ASPECTS OF THE MORPHOLOGICAL CHARACTER AND STABILITY OF TWO-PHASE STATES IN NON-ELLIPTIC SOLIDS

Thesis by

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ABSTRACT

Part I. This work focuses on the construction of equilibrated two-phase antiplane shear deformations of a *non-elliptic isotropic* and *incompressible hyperelastic* material. It is shown that this material can sustain *metastable* two-phase equilibria which are neither piecewise homogeneous nor axisymmetric, but, rather, involve non-planar interfaces which completely segregate inhomogeneously deformed material in distinct elliptic phases. These results are obtained by studying a constrained boundary value problem involving an interface across which the deformation gradient jumps. The boundary value problem is recast as an integral equation and conditions on the interface sufficient to guarantee the existence of a solution to this equation are obtained. The contraints, which enforce the segregation of material in the two elliptic phases, are then studied. Sufficient conditions for their satisfaction are also secured. These involve additional restrictions on the interface across which the deformation gradient jumps—which, with all restrictions satisfied, constitutes a *phase boundary*. An uncountably infinite number of such phase boundaries are shown to exist. It is demonstrated that, for each of these, there exists a solution—unique up to an additive constant—for the constrained boundary value problem. As an illustration, approximate solutions which correspond to a particular class of phase boundaries are then constructed. Finally, the kinetics and stability of an arbitrary element within this class of phase boundaries are analyzed in the context of a quasistatic motion.

Part II. This work investigates the linear stability of an antiplane shear motion which involves a planar phase boundary in an arbitrary element of a wide class of *non-elliptic* generalized neo-Hookean materials which have two distinct elliptic phases. It is shown, via a *normal mode analysis*, that, in the absence of inertial effects, such a process is linearly unstable with respect to a large class of disturbances if and only if the *kinetic response function*—a constitutively supplied entity which gives the normal velocity of a phase boundary in terms of the driving traction which acts on it or *vice versa*—is locally decreasing as a function

of the appropriate argument. An alternate analysis, in which the linear stability problem is recast as a functional equation for the interface position, allows the interface to be tracked subsequent to perturbation. A particular choice of the initial disturbance is used to show that, in the case of an unstable response, the morphological character of the phase boundary evolves to qualitatively resemble the *plate-like* structures which are found in displacive solid-solid phase transformations. In the presence of inertial effects a combination of normal mode and energy analyses are used to show that the condition which is necessary and sufficient for instability with respect to the relevant class of perturbations in the absence of inertia remains necessary for the entire class of perturbations and sufficient for all but a very special, and physically unrealistic, subclass of these perturbations. The linear stability of the relevant process depends, therefore, entirely upon the transformation kinetics intrinsic to the kinetic response function.

Part III. This investigation is directed toward understanding the role of coupled mechanical and thermal effects in the linear stability of an isothermal antiplane shear motion which involves a single planar phase boundary in a *non-elliptic* thermoelastic material which has multiple elliptic phases. When the relevant process is static—so that the phase boundary does not move prior to the imposition of the disturbance—it is shown to be linearly stable. However, when the process involves a steadily propagating phase boundary it may be linearly unstable. Various conditions sufficient to guarantee the linear instability of the process are obtained. These conditions depend on the monotonicity of the *kinetic response function*—a constitutively supplied entity which relates the *driving traction* acting on a phase boundary to the local absolute temperature and the normal velocity of the phase boundary—and, in certain cases, on the spectrum of wave-numbers associated with the perturbation to which the process is subjected. Inertia is found to play an insignificant role in the qualitative features of the aforementioned sufficient conditions. It is shown, in particular, that instability can arise even when the normal velocity of the phase boundary is an increasing function of the driving traction if the temperature dependence in the kinetic response function is of a suitable nature. Significantly, the instability which is present in this setting occurs only in the *long waves* of the Fourier decomposition of the moving phase boundary, implying that the interface prefers to be highly wrinkled.

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ON THE CONSTRUCTION OF TWO-PHASE EQUILIBRIA IN A NON-ELLIPTIC HYPERELASTIC MATERIAL

1. INTRODUCTION

Finite elastic equilibria with discontinuous deformation gradients have figured prominently in recent continuum mechanical treatments of displacive solid-solid phase transformations. Models of this sort are pertinent to the investigation of *shape memory*, *twinning* and *transformation toughening* in solids—all three of which occur in both metallic and ceramic alloys. Micrographs of multiphase equilibrium states in alloys, such as those presented by ZACKAY, JUSTUSSON, & SCHMATZ [30] and PORTER & HEUER [24], often display configurations wherein the various phases are segregated by geometrically complicated interfaces. One question which arises regarding the aformentioned continuum mechanical idealizations of such materials is whether they are capable of capturing the morphological complexity of such deformations. As a first step toward answering this question, this work focuses, within the context of a particular class of hypothetical materials, on the construction of equilibria involving coexistent phases segregated by surfaces which—although not as morphologically complex as those displayed in [24] and [30]—are, at least, non-planar.

In a *homogeneous*, *hyperelastic* material discontinuous deformation gradients occur only if the relevant *elastic potential* allows for a loss of ellipticity, at certain values of the deformation gradient, in the associated displacement equations of equilibrium.[1] Materials characterized by elastic potentials which allow such a loss of ellipticity are referred to as *non-elliptic*. Of particular importance in this work are non-elliptic materials which have at least two disjoint elliptic phases. Examples of such materials are provided by ERICKSEN [12] in the context of a one-dimensional bar theory, FOSDICK & MACSITHIGH [14] in their work on the helical shear of a cicular elastic tube, and by ABEYARATNE [1] in his study involving a special class of *incompressible*, *isotropic* materials. ABEYARATNE [1], BALL & JAMES [10], GURTIN [18], and SILLING [28] have demonstrated that materials of this sort support equilibrium states which display coexistent elliptic phases and,

[1] For a discussion of this issue see, for instance, ROSAKIS [26].

in addition, minimize the relevant energy functional. As a result of the latter property these states are referred to as *mechanically stable.* The associated deformation fields in all of the foregoing works are either piecewise homogeneous or axisymmetric. In the case of equilibrated piecewise homogeneous deformations the associated phase boundaries must be planar. BALL & JAMES [10] and SILLING [28] have shown, however, that *energy-minimizing sequences* of piecewise homogeneous mechanically stable deformations may possess limits which are *metastable* as opposed to mechanically stable and, moreover, involve non-planar phase boundaries. On the other hand, ABEYARATNE [2] and SILLING [28] have constructed, respectively, asymptotic and numerical solutions to a boundary value problem involving a mode III crack in a particular subclass of incompressible, isotropic non-elliptic materials. These solutions are not mechanically stable and are neither piecewise homogeneous nor axisymmetric; furthermore, they include the non-elliptic material phase and, in addition, transitions between the elliptic and non-elliptic phases which do not involve jumps in the deformation gradient. These solutions do, however, involve surfaces which separate the two elliptic phases present in the deformation. The relevant interfaces are, moreover, non-planar. ROSAKIS [27] has recently shown that a special class of *anisotropic* non-elliptic materials is capable of sustaining equilibria in which a family of cusped *lenticular* inclusions of one elliptic phase reside in a matrix of another elliptic phase. These states are, in general, metastable.

As yet there are no results pertaining to the existence, in non-elliptic isotropic hyperelastic materials, of multiphase equilibrium states which are free of the non-elliptic phase and are neither piecewise homogeneous nor axisymmetric. The primary objective of this investigation is to prove constructively that a class of non-elliptic isotropic incompressible hyperelastic materials is capable of sustaining deformations of this type. These deformations will typically be metastable—like those associated with the limits of the aforementioned minimizing sequences of piecewise homogeneous deformations and the states constructed by ROSAKIS [27].

Although non-planar phase interfaces may, in reality, reflect anisotropic effects, these results show that they can exist within the context of a model which does not take anisotropy into consideration. Isotropic materials may, consequently, be useful in preliminary studies of the kinetics and stabilty of interfaces between phases. These issues are taken up briefly in the final section of this work and, more thoroughly, by FRIED [16] in a linear stability analysis of states involving planar phase interfaces for a class of non-elliptic isotropic materials.

Chapter 2 is devoted to preliminaries. After a brief overview of the notation to be used, Section 2.1 introduces the kinematics and fundamental balance principles which will be needed in the following. Section 2.2 explains the constitutive restrictions which will be adhered to throughout this work. Section 2.3 begins by introducing the concept of a *quasistatic motion.* It then discusses the notions of mechanical dissipation and driving traction associated with surfaces across which the deformation gradient jumps; these lead naturally to the consideration of a *kinetic relation* and the associated *kinetic response function.* In the final section of Chapter 2, the kinematics are specialized to those of *antiplane shear*.

Chapter 3 focuses upon the solution of a particular constrained boundary value problem, in antiplane shear, involving the field equations and jump conditions put forth in Section 2.4. After formulating the problem in Section 3.1, a representation for the solution of the boundary value problem is presented in Section 3.2. This representation is indeterminate in that it involves the unknown jump in the normal derivative of the displacement field over an interface across which the deformation gradient is discontinuous. In Section 3.3 an integral equation is derived for the unknown jump in the normal derivative of displacement in terms of a parameterization of the interface. Sufficient conditions for the existence of a unique solution of this integral equation are then obtained. These constitute analytical restrictions on the interface geometry. It transpires that there exist an uncountably infinite number of interfaces which comply with these restrictions. It is then shown that for each of these interfaces there exists a solution, unique up to an additive constant, to the boundary value problem stated in Section 3.1. In Section 3.4 the constraints which enforce segregation of the elliptic phases are analyzed. These impose further analytical restrictions on the interface geometry. It is shown that, within the set of interfaces which allow a solution to the boundary value problem, there exist an uncountably infinite number of interfaces which also satisfy these restrictions and, hence, allow a solution to the constrained boundary value problem. Each of these solutions involves a non-planar and non-axisymmetric phase interface which separates elliptic phases subjected to inhomogeneous deformations. Section 3.5 illustrates the results of the two preceeding sections in determining a particular class of non-planar surfaces for which the constrained boundary value problem can be solved. Approximations for the strain and displacement fields corresponding to the solutions of the appropriate family of constrained boundary value problems are then constructed.

The last chapter is concerned with the kinetics and stability of slowly propagating phase boundaries. In Section 4.1 the distribution of driving traction along a phase interface of the kind constructed in Section 3.5 is calculated. Section 4.2 is concerned with observations pertaining to the kinetics of such a surface. Ingredients crucial to this analysis are the kinetic relation and response function introduced in Section 2.3. This section concludes with results that relate the monotonicity of the kinetic response function for a particular material and the *kinetic stability* of that material. These final results are consistent with those obtained in [16].

2. PRELIMINARIES

2.1. Notation, kinematics and balance principles. In the following $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. The intervals $(0,\infty)$ and $[0,\infty)$ are represented by $\mathbb{R}_+$ and $\bar{\mathbb{R}}_+$. The symbol $\mathbb{R}^n$, with n equal to 2 or 3, represents real n-dimensional space equipped with the standard Euclidean norm. If U is a set, then its closure, interior and boundary are designated by $\bar{U}$, $\mathring{U}$, and ∂U, respectively. The complement of a set V with respect to U is written as $U \setminus V$. Given a function $\psi : U \to W$ and a subset V of U, $\psi(V)$ stands for the image of V under the map ψ.

Vectors and linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ (referred to herein as *tensors*) are distinguished from scalars with the aid of boldface type—lower and upper case for vectors and tensors, respectively. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$, their inner product is then written as $\mathbf{a} \cdot \mathbf{b}$; the Euclidean norm of $\mathbf{a}$ is, further, written as $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. The set of unit vectors—that is, vectors with unit Euclidean norm—in $\mathbb{R}^3$ is designated by $\mathcal{N}$. The symbol $\mathcal{L}$ refers to the set of tensors, $\mathcal{L}_+$ denotes the set of all tensors with positive determinant, and $\overset{+}{\mathcal{S}}$ stands for the collection of all symmetric positive definite tensors. If $\mathbf{F}$ is in $\mathcal{L}$ then $\mathbf{F}^T$ represents its transpose; if, moreover, $\det \mathbf{F} \neq 0$, then the inverse of $\mathbf{F}$ and its transpose are written as $\mathbf{F}^{-1}$ and $\mathbf{F}^{-T}$, respectively. The notation $\mathbf{a} \otimes \mathbf{b}$ refers to the tensor $\mathbf{A}$, formed by the outer product of $\mathbf{a}$ with $\mathbf{b}$, defined such that $\mathbf{A}\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ for any vector $\mathbf{c}$ in $\mathbb{R}^3$. If $\mathbf{A}$ and $\mathbf{B}$ are tensors then their inner product is written as $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr} \mathbf{A}\mathbf{B}^T$.

When component notation is used, Greek indices range only over $\{1,2\}$; summation of repeated indices over the appropriate range is implicit. A subscript preceded by a comma denotes partial differentiation with respect to the corresponding coordinate. Also, a superposed dot signifies partial differentiation with respect to time.

Let q lie in $[1,\infty)$. Then, a function $\psi : \mathbb{R} \to \mathbb{R}$ is an element of $L^q(\mathbb{R})$ if it is q integrable on $\mathbb{R}$—that is, if its L^q norm over $\mathbb{R}$,

$$\|\psi\|_{L^q(\mathbb{R})} = \Big(\int_{-\infty}^{+\infty} |\psi(\xi)|^q \, d\xi \Big)^{\frac{1}{q}},$$

is defined. Similarly, ψ is an element of $L^\infty(\mathbb{R})$ if ψ is bounded on $\mathbb{R}$—that is, if its L^∞ norm over $\mathbb{R}$,

$$\|\psi\|_{L^\infty(\mathbb{R})} = \sup_{\xi \in \mathbb{R}} |\psi(\xi)|,$$

exists. A function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ is an element of $L^q(\mathbb{R}^2)$ or $L^\infty(\mathbb{R}^2)$, respectively, if the analogous L^q or L^∞ norm over $\mathbb{R}^2$ exists.

Consider, now, a body $\mathcal{B}$ which, in a reference configuration, occupies a region $\mathcal{R}$ contained in $\mathbb{R}^3$. Let the invertible mapping $\hat{\mathbf{y}} : \mathcal{R} \to \mathcal{R}_*$, with

$$\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{R}, \tag{2.1.1}$$

characterize a *deformation* of $\mathcal{B}$ from the reference configuration onto a configuration that occupies the region $\mathcal{R}_*$ in $\mathbb{R}^3$. Assume that the deformation $\hat{\mathbf{y}}$, or equivalently the *displacement* $\mathbf{u}$, is continuous and possesses piecewise continuous first and second gradients on $\mathcal{R}$. Let S be the set of points contained in $\mathcal{R}$ defined so that $\hat{\mathbf{y}}$ is differentiable on the set $\mathcal{R} \setminus S$. Introduce the *deformation gradient tensor* $\mathbf{F} : R \setminus S \to \mathcal{L}$ by

$$\mathbf{F}(\mathbf{x}) = \nabla \hat{\mathbf{y}}(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{R} \setminus S, \tag{2.1.2}$$

where the associated *Jacobian determinant*, $J : \mathcal{R} \setminus S \to \mathbb{R}$, of $\hat{\mathbf{y}}$ is restricted to be strictly positive on its domain of definition:

$$J(\mathbf{x}) = \det \mathbf{F}(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \mathcal{R} \setminus S. \tag{2.1.3}$$

Hence, $\mathbf{F} : R \setminus S \to \mathcal{L}_+$. The *left Cauchy-Green tensor* $\mathbf{G} : \mathcal{R} \setminus S \to \overset{+}{\mathcal{S}}$ corresponding to the deformation $\hat{\mathbf{y}}$ is given by

$$\mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{x})\mathbf{F}^T(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{R} \setminus S. \tag{2.1.4}$$

The *deformation invariants* associated with $\hat{\mathbf{y}}$ exist on $\mathcal{R} \setminus S$ and are supplied through the fundamental scalar invariants of $\mathbf{G}$:

$$I_1(\mathbf{G}) = \operatorname{tr}\mathbf{G}, \quad I_2(\mathbf{G}) = \tfrac{1}{2}\left((\operatorname{tr}\mathbf{G})^2 - \operatorname{tr}(\mathbf{G}^2)\right), \quad I_3(\mathbf{G}) = \det\mathbf{G}. \tag{2.1.5}$$

With the above kinematic antecedents in place introduce the *nominal mass density* $\rho : \mathcal{R} \to I\!R_+$, the *nominal body force per unit mass* $\mathbf{b} : \mathcal{R} \to I\!R^3$, and the *nominal stress tensor* $\mathbf{S} : \mathcal{R} \setminus S \to \mathcal{L}$, and suppose that ρ and $\mathbf{b}$ are continuous on $\mathcal{R}$, while $\mathbf{S}$ is piecewise continuous on $\mathcal{R}$, continuous on $\mathcal{R} \setminus S$, and also has a piecewise continuous gradient on $\mathcal{R}$. In the absence of constitutive assumptions relating the stress to the deformation gradient, the sets over which $\mathbf{S}$ and $\mathbf{F}$ suffer jumps need not be equivalent. The scope of this investigation is limited, however, to elastic materials wherein stress is continuously related to strain—hence, $\mathbf{S}$ is assumed to obey the above smoothness criteria. Let ρ_* be the mass density in the deformed configuration associated with $\hat{\mathbf{y}}$. Given a *regular* subregion $\mathcal{P}$ of $\mathcal{R}$, let $\mathbf{m} : \partial\mathcal{P} \to \mathcal{N}$ denote the unit outward normal to $\partial\mathcal{P}$. Then the global balance laws of mass, and—in the absence of inertia—force and moment equilibrium require that

$$\int_{\mathcal{P}} \rho \, dV = \int_{\hat{\mathbf{y}}(\mathcal{P})} \rho_* \, dV, \tag{2.1.6}$$

$$\int_{\partial\mathcal{P}} \mathbf{S}\mathbf{m} \, dA + \int_{\mathcal{P}} \rho\mathbf{b} \, dV = \mathbf{0}, \tag{2.1.7}$$

and

$$\int_{\partial\mathcal{P}} \hat{\mathbf{y}} \wedge \mathbf{S}\mathbf{m} \, dA + \int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho\mathbf{b} \, dV = \mathbf{0}, \tag{2.1.8}$$

respectively, for every regular subregion $\mathcal{P}$ contained in $\mathcal{R}$.

Localization of the balance laws (2.1.6)–(2.1.8) at an arbitrary point contained in the interior of $\mathcal{R} \setminus S$ yields the following familiar field equations:

$$\begin{aligned} \rho = \rho_*(\hat{\mathbf{y}})J \quad &\text{on} \quad \mathcal{R} \setminus S, \\ \nabla \cdot \mathbf{S} + \rho\mathbf{b} = \mathbf{0} \quad &\text{on} \quad \mathcal{R} \setminus S, \\ \mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T \quad &\text{on} \quad \mathcal{R} \setminus S. \end{aligned} \tag{2.1.9}$$

Suppose, from now on, that the set S is a regular surface. Then, localization of (2.1.6)–(2.1.8) at an arbitrary point in S yields the following jump conditions:

$$\begin{aligned} [\![\rho_*(\hat{\mathbf{y}})J]\!] &= 0 \quad \text{on} \quad S, \\ [\![\mathbf{Sn}]\!] &= \mathbf{0} \quad \text{on} \quad S, \end{aligned} \tag{2.1.10}$$

where, given a generic field quantity $g : \mathcal{R} \setminus S \to I\!R$ which jumps across S, $[\![g]\!]$ is defined through

$$[\![g(\mathbf{x})]\!] = \lim_{h \searrow 0} \big(g\big(\mathbf{x} + h\mathbf{n}(\mathbf{x})\big) - g\big(\mathbf{x} - h\mathbf{n}(\mathbf{x})\big)\big) \quad \forall \mathbf{x} \in S, \tag{2.1.11}$$

with $\mathbf{n} : S \to \mathcal{N}$ a unit normal to S. Observe from the jump condition $(2.1.10)_1$ that the mass denity in the deformed state ρ_* is only defined on $\hat{\mathbf{y}}(\mathcal{R} \setminus S)$. Evidently, equations $(2.1.9)_1$ and $(2.1.10)_1$ are completely decoupled from equations $(2.1.9)_{2,3}$ and $(2.1.10)_2$; that is, given a solution to a boundary value problem involving the latter set of equations, ρ_* can be calculated *a posteriori*. For this reason equations $(2.1.9)_1$ and $(2.1.10)_1$ will be disregarded in the subsequent analysis.

In addition to the jump conditions given in (2.1.10), the stipulated continuity of $\hat{\mathbf{y}}$ gives the following kinematic jump condition

$$[\![\mathbf{u}]\!] = \mathbf{0} \quad \text{on} \quad S. \tag{2.1.12}$$

2.2. Constitutive assumptions. Let $\mathcal{B}$ be composed of a hyperelastic material which is homogeneous, isotropic and incompressible. Since $\mathcal{B}$ is hyperelastic its mechanical response is governed by an elastic potential or *strain energy per unit reference volume.* The homogeneity of $\mathcal{B}$ implies that the elastic potential does not depend explicitly on position in the reference configuration. Furthermore, because $\mathcal{B}$ is isotropic the elastic potential can depend on the deformation gradient $\mathbf{F}$ only through the deformation invariants $I_k(\mathbf{G})$ defined in (2.1.7). The incompressibility of $\mathcal{B}$ requires that the deformation $\hat{\mathbf{y}}$ be *isochoric*, *i.e.*,

$$I_3\big(\mathbf{G}(\mathbf{x})\big) = J^2(\mathbf{x}) = 1 \quad \forall \mathbf{x} \in \mathcal{R} \setminus S. \tag{2.2.1}$$

An additional consequence of isotropy is, therefore, that the elastic potential can be expressed as a function solely of the first two deformation invariants. It can also be demonstrated via (2.1.5) that, when (2.1.1) holds, $I_\alpha(\mathbf{G}(\mathbf{x})) \geq 3$ for all $\mathbf{x}$ contained in $\mathcal{R} \setminus S$. Now, let $\tilde{W} : [3, \infty) \times [3, \infty) \to I\!R$ denote an elastic potential which characterizes $\mathcal{B}$ and assume that $\tilde{W}$ is continuously differentiable with piecewise continuous second derivatives on its domain of definition. The nominal stress response of $\mathcal{B}$ is then determined through $\tilde{W}$ up to an arbitrary pressure $p : \mathcal{R} \setminus S \to I\!R$ required to accomodate the kinematic constraint (2.2.1) imposed by the incompressibility of $\mathcal{B}$: *viz.*,

$$\mathbf{S} = 2\left(\tilde{W}_{I_1}(I)\mathbf{F} + \tilde{W}_{I_2}(I)\big(I_1(\mathbf{G})\mathbf{1} - \mathbf{G}\big)\mathbf{F}\right) - p\mathbf{F}^{-T} \quad \text{on} \quad \mathcal{R} \setminus S, \tag{2.2.2}$$

where $I : \mathcal{R} \setminus S \to [3, \infty) \times [3, \infty)$ is given by

$$I(\mathbf{x}) = \big(I_1\big(\mathbf{G}(\mathbf{x})\big), I_2\big(\mathbf{G}(\mathbf{x})\big)\big) \quad \forall \mathbf{x} \in \mathcal{R} \setminus S.$$

Following GURTIN [18], let the class of *generalized neo-Hookean* materials refer to that subset of hyperelastic materials, first introduced by KNOWLES [21], which are homogeneous, isotropic and incompressible with elastic potential independent of the second deformation invariant $(2.1.5)_2$. Assume, henceforth, that $\mathcal{B}$ is composed of a generalized neo-Hookean material with elastic potential $W : [3, \infty) \to I\!R$, where W is continuously differentiable with piecewise continuous derivative on $[3, \infty)$. Then, by (2.2.2), the nominal stress response of $\mathcal{B}$ is determined by

$$\mathbf{S} = 2W'\big(I_1(\mathbf{G})\big)\mathbf{F} - p\mathbf{F}^{-T} \quad \text{on} \quad \mathcal{R} \setminus S. \tag{2.2.3}$$

Suppose also that the elastic potential is *normalized* so that

$$W(3) = 0. \tag{2.2.4}$$

Choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and consider the response of the material at hand to a simple shear deformation $\hat{\mathbf{y}}$ given by

$$\hat{\mathbf{y}}(\mathbf{x}, t) = (\mathbf{1} + \gamma \mathbf{e}_3 \otimes \mathbf{e}_1)\mathbf{x} \quad \forall (\mathbf{x}, t) \in \mathcal{M}, \tag{2.2.5}$$

where the constant γ—assumed non-negative without loss of generality—denotes the amount of shear. From (2.1.2), (2.2.3) and (2.2.5) the nominal shear stress corresponding to the deformation $\hat{\mathbf{y}}$ is, for each γ in $\bar{\mathbb{R}}_+$, found to be

$$\mathbf{e}_3 \cdot \mathbf{S}\mathbf{e}_1 = 2\gamma W'(3 + \gamma^2) =: \tau(\gamma). \tag{2.2.6}$$

In [21–22] Knowles demonstrates that the 31 and 32 components of nominal and Cauchy shear stress are, in the present setting, equal. The function $\tau : \bar{\mathbb{R}}_+ \to \mathbb{R}$ is, hence, referred to as the *shear stress response function* of the generalized neo-Hookean material, characterized by W, in simple shear. An immediate consequence of (2.2.4) and (2.2.6) is

$$W(I_1) = \int_0^{\sqrt{I_1 - 3}} \tau(\gamma)\, d\gamma \quad \forall I_1 \in [3, \infty), \tag{2.2.7}$$

so that the response of a generalized neo-Hookean material, in *all* three dimensional deformations, is, up to a hydrostatic pressure, completely characterized by specifying the shear stress response function τ. Define the *secant modulus in shear* $M : \bar{\mathbb{R}}_+ \to \mathbb{R}$ of a generalized neo-Hookean material with elastic potential W by

$$M(\gamma) = 2W'(3 + \gamma^2) \quad \forall \gamma \in \bar{\mathbb{R}}_+, \tag{2.2.8}$$

and assume that, in compliance with the *Baker-Ericksen inequality,*

$$M(\gamma) > 0 \quad \forall \gamma \in \mathbb{R}_+. \tag{2.2.9}$$

Assume, also, that $M(0) > 0$ so that the infinitesimal shear modulus of the material at hand is positive. Note from (2.2.6) and (2.2.8) that the shear stress response function τ must also satisfy

$$\tau(0) = 0, \quad \tau'(0) = M(0). \tag{2.2.10}$$

Observe, also, that the stipulated smoothness of W guarantees that both τ and M are piecewise continuously differentiable on $\bar{\mathbb{R}}_+$.

It is worth remarking that, despite the significant restrictions which have been placed upon the class of materials which will be considered in this investigation, no presuppositions have been made regarding the sign of the derivative—where it exists—of the shear stress response function corresponding to the generalized neo-Hookean material through (2.2.6). In [22] KNOWLES shows that the monotonicity of the shear stress response function τ is related directly to the ellipticity of the generalized neo-Hookean material which it characterizes: if τ is not a monotonically increasing function on its domain of definition then the associated material is non-elliptic. This investigation will make use of a particular subclass of non-elliptic generalized neo-Hookean materials, first suggested by ABEYARATNE [1]; this class of materials is characterized by the set of shear stress response functions τ which are continuous on $\bar{\mathbb{R}}_+$ and piecewise continuously differentiable on $\bar{\mathbb{R}}_+ \setminus \{\underset{*}{\gamma}, \overset{*}{\gamma}\}$, where $0 < \underset{*}{\gamma} < \overset{*}{\gamma}$, such that

$$\tau' > 0 \quad \text{on} \quad \bar{\mathbb{R}}_+ \setminus [\underset{*}{\gamma}, \overset{*}{\gamma}], \quad \tau' < 0 \quad \text{on} \quad (\underset{*}{\gamma}, \overset{*}{\gamma}). \tag{2.2.11}$$

The sets of shear strains lying in the intervals $[0, \underset{*}{\gamma})$ and $(\overset{*}{\gamma}, \infty)$ are referred to as the *high* and *low strain phases* of the generalized neo-Hookean material specified by the shear stress response function τ. Together the high and low strain phases of such a material comprise its *elliptic phases.* A generalized neo-Hookean material characterized by a shear stress response function of this type will be referred to herein as a *three-phase* material. See Figure 1 for a graph of a shear stress response function typical of those which specify three-phase materials. Within the class of three-phase materials special attention will be given those materials (proposed by ABEYARATNE in [2]) for which

$$\tau_p(\gamma) = \begin{cases} \mu_1\gamma & \text{if} \quad \gamma \in [0, \underset{*}{\gamma}], \\ d(\gamma) & \text{if} \quad \gamma \in [\underset{*}{\gamma}, \overset{*}{\gamma}], \\ \mu_2\gamma & \text{if} \quad \gamma \in [\overset{*}{\gamma}, \infty), \end{cases} \tag{2.2.12}$$

where the function $d : [\underset{*}{\gamma}, \overset{*}{\gamma}] \rightarrow I\!R$ is linear in its argument. Observe that, in accordance with $(2.2.11)_2$, d is required to decrease on $(\underset{*}{\gamma}, \overset{*}{\gamma})$. A further consequence of (2.2.11) is that μ_1 must be greater than μ_2 which must itself be positive. Figure 2 shows the graph of τ_p.

2.3. Dissipation, driving traction and the kinetic relation. For the purposes of this section it is necessary to consider a one parameter family of deformations $\hat{\mathbf{y}}(\cdot, t) : \mathcal{R} \rightarrow \mathcal{R}_t$ where t, which denotes time, increases from t_0 to t_1. It is assumed that $\hat{\mathbf{y}}(\mathbf{x}, \cdot)$ is continuous with piecewise continuous first and second derivatives for each fixed $\mathbf{x}$ in $\mathcal{R}$. Let S_t be a regular surface, with unit normal $\mathbf{n}(\cdot, t) : S_t \rightarrow \mathcal{N}$, contained in $\mathcal{R}_t$ for each value of t in $[t_0, t_1]$. The fields $\mathbf{u}(\cdot, t) : \mathcal{R} \rightarrow I\!R^3$, $\mathbf{F}(\cdot, t) : \mathcal{R} \setminus S_t \rightarrow \mathcal{L}_+$, $\mathbf{b}(\cdot, t) : \mathcal{R} \rightarrow I\!R^3$, and $\mathbf{S}(\cdot, t) : \mathcal{R} \setminus S_t \rightarrow \mathcal{L}$ are, at each t contained in $[t_0, t_1]$, the obvious counterparts of those introduced in Section 2.1. A one parameter family of deformations of this sort is referred to as a *quasistatic motion* if the above quantities and the nominal mass density satisfy the field equations

$$\begin{aligned} \nabla \cdot \mathbf{S}(\cdot, t) + \rho\mathbf{b}(\cdot, t) = \mathbf{0} \quad \text{on} \quad \mathcal{R} \setminus S_t \quad \forall t \in [t_0, t_1], \\ \mathbf{S}(\cdot, t)\mathbf{F}^T(\cdot, t) = \mathbf{F}(\cdot, t)\mathbf{S}^T(\cdot, t) \quad \text{on} \quad \mathcal{R} \setminus S_t \quad \forall t \in [t_0, t_1], \end{aligned} \tag{2.3.1}$$

the jump condition

$$[\![\mathbf{S}(\cdot,t)\mathbf{n}(\cdot,t)]\!] = \mathbf{0} \quad \text{on} \quad S_t \quad \forall t \in [t_0, t_1], \tag{2.3.3}$$

and the kinematic condition of displacement continuity

$$[\![\mathbf{u}(\cdot,t)]\!] = \mathbf{0} \quad \text{on} \quad S_t \quad \forall t \in [t_0, t_1]. \tag{2.3.4}$$

KNOWLES [23] has shown that, in a quasistatic motion, the presence of a moving surface of discontinuity S_t of the type considered here has an effect on the balance of mechanical energy. Let $\mathcal{P}$ be a regular subregion contained in $\mathcal{R}$. In [23] it is demonstrated that the difference in the rate of work of the mechanical forces external to $\mathcal{P}$ and the rate at which energy is stored in $\mathcal{P}$ is given by

$$\delta_s(t;\mathcal{P}) = \int_{S_t \cap \mathcal{P}} f(\mathbf{x},t) V_n(\mathbf{x},t)\, dA \quad \forall t \in [t_0, t_1], \tag{2.3.5}$$

where, for each t in $[t_0, t_1]$, $f(\cdot,t) : S_t \rightarrow I\!R$ is the scalar *driving traction* and $V_n(\cdot,t) : S_t \rightarrow I\!R$ is the *normal velocity* of the interface (in the reference configuration). The function $\delta_s(\cdot;\mathcal{P}) : [t_0, t_1] \rightarrow I\!R$ is referred to as the *rate of dissipation* of mechanical energy associated with the region $\mathcal{P}$. It has been shown by YATOMI & NISHIMURA [29] as well as ABEYARATNE & KNOWLES [8] that the form of the driving traction for a hyperelastic material is, in the quasistatic setting, supplied by

$$f(\cdot,t) = [\![W(\mathbf{F}(\cdot,t))]\!] - \overset{\pm}{\mathbf{S}}(\cdot,t) \cdot [\![\mathbf{F}(\cdot,t)]\!] \quad \text{on} \quad S_t \quad \forall t \in [t_0, t_1], \tag{2.3.6}$$

where $\overset{+}{\mathbf{S}}(\cdot,t)$ (*resp.*, $\overset{-}{\mathbf{S}}(\cdot,t)$) is the limiting value of the field $\mathbf{S}(\cdot,t)$ on the side of the interface into which the unit normal $\mathbf{n}(\cdot,t)$ is (*resp.*, is not) directed at t in $[t_0, t_1]$.

When treated from a *thermomechanical* perspective, the dissipation rate can be shown to be identical to the product of the temperature and the rate of entropy production—provided that the temperature is spatially uniform and independent of time.[2] The *Clausius-Duhem inequality* then requires that the dissipation rate associated with a quasistatic motion of the kind envisioned here be non-negative, that is

$$\delta_s(t;\mathcal{P}) \geq 0 \quad \forall t \in [t_0, t_1], \tag{2.3.7}$$

for every regular subregion $\mathcal{P}$ contained in $\mathcal{R}$. A localization of (2.3.5) at an arbitrary point on the interface therefore yields the inequality

$$f(\cdot,t)V_n(\cdot,t) \geq 0 \quad \text{on} \quad S_t \quad \forall t \in [t_0, t_1] \tag{2.3.8}$$

as a condition imposed for the *admissibility* of the quasistatic motion.

