, B, C)] of the components of the stress-energy tensor for freely-propagating massless-particle gases in the diagonal metric of equation (III.A.1). In general, the components of this stress-energy tensor were expressible in terms of elliptic integrals, but we found that we could carry out the integrations in terms of elementary functions in the axisymmetric (i.e., A = B) case. In one limiting axisymmetric case we presented an analytical solution of the field equations due to Doroshkevich et al. (1967), and in another limiting axisymmetric case we derived a new anisotropic solution to the Einstein field equations. In general, we discussed the dynamical effects due to the anisotropic stresses of non-interacting massless-particle gases.

In subsection II.C.3) we considered the decoupling of gravitons, neutrinos, and photons in great detail. We found that the dynamical effects peculiar to non-interacting photons (and massive particles) are negligible, since they decouple only after the photon temperature falls below about 3000 ^{o}K. We were forced to neglect the effects of decoupled gravitons because of the great lack of present-day knowledge concerning the physics of graviton decoupling. The dynamical effects associated with decoupled muon-neutrinos (ν_{μ}) were found to depend critically upon the physical processes occurring during electron-neutrino (ν_e) decoupling. The concept of $\underline{\nu_e}$ viscosity, first introduced by Misner (1967, 1968), was used in a new, more realistic, analysis of the viscous damping of expansion anisotropies near temperatures of about 10^{10} ^{o}K. We found that expansion anisotropies are damped to essentially negligible levels by the time that temperatures are below 10^{9} ^{o}K, and that the "standard" isotropic cosmological models are an adequate representation of our Universe after this point!

In an effort to better understand the structure of the Einstein field equations and the possible effects of large expansion anisotropies, we totally neglected all effects due to anisotropic neutrino stresses in §§ II.D. and II.E.. We derived a large number of analytical and numerical solutions to the Einstein field equations for anisotropic Bianchi Type I cosmologies. In § II.D. our stress-energy tensor was that of perfect-fluid matter with the barotropic equation of state $p_m = \gamma \rho_m$ $(0 \leq \gamma \leq 1)$. We found the general analytical solution for all γ for this case: the PERFECT-FLUID case. We presented the explicit solutions (in terms of elementary functions) and discussed the properties of the following: (1) the DUST solution $(\gamma = 0)$, (2) the RADIATION solution $(\gamma = 1/3)$, (3) two infinite sequences of HARD solutions $(1/3 < \gamma < 1)$, (4) the ZEL'DOVICH solution $(\gamma = 1)$, and (5) the DUST-PLUS-RADIATION solution $(\rho = \rho_d + \rho_r,\ p = \rho_r/3)$. We used these analytical solutions to construct semi-realistic, anisotropic, DUST-PLUS-RADIATION cosmological models of our Universe. In § II.E. our stress-energy tensor consisted of perfect-fluid matter with the barotropic equation of state, and a uniform comoving magnetic field, with energy-density ρ_b, aligned along the z-axis. We called these the PERFECT-FLUID-MAGNETIC cosmologies. We derived several new analytical solutions [in particular, the PURE-MAGNETIC solution $(\rho_m = 0,\ \rho_b \neq 0)$ and the ZEL'DOVICH-MAGNETIC solution $(\rho_m$ and $\rho_b \neq 0,\ \gamma = 1)$], found the singularity behavior of all remaining PERFECT-FLUID-MAGNETIC cases, and studied these remaining cases by numerical integration of the Einstein field equations. We then discussed the properties of heuristic, semi-realistic, anisotropic, DUST-PLUS-RADIATION-MAGNETIC cosmological models of our Universe.

In all of the semi-realistic, anisotropic, cosmological models we constructed we studied the possible effects of expansion anisotropies and of a uniform primordial magnetic field upon the following: (a) the type of initial physical singularity, (b) the thermal history and temporal evolution of our Universe, (c) primordial element formation, (d) the time when expansion anisotropies become small, and (e) the temperature isotropy of the observed 2.7 $^{\circ}$K cosmic microwave radiation.

In conclusion, we can say the following: We have gained an excellent understanding of the possible effects of expansion anisotropies in Bianchi Type I cosmologies and cosmological models. Our many new analytical solutions have given us useful insights into the structure and properties of the Einstein field equations. We have found that non-interacting massless particles and the viscosity associated with ν_e decoupling are the fundamental physical factors which determine the possible evolution of Bianchi Type I cosmologies for temperatures less than about 10^9 $^{\circ}$K. Expansion anisotropies are essentially negligible by the time that primordial element formation begins, and the "standard" isotropic cosmological models are an adequate representation of our Universe thereafter. We see that we can now reasonably say this, that the observational consequences of Bianchi

Type I cosmological models are practically identical to those of the "standard" isotropic Lemaître models of our Universe.

III. B. THE OUTLOOK FOR THE FUTURE

In this thesis we have learned a great deal about anisotropic Bianchi Type I cosmologies. Our investigation, though extensive, is not complete. We can see many interesting areas of research in Bianchi Type I cosmologies which should be studied in the future. Here we will end this thesis by listing some of the more important research problems which remain in Bianchi Type I cosmologies:

(a) An accurate analysis of the physical and dynamical effects associated with the decoupling of massless particles near the initial physical singularity is needed. In particular, the viscous damping of expansion anisotropies by the decoupling of gravitons, muon-neutrinos, and electron-neutrinos warrants more detailed study than that given here. The dynamical properties associated with particles decoupling in a primordial magnetic field should also be investigated.

(b) When sourceless electromagnetic fields are present in a Bianchi Type I cosmology, the metric cannot generally be diagonalized to the form of equation (III.A.1). An investigation of the properties of such non-diagonal Bianchi Type I metrics and of such general sourceless electromagnetic fields would be very useful now. We are especially interested in the present-day observational consequences of such metrics and fields.

(c) More analytical solutions to the Einstein field equations for the PERFECT-FLUID-MAGNETIC cosmologies of § II.E. are needed. The general RADIATION-MAGNETIC solution is of greatest interest here, especially with respect to the possible effects of a uniform primordial magnetic field upon primordial element formation. Any new analytical solution will be of great interest, to the extent that it further elucidates the structure and properties of these field equations.

(d) The effects of the viscous damping of anisotropy upon the growth of density perturbations in anisotropic Bianchi Type I cosmologies (see Doroshkevich 1966) have not yet been considered. An accurate analysis of these effects is now needed.

When the above research problems have been satisfactorily solved, our knowledge of all physically permissible Bianchi Type I cosmologies will be practically complete. Then more difficult cosmologies, such as those of Bianchi Type V and IX, may be approached with greater confidence.

APPENDIX A

GASES OF MASSLESS PARTICLES IN THERMAL EQUILIBRIUM

We desire to evaluate equations (II.C.17) and (II.C.18) in the case of thermal equilibrium. Using equations (II.C.23), (II.C.24), and (II.C.25), we write equation (II.C.18) as:

$$n(\pm) = \tilde{\mu}h^{-3} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} (P^0)^2 \{\exp[(P^0/T)-D] \pm 1\}^{-1} \, dP^0. \qquad (A.1)$$

Evaluating the angular integrals and making the change of variable

$$x \equiv P^0/T \quad , \qquad (A.2)$$

we obtain the result indicated in equations (II.C.26) and (II.C.27):

$$n(\pm) = (4\pi\tilde{\mu}h^{-3}) \, T^3 \int_0^{\infty} x^2 \, [\exp(x-D) \pm 1]^{-1} \, dx \quad . \qquad (A.3)$$

We obtain $T^0{}_0$ by substituting equations (II.C.16), (II.C.23), and (I.C.25) into equation (II.C.17):

$$T^0{}_0 \, (\pm) = \tilde{\mu}h^{-3} \iiint (P^0)^3 \, \{\exp[(P^0/T)-D] \pm 1\}^{-1} \, dP^0 \, d\Omega \quad . \qquad (A.4)$$

Evaluating the angular integrals indicated by equation (II.C.24) and changing variables as in equation (A.2), we obtain the result displayed in equations (II.C.26) and (II.C.27):

$$T^0{}_0 \, (\pm) = (4\pi\tilde{\mu}h^{-3}) \, T^4 \int_0^{\infty} x^3 \, [\exp(x-D) \pm 1]^{-1} \, dx \quad . \qquad (A.5)$$

The $T^i{}_i$ (no sum) are evaluated in the same way as was $T^0{}_0$, except that equations (I.C.22) must now be used and the angular integrals are more interesting. We obtain:

$$\left.\begin{aligned} T^i{}_i\,(\pm) \\ &= \tilde{\mu}h^{-3} \iiint P^i P_i \{\exp[(P^0/T)-D] \pm 1\}^{-1} P^0 dP^0\, d\Omega \quad \text{(no sum)} \\ &= \tilde{\mu}h^{-3} \iiint g_{ii}(P^i)^2 \{\exp[(P^0/T)-D] \pm 1\}^{-1} P^0 dP^0\, d\Omega \quad \text{(no sum)} \\ &= -\tilde{\mu}h^{-3} \iint_{\Omega} F^2(\theta,\varphi)\, d\Omega \int_0^{\infty} (P^0)^3 \{\exp[(P^0/T)-D] \pm 1\}^{-1} dP^0 \end{aligned}\right\}, \quad (A.6)$$

where

$$F(\theta,\varphi) \equiv (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta) \text{ for } i = (1,2,3). \qquad (A.7)$$

Using equation (A.2) and doing the indicated angular integrations, we find the results given in equations (II.C.26) and (II.C.27):

$$\left.\begin{aligned} T^1{}_1(\pm) = T^2{}_2(\pm) = T^3{}_3(\pm) &= -(4\pi\tilde{\mu}/3h^3)\, T^4 \int_0^{\infty} x^3[\exp(x-D) \pm 1]^{-1} dx \\ &= -(1/3)\, T^0{}_0(\pm) \end{aligned}\right\} (A.8)$$

Since $T^{\mu}{}_{\nu}$ is a symmetric tensor, it has six independent off-diagonal terms. All of these off-diagonal terms vanish in the case of thermal equilibrium because:

$$\left.\begin{aligned}
T^0{}_1(\pm) &\propto \int_0^{2\pi} \cos\varphi \, d\varphi = 0 \\
T^0{}_2(\pm) &\propto \int_0^{2\pi} \sin\varphi \, d\varphi = 0 \\
T^0{}_3(\pm) &\propto \int_0^{\pi} \sin\Theta \cos\Theta \, d\Theta = 0 \\
T^1{}_2(\pm) &\propto \int_0^{2\pi} \sin\varphi \cos\varphi \, d\varphi = 0 \\
T^1{}_3(\pm) &\propto \int_0^{2\pi} \cos\varphi \, d\varphi = 0 \\
T^2{}_3(\pm) &\propto \int_0^{2\pi} \sin\varphi \, d\varphi = 0
\end{aligned}\right\} \quad . \qquad (A.9)$$

Therefore

$$T^\mu{}_\nu(\pm) = 0 \quad \text{for } (\mu \neq \nu) \quad . \qquad (A.10)$$

APPENDIX B

ENERGY INTEGRALS FOR GASES OF MASSLESS BOSONS AND FERMIONS

In equation (II.C.27) we have the class of energy integrals

$$f_\ell^{(\pm)}(D) \equiv \int_0^\infty x^{\ell+1}\,[\exp(x-D) \pm 1]^{-1}\,dx \quad . \qquad \text{(B.1)}$$

Here we will investigate some of the pertinent properties of these integrals.

For massless bosons (photons and gravitons) we need only consider $f_\ell^{(-)}(0)$ (see Chiu 1967). In this case, the integrals may be evaluated as follows:

$$f_\ell^{(-)}(0) \equiv \int_0^\infty x^{\ell+1}[\exp(x)-1]^{-1}\,dx = \int_0^\infty x^{\ell+1}\exp(-x)[1-\exp(-x)]^{-1}dx$$

$$= \sum_{k=1}^{\infty} \int_0^\infty x^{\ell+1}\exp(-kx)\,dx = (\ell+1)!\sum_{k=1}^{\infty} k^{-(\ell+2)}$$

$$= (\ell+1)!\quad \zeta(\ell+2) \quad , \qquad \text{(B.2)}$$

where $\zeta(\ell)$ is the Riemann zeta-function [see equation (II.C.30)]. The first few values of $\zeta(\ell)$ are:

$$\zeta(0) = -\tfrac{1}{2}\,,\ \zeta(1) = \infty,\ \ \zeta(2) = \pi^2/6,\ \ \zeta(3) = 1.202\ldots,\zeta(4) = \pi^4/90,\ldots \qquad \text{(B.3)}$$

In a non-degenerate (D = 0) massless fermion gas (neutrinos and anti-neutrinos) we must evaluate $f_\ell^{(+)}(0)$. Using the procedure resulting in equation (B.2), we readily find:

$$\left.\begin{aligned} f_{\ell}^{(+)}(0) &= (\ell+1)!\,[1-2^{-(\ell+1)}]\;\zeta(\ell+2) \\ &= [1-2^{-(\ell+1)}]\,f_{\ell}^{(-)}(0) \end{aligned}\right\} \quad . \tag{B.4}$$

The only case remaining to be studied is that of a degenerate ($D \neq 0$) gas of massless fermions. We must consider the properties of $f_{\ell}^{(+)}(D)$. Since these integrals cannot be evaluated in terms of elementary functions, we shall investigate their behavior for both small and large values of D.