In the context of a motion which involves such an interface it is necessary (see [3–8]) to supplement, in some fashion, the constitutive information which relates the stress and strain fields. An approach to this taken by ABEYARATNE & KNOWLES [8] entails the provision of a *kinetic relation* which gives the normal velocity of the interface in terms of the driving traction which acts thereon or *vice versa*. In the former case one specifies a function $\tilde{V} : \mathbb{R} \to \mathbb{R}$ such that

$$V_n = \tilde{V}(f) \quad \forall f \in \mathbb{R}. \tag{2.3.9}$$

Here $\tilde{V}$ is referred to as the *kinetic response function.* If the function $\tilde{V}$ is such that $\tilde{V}(f)f \geq 0$ on $\mathbb{R}$ then (2.3.8) is automatically satisfied and the kinetic response function is itself referred to as *admissible.* If an admissible kinetic response function is continuous on $\mathbb{R}$, then it must satisfy $\tilde{V}(0) = 0$. If, in addition to being admissible, $\tilde{V}$ is continuously differentiable on $\mathbb{R}$, then $\tilde{V}'(0) \geq 0$. Otherwise admissibility implies nothing with regard to the sign of the derivative of a smooth

[2] For a detailed discussion of the these issues see ABEYARATNE & KNOWLES [8].

kinetic response function $\tilde{V}$. All kinetic response functions considered herein are assumed to be admissible.

In the work of ABEYARATNE [1], BALL & JAMES [10], GURTIN [18], GURTIN & TEMAM [19], FOSDICK & MACSITHIGH [14], and SILLING [28] the necessary additional constitutive information is provided by setting the driving traction equal to zero on S_t for all t in $[t_0, t_1]$. This amounts to prescribing a particular rate independent kinetic relation whereby energy is conserved; it is, furthermore, a necessary consequence of requiring that a suitable *energy functional* be minimized at each t in $[t_0, t_1]$ (see ABEYARATNE [3]).

2.4. Antiplane shear of a generalized neo-Hookean material. Suppose, from now on, that $\mathcal{R}$ is a cylindrical region and choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ so that the unit base vector $\mathbf{e}_3$ is parallel to the generatrix of $\mathcal{R}$. The deformation $\hat{\mathbf{y}}$ defined through (2.1.1) consists of an antiplane shear normal to the plane spanned by the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ if it is of the form

$$\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + u(x_1, x_2)\mathbf{e}_3 \quad \forall \mathbf{x} \in \mathcal{R}. \tag{2.4.1}$$

Observe that the displacement field intrinsic to such an antiplane shear deformation has only one nonzero component which lies in the $\mathbf{e}_3$ direction and is independent of the x_3–coordinate. In (2.4.1) $x_\alpha = \mathbf{x} \cdot \mathbf{e}_\alpha$ for each $\mathbf{x}$ contained in $\mathcal{R}$. The function u will be referred to as the *out-of-plane* displacement field. Inspection of (2.4.1) reveals that any discontinuities in the gradient of $\hat{\mathbf{y}}$ must be due to discontinuities in the out-of-plane displacement field and, hence, occur across surfaces which do not vary with the x_3–coordinate.

KNOWLES [21] has demonstrated that, although not every hyperelastic, isotropic and incompressible material can sustain antiplane shear deformations, all generalized neo-Hookean materials are capable of doing so. It has been shown (KNOWLES [21–22]) that for such materials the local balance equations $(2.1.9)_{2,3}$ reduce, in the absense of body forces and under the asssumption that the nominal stress tensor is independent of the x_3-coordinate, to the scalar equation

$$(M(\gamma)u_{,\alpha})_{,\alpha} = 0 \quad \text{on} \quad \mathcal{D}\setminus C. \tag{2.4.2}$$

See Fosdick & Serrin [15] and Fosdick & Kao [13] for a general discussion of circumstances under which the field equations $(2.1.9)_{2,3}$ reduce to a single scalar equation. In (2.4.2) M is the secant modulus in shear as defined in (2.2.8), $\gamma : \mathcal{D}\setminus C \to I\!R$ is the *shear strain field* given by

$$\gamma(x_1,x_2) = \sqrt{u_{,\alpha}(x_1,x_2)u_{,\alpha}(x_1,x_2)} \quad \forall (x_1,x_2) \in \mathcal{D}\setminus C, \tag{2.4.3}$$

$\mathcal{D}$ is a plane region with shape determined by a generic cross section of $\mathcal{R}$, and C is a curve contained in $\mathcal{D}$ and determined similarly by a cross section of the surface across which the deformation gradient jumps. Furthermore, the jump condition $(2.1.10)_2$ reduces, for a generalized neo-Hookean material subjected to antiplane shear, to

$$[\![M(\gamma)u_{,\alpha}n_\alpha]\!] = 0 \quad \text{on} \quad C, \tag{2.4.4}$$

where $\mathbf{n} : C \to \mathcal{N}$ is a unit normal to C, while (2.1.15) becomes

$$[\![u]\!] = 0 \quad \text{on} \quad C. \tag{2.4.5}$$

It is also readily shown that the driving traction f, introduced in Section 2.3, for a generalized neo-Hookean material subjected to an antiplane shear deformation involving a discontinuity in the gradient of displacement across a curve C is given by

$$f = [\![W(3+\gamma^2)]\!] - M(\overset{\pm}{\gamma})\overset{\pm}{u}_{,\alpha}[\![u_{,\alpha}]\!] \quad \text{on} \quad C. \tag{2.4.6}$$

In (2.4.6) $\overset{+}{u}_{,\alpha}$ and $\overset{-}{u}_{,\alpha}$ refer to the limiting values of the gradient of the out-of-plane displacement field on the side of the curve C into which and out of which the unit normal $\mathbf{n}$ points, respectively. Evidently, $\overset{+}{\gamma}$ and $\overset{-}{\gamma}$ are given in terms of $\overset{+}{u}_{,\alpha}$ and $\overset{-}{u}_{,\alpha}$ by (2.4.3).

3. STUDY OF A CONSTRAINED BOUNDARY VALUE PROBLEM IN THE ANTIPLANE SHEAR OF A THREE-PHASE MATERIAL

This chapter focuses, in the context of antiplane shear, on the construction of two-phase equilibria of a body composed of the particular non-elliptic generalized neo-Hookean material with shear stress response function τ_p defined via (2.2.12). These equilibria will involve non-planar interfaces which segregate material in different elliptic phases. The interfaces will be described by surfaces $\mathcal{Q}_s$ of the form

$$\mathcal{Q}_s = \{\mathrm{x} \in \mathbb{R}^3 \,|\, x_1 = s(x_2), x_2 \in \mathbb{R}, x_3 \in \mathbb{R}\}$$

where s is twice continuously differentiable on $\mathbb{R}$, $s^{(n)}$ is in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ for $n = 0, 1, 2$, and

$$\lim_{x_2 \to \pm\infty} s(x_2) = 0.$$

In Section 3.1 a boundary value problem for the out-of-plane displacement field associated with two-phase antiplane shear deformations of a three-phase material is formulated and specialized to the case of the material with shear stress response τ_p. This boundary value problem is supplemented by a set of constraints which require that the non-elliptic phase of the relevant material is absent and, moreover, that the elliptic phases are segregated. In Section 3.2 the boundary value problem is converted into an integral equation for the jump in the normal derivative of the out-of-plane displacement field across $\mathcal{Q}_s$. In Section 3.3 it is shown that there exists a unique solution to this integral equation for every $\mathcal{Q}_s$ defined by a function s which, in addition to the restrictions delineated above, satisfies

$$\Big(\int_{-\infty}^{+\infty} |s'(x_2)||s''(x_2)|\, dx_2\Big)^{\frac{1}{2}} < \sqrt{\tfrac{3\pi}{2}}\,\frac{\mu_1 + \mu_2}{\mu_1 - \mu_2},$$

where μ_1 and μ_2 are the moduli associated with the elliptic phases of the material defined by τ_p. It is then demonstrated that for each such s there exists a unique (up to an arbitrary additive constant) solution to the boundary value problem stated in Section 3.1. Since, however, the constraints mentioned above are not necessarily satisfied by any of these solutions, the deformation associated with a given solution does not necessarily constitute a two-phase equilibrium state of the type sought after here. In Section 3.4 it is shown that, provided a certain *functional* of s is sufficiently small—in a sense to be made precise—then there exists a unique solution to the constrained boundary value problem and, hence, a two-phase equilibrium state of the type sought after here. The concluding section of this chapter is concerned with the construction of a class of two-phase states which involve non-planar interfaces separating material in distinct elliptic phases.

3.1. Formulation and reduction of the boundary value problem and phase segregation requirements. Suppose that $\mathcal{B}$ is composed of a three-phase material and that the cylinder $\mathcal{R}$ degenerates to occupy all of $I\!R^3$. Let the rectangular Cartesian frame X be as in Section 2.4. Consider the effect of subjecting $\mathcal{R}$ to a particular antiplane loading whereby, independent of the x_2–coordinate, the shear strain approaches uniform values of γ_l as x_1 tends to $-\infty$ and γ_r as x_1 tends to $+\infty$. Assume that γ_l is greater than $\overset{*}{\gamma}$ and that γ_r lies strictly between 0 and $\underset{*}{\gamma}$; the prescribed remote shear strains associated with the loading are, thus, in the high and low strain phases of the material at hand as x_1 approaches $-\infty$ and $+\infty$, respectively. If γ_l and γ_r are chosen so that the corresponding remote shear stresses $\tau(\gamma_l)$ and $\tau(\gamma_r)$ are equal then—for every three-phase material—there exists, modulo an arbitrary additive constant, a unique one parameter family of pairwise homogeneous out-of-plane displacement fields $u_a : I\!R \to I\!R$ which satisfy the equilibrium equation in (2.4.2) on $I\!R^2 \setminus C_a$, with the straight line C_a given by $\{x_\alpha \mathbf{e}_\alpha \in I\!R^2 \,|\, x_1 = a, x_2 \in I\!R\}$, the jump conditions in (2.4.4) and (2.4.5) on C_a and, of course, the decay requirements associated with the prescribed conditions at $x_1 = \pm\infty$. The function u_a is given by

$$u_a(x_1) = \begin{cases} \gamma_l(x_1 - a) & \text{if} \quad x_1 < a, \\ \gamma_r(x_1 - a) & \text{if} \quad x_1 > a. \end{cases} \tag{3.1.1}$$

Here a determines the point of intersection of the plane surface $\mathcal{Q}_a$, given by $\mathcal{C}_a \times \mathbb{R}$, with the x_1-axis. Note that, for each fixed a, the pairwise homogeneous deformation associated with (3.1.1) through (2.4.1) involves exclusively the elliptic phases of the material under consideration and that these are segregated by $\mathcal{Q}_a$; the deformation associated with (3.1.1) will be referred to as a *globally elliptic pairwise homogeneous equilibrium state.* The interface $\mathcal{Q}_a$ associated with such a state will, in turn, be referred to as a *phase boundary.* Observe that the qualitative character of the equilibrium state associated with u_a is clearly unaffected by the value of a.

Envision a generalization of the globally elliptic pairwise homogeneous equilibrium state wherein the kinematics remain those of antiplane shear and the loading conditions are as described at the outset of this section but the planar phase boundary is replaced by a non-planar interface $\mathcal{Q}_s$ with cross section $\mathcal{C}_s$ where, for simplicity,

$$\mathcal{Q}_s = \mathcal{C}_s \times \mathbb{R} \tag{3.1.2}$$

with

$$\mathcal{C}_s = \{x_\alpha \mathbf{e}_\alpha \in \mathbb{R}^2 \,|\, x_1 = s(x_2), x_2 \in \mathbb{R}\}. \tag{3.1.3}$$

Assume that the state is equilibrated in the sense that the balance equation in (2.4.2) holds on $\mathbb{R}^2 \setminus \mathcal{C}_s$ while the jump conditions in (2.4.4) and (2.4.5) are satisfied on $\mathcal{C}_s$. Clearly, if such a state exists, the deformation field intrinsic to it must be inhomogeneous on either side of the interface $\mathcal{Q}_s$. Observe that even if a three-phase material is capable of sustaining a deformation of this kind the shear strain field may not, in general, be distributed so that only the elliptic phases of the material are present; if, however, this is the case and, furthermore, the high and low strain phases of the relevant material are segregated by the interface $\mathcal{Q}_s$ then the deformation will be said to constitute a *globally elliptic inhomogeneous two-phase equilibrium state* with phase boundary $\mathcal{Q}_s$.

Consider, now, the geometry of the curve $\mathcal{C}_s$ which determines the phase boundary $\mathcal{Q}_s$ essential to a globally elliptic inhomogeneous two-phase equilibrium state. Since the shear strain field is constant as x_1 approaches $\pm\infty$, it is clear that s must be bounded on $\mathbb{R}$ in order for $\mathcal{C}_s$ to qualify as a cross section of the phase boundary $\mathcal{Q}_s$. The kinematics and boundary conditions place no further restrictions on the geometry of $\mathcal{C}_s$.

Observe that if, in addition to being bounded and continuous on $\mathbb{R}$, s satisfies one or both of

$$\lim_{x_2 \to -\infty} s(x_2) = \overset{-}{c}, \quad \lim_{x_2 \to +\infty} s(x_2) = \overset{+}{c}, \tag{3.1.4}$$

where $\overset{-}{c}$ and $\overset{+}{c}$ are real constants, then the loading must be restricted so that the far field shear stresses $\tau(\gamma_l)$ and $\tau(\gamma_r)$ are equal. To see this suppose that $(3.1.4)_1$ holds. Then, as x_2 approaches $-\infty$ the phase boundary becomes planar and the local character of the deformation begins to resemble a pairwise homogeneous state. Since the far field shear strains γ_l and γ_r are constant the local shear strains must match these appropriately on either side of $\mathcal{C}_s$ as x_1 approaches $-\infty$. Hence, the local shear stresses must match their far field counterparts $\tau(\gamma_l)$ and $\tau(\gamma_r)$, and, by the jump condition in (2.4.4) which holds on $\mathcal{C}_s$, $\tau(\gamma_l) = \tau(\gamma_r)$. A completely analogous argument can be constructed if $(3.1.4)_2$ holds instead of $(3.1.4)_1$. Certainly, if both of (3.1.4) hold, the result is still true. Note, however, that if neither of (3.1.4) hold, and, hence, the curve $\mathcal{C}_s$ is merely bounded, there is no reason to rule out—*a priori*—loading conditions wherein the far field stresses are unequal.

Assume, henceforth, that (3.1.4) holds with $\overset{-}{c}$ and $\overset{+}{c}$ equal to, say, c. Recalling the role of a in (3.1.1), there is certainly no additional loss in generality incurred by taking $c = 0$. In this case (3.1.4) becomes

$$\lim_{x_2 \to \pm\infty} s(x_2) = 0. \tag{3.1.5}$$

Let $\mathcal{U}$ be the set of functions defined by

$$\mathcal{U} = \{\psi : \mathbb{R} \to \mathbb{R} \,|\, \psi \in C(\mathbb{R}), \lim_{x_2 \to \pm\infty} \psi(x_2) = 0, \psi \not\equiv 0 \text{ on } \mathbb{R}\}.$$

Assume, henceforth, that s is an element of the set

$$\mathcal{A} = \mathcal{U} \cap \mathcal{V} \cap \mathcal{W}, \tag{3.1.6}$$

where $\mathcal{V}$ and $\mathcal{W}$ are defined by

$$\mathcal{V} = \{\psi : I\!R \to I\!R \,|\, \psi \in C^2(I\!R), \psi^{(n)} \in L^2(I\!R), n = 0, 1, 2\}, \tag{3.1.7}$$

and

$$\mathcal{W} = \{\psi : I\!R \to I\!R \,|\, \psi \in C^2(I\!R), \psi^{(n)} \in L^\infty(I\!R), n = 0, 1, 2\}, \tag{3.1.8}$$

respectively.

Given an element s of $\mathcal{A}$ which describes an interface $\mathcal{Q}_s$ it is convenient to define plane sets $\mathcal{D}_s^l$ and $\mathcal{D}_s^r$ by

$$\mathcal{D}_s^l = \{x_\alpha \mathrm{e}_\alpha \in I\!R^2 \,|\, x_1 \leq s(x_2), x_2 \in I\!R\}, \quad \mathcal{D}_s^r = I\!R^2 \setminus \mathring{\mathcal{D}}_s^l. \tag{3.1.9}$$

Clearly, the union and intersection of $\mathcal{D}_s^l$ and $\mathcal{D}_s^r$ form generic cross-sections of the cylinder $\mathcal{R}$ and the phase boundary $\mathcal{Q}_s$, respectively. Note, also, that if s is an element of $\mathcal{A}$ then, by (3.1.3) and its assumed smoothness, a unit normal to $\mathcal{Q}_s$ exists everywhere on $\mathcal{Q}_s$ and depends only on the x_2–coordinate. Let $\mathrm{n} : I\!R \to \mathcal{N}$ designate the unit normal to $\mathcal{Q}_s$ which points into the region of low strain—$\mathring{\mathcal{D}}_s^r \times I\!R$. Then the representation for n is computed easily from the definitions of $\mathcal{Q}_s$ and $\mathcal{C}_s$ and is given by

$$\mathrm{n}(x_2) = \frac{\mathrm{e}_1 - s'(x_2)\mathrm{e}_2}{\sqrt{1 + s'(x_2)^2}} \quad \forall x_2 \in I\!R. \tag{3.1.10}$$

Now, if a three-phase material is capable of sustaining a globally elliptic inhomogeneous two-phase equilibrium state of antiplane shear with phase boundary $\mathcal{Q}_s$ then the out-of-plane displacement field u associated with the deformation through (2.4.1) must, by virtue of (2.4.2), (2.4.4) and (2.4.5), satisfy the following field equation and jump conditions:

$$
\begin{aligned}
(M(\gamma)u_{,\alpha})_{,\alpha} &= 0 \quad \text{on} \quad I\!R^2 \setminus \mathcal{C}_s, \\
[\![M(\gamma)u_{,\alpha}n_\alpha]\!] &= 0 \quad \text{on} \quad \mathcal{C}_s, \\
[\![u]\!] &= 0 \quad \text{on} \quad \mathcal{C}_s,
\end{aligned}
\tag{3.1.11}
$$

with $\mathbf{n}$ as indicated in (3.1.10); in order to comply with the prescribed loading it suffices to require that the gradient of u satisfies the following asymptotic decay conditions:

$$
u_{,\alpha}(x_1,\cdot)\mathbf{e}_\alpha = \begin{cases} \gamma_l\,\mathbf{e}_1 + o(1) & \text{as} \quad x_1 \to -\infty, \\ \gamma_r\mathbf{e}_1 + o(1) & \text{as} \quad x_1 \to +\infty, \end{cases} \quad \text{on} \quad I\!R; \tag{3.1.12}
$$

moreover, in order to assure that only the elliptic phases of the material at hand are present and are segregated by $\mathcal{Q}_s$, the shear strain field γ, given in terms of the gradient of u by (2.4.3), must conform to the following inequalities:

$$
\gamma \in (\overset{*}{\gamma}, \infty) \quad \text{on} \quad \mathring{\mathcal{D}}_s^l, \qquad \gamma \in [0, \underset{*}{\gamma}) \quad \text{on} \quad \mathring{\mathcal{D}}_s^r, \tag{3.1.13}
$$

where $\mathcal{D}_s^l$ and $\mathcal{D}_s^r$ are given by (3.1.9). These inequalities will be referred to as the *phase segregation requirement.*

Given a three-phase material, (3.1.11)–(3.1.12) comprise, for each fixed s contained in $\mathcal{A}$, a boundary value problem in the out-of-plane displacement field u, while (3.1.13) acts as a system of constraints thereon. Together (3.1.11)–(3.1.13) will be referred to as the constrained boundary value problem in u for the three-phase material with secant modulus in shear M. Given a particular three-phase material the constrained boundary value problem need not have a solution for any function s in $\mathcal{A}$. The study of (3.1.11)–(3.1.13) for a specific material may, however, serve as a means to determine a subset of $\mathcal{A}$ for which the constrained boundary value problem is soluble.

Before proceeding note that the jump conditions $(3.1.11)_{2,3}$ holding across C_s can be recast to read

$$M\left(\gamma\left(s(\cdot)+,\cdot\right)\right)\frac{\partial u}{\partial n}\left(s(\cdot)+,\cdot\right) = M\left(\gamma\left(s(\cdot)-,\cdot\right)\right)\frac{\partial u}{\partial n}\left(s(\cdot)-,\cdot\right) \quad \text{on} \quad \mathbb{R},$$
$$u\left(s(\cdot)+,\cdot\right) = u\left(s(\cdot)-,\cdot\right) \quad \text{on} \quad \mathbb{R}, \tag{3.1.14}$$

where the $+$ and $-$ symbols indicate the limiting values of the appropriate quantities on the high and low strain sides of the interface, respectively.

For simplicity attention will, for the remainder of this work, be restricted to the constrained boundary value problem for the material characterized by the shear stress response function τ_p defined in (2.2.12). In this case the form of the shear stress response function is such that the secant modulus in shear M is constant in both the high and low strain elliptic phases; hence, $(3.1.11)_1$ and $(3.1.14)_1$ reduce to

$$u_{,\alpha\alpha} = 0 \quad \text{on} \quad \mathbb{R}^2 \setminus C_s,$$
$$\mu_1 \frac{\partial u}{\partial n}\left(s(\cdot)+,\cdot\right) = \mu_2 \frac{\partial u}{\partial n}\left(s(\cdot)-,\cdot\right) \quad \text{on} \quad \mathbb{R}. \tag{3.1.15}$$

The analytical difficulties of the special constrained boundary value problem posed by (3.1.15), $(3.1.14)_2$, (3.1.12) and (3.1.13) are certainly less daunting than those encountered in the analogous problem for a more general three-phase material. For each fixed s in $\mathcal{A}$ the only non-linearity which encumbers the problem associated with τ_p is that imposed by the strain constraints (3.1.13). In the present absence of results pertaining to the existence of globally elliptic two-phase equilibria in arbitrary three-phase materials, any results which can be obtained for this particular material constitute progress toward a qualitative understanding of the more general issue.

As a first step in analyzing the constrained boundary value problem comprised by (3.1.15), $(3.1.14)_2$, (3.1.12) and (3.1.13) it is convenient to introduce a *reduced* out-of-plane displacement field $v : \mathbb{R}^2 \setminus C_s \to \mathbb{R}$ specified via

$$v(x_1, x_2) = u(x_1, x_2) - u_0(x_1, x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2 \setminus C_s, \tag{3.1.16}$$

where $u_0 : I\!R^2 \setminus \mathcal{C}_s \to I\!R$ is furnished by

$$u_0(x_1, x_2) = H\left(s(x_2) - x_1\right)\gamma_l x_1 + H\left(x_1 - s(x_2)\right)\gamma_r x_1 \quad \forall (x_1, x_2) \in I\!R^2 \setminus \mathcal{C}_s,$$

and $H : I\!R \setminus \{0\} \to I\!R$ is the *Heaviside function*:

$$H(x_1) = \begin{cases} 0 & \text{if} \quad x_1 < 0, \\ 1 & \text{if} \quad x_1 > 0. \end{cases}$$

Solving for u in (3.1.16) and inserting the result in (3.1.15), $(3.1.14)_2$ and (3.1.12) shows, with the aid of the definitions of u_0 and H, that the reduced out-of-plane displacement field v must satisfy the following boundary value problem:

$$\begin{aligned} v_{,\alpha\alpha} &= 0 \quad \text{on} \quad I\!R^2 \setminus \mathcal{C}_s, \\ \mu_1 \frac{\partial v}{\partial n}\big(s(\cdot)+, \cdot\big) &= \mu_2 \frac{\partial v}{\partial n}\big(s(\cdot)-, \cdot\big) \quad \text{on} \quad I\!R, \\ v\big(s(\cdot)+, \cdot\big) - v\big(s(\cdot)-, \cdot\big) &= (\gamma_l - \gamma_r)s \quad \text{on} \quad I\!R, \\ v_{,\alpha}(x_1, \cdot)\mathbf{e}_\alpha &= \mathrm{o}(1) \quad \text{as} \quad x_1 \to \pm\infty \quad \text{on} \quad I\!R. \end{aligned} \tag{3.1.17}$$

Note that in deriving $(3.1.17)_2$ use has been made of the equality of remote shear stresses—which, as shown at the beginning of this section, is a necessary consequence of (3.1.5). The phase segregation requirement (3.1.13) can be written—after appropriate substitution for u—in terms of the components of the gradient of v as follows:

$$\begin{aligned} \overset{*}{\gamma}{}^2 < v_{,\alpha} v_{,\alpha} + \gamma_l(2v_{,1} + \gamma_l) & \qquad \text{on} \quad \mathring{\mathcal{D}}{}^l_s, \\ 0 \le v_{,\alpha} v_{,\alpha} + \gamma_r(2v_{,1} + \gamma_r) < \underset{*}{\gamma}{}^2 & \qquad \text{on} \quad \mathring{\mathcal{D}}{}^r_s. \end{aligned} \tag{3.1.18}$$

The boundary value problem in (3.1.17) will be referred to as the *reduced boundary value problem* with the implicit understanding that it is in the reduced out-of-plane displacement field v and for the special three-phase material with shear stress response function τ_p. The system of inequalities in (3.1.18) will be labelled the *reduced phase segregation requirement.* It is clear from the simple relation between the primitive out-of-plane displacement field u and its reduced counterpart v that any solution to the reduced problem yields a solution to the original problem. The next section will focus on obtaining a representation for the solution to the reduced boundary value problem with the reduced phase segregation requirement held in abeyance. This representation will lead to an integral equation which, for each fixed s in $\mathcal{A}$, may be analyzed in place of the associated reduced boundary value problem.

3.2. Reformulation of the reduced boundary value problem as an integral equation. Let s be an arbitrary element of $\mathcal{A}$. Since, by $(3.1.17)_1$, the reduced out-of-plane displacement field v is *harmonic* on $\mathbb{R}^2 \setminus \mathcal{C}_s$ the jump conditions $(3.1.17)_2$ and $(3.1.17)_3$ suggest that v can be represented, modulo an arbitrary additive constant, as the sum of a *single-* and a *double-layer potential* along the curve $\mathcal{C}_s$.[3] The densities of the appropriate single- and double-layer potentials are given, respectively, in terms of the jumps in the normal derivative of v and of v itself across the curve $\mathcal{C}_s$. From the jump condition $(3.1.17)_3$ it is clear that the density of the double-layer potential must be given by $(\gamma_l - \gamma_r)s$ on $\mathbb{R}$. Since, by the definition of the shear stress response function τ_p, the moduli μ_1 and μ_2 which appear in $(3.1.17)_2$ are required to be unequal, this jump condition does not yield direct information regarding the form of the density of the single-layer potential. It is, therefore, necessary to designate the jump in the normal derivative of v across $\mathcal{C}_s$ in terms of an unknown function—say $(\gamma_l - \gamma_r)\phi$, where it is assumed, until demonstrated otherwise, that $\phi : \mathbb{R} \to \mathbb{R}$ does not vanish identically on $\mathbb{R}$. Hence, the proposed representation for the reduced out-of-plane displacement field v takes the form

$$v(x_1, x_2) = \frac{\gamma_l - \gamma_r}{2\pi} \left(S_\phi(x_1, x_2) + D_s(x_1, x_2) \right) \quad \forall (x_1, x_2) \in \mathbb{R}^2 \setminus \mathcal{C}_s, \qquad (3.2.1)$$

where the functions $S_\phi : \mathbb{R}^2 \to \mathbb{R}$ and $D_s : \mathbb{R}^2 \setminus \mathcal{C}_s \to \mathbb{R}$ issue, respectively,

[3] For an overview of the relevant potential theory see COURANT & HILBERT [11].

from the single- and double-layer potentials on $\mathcal{C}_s$ with densities ϕ and s and are given by

$$S_\phi(x_1,x_2) = \int_{-\infty}^{+\infty} G_1^s(x_1,x_2,\xi)\phi(\xi)\sqrt{1+s'(\xi)^2}\,d\xi \quad \forall (x_1,x_2) \in I\!R^2, \tag{3.2.2}$$

and

$$D_s(x_1,x_2) = \int_{-\infty}^{+\infty} G_2^s(x_1,x_2,\xi)s(\xi)\,d\xi \quad \forall (x_1,x_2) \in I\!R^2 \setminus \mathcal{C}_s. \tag{3.2.3}$$

The kernels $G_1^s : (I\!R^2 \setminus \mathcal{C}_s) \times I\!R \to I\!R$ and $G_2^s : I\!R^2 \times I\!R \to I\!R$ which appear in (3.2.2) and (3.2.3) are given, for each ξ contained in $I\!R$, by

$$G_1^s(x_1,x_2,\xi) = \ln\sqrt{(x_1-s(\xi))^2+(x_2-\xi)^2} \quad \forall (x_1,x_2) \in I\!R^2 \setminus \mathcal{C}_s, \tag{3.2.4}$$

and

$$G_2^s(x_1,x_2,\xi) = \frac{(x_1-s(\xi))-(x_2-\xi)s'(\xi)}{(x_1-s(\xi))^2+(x_2-\xi)^2} \quad \forall (x_1,x_2) \in I\!R^2. \tag{3.2.5}$$

Consider, now, the issue of verifying the status of the representation (3.2.1) as a solution to the boundary value problem (3.1.17). Since the single- and double-layer potentials are harmonic, by construction, on $I\!R^2 \setminus \mathcal{C}_s$, it is evident that the function v given by (3.2.1)–(3.2.5) satisfies $(3.1.17)_1$. A series of direct calculations too long to display here show that $v_{,1}(x_1,\cdot)$ and $v_{,2}(x_1,\cdot)$ both behave asymptotically like $O(1/x_1)$ as x_1 approaches $\pm\infty$ on $I\!R$ so that the representation (3.2.1)–(3.2.5) complies with $(3.1.17)_4$ and, hence, the loading conditions. Since the single-layer term (3.2.2) is continuous on $I\!R^2$ and the double-layer term (3.2.3) has been constructed so that it possesses a jump of $2\pi s$ across the curve $\mathcal{C}_s$ it is also clear that (3.2.1)–(3.2.5) furnishes a representation of v which satisfies the jump condition in $(3.1.17)_3$. The only remaining requirement which must be satisfied by (3.2.1)–(3.2.5) in order for it to provide a solution to the reduced boundary value problem is the jump condition $(3.1.17)_2$ involving the normal derivative of v. A straightforward but tedious calculation using standard results from potential theory delivers the limits of the normal derivative of v on either side of C_s in the form

$$\begin{aligned}\frac{\partial v}{\partial n}(s(x_2)\pm, x_2) = &-\frac{\gamma_l - \gamma_r}{2\pi\sqrt{1+s'(x_2)^2}} \fint_{-\infty}^{+\infty} I_s(x_2,\xi)s'(\xi)\,d\xi \\ &+\frac{\gamma_l - \gamma_r}{2\pi\sqrt{1+s'(x_2)^2}} \int_{-\infty}^{+\infty} K_s(x_2,\xi)\phi(\xi)\sqrt{1+s'(\xi)^2}\,d\xi \\ &\pm\frac{\gamma_l-\gamma_r}{2}\phi(x_2) \quad \forall x_2 \in I\!R, \end{aligned} \tag{3.2.6}$$

where, for each fixed x_2 in $I\!R$, $I_s(x_2,\cdot) : I\!R \setminus \{x_2\} \to I\!R$ and $K_s(x_2,\cdot) : I\!R \to I\!R$ are given, repectively, by

$$I_s(x_2,\xi) = \frac{(s(x_2)-s(\xi))s'(x_2) + (x_2-\xi)}{(s(x_2)-s(\xi))^2 + (x_2-\xi)^2} \quad \forall \xi \in I\!R \setminus \{x_2\}, \tag{3.2.7}$$

and

$$K_s(x_2,\xi) = \frac{(s(x_2)-s(\xi)) - (x_2-\xi)s'(x_2)}{(s(x_2)-s(\xi))^2 + (x_2-\xi)^2} \quad \forall \xi \in I\!R. \tag{3.2.8}$$

Observe from (3.2.8) that as ξ approaches x_2, $I_s(x_2,\xi)$ is singular for each x_2 in $I\!R$ in that

$$I_s(x_2,\xi) \sim \frac{1}{x_2-\xi} \quad \text{as} \quad \xi \to x_2 \quad \forall x_2 \in I\!R.$$

Hence, the integral involving I_s in (3.2.6) must, as indicated, be taken in the sense of the *Cauchy principal value.* One also finds that, as x_2 approaches ξ, $I_s(x_2,\xi)$ is singular for each ξ in $I\!R$ in a manner entirely analogous to that displayed above. On the other hand, an examination of (3.2.6) reveals that the behavior of $K_s(x_2,\xi)$ as either ξ approaches a fixed x_2 in $I\!R$ or as x_2 approaches a fixed ξ in

$I\!R$ is regular; in fact, since s is an element of $\mathcal{A}$, the limits

$$\lim_{\xi \to x_2} K_s(x_2, \xi) = -\frac{s''(x_2)}{2(1 + s'(x_2)^2)} \quad \forall x_2 \in I\!R,$$

and

$$\lim_{x_2 \to \xi} K_s(x_2, \xi) = -\frac{s''(\xi)}{2(1 + s'(\xi)^2)} \quad \forall \xi \in I\!R,$$

both exist and are finite.

Recall, now, that the function ϕ which appears in the second term on the right hand side of (3.2.6) is unknown. The jump condition $(3.1.17)_2$ serves, therefore, as a device by which this function can be determined. An appropriate substitution of (3.2.6) into $(3.1.17)_2$ yields—after collecting terms and dropping a non-vanishing common factor—the following equation:

$$\begin{aligned}(\mu_1 + \mu_2)\phi + \frac{\mu_1 - \mu_2}{\pi\sqrt{1 + (s')^2}} \int_{-\infty}^{+\infty} K_s(\cdot, \xi)\phi(\xi)\sqrt{1 + s'(\xi)^2}\, d\xi \\ = \frac{\mu_1 - \mu_2}{\pi\sqrt{1 + (s')^2}} \fint_{-\infty}^{+\infty} I_s(\cdot, \xi)s'(\xi)\, d\xi \quad \text{on} \quad I\!R.\end{aligned} \tag{3.2.9}$$

Observe that (3.2.9) constitutes, for each fixed s contained in $\mathcal{A}$, a linear integral equation to be solved for ϕ on $I\!R$. The integral equation in (3.2.9) can be simplified by making a few modest substitutions; toward this end define $\varphi : I\!R \to I\!R$ by

$$\varphi(x_2) = \phi(x_2)\sqrt{1 + s'(x_2)^2} \quad \forall x_2 \in I\!R, \tag{3.2.10}$$

and introduce a real constant λ through the relation

$$\lambda = \frac{\mu_1 - \mu_2}{\pi(\mu_1 + \mu_2)}. \tag{3.2.11}$$

Recall from the definition of τ_p that the moduli μ_1 and μ_2 satisfy $0 < \mu_2 < \mu_1$; λ must, consequently, lie strictly between 0 and $1/\pi$. Continuing with the simplification of (3.2.9), multiply and then divide both sides of the integral equation by

$\sqrt{1+(s')^2}$ and $(\mu_1+\mu_2)$, respectively, to obtain, with the aid of the definitions (3.2.10) and (3.2.11) the following alternative to (3.2.9):

$$\varphi+\lambda\int_{-\infty}^{+\infty} K_s(\cdot,\xi)\varphi(\xi)\,d\xi=\lambda\fint_{-\infty}^{+\infty} I_s(\cdot,\xi)s'(\xi)\,d\xi \quad \text{on} \quad I\!R. \tag{3.2.12}$$

For the purpose of facilitating the forthcoming discussion introduce, for each function s contained in $\mathcal{A}$, an operator $\mathcal{M}_s$ such that, for each function ψ the function $\mathcal{M}_s\psi$ is given by

$$\mathcal{M}_s\psi=\int_{-\infty}^{+\infty} K_s(\cdot,\xi)\psi(\xi)\,d\xi \quad \text{on} \quad I\!R. \tag{3.2.13}$$

In addition, let a function $f_s: I\!R\to I\!R$ be defined for each s in $\mathcal{A}$ via

$$f_s=\fint_{-\infty}^{+\infty} I_s(\cdot,\xi)s'(\xi)\,d\xi \quad \text{on} \quad I\!R. \tag{3.2.14}$$

With the aid of (3.2.13) and (3.2.14), (3.2.12) can be recast to read

$$\varphi+\lambda\mathcal{M}_s\varphi=\lambda f_s \quad \text{on} \quad I\!R. \tag{3.2.15}$$

Evidently a solution φ to (3.2.15) provides, through (3.2.10), a solution to (3.2.9). However, it is also clear from (3.2.1)–(3.2.3) and (3.2.10) that, given φ, v can be obtained directly in the form

$$v(x_1,x_2)=\frac{\gamma_l-\gamma_r}{2\pi}\int_{-\infty}^{+\infty}\left(G_1^s(x_1,x_2,\xi)\varphi(\xi)+G_2^s(x_1,x_2,\xi)s(\xi)\right)d\xi$$
$$\forall(x_1,x_2)\in I\!R^2\setminus\mathcal{C}_s, \tag{3.2.16}$$

which obviates the need to consider ϕ. Hence, for each s in $\mathcal{A}$ the task of constructing a solution v to the corresponding reduced boundary value problem

(3.1.17) is altered, via potential theory, to one of constructing a solution to the corresponding integral equation (3.2.15). The task of the next section is to determine a set of conditions upon s—in addition to requiring that it to be an element of $\mathcal{A}$—which are sufficient to guarantee the existence of a solution to (3.2.15).