Recall that

$$f_{\ell}^{(+)}(D) \equiv \int_{0}^{\infty} x^{\ell+1}\,[\exp(x-D)+1]^{-1}\,dx \quad . \tag{B.5}$$

Let us make the change of variable

$$y \equiv x-D \quad . \tag{B.6}$$

Then equation (B.5) becomes:

$$f_{\ell}^{(+)}(D) = \int_{-D}^{\infty} (y+D)^{\ell+1}\,[\exp(y)+1]^{-1}\,dy \quad . \tag{B.7}$$

Using the binomial expansion, we reduce equation (B.7) to:

$$f_\ell^{(+)}(D) = D^{\ell+1} \int_{-D}^{\infty} [\exp(y) + 1]^{-1}\, dy$$

$$+ \sum_{r=o}^{\ell} \binom{\ell+1}{r} D^r \left\{ \int_o^{\infty} \frac{y^{\ell+1-r}\, dy}{[\exp(y) + 1]} + \int_{-D}^{o} \frac{y^{\ell+1-r}\, dy}{[\exp(y) + 1]} \right\} . \quad (B.8)$$

Here the binomial coefficient $\binom{n}{m}$ is equal to

$$\binom{n}{m} = n!\, / [(n-m)!m!] \; . \quad (B.9)$$

The first and second integrals on the right-hand side of equation (B.8) are straightforward, and we obtain:

$$f_\ell^{(+)}(D) = D^{\ell+1}\, \ell n\, [1+ \exp(D)] + \sum_{r=o}^{\ell} \frac{(\ell+1)!}{r!} [1-2^{-(\ell+1-r)}]\, \zeta(\ell+2-r)\, D^r$$

$$+ \sum_{r=o}^{\ell} \binom{\ell+1}{r} D^r \int_{-D}^{o} y^{\ell+1-r} [1 + \exp(y)]^{-1}\, dy \; . \quad (B.10)$$

When D = 0 in equation (B.10) we see that we recover equation (B.4) immediately.

Let us now consider the asymptotic behavior of the sole remaining integral in equation (B.10):

$$I(D) \equiv \int_{-D}^{o} y^{\ell+1-r} [1 + \exp(y)]^{-1}\, dy \quad . \quad (B.11)$$

Our analysis here is based upon pages 127 and 310 of Gradshteyn and Ryzhik (1965). For small D we proceed as follows:

$$I(D) = \int_{-D}^{0} y^{\ell+1-r} \left[\frac{\exp(-y/2)}{\exp(y/2) + \exp(-y/2)}\right] dy$$

$$= \int_{-D}^{0} y^{\ell+1-r} \quad (1/2)[1-\tanh\,(y/2)]\,dy$$

$$= (1/2)\left[(-1)^{\ell+1-r}(\ell+2-r)^{-1}\,D^{\ell+2-r} - \int_{-D}^{0} y^{\ell+1-r}\tanh\,(y/2)dy\right] \quad .(B.12)$$

Now make the change of variable

$$z \equiv -y/2 \quad , \qquad (B.13)$$

to obtain, finally:

$$-(1/2)\int_{-D}^{0} y^{\ell+1-r}\tanh\,(y/2)\,dy = (-2)^{\ell+1-r}\int_{0}^{D/2} z^{\ell+1-r}\tanh\,z\,dz$$

$$= (-1)^{\ell+1-r}\sum_{m=1}^{\infty}\frac{(2^{2m}-1)B_{2m}}{(2m+\ell+1-r)(2m)!}\,D^{(2m+\ell+1-r)} \quad .(B.14)$$

Here B_n are the well-known Bernoulli numbers, and equation (B.14) is valid for $0 \leq D < \pi$. Therefore, for $0 \leq D < \pi$ we have:

$$I(D) = (-1)^{\ell+1-r}\,D^{\ell+2-r}\left\{[2(\ell+2-r)]^{-1} + \sum_{m=1}^{\infty}\frac{(2^{2m}-1)B_{2m}}{(2m+\ell+1-r)(2m)!}\,D^{(2m-1)}\right\} .(B.15)$$

The asymptotic behavior of $f_\ell^{(+)}(D)$ for <u>small</u> D now follows from equations (B.10), (B.11), and (B.15). For example, retaining only the first-order correction term for D we have (with $\ell > 0$):

$$f_\ell^{(+)}(D) = f_\ell^{(+)}(0) + (\ell+1)!\ [1-2^{-\ell}]\ \zeta(\ell+1)\ D + \text{Order}(D^2) \quad . \qquad (B.16)$$

Finally, we consider the asymptotic behavior of $f_\ell^{(+)}(D)$ for large D. Our analysis above is again applicable down to equation (B.11). Let us change our variables in equation (B.11) via

$$u \equiv - y \quad . \qquad (B.17)$$

Then we have:

$$I(D) = (-1)^{\ell+1-r} \int_0^D u^{\ell+1-r}\ [1 + \exp(-u)]^{-1}\ du \quad . \qquad (B.18)$$

Proceeding now as we did in equation (B.2) we have:

$$\begin{aligned} I(D) &= (-1)^{\ell+1-r} \sum_{k=0}^{\infty} (-1)^k \int_0^D u^{\ell+1-r} \exp(-ku)\ du \\ &\approx (-1)^{\ell+1-r} \left\{ \int_0^D u^{\ell+1-r}\ du + \sum_{k=1}^{\infty} (-1)^k \int_0^{\infty} u^{\ell+1-r} \exp(-ku) du \right\} \\ &\approx (-1)^{\ell+1-r} \left[(\ell+2-r)^{-1}\ D^{\ell+2-r} + \sum_{k=1}^{\infty} (-1)^k\ (\ell+1-r)!\ k^{-(\ell+2-r)} \right] . \end{aligned} \qquad (B.19)$$

Equation (B.19) is approximate, because we let one limit of integration in the second line go to infinity. Our result, however, is very accurate for $D \gtrsim 5$. Therefore, for $D \gtrsim 5$ we may combine equations (B.10), (B.11), and (B.19) to find that the asymptotic behavior of $f_\ell^{(+)}(D)$ is:

$$f_{\ell}^{(+)}(D) \approx D^{\ell+2} \left[1 + (-1)^{\ell+1} (\ell+1)! \sum_{r=o}^{\ell} \frac{(-1)^{-r}}{r!(\ell+2-r)!} \right] . \quad (B.20)$$

This concludes our investigation of the energy integrals [equation (B.1)].

APPENDIX C

A NEW ANISOTROPIC SOLUTION OF THE EINSTEIN FIELD EQUATIONS FOR MASSLESS, NON-INTERACTING PARTICLES

Consider a diagonal stress-energy tensor of the form:

$$T^{\mu}_{\ \nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -\rho/2 & 0 & 0 \\ 0 & 0 & -\rho/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \qquad (C.1)$$

in the diagonal metric of equation (II.B.2). The conservation equation (II.C.12) becomes:

$$(\rho V)^{\cdot} + (1/2)(a + b)\,(\rho V) = 0 \quad , \qquad (C.2)$$

which immediately implies:

$$(\rho/\rho_*) = (AB/A_*B_*)^{-1/2}(V/V_*)^{-1} \quad . \qquad (C.3)$$

Here, all quantities with the subscript * are constants. The Einstein field equations (II.C.17) become:

$$ab + ac + bc = 8\pi\rho \quad , \qquad (C.4.a)$$

$$(\dot{b} + \dot{c}) + b^2 + c^2 + bc = -4\pi\rho \quad , \qquad (C.4.b)$$

$$(\dot{a} + \dot{c}) + a^2 + c^2 + ac = -4\pi\rho \quad , \qquad (C.4.c)$$

$$(\dot{a} + \dot{b}) + a^2 + b^2 + ab = 0 \quad , \qquad (C.4.d)$$

and because of equation (C.2) only three of equations (C.4) are independent.