3.3. Analysis of the integral equation. Suppose that s is contained in $\mathcal{A}$ and consider the kernel K_s associated with it by (3.2.8). Observe that the stipulated smoothness of s implies that K_s is a continuous function on $\mathbb{R}^2$. Moreover, since

$$|K_s(x_2,\xi)| \leq \frac{|s(x_2)-s(\xi)-(x_2-\xi)s'(x_2)|}{(x_2-\xi)^2} \quad \forall (x_2,\xi)\in\mathbb{R}^2; \tag{3.3.1}$$

the boundedness of s'' on $\mathbb{R}$ and Taylor's theorem imply the following global estimate for the modulus of K_s:

$$|K_s(x_2,\xi)| \leq \frac{1}{2}\sup_{\zeta\in\mathbb{R}}|s''(\zeta)| = \frac{1}{2}\|s''\|_{L^\infty(\mathbb{R})} \quad \forall (x_2,\xi)\in\mathbb{R}^2. \tag{3.3.2}$$

Hence, the kernel K_s corresponding to any s in $\mathcal{A}$ is continuous and bounded on $\mathbb{R}$; furthermore, the bound is given explicitly in terms of a functional of s—the L^∞ norm of s'' over $\mathbb{R}$. If the integral equation held over a compact domain then the bound (3.3.2) would lead, for each fixed λ in $(0,1/\pi)$, to sufficient conditions in terms of the size of $\|s''\|_{L^\infty(\mathbb{R})}$ which would allow the construction of a unique solution to the integral equation via a uniformly convergent *Neumann series*. Since the integral equation in (3.2.15) holds over $\mathbb{R}$, it will be convenient to determine conditions on functionals of s other than its L^∞ norm which are sufficient to guarantee an analogous result. Toward this end consider the Neumann series for this integral equation. This series is readily obtained via the method of successive substitutions and is given by $\Phi:\mathbb{R}\to\mathbb{R}$ as defined below:

$$\Phi = \lambda\sum_{n=0}^{\infty}(-\lambda)^n\mathcal{M}_s^n f_s \quad \text{on} \quad \mathbb{R}. \tag{3.3.3}$$

Observe that, with the aid of (3.2.13) and a formal interchange of summation and integration, Φ satisfies

$$\Phi + \lambda \mathcal{M}_s \Phi = \lambda f_s \quad \text{on} \quad I\!R.$$

That is, provided the formal operations performed above can be justified, Φ furnishes a solution to the integral equation. If the Neumann series converges uniformly then this is certainly the case. Consider the following geometric series:

$$g = \lambda \|f_s\|_{L^2(I\!R)} \sum_{n=0}^{\infty} (\lambda)^n \|K_s\|^n_{L^2(I\!R^2)} \,. \tag{3.3.4}$$

If K_s and f_s are elements of $L^2(I\!R^2)$ and $L^2(I\!R)$, respectively, and the L^2 norm of K_s over $I\!R^2$ satisfies

$$\lambda \|K_s\|_{L^2(I\!R^2)} < 1 \tag{3.3.5}$$

then (3.3.4) will converge. Note that the Neumann series is majorized by the geometric series. Conditions sufficient to guarantee the convergence of (3.3.4) are, accordingly, sufficient to assure that the Neumann series converges uniformly on its domain of definition and, therefore, as alluded to above, that Φ supplies a solution to (3.2.15). Provided these sufficient conditions are in force, the operator $\mathcal{M}_s$ is, moreover, a *Fredholm integral operator* with domain and range $L^2(I\!R)$. Hence, the *Fredholm alternative* holds and it can be shown that the solution Φ to the integral equation provided by the Neumann series is unique.[4]

At present the aforementioned sufficient conditions are only of value if there exist functions s in the set $\mathcal{A}$ defined by (3.1.6)–(3.1.8) for which they hold. It will now be demonstrated that the first two conditions are satisfied for every s in $\mathcal{A}$ and that the third holds for every s contained in the proper subset $\mathcal{I}$ of $\mathcal{A}$ defined by

$$\mathcal{I} = \{s \in \mathcal{A} \,|\, \lambda \sqrt{\tfrac{2\pi}{3}} \Big(\int_{-\infty}^{+\infty} |s'(x_2)||s''(x_2)| \, dx_2 \Big)^{\frac{1}{2}} < 1\}. \tag{3.3.6}$$

[4] See GARABEDIAN [17] for a discussion of Neumann series and the foregoing results.

First suppose that s is an element of $\mathcal{A}$. Show that the kernel K_s must consequentially be square integrable on its domain of definition. Let $k_s : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$k_s(x_2,\eta) = \frac{s(x_2) - s(x_2+\eta) + \eta s'(x_2)}{\eta^2} \quad \forall (x_2,\eta) \in \mathbb{R}^2. \tag{3.3.7}$$

Note, from (3.3.1), (3.3.7) that

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_s^2(x_2,\xi)\,d\xi\,dx_2 &\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{|s(x_2)-s(\xi)-(x_2-\xi)s'(x_2)|^2}{(x_2-\xi)^4}\,d\xi\,dx_2 \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\,d\eta\,dx_2. \end{aligned} \tag{3.3.8}$$

Hence, to demonstrate that K_s is contained in $L^2(\mathbb{R}^2)$ it suffices to show that k_s as defined in (3.3.7) is square integrable on $\mathbb{R}^2$. Now, with a formal change in the order of integration and the use of Parseval's identity the far right-hand-side of (3.3.8) can be recast as

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\,d\eta\,dx_2 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\,dx_2\,d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\mathcal{F}\{k_s\}(\omega,\eta)|^2\,d\omega\,d\eta. \end{aligned} \tag{3.3.9}$$

The function $\mathcal{F}\{k_s\}(\cdot,\eta) : \mathbb{R} \to \mathbb{C}$ which appears in (3.3.9) represents, for each η in $\mathbb{R}$, the Fourier transform of $k_s(\cdot,\eta)$. This is supplied by

$$\begin{aligned}\mathcal{F}\{k_s\}(\omega,\eta) &= \int_{-\infty}^{+\infty} k_s(x_2,\eta)e^{-i\omega x_2}\,dx_2 \\ &= \hat{s}(\omega)\frac{1+i\omega\eta - e^{i\omega\eta}}{\eta^2} \quad \forall (\omega,\eta) \in \mathbb{R}^2, \end{aligned} \tag{3.3.10}$$

where $\hat{s} : \mathbb{R} \to \mathbb{C}$, in turn, is the Fourier transform of s:

$$\hat{s}(\omega) = \int_{-\infty}^{+\infty} s(x_2) e^{-i\omega x_2} \, dx_2 \quad \forall \omega \in \mathbb{R}. \tag{3.3.11}$$

A formal change in the order of integration on the far right-hand-side of (3.3.9) yields

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k_s^2(x_2, \eta) \, d\eta \, dx_2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\mathcal{F}\{k_s\}(\omega, \eta)|^2 \, d\eta \, d\omega,$$

so that, with the aid of (3.3.10) and (3.3.11),

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k_s^2(x_2, \eta) \, d\eta \, dx_2 &= \int_{-\infty}^{+\infty} |\hat{s}(\omega)|^2 \Big(\int_{-\infty}^{+\infty} \frac{|1 + i\eta\omega - e^{i\eta\omega}|^2}{2\pi \, \eta^4} \, d\eta \Big) \, d\omega \\ &= \int_{-\infty}^{+\infty} |\omega|^3 |\hat{s}(\omega)|^2 \, d\omega \int_{-\infty}^{+\infty} \frac{|1 + i\zeta - e^{i\zeta}|^2}{2\pi \, \zeta^4} \, d\zeta. \end{aligned} \tag{3.3.12}$$

Next, a straightforward application of contour integration yields the identity

$$\int_{-\infty}^{+\infty} \frac{|1 + i\zeta - e^{i\zeta}|^2}{2\pi \, \zeta^4} \, d\zeta = \frac{1}{3},$$

so that (3.3.12) implies that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k_s^2(x_2, \eta) \, d\eta \, dx_2 = \frac{1}{3} \int_{-\infty}^{+\infty} |\omega|^3 |\hat{s}(\omega)|^2 \, d\omega. \tag{3.3.13}$$

Note that, provided the integral on the right-hand-side of (3.3.13) exists, the two formal changes in the order of integration performed above are justified by Fubini's theorem.[5] Now, by (3.3.12), elementary identities involving the Fourier

[5] See HALMOS [20] for a statement and proof of Fubini's theorem.

transform of first and second derivatives, and Parseval's identity, (3.3.13) gives

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\,d\eta\,dx_2 = \frac{2\pi}{3}\int_{-\infty}^{+\infty}|s'(x_2)||s''(x_2)|\,dx_2. \tag{3.3.14}$$

Hence, provided s is an element of $\mathcal{A}$ it is apparent from (3.3.8), (3.3.14) and the Cauchy-Schwarz inequality that the kernel K_s is square integrable on $\mathbb{R}^2$ and, moreover, that $\|K_s\|_{L^2(\mathbb{R}^2)}$ can be estimated as follows:

$$\|K_s\|^2_{L^2(\mathbb{R}^2)} \leq \frac{2\pi}{3}\int_{-\infty}^{+\infty}|s'(x_2)||s''(x_2)|\,dx_2 \leq \tfrac{2\pi}{3}\,\|s'\|_{L^2(\mathbb{R})}\,\|s''\|_{L^2(\mathbb{R})}\,. \tag{3.3.15}$$

Observe that while the membership of s in $\mathcal{A}$ is certainly sufficient to ensure that K_s is an element of $L^2(\mathbb{R}^2)$ it is not necessary. An application of Hölder's inequality to (3.3.14) shows, for instance, that in order for K_s to be square integrable on $\mathbb{R}^2$ it is sufficient to require that s' and s'' be elements of $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, respectively, for some p in $[1,\infty)$ and conjugate exponent $q = p/(p-1)$. The choice $p = q = 2$ clearly leads to the ultimate estimate in (3.3.14). It is, moreover, clear that, provided s is in $\mathcal{A}$, the domain and range of the operator $\mathcal{M}_s$ introduced in (3.2.13) can both be taken as $L^2(\mathbb{R})$. An immediate consequence of this observation is that if ψ is in $L^2(\mathbb{R})$ then so also is $\mathcal{M}_s^n\psi$ for any natural number n.

Next, given that K_s is an element of $L^2(\mathbb{R}^2)$ for every function s in $\mathcal{A}$, consider the issue of proving that the forcing f_s is similarly square integrable on its domain of definition. Observe, first, that the singular behavior of I_s which appears in the definition (3.2.15) suggests that f_s can be linearly decomposed into a regular part and a Cauchy principal value part as follows:

$$\begin{aligned} f_s(x_2) &= \int_{-\infty}^{+\infty}\Big(I_s(x_2,\xi) - \frac{1}{x_2-\xi}\Big)s'(\xi)\,d\xi + \fint_{-\infty}^{+\infty}\frac{s'(\xi)\,d\xi}{x_2-\xi} \\ &= -\int_{-\infty}^{+\infty}K_s(x_2,\xi)\frac{s(x_2)-s(\xi)}{x_2-\xi}s'(\xi)\,d\xi + \fint_{-\infty}^{+\infty}\frac{s'(\xi)\,d\xi}{x_2-\xi} \quad \forall x_2 \in \mathbb{R}. \end{aligned} \tag{3.3.16}$$

It is now convenient to define functions $g_s : \mathbb{R} \to \mathbb{R}$ and $h_s : \mathbb{R} \to \mathbb{R}$ by

$$g_s(x_2) = -\int_{-\infty}^{+\infty} K_s(x_2,\xi)\frac{s(x_2)-s(\xi)}{x_2-\xi}s'(\xi)\,d\xi \quad \forall x_2 \in \mathbb{R}, \tag{3.3.17}$$

and

$$h_s(x_2) = \fint_{-\infty}^{+\infty} \frac{s'(\xi)\,d\xi}{x_2-\xi} \quad \forall x_2 \in \mathbb{R}, \tag{3.3.18}$$

respectively. Consider the term of the decomposition involving the function g_s. From the assumed smoothness of s, the difference quotient which appears in the integrand on the right hand side of (3.3.17) satisfies

$$\left|\frac{s(x_2)-s(\xi)}{x_2-\xi}\right| \le \|s'\|_{L^\infty(\mathbb{R})} \quad \forall (x_2,\xi) \in \mathbb{R}^2,$$

and, hence, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}\int_{-\infty}^{+\infty} g_s^2(x_2)\,dx_2 &\le \|s'\|^2_{L^\infty(\mathbb{R})} \int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty} K_s(x_2,\xi)s'(\xi)\,d\xi\Big)^2\,dx_2 \\ &\le \|s'\|^2_{L^\infty(\mathbb{R})} \int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty} K_s^2(x_2,\xi)\,d\xi\Big)\Big(\int_{-\infty}^{+\infty} |s'(\xi)|^2\,d\xi\Big)\,dx_2 \\ &\le \|s'\|^2_{L^\infty(\mathbb{R})}\,\|K_s\|^2_{L^2(\mathbb{R}^2)}\,\|s'\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{3.3.19}$$

Therefore, since s' is contained in $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$, (3.3.19) and the bound (3.3.15) on $\|K_s\|_{L^2(\mathbb{R}^2)}$ guarantee that g_s is square integrable on $\mathbb{R}$ and, furthermore, deliver the estimate

$$\|g_s\|_{L^2(\mathbb{R})} \le \sqrt{\tfrac{2\pi}{3}}\,\|s'\|_{L^\infty(\mathbb{R})}\,\|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\,\|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\,. \tag{3.3.20}$$

Next, observe from (3.3.18) that h_s is a scalar multiple of the *Hilbert transform* of s' and recall that the Hilbert transform maps the space $L^q(\mathbb{R})$ into itself for

q in $(1,\infty)$.[6] Consequently, the square integrability of s' on $\mathbb{R}$ shows that h_s is also an element of $L^2(\mathbb{R})$. Since the space of square integrable functions on $\mathbb{R}$ is linear it is clear that f_s, as the sum of two functions contained in $L^2(\mathbb{R})$, is itself an element of $L^2(\mathbb{R})$. Hence, under the assumption that s is a member of the set $\mathcal{A}$ the forcing f_s must necessarily be square integrable on $\mathbb{R}$. Note that—based on the last statement and earlier remarks pertaining to the domain and range of the integral operator $\mathcal{M}_s$—the function $\mathcal{M}_s^n f_s$ is contained in $L^2(\mathbb{R})$ for every non-negative integer n. Hence, if the Neumann series (3.3.3) is uniformly convergent then φ must also be square integrable on $\mathbb{R}$.

Up to this point it has been shown, as proposed above, that K_s and f_s are square integrable on their domains of definition for every s contained in $\mathcal{A} \supset \mathcal{I}$. Finally, it is readily apparent from the primary estimate of $\|K_s\|_{L^2(\mathbb{R}^2)}$ given in (3.3.15) that if s is an element of the set $\mathcal{I}$ introduced in (3.3.6) then inequality (3.3.5) must hold. Hence, (3.3.5) is satisfied for every element s of $\mathcal{I}$. To recapitulate, observe that if s is in $\mathcal{I}$ then there exists a unique solution to the corresponding integral equation (3.2.14) given by the appropriate Neumann series (3.3.3).

There may exist solutions to (3.2.15) which are not obtainable via the Neumann series construction. Since, however, the solution to the integral equation obtained via this construction is unique for each s in $\mathcal{I}$ it is apparent from the above discussion that, should there exist any solutions to (3.2.15) which can be aquired by alternate means, these must correspond to curves $\mathcal{C}_s$ described by functions s which do not belong to $\mathcal{I}$ (and may not even belong to $\mathcal{A}$). It is interesting to speculate on whether some of these solutions might correspond to states wherein the phase boundaries manifest large slopes and/or curvatures akin to those exhibited by the *fingers* found in studies of porous media, solidification, and crystal growth.

Prior to concluding this section a few comments regarding the uniqueness of

[6] This fact is established in RIESZ [25].

the solution to the reduced boundary value problem are in order. Let s be an element of $\mathcal{I}$. Then, by the foregoing results, the related reduced boundary value problem has a solution given by the appropriate Neumann series (3.3.3). It is known that the solution to the integral equation which issues from the reduced boundary value problem is unique. The uniqueness of the solution to the reduced boundary value problem is, however, still in question. It will now be shown that the solution of the reduced boundary value problem is unique—just as with the globally elliptic pairwise homogeneous equilibrium states—up to an arbitrary additive constant. To see this suppose that $v_1 : I\!R^2 \backslash \mathcal{C}_s \to I\!R$ and $v_2 : I\!R^2 \backslash \mathcal{C}_s \to I\!R$ are both solutions to the reduced boundary value problem corresponding to s in $\mathcal{I}$; define $w : I\!R^2 \setminus \mathcal{C}_s \to I\!R$ by their difference $(v_1 - v_2)$ on $I\!R^2 \setminus \mathcal{C}_s$. Then, from (3.1.17), w clearly satisfies the following boundary value problem:

$$\begin{gathered} w,_{\alpha\alpha} = 0 \quad \text{on} \quad I\!R^2 \setminus \mathcal{C}_s, \\ \mu_1 \frac{\partial w}{\partial n}\big(s(\cdot)+,\cdot\big) = \mu_2 \frac{\partial w}{\partial n}\big(s(\cdot)-,\cdot\big) \quad \text{on} \quad I\!R, \\ w\big(s(\cdot)+,\cdot\big) = w\big(s(\cdot)-,\cdot\big) \quad \text{on} \quad I\!R, \\ w,_\alpha(x_1,\cdot)\mathbf{e}_\alpha = o(1) \quad \text{as} \quad x_1 \to \pm\infty \quad \text{on} \quad I\!R. \end{gathered} \tag{3.3.21}$$

From (3.2.16) it is readily apparent that a solution to (3.3.21) is provided, modulo an arbitrary additive constant, by

$$w(x_1,x_2) = \frac{\gamma_l - \gamma_r}{2\pi} \int_{-\infty}^{+\infty} G_1^s(x_1,x_2,\xi)\psi(\xi)\, d\xi \quad \forall (x_1,x_2) \in I\!R^2. \tag{3.3.22}$$

where $G_1^s : (I\!R^2 \setminus \mathcal{C}_s) \times I\!R \to I\!R$ is given by $(3.2.3)_1$, and $\psi : I\!R \to I\!R$ satisfies

$$\psi + \lambda \mathcal{M}_s \psi = 0 \quad \text{on} \quad I\!R. \tag{3.3.23}$$

It is clear, based on the assumption that s is in $\mathcal{I}$, that λ cannot be an eigenvalue of the operator $\mathcal{M}_s$. Hence, (3.3.23) has only the zero solution. Now, since the representation (3.3.22) for w is modulo an arbitrary additive constant, the functions v_1 and v_2 can differ at most, as stated above, by a constant.

During the course of this section it has been shown that, for each function s contained in $\mathcal{I}$ there exists, up to an arbitrary additive constant, a unique solution to the reduced boundary value problem (3.1.17). This solution corresponds to a deformation involving a non-planar interface $\mathcal{Q}_s$. Inasmuch as the reduced phase segregation requirement (3.1.18) has not yet been applied it is still unclear whether any of the aforementioned solutions give rise to globally elliptic two-phase equilibrium states. The next section will, therefore, focus on characterizing a subset of $\mathcal{I}$ for which there exist solutions to the reduced boundary value problem augmented by the (reduced) constraints of phase segregation. If a function s belongs to this subset of $\mathcal{I}$ the interface $\mathcal{Q}_s$ will qualify as a phase boundary.

3.4. Implementation and satisfaction of the reduced phase segregation requirement. Let s be an element of $\mathcal{I}$ and suppose that φ and (up to an additive constant) v are the corresponding solutions to the integral equation (3.2.15) and the reduced boundary value problem (3.1.17). If v is to provide—through (3.1.16)—a solution u to the constrained boundary value problem its gradient must comply with the reduced strain constraints (3.1.18). Let $\kappa : \mathbb{R}^2 \setminus \mathcal{C}_s \to \bar{\mathbb{R}}_+$ denote the *reduced shear strain field* given by

$$\kappa = v_{,\alpha} v_{,\alpha} \quad \text{on} \quad \mathbb{R}^2 \setminus \mathcal{C}_s. \qquad (3.4.1)$$

Certainly $|v_{,1}|$ must be less than or equal to κ on $\mathbb{R}^2 \setminus \mathcal{C}_s$; hence, if the reduced shear strain field complies with

$$\kappa^{\frac{1}{2}} < \min\{\underset{*}{\gamma} - \gamma_r, \gamma_l - \overset{*}{\gamma}\} \quad \text{on} \quad \mathbb{R}^2 \setminus \mathcal{C}_s, \qquad (3.4.2)$$

then both of the inequalities which comprise the reduced phase segregation requirement (3.1.18) will be satisfied. Notice that the foregoing condition is sufficient but not necessary to ensure the segregation of phases. It may, consequently, lead to overly conservative restrictions. Despite the strong restrictions which may be imposed by enforcing (3.4.2) in lieu of (3.1.18), it will be demonstrated that there exists a non-empty subset of $\mathcal{I}$ each element of which gives rise to a soluble reduced constrained boundary value problem with a reduced shear strain field κ that allows their satisfaction.

The following simple calculation shows that κ is subharmonic on $I\!R^2 \setminus \mathcal{C}_s$:

$$\begin{aligned}\kappa,_{\alpha\alpha} &= (v,_\beta v,_\beta),_{\alpha\alpha} = 2(v,_{\alpha\beta} v,_\beta),_\alpha \\ &= 2(v,_{\alpha\beta} v,_{\alpha\beta} + v,_{\alpha\alpha\beta} v,_\beta) \\ &= 2v,_{\alpha\beta} v,_{\alpha\beta} \geq 0 \quad \text{on} \quad I\!R^2 \setminus \mathcal{C}_s.\end{aligned}$$

The subharmonicity of κ on $I\!R^2 \setminus \mathcal{C}_s$ implies, given the decay properties of the gradient of v embodied by $(3.1.17)_4$, that its maximum values on $\mathring{\mathcal{D}}_s^l$ and $\mathring{\mathcal{D}}_s^r$ must occur in the limits approaching the curve $\mathcal{C}_s$ from the high and low strain sides, respectively. Hence, in determining whether the reduced shear strain field κ satisfies (3.4.2) it is sufficient to analyze its limiting behavior on either side of the curve $\mathcal{C}_s$. A convenient approach to this is afforded by examining the limits of the normal and tangential derivatives of v on either side of $\mathcal{C}_s$. From (3.2.6), (3.2.11) and (3.2.12) it is evident that the limiting values of the normal derivative of the the reduced out-of-plane displacement field are given by

$$\frac{\partial v}{\partial n}(s(x_2)-, x_2) = -\frac{\gamma_l \varphi(x_2)}{\sqrt{1+s'(x_2)^2}} \quad \forall x_2 \in I\!R \tag{3.4.3}$$

on the high strain side of $\mathcal{C}_s$, and

$$\frac{\partial v}{\partial n}(s(x_2)+, x_2) = -\frac{\gamma_r \varphi(x_2)}{\sqrt{1+s'(x_2)^2}} \quad \forall x_2 \in I\!R \tag{3.4.4}$$

on the low strain side of $\mathcal{C}_s$.

Let $\mathbf{l} : I\!R \to \mathcal{N}$ designate the unit tangent vector to $\mathcal{Q}_s$ defined by

$$\mathbf{l}(x_2) = \frac{s'(x_2)\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{1+s'(x_2)^2}} \quad \forall x_2 \in I\!R. \tag{3.4.5}$$

Then a calculation very similar to that used in obtaining (3.2.4) yields

$$\begin{aligned}\frac{\partial v}{\partial l}(s(x_2)\pm,x_2) = {} & \frac{\gamma_l-\gamma_r}{2\pi\sqrt{1+s'(x_2)^2}}\int_{-\infty}^{+\infty} K_s(x_2,\xi)s'(\xi)\,d\xi \\ & + \frac{\gamma_l-\gamma_r}{2\pi\sqrt{1+s'(x_2)^2}}\,\text{⨍}_{-\infty}^{+\infty} I_s(x_2,\xi)\varphi(\xi)\,d\xi \\ & \pm \frac{\gamma_l-\gamma_r}{2\sqrt{1+s'(x_2)^2}}s'(x_2) \quad \forall x_2\in \mathbb{R} \end{aligned} \tag{3.4.6}$$

for the limits of the tangential derivative of v on either side of the curve C_s. Here, as in (3.2.4), $I_s(x_2,\cdot)$ and $K_s(x_2,\cdot)$ are given, for each x_2 in $\mathbb{R}$, by (3.2.5).

Turn now to the estimation of (3.4.3), (3.4.4) and $(3.4.6)_\pm$. Consider the limits of the normal derivative first. The following pair of inequalities follow immediately from (3.4.3) and (3.4.4):

$$\begin{aligned}\left|\frac{\partial v}{\partial n}(s(x_2)-,x_2)\right| &\le \gamma_l\,|\varphi(x_2)| \le \gamma_l\,\|\varphi\|_{L^\infty(\mathbb{R})} \quad \forall x_2\in\mathbb{R},\\ \left|\frac{\partial v}{\partial n}(s(x_2)+,x_2)\right| &\le \gamma_r\,|\varphi(x_2)| \le \gamma_r\,\|\varphi\|_{L^\infty(\mathbb{R})} \quad \forall x_2\in\mathbb{R}.\end{aligned} \tag{3.4.7}$$

Hence, in order to bound the limits of the normal derivative of the reduced displacement field on either side of C_s it is only necessary to estimate the L^∞ norm of φ over $\mathbb{R}$. In Appendix A it is shown that if $\psi:\mathbb{R}\to\mathbb{R}$ is an element of the set $\mathcal{V}$ defined in (3.1.7) then $\|\psi\|_{L^\infty(\mathbb{R})}$ exists and can be bounded as follows:

$$\|\psi\|_{L^\infty(\mathbb{R})} \le 2(\tfrac{\pi}{2})^{\frac14}\,\|\psi\|^{\frac12}_{L^2(\mathbb{R})}\,\|\psi'\|^{\frac12}_{L^2(\mathbb{R})}\,. \tag{3.4.8}$$

Recall from Section 3.3 that if s is contained in $\mathcal{I}$ then φ is square integrable on $\mathbb{R}$. Therefore, if φ' exists and is square integrable on $\mathbb{R}$ the inequality displayed in (3.4.8) can be used—with ψ replaced by φ—to obtain an estimate for the L^∞ norm of φ over $\mathbb{R}$. Suppose, from now on, that s is a three times continuously differentiable element of $\mathcal{I}$ with a square integrable third derivative on $\mathbb{R}$. It can be readily shown that this is sufficient to guarantee that φ' exists and is an element of $L^2(\mathbb{R})$. In Appendix B it is demonstrated that the L^2 norms of φ and φ' over $\mathbb{R}$ can be estimated, respectively, by

$$\|\varphi\|_{L^2(\mathbb{R})} \leq \frac{c_1(1+\|s'\|_{L^2(\mathbb{R})}\|s''\|_{L^2(\mathbb{R})})\|s'\|_{L^2(\mathbb{R})}}{1-\lambda\sqrt{\frac{2\pi}{3}}\|s'\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}}, \tag{3.4.9}$$

and

$$\begin{aligned}\|\varphi'\|_{L^2(\mathbb{R})} \leq c_2\big[&\|s''\|_{L^2(\mathbb{R})} + \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\|s''\|_{L^2(\mathbb{R})}\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \\ &+ \|s'\|^{2}_{L^2(\mathbb{R})}\|s''\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \\ &+ \|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})} \\ &+ \|s'\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|s''\|_{L^2(\mathbb{R})}\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})}\big].\end{aligned} \tag{3.4.10}$$

The constants c_1 and c_2 which appear in (3.4.9) and (3.4.10) are positive real numbers entirely independent of s. Note that the denominator in (3.4.9) is strictly positive since s is an element of $\mathcal{I}$. With the aid of (3.4.9) the estimate (3.4.10) for $\|\varphi'\|_{L^2(\mathbb{R})}$ can be expressed completely in terms of $\|s'\|_{L^2(\mathbb{R})}$, $\|s''\|_{L^2(\mathbb{R})}$ and $\|s'''\|_{L^2(\mathbb{R})}$. Hence, (3.4.9), (3.4.10), (3.4.8) and (3.4.7) give estimates for the moduli of the limiting values of the normal component of the gradient of v on either side of $\mathcal{C}_s$ in terms of the L^2 norms of the first three derivatives of s over $\mathbb{R}$.

To provide an estimate for κ it remains to obtain bounds on the limiting values of the tangential derivative of v. Toward this objective, introduce a function $\Lambda : \mathbb{R} \rightarrow \bar{\mathbb{R}}_+$ by

$$\Lambda(x_2) = \int_{-\infty}^{+\infty} K_s(x_2,\xi)s'(\xi)\,d\xi + \fint_{-\infty}^{+\infty} I_s(x_2,\xi)\varphi(\xi)\,d\xi + \pi s'(x_2) \quad \forall x_2 \in \mathbb{R}. \tag{3.4.11}$$

Then it is clear from (3.4.6) and (3.4.11) that

$$\left|\frac{\partial v}{\partial l}\left(s(x_2)\pm, x_2\right)\right| \leq \frac{\gamma_l - \gamma_r}{2\pi}|\Lambda(x_2)| \leq \frac{\gamma_l - \gamma_r}{2\pi}\|\Lambda\|_{L^\infty(\mathbb{R})} \quad \forall x_2 \in \mathbb{R}. \tag{3.4.12}$$

Under the current assumption that s''' exists and is square integrable on $\mathbb{R}$ it can be shown that both Λ and Λ' are elements of $L^2(\mathbb{R})$. Inequality (3.4.8) can, therefore, be applied with ψ replaced by Λ; this leads, through (3.4.12), to a bound on the limiting values of the tangential component of the gradient of v on either side of C_s. Given the bounds (3.4.9) and (3.4.10) for $\|\varphi\|_{L^2(\mathbb{R})}$ and $\|\varphi'\|_{L^2(\mathbb{R})}$ it is straightforward to derive estimates for the L^2 norms of Λ and Λ' over $\mathbb{R}$ in the form

$$\begin{aligned}\|\Lambda\|_{L^2(\mathbb{R})} \leq c_3\Big[&\|s'\|_{L^2(\mathbb{R})} + \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} + \|\varphi\|_{L^2(\mathbb{R})} \\ &+ \|s'\|_{L^2(\mathbb{R})}\|s''\|_{L^2(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})}\Big],\end{aligned} \tag{3.4.13}$$

and

$$\begin{aligned}\|\Lambda'\|_{L^2(\mathbb{R})} \leq c_4\Big[&\|s''\|_{L^2(\mathbb{R})} + \|s'\|_{L^2(\mathbb{R})}\|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \\ &+ \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\|s''\|_{L^2(\mathbb{R})}\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \\ &+ \|s'\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|s''\|_{L^2(\mathbb{R})}\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})} \\ &+ \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\|s''\|^{2}_{L^2(\mathbb{R})}\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})} \\ &+ \|\varphi\|_{L^2(\mathbb{R})}\Big].\end{aligned} \tag{3.4.14}$$

respectively. Here c_3 and c_4 are positive real numbers which, like c_1 and c_2, are independent of s. Note that the L^2 norm of φ over $\mathbb{R}$ appears in the estimates for both $\|\Lambda\|_{L^2(\mathbb{R})}$ and $\|\Lambda'\|_{L^2(\mathbb{R})}$. Hence, with the aid of (3.4.9) these can be rewritten solely in terms of the L^2 norms of s', s'' and s''' over $\mathbb{R}$. As with the normal component of the gradient of v, a bound on the limiting values of the tangential component of the gradient of v on either side of C_s is given, from

(3.4.13), (3.4.14), (3.4.11), (3.4.12), (3.4.9) and (3.4.8) in terms of the L^2 norms of the first three derivatives of s over $I\!R$.

From the preceeding discussion it is clear that κ can be made arbitrarily small by reducing the size of $\|s'\|_{L^2(I\!R)}$, $\|s''\|_{L^2(I\!R)}$ and $\|s'''\|_{L^2(I\!R)}$. More specifically, (3.4.9), (3.4.10), (3.4.13) and (3.4.14) can be substituted appropriately into (3.4.8) to compute, using (3.4.7) and (3.4.12), an upper bound for κ on $\mathring{\mathcal{D}}_s^l \cup \mathring{\mathcal{D}}_s^r$ in the form

$$\kappa \leq \Gamma^2\left(\|s'\|_{L^2(I\!R)}, \|s''\|_{L^2(I\!R)}, \|s'''\|_{L^2(I\!R)}\right) =: \Gamma_s^2. \tag{3.4.15}$$

Define a set of functions $\mathcal{J}$ by

$$\mathcal{J} = \{s \in \mathcal{A} \mid \Gamma_s < \min\{\underset{*}{\gamma} - \gamma_r, \gamma_l - \overset{*}{\gamma}\}\} \cap \mathcal{X}, \tag{3.4.16}$$

where $\mathcal{X}$ is given by

$$\mathcal{X} = \{\psi : I\!R \to I\!R \mid \psi \in C^3(I\!R), \psi^{(n)} \in L^2(I\!R), n = 0, 1, 2, 3\}. \tag{3.4.17}$$

Then, from (3.4.2), (3.4.15) and (3.4.16), provided s is an element of the set Π defined by

$$\Pi = \mathcal{I} \cap \mathcal{J}, \tag{3.4.18}$$

there exists a solution—unique up to an additive constant—to the associated constrained boundary value problem (3.1.17)–(3.1.18) with phase boundary $\mathcal{Q}_s$. This solution defines a globally elliptic inhomogeneous two-phase equilibrium state and, therefore, establishes the existence result sought after here. Note—from the definition of Π—that the approach delineated above provides a means by which an uncountably infinite number of such states can be constructed. It is significant that the loading conditions related at the outset of Section 3.1 give rise to not only a globally elliptic pairwise homogeneous equilibrium state but also an uncountably infinite number of globally elliptic inhomogeneous two-phase equilibria. This result clearly reflects the underlying non-linearity of the problem.

As remarked earlier, there may exist globally elliptic inhomogeneous two-phase equilibrium states which cannot be constructed via the approach taken above and, thus, do not correspond to phase boundaries in the set Π. Under relaxed smoothness assumptions on s there may, however, exist still other globally elliptic inhomogeneous two-phase equilibrium states which can be constructed via Neumann series. In particular, under such relaxed circumstances, it may be possible to demonstrate the existence of states wherein the associated phase boundaries exhibit geometrical irregularities such as corners or cusps (recall that the existence of equilibria involving cusped phase boundaries has been established by ROSAKIS [27] in his work involving a special anisotropic material).

3.5. An example. Given the results of Sections 3.3 and 3.4 it is illuminating to consider a particular class of functions in the set $\mathcal{A}$ defined by (3.1.6) and determine a subset of this class of functions which are also contained in the set Π defined by (3.4.18). Toward this end, let s is given by

$$s(x_2) = \frac{h}{1 + (\frac{x_2}{\ell})^2} \quad \forall x_2 \in \mathbb{R}, \tag{3.5.1}$$

where h and ℓ are both positive constants. A representative graph of s is displayed in Figure 3. Note that s is clearly an infinitely differentiable element of the set $\mathcal{A}$ for all values of the parameters h and ℓ. Let the ratio of h to ℓ be denoted by ϵ. The kernel K_s associated with s must, as a consequence of the results of Section 3.3, be square integrable on $\mathbb{R}^2$. In fact, from (3.3.15) one finds, after a bit of calculation, that the L^2 norm of K_s over $\mathbb{R}^2$ can be bounded as follows:

$$\|K_s\|_{L^2(\mathbb{R}^2)} \leq \|k_s\|_{L^2(\mathbb{R}^2)} = \frac{\pi}{2}\epsilon. \tag{3.5.2}$$

In the latter, k_s is as defined in (3.3.7). Hence, it is clear from (3.2.10) and (3.3.6) that if—for a given choice of the moduli μ_1 and μ_2 which define the elliptic phases of the three-phase material with shear stress response function

τ_p—the parameter ϵ satisfies

$$\epsilon < 2\,\frac{\mu_1 + \mu_2}{\mu_1 - \mu_2}, \tag{3.5.3}$$

then the function s introduced above will be an element of $\mathcal{I} \cap C^\infty(\mathbb{R})$. Assume, from now on, that s as defined in (3.5.1) is such that (3.5.3) holds. Then the Neumann series (3.3.3) converges uniformly on $\mathbb{R}$ to a solution of the integral equation. With the aid of the decomposition (3.3.16) of the forcing f_s, the solution of the integral equation can be expressed as

$$\varphi(x_2) = \lambda \sum_{n=0}^{\infty} (-\lambda)^n (\mathcal{M}_s^n h_s)(x_2) + \lambda \sum_{n=0}^{\infty} (-\lambda)^n (\mathcal{M}_s^n g_s)(x_2) \quad \forall x_2 \in \mathbb{R}, \tag{3.5.4}$$

where $\mathcal{M}_s$ is as defined in (3.2.13). Given τ_p and, hence, the moduli μ_1 and μ_2, it can be readily shown that, for every ϵ which satisfies (3.5.3), the following order relations hold for each non-negative integer n:

$$\begin{aligned} (\mathcal{M}_s^n h_s)(x_2) &= O(\epsilon^{2n+1}) \quad \forall x_2 \in \mathbb{R}, \\ (\mathcal{M}_s^n g_s)(x_2) &= O(\epsilon^{2n+2}) \quad \forall x_2 \in \mathbb{R}, \end{aligned} \tag{3.5.5}$$

Therefore, facilitated by (3.5.4) and (3.5.5), φ can be represented in the form

$$\varphi(x_2) = 2\lambda h \ell^2 \int_{-\infty}^{+\infty} \frac{\xi\, d\xi}{(l^2 + \xi^2)^2 (\xi - x_2)} + O(\epsilon^2) \quad \forall x_2 \in \mathbb{R}. \tag{3.5.6}$$

An application of contour integration yields

$$\int_{-\infty}^{+\infty} \frac{\xi\, d\xi}{(l^2 + \xi^2)^2 (\xi - x_2)} = \frac{\pi}{2\ell^3} \frac{1 - (\frac{x_2}{\ell})^2}{(1 + (\frac{x_2}{\ell})^2)^2} \quad \forall x_2 \in \mathbb{R},$$

so that (3.5.6) becomes, with the aid of (3.2.11),

$$\varphi(x_2) = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \frac{1 - (\frac{x_2}{\ell})^2}{(1 + (\frac{x_2}{\ell})^2)^2}\, \epsilon + O(\epsilon^2) \quad \forall x_2 \in \mathbb{R}. \tag{3.5.7}$$

Observe that (3.5.7) and (3.5.1) can be used in (3.2.16) and (3.1.16) to construct, for each appropriate pair (h, ℓ), an approximate solution to the reduced boundary value problem (3.1.17). Now, substitution of (3.5.1) and (3.5.7) into (3.4.3) and (3.4.4) delivers the following formulae for the limiting values of the normal derivative of v on either side of $\mathcal{C}_s$:

$$\begin{aligned}
\frac{\partial v}{\partial n}(s(x_2)-, x_2) &= -\frac{\mu_1(\gamma_l - \gamma_r)}{\mu_1 + \mu_2} \frac{1 - (\frac{x_2}{\ell})^2}{(1 + (\frac{x_2}{\ell})^2)^2}\, \epsilon + O(\epsilon^2) \quad \forall x_2 \in I\!R, \\
\frac{\partial v}{\partial n}(s(x_2)+, x_2) &= -\frac{\mu_2(\gamma_l - \gamma_r)}{\mu_1 + \mu_2} \frac{1 - (\frac{x_2}{\ell})^2}{(1 + (\frac{x_2}{\ell})^2)^2}\, \epsilon + O(\epsilon^2) \quad \forall x_2 \in I\!R.
\end{aligned} \tag{3.5.8}$$

Similarly, substitution of (3.5.1) and (3.5.7) into (3.4.6) gives rise to the following expressions for the limiting values of the tangential derivative of v on either side of $\mathcal{C}_s$:

$$\begin{aligned}
\frac{\partial v}{\partial l}(s(x_2)-, x_2) &= \frac{2\mu_1(\gamma_l - \gamma_r)}{\mu_1 + \mu_2} \frac{\frac{x_2}{\ell}}{(1 + (\frac{x_2}{\ell})^2)^2}\, \epsilon + O(\epsilon^2) \quad \forall x_2 \in I\!R, \\
\frac{\partial v}{\partial l}(s(x_2)+, x_2) &= -\frac{2\mu_2(\gamma_l - \gamma_r)}{\mu_1 + \mu_2} \frac{\frac{x_2}{\ell}}{(1 + (\frac{x_2}{\ell})^2)^2}\, \epsilon + O(\epsilon^2) \quad \forall x_2 \in I\!R.
\end{aligned} \tag{3.5.9}$$

The expansions in (3.5.8) and (3.5.9) show the dependence on ϵ of the limiting values of the normal and tangential derivatives of v on either side of $\mathcal{C}_s$. They readily imply that, for the function s indicated in (3.5.1), the associated reduced shear strain field κ introduced via (3.4.1) satisfies

$$\kappa = O(\epsilon^2) \quad \text{on} \quad I\!R \setminus \mathcal{C}_s. \tag{3.5.10}$$

An immediate consequence of (3.5.10) is that if ϵ is made sufficiently small the reduced phase segregation requirement (3.1.18) will then be satisfied and the relevant function s will be contained in Π.

Note, alternatively, that the L^2 norms, over $I\!R$, of the first three derivatives of the function s defined in (3.5.1) can be computed directly to give

$$\|s'\|_{L^2(I\!R)} = \sqrt{\tfrac{\pi}{4}}\frac{h}{\ell^{\frac{1}{2}}}, \quad \|s''\|_{L^2(I\!R)} = \sqrt{\tfrac{3\pi}{4}}\frac{h}{\ell^{\frac{3}{2}}}, \quad \|s'''\|_{L^2(I\!R)} = \sqrt{\tfrac{45\pi}{8}}\frac{h}{\ell^{\frac{5}{2}}}.$$

If the foregoing are substituted in the estimates (3.4.9), (3.4.10), (3.4.13), and (3.4.14) then it is straightforward to show, with appropriate use of (3.4.7), that the quantity $\mathit{\Gamma}_s$ defined in (3.4.15) is of order ϵ—which corroborates the asymptotic results obtained above. Hence, if the parameters h and ℓ which appear in the definition of s are chosen so that ϵ is sufficiently small, $\mathit{\Gamma}_s$ will satisfy

$$\mathit{\Gamma}_s < \min\{\underset{*}{\gamma} - \gamma_r, \gamma_l - \overset{*}{\gamma}\}, \tag{3.5.11}$$

and s will be an element of $\mathit{\Pi}$.

In either case a class of phase boundaries $\mathcal{Q}_s$ for which $\epsilon = h/\ell$ is sufficiently small emerges from the class of functions s given by (3.5.1). The reduced out-of-plane displacement field corresponding to each such s, is by using (3.5.1) and (3.5.4)–(3.5.7) in (3.2.16), given approximately by

$$\begin{aligned} v(x_1, x_2) \sim \; & \frac{\gamma_l - \gamma_r}{2\pi} \frac{h x_2}{\ell} \int_{-\infty}^{+\infty} \frac{d\xi}{(1+\xi^2)((\frac{x_1}{\ell})^2 + (\frac{x_2}{\ell} - \xi)^2)} \\ & + \frac{\gamma_l - \gamma_r}{2\pi} h \int_{-\infty}^{+\infty} \frac{1-\xi^2}{(1+\xi^2)^2} \ln\sqrt{(\tfrac{x_1}{\ell})^2 + (\tfrac{x_2}{\ell} - \xi)^2}\, d\xi \\ & \qquad \forall (x_1, x_2) \in \mathbb{R}^2 \setminus \mathcal{C}_s. \end{aligned} \tag{3.5.12}$$

An approximation to the corresponding primitive out-of-plane displacement field u is then calculated easily by substituting (3.5.12) appropriately into (3.1.16).

4. STUDY OF PHASE BOUNDARY KINETICS AND STABILITY

This chapter relies on the concept of a quasistatic motion introduced in Section 2.3. Recall that, in addition to the shear stress response function τ_p, the constitutive characterization of the material at hand includes a kinetic response function $\tilde{V}$ which, in the setting of a quasistatic motion, dictates the dependence of the normal velocity of a particle located at a point on a phase boundary on the driving traction acting at that point. Given the distribution of driving traction on a particular phase boundary it is therefore—through the kinetic relation—possible to discuss the kinetics and stability of that phase boundary in slow motions. For illustrative purposes this is done below in the context of the specific class of phase boundaries studied in Section 3.5. A similar analysis could, in principle, be performed for any function s in Π.

In Section 4.1 the driving traction f which acts on such a phase boundary is derived. It is demonstrated that f is composed of the sum of an *ambient* term f_0 which corresponds to the constant driving traction which would act on a planar phase boundary corresponding to a suitable globally elliptic pairwise homogeneous equilibrium state and higher order terms which represent the increment to the driving traction resulting from the non-planarity of the surface $\mathcal{Q}_s$.

In Section 4.2 the ambient term f_0 and the most significant non-constant term in the driving traction f are used in conjunction with $\tilde{V}$ and the kinetic relation to address phase boundary kinetics and stability.

4.1. The driving traction acting on an arbitrary element of a specific class of phase boundaries. Let s be given by (3.5.1) with $\epsilon = h/\ell$ chosen so that (3.5.11) is fulfilled. As discussed in Section 3.5, $\mathcal{Q}_s$ is then a phase boundary. Consider the computation of the driving traction acting on $\mathcal{Q}_s$. The simple manner in which $\mathcal{Q}_s$ can be parameterized implies that—in the present context of antiplane shear—the expression for the driving traction provided in (2.4.6) can be written as a function of one variable. Furthermore, it can be shown without difficulty that, for the special three-phase material with shear stress response function τ_p, the driving traction f is given by

$$f = \frac{\mu_1 - \mu_2}{2}\left(u_{,\alpha}(s(\cdot)+,\cdot)u_{,\alpha}(s(\cdot)-,\cdot) - \overset{*}{\gamma}\underset{*}{\gamma}\right) \quad \text{on} \quad \mathbb{R}. \tag{4.1.1}$$

The definition of v supplied in (3.1.16) readily furnishes the following expressions for $u_{,\alpha}(s(\cdot)\pm,\cdot)\mathbf{e}_\alpha$ on $\mathbb{R}$:

$$\begin{aligned} u_{,\alpha}(s(\cdot)-,\cdot)\mathbf{e}_\alpha &= \gamma_l \mathbf{e}_1 + v_{,\alpha}(s(\cdot)-,\cdot)\mathbf{e}_\alpha \\ &= \gamma_l \mathbf{e}_1 + \frac{\partial v}{\partial n}(s(\cdot)-,\cdot)\mathbf{n} + \frac{\partial v}{\partial l}(s(\cdot)-,\cdot)\mathbf{l} \quad \text{on} \quad \mathbb{R}, \\ u_{,\alpha}(s(\cdot)+,\cdot)\mathbf{e}_\alpha &= \gamma_r \mathbf{e}_1 + v_{,\alpha}(s(\cdot)+,\cdot)\mathbf{e}_\alpha \\ &= \gamma_r \mathbf{e}_1 + \frac{\partial v}{\partial n}(s(\cdot)+,\cdot)\mathbf{n} + \frac{\partial v}{\partial l}(s(\cdot)+,\cdot)\mathbf{l} \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{4.1.2}$$

It is possible to show, from (3.5.1), (3.1.7) and (3.4.5), that the unit normal and tangent vectors to $\mathcal{Q}_s$ satisfy the following order relations:

$$\mathbf{n}\cdot\mathbf{e}_1 = 1 + O(\epsilon) \quad \text{on} \quad \mathbb{R}, \quad \mathbf{l}\cdot\mathbf{e}_1 = O(\epsilon) \quad \text{on} \quad \mathbb{R}. \tag{4.1.3}$$

Hence, (3.1.16), (4.1.2) and (4.1.3) yield

$$\begin{aligned} u_{,\alpha}(s(\cdot)+,\cdot)u_{,\alpha}(s(\cdot)-,\cdot) = \gamma_l\gamma_r + \gamma_l \frac{\partial v}{\partial n}(s(\cdot)+,\cdot) + \gamma_r \frac{\partial v}{\partial n}(s(\cdot)-,\cdot) \\ + O(\epsilon^2) \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{4.1.4}$$

Now, if (3.5.8) is inserted appropriately in (4.1.4) and the result is substituted into (4.1.1) the driving traction along the phase boundary $\mathcal{Q}_s$ can be expanded in powers of ϵ as follows

$$f(x_2) = f_0 - \nu \frac{1 - (\frac{x_2}{\ell})^2}{(1 + (\frac{x_2}{\ell})^2)^2}\,\epsilon + O(\epsilon^2) \quad \forall x_2 \in \mathbb{R}, \tag{4.1.5}$$

with the constants f_0 and ν given by

$$f_0 = \frac{\mu_1 - \mu_2}{2}(\gamma_l\gamma_r - \overset{*}{\gamma}\underset{*}{\gamma}), \quad \nu = \frac{(\mu_1 - \mu_2)^2}{(\mu_1 + \mu_2)}\gamma_l\gamma_r. \tag{4.1.6}$$

f_0 is the ambient or *base* driving traction which would hold on a planar phase boundary associated with a globally elliptic pairwise homogeneous equilibrium state with displacement field (3.1.1). Observe that since

$$-\frac{1}{8} \leq \frac{1-(\frac{x_2}{\ell})^2}{(1+(\frac{x_2}{\ell})^2)^2} \leq 1 \quad \forall x_2 \in I\!R,$$

the coefficient of the $O(\epsilon)$ term in the expansion of f provides a bounded correction to the base term f_0. This term is, for small ϵ, the most significant contribution to f which results from the deviation in the geometry of $\mathcal{Q}_s$ from planar. Note that f_0 can take on any real value whereas ν must be positive. Furthermore, because the difference $(\mu_1 - \mu_2)$ is squared in $(4.1.6)_2$, the positivity of ν holds even if the affiliations of the moduli μ_1 and μ_2 are reversed so as to be associated with the high and low strain phases of the material at hand. From (4.1.5) it is apparent that the $O(\epsilon)$ contribution to the distribution of driving traction on the phase boundary $\mathcal{Q}_s$ under consideration is a complicated function of position. See Figure 4 for the graph of the function corresponding to the $O(\epsilon)$ term in the expansion of f. The results of ABEYARATNE [3], imply that the equilibria at hand must, in general, constitute metastable states.

4.2. Kinetics and stability of an arbitrary element of a particular class of phase boundaries. Given the expansion (4.1.5) consider the issue of analyzing the kinetics and stability of an arbitrary element of the class of phase boundaries at hand. Suppose that $\tilde{V}$ is twice continuously differentiable on $I\!R$. Then, as noted in the remarks following (2.3.9), the admissibility of $\tilde{V}$ requires that $\tilde{V}(0) = 0$ and $\tilde{V}'(0) \geq 0$. Assume, for the purposes of this discussion, that f_0 is not a critical point of $\tilde{V}$; note that this requires, in particular, that if $f_0 = 0$ then $\tilde{V}'(0) > 0$. See Figure 5 for examples of graphs of monotonic and non-monotonic kinetic response functions.

Let $\mathcal{Q}_s^c$ and $\mathcal{Q}_s^f$ be those subsets of the phase boundary $\mathcal{Q}_s$ defined as follows:

$$\mathcal{Q}_s^c = \{\mathbf{x} \in \mathcal{Q}_s \,|\, x_2 \in (-\ell, \ell)\}, \quad \mathcal{Q}_s^f = \mathcal{Q}_s \setminus \bar{\mathcal{Q}}_s^c.$$

Observe, from (4.1.5), that $\mathcal{Q}_s^c$ and $\mathcal{Q}_s^f$ correspond to the portions of $\mathcal{Q}_s$ upon which the $O(\epsilon)$ correction to f_0 is negative and positive, respectively. Note, also, that $\mathcal{Q}_s^c$ is the subset of $\mathcal{Q}_s$ whose geometry deviates most significantly from planar—that is, roughly speaking, the major portion of the bump which is associated with the graph of the function s given by (3.5.1) corresponds to the image of $(-\ell, \ell)$ under s. See Figure 4.

From the assumed smoothness of $\tilde{V}$, (4.1.5) and Taylor's theorem the normal velocity at a point on the phase boundary is given by

$$V_n(\mathbf{x}) = \tilde{V}(f(x_2)) = \tilde{V}(f_0) - \nu\tilde{V}'(f_0)\frac{1-(\frac{x_2}{\ell})^2}{(1+(\frac{x_2}{\ell})^2)^2}\,\epsilon + O(\epsilon^2) \quad \forall \mathbf{x} \in \mathcal{Q}_s. \quad (4.2.1)$$

In determining the kinetic tendencies of $\mathcal{Q}_s$ it is now convenient to consider two cases. These are $f_0 = 0$ and $f_0 \neq 0$. Note that the base globally elliptic pairwise homogeneous equilibrium state is mechanically stable only in the first of these two cases.

Consider the case $f_0 = 0$. Then, since $\tilde{V}(0) = 0$, (4.2.1) imples that

$$V_n(\mathbf{x}) = -\nu\tilde{V}'(0)\frac{1-(\frac{x_2}{\ell})^2}{(1+(\frac{x_2}{\ell})^2)^2}\,\epsilon + O(\epsilon^2) \quad \forall \mathbf{x} \in \mathcal{Q}_s. \quad (4.2.2)$$

Since $\tilde{V}'(0)$ and ν are positive, it is apparent from (4.2.2) and $(4.1.3)_1$ that, to most significant order in ϵ, all points on $\mathcal{Q}_s^c$ tend to move in the $-\mathbf{e}_1$ direction while all points on $\mathcal{Q}_s^f$ tend to move in the $\mathbf{e}_1$ direction. That is, if $f_0 = 0$ then the phase boundary displays a proclivity to become planar.

Now consider the case where $f_0 \neq 0$. Suppose, first, that f_0 is positive. As such the dominant contribution to the normal velocity is, at all points on $\mathcal{Q}_s$, in the $\mathbf{e}_1$ direction. Recall that f_0 is assumed not to be a critical point of $\tilde{V}$; hence, since $f_0 \neq 0$, $\tilde{V}'(f_0)$ can be either positive or negative. If $\tilde{V}'(0) > 0$ then, since $\nu > 0$, the normal velocity of points on $\mathcal{Q}_s^c$ and $\mathcal{Q}_s^f$ will decrease and increase, respectively, on top of the ambient value $\tilde{V}(f_0)$. This, as in the case where $f_0 = 0$, indicates a tendency for the phase boundary to straighten out. If, however,

$\tilde{V}'(f_0) < 0$ then, since $\nu > 0$, the exact opposite occurs—the normal velocity of points on $\mathcal{Q}_s^c$ and $\mathcal{Q}_s^f$ will add positive and negative increments, respectively, to the ambient value $\tilde{V}(f_0)$. The *protruding* part of the phase boundary, if $\tilde{V}'(f_0) < 0$, portrays a tendency to *grow* while the *flat* part *lags* behind. The subcase where f_0 is negative is yields a completely analogous result. That is, when $f_0 < 0$ the phase boundary shows a propensity to become planar or develop a larger protrusion depending upon whether $\tilde{V}'(f_0)$ is positive or negative, respectively.

The foregoing discussion shows that the kinetics of a phase boundary $\mathcal{Q}_s$ in the class at hand are, to first order in ϵ, stable or unstable depending upon whether the kinetic response function is locally increasing or decreasing at the ambient driving traction f_0. If the constitutive description of a three-phase material with shear stress response function τ_p also includes a monotonically increasing kinetic response function such as that depicted in Figure 5a it is clear that phase boundaries of the class under consideration will always be stable. If, on the other hand, the constitutive description includes a non-monotonic kinetic response function like that depicted in Figure 5b it is always possible to choose γ_l and γ_r so that the phase boundary is unstable. These results suggest that it may be reasonable to classify those three-phase materials with shear stress response function τ_p as *kinetically stable* and *kinetically unstable* depending on whether the kinetic response function $\tilde{V}$ is a monotonic or non-monotonic function of its argument. Such a classification is consistent with that found by FRIED [16] in a linear stability analysis of planar phase boundaries in arbitrary three-phase materials subjected to a class of perturbations which encompasses the set of phase boundaries Π determined in Chapter 3.

APPENDICES

Appendix A. In this appendix inequality (3.4.8) is established for all functions ψ contained in the set $\mathcal{V}$ defined in (3.1.7). Let $\chi : \mathbb{R} \to \mathbb{R}$ be an element of $\mathcal{V}$ with compact support about the origin; suppose, further, that $\chi(0) = 1$. Then, if ψ is contained in $\mathcal{V}$, one has the following inequality:[7]

$$\|\psi\|_{L^\infty(\mathbb{R})} \le \|\chi\|_{\mathcal{V}} \, \|\psi\|_{\mathcal{V}} \, , \tag{A.1}$$

where

$$\|\psi\|_{\mathcal{V}} = \|\psi\|_{L^2(\mathbb{R})} + \|\psi'\|_{L^2(\mathbb{R})} \, .$$

Hence, (A.1) shows that all elements ψ of $\mathcal{V}$ are bounded on $\mathbb{R}$. The limit

$$\lim_{x_2 \to -\infty} \psi(x_2) e^{-\frac{1}{2}(\frac{x_2}{\ell})^2} = 0 \tag{A.2}$$

must, consequentially, hold for every (without loss of generality) positive real number ℓ and every function ψ in $\mathcal{V}$. Evidently, then, such a function ψ can be expressed as follows:

$$\psi(x_2) = \int_{-\infty}^{x_2} \frac{\partial}{\partial \xi} \left(\psi(\xi) e^{-\frac{1}{2}(\frac{x_2 - \xi}{\ell})^2} \right) d\xi \quad \forall x_2 \in \mathbb{R}. \tag{A.3}$$

Thus, from (A.3) and the Cauchy-Schwarz inequality it is clear that

$$\begin{aligned} |\psi(x_2)| &\le \int_{-\infty}^{+\infty} \frac{|x_2 - \xi|}{\ell^2} |\psi(\xi)| e^{-\frac{1}{2}(\frac{x_2-\xi}{\ell})^2} \, d\xi + \int_{-\infty}^{+\infty} |\psi'(\xi)| e^{-\frac{1}{2}(\frac{x_2-\xi}{\ell})^2} \, d\xi \\ &\le \pi^{\frac{1}{4}} \left(\tfrac{1}{\sqrt{2\ell}} \|\psi\|_{L^2(\mathbb{R})} + \sqrt{\ell} \, \|\psi'\|_{L^2(\mathbb{R})} \right) \quad \forall (x_2, \ell) \in \mathbb{R} \times \mathbb{R}_+ . \end{aligned} \tag{A.4}$$

It is then obvious from (A.4) that

$$\|\psi\|_{L^\infty(\mathbb{R})} \le \pi^{\frac{1}{4}} \left(\tfrac{1}{\sqrt{2\ell}} \|\psi\|_{L^2(\mathbb{R})} + \sqrt{\ell} \, \|\psi'\|_{L^2(\mathbb{R})} \right) \quad \forall \ell \in \mathbb{R}_+ . \tag{A.5}$$

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[7] See AUBIN [9] for a demonstration of this fact.

Now, minimize the expression on the right hand side of the inequality in (A.5) with respect to ℓ to obtain (3.4.8). Note that the constant $2(\frac{\pi}{2})^{\frac{1}{4}}$ in (3.4.8) may not be the sharpest possible one for an estimate of this type. That is, there may exist a function ψ in $\mathcal{V}$ more optimal than the Gaussian used in (A.3)–(A.5).

Appendix B. In this appendix the estimates (3.4.9) and (3.4.10) for $\|\varphi\|_{L^2(\mathbb{R})}$ and $\|\varphi'\|_{L^2(\mathbb{R})}$ are established. First consider (3.4.9). From the integral equation in (3.2.15), the Minkowski inequality and the Cauchy-Schwarz inequality it is clear that

$$\|\varphi\|_{L^2(\mathbb{R})} \leq \lambda \|K_s\|_{L^2(\mathbb{R}^2)} \|\varphi\|_{L^2(\mathbb{R})} + \lambda \|f_s\|_{L^2(\mathbb{R})} . \tag{B.1}$$

With the aid of the decomposition of f_s provided in (3.3.17), the bound (3.3.21), and the fact $\|h_s\|_{L^2(\mathbb{R})} = \pi \|s'\|_{L^2(\mathbb{R})}$ (B.1) implies that

$$\|\varphi\|_{L^2(\mathbb{R})} \leq \frac{\lambda(\pi + \|s'\|_{L^\infty(\mathbb{R})} \|K_s\|_{L^2(\mathbb{R}^2)}) \|s'\|_{L^2(\mathbb{R})}}{1 - \lambda \|K_s\|_{L^2(\mathbb{R}^2)}} . \tag{B.2}$$

Now, use (3.4.8) and (3.4.9) in (B.2) to give (3.4.9).

Next consider (3.4.10). Recall that in order to obtain an estimate for the L^2 norm of φ' over $\mathbb{R}$ it is sufficient to require that s be an element of $\mathcal{I} \cap \mathcal{X}$, where $\mathcal{X}$ is given by (3.4.17). Suppose that this is the case. Then it is permissible to differentiate the integral equation in (3.2.15) to obtain

$$\varphi' + \lambda \int_{-\infty}^{+\infty} \tilde{K}_s(\cdot, \xi) \varphi(\xi) \, d\xi = \lambda f_s' \quad \text{on} \quad \mathbb{R}, \tag{B.3}$$

where

$$\tilde{K}_s(x_2, \xi) = 2K_s^2(x_2, \xi) \frac{s(x_2) - s(\xi)}{x_2 - \xi} + 2L_s(x_2, \xi) \quad \forall (x_2, \xi) \in \mathbb{R}^2, \tag{B.4}$$

and

$$L_s(x_2,\xi) = \frac{s(x_2) - s(\xi) - (x_2-\xi)s'(x_2) + \frac{1}{2}(x_2-\xi)^2 s''(x_2)}{(x_2-\xi)((s(x_2)-s(\xi))^2 + (x_2-\xi)^2)}$$

$$\forall (x_2,\xi) \in \mathbb{R}^2. \tag{B.5}$$

Clearly, with the aid of the Cauchy-Schwarz and Minkowski inequalities, (B.3) implies the following estimate for $\|\varphi'\|_{L^2(\mathbb{R})}$:

$$\|\varphi'\|_{L^2(\mathbb{R})} \leq \lambda \|\tilde{K}_s\|_{L^2(\mathbb{R}^2)} \|\varphi\|_{L^2(\mathbb{R})} + \lambda \|f_s'\|_{L^2(\mathbb{R})}. \tag{B.6}$$

So, given bounds for $\|\tilde{K}_s\|_{L^2(\mathbb{R}^2)}$ and $\|f_s'\|_{L^2(\mathbb{R})}$ in terms of the L^2 norms, over $\mathbb{R}$, of the first three derivatives of s, (B.6) will provide, in conjunction with (3.4.9), an estimate for $\|\varphi'\|_{L^2(\mathbb{R})}$. A bound for the L^2 norm of L_s over $\mathbb{R}^2$ can be obtained in exactly the same manner as that established for K_s in Section 3.3. This bound is

$$\|L_s\|_{L^2(\mathbb{R}^2)} \leq \frac{\sqrt{\pi}}{6} \|s''\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|s'''\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{B.7}$$

Now, from (B.4), (B.5), (B.7), (3.3.2) the Cauchy-Schwarz inequality and the Minkowski inequality one obtains the following estimate for $\|\tilde{K}_s\|_{L^2(\mathbb{R}^2)}$:

$$\|\tilde{K}_s\|_{L^2(\mathbb{R}^2)} \leq \sqrt{\tfrac{2\pi}{3}} \|s'\|_{L^\infty(\mathbb{R})} \|s''\|_{L^\infty(\mathbb{R})} \|s'\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|s''\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}$$
$$+ \tfrac{\sqrt{\pi}}{6} \|s''\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|s'''\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{B.8}$$

It is also clear that

$$\|f_s'\|_{L^2(\mathbb{R})} \leq \pi \|s''\|_{L^2(\mathbb{R})} + \|s'\|_{L^\infty(\mathbb{R})} \|\tilde{K}_s\|_{L^2(\mathbb{R}^2)} \|s'\|_{L^2(\mathbb{R})}$$
$$+ \tfrac{1}{2} \|s''\|_{L^\infty(\mathbb{R})} \|K_s\|_{L^2(\mathbb{R}^2)} \|s'\|_{L^2(\mathbb{R})}. \tag{B.9}$$

Using (3.4.8), inequalities (B.8) and (B.9) become

$$\|\tilde{K}_s\|_{L^2(\mathbb{R}^2)} \leq \tfrac{4\pi}{\sqrt{3}} \|s'\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|s''\|_{L^2(\mathbb{R})} \|s'''\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}$$
$$+ \tfrac{\sqrt{\pi}}{3} \|s''\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|s'''\|_{L^2(\mathbb{R})}^{\frac{1}{2}}, \tag{B.10}$$

and

$$\|f_s'\|_{L^2(\mathbb{R})} \leq \pi \, \|s''\|_{L^2(\mathbb{R})} + 2(\tfrac{\pi}{2})^{\frac{1}{4}} \, \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})} \, \|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \, \|\tilde{K}_s\|_{L^2(\mathbb{R}^2)}$$
$$+ (\tfrac{\pi}{2})^{\frac{1}{4}} \, \|s'\|_{L^2(\mathbb{R})} \, \|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \, \|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \, \|K_s\|_{L^2(\mathbb{R}^2)} \, , \qquad \text{(B.11)}$$

respectively. Combining (B.6), (B.10), (B.11) and (3.3.15) leads, after a bit of algebra, to the desired estimate (3.4.10).

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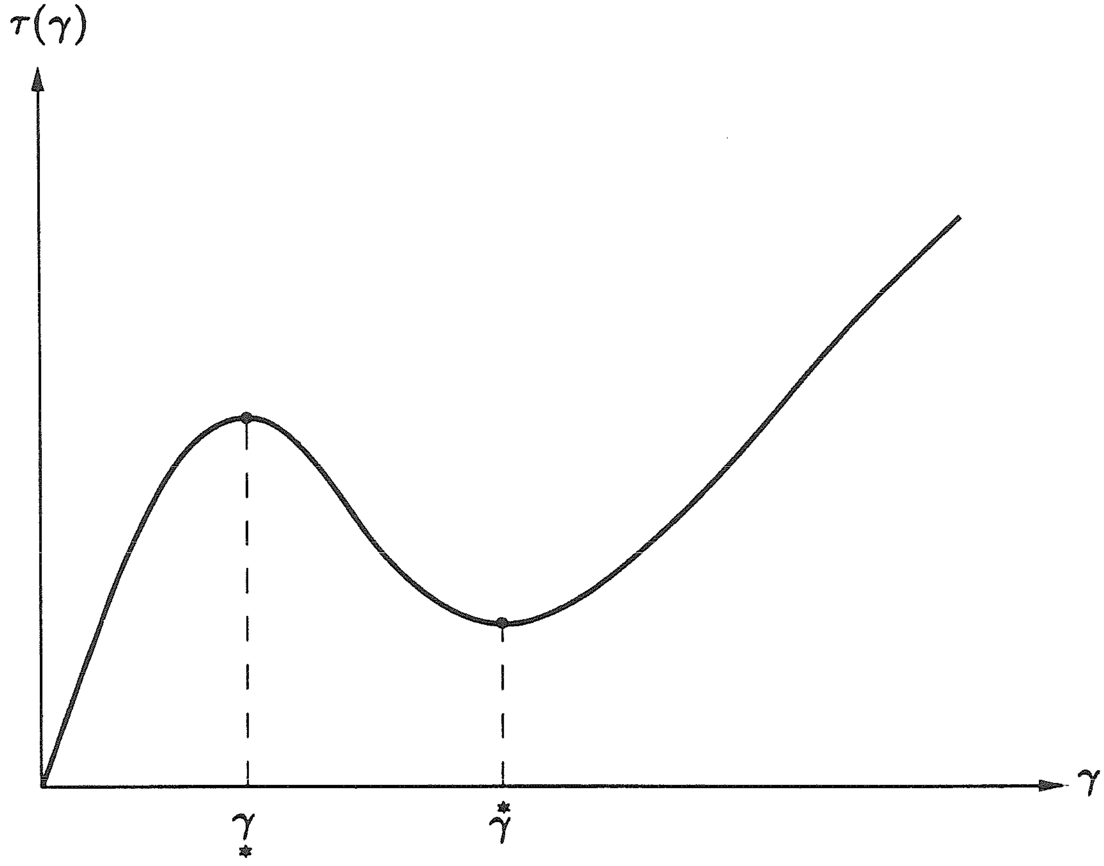

Figure 1: Graph of the shear stress response function τ.