Subtracting equation (C.4.c) from equation (C.4.b) gives the first integral:

$$(a - b)\ (V/V_*) = K_1 = \text{constant} \quad . \tag{C.5}$$

Similar simple manipulations of equations (C.4) lead to the system of equations:

$$\left.\begin{aligned} ab + ac + bc &= 8\pi\rho \\ c(V/V_*) = K_2 &= \text{constant} \\ (a - b)\ (V/V_*) = K_1 &= \text{another constant} \end{aligned}\right\} \quad . \tag{C.6}$$

Let us change our time-variable via:

$$dt = (V/V_*)\ d\bar{\eta} \quad . \tag{C.7}$$

Using equations (II.A.30) and (C.7), the last two of equations (C.6) may be integrated to:

$$(C/C_*) = \exp(K_2\ \bar{\eta}) \quad , \tag{C.8.a}$$

$$(A/A_*) = (B/B_*)\ \exp(K_1\ \bar{\eta}) \quad . \tag{C.8.b}$$

The first of equations (C.6), with equation (C.3), gives:

$$(1/f)\ (df/d\bar{\eta}) = \pm\ [K_2{}^2 + (K_1/2)^2 + 8\pi\rho_* f]^{1/2} \quad , \tag{C.9}$$

where we have set:

$$(B/B_*) \equiv f(\bar{\eta})\ \exp[-\ (K_2 + \tfrac{1}{2}\ K_1)\bar{\eta}] \quad . \tag{C.10}$$

Integrating equation (C.9) gives us:

$$f = 4\varkappa(\alpha/\beta)^2\ \exp(\alpha\bar{\eta})\ [1 - \varkappa\exp(\alpha\bar{\eta})]^{-2} \quad , \tag{C.11}$$

where

$$\left.\begin{aligned} \varkappa &\equiv \left[\frac{(\alpha^2 + \beta^2)^{1/2} - \alpha}{(\alpha^2 + \beta^2)^{1/2} + \alpha}\right] \\ \alpha^2 &\equiv K_2{}^2 + (K_1/2)^2 \\ \beta^2 &= 8\pi\rho_* \end{aligned}\right\} \quad . \tag{C.12}$$

Let us now make the following simplifying notational changes:

$$K_2 \equiv t_*{}^{-1}\ ,\quad \eta \equiv K_2\bar{\eta}\ ,\quad 8\pi\rho_* t_*{}^2 = 1\ ,\quad k \equiv (K_1/2K_2). \tag{C.13}$$

Then, our general solution takes the form:

$$\left.\begin{aligned} (A/A_*,\ B/B_*,\ C/C_*) &= \{f\ \exp[-(1-k)\eta],\ f\ \exp[-(1+k)\eta],\ \exp(\eta)\} \\ (V/V_*) &\equiv (ABC/A_*B_*C_*) = f^2\ \exp(-\eta) \\ 8\pi\rho &= (t_*{}^2\ f^3)^{-1}\ \exp(2\eta) \\ (t/t_*) &= \int_0^{\eta} f^2\ \exp(-\eta)d\eta \end{aligned}\right\}, \tag{C.14}$$

with

$$f \equiv 4\varkappa(1+k^2)\exp[(1+k^2)^{1/2}\eta]\{1-\varkappa\exp[(1+k^2)^{1/2}\eta]\}^{-2} \quad , \qquad (C.15.a)$$

and

$$\varkappa \equiv [(2+k^2)^{1/2}-(1+k^2)^{1/2}]^2 \quad . \qquad (C.15.b)$$

Note that this general solution is valid only in the η-range

$$0 \leq \eta \leq (1+k^2)^{-1/2}\,\ell n(1/\varkappa) \quad , \qquad (C.16.a)$$

and the k-range

$$-1 \leq k \leq +1 \quad . \qquad (C.16.b)$$

APPENDIX D

A STUDY OF THE AXISYMMETRIC RADIATION-MAGNETIC CASE

The axisymmetric RADIATION-MAGNETIC case is characterized by $\mu \neq 0$, $\beta \neq 0$, $\gamma = 1/3$, and $A(t) \equiv B(t)$. In this case, the field equations (II.E.7) take the form:

$$a^2 + 2ac = + V^{-4/3} + \delta A^{-4} \quad , \qquad \text{(D.1.a)}$$

$$2a' + 3a^2 = (-1/3)V^{-4/3} + \delta A^{-4} \quad , \qquad \text{(D.1.b)}$$

where all derivatives are with respect to normalized time:

$$\tau \equiv \mu^{1/2}\, t \quad , \qquad \text{(D.2)}$$

and where

$$\left.\begin{aligned} V &\equiv A^2 C \\ \delta &\equiv \beta/\mu \end{aligned}\right\} . \qquad \text{(D.3)}$$

Now make the change of variables

$$\left.\begin{aligned} x &\equiv (C/A) \\ y &\equiv (A/x)(dx/dA) \end{aligned}\right\} . \qquad \text{(D.4)}$$

Equation (D.1.a) immediately leads to "quadratures" for the time dependence of A, if we know C as a function of A:

$$\tau = \int x^{2/3} \left[\frac{3+2y}{1+\delta x^{4/3}} \right]^{1/2} A\, dA \quad . \qquad \text{(D.5)}$$

Now we must proceed to find C as a function of A (note that this was how the axisymmetric DUST-MAGNETIC case was solved). Dividing equation (D.1.b) by equation (D.1.a) we find:

$$\left[\frac{3+2y(x/a)(da/dx)}{3+2y} \right] = \left[\frac{-1+3\delta x^{4/3}}{3+3\delta x^{4/3}} \right] \quad . \qquad \text{(D.6)}$$

Making the change of variable

$$z \equiv \delta x^{4/3} \quad , \qquad \text{(D.7)}$$

we find, after much tedious algebra:

$$[(1+2z)/(3+2y)]\, 4z(dy/dz) + (12z/y) + [(1+9z+6z^2)/(1+z)] = 0 \quad . \text{(D.8)}$$

A final change of variable

$$\zeta \equiv (2y)^{-1} \quad , \qquad \text{(D.8)}$$

and a great deal more algebra yields:

$$d\zeta/dz = A_1(z)\zeta + A_2(z)\zeta^2 + A_3(z)\zeta^3 \quad , \qquad \text{(D.9)}$$

where

$$(2z)(1+z)(1+2z)[A_1,A_2,A_3] \equiv [1+9z+6z^2,\ 3+51z+42z^2,\ 72z+72z^2] \quad . \text{(D.10)}$$

From page 74 of Davis (1962) we see that equation (D.9) is an Abel first-order nonlinear differential equation. Analytical solutions to such an equation can be readily obtained only if there exist certain very special relationships between the functions A_1, A_2, and A_3. Our equation (D.10) does not satisfy these relationships. The best that we can do is to obtain a transcendental solution for $\zeta(z)$. I am indebted to Professor Alfonso Campolattaro for showing me the following transcendental solution to equations (D.9) and (D.10).