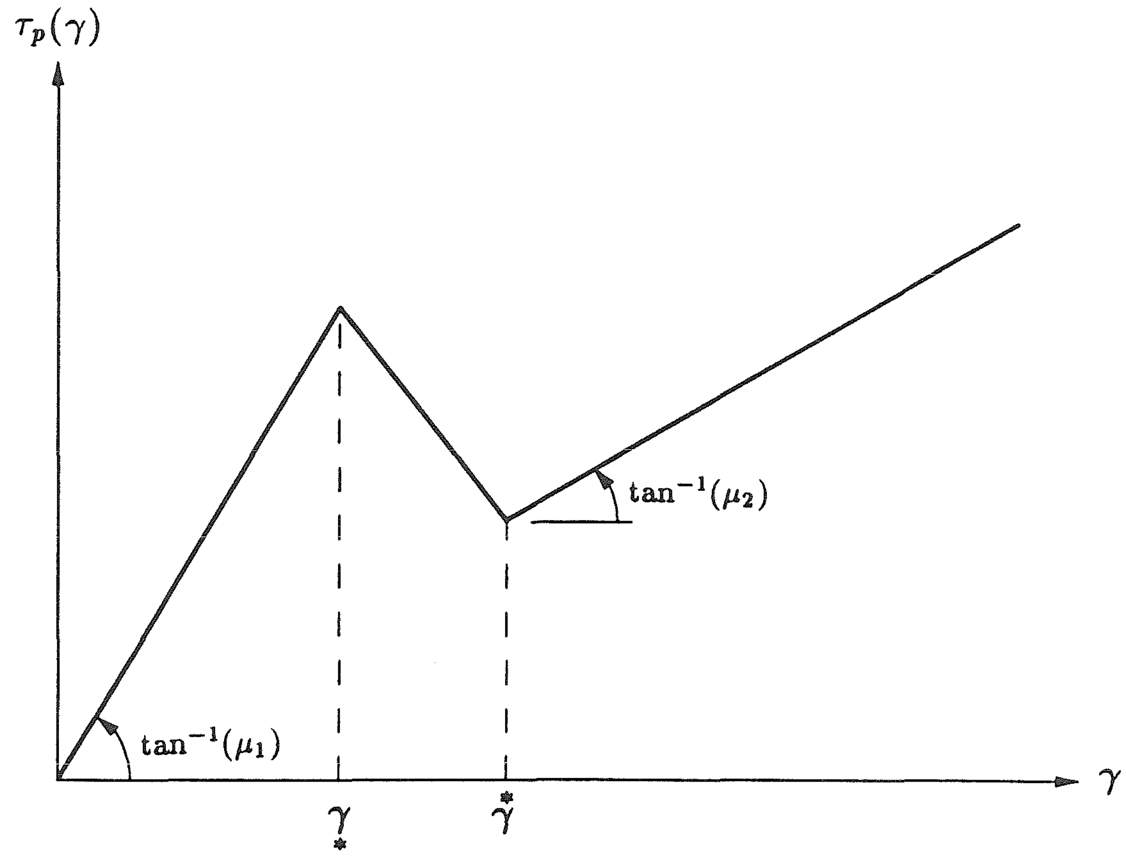

Figure 2: Graph of the shear stress response function τ_p.

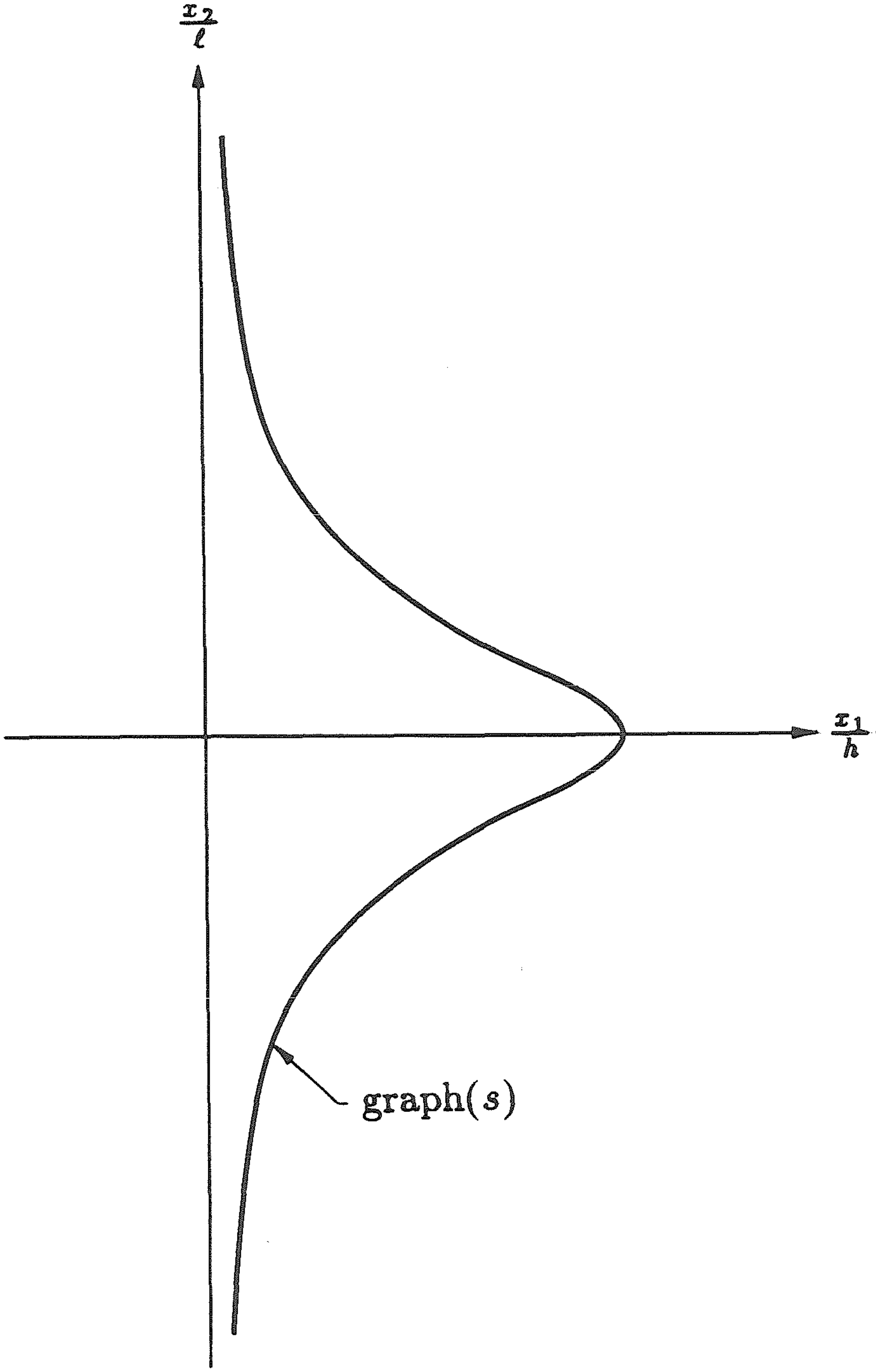

Figure 3: Graph of s.

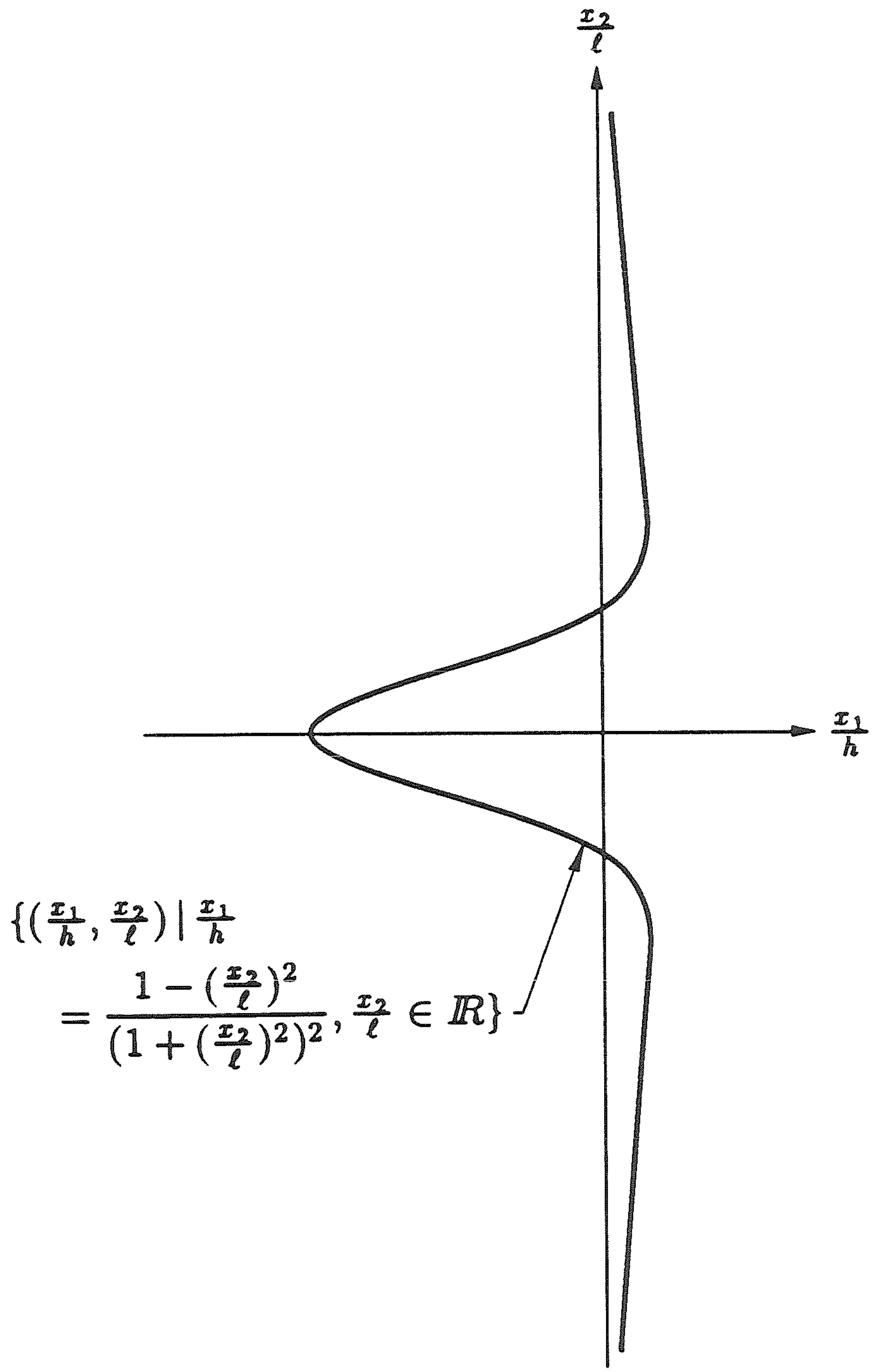

Figure 4: Graph of the first order correction to the driving traction.

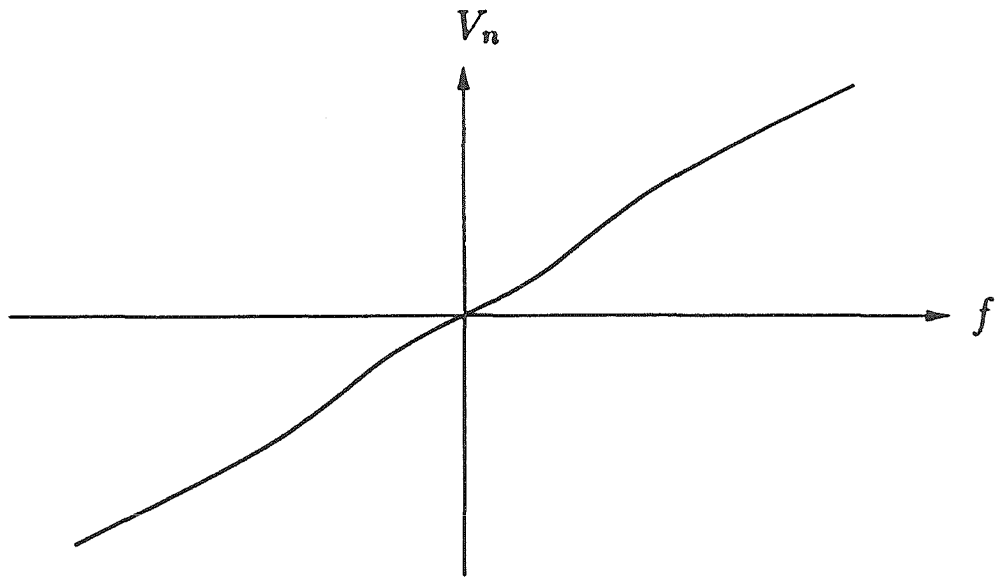

Figure 5a: Graph of a monotone increasing admissible kinetic response function.

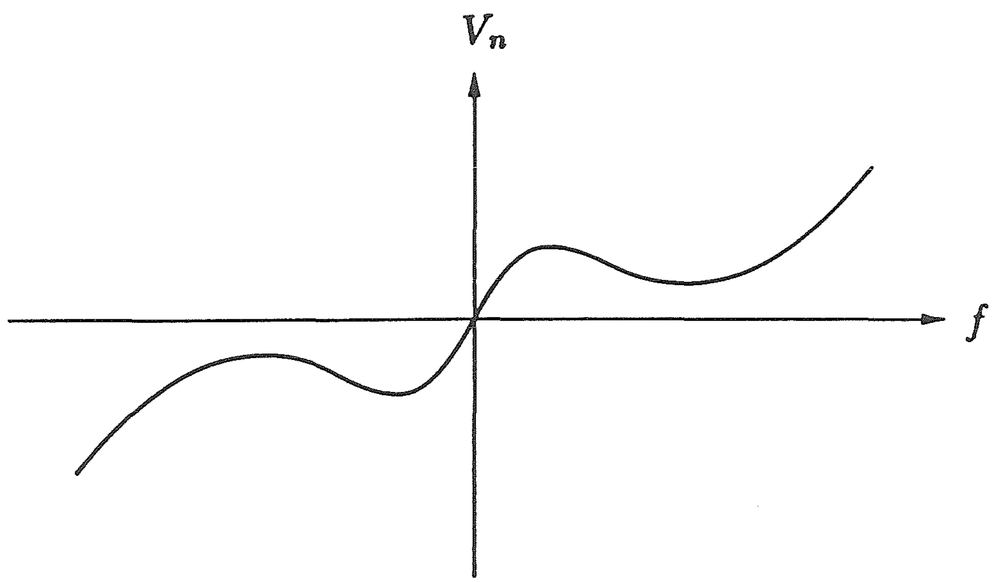

Figure 5b: Graph of a non-monotone admissible kinetic response function.

LINEAR STABILITY OF A TWO-PHASE PROCESS INVOLVING A STEADILY PROPAGATING PLANAR PHASE BOUNDARY IN A SOLID: PART 1. PURELY MECHANICAL CASE

1. INTRODUCTION

Displacive solid-solid phase transformations occur in a wide variety of metallic and ceramic alloys. The different phases of a material capable of undergoing such a transformation are generally distinguished by distinct crystal structures. Transformations involving such materials are characterized by the existence of interfaces, or *phase boundaries*, which segregate material in different phases. Of particular interest is the growth stage of a displacive solid-solid phase transformation which is directly related to the kinetics governing the motion of phase boundaries. Some experimental work directed at understanding the growth stage of these transformations has been performed. NISHIYAMA [25] has separated the transformation kinetics of relevant materials into three classes based upon the speed at which they occur. Depending on how they are loaded, some materials may exhibit kinetics which fall into any or all of these three classes. In the fastest of these, phase boundaries propagate at velocities which are of the same order of magnitude as the velocity of shear wave propagation; in the remaining two classes the velocities with which phase boundaries propagate are many orders of magnitude smaller the the shear wave speed. The work of GRUJICIC, OLSON & OWEN [16] and CLAPP & YU [10] suggests that slowly propagating phase boundaries are most often observed to be planar in structure, while those which propagate rapidly often display highly complex geometries involving *plate-like* or *dendritic* structures. These complicated structures are reminiscent of those which occur in crystal growth, that are known to evolve from states involving planar interfaces which separate solidified crystal material from liquid melt.[1] It is, therefore, natural to speculate as to whether the complicated plate-like morpholgies observed in rapid displacive solid-solid phase transformations can emerge in an analogous fashion from their slow counterparts.

Finite elastic dynamical processes in materials capable of sustaining equilibria with discontinuous deformation gradients have figured prominently in recent

[1] LANGER [22] provides an overview of such phenomena.

continuum mechanical treatments of displacive solid-solid phase transformations.[2] In a *homogeneous*, *hyperelastic* material equilibrium states with discontinuous deformation gradients occur only if the relevant *elastic potential* allows for a loss of ellipticity—at certain values of the deformation gradient—in the associated displacement equations of *equilibrium*.[3] Materials characterized by elastic potentials which allow such a loss of ellipticity are referred to as *non-elliptic*. Of particular importance in most of the work that has been done in this area are non-elliptic materials which have two disjoint elliptic phases. Such hypothetical materials serve as models for actual materials which can sustain displacive solid-solid phase transformations; surfaces which, in either equilibrium or dynamics, separate the different phases of a non-elliptic material function as models for the phase boundaries which occur in actual materials and, hence, are referred to as such.

Despite the apparent dearth of experimental information regarding the issue of whether the growth stage of displacive solid-solid phase transformations can involve the emergence of complicated dendritic structures from planar phase boundaries, it is legitimate to examine this topic from an analytical perspective in the foregoing continuum mechanical context. Except for the work of SILLING [30], the bulk of the continuum mechanical investigations which consider dynamical processes are confined to one-dimensional bar theory and, hence, are not of direct bearing on the issue of phase boundary morphology. SILLING [30] has demonstrated, through an asymptotic analysis, that a particular *generalized neo-Hookean* material is capable of sustaining a motion which involves a steadily propagating cusped surface of discontinuity which segregates distinct elliptic phases of the relevant material. This cusped phase boundary can be thought of as a model for one which would accompany a single plate-like structure in an actual displacive solid-solid phase transformation. SILLING [30] also performs numerical

[2] See, *e.g.*, ABEYARATNE & KNOWLES [4–6], JAMES [18], PENCE [27] and SILLING [30].

[3] For a discussion of this issue consult, for instance, ROSAKIS [28].

calculations which seem to support his asymptotic results. It is important to note that this work does not consider the issue of the emergence of the cusped phase boundary from a planar one.

In analogy to the large body of analytical work which has been directed at modeling the emergence of dendritic structures from planar interfaces in the process of crystal growth,[4] it seems reasonable—as a first step in addressing the issue at hand—to investigate the stability of a two-phase process involving a steadily propagating planar phase boundary in a non-elliptic material. In a study which focuses primarily on contructively establishing the existence of two-phase equilibria in a special non-elliptic generalized neo-Hookean material FRIED [14] also analyzes—in an *inertia-free* setting—the stability of such a state with respect to a particular class of perturbations.[5] Together, the narrow class of perturbations which is considered, the absence of inertial effects, and the constitutive specialization which is adhered to severely restrict the generality of the results which are obtained in [14]. The objective of the present inquiry is, therefore, to perform a more general stability analysis where a two-phase process involving a steadily propagating planar phase boundary in a wide class of non-elliptic generalized neo-Hookean materials is subjected to a broad class of disturbances and inertial effects are taken into consideration. It will transpire, however, that the stability results which are obtained are consistent with those secured by FRIED [14].

Chapter 2 is devoted to preliminaries. After a brief overview of the notation to be used, Section 2.1 introduces the kinematics and fundamental balance principles which will be needed in the following. Section 2.2 explains the constitutive restrictions which will be adhered to throughout this work. Section 2.3 is concerned with the notions of *mechanical dissipation* and *driving traction* which are associated with phase boundaries; these lead naturally to the consideration of a *kinetic relation*—which gives the normal velocity of a phase boundary in terms

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[4] See, for example, LANGER [22], MULLINS & SEKERKA [24] and STRAIN [31].

[5] The relevant material is also that used by SILLING in [29–30].

of the driving traction which acts on it or *vice versa*—and an associated *kinetic response function*. In the final section of Chapter 2, the kinematics are specialized to those of *antiplane shear*.

Chapter 3 concentrates upon a linear stability analyis of a two-phase process involving a steadily propagating planar phase boundary in a non-elliptic generalized neo-Hookean material which obeys the Baker-Ericksen inequality. The process to be perturbed, which involves an antiplane shear deformation, is introduced in Section 3.1. In Section 3.2 the class of perturbations which will be applied to the base process are then introduced. Each admissible perturbation involves, in general, a disturbance of the configuration of the phase boundary and of the displacement and velocity fields in a small neighborhood of the phase boundary—all of which are assumed to be small in some appropriate sense. The kinematics of the perturbation are also restricted to those of antiplane shear. It is assumed, furthermore, that the post-perturbation deformation remains an antiplane shear and involves only one phase boundary. Section 3.3 is devoted to the linearization about the base process of the field equations, which hold away from the phase boundary, about the base process introduced in Section 3.1. In a similar manner, Section 3.4 is concerned with the linearization about the base process of the jump conditions and kinetic relation which hold on the phase boundary. A summary of the complete linearized system of field equations, jump conditions, kinetic relation, boundary and initial conditions which describe the process generated by the perturbation is presented in Section 3.5. Included are both the inertial and inertia-free cases. In Section 3.6 a *normal mode analysis* is performed in the absence of inertial effects. A condition necessary and sufficient for the base process to be unstable with respect to any perturbation of the type introduced in Section 3.2 is obtained. This condition involves only the local behavior of the derivative of the (essentially arbitrary) kinetic response function introduced in Section 2.3. An alternative to the normal mode analysis of Section 3.6 is performed in Section 3.7. Here the relevant initial boundary value problem is converted into a functional initial value problem for the correction to the interface position arising from the perturbation. Analysis of this problem yields identical stability criteria to those which are obtained in Section 3.6. Moreover, when instability is present, its manifestation can be tracked over a finite time interval upon which the linearization performed in the foregoing remains valid. The results suggest the emergence of plate-like or dendritic structures. Section 3.8 contains both a normal mode and an energy analyis in the case where inertial effects are included. First, the normal mode analysis leads to a necessary and sufficient condition for the base process to be unstable with respect to a special subset of the class of perturbations introduced in Section 3.2. The operative stability criterion is identical to that which holds for all initial disturbances in the inertia-free case. An argument based upon a Fourier-Laplace transform analysis of the relevant initial boundary value problem is then used to show that all but a very special subset of the full class of perturbations introduced in Section 3.2 are covered by the normal mode analysis. Section 3.8 is concluded with an energy analysis which is used to show that the sufficient condition referred to above is also necessary for the base process to be unstable with respect to any perturbation of the class introduced in Section 3.2. Hence, it is shown that the presence of inertia does not qualitatively alter the linear stability of the base process of interest. Finally, in Section 3.9, a discussion which focusses on the physical reasonableness of admissible non-monotonic kinetic response functions is undertaken.

2. PRELIMINARIES

2.1. Notation, kinematics and balance principles. In the following $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. The intervals $(0,\infty)$ and $[0,\infty)$ are represented by $\mathbb{R}_+$ and $\bar{\mathbb{R}}_+$. The symbol $\mathbb{R}^n$, with n equal to 2 or 3, represents real n-dimensional space equipped with the standard Euclidean norm. If U is a set, then its closure, interior and boundary are designated by $\bar{U}$, $\overset{\circ}{U}$, and ∂U, respectively. The complement of a set V with respect to U is written as $U \setminus V$. Given a function $\psi : U \to W$ and a subset V of U, $\psi(V)$ stands for the image of V under the map ψ.

Vectors and linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ (referred to herein as *tensors*) are distinguished from scalars with the aid of boldface type—lower and upper case for vectors and tensors, respectively. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$, their inner product is then written as $\mathbf{a}\cdot\mathbf{b}$; the Euclidean norm of $\mathbf{a}$ is, further, written as $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$. The set of unit vectors—that is, vectors with unit Euclidean norm—in $\mathbb{R}^3$ is designated by $\mathcal{N}$. The symbol $\mathcal{L}$ refers to the set of tensors, $\mathcal{L}_+$ denotes the set of all tensors with positive determinant, and $\overset{+}{\mathcal{S}}$ stands for the collection of all symmetric positive definite tensors. If $\mathbf{F}$ is in $\mathcal{L}$ then $\mathbf{F}^T$ represents its transpose; if, moreover, $\det\mathbf{F} \neq 0$, then the inverse of $\mathbf{F}$ and its transpose are written as $\mathbf{F}^{-1}$ and $\mathbf{F}^{-T}$, respectively. The notation $\mathbf{a}\otimes\mathbf{b}$ refers to the tensor $\mathbf{A}$, formed by the outer product of $\mathbf{a}$ with $\mathbf{b}$, defined such that $\mathbf{A}\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$ for any vector $\mathbf{c}$ in $\mathbb{R}^3$. If $\mathbf{A}$ and $\mathbf{B}$ are tensors then their inner product is written as $\mathbf{A}\cdot\mathbf{B} = \operatorname{tr}\mathbf{A}\mathbf{B}^T$.

When component notation is used, Greek indices range only over $\{1,2\}$; summation of repeated indices over the appropriate range is implicit. A subscript preceded by a comma denotes partial differentiation with respect to the corresponding coordinate. Also, a superposed dot signifies partial differentiation with respect to time.

Consider, now, a body $\mathcal{B}$ which, in a reference configuration, occupies a region $\mathcal{R}$ contained in $\mathbb{R}^3$. A motion of $\mathcal{B}$ on a time interval $\mathcal{T} \subset \mathbb{R}$ is characterized by a one-parameter family of invertible mappings $\hat{\mathbf{y}}(\cdot, t) : \mathcal{R} \to \mathcal{R}_t$, with

$$\hat{\mathbf{y}}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x}, t) \quad \forall(\mathbf{x}, t) \in \mathcal{M}, \tag{2.1.1}$$

where $\mathcal{M} = \mathcal{R} \times \mathcal{T}$ represents the *trajectory* of the motion. Assume that the *deformation* $\hat{\mathbf{y}}$, or equivalently the *displacement* $\mathbf{u}$, is continuous and possesses piecewise continuous first and second partial derivatives on $\mathcal{M}$. Let S_t be the set of points contained in $\mathcal{R}$ defined so that, at each instant t in $\mathcal{T}$, $\hat{\mathbf{y}}(\cdot, t)$ is twice continuously differentiable on the set $\mathcal{R} \setminus S_t$. Let the set Σ be defined by

$$\Sigma = \{(\mathbf{x}, t) | \, \mathbf{x} \in S_t, t \in \mathcal{T}\}. \tag{2.1.2}$$

Introduce the *deformation gradient tensor* $\mathbf{F} : \mathcal{M} \setminus \Sigma \to \mathcal{L}$ by

$$\mathbf{F}(\mathbf{x}, t) = \nabla \hat{\mathbf{y}}(\mathbf{x}, t) \quad \forall(\mathbf{x}, t) \in \mathcal{M} \setminus \Sigma, \tag{2.1.3}$$

where the associated *Jacobian determinant*, $J : \mathcal{M} \setminus \Sigma \to I\!R$, of $\hat{\mathbf{y}}$ is restricted to be strictly positive on its domain of definition:

$$J(\mathbf{x}, t) = \det \mathbf{F}(\mathbf{x}, t) > 0 \quad \forall(\mathbf{x}, t) \in \mathcal{M} \setminus \Sigma.$$

Hence, $\mathbf{F} : \mathcal{M} \setminus \Sigma \to \mathcal{L}_+$. The *left Cauchy-Green tensor* $\mathbf{G} : \mathcal{M} \setminus \Sigma \to \overset{+}{\mathcal{S}}$ corresponding to the deformation $\hat{\mathbf{y}}$ is given by

$$\mathbf{G}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, t)\mathbf{F}^T(\mathbf{x}, t) \quad \forall(\mathbf{x}, t) \in \mathcal{M} \setminus \Sigma. \tag{2.1.4}$$

The *deformation invariants* associated with $\hat{\mathbf{y}}$ exist on $\mathcal{M} \setminus \Sigma$ and are supplied through the fundamental scalar invariants of $\mathbf{G}$:

$$I_1(\mathbf{G}) = \operatorname{tr} \mathbf{G}, \quad I_2(\mathbf{G}) = \tfrac{1}{2}\left((\operatorname{tr} \mathbf{G})^2 - \operatorname{tr}(\mathbf{G}^2)\right), \quad I_3(\mathbf{G}) = \det \mathbf{G}. \tag{2.1.5}$$

With the above kinematic antecedents in place introduce the *nominal mass density* $\rho : \mathcal{R} \to I\!R_{+}$, the *nominal body force per unit mass* $\mathbf{b} : \mathcal{M} \to I\!R^{3}$, and the *nominal stress tensor* $\mathbf{S} : \mathcal{M} \setminus \Sigma \to \mathcal{L}$, and suppose that ρ is constant on $\mathcal{R}$ and $\mathbf{b}$ is continuous on $\mathcal{M}$, while $\mathbf{S}$ is piecewise continuous on $\mathcal{M}$, continuous on $\mathcal{M} \setminus \Sigma$, and has a piecewise continuous gradient on $\mathcal{M}$. Let ρ_{*} be the mass density in the deformed configuration associated with $\hat{\mathbf{y}}$. Given a *regular* subregion $\mathcal{P}$ of $\mathcal{R}$, with $\partial\mathcal{P} \cap S_t$ a set of measure zero in $\partial\mathcal{P}$ for each t in $\mathcal{T}$, let $\mathbf{m} : \partial\mathcal{P} \to \mathcal{N}$ denote the unit outward normal to $\partial\mathcal{P}$. Then the global balance laws of mass, linear momentum, and angular momentum require that

$$\int_{\mathcal{P}} \rho \, dV = \int_{\hat{\mathbf{y}}(\mathcal{P})} \rho_{*} \, dV \quad \text{on} \quad \mathcal{T}, \tag{2.1.6}$$

$$\int_{\partial\mathcal{P}} \mathbf{S}\mathbf{m} \, dA + \int_{\mathcal{P}} \rho \mathbf{b} \, dV = \overline{\int_{\mathcal{P}} \rho \dot{\mathbf{u}} \, dV}^{\,\cdot} \quad \text{on} \quad \mathcal{T}, \tag{2.1.7}$$

and

$$\int_{\partial\mathcal{P}} \hat{\mathbf{y}} \wedge \mathbf{S}\mathbf{m} \, dA + \int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho \mathbf{b} \, dV = \overline{\int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho \dot{\mathbf{u}} \, dV}^{\,\cdot} \quad \text{on} \quad \mathcal{T}, \tag{2.1.8}$$

respectively, for every such regular subregion $\mathcal{P}$ contained in $\mathcal{R}$.

Localization of the balance laws (2.1.6)–(2.1.8) at an arbitrary point contained in the interior of $\mathcal{M} \setminus \Sigma$ yields the following familiar field equations:

$$\begin{aligned} \rho = \rho_{*}(\hat{\mathbf{y}}) J \quad &\text{on} \quad \mathcal{M} \setminus \Sigma, \\ \nabla \cdot \mathbf{S} + \rho \mathbf{b} = \rho \ddot{\mathbf{u}} \quad &\text{on} \quad \mathcal{M} \setminus \Sigma, \\ \mathbf{S}\mathbf{F}^{T} = \mathbf{F}\mathbf{S}^{T} \quad &\text{on} \quad \mathcal{M} \setminus \Sigma. \end{aligned} \tag{2.1.9}$$

Suppose, from now on, that the set S_t is a regular surface for every t in $\mathcal{T}$. The set Σ then represents the trajectory of a surface of discontinuity in $\mathbf{F}$ and $\mathbf{S}$. Let $g(\cdot, t)$ denote a generic field quantity $g(\cdot, t) : S_t \to I\!R$ which is discontinuous across S_t at the instant t in $\mathcal{T}$. Define the *jump* $[\![g(\cdot,t)]\!]$ of $g(\cdot,t)$ across S_t by

$$[\![g(\mathbf{x},t)]\!] = \lim_{h\searrow 0}\big(g\big(\mathbf{x}+h\mathbf{n}(\mathbf{x},t),t\big) - g\big(\mathbf{x}-h\mathbf{n}(\mathbf{x},t),t\big)\big) \quad \forall(\mathbf{x},t)\in\Sigma, \tag{2.1.10}$$

where $\mathbf{n}(\cdot,t): S_t \to \mathcal{N}$ is a unit normal to S_t at each t in $\mathcal{T}$. Then, localization of (2.1.6)–(2.1.8) at an arbitrary point in Σ yields the following jump conditions

$$\begin{aligned} [\![\rho_*(\hat{\mathbf{y}})J]\!] &= 0 \quad \text{on} \quad \Sigma, \\ [\![\mathbf{S}\mathbf{n}]\!] + \rho V_n[\![\dot{\mathbf{u}}]\!] &= \mathbf{0} \quad \text{on} \quad \Sigma, \end{aligned} \tag{2.1.11}$$

where $V_n(\cdot,t): S_t \to I\!R$ denotes the component of the velocity of the surface S_t in the direction of $\mathbf{n}(\cdot,t)$ at the instant t in $\mathcal{T}$.

Equations $(2.1.9)_1$ and $(2.1.11)_1$ are, evidently, completely decoupled from equations $(2.1.9)_{2,3}$ and $(2.1.11)_2$; that is, given a solution to, say, a boundary value problem involving $(2.1.9)_{2,3}$ and $(2.1.10)_2$, ρ_* can be calculated *a posteriori*. For this reason equations $(2.1.9)_1$ and $(2.1.11)_1$ will be disregarded in the subsequent analysis.

In this investigation an *inertia-free* motion is defined as one wherein the inertial terms on the right hand sides of the global balance equations (2.1.7) and (2.1.8) are replaced by the zero vector. In the context of an inertia-free motion the field equation $(2.1.9)_2$ simplifies to read

$$\nabla\cdot\mathbf{S} + \rho\mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathcal{M}\setminus\Sigma, \tag{2.1.12}$$

and the jump condition $(2.1.11)_2$ becomes

$$[\![\mathbf{S}\mathbf{n}]\!] = \mathbf{0} \quad \text{on} \quad \Sigma. \tag{2.1.13}$$

Equations $(2.1.9)_{1,3}$ and $(2.1.11)_1$ remain, of course, unaltered.

In addition to the jump conditions given in (2.1.11) in the inertial case or $(2.1.10)_1$ and (2.1.13) in the inertia-free case, the stipulated continuity of $\hat{\mathbf{y}}$ gives the following *kinematic* jump condition

$$[\![\mathbf{u}]\!] = \mathbf{0} \quad \text{on} \quad \Sigma. \tag{2.1.14}$$

2.2. Constitutive assumptions. Let $\mathcal{B}$ be composed of a hyperelastic material which is homogeneous, isotropic and incompressible. Since $\mathcal{B}$ is hyperelastic its mechanical response is governed by an elastic potential or *strain energy per unit reference volume.* The homogeneity of $\mathcal{B}$ implies that the elastic potential does not depend explicitly on position in the reference configuration. Furthermore, because $\mathcal{B}$ is isotropic the elastic potential can depend on the deformation gradient $\mathbf{F}$ only through the deformation invariants $I_k(\mathbf{G})$ defined in (2.1.5). The incompressibility of $\mathcal{B}$ requires that the deformation $\hat{\mathbf{y}}$ be *isochoric*, *i.e.*,

$$I_3\big(\mathbf{G}(\mathbf{x},t)\big) = J^2(\mathbf{x},t) = 1 \quad \forall (\mathbf{x},t) \in \mathcal{M} \setminus \Sigma. \tag{2.2.1}$$

An additional consequence of isotropy is, therefore, that the elastic potential can be expressed as a function solely of the first two deformation invariants. It can also be demonstrated via (2.1.5) that, when (2.2.1) holds, $I_\alpha(\mathbf{G}(\mathbf{x},t)) \geq 3$ for all $(\mathbf{x},t)$ contained in $\mathcal{M} \setminus \Sigma$. Now, let $\tilde{W} : [3,\infty) \times [3,\infty) \to I\!R$ denote an elastic potential which characterizes $\mathcal{B}$ and assume that $\tilde{W}$ is continuously differentiable with piecewise continuous second derivatives on its domain of definition. The nominal stress response of $\mathcal{B}$ is then determined through $\tilde{W}$ up to an arbitrary pressure $p : \mathcal{M} \setminus \Sigma \to I\!R$ required to accomodate the kinematic constraint (2.2.1) imposed by the incompressibility of $\mathcal{B}$: *viz.*,

$$\mathbf{S} = 2\left(\tilde{W}_{I_1}(I)\mathbf{F} + \tilde{W}_{I_2}(I)\big(I_1(\mathbf{G})\mathbf{1} - \mathbf{G}\big)\mathbf{F}\right) - p\mathbf{F}^{-T} \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \tag{2.2.2}$$

where $I : \mathcal{M} \setminus \Sigma \to [3,\infty) \times [3,\infty)$ is given by

$$I(\mathbf{x},t) = \Big(I_1\big(\mathbf{G}(\mathbf{x},t)\big), I_2\big(\mathbf{G}(\mathbf{x},t)\big)\Big) \quad \forall (\mathbf{x},t) \in \mathcal{M} \setminus \Sigma.$$

Following GURTIN [17], let the class of *generalized neo-Hookean* materials refer to that subset of hyperelastic materials, first introduced by KNOWLES [19], which are homogeneous, isotropic and incompressible with elastic potential independent of the second deformation invariant $(2.1.5)_2$. Assume, henceforth, that $\mathcal{B}$ is composed of a generalized neo-Hookean material with elastic potential $W : [3, \infty) \rightarrow I\!R$, where W is continuously differentiable with piecewise continuous derivative on $[3, \infty)$. Then, by (2.2.2), the nominal stress response of $\mathcal{B}$ is determined by

$$\mathbf{S} = 2W'(I_1(\mathbf{G}))\mathbf{F} - p\mathbf{F}^{-T} \quad \text{on} \quad \mathcal{M} \setminus \Sigma. \tag{2.2.3}$$

Suppose also that the elastic potential is *normalized* so that

$$W(3) = 0. \tag{2.2.4}$$

Choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and consider the response of the material at hand to a simple shear deformation $\hat{\mathbf{y}}$ given by

$$\hat{\mathbf{y}}(\mathbf{x}, t) = (\mathbf{1} + \gamma \mathbf{e}_3 \otimes \mathbf{e}_1)\mathbf{x} \quad \forall(\mathbf{x}, t) \in \mathcal{M}, \tag{2.2.5}$$

where the constant γ—assumed non-negative without loss of generality—denotes the amount of shear. From (2.1.3), (2.2.3) and (2.2.5) the nominal shear stress corresponding to the deformation $\hat{\mathbf{y}}$ is, for each γ in $\bar{I\!R}_+$, found to be

$$\mathbf{e}_3 \cdot \mathbf{S}\mathbf{e}_1 = 2\gamma W'(3 + \gamma^2) =: \tau(\gamma). \tag{2.2.6}$$

In [19–20] KNOWLES demonstrates that the 31 and 32 components of nominal and Cauchy shear stress are, in the present setting, equal. The function $\tau : \bar{I\!R}_+ \rightarrow I\!R$ is, hence, referred to as the *shear stress response function* of the generalized neo-Hookean material, characterized by W, in simple shear. An immediate consequence of (2.2.4) and (2.2.6) is

$$W(I_1) = \int_0^{\sqrt{I_1 - 3}} \tau(\gamma)\, d\gamma \quad \forall I_1 \in [3, \infty), \tag{2.2.7}$$

so that the response of a generalized neo-Hookean material, in *all* three dimensional deformations, is, up to a hydrostatic pressure, completely characterized by specifying the shear stress response function τ. Define the *secant modulus in shear* $M : \bar{\mathbb{R}}_+ \to \mathbb{R}$ of a generalized neo-Hookean material with elastic potential W by

$$M(\gamma) = 2W'(3+\gamma^2) \quad \forall \gamma \in \bar{\mathbb{R}}_+, \tag{2.2.8}$$

and assume that, in compliance with the *Baker-Ericksen inequality,*

$$M(\gamma) > 0 \quad \forall \gamma \in \mathbb{R}_+. \tag{2.2.9}$$

Assume, also, that $M(0) > 0$ so that the infinitesimal shear modulus of the material at hand is positive. Note from (2.2.6) and (2.2.8) that the shear stress response function τ must also satisfy

$$\tau(0) = 0, \quad \tau'(0) = M(0). \tag{2.2.10}$$

Observe, in addition, that the stipulated smoothness of W guarantees that both τ and M are piecewise continuously differentiable on $\bar{\mathbb{R}}_+$.

Despite the significant restrictions which have been placed upon the class of materials which will be considered in this investigation, no presuppositions have been made regarding the sign of the derivative—where it exists—of the shear stress response function corresponding to the generalized neo-Hookean material defined through (2.2.6). In [20] Knowles shows that the monotonicity of the shear stress response function τ is related directly to the ellipticity of the generalized neo-Hookean material which it characterizes: if τ is not a monotonically increasing function on its domain of definition then the associated material is non-elliptic. This investigation will make use of a particular subclass of non-elliptic generalized neo-Hookean materials, first suggested by Abeyaratne [1]; this class of materials is characterized by the set of shear stress response functions τ which are continuous on $\bar{\mathbb{R}}_+$ and piecewise continuously differentiable on $\bar{\mathbb{R}}_+ \setminus \{\underset{*}{\gamma}, \overset{*}{\gamma}\}$, where $0 < \underset{*}{\gamma} < \overset{*}{\gamma}$, such that

$$\begin{aligned} \tau' > 0 \quad &\text{on} \quad \bar{\mathbb{R}}_+ \setminus [\underset{*}{\gamma}, \overset{*}{\gamma}], \\ \tau' < 0 \quad &\text{on} \quad (\underset{*}{\gamma}, \overset{*}{\gamma}). \end{aligned} \tag{2.2.11}$$

The sets of shear strains lying in the intervals $[0, \underset{*}{\gamma})$ and $(\overset{*}{\gamma}, \infty)$ are referred to as the *high* and *low strain phases* of the generalized neo-Hookean material specified by the shear stress response function τ. Together the high and low strain phases of such a material comprise its *elliptic phases.* A generalized neo-Hookean material characterized by a shear stress response function of this type will be referred to herein as a *three-phase* material. See Figure 1 for a graph of a shear stress response function typical of those which specify three-phase materials.

2.3. Dissipation, driving traction and the kinetic relation. Let $\mathcal{P}$ be a regular subregion contained in $\mathcal{R}$ chosen so that $\partial\mathcal{P} \cap \Sigma$ is a set of measure zero in $\partial\mathcal{P}$. The total mechanical energy contained in $\mathcal{P}$ at an instant t contained in $\mathcal{T}$ is given, under the present constitutive assumptions, by

$$E(t;\mathcal{P}) = \int_{\mathcal{P}} \left(W\big(I_1(\mathbf{G}(\mathbf{x},t))\big) + \tfrac{1}{2}\rho|\dot{\mathbf{u}}(\mathbf{x},t)|^2\right) dV \quad \forall t \in \mathcal{T}. \tag{2.3.1}$$

A standard calculations then shows, with the aid of (2.2.3), that

$$\dot{E}(t;\mathcal{P}) = \int_{\partial\mathcal{P}} \mathbf{S}(\mathbf{x},t)\mathbf{n}(\mathbf{x},t) \cdot \dot{\mathbf{u}}(\mathbf{x},t)\, dA - \int_{\mathcal{P}\cap S_t} f(\mathbf{x},t) V_n(\mathbf{x},t)\, dA \quad \forall t \in \mathcal{T}, \tag{2.3.2}$$

where $f : \Sigma \to \mathbb{R}$ is the scalar *driving traction* given by

$$f(\cdot,t) = [\![W\big(I_1(\mathbf{G}(\cdot,t))\big)]\!] - \langle\!\langle \mathbf{S}(\cdot,t)\rangle\!\rangle \cdot [\![\mathbf{F}(\cdot,t)]\!] \quad \text{on} \quad S_t \quad \forall t \in \mathcal{T}, \tag{2.3.3}$$

and, given a generic field quantity $g(\cdot,t) : S_t \to \mathbb{R}$ which jumps across S_t at the instant t in $\mathcal{T}$, $\langle\!\langle g(\cdot,t)\rangle\!\rangle$ is defined through

$$\langle\!\langle g(\mathbf{x},t)\rangle\!\rangle = \lim_{h\searrow 0} \tfrac{1}{2}\left(g\big(\mathbf{x}+h\mathbf{n}(\mathbf{x},t),t\big) + g\big(\mathbf{x}-h\mathbf{n}(\mathbf{x},t),t\big)\right) \quad \forall (\mathbf{x},t) \in \Sigma, \tag{2.3.4}$$

In [21] KNOWLES derives (2.3.2) in the absence of inertial effects; following either YATOMI & NISHIMURA [32] or ABEYARATNE & KNOWLES [3] it can be shown that (2.3.3) reduces, in this setting, to

$$f(\cdot,t) = [\![W(I_1(\mathbf{G}(\cdot,t)))]\!] - \overset{\pm}{\mathbf{S}}(\cdot,t) \cdot [\![\mathbf{F}(\cdot,t)]\!] \quad \text{on} \quad S_t \quad \forall t \in \mathcal{T}, \tag{2.3.5}$$

where $\overset{+}{\mathbf{S}}(\cdot,t)$ (*resp.*, $\overset{-}{\mathbf{S}}(\cdot,t)$) is the limiting value of the field $\mathbf{S}(\cdot,t)$ on the side of the interface into which the unit normal $\mathbf{n}(\cdot,t)$ is (*resp.*, is not) directed at the instant t in $\mathcal{T}$.

From (2.3.2) it is clear that the presence of a moving surface of discontinuity S_t of the type considered here may effect the balance of mechanical energy. Let the difference in the rate of work of the mechanical forces external to $\mathcal{P}$ and the rate at which energy is stored in $\mathcal{P}$ be referred to as the *rate of dissipation* of mechanical energy associated with the region $\mathcal{P}$. When treated from a *thermomechanical* perspective, the dissipation rate can be shown to be identical to the product of the temperature and the rate of entropy production—provided that the temperature is spatially uniform and independent of time.[6] The *Clausius-Duhem inequality* then requires that the dissipation rate associated with a motion of the kind envisioned here be non-negative, *i.e.*,

$$\int_{\mathcal{P}\cap S_t} f(\mathbf{x},t)V_n(\mathbf{x},t)\,dA \geq 0 \quad \forall t \in \mathcal{T}, \tag{2.3.6}$$

for every regular subregion $\mathcal{P}$, with $\partial\mathcal{P} \cap S_t$ a set of measure zero in $\partial\mathcal{P}$ at t in $\mathcal{T}$, contained in $\mathcal{R}$. A localization of (2.3.6) at an arbitrary point on the interface therefore yields the inequality

$$fV_n \geq 0 \quad \text{on} \quad \Sigma \tag{2.3.7}$$

as a condition imposed for the *admissibility* of the motion.

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[6] For a detailed discussion of the these issues see ABEYARATNE & KNOWLES [3].

Assume that the surface S_t separates, for each t in $\mathcal{T}$, low and high strain elliptic phases of the three phase material at hand. In the context of a motion which involves such an interface it is necessary (see [3–6]) to supplement, in some fashion, the constitutive information which relates the stress and strain fields. An approach to this taken by ABEYARATNE & KNOWLES [3] entails the provision of a *kinetic relation* which gives the normal velocity of the interface in terms of the driving traction that acts on it or *vice versa*. In the former case a constitutive $\tilde{V} : \mathbb{R} \to \mathbb{R}$ is provided so that

$$V_n = \tilde{V}(f) \quad \forall f \in \mathbb{R}, \tag{2.3.8}$$

while, in the latter case a constitutive function $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ is given so that

$$f = \tilde{\varphi}(V_n) \quad \forall V_n \in \mathbb{R}. \tag{2.3.9}$$

The functions $\tilde{V}$ and $\tilde{\varphi}$ are referred to as the *kinetic response functions.* Both varieties of kinetic response functions will be considered in this investigation. If $\tilde{V}$ is such that $\tilde{V}(f)f \geq 0$ for all f in $\mathbb{R}$ then (2.3.8) is automatically satisfied and $\tilde{V}$ is referred to as *admissible.* If $\tilde{\varphi}(V)V \geq 0$ for all V in $\mathbb{R}$, $\tilde{\varphi}$ is, similarly, referred to as admissible. If an admissible kinetic response function $\tilde{V}$ (or $\tilde{\varphi}$) is continuous on $\mathbb{R}$, then it must satisfy $\tilde{V}(0) = 0$ (or $\tilde{\varphi}(0) = 0$). If, in addition, to being admissible, $\tilde{V}$ (or $\tilde{\varphi}$) is continuously differentiable on $\mathbb{R}$, then $\tilde{V}'(0) \geq 0$ (or $\tilde{\varphi}'(0) \geq 0$). Otherwise, admissiblility implies nothing with regard to the sign of the derivative of a smooth kinetic response function. All kinetic response functions considered herein are assumed to be admissible. See Figure 2 and Figure 3 for graphs of such kinetic response functions.

ABEYARATNE [1], BALL & JAMES [9], GURTIN [17], FOSDICK & MACSITHIGH [12], and SILLING [29] consider either equilibrium states or inertia-free motions and require that the driving traction, f, be identically equal to zero on Σ. This is equivalent to prescribing a supplementary kinetic relation in the form (2.3.9) with

$\tilde{\varphi}$ identically zero on $I\!R$. Provision of such a kinetic relation is, furthermore, a necessary consequence of requiring that a suitable *energy functional* be minimized at each t in $\mathcal{T}$ (see ABEYARATNE [2]).

2.4. Antiplane shear motions of a generalized neo-Hookean material. Suppose, from now on, that $\mathcal{R}$ is a cylindrical region and choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ so that the unit base vector $\mathbf{e}_3$ is parallel to the generatrix of $\mathcal{R}$. The deformation $\hat{\mathbf{y}}$ defined through (2.1.1) consists of an antiplane shear normal to the plane spanned by the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ if it is of the form

$$\hat{\mathbf{y}}(\mathbf{x}, t) = \mathbf{x} + u(x_1, x_2, t)\mathbf{e}_3 \quad \forall(\mathbf{x}, t) \in \mathcal{M}. \tag{2.4.1}$$

Observe that the displacement field intrinsic to an antiplane shear deformation of this type has only one nonzero component which lies in the $\mathbf{e}_3$ direction and is independent of the x_3–coordinate. In (2.4.1) $x_\alpha = \mathbf{x} \cdot \mathbf{e}_\alpha$ for each $\mathbf{x}$ contained in $\mathcal{R}$. The function u will be referred to as the *out-of-plane* displacement field. Inspection of (2.4.1) reveals that any discontinuities in the gradient of $\hat{\mathbf{y}}$ must result from discontinuities in the out-of-plane displacement field and, hence, occur across surfaces which do not vary with the x_3–coordinate. Let S_t denote such a surface at the instant t in $\mathcal{T}$ and let Σ be defined as in (2.1.2).

It is possible to show, following the work of KNOWLES [20] in the inertia-free context, that, although not every hyperelastic, isotropic and incompressible material can sustain nontrivial antiplane shear motions, all generalized neo-Hookean materials are capable of doing so. It is easily shown that for such materials the local balance equations $(2.1.9)_{2,3}$ reduce, in the absense of body forces and under the asssumption that the nominal stress tensor is independent of the x_3-coordinate, to the scalar equation

$$\big(M(\gamma)u_{,\alpha}\big)_{,\alpha} = \rho\ddot{u} \quad \text{on} \quad \mathcal{X} \setminus \Gamma, \tag{2.4.2}$$

where $\mathcal{X}$ is given by $\mathcal{D} \times \mathcal{T}$, $\mathcal{D}$ is a generic cross section of $\mathcal{R}$, and $\Gamma = \{(x_1, x_2, t) | \, (x_1, x_2) \in C_t, t \in \mathcal{T}\}$ with $C_t = \mathcal{D} \cap S_t$ at each t in $\mathcal{T}$. See FOSDICK

& SERRIN [13] and FOSDICK & KAO [11] for a general discussion of circumstances under which the field equations $(2.1.9)_{2,3}$ reduce to a single scalar equation. In (2.4.2) M is the secant modulus in shear as defined in (2.2.8) and $\gamma : \mathcal{X} \setminus \Gamma \to I\!R$ is the *shear strain field* given by

$$\gamma(x_1, x_2, t) = \sqrt{u_{,\alpha}(x_1, x_2, t) u_{,\alpha}(x_1, x_2, t)} \quad \forall (x_1, x_2, t) \in \mathcal{X} \setminus \Gamma. \tag{2.4.3}$$

For a generalized neo-Hookean material subjected to antiplane shear, the jump condition $(2.1.11)_2$ reduces to

$$[\![M(\gamma) u_{,\alpha} n_\alpha]\!] + \rho V_n [\![\dot{u}]\!] = 0 \quad \text{on} \quad \Gamma, \tag{2.4.4}$$

where $\Gamma = \{(\mathbf{x}, t) | \, \mathbf{x} \in C_t, t \in \mathcal{T}\}$ and $\mathbf{n}(\cdot, t) : C_t \to \mathcal{N}$ is a unit normal to C_t, while the kinematic jump condition (2.1.14) becomes

$$[\![u]\!] = 0 \quad \text{on} \quad \Gamma. \tag{2.4.5}$$

It is also readily shown that the driving traction f, introduced in Section 2.3, for a generalized neo-Hookean material subjected to an antiplane shear deformation involving a discontinuity in the gradient and, perhaps, the partial derivative with respect to time of out-of-plane displacement field across a moving curve C_t is given, with the aid of (2.3.5), by

$$f = \int_{\overset{-}{\gamma}}^{\overset{+}{\gamma}} \tau(\gamma)\, d\gamma - \langle\!\langle M(\gamma) u_{,\alpha} \rangle\!\rangle [\![u_{,\alpha}]\!] \quad \text{on} \quad \Gamma. \tag{2.4.6}$$

With reference to (2.1.12), (2.1.13) and (2.3.5) it is easily demonstrated that, in the absence of inertial effects, (2.4.2) is replaced by

$$(M(\gamma) u_{,\alpha})_{,\alpha} = 0 \quad \text{on} \quad \mathcal{X} \setminus \Gamma, \tag{2.4.7}$$

while (2.4.4) becomes

$$[\![M(\gamma) u_{,\alpha}\, n_\alpha]\!] = 0 \quad \text{on} \quad \Gamma, \tag{2.4.8}$$

and (2.4.6) simplifies to

$$f = \int_{\bar{\gamma}}^{\overset{+}{\gamma}} \tau(\gamma)\, d\gamma - M(\overset{\pm}{\gamma}) \overset{\pm}{u}_{,\alpha}\, [\![u_{,\alpha}]\!] \quad \text{on} \quad \Gamma. \tag{2.4.9}$$

Observe that, within the context of an antiplane shear deformation of the type described above, no generality is lost by focussing exclusively upon the motion on a cross-section $\mathcal{D}$ of the cylinder $\mathcal{R}$ and the dynamics of the curve $C_t = \mathcal{D} \cap S_t$. In the following, curves C_t across which the gradient and, perhaps, the partial derivative with respect to time of the out-of-plane displacement field $u(\cdot, \cdot, t)$ jump, at some instant t in $\mathcal{T}$, and which segregate the high and low strain phases of the material at hand will, therefore, be referred to as *phase boundaries.*

3. LINEAR STABILITY OF A PROCESS INVOLVING A STEADILY MOVING PLANAR PHASE BOUNDARY IN A THREE-PHASE MATERIAL

3.1. Description of the base process. Suppose that $\mathcal{B}$ is composed of a three-phase material and that the cylinder $\mathcal{R}$ degenerates so as to occupy all of $I\!R^3$. Let the rectangular Cartesian frame X be as in Section 2.4. Consider an antiplane shear motion on the time interval $(-\infty, 0)$ with an out-of-plane displacement field $u_0(\cdot, t) : I\!R \to I\!R$ given by

$$u_0(x_1, t) = \begin{cases} \gamma_l x_1 + v_l t & \text{if} \quad x_1 < v_0 t, \\ \gamma_r x_1 + v_r t & \text{if} \quad x_1 > v_0 t, \end{cases} \tag{3.1.1}$$

for each t in $(-\infty, 0)$, where the shear strains γ_l and γ_r satisfy one of the following:

$$0 < \gamma_r < \underset{*}{\gamma} < \overset{*}{\gamma} < \gamma_l, \quad 0 < \gamma_l < \underset{*}{\gamma} < \overset{*}{\gamma} < \gamma_r. \tag{3.1.2}$$

Since one of (3.1.2) must hold, there is no loss in generality incurred by assuming that the base interface normal velocity v_0 is non-negative; that is,

$$v_0 \geq 0. \tag{3.1.3}$$

It is clear that u_0 satisfies the differential equation in (2.4.2) on the set $(I\!R^2 \times (-\infty, 0)) \setminus \Gamma_0$ with Γ_0 given by $\{(x_1, x_2, t) | \, (x_1, x_2) \in A_t, t \in (-\infty, 0)\}$ and $A_t = \{(x_1, x_2) | \, x_1 = v_0 t, x_2 \in I\!R\}$ for each t in $(-\infty, 0)$. The moving line A_t is, for each t in $(-\infty, 0)$, a phase boundary.

Assume, in order to comply with the jump conditions in (2.4.4) and (2.4.5) on Γ_0, that the constants γ_l, γ_r, v_l, v_r, and v_0 associated with (3.1.1) are restricted to satisfy the following equations:

$$\begin{aligned} v_r - v_l + v_0(\gamma_r - \gamma_l) = 0, \\ \tau(\gamma_r) - \tau(\gamma_l) + \rho v_0(v_r - v_l) = 0. \end{aligned} \tag{3.1.4}$$

Assume that the normal velocity of the phase boundary in the base process is *locally subsonic* so that v_0 satisfies the following inequality:

$$v_0 < \min\left\{\sqrt{\tau'(\gamma_l)/\rho}, \sqrt{\tau'(\gamma_r)/\rho}\right\}. \tag{3.1.5}$$

It is then permissible[7] to impose the kinetic relation of the form (2.3.8) or (2.3.9) on Γ_0 and require that the parameters γ_l, γ_r, v_l, v_r, and v_0 satisfy one of

$$v_0 = \tilde{V}(f_0), \quad f_0 = \tilde{\varphi}(v_0), \tag{3.1.6}$$

depending upon whether a kinetic relation of the form (2.3.8) or (2.3.9) is provided. In (3.1.6) the base driving traction f_0 is given, with the aid of (2.4.6), by

$$f_0 = \int_{\gamma_l}^{\gamma_r} \tau(\gamma)\, d\gamma - \tfrac{1}{2}(\tau(\gamma_r) + \tau(\gamma_l))(\gamma_r - \gamma_l). \tag{3.1.7}$$

Observe, as a consequence of (3.1.3) and (2.3.7), that f_0 must satisfy

$$f_0 \geq 0. \tag{3.1.8}$$

In a coordinate frame moving with the phase boundary, the base process described involves a piecewise homogeneous shear strain field. If γ_l and γ_r are consistent with $(3.1.2)_1$ then (3.1.3) implies that the base process is one wherein the high strain elliptic phase of the material at hand *grows* at the expense of the low strain elliptic phase; whereas, if γ_l and γ_r comply with $(3.1.2)_2$ then (3.1.3) implies that the base process is such that the low strain elliptic phase of the material at hand *grows* at the expense of the high strain elliptic phase. In either case the discontinuity involved is, for the duration of the motion, a *normal* phase boundary—that is, the angle between the limiting values of the gradient of the

[7] See Abeyaratne & Knowles [4].

out-of-plane displacement field on either side of the phase boundary is zero at every point of the phase boundary over the time interval $(-\infty, 0)$.

Suppose, in addition to all the above, that the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ is chosen so that its derivative is non-zero at the base driving traction f_0; that is, assume that one of the following—as is appropriate to either (2.3.8) or (2.3.9)—must hold :

$$\tilde{V}'(f_0) \neq 0, \quad \tilde{\varphi}'(v_0) \neq 0. \tag{3.1.9}$$

This assumption is made in order to preclude the necessity of going to higher order in the context of the forthcoming stability analysis. See Figure 2 for the graph of a smooth admissible kinetic reponse function which satisfies (3.1.9).

When inertial effects are ignored it is clear that u_0 as defined in (3.1.1) also satisfies the field equation in (2.4.7) on $(\mathbb{R}^2 \times (-\infty, 0)) \setminus \Gamma_0$. Equation $(3.1.4)_1$ is, in this context, still sufficient to satisfy (2.4.5) on Γ_0. In place of $(3.1.4)_2$, the constants γ_l, γ_r, v_l, v_r, and v_0 must, however, satisfy

$$\tau(\gamma_r) - \tau(\gamma_l) = 0, \tag{3.1.10}$$

in order for the jump condition in (2.4.8) to hold on Γ_0. Although the expression for the base driving traction f_0 given in (3.1.7) remains valid in the inertia-free setting, (3.1.10) can be used to show that

$$f_0 = \int_{\gamma_l}^{\gamma_r} \tau(\gamma)\, d\gamma - \tau_*(\gamma_r - \gamma_l), \tag{3.1.11}$$

where $\tau_* = \tau(\gamma_l) = \tau(\gamma_r)$.

Given a shear stress response function τ which describes a particular three-phase material and an arbitrary kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ which describes the dynamics of phase boundaries which may occur therein, there may or may not, in general, exist constants γ_l, γ_r, v_l, v_r, and v_0 which satisfy one of $(3.1.2)_1$ or $(3.1.2)_2$ and are consistent with the restrictions embodied by (3.1.4), (3.1.5),

$(3.1.6)_1$ or $(3.1.6)_2$, and $(3.1.9)_1$ or $(3.1.9)_2$, or in the inertia-free case, $(3.1.4)_1$, (3.1.10), (3.1.5), $(3.1.6)_1$ or $(3.1.6)_2$, (3.1.11) and $(3.1.9)_1$ or $(3.1.9)_2$. Within the context of this investigation it will be assumed, however, that $\tilde{V}$ or $\tilde{\varphi}$ is chosen so that a non-trivial base process exists.

3.2. Perturbation of the base process. Suppose that at the instant $t = 0$ the out-of-plane displacement and velocity fields and the configuration of the phase boundary associated with the motion specified in Section 3.1 are subjected to a perturbation. Let this perturbation be such that the phase boundary can be, at $t = 0+$, described by the graph C_0 of a continuous function $h : I\!R \to I\!R$ of the x_2–coordinate, and segregates elliptic phases of the three-phase material at hand in a sense consistent with that which was present for t in $(-\infty, 0)$. Let the out-of-plane displacement and velocity fields linked to this perturbation be given, respectively, by a once continuously differentiable function $\eta : I\!R^2 \to I\!R$ and a continuous function $\varpi : I\!R^2 \to I\!R$. Assume that h, η and ϖ represent small deviations, in some appropriate sense, from their counterparts in the base process. In particular, suppose that h, η, $\eta_{,\alpha}$, and ϖ are all square integrable on their domains of definition. Furthermore, require that the components of the gradient of η allow the satisfaction of

$$\lim_{x_1^2+x_2^2 \to \infty} \eta_{,\alpha}(x_1, x_2)\eta_{,\alpha}(x_1, x_2) = 0, \tag{3.2.1}$$

while ϖ complies with

$$\lim_{x_1^2+x_2^2 \to \infty} \varpi(x_1, x_2) = 0, \tag{3.2.2}$$

so that the disturbance is *localized* in a neighborhood of the phase boundary associated with the base state at $t = 0$.

The perturbation at $t = 0$ will initiate a new process involving an out-of-plane displacement field $u : I\!R^2 \times \bar{I\!R}_+ \to I\!R$ which is, in general, a *weak solution* of the field equation (2.4.2) and satisfies the jump conditions in (2.4.4) and (2.4.5) at all discontinuities in its gradient, the kinetic relation (2.3.8) or (2.3.9) at all phase boundaries, and the initial conditions

$$u(\cdot,\cdot,0+) = u_0(\cdot,0+) + \eta \quad \text{on} \quad \mathbb{R}^2,$$
$$\dot{u}(\cdot,\cdot,0+) = \dot{u}_0(\cdot,0+) + \varpi \quad \text{on} \quad \mathbb{R}^2. \tag{3.2.3}$$

Since the perturbation is small, assume that the subsequent process involves only a single phase boundary $C_t = \{(x_1, x_2, t)|\, x_1 = \varsigma(x_2, t), x_2 \in \mathbb{R}\}$ for each t in $\mathbb{R}_+$, with $\varsigma : \mathbb{R} \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ continuously differentiable on its domain of definition and defined so that it complies with the initial condition

$$\varsigma(\cdot, 0+) = h \quad \text{on} \quad \mathbb{R}. \tag{3.2.4}$$

With the intent of linearizing the field equation in (2.4.2) about the base process, write, for each t in $\bar{\mathbb{R}}_+$,

$$u(x_1, x_2, t) = u_0(x_1, t) + w(x_1, x_2, t) \quad \forall (x_1, x_2) \in \mathcal{D} \setminus C_t, \tag{3.2.5}$$

where w and its derivatives are assumed to represent small departures from the relevent quantities in the base process. Assume that the components of the gradient of w satisfy the following limits:

$$\lim_{x_1 \to \pm\infty} w_{,1}(x_1, \cdot, \cdot) = 0 \quad \text{on} \quad \mathbb{R} \times \bar{\mathbb{R}}_+,$$
$$\lim_{x_2 \to \pm\infty} w_{,2}(\cdot, x_2, \cdot) = 0 \quad \text{on} \quad \mathbb{R} \times \bar{\mathbb{R}}_+. \tag{3.2.6}$$

From (3.2.3) and (3.2.5) it is clear, moreover, that—when inertial effects are not ignored—the increment w to the out-of-plane displacement field must satisfy the following initial conditions:

$$w(\cdot,\cdot,0+) = \eta \quad \text{on} \quad \mathbb{R}^2,$$
$$\dot{w}(\cdot,\cdot,0+) = \varpi \quad \text{on} \quad \mathbb{R}^2. \tag{3.2.7}$$

It is important to emphasize that these can not be imposed in the inertia-free setting.

Next, define $s : I\!R \times \bar{I\!R}_{+} \to I\!R$, the correction to the interface position due to the perturbation, via

$$\varsigma(\cdot, t) = v_0 t + s(\cdot, t) \quad \text{on} \quad I\!R \quad \forall t \in \bar{I\!R}_{+}. \tag{3.2.8}$$

Note, from (3.2.4) that the increment s to the phase boundary position must satisfy the initial condition

$$s(\cdot, 0+) = h \quad \text{on} \quad I\!R. \tag{3.2.9}$$

Observe that the unit normal vectors $\mathbf{n}_{\pm}(\cdot, t) : I\!R \to \mathcal{N}$ to C_t are given by

$$\mathbf{n}_{\pm}(\cdot, t) = \pm \frac{\mathbf{e}_1 - s_{,2}(\cdot, t)\mathbf{e}_2}{\sqrt{1 + s_{,2}^2(\cdot, t)}} \quad \text{on} \quad I\!R \quad \forall t \in I\!R_{+}. \tag{3.2.10}$$

For the remainder of this work, choose the unit normal vector associated with the plus sign in (3.2.10) and drop this sign when referring to it. The normal velocity $V_n(\cdot, t) : I\!R \to I\!R$ of C_t is given, for each t in $I\!R_{+}$, by

$$V_n(\cdot, t) = \frac{v_0 + \dot{s}(\cdot, t)}{\sqrt{1 + s_{,2}^2(\cdot, t)}} \quad \text{on} \quad I\!R \quad \forall t \in I\!R_{+}. \tag{3.2.11}$$

3.3. Linearization of the field equations associated with the process initiated by the perturbation. Let $\mathcal{D}_t^l$ and $\mathcal{D}_t^r$ denote, for each t in $I\!R_{+}$, plane sets defined by

$$\mathcal{D}_t^l = \{(x_1, x_2) |\, x_1 \leq \varsigma(x_2, t)\}, \quad \mathcal{D}_t^r = I\!R^2 \setminus \mathring{\mathcal{D}}_t^l. \tag{3.3.1}$$

Let $\mathcal{X}_l$ and $\mathcal{X}_r$ be given, in turn, by

$$\mathcal{X}_l = \{(x_1, x_2, t) |\, (x_1, x_2) \in \mathcal{D}_t^l, t \in I\!R_{+}\}, \tag{3.3.2}$$

and

$$\mathcal{X}_r = \{(x_1, x_2, t) | \, (x_1, x_2) \in \mathcal{D}_t^r, t \in I\!R_+\}. \tag{3.3.3}$$

The field equations which hold on $\mathring{\mathcal{X}}_l$ and $\mathring{\mathcal{X}}_r$ can be obtained by linearizing the partial differential equation in (2.4.2) about γ_l and γ_r, respectively. First, consider the derivation of the field equation which holds on $\mathring{\mathcal{X}}_l$. From (2.4.3), (3.2.5) and the assumption regarding the magnitude of the spatial gradient of the increment w to the out-of-plane displacement field it is clear that

$$\begin{aligned} \gamma &= \sqrt{(\gamma_l + w_{,1})^2 + w_{,2}^2} = \sqrt{\gamma_l^2 + 2\gamma_l w_{,1} + w_{,\alpha} w_{,\alpha}} \\ &\cong \gamma_l + w_{,1} \quad \text{on} \quad \mathring{\mathcal{X}}_l. \end{aligned} \tag{3.3.4}$$

From (2.2.8), (3.3.4) and Taylor's theorem it is further evident that

$$M(\gamma) \cong M(\gamma_l + w_{,1}) \cong M(\gamma_l) + M'(\gamma_l) w_{,1} \quad \text{on} \quad \mathring{\mathcal{X}}_l. \tag{3.3.5}$$

Next, using (3.2.5) and (3.3.5) in the left-hand-side of the partial differential equation in (2.4.2) gives

$$\begin{aligned} (M(\gamma) u_{,\alpha})_{,\alpha} &\cong [(M(\gamma_l) + M'(\gamma_l) w_{,1}) u_{,\alpha}]_{,\alpha} \\ &\cong [(M(\gamma_l) + M'(\gamma_l) w_{,1})(\gamma_l + w_{,1})]_{,1} \\ &\qquad + [(M(\gamma_l) + M'(\gamma_l) w_{,1}) w_{,2}]_{,2} \\ &\cong \tau'(\gamma_l) w_{,11} + M(\gamma_l) w_{,22} \quad \text{on} \quad \mathring{\mathcal{X}}_l \end{aligned} \tag{3.3.6}$$

Note that, in deriving (3.3.6), the smoothness of τ and hence M, the identity

$$\tau(\gamma) = M(\gamma)\gamma \quad \forall \gamma \in \bar{I\!R}_+,$$

which follows from (2.2.6) and (2.2.8) and its consequence

$$\tau'(\gamma) = M(\gamma) + M'(\gamma)\gamma \quad \forall \gamma \in \bar{I\!R}_+ \setminus \{\underset{*}{\gamma}, \overset{*}{\gamma}\}$$

have been used. Observe, also, that $\tau'(\gamma_l) > 0$ by whichever of (3.1.2) is appropriate, and $M(\gamma_l) > 0$ by (2.2.9). From (3.1.1), (3.2.5) and (3.3.6), the linearized field equation which holds on $\mathring{\mathcal{X}}_l$ is

$$a_l^2 w_{,11} + b_l^2 w_{,22} = \ddot{w}, \tag{3.3.7}$$

where the positive constants a_l and b_l are defined by

$$a_l = \sqrt{\tau'(\gamma_l)/\rho}, \quad b_l = \sqrt{M(\gamma_l)/\rho}. \tag{3.3.8}$$

Similarly, the linearized field equation which holds on $\mathring{\mathcal{X}}_r$ is

$$a_r^2 w_{,11} + b_r^2 w_{,22} = \ddot{w}, \tag{3.3.9}$$

where the positive constants a_r and b_r are defined by

$$a_r = \sqrt{\tau'(\gamma_r)/\rho}, \quad b_r = \sqrt{M(\gamma_r)/\rho}. \tag{3.3.10}$$

In writing (3.3.10), the positivity of $\tau'(\gamma_r) > 0$ and $M(\gamma_r) > 0$, which are results of (2.2.9) and whichever of $(3.1.2)_{1,2}$ is appropriate, have been used.