We can write equation (D.9) in the form:

$$d\zeta/dz = A_3\zeta[\zeta - B_1(z)]\,[\zeta - B_2(z)] \quad , \qquad \text{(D.11)}$$

where B_1 and B_2 are the roots of the quadratic equation:

$$B^2 + (A_2/A_3)\,B + (A_1/A_3) = 0 \quad . \qquad \text{(D.12)}$$

From equation (D.10) we easily find:

$$(B_1,B_2) = -\,\{1/3,\ [(1+9z+6z^2)/(24z)(1+z)]\} \quad . \qquad \text{(D.13)}$$

In general, equation (D.11) may be written:

$$\frac{d\zeta}{\zeta(\zeta-B_1)(\zeta-B_2)} = A_3 dz \quad . \qquad \text{(D.14)}$$

We now make a partial fraction expansion:

$$\frac{1}{\zeta(\zeta-B_1)(\zeta-B_2)} \equiv \frac{F(z)}{\zeta} + \frac{G(z)}{(\zeta-B_1)} + \frac{H(z)}{(\zeta-B_2)} \quad . \qquad \text{(D.15)}$$

Let us define:

$$J(\zeta,z) \equiv F(z)\ln\zeta + G(z)\ \ln(\zeta-B_1) + H(z)\ \ln(\zeta-B_2) = K = \text{constant}. \qquad \text{(D.16)}$$

Taking the total derivative of equation (D.16) and using equations (D.14) and (D.15), we have the system of equations:

$$\left.\begin{aligned} F'\ln\zeta + G'\ln(\zeta-B_1) + H'\ln(\zeta-B_2) - \left(\frac{G\,B_1'}{\zeta-B_1}\right) - \left(\frac{H\,B_2'}{\zeta-B_2}\right) + A_3 &= 0 \\ F\ln\zeta + G\ln(\zeta-B_1) + H\ln(\zeta-B_2) - K &= 0 \end{aligned}\right\}, \tag{D.17}$$

where a prime (') denotes differentiation with respect to z in equation (D.17). Multiplying the first (second) line in equation (D.17) by H (H'), subtracting, and exponentiating yields the transcendental solution:

$$\zeta^{(HF'-FH')}(\zeta-B_1)^{(HG'-GH')}\exp\left[-\left(\frac{HGB_1'}{\zeta-B_1}\right) - \left(\frac{H^2B_2'}{\zeta-B_2}\right)\right] = \exp(-HA_3-H'K)\ . \tag{D.18}$$

Since we desire only to illustrate the complexity of the axisymmetric RADIATION-MAGNETIC case here, we shall not proceed further.

REFERENCES

Abell, G. O. 1965, "Clustering of Galaxies", Annual Review of Astronomy and Astrophysics, 3, 1.

Abramowitz, M., and Stegun, I. A. 1965, Handbook of Mathematical Functions (Dover Publications, New York).

Alpher, R. A., Gamow, G., and Herman, R. C. 1967, "Thermal Cosmic Radiation and the Formation of Protogalaxies", Proceedings of the National Academy of Science, 58, 2179.

Bahcall, J. N., and Salpeter, E. E. 1966, "Absorption Lines in the Spectra of Distant Sources", Astrophysical Journal, 144, 847.

Bardeen, J. M. 1965, "Stability and Dynamics of Spherically Symmetric Masses in General Relativity", Doctoral Dissertation in Physics, California Institute of Technology.

Bianchi, L. 1918, Capitolo XIII, "Appicazioni alla teoria degli spazii pluridimensionali con un gruppo continuo di movimenti", § 198, Lezioni Sulla Teoria Dei Gruppi Continui Finiti Di Trasformazioni (E. Spoerri, Pisa)[see also Nicola Zanichelli, Bologna (1928) edition].

Bondi, H. 1961, Cosmology (Cambridge University Press, Cambridge).

Bondi, H., and Gold, T. 1948, "The Steady State Theory of the Expanding Universe", Monthly Notices of the Royal Astronomical Society, 108, 252.

Brans, C., and Dicke, R. H. 1961, "Mach's Principle and a Relativistic Theory of Gravitation", Physical Review, 124, 925.

Brecher, K., and Morrison, P. 1967, "Cosmology, Black-Body Radiation, and the Diffuse X-Ray Background", Astrophysical Journal, 150, L61.

Cameron, A. G. W. 1967, "The Generation of Cosmic Magnetic Fields", Astrophysical Letters 1, 9.

Cayrel, R. 1968, "The Location of a Few Subdwarfs in the Theoretical H-R Diagram and the Helium Content of Population II", Astrophysical Journal, 151, 997.

Chernikov, N. A. 1962a, Doklady Akademii Nauk SSSR, 144, 89 [English translation is "Kinetic Equation For a Relativistic Gas in an Arbitrary Gravitational Field", Soviet Physics-Doklady, 7, 397 (1962)].

__________ 1962b, ibid, 144, 314 [English translation is "Flux Vector and Mass Tensor of a Relativistic Ideal Gas", ibid, 7, 414 (1962)].

__________ 1962c, ibid, 144, 544 [English translation is "Relativistic Maxwell-Boltzmann Distribution and the Integral Form of the Conservation Laws", ibid, 7, 428 (1962)].

__________ 1963, "The Relativistic Gas in the Gravitational Field", Acta Physica Polonica, 23, 629.

Chernin, A. D., 1965, Astronomicheskii Zhurnal, **42**, 1124 [English translation is "A Model of a Universe Filled by Radiation and Dustlike Matter", Soviet Astronomy-AJ, **9**, 871 (1966)].

Chiu, H-Y. 1967, "A Cosmological Model for Our Universe. I", Annals of Physics, **43**, 1.

Clemence, G. M. 1957, "Astronomical Time", Reviews of Modern Physics, **29**, 2.

Cohen, J. M. 1967, "Friedman Cosmological Models with Both Radiation and Matter", Nature, **216**, 249.

Conference on the Instability of Systems of Galaxies 1961, Astronomical Journal, **66**, 533.

Conklin, E. K., and Bracewell, R. N. 1967a, "Isotropy of Cosmic Background Radiation at 10,690 MHz", Physical Review Letters **18**, 614.

________________ 1967b, "Limits on Small Scale Variations in the Cosmic Background Radiation", Nature **216**, 777.

Dautcourt, G., Papapetrou, A., and Treder, H. 1962, "Eindimensionale Gravitationsfelder", Annalen der Physik, **9**, 330.

Davis, H. T. 1962, Introduction to Nonlinear Differential and Integral Equations (Dover Publications, New York).

Dicke, R. H. 1964, The Theoretical Significance of Experimental Relativity (Gordon and Breach, New York).

__________ 1967, "Gravitational Theory and Observation", Physics Today, 20, # 1, 55.

__________ 1968, "Scalar-Tensor Gravitation and the Cosmic Fireball", Astrophysical Journal, 152, 1.

Dicke, R. H., Peebles, P. J. E., Roll, P. G., and Wilkinson, D. T. 1965, "Cosmic Black-Body Radiation", Astrophysical Journal, 142, 414.

Doroshkevich, A. G. 1965, Astrofizika, 1, 255 [English translation is "Model of a Universe With a Uniform Magnetic Field", Astrophysics, 1, 138 (1965)].

__________ 1966, Astrofizika, 2, 37 [English translation is "The Gravitational Instability of Anisotropic Homogeneous Solutions", Astrophysics, 2, 15 (1966)].

Doroshkevich, A. G., Zel'dovich, Ya. B., and Novikov, I.D. 1967, Zhurnal Éksperimental'noi i Teoreticheskoi Fiziki-Pis'ma v Redaktsiyu, 5, 119 [English translation is "Neutrinos and Gravitons in the Anisotropic Model of the Universe", JETP Letters, 5, 96 (1967)].

Einstein, A. 1915, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, p. 778, "Zur Allgemeinen Relativitätstheorie"; p. 799, "Zur Allgemeinen Relativitätstheorie (Nachtrag)"; p. 844, "Die Feldgleichungen der Gravitation".

Einstein, A., and de Sitter, W., 1932, "On the Relation Between the Expansion and the Mean Density of the Universe", Proceedings of the National Academy of Science, 18, 213.

Eisenhart, L. P. 1926, Riemannian Geometry (Princeton University Press, Princeton).

Ellis, G. F. R. 1967, "Dynamics of Pressure-Free Matter in General Relativity", Journal of Mathematical Physics, 8, 1171.

Ellis, G. F. R., and MacCallum, M. A. H. 1968, "A Class of Homogeneous Cosmological Models", Cambridge University preprint (submitted for publication to the Journal of Mathematical Physics).

Epstein, E. E. 1967, "On the Small-Scale Distribution of 3.4 mm Wavelength of the Reported 3 °K Background Radiation", Astrophysical Journal, 148, L157.

Estabrook, F. B., Wahlquist, H. D., and Behr, C. G. 1968, "Dyadic Analysis of Spatially Homogeneous World Models", Journal of Mathematical Physics, 9, 497.

Ewing, M. S., Burke, B. F., and Staelin, D. H. 1967, "Cosmic Background Measurement at a Wavelength of 9.24 mm", Physical Review Letters, 19, 1251.

Faulkner, J. 1967, "Quasi-Homology Relations and the Eggen-Sandage Residue", Astrophysical Journal, 147, 617.

Faulkner, J., and Iben, Jr., I. 1966, "The Evolution of Population II Stars", Astrophysical Journal, 144, 995.

Felten, J. E. 1966, "Energy Density of Starlight in the Metagalaxy", Astrophysical Journal, 144, 241.

Felten, J. E., Gould, R. J., Stein, W. A., and Woolf, N. J. 1966, "X-Rays From the Coma Cluster of Galaxies", Astrophysical Journal, 146, 955.

Felten, J. E., and Morrison, P. 1966, "Omnidirectional Inverse Compton and Synchrotron Radiation From Cosmic Distributions of Fast Electrons and Thermal Photons", Astrophysical Journal, 146, 686.

Field, G. 1962, "Absorption by Intergalactic Hydrogen", Astrophysical Journal, 135, 684.

Field, G., and Henry, R. 1964, "Free-Free Emission by Intergalactic Hydrogen", Astrophysical Journal, 140, 1002.

Field, G. B., and Hitchcock, J. L. 1966a, "Cosmic Black-Body Radiation at $\lambda = 2.6$ mm", Physical Review Letters, 16, 817.

__________ 1966b, "The Radiation Temperature of Space at 2.6 mm and the Excitation of Interstellar CN", Astrophysical Journal, 146, 1.

Field, G. B., Solomon, P. M., and Wampler, E. J. 1966, "The Density of Intergalactic Hydrogen Molecules", Astrophysical Journal, 145, 351.

Friedmann, A. 1922, "Über die Krümmung des Raumes", Zeitschrift für Physik, 10, 377.

__________ 1924, "Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes", ibid, 21, 326.

Gould, R. J., and Ramsay, W. 1966, "The Temperature of Intergalactic Matter", Astrophysical Journal, 144, 587.

Gradshteyn, I. S., and Ryzhik, I. M. 1965, Tables of Integrals, Series, and Products, 4th edition (Academic Press, New York).

Greenstein, J. L., and Münch, G. 1966, "The Weakness of Helium Lines in Globular Clusters and Halo B Stars", Astrophysical Journal, 146, 618.

Grishchuk, L. P. 1967, Astronomicheskii Zhurnal, 44, 1097 [English translation is "Cosmological Models and Spatial-Homogeneity Criteria", Soviet Astronomy-AJ, 11, 881 (1968)].

Gröbner, W., and Hofreiter, N. 1949, Integraltafel-Erster Teil (Springer-Verlag, Wien und Innsbruck).

__________ 1950, Integraltafel-Zweiter Teil (Springer-Verlag, Wien und Innsbruck).

Gunn, J. E., and Peterson, B. A. 1965, "On the Density of Neutral Hydrogen in Intergalactic Space", Astrophysical Journal, 142, 1633.

Harrison, B. K., Thorne, K. S., Wakano, M., and Wheeler, J. A. 1965, Gravitation Theory and Gravitational Collapse (University of Chicago Press, Chicago).

Harrison, E. R. 1965, "Equation of State of Matter at Supernuclear Density", Astrophysical Journal, 142, 1643.

____________ 1967, "Classification of Uniform Cosmological Models", Monthly Notices of the Royal Astronomical Society, 137, 69.

____________ 1968 , "Improved Friedmann Models" (preprint received January 1968; to be published).

Hawking, S. W., and Ellis, G. F. R. 1968, "The Cosmic Black-Body Radiation and the Existence of Singularities in Our Universe", Astrophysical Journal, 152, 25.

Hawking, S. W., and Tayler, R. J. 1966, "Helium Production in an Anisotropic Big-Bang Cosmology", Nature, 209, 1278.

Heckmann, O., and Schücking, E. 1962, Chapter 11 on "Relativistic Cosmology", in Gravitation: An Introduction to Current Research, ed. by L. Witten (John Wiley and Sons, New York).

Henry, R. C., Fritz, G., Meekins, J. F., Friedman, H., and Byram, E.T. 1968, "Possible Detection of a Dense Intergalactic Plasma", Astrophysical Journal, 153, L11.

Holden, D. J. 1966, "An Investigation of the Clustering of Radio Sources", Monthly Notices of the Royal Astronomical Society, 133, 225.

Howell, T. F., and Shakeshaft, J. R. 1966, "Measurement of the Minimum Cosmic Background Radiation at 20.7 cm Wavelength", Nature, 210, 1318.

Hoyle, F. 1958, "The Steady State Theory", in La Structure et L'Evolution de L'Univers, ed. by R. Stoops (Eleventh Solvang Conference, Brussels), p. 53.

Hughes, R. G., and Longair, M. S. 1967, "Evidence on the Isotropy of Faint Radio Sources", Monthly Notices of the Royal Astronomical Society, **135**, 131.

Iben, Icko, Jr., and Faulkner, J. 1968, "On the Age and Initial Helium Abundance of Extreme Population II Stars", Astrophysical Journal, **153**, 101.

Jacobs, K. C. 1967, "Friedmann Cosmological Model With Both Radiation and Matter", Nature, **215**, 1156.

__________ 1968, "Spatially Homogeneous and Euclidean Cosmological Models With Shear", Astrophysical Journal, **153**, 661.

__________ 1969, "Bianchi Type I Cosmologies With a Uniform Magnetic Field", Astrophysical Journal (to be published in February, 1969).

Kantowski, R., and Sachs, R. K. 1966, "Some Spatially Homogeneous Anisotropic Relativistic Cosmological Models", Journal of Mathematical Physics, **7**, 443.

Kasner, E. 1921, "Geometrical Theorems on Einstein's Cosmological Equations", American Journal of Mathematics, **43**, 217.

Koehler, J. A. 1966, "Intergalactic Atomic Neutral Hydrogen. II", Astrophysical Journal, **146**, 504.