From (2.4.7) and (3.3.6) it is clear that, in the inertia-free setting, equations (3.3.7) and (3.3.9) are supplanted by

$$a_l^2 w_{,11} + b_l^2 w_{,22} = 0, \tag{3.3.11}$$

and

$$a_r^2 w_{,11} + b_r^2 w_{,22} = 0, \tag{3.3.12}$$

which hold on $\mathring{\mathcal{X}}_l$ and $\mathring{\mathcal{X}}_r$, repectively.

3.4. Linearization of the jump conditions and kinetic relation associated with the process initiated by the perturbation. Since the set

$\Gamma = \{(x_1, x_2, t) | (x_1, x_2) \in C_t, t \in \mathbb{R}_+\}$ represents the post-disturbance trajectory of the phase boundary, the jump conditions in (2.4.4) and (2.4.5) and the kinetic balance equation in (2.3.8) or (2.3.9)—with V_n and f given, respectively, by (3.2.11) and (2.4.6)—must hold on it. Assume, henceforth, that the function s introduced via (3.2.8) and its derivatives are small in the same sense that w is small. Note, first, that this assumption implies, using (3.2.10) and (3.2.11), the following approximations for $\mathbf{n}$ and V_n on Γ:

$$\mathbf{n} \cong \mathbf{e}_1 - s_{,2}\,\mathbf{e}_2 \quad \text{on} \quad \Gamma, \qquad V_n \cong v_0 + \dot{s} \quad \text{on} \quad \Gamma. \tag{3.4.1}$$

It will now be shown that a further consequence of the above stipulation regarding the size of s and its derivatives is that, within the error associated with the linearization, the jump conditions in (2.4.4) and the kinetic relation can be enforced on an undisturbed continuation of the phase boundary intrinsic to the base process I given by

$$I = \{(x_1, x_2, t) | x_1 = v_0 t, x_2 \in \mathbb{R}, t \in \mathbb{R}_+\}, \tag{3.4.2}$$

but, when V_n appears in any of these, the contribution due to $\dot{s}$ from $(3.4.1)_2$ must be retained. To see this consider, for example, the limiting values of the x_1–component of the gradient of the out-of-plane displacement field u on either side of the phase boundary; note that by Taylor's theorem, (3.1.1), and (3.2.8),

$$\begin{aligned} u_{,1}(\varsigma(x_2,t)-, x_2, t) &:= \lim_{h \searrow 0} u_{,1}(v_0 t - h + s(x_2,t), x_2 + h s_{,2}(x_2,t), t) \\ &\cong \gamma_l + w_{,1}((v_0 t + s(x_2,t))-, x_2, t) \\ &\cong \gamma_l + w_{,1}(v_0 t-, x_2, t) + w_{,11}(v_0 t-, x_2, t) s(x_2,t) \\ &= u_{,1}(v_0 t-, x_2, t) + w_{,11}(v_0 t-, x_2, t) s(x_2,t) \\ &\qquad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+, \end{aligned} \tag{3.4.3}$$

and, similarly,

$$
\begin{aligned}
u,_1(\varsigma(x_2,t)+,x_2,t) &:= \lim_{h\searrow 0} u,_1(v_0t+h+s(x_2,t),x_2+hs,_2(x_2,t),t) \\
&\cong \gamma_r + w,_1((v_0t+s(x_2,t))+,x_2,t) \\
&\cong \gamma_r + w,_1(v_0t+,x_2,t) + w,_{11}(v_0t+,x_2,t)s(x_2,t) \\
&= u,_1(v_0t+,x_2,t) + w,_{11}(v_0t+,x_2,t)s(x_2,t) \\
&\qquad \forall (x_2,t) \in I\!R \times I\!R_+. \qquad (3.4.4)
\end{aligned}
$$

Since both s and the derivatives of w are assumed small and of the same order, the quadratic terms on the right-hand-sides of (3.4.3) and (3.4.4) can be neglected. This produces

$$
u,_1(\varsigma(x_2,t)\pm,x_2,t) \cong u,_1(v_0t\pm,x_2,t) \quad \forall (x_2,t) \in I\!R \times I\!R_+. \qquad (3.4.5)
$$

Within the scope of the linearization, $u,_1(\varsigma(x_2,t)\pm,x_2,t)$ is, therefore, obtained, for each (x_2,t) in $I\!R \times I\!R_+$, by evaluating $u,_1(\cdot,x_2,t)$ at $v_0t\pm$. Analogous remarks also hold for the x_2–component of the gradient of the out-of-plane displacement field, the out-of-plane velocity field and the shear strain field. Hence, with the aid of (3.4.1), (2.4.4) implies that

$$
\begin{aligned}
0 =& \tau(\gamma_r) - \tau(\gamma_l) + \rho v_0(v_r - v_l) + \rho v_0(\gamma_r - \gamma_l)\dot{s}(x_2,t) \\
&+ \rho\left(a_r^2 w,_1(v_0t+,x_2,t) - a_l^2 w,_1(v_0t-,x_2,t)\right) \\
&+ \rho v_0\big(\dot{w}(v_0t+,x_2,t) - \dot{w}(v_0t-,x_2,t)\big) \quad \forall (x_2,t) \in I\!R \times I\!R_+. \qquad (3.4.6)
\end{aligned}
$$

From $(3.1.4)_2$, the constant term on the right-hand-side of (3.4.6) is zero and, hence, the linearization of the jump condition which enforces the balance of linear momentum across the phase boundary leads to

$$
\begin{aligned}
0 =& a_r^2 w,_1(v_0t+,\cdot,t) - a_l^2 w,_1(v_0t-,x_2,t) + v_0(\gamma_r - \gamma_l)\dot{s}(x_2,t) \\
&+ v_0\left(\dot{w}(v_0t+,x_2,t) - \dot{w}(v_0t-,x_2,t)\right) \quad \forall (x_2,t) \in I\!R \times I\!R_+. \qquad (3.4.7)
\end{aligned}
$$

Prior to deriving the linearized kinetic relation it is convenient linearize the driving traction f. From (3.2.4), (2.4.3) and the foregoing discussion one finds that

$$f(x_2,t) \cong f_0 + \tfrac{1}{2}\rho(\gamma_l - \gamma_r)(a_r^2 w_{,1}(v_0t+, x_2, t) + a_l^2 w_{,1}(v_0t-, x_2, t))$$
$$+ \tfrac{1}{2}(\tau(\gamma_r) - \tau(\gamma_l))(w_{,1}(v_0t+, x_2, t) - w_{,1}(v_0t-, x_2, t))$$
$$\forall (x_2,t) \in I\!R \times I\!R_+, \tag{3.4.8}$$

where the base driving traction f_0 is given by (3.1.7). From $(3.1.4)_2$ it is clear, furthermore, that (3.4.8) simplifies to read

$$f(x_2,t) \cong f_0 + \tfrac{1}{2}\rho(\gamma_l - \gamma_r)((a_r^2 - v_0^2)w_{,1}(v_0t+, x_2, t) + (a_l^2 - v_0^2)w_{,1}(v_0t-, x_2, t))$$
$$\forall (x_2,t) \in I\!R \times I\!R_+. \tag{3.4.9}$$

If the kinetic relation is of the form given in (2.3.8), then $(3.4.1)_2$, (3.4.9) and Taylor's theorem lead to

$$v_0 + \dot{s}(x_2,t) = \tilde{V}(f_0)$$
$$+ \tfrac{1}{2}\rho\tilde{V}'(f_0)(\gamma_l - \gamma_r)((a_r^2 - v_0^2)w_{,1}(v_0t+, x_2, t) + (a_l^2 - v_0^2)w_{,1}(v_0t-, x_2, t))$$
$$\forall (x_2,t) \in I\!R \times I\!R_+. \tag{3.4.10}$$

If, on the other hand, the kinetic relation is provided in the form (2.3.9), then $(3.4.1)_2$, (3.4.9) and Taylor's theorem give, similarly,

$$\tilde{\varphi}(v_0) + \tilde{\varphi}'(v_0)\dot{s}(x_2,t) = f_0$$
$$+ \tfrac{1}{2}\rho(\gamma_l - \gamma_r)((a_r^2 - v_0^2)w_{,1}(v_0t+, x_2, t) + (a_l^2 - v_0^2)w_{,1}(v_0t-, x_2, t))$$
$$\forall (x_2,t) \in I\!R \times I\!R_+. \tag{3.4.11}$$

Use of the $(3.1.6)_1$ and $(3.1.6)_2$ in (3.4.10) and (3.4.11), respectively, results in the linearized kinetic relation

$$\begin{aligned} \dot{s}(x_2,t) =& \frac{\gamma_l-\gamma_r}{2v_*}((a_r^2-v_0^2)w,_1(v_0t+,x_2,t)+(a_l^2-v_0^2)w,_1(v_0t-,x_2,t)) \\ & \forall(x_2,t)\in I\!R\times I\!R_+, \end{aligned} \tag{3.4.12}$$

where the constant v_* is defined by either

$$v_* = \frac{1}{\rho\tilde{V}'(f_0)}, \tag{3.4.13}$$

if the kinetic relation is furnished in the form (2.3.8), or

$$v_* = \frac{\tilde{\varphi}'(v_0)}{\rho}, \tag{3.4.14}$$

if the kinetic relation is supplied in the form (2.3.9). Note, from (3.1.9), that v_* is a real—but nonzero—constant.

Consider, now, the task of linearizing the jump condition in (2.4.5). Note, from (3.2.5), that (2.4.5) implies

$$\begin{aligned} 0 =& u_0(\zeta(x_2,t)+,t)-u_0(\zeta(x_2,t)-,t)+w(\zeta(x_2,t)+,x_2,t)-w(\zeta(x_2,t)-,x_2,t) \\ & \forall(x_2,t)\in I\!R\times I\!R_+, \end{aligned} \tag{3.4.15}$$

Certainly, from (3.1.1), (3.2.8) and $(3.1.4)_1$,

$$u_0(\zeta(x_2,t)+,t)-u_0(\zeta(x_2,t)-,t) = (\gamma_l-\gamma_r)s(x_2,t) \quad \forall(x_2,t)\in I\!R\times I\!R_+; \tag{3.4.16}$$

furthermore, from (3.2.8), Taylor's theorem, and the assumption regarding the small magnitude of products involving s and the derivatives of w,

$$\begin{aligned} w(\varsigma(x_2,t)\pm,x_2,t) &\cong w(v_0t\pm,x_2,t)+w,_1(v_0t\pm,x_2,t)s(x_2,t) \\ &\cong w(v_0t\pm,x_2,t) \quad \forall(x_2,t)\in I\!R\times I\!R_+. \end{aligned} \tag{3.4.17}$$

Hence, from (3.4.15)–(3.4.17), the linearization of the jump condition which enforces the continuity of displacement across the phase boundary yields

$$w(v_0t+, x_2, t) - w(v_0t-, x_2, t) = (\gamma_l - \gamma_r)s(x_2, t) \quad \forall (x_2, t) \in I\!R \times I\!R_+. \quad (3.4.18)$$

Differentiation of (3.4.18) with respect to time then results in the following identity:

$$\begin{aligned} 0 =& \dot{w}(v_0t+, x_2, t) - \dot{w}(v_0t-, x_2, t) + (\gamma_r - \gamma_l)\dot{s}(x_2, t) \\ &+ v_0(w_{,1}(v_0t+, x_2, t) - w_{,1}(v_0t-, x_2, t)) \\ &\qquad \forall (x_2, t) \in I\!R \times I\!R_+, \end{aligned} \quad (3.4.19)$$

Appropriate substitution of (3.4.19) into the linearization of the jump condition which enforces the balance of linear momentum across the phase boundary (3.4.7) then gives rise to

$$\begin{aligned} 0 =& (a_r^2 - v_0^2)w_{,1}(v_0t+, x_2, t) - (a_l^2 - v_0^2)w_{,1}(v_0t-, x_2, t) \\ &+ 2v_0(\gamma_l - \gamma_r)\dot{s}(x_2, t) \quad \forall (x_2, t) \in I\!R \times I\!R_+. \end{aligned} \quad (3.4.20)$$

By virtue of the foregoing calculations it is crucial to note that, within the scope of the linearization, it is legitimate to enforce the partial differential equations in (3.3.7) and (3.3.9) on the interiors of the sets Ω_l and Ω_r defined by

$$\Omega_l = \{(x_1, x_2, t) | \, (x_1, x_2) \in \Pi_t^l, t \in I\!R_+\}, \quad (3.4.21)$$

with $\Pi_t^l = \{(x_1, x_2) | \, x_1 \leq v_o t, x_2 \in I\!R\}$ for each t in $I\!R_+$, and

$$\Omega_r = \{(x_1, x_2, t) | \, (x_1, x_2) \in \Pi_t^r, t \in I\!R_+\}, \quad (3.4.22)$$

with $\Pi_t^r = \{(x_1, x_2) | \, x_1 \geq v_o t, x_2 \in I\!R\}$ for each t in $I\!R_+$, instead of the sets $\mathring{\mathcal{X}}_l$ and $\mathring{\mathcal{X}}_r$ (recall (3.3.2) and (3.3.3)).

In the inertia-free case it is readily shown that, while (3.4.18) continues to hold, (3.4.17) is replaced by

$$a_r^2 w,_1 (v_0 t+, x_2, t) - a_l^2 w,_1 (v_0 t-, x_2, t) = 0 \quad \forall (x_2, t) \in I\!R \times I\!R_+, \qquad (3.4.23)$$

and (3.4.11) simplifies to read

$$\dot{s}(x_2, t) = \frac{\gamma_l - \gamma_r}{2v_*} (a_r^2 w,_1 (v_0 t+, x_2, t) + a_l^2 w,_1 (v_0 t-, x_2, t)) \quad \forall (x_2, t) \in I\!R \times I\!R_+. \qquad (3.4.24)$$

Finally, remarks analogous to those made regarding the enforcement of the partial differential equations in (3.3.7) and (3.3.9) on $\mathring{\Omega}_l$ and $\mathring{\Omega}_r$ apply also to those in (3.3.11) and (3.3.12).

3.5. Linearized description of the post perturbation process. In this section the linearized field equations, jump conditions, kinetic relation, initial conditions (where appropriate), and far field decay conditions satisfied by the increments w and s to the out-of-plane displacement field and the interface position are listed in both the inertial and inertia-free cases.

In the inertial case, (3.3.7) and (3.3.9) give the following linearized field equations

$$\begin{aligned} a_l^2 w,_{11} + b_l^2 w,_{22} &= \ddot{w} \quad \text{on} \quad \mathring{\Omega}_l, \\ a_r^2 w,_{11} + b_r^2 w,_{22} &= \ddot{w} \quad \text{on} \quad \mathring{\Omega}_r. \end{aligned} \qquad (3.5.1)$$

In addition, from (3.4.20) and (3.4.18), the following jump conditions hold

$$\begin{aligned} [\![(a^2 - v_0^2) w,_1]\!] &= 2v_0(\gamma_r - \gamma_l)\dot{s} \quad \text{on} \quad I, \\ [\![w]\!] &= (\gamma_l - \gamma_r) s \quad \text{on} \quad I, \end{aligned} \qquad (3.5.2)$$

where

$$a_+^2 = a_r^2, \quad a_-^2 = a_l^2. \qquad (3.5.3)$$

Next, from (3.4.12) the following linearized kinetic relation holds:

$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle (a^2 - v_0^2) w,_1 \rangle\!\rangle \quad \text{on} \quad I. \tag{3.5.4}$$

The initial conditions satisfied by w and s are, from (3.2.4) and (3.2.7),

$$\begin{aligned} w(\cdot,\cdot,0+) &= \eta \quad \text{on} \quad \mathbb{R}^2, \\ \dot{w}(\cdot,\cdot,0+) &= \varpi \quad \text{on} \quad \mathbb{R}^2, \\ s(\cdot,0+) &= h \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{3.5.5}$$

Finally, from (3.2.6), it is assumed that the following far field conditions hold

$$\begin{aligned} \lim_{x_1 \to \pm\infty} w,_1 (x_1, \cdot, t) &= 0 \quad \text{on} \quad \mathbb{R}, \\ \lim_{x_2 \to \pm\infty} w,_2 (\cdot, x_2, t) &= 0 \quad \text{on} \quad \mathbb{R}, \end{aligned} \tag{3.5.6}$$

for each t in $\mathbb{R}_+$.

In the inertia-free case, (3.5.1) is replaced by the following:

$$\begin{aligned} a_l^2 w,_{11} + b_l^2 w,_{22} &= 0 \quad \text{on} \quad \mathring{\Omega}_l, \\ a_r^2 w,_{11} + b_r^2 w,_{22} &= 0 \quad \text{on} \quad \mathring{\Omega}_r. \end{aligned} \tag{3.5.7}$$

Furthermore, the jump condition $(3.5.2)_1$ is, by virtue of (3.4.23), replaced by

$$[\![a^2 w,_1]\!] = 0 \quad \text{on} \quad I, \tag{3.5.8}$$

while $(3.5.2)_2$ continues to hold. Following (3.4.24), the linearized kinetic relation (3.5.4) is superseded by

$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle a^2 w,_1 \rangle\!\rangle \quad \text{on} \quad I, \tag{3.5.9}$$

In the absence of inertial effects initial conditions cannot be given for the increments to the out-of-plane displacement and velocity fields w and $\dot{w}$; the initial condition $(3.5.5)_3$ pertaining to s still, however, continues to be applicable. The decay conditions (3.5.6) also still hold.

3.6. Normal mode analysis in the inertia-free setting. An approximate means for analyzing the linear stability of the base process described in Section 3.1 is afforded by the study of the inertia-free initial value problem consisting of (3.5.7)–(3.5.9), $(3.5.2)_2$, $(3.5.5)_3$ and (3.5.6). Observe that, by virtue of the linearization, the relevant partial differential equations, jump conditions and kinetic relation are all linear with constant coefficients; note, also, that the domains $\mathring{\Pi}^l_t$ and $\mathring{\Pi}^r_t$ are, for each t in $\mathbb{R}_+$, rectangular. It is therefore possible to find a solution to the linearized partial differential equations, jump conditions and kinetic relation in the form

$$w(x_1, x_2, t) = \begin{cases} W_l e^{+\xi_l(x_1 - v_0 t)} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}^l_t, \quad t \in \mathbb{R}_+, \\ W_r e^{-\xi_r(x_1 - v_0 t)} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}^r_t, \quad t \in \mathbb{R}_+, \end{cases} \tag{3.6.1}$$
$$s(x_2, t) = S e^{i\kappa x_2} e^{pt} \quad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+,$$

where the *amplitudes* W_l, W_r and S, *wave-numbers* ξ_l, ξ_r and κ, and *growth-rate* p are all constants. To comply with the decay condition $(3.5.6)_1$ it is clear that $\Re(\xi_l)$ and $\Re(\xi_r)$ must both be positive. The *Ansatz* (3.6.1) is not, in general, consistent with the initial condition $(3.5.5)_3$ or the decay condition $(3.5.6)_2$; since the initial disturbance h is stipulated to be square integrable on $\mathbb{R}$, and hence can be represented as a Fourier integral—

$$h(x_2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{h}(\kappa) e^{i\kappa x_2} \, d\kappa \quad \forall x_2 \in \mathbb{R}, \tag{3.6.2}$$

it is reasonable to expect that stability results can be obtained by a *normal-mode analysis*; such an analysis entails substitution of (3.6.1) into (3.5.7)–(3.5.9) to determine the growth-rate p as a function of the positive wave-numbers ξ_l and ξ_r and the real wave-number κ. In the context of such an undertaking, the amplitude S and wave-number κ are regarded as given and non-zero, while, along with the growth-rate p, the amplitudes W_l and W_r and wave-numbers ξ_l and ξ_r are—due to the present lack of inertial effects—to be determined. If there exists a complex growth-rate p with positive real part which arises as a solution to the aforementioned problem then the base process will be referred to as *linearly unstable.* Otherwise, the base process will be called *linearly stable.*

Substitution of (3.6.1) into (3.5.7)–(3.5.9) and $(3.5.2)_2$ yields the following system of five equations in the five unknowns W_l, W_r, ξ_l, ξ_r and p:

$$
\begin{aligned}
(a_l^2\xi_l^2 - b_l^2\kappa^2)W_l &= 0,\\
(a_r^2\xi_l^2 - b_r^2\kappa^2)W_r &= 0,\\
a_r^2\xi_r W_r + a_l^2\xi_l W_l &= 0,\\
W_r - W_l - (\gamma_r - \gamma_l)S &= 0,\\
\frac{\gamma_l - \gamma_r}{2v_*}(a_r^2\xi_r W_r - a_l^2\xi_l W_l) + Sp &= 0.
\end{aligned}
\tag{3.6.3}
$$

First, $(3.6.3)_{1,2}$ give, recalling that ξ_l and ξ_r must be positive,

$$
\xi_l = \frac{b_l}{a_l}|\kappa|, \quad \xi_r = \frac{b_r}{a_r}|\kappa|. \tag{3.6.4}
$$

In particular, (3.6.4) implies that, the wave-numbers ξ_l and ξ_r must be real and, further, that, of ξ_l, ξ_r and κ, the growth-rate p will depend only on κ. Next, $(3.6.3)_{3,4,5}$ can be solved for the remaining unknowns W_l, W_r and p to yield

$$
\begin{aligned}
W_l &= -\frac{\nu^2|\kappa|}{a_l b_l(\gamma_l - \gamma_r)}S,\\
W_r &= -\frac{\nu^2|\kappa|}{a_r b_r(\gamma_r - \gamma_l)}S,\\
p &= -\frac{\nu^2}{v_*}|\kappa|,
\end{aligned}
\tag{3.6.5}
$$

where the constant ν^2 is defined as follows:

$$
\nu^2 = \frac{a_l b_l a_r b_r(\gamma_l - \gamma_r)^2}{a_l b_l + a_r b_r}. \tag{3.6.6}
$$

Equation $(3.6.5)_3$ which gives the growth-rate p in terms of the wave-number κ and various physical parameters intrinsic to the problem at hand will be referred to as the *dispersion relation.* It is clear from $(3.6.5)_3$ that the growth-rate p is real. Since $\kappa \neq 0$ by assumption and $v_* \neq 0$ by (3.1.10), it is clear, moreover, that the signs of p and v_* are opposite: *viz.*,

$$\operatorname{sgn}(p) = \operatorname{sgn}(-v_*). \tag{3.6.7}$$

The linear stability of the base process, in the absence of inertial effects, thus depends entirely upon the sign of v_*. Significantly, the wave-number κ plays no role in determining stability. Moreover, the local mechanical properties of the high and low strain elliptic phases of the three-phase material at hand do not effect the stability criteria. If $v_* > 0$ then the base process is linearly stable with respect to all initial disturbances h of the type under consideration. If, alternatively, $v_* < 0$ then the base process is linearly unstable with respect to all disturbances h of the type considered here. To summarize, in the absence of inertial effects, the criterion $v_* < 0$ is necessary and sufficient for the base process to be unstable with respect to any perturbation of the type put into consideration in Section 3.2. Thus, if the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$—as is appropriate to whether a kinetic relation of the form (2.3.8) or (2.3.9) is prescribed—is non-monotonic, the base process may be linearly unstable with respect to any initial disturbance h of the type being considered. Discussion regarding the physical reasonableness of a non-monotonic kinetic response function is left until Section 3.9.

Suppose, now, that the normal velocity v_0 of the phase boundary in the base process is zero. Note that this is equivalent to requiring that the base process be mechanically equilibrated. Then, by the admissibility of the kinetic response function, one or both of f_0 and v_* must be zero. Since the latter contradicts (3.1.10), $v_0 = 0$ implies, at present, that $f_0 = 0$ and, furthermore, either $\tilde{V}'(0) > 0$ or $\tilde{\varphi}'(0) > 0$. Thus the foregoing stability dichotomy implies that a mechanically equilibrated base process of the type defined in Section 3.1 must be linearly stable with respect to all initial disturbances h of the type considered here.

Note that the foregoing results are consistent with those presented by FRIED [14] in a study of the distribution of driving traction along a particular non-planar phase boundary. The latter work shows, roughly speaking, that the driving traction is less in regions of the interface where the curvature is larger. Specifically, if small deviations of the relevant type from a planar interface with constant base driving traction f_* are considered and a kinetic response function $\tilde{V}$ is provided so that $\tilde{V}'(f_*) < 0$ then the normal velocity of the interface in regions of higher curvature exceeds that in regions of lesser curvature. The interface, then, has a marked tendency to evolve in a manner wherein its curved portions move ahead of its flat portions—and, hence, become less planar. Such a response is intuitively *unstable.* If, however, f_* is such that $\tilde{V}'(f_*) > 0$, flat portions of the interface tend to *catch up* with the curved portions so that the interface regains its planar shape—hence, the planar interface is *stable.*

3.7. An alternative to normal mode analysis in the inertia-free setting. The analysis performed in Section 3.6 resulted in a necesary and sufficient condition for the base process to be unstable with respect to an arbitrary perturbation of the type discussed in Section 3.2; it did not, however, encompass a means for *tracking* the evolution of either a stable or unstable response to perturbation. This section will focus on an analysis which does allow the post-perturbation evolution of the phase boundary to be followed in the linear regime. Since the analysis is performed in the linear realm it is, in the case of an unstable response, only of value in a short time interval following the perturbation at $t = 0$. Consider, again, the inertia-free initial boundary value problem consisting of (3.5.7)–(3.5.9), $(3.5.2)_2$, $(3.5.5)_3$ and (3.5.6). Techniques from potential theory are used in the appendix of this work to show that the increment $w(\cdot, \cdot, t)$ to the out-of-plane displacement field can be represented, for each t in $\mathbb{R}_+$, in terms of the sum of a single-layer potential, with density proportional to $\dot{s}$, and a double-layer potential potential, with density proportional to s—each on the line

$x_1 = v_0 t$—as follows:

$$w(x_1, x_2, t) = \int_{-\infty}^{+\infty} K_1(x_1 - v_0 t, x_2 - \zeta, \tfrac{b_d}{a_d})\dot{s}(\zeta, t)\, d\zeta$$
$$+ \int_{-\infty}^{+\infty} K_2(x_1 - v_0 t, x_2 - \zeta, \tfrac{b_d}{a_d}) s(\zeta, t)\, d\zeta \quad \forall (x_1, x_2) \in \mathring{\Pi}_t^d. \quad (3.7.1)$$

The sub- and superscripts d in (3.7.1) are to be replaced by either l or r—as appropriate, while the kernels $K_\alpha(\cdot, \cdot, c) : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ are defined via

$$\begin{aligned} K_1(x, y, c) &= \frac{1}{2\pi} \frac{v_*}{\nu^2(\gamma_l - \gamma_r)} \ln \sqrt{c^2 x^2 + y^2} \quad \forall (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ K_2(x, y, c) &= \frac{1}{2\pi} \frac{cx}{c^2 x^2 + y^2} \quad \forall (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \end{aligned} \qquad (3.7.2)$$

for each c in $\mathbb{R}_+$, and s, introduced in (3.2.8), satisfies the following functional initial value problem

$$\begin{aligned} \dot{s}(x_2, t) &= \frac{\nu^2}{\pi v_*} \int_{-\infty}^{+\infty} \frac{s_{,2}(\zeta, t)\, d\zeta}{x_2 - \zeta} \quad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+, \\ s(x_2, 0) &= h(x_2) \quad \forall x_2 \in \mathbb{R}. \end{aligned} \qquad (3.7.3)$$

The integral on the right-hand-side of $(3.7.3)_1$ is, as indicated, of the Cauchy principal value type.

The stability of the base process rests on the stability of the solution s to the foregoing functional initial value problem. To investigate this issue, assume—because of the square integrability of h—that s can be represented in the form

$$s(x_2, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{s}(\kappa, t) e^{i\kappa x_2}\, d\kappa \quad \forall (x_2, t) \in \mathbb{R} \times \bar{\mathbb{R}}_+, \qquad (3.7.4)$$

for some function $\hat{s} : \mathbb{R} \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ which satisfies

$$\hat{s}(\kappa, 0) = \int_{-\infty}^{+\infty} h(x_2) e^{-i\kappa x_2}\, dx_2 =: \hat{h}(\kappa) \quad \forall \kappa \in \mathbb{R}. \qquad (3.7.5)$$

Now, from (3.7.3)–(3.7.5) it is possible to derive, for each fixed κ in $I\!R$, the following initial value problem:

$$\dot{\hat{s}}(\kappa,\cdot)+\frac{\nu^2}{v_*}|\kappa|\hat{s}(\kappa,\cdot)=0 \quad \text{on} \quad I\!R, \qquad \hat{s}(\kappa,0)=\hat{h}(\kappa). \tag{3.7.6}$$

Observe that the Laplace transform could be applied to (3.7.6) to derive the dispersion relation $(3.6.5)_3$. The linear stability results obtained in Section 3.6 would, then, follow immediately. Instead, note that (3.7.6) can be solved, formally, to yield the following expression for $\hat{s}$:

$$\hat{s}(\kappa,t)=\hat{h}(\kappa)e^{-\frac{\nu^2}{v_*}|\kappa|t} \quad \forall(\kappa,t)\in I\!R\times \bar{I\!R}_+. \tag{3.7.7}$$

Inspection of (3.7.7) reveals that, as demonstrated in Section 3.6, the linear stability of the base process is decided entirely by the sign of v_*. Note that if $v_*>0$ then, by (3.7.4) and (3.7.7),

$$s(x_2,t)=\frac{\nu^2 t}{\pi v_*}\int_{-\infty}^{+\infty}\frac{h(\zeta)\,d\zeta}{(x_2-\zeta)^2+(\frac{\nu^2 t}{v_*})^2} \quad \forall(x_2,t)\in I\!R\times \bar{I\!R}_+. \tag{3.7.8}$$

Hence, under these circumstances, it is clear that

$$\lim_{t\to\infty} s(x_2,t)=0 \quad \forall x_2\in I\!R, \tag{3.7.9}$$

for all square integrable initial disturbances h. The base process is, therefore, linearly stable with respect to all such disturbances.

Suppose, now, that $v_*<0$—which, as remarked in Section 3.6, can occur only if $v_0>0$. It is then instructive to consider a special example where the initial disturbance h is given by

$$h(x_2)=\frac{1}{\pi}\frac{A}{1+(\frac{x_2}{\ell})^2} \quad \forall x_2\in I\!R, \tag{3.7.10}$$

with ℓ taken, without loss of generality, to be a positive constant. See Figure 2 for a representative graph of h. Note that since $(3.7.3)_1$ is linear it is, in general, possible to consider an initial disturbance which involves a linear superposition of disturbances of the type defined in (3.7.10), *e.g.*, $H : \mathbb{R} \to \mathbb{R}$ given by

$$H(x_2) = \frac{1}{\pi}\sum_{n=1}^{N} \frac{A_n}{1+(\frac{x_2-y_n}{\ell_n})^2} \quad \forall x_2 \in \mathbb{R},$$

where N is a natural number and A_n, y_n and ℓ_n are real constants for each n in $\{1, 2, \ldots, N\}$. The results which follow are easily extended to apply to such a generalization of the initial disturbance. Now, the Fourier transform $\hat{h}$ of the function h defined in (3.7.10) is given by

$$\hat{h}(\kappa) = A\ell e^{-\ell|\kappa|} \quad \forall \kappa \in \mathbb{R}. \tag{3.7.11}$$

Next, (3.7.4), (3.7.7) and (3.7.11) imply that

$$s(x_2, t) = \frac{A\ell}{2\pi}\int_{-\infty}^{+\infty} e^{-(\ell-\frac{\nu^2 t}{|v_*|})|\kappa|} e^{i\kappa x_2}\, d\kappa \quad \forall (x_2, t) \in \mathbb{R} \times [0, t_c), \tag{3.7.12}$$

where the *critical* time t_c is chosen small enough so that the amplitude of s does not contradict the assumptions necessary for the linearization to remain valid over the time interval $[0, t_c)$; *i.e.*,

$$t_c \ll \frac{|v_*|\ell}{\nu^2} =: t_\infty. \tag{3.7.13}$$

From (3.7.12) it transpires, further, that

$$s(x_2, t) = \frac{1}{\pi}\frac{At_\infty(t_\infty - t)}{(t_\infty - t)^2 + t_\infty^2(\frac{x_2}{\ell})^2} \quad \forall (x_2, t) \in \mathbb{R} \times [0, t_c). \tag{3.7.14}$$

The expression for $s(\cdot, t)$ on $\mathbb{R}$ given in (3.7.14) shows that the amplitude and wavelength of the hump associated with the special initial disturbance under consideration increase and decrease, respectively, as t approaches t_c. Hence, up until some critical time, the assumption $v_* < 0$ implies that the hump associated with the initial disturbance grows in an unstable fashion. See Figure 4 for a depiction of this unstable evolution. Observe, moreover, that, as t approaches t_∞, the function s given by (3.7.14) forms a *delta sequence* so that

$$\lim_{t \to t_\infty} s(\cdot, t) = A\ell\delta \quad \text{on} \quad \mathbb{R}, \tag{3.7.15}$$

where δ is the Dirac delta distribution.