Koehler, J. A., and Robinson, B. J. 1966, "Intergalactic Atomic Neutral Hydrogen. I", Astrophysical Journal, 146, 488.

Kompaneets, A. S., and Chernov, A. S. 1964, Zhurnal Éksperimental'noi i Teoreticheskoi Fiziki, 47, 1939 [English translation is "Solution of the Gravitation Equations for a Homogeneous Anisotropic Model", Soviet Physics-JETP, 20, 1303 (1965)].

Kristian, J., and Sachs, R. K. 1966, "Observations in Cosmology", Astrophysical Journal, 143, 379.

Kubo, R. 1965, in Chapter 4 of Statistical Mechanics (Interscience, New York).

Landau, L. D., and Lifshitz, E. M. 1962, The Classical Theory of Fields, second edition (Addison-Wesley; Reading, Massachusetts).

Lemaître, G. 1927, Annales de'la Société Scientifique de Bruxelles, A47, 49.

__________ 1930, "On the Random Motion of Material Particles in the Expanding Universe. Explanation of a Paradox", Bulletin of the Astronomical Institutes of the Netherlands, 5, No. 200, p. 273.

__________ 1931, "A Homogeneous Universe of Constant Mass and Increasing Radius Accounting for the Radial Velocity of Extra-Galactic Nebulae", Monthly Notices of the Royal Astronomical Society, 91, 483.

Lifshitz, E. M., and Khalatnikov, I. M. 1960, Zhurnal Éksperimental'noi i Teoreticheskoi Fiziki, 39, 800 [English translation is "On the Singularities of Cosmological Solutions of the Gravitational Equations", Soviet Physics-JETP, 12, 558 (1961)].

________________ 1963a, Uspekhi Fizicheskikh Nauk, 80, 391 [English translation is "Problems in Relativistic Cosmology", Soviet Physics-Uspekhi, 6, 495 (1964)].

________________ 1963b, "Investigations in Relativistic Cosmology", Advances in Physics, 12, 185.

Lindquist, R. W. 1966, "Relativistic Transport Theory", Annals of Physics, 37, 487.

Longair, M. S. 1966, "On the Interpretation of Radio Source Counts", Monthly Notices of the Royal Astronomical Society, 133, 421.

Matzner, R. A. 1967a, "On the Present Temperature of Primordial Black-body Gravitational Radiation", University of Maryland Technical Report No. 735 (College Park).

________ 1967b, "Neutrino Sources For Bianchi Type V Cosmologies", University of Texas preprint of late 1967 (Austin).

McCrea, W. H. 1968, "Cosmology After Half a Century", Science, 160, 1295.

McIntosh, C. B. G. 1968, "Cosmological Models With Both Radiation and Matter", Monthly Notices of the Royal Astronomical Society, 138, 423.

McVittie, G. C. 1949, Cosmological Theory, second edition (Methuen & Co. Ltd., London).

Misner, C. W. 1967, "Neutrino Viscosity and the Isotropy of Primordial Black-Body Radiation", Physical Review Letters, 19, 533.

____________ 1968, "The Isotropy of the Universe", Astrophysical Journal, 151, 431.

Morse, P. M. 1962, Thermal Physics (W. A. Benjamin, New York).

Narlikar, V. V., and Karmarkar, K. K. 1946, "On a Curious Solution of Relativistic Field Equations", Current Science, 15, 69.

North, J. D. 1965, The Measure of the Universe - A History of Modern Cosmology (Clarendon Press, Oxford).

Novikov, I. D., and Zel'dovich, Ya. B. 1967, "Cosmology", in Annual Review of Astronomy and Astrophysics, Vol. 5, 627.

O'Connell, R. F. 1968, "Motion of a Relativistic Electron With An Anomalous Magnetic Moment in a Constant Magnetic Field", Physics Letters, 27A, 391.

Oort, J. H. 1958, "Distribution of Galaxies and the Density in the Universe", in La Structure et L'Evolution de L'Univers, ed. by R. Stoops (Eleventh Solvang Conference, Brussels), p. 163.

____________ 1965, in Chapter 21 on "Stellar Dynamics", in Galactic Structure, Volume V, ed. by A. Blaauw and M. Schmidt (University of Chicago Press, Chicago and London).

Oszvath, I. 1962, "Lösungen der Einsteinschen Feldgleichungen mit einfach transitiver Bewegungsgruppe", Akademie der Wissenschaften und der Literatur in Mainz --- Abhandlungen der Mathematisch -- Naturwissenschaftlichen Klasse, 13, 1001.

Partridge, R. B., and Wilkinson, D. T. 1967, "Isotropy and Homogeneity of the Universe From Measurements of the Cosmic Microwave Background", Physical Review Letters, 18, 557.

Peebles, P. J. E. 1967 (in an unpublished paper based on lecture notes for the June Institute of Astronomy, University of Toronto).

Penzias, A. A., and Scott III, E. H. 1968, "Intergalactic H I Absorption at 21 Centimeters", Astrophysical Journal, 153, L7.

Penzias, A. A., and Wilson, R. W. 1965, "A Measurement of Excess Antenna Temperature at 4080 Mc/s.", Astrophysical Journal, 142, 419.

________________ 1967, "A Measurement of the Background Temperature at 1415 MHz", Astronomical Journal, 72, 315.

Petrov, A. Z. 1961, Prostranstva Einshteina (Mockba) [see the German translation Einstein-Räume, 1964 (Akademie-Verlag, Berlin)].

Pooley, G. G., and Ryle, M. 1968, "The Extension of the Number-Flux Density Relation for Radio Sources to Very Small Flux Densities", Monthly Notices of the Royal Astronomical Society, 139, 515.

Puzanov, V. I., Salomonovich, A. E., and Stankevich, K.S. 1967, Astronomicheskii Zhurnal, 44, 1129 [English translation is "Measurements of the Temperature of the Primordial Background Radiation at 8.2 mm Wavelength", Soviet Astronomy-AJ, 11, 905 (1968)].

Raychaudhuri, A. K. 1955, "Relativistic Cosmology. I", Physical Review, 98, 1123.

____________ 1957, "Singular State in Relativistic Cosmology", Physical Review, 106, 172.

____________ 1958, "An Anisotropic Cosmological Solution in General Relativity", Proceedings of the Physical Society of London, 72, 263.

Rees, M. J., and Sciama, D. W. 1967, "The Detection of Heavy Elements in Intergalactic Space", Astrophysical Journal, 147, 353.

Robinson, B. B. 1961, "Relativistic Universes With Shear", Proceedings of the National Academy of Science, 47, 1852.

Roll, P. G., and Wilkinson, D. T. 1966, "Cosmic Background Radiation of 32 cm - Support for Cosmic Black-Body Radiation", Physical Review Letters, 16, 405.

Rosen, G. 1962, "Symmetries of the Einstein-Maxwell Equations", Journal of Mathematical Physics, 3, 313.

______ 1964, "Spatially Homogeneous Solutions to the Einstein-Maxwell Equations", Physical Review, 136, B297.

Sandage, A. 1961a, "The Ability of the 200-inch Telescope to Discriminate Between Selected World Models", Astrophysical Journal, 133, 355.

__________ 1961b, "The Light Travel Time and the Evolutionary Correction to Magnitudes of Distant Galaxies", Astrophysical Journal, 134, 916.

__________ 1962, "The Ages of M67, NGC 188, M3, M5, and M13 According to Hoyle's 1959 Models", Astrophysical Journal, 135, 349.

__________ 1968, "A New Determination of the Hubble Constant From Globular Clusters in M87", Astrophysical Journal, 152, L149.

Sargent, W. L. W. 1967, "The Spectra of Blue Horizontal-Branch Stars in Three Northern Globular Clusters", Astrophysical Journal, 148, L147.

Sargent, W. L. W., and Searle, L. 1966, "Spectroscopic Evidence on the Helium Abundance of Stars in the Galactic Halo", Astrophysical Journal, 145, 652.

____________________ 1967, "The Interpretation of the Helium Weakness in Halo Stars", Astrophysical Journal, 150, L33.

____________________ 1968, "A Quantitative Description of the Spectra of the Brighter Feige Stars", Astrophysical Journal, 152, 443.

Saunders, P. T. 1967, "Non-Isotropic Model Universes", Doctoral

Dissertation in Physics, King's College, University of London.

Schücking, E. L. 1966, in Chapter on "Observational Cosmology", High Energy Astrophysics (Les Houches lectures at École d'été de Physique Théorique), 1, 223, edited by C. De Witt, E. Schatzman, and P. Véron (Gordon and Breach, New York).

Schücking, E., and Heckmann, O. 1958, "World Models", La Structure et L'Evolution de L'Univers, ed. by R. Stoops (Eleventh Solvang Conference, Brussels), p. 149.

Sciama, D. W. 1967, "Peculiar Velocity of the Sun and the Cosmic Microwave Background", Physical Review Letters, 18, 1065.

Shepley, L. C. 1965, "SO(3,R) - Homogeneous Cosmologies", Doctoral Dissertation in Physics, Princeton University.

Shikin, I. S. 1966, Doklady Akademii Nauk SSSR, 171, 73 [English translation is "A Uniform Anisotropic Cosmological Model With a Magnetic Field", Soviet Physics-Doklady, 11, 944 (1967)].

__________ 1967, ibid, 176, 1048 [English translation is "A Uniform Axisymmetrical Cosmological Model in the Ultrarelativistic Case", ibid, 12, 950 (1968)].

Silk, J. 1968, "The Diffuse X-Ray Background", Astrophysical Journal, 151, L19.

Stewart, J. M., and Ellis, G. F. R. 1968, a paper concerning "Local

Rotational Symmetry", Journal of Mathematical Physics (in press).

Stokes, R. A., Partridge, R. B., and Wilkinson, D. T. 1967, "New Measurements of the Cosmic Microwave Background at $\lambda = 3.2$ cm and $\lambda = 1.58$ cm - Evidence in Support of a Blackbody Spectrum", Physical Review Letters, 19, 1199.

Strom, S. E., and Strom, K. M. 1967, "The Helium Content of Subdwarfs", Astrophysical Journal, 150, 501.

Taub, A. H. 1951, "Empty Space-Times Admitting a Three Parameter Group of Motions", Annals of Mathematics, 53, 472.

Tauber, G. E., and Weinberg, J. W. 1961, "Internal State of a Gravitating Gas", Physical Review, 122, 1342.

Tayler, R. J. 1968, "Half Life of the Neutron and Cosmological Helium Production", Nature, 217, 433.

Thaddeus, P., and Clauser, J. F. 1966, "Cosmic Microwave Radiation at 2.63 mm From Observations of Interstellar CN", Physical Review Letters, 16, 819.

Thorne, K. S. 1965, "Geometrodynamics of Cylindrical Systems", Doctoral Dissertation in Physics, Princeton University.

__________ 1966, Appendix B of the section on "Relativistic Stellar Structure and Dynamics", in High Energy Astrophysics (Grenoble: Ecole d'été de Physique Théorique), Volume 3, edited by C. De Witt, E. Schatzman and P. Véron (Gordon and Breach, New York).

____________ 1967, "Primordial Element Formation, Primordial Magnetic Fields, and the Isotropy of the Universe", Astrophysical Journal, 148, 51.

Tolman, R. C. 1934, in Chapter X on "Applications to Cosmology", Relativity Thermodynamics and Cosmology (Clarendon Press, Oxford).

Tomita, K. 1968, "Theoretical Relations Among Observable Quantities in an Anisotropic and Homogeneous Universe", Hiroshima University preprint.

Van den Bergh, S. 1961, "The Luminosity Function of Galaxies", Zeitschrift für Astrophysik, 53, 219.

Vignon, B. 1966, "L'équation de Boltzmann en Relativité", Comptes Rendus (des Séances de L'Académie Des Sciences) 262A, 795.

Wagoner, R. V., Fowler, W. A., and Hoyle, F. 1967, "On the Synthesis of Elements at Very High Temperatures", Astrophysical Journal, 148, 3.

Wayman, P. A. 1966, "Determination of the Inertial Frame of Reference", Quarterly Journal of the Royal Astronomical Society, 7, 138.

Welch, W. J., Keachie, S., Thornton, D.D., and Wrixon, G. 1967, "Measurement of the Cosmic Microwave Background Temperature at 1.5 cm Wavelength", Physical Review Letters, 18, 1068.

Weymann, R. 1966, "The Energy Spectrum of Radiation in the Expanding Universe", Astrophysical Journal, 145, 560.

_________ 1967, "Possible Thermal Histories of Intergalactic Gas", Astrophysical Journal, 147, 887.

Wilkinson, D. T. 1967, "Measurement of the Cosmic Microwave Background at 8.56 mm Wavelength", Physical Review Letters, 19, 1195.

Wilkinson, D.T., and Partridge, R. B. 1967, "Large Scale Density Inhomogeneities in the Universe", Nature, 215, 719.

Wilson, R. W., and Penzias, A. A. 1967, "Isotropy of Cosmic Background Radiation at 4080 Megahertz", Science, 156, 1100.

Woolf, N. J. 1967, "On the Stabilization of Clusters of Galaxies by Ionized Gas", Astrophysical Journal, 148, 287.

Zel'dovich, Ya. B. 1961, Zhurnal É'ksperimental'noi i Teoreticheskoi Fiziki, 41, 1609 [English translation is "The Equation of State at Ultrahigh Densities and its Relativistic Limitations", Soviet Physics-JETP, 14, 1143 (1962)].

_________ 1964, Astronomicheskii Zhurnal, 41, 873 [English translation is "Newtonian and Einsteinian Motion of Homogeneous Matter", Soviet Astronomy-AJ, 8, 700 (1965)].

_________ 1965a, "Survey of Modern Cosmology", in Advances in Astronomy and Astrophysics, Vol. 3, 241.

_________ 1965b, Zhurnal Éksperimental'noi i Teoreticheskoi Fiziki, 48, 986 [English translation is "Magnetic Model of the Universe", Soviet Physics-JETP, 21, 656 (1965)].

_______________ 1965c, "On Anisotropic Motion of Homogeneous Matter", Monthly Notices of the Royal Astronomical Society, 129, 19.

_______________ 1966, Uspekhi Fizicheskikh Nauk, 89, 647 [English translation is "The 'Hot' Model of the Universe", Soviet Physics-Uspekhi, 9, 602 (1967)].