3.8. Normal mode and energy analysis with inertial effects included. The applicability of the stability criterion obtained in Section 3.6, and recovered in Section 3.7, is, so far, limited to contexts where inertial effects are insignificant. This section is concerned with the extension, as far as possible, of the inertia-free stability dichotomy based upon the sign of v_* into the inertial realm. As a first step toward achieving this goal a normal mode analysis analogous to that performed in Section 3.6 is undertaken with inertial terms present. The conclusion of this analysis is that $v_* < 0$ is necessary and sufficient to guarantee the linear instability of the base process with respect to a particular subset of the class of perturbations put into consideration in Section 3.2. It is then argued that the remaining class of perturbations not covered by the normal mode analysis is, in fact, very small. Next an energy argument is used to show that $v_* < 0$ is a necessary condition for the base process to be linearly unstable with respect to an arbitrary disturbance within the full class of perturbations introduced in Section 3.2.

Consider, now, the initial boundary value problem composed by (3.5.1), (3.5.2) and (3.5.4)–(3.5.6). It is, once again, possible to find a solution to the system formed by the linearized field equations, jump conditions and kinetic relation in the form of (3.6.1). Remarks regarding the decomposition of the initial data—which now includes η and ϖ as well as h—and the satisfaction of $(3.5.6)_2$

akin to those made in Section 3.6 are pertinent. Observe that, in the inertial case, the amplitudes W_r, W_l and S and wave-numbers ξ_l, ξ_r and κ must be viewed as given for the normal mode analysis to be effective in determining necessary and sufficient conditions—via a dispersion relation analogous to $(3.6.5)_3$—for instability with respect to arbitrary disturbances of the out-of-plane displacement and velocity fields and the interface within the class of perturbations put forth in Section 3.2. Substitution, however, of (3.6.1) into (3.5.1), (3.5.2) and (3.5.4) produces the following relations

$$\begin{aligned}
\xi_l &= \frac{f_l(\kappa,p) - v_0 p}{a_l^2 - v_0^2}, \\
\xi_r &= \frac{f_r(\kappa,p) + v_0 p}{a_r^2 - v_0^2}, \\
W_l &= \frac{(\gamma_l - \gamma_r)(v_0 p - f_r(\kappa,p))}{f_l(\kappa,p) + f_r(\kappa,p)} S, \\
W_r &= \frac{(\gamma_l - \gamma_r)(v_0 p + f_l(\kappa,p))}{f_l(\kappa,p) + f_r(\kappa,p)} S, \\
p &= -\frac{(\gamma_l - \gamma_r)^2 (f_l(\kappa,p) f_r(\kappa,p) + v_0^2 p^2)}{v_*(f_l(\kappa,p) + f_r(\kappa,p))},
\end{aligned} \tag{3.8.1}$$

where $f_l : I\!R \times \mathbb{C} \to \mathbb{C}$ and $f_r : I\!R \times \mathbb{C} \to \mathbb{C}$ are given by

$$\begin{aligned}
f_l(\kappa,p) &= \sqrt{(a_l^2 - v_0^2) b_l^2 \kappa^2 + a_l^2 p^2} \quad \forall (\kappa,p) \in I\!R \times \mathbb{C}, \\
f_r(\kappa,p) &= \sqrt{(a_r^2 - v_0^2) b_r^2 \kappa^2 + a_r^2 p^2} \quad \forall (\kappa,p) \in I\!R \times \mathbb{C}.
\end{aligned} \tag{3.8.2}$$

The square roots in (3.8.2) are defined so that when their respective arguments are real and positive the resulting square root is also real and positive.

It is clear from $(3.8.1)_{1,2,3,4}$ that, for (3.6.1) to represent a solution to (3.5.1), (3.5.2) and (3.5.4), the amplitudes W_l and W_r and wave-numbers ξ_l and ξ_r cannot be chosen independently of S and κ. Hence, the normal mode procedure is only capable of analyzing the linear stability of the base process with respect to a certain class of perturbations; that is, it is only possible—via this analysis—to determine conditions necessary and sufficent for the instability of the base process with respect to a subset of the class of perturbations put into consideration in Section 3.2. To do this it suffices to analyze the zero structure of the dispersion relation $(3.8.1)_5$ as a function of the growth-rate p for fixed values of the wave-number κ and the parameters γ_l, γ_r, v_0, a_l, a_r, b_l, b_r, ρ and v_*. This is done below.

If $v_* < 0$ it is evident, by inspection, that there exists a real positive root p to $(3.8.1)_5$ for all admissible values of κ, γ_l, γ_r, v_0, a_l, a_r, b_l, b_r and ρ. Hence, $v_* < 0$ is sufficient—indepedent of the value of the wave-number κ—to guarantee that the base process is unstable with respect to the narrowed class of initial disturbances at hand. To establish the converse, show that the condition $v_* > 0$ is sufficent to guarantee that there cannot exist unstable zeros p to $(3.8.1)_5$. Let $F(\kappa, \cdot) : \mathbb{C} \rightarrow \mathbb{C}$ be given, for each κ in $\mathbb{R}$, by

$$F(\kappa, p) = \frac{(\gamma_l - \gamma_r)^2 (f_l(p,\kappa) f_r(p,\kappa) + v_0^2 p^2)}{f_l(p,\kappa) + f_r(p,\kappa)} \quad \forall p \in \mathbb{C}. \tag{3.8.3}$$

Now, it is easy to show that if $\Re(p) > 0$ then $\Re(F(\kappa,p)) > 0$ for every κ in $\mathbb{R}$. Hence, if $v_* > 0$ then all roots p to $(3.8.1)_5$ must have non-positive real parts. Thus, $v_* < 0$ is also a necessary condition for the base process to be unstable with respect to an arbitrary element of the class of perturbations which can be tested via the normal mode analysis. Observe that this conclusion is independent of the wave-number κ associated with (3.6.1).

If, in place of the foregoing normal mode analysis, a full-fledged Fourier-Laplace transform analysis of (3.5.1), (3.5.2) and (3.5.4)–(3.5.6) is performed, then the narrowing of the class of initial data necessitated by the normal mode analysis does not occur. Furthermore, in this case it transpires that the Fourier-Laplace transform of s can be expressed in the form

$$S(\kappa, p) = \frac{\hat{h}(\kappa) + \frac{\gamma_l - \gamma_r}{v_*} H(\kappa, p)}{p + \frac{1}{v_*} F(\kappa, p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C}, \tag{3.8.4}$$

where $\hat{h}$ is the Fourier transform of h and, for each (κ, p) in $\mathbb{R} \times \mathbb{C}$, $H(\kappa, p)$ is a functional of the initial data η and ϖ. Evidently, it is possible that there exist combinations of η, ϖ and h which would allow the cancellation of an unstable zero in the denominator of the expression on the right-hand-side of (3.8.4) by a zero in its numerator. It is equally clear, however, that the set of such initial data constitutes a very small one within the full class of initial data purveyed in Section 3.2. Hence, except in response to very special initial perturbations the condition $v_* < 0$ is necessary and sufficent for the base state to be linearly unstable. The normal mode analysis thus shows that $v_* < 0$ is necessary and sufficient for the base process to be unstable with respect to all but a very small subset of the initial disturbances under consideration.

Now, to show that $v_* < 0$ is a necessary condition for the base state to be linearly unstable with respect to any perturbation in the full class introduced in Section 3.2 consider the dependence of the following energy $\mathcal{E}$ on time:

$$\begin{aligned}\mathcal{E}(t) =& \frac{\rho\ell}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\left(\dot{v}^2(x_1,x_2,t) + (a_l^2 - v_0^2)v,_1^2(x_1,x_2,t) + b_l^2 v,_2^2(x_1,x_2,t)\right)dx_1 dx_2 \\ &+ \frac{\rho\ell}{2}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\left(\dot{v}^2(x_1,x_2,t) + (a_r^2 - v_0^2)v,_1^2(x_1,x_2,t) + b_r^2 v,_2^2(x_1,x_2,t)\right)dx_1 dx_2 \\ & \forall t \in [0, t_*). \end{aligned} \tag{3.8.5}$$

In (3.8.5) ℓ is a positive constant which carries units of length and the function $v : \mathbb{R}^2 \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ is defined via

$$w(x_1,x_2,t) = v(x_1 - v_0 t, x_2, t) \quad \forall (x_1,x_2,t) \in \mathbb{R}^2 \times \bar{\mathbb{R}}_+. \tag{3.8.6}$$

By $(3.5.5)_{1,2}$ and the stipulated square integrability of $\eta,_\alpha$ and ϖ it is evident that $\mathcal{E}(0)$ exists. Assume, then, that there exists a positive time t_*, which may be very small, such that the integrals which define $\mathcal{E}$ exist on the time interal

$(0, t_*)$. Recalling (3.1.5), (3.3.8) and (3.3.10), it is clear that $\mathcal{E}$ is non-negative on its domain of definition. Now, an alternate definition of linear stability in terms of the energy $\mathcal{E}$ is that it remain bounded on $\bar{\mathbb{R}}_+$. A series of calculations which use (3.5.2) and (3.5.4)–(3.5.6) then show that the *power* $\dot{\mathcal{E}}$ is given by

$$\dot{\mathcal{E}}(t) = -\rho \ell v_* \int_{-\infty}^{+\infty} \dot{s}^2(x_2, t)\, dx_2 \quad \forall t \in [0, t_*). \tag{3.8.7}$$

Certainly, if $v_* > 0$ then $\dot{\mathcal{E}}(t) \leq 0$ for all t in $[0, t_*)$; furthermore, under these circumstances, the interval over which $\mathcal{E}$ is defined can be extended in increments to $\bar{\mathbb{R}}_+$ leading to the following inequality

$$\dot{\mathcal{E}}(t) \leq 0 \quad \forall t \in \bar{\mathbb{R}}_+. \tag{3.8.8}$$

The condition $v_* > 0$ is, therefore, sufficient to ensure that the energy $\mathcal{E}$ remains bounded for all time and, hence—according to the above definition of linear stability—that the base process is stable with respect to all perturbations of the type introduced in Section 3.2. Now, since $v_* \neq 0$ by assumption, it is clear that $v* < 0$ is a necessary condition for the base process to become unstable with respect to any perturbation of the type under consideration.

From the foregoing discussion it is clear that admissibility, the assumed smoothness of $\tilde{V}$ or $\tilde{\varphi}$ and (3.1.9) imply that when $v_0 = 0$, the base driving traction $f_0 = 0$ and $\tilde{V}'(0) > 0$ or $\tilde{\varphi}'(0) > 0$ and, hence, a mechanically equilibrated base process of the kind introduced in Section 3.1 is—as in the inertia-free case—stable with respect to all perturbations considered in this work.

To recapitulate, the calculations performed this section show that $v_* < 0$ is a necessary condition for the base process to be unstable with respect to any initial disturbance of the type under consideration and a sufficient condition for the base process to be unstable to all but a small subset of these initial disturbances. Comparing the results obtained in Section 3.6 with those obtained here, it is apparent that inertial effects do not significantly effect the linear stability criteria.

3.9. Discussion. The analysis of Sections 3.6 and 3.8 shows that—regardless of whether intertial effects are included or not—$v_* < 0$ is necessary for the base process to be linearly unstable with respect to any perturbation of the type purveyed in Section 3.2 and sufficient for the base process to be unstable with respect to all but a small subset of these perturbations. In Section 3.7 it is shown, in the absence of inertial effects, that if $v_* < 0$ then the linear instability will manifest itself in a manner whereby the morphology of the interface evolves so as to develop plate-like or dendritic structures. Recall, from the alternate definitions (3.4.13) and (3.4.14) of v_*, that $v_* < 0$ can occur only if the relevant kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ is locally decreasing at f_0 or v_0, respectively. Is it physically plausible for $\tilde{V}$ or $\tilde{\varphi}$ to display such non-monotonicity? Recall from Section 2.3 that admissibility—from the perspective of the Clausius-Duhem inequality—does not restrict the monotonicity of the kinetic response function. OWEN, SCHOEN & SRINIVASAN [26] suggest, moreover, that *unstable kinetics* of the sort where $\tilde{\varphi}$ has a single maximum as a function of V_n—and thus must be a non-monotonic function of V_n—may be responsible for the rapid growth of plate-like structures which is observed experimentally. Furthermore, there exist other physical contexts, the most notable of which include unstable crack growth and the *slip-stick* peeling of tape,[8] where the analogues of such non-monotonic kinetic response functions are considered physically acceptable.

It is reasonable, based on the foregoing discussion, to refer to the type of linear instability which occurs when $v_* < 0$ as *kinetic instablity.* This investigation has demonstrated that, under the current kinematical and constitutive restrictions, in a purely mechanical context, independent of whether inertial effects are accounted for or not, the only means by which a linear instability involving the emergence of plate-like or dendritic structures from a planar interface can occur is if a kinetic instability is present. It is possible that, in a broader context, other brands of instability may be present. This may, in particular, be true when thermal effects

[8] See AIFANTIS [7], AUBREY, WELDING & WONG [8] and MAUGIS & BARQUINS [23].

are taken into consideration. An investigation which takes both mechanical and thermal effects into consideration is performed by FRIED [15].

APPENDIX

In this appendix (3.7.1) and (3.7.3) are established. Consider the inertia-free initial value problem comprised of (3.5.7)–(3.5.9), $(3.5.2)_2$, $(3.5.5)_3$ and (3.5.6). Define $v : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ by

$$w(x_1, x_2, t) = v(\hat{\eta}_1(x_1, t), \hat{\eta}_2(x_2), t) \quad \forall (x_1, x_2, t) \in \mathbb{R}^2 \times \mathbb{R}_+, \tag{A.1}$$

where $\hat{\eta}_1(\cdot, t) : \mathbb{R} \to \mathbb{R}$ and $\hat{\eta}_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$\hat{\eta}_1(x_1, t) = \begin{cases} \frac{b_l}{a_l}(x_1 - v_0 t) & \text{if} \quad x_1 < v_0 t, \\ \frac{b_r}{a_r}(x_1 - v_0 t) & \text{if} \quad x_1 > v_0 t, \end{cases} \tag{A.2}$$

for each t in $\bar{\mathbb{R}}_+$, and

$$\hat{\eta}(x_2) = x_2 \quad \forall x_2 \in \mathbb{R}. \tag{A.3}$$

Then, in terms of v, (3.5.7)–(3.5.9) yield, for each t in $\mathbb{R}_+$,

$$\begin{aligned} v,_{\alpha\alpha} &= 0 \quad \text{on} \quad ((-\infty, 0) \cup (0, \infty)) \times \mathbb{R}, \\ a_r b_r v,_1(0+, \cdot, t) - a_l b_l v,_1(0-, \cdot, t) &= 0 \quad \text{on} \quad \mathbb{R}, \\ v(0+, \cdot, t) - v(0-, \cdot, t) &= (\gamma_l - \gamma_r) s(\cdot, t) \quad \text{on} \quad \mathbb{R} \\ a_r b_r v,_1(0+, \cdot, t) + a_l b_l v,_1(0-, \cdot, t) &= \frac{2v_*}{(\gamma_l - \gamma_r)} \dot{s}(\cdot, t) \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{A.4}$$

Now, from $(A.4)_{2,4}$ it is clear that

$$v,_1(0+, \cdot, t) - v,_1(0-, \cdot, t) = \frac{v_*(a_l b_l - a_r b_r)}{a_l b_l a_r b_r (\gamma_l - \gamma_r)} \dot{s}(\cdot, t) \quad \text{on} \quad \mathbb{R}, \tag{A.5}$$

for each t in $\mathbb{R}_+$ Consider, next, $(A.4)_{1,3}$ and (A.5). Since $v(\cdot, \cdot, t)$ is harmonic on $((-\infty, 0) \cup (0, \infty)) \times \mathbb{R}$ for each t in $\bar{\mathbb{R}}_+$, the jumps in the normal derivative of v and in v itself across the line $l = \{(\eta_1, \eta_2) | \, \eta_1 = 0, \eta_2 \in \mathbb{R}\}$ indicate that it can be represented by the sum of appropriate single- and double-layer potentials with densities proportional to $\dot{s}$ and s, respectively. That is, for each t in $I\!R_+$,

$$v(\eta_1,\eta_2,t) = \int_{-\infty}^{+\infty} \left(L_1(\eta_1,\eta_2-\zeta)\dot{s}(\zeta,t) + L_2(\eta_1,\eta_2-\zeta)s(\zeta,t)\right)\, d\zeta$$
$$\forall(\eta_1,\eta_2)\in((-\infty,0)\cup(0,\infty))\times I\!R, \qquad \text{(A.6)}$$

where $L_\alpha : I\!R^2\setminus\{(0,0)\}\times I\!R_+ \to I\!R$ are given by

$$\begin{aligned} L_1(\eta_1,\eta_2) &= \frac{1}{2\pi}\frac{v_*}{\nu^2(\gamma_l-\gamma_r)}\ln\sqrt{\eta_1^2+\eta_2^2} \quad \forall(\eta_1,\eta_2)\in I\!R^2\setminus\{(0,0)\}, \\ L_2(\eta_1,\eta_2) &= \frac{1}{2\pi}\frac{\eta_1}{\eta_1^2+\eta_2^2} \quad \forall(\eta_1,\eta_2)\in I\!R^2\setminus\{(0,0)\}, \end{aligned} \qquad \text{(A.7)}$$

and where ν^2 is as defined in (3.6.6). With the aid of (A.1)–(A.3), (A.6) gives the representation (3.7.1) of w. Next, use of standard identities from potential theory gives the following expression for the limiting values of $v_{,1}(\cdot,\cdot,t)$ on either side of l

$$v_{,1}(0\pm,\eta_2,t) = \frac{1}{2\pi}\,⨍_{-\infty}^{+\infty}\frac{s_{,2}(\zeta,t)\,d\zeta}{\eta_2-\zeta} \pm \frac{v_*(a_lb_l-a_rb_r)}{2a_lb_la_rb_r(\gamma_l-\gamma_r)^2}\dot{s}(\eta_2,t) \quad \forall\eta_2\in I\!R, \quad \text{(A.8)}$$

for each t in $I\!R_+$. Substitution of (A.8) into either $(A.4)_2$ or $(A.4)_4$ then yields the functional equation $(3.7.3)_1$. The initial condition $(3.7.3)_2$ follows directly from $(3.5.5.)_3$.

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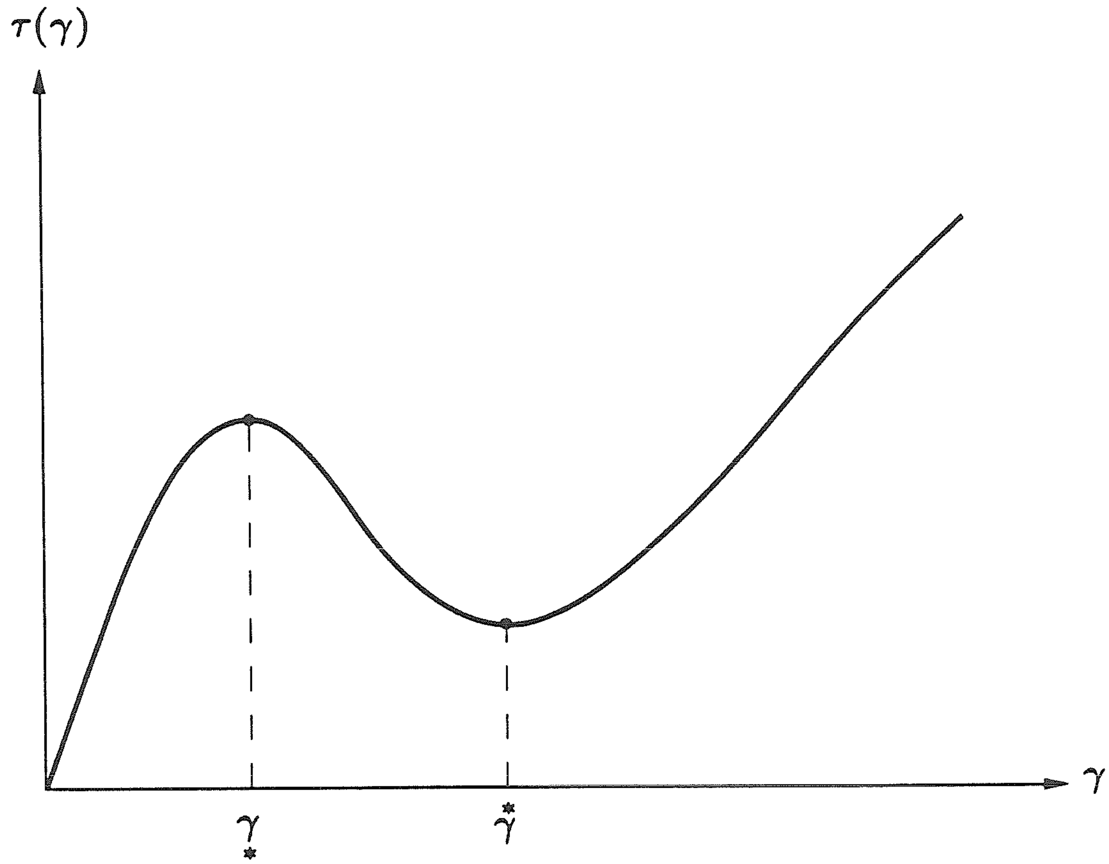

Figure 1: Graph of the shear stress response function τ.

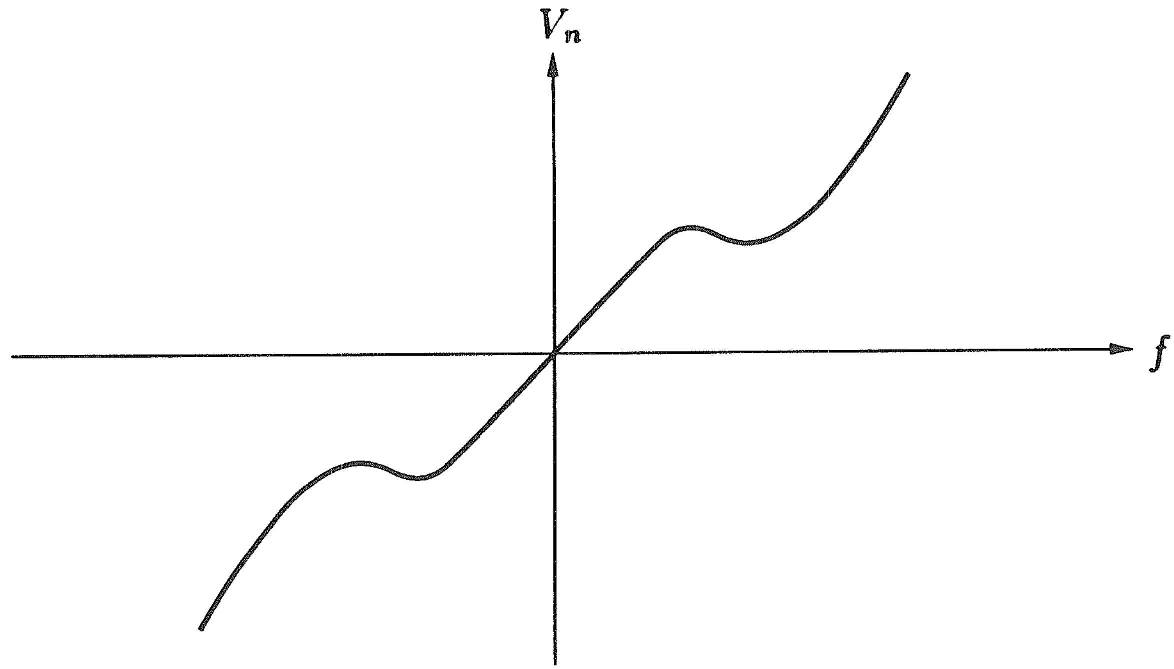

Figure 2: Graph of the kinetic response function $\tilde{V}$.

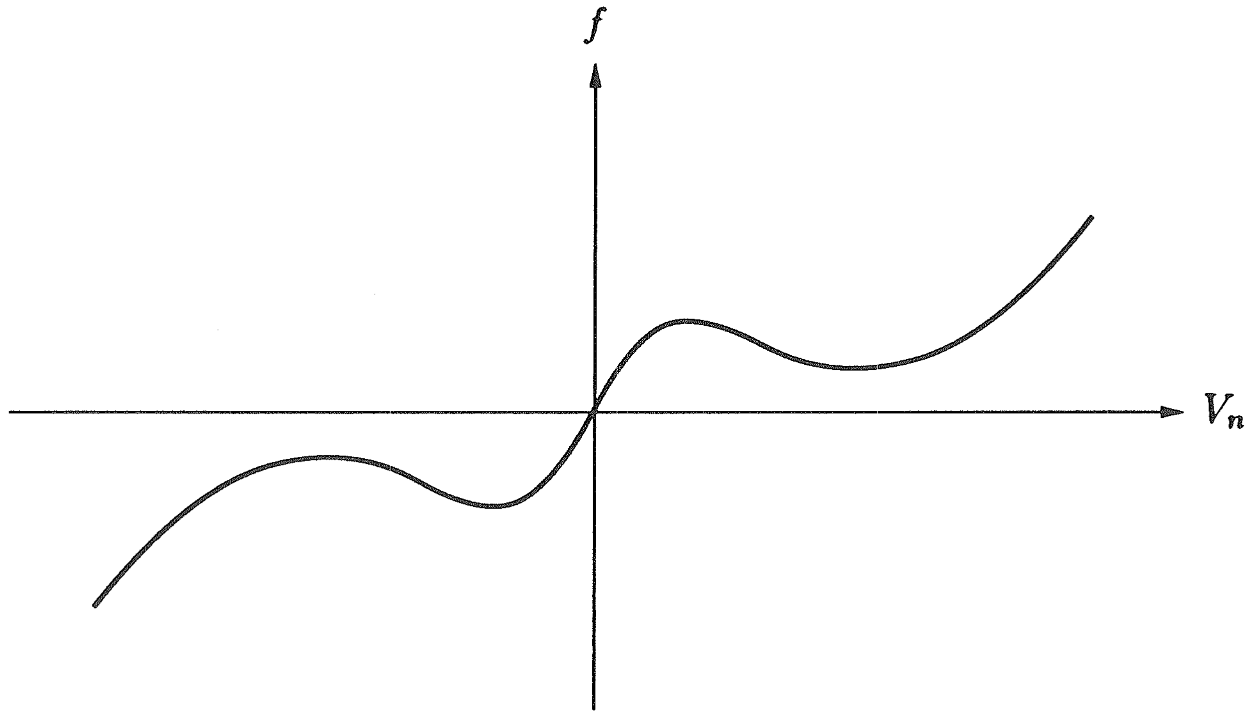

Figure 3: Graph of the kinetic response function $\tilde{\varphi}$.

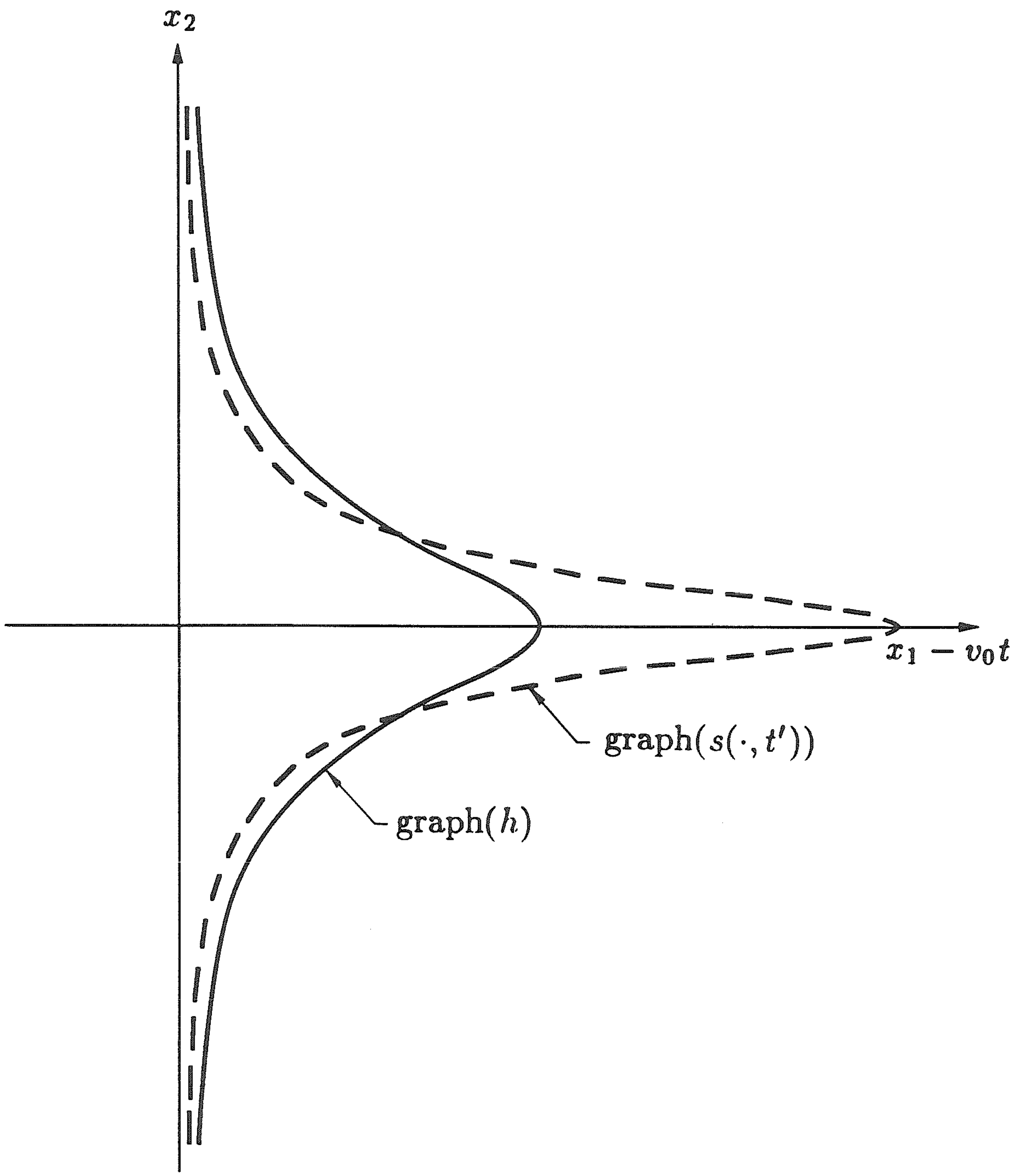

Figure 4: Superimposed graphs of the h and $s(\cdot, t')$ for some $t' \ll t_c$.

LINEAR STABILITY OF A TWO-PHASE PROCESS INVOLVING A STEADILY PROPAGATING PLANAR PHASE BOUNDARY IN A SOLID: PART 2. THERMOMECHANICAL CASE

1. INTRODUCTION

Recently [9], motivated by a desire to determine whether continuum mechanical models for displacive solid-solid phase transformations can predict the emergence of *plate-like* or *dendritic* structures from states involving planar phase boundaries, a purely mechanical two-phase dynamical process in a *non-elliptic* generalized neo-Hookean material was considered. The process involved an antiplane shear motion with a single steadily propagating planar *phase boundary* separating *high* and *low* strain elliptic phases of the relevant material. In a frame moving with the phase boundary, the shear strain field was piecewise homogeneous and the angle between the limiting values of the gradient of the out-of-plane displacement field on either side of the phase boundary was zero—so that the phase boundary was, for each instant of the motion, of *normal* type. The linear stability of this process with respect to a broad class of perturbations was then investigated. It was shown that a necessary and sufficient condition for the process to be linearly stable was that the kinetic response function—which gives the *driving traction* acting on a phase boundary in terms of the normal velocity of the phase boundary, or *vice-versa*—be a locally increasing function of its argument at the value corresponding to the base process. A necessary consequence of this stability criterion is that, in order for the process to be unstable, the kinetic response function must exhibit a non-monotonic dependence on its argument. Non-monotonic kinetic response functions are admissible under the *Clausius-Duhem* version of the second law of thermodynamics (specialized to isothermal conditions for the purposes of the purely mechanical process discussed in [9]); the work of Owen, Schoen & Srinivasan [15] implies, furthermore, that a non-monotonic relation between interfacial driving traction and normal velocity may be operative in the *unstable kinetics* which are observed to accompany the emergence and growth of plate-like structures. Under such kinetics, the results obtained [9] suggest than an evolution from a planar to a plate-like phase boundary morphology might be possible with the confines of a purely mechanical theory.

Thermal effects are manifestly absent from the purely mechanical investigation in [9]. The experimental work of Clapp & Yu [5], Grujicic, Olson & Owen [10] and Cong Dahn, Morphy & Rajan [6] indicates that temperature effects do play an intrinsic, if not entirely understood, part in the kinetics of phase boundaries in displacive solid-solid phase transformations. The investigation which follows is, therefore, directed toward understanding the outcome, with regard to the morphological stability of states involving planar phase boundaries, when thermal as well as mechanical effects are taken into consideration in a model for displacive solid-solid phase transformations. Of particular interest is the question of whether thermal effects allow for an evolution from planar to plate-like phase boundary morphology under kinetics which are mechanically stable in the sense of [9]. The paper is organized as follows.

Chapter 2 is dedicated to preliminaries. Following a synopsis of the notation to be used, Section 2.1 introduces the kinematics and fundamental balance principles which will be needed thereafter. Section 2.2 focuses on the *rate of entropy production* due to the presence of phase boundaries and introduces the associated notion of the *driving traction* acting on a phase boundary. In Section 2.3 a *thermoelastic material* is defined and in Section 2.4 the particular class of thermoelastic materials which will be used in the forthcoming analysis is introduced. Section 2.5 is concerned with the *kinetic relation* and allied *kinetic response function.* In the final section of Chapter 2 the kinematics are specialized to those of antiplane shear and a thermoelastic antiplane shear motion is defined.

Chapter 3 is devoted to the linear stability analysis of an isothermal two-phase process which involves a steadily propagating planar phase boundary in an arbitrary thermoelastic material within the class introduced in Section 3.4. The relevant process, which is a straightforward generalization of that used in the purely mechanical investigation [9], is introduced in Section 3.1, while the class of perturbations to which it will be subjected is put forth in Section 3.2. Each admissible perturbation involves, in general, a disturbance of the configuration of the phase boundary and of the displacement, velocity and temperature fields in a small neighborhood of the phase boundary. All disturbances are assumed to be small in some appropriate sense. The kinematics of the perturbation are restricted to those of antiplane shear. It is assumed that the post-perturbation process is a thermoelastic antiplane shear and involves only one phase boundary. Sections 3.3 and 3.4 address, respectively, the linearization—about the unperturbed process—of the field equations which hold away from the phase boundary and the jump conditions and kinetic relation which hold on the phase boundary. After a specialization of the base process, a summary of the complete linearized system of field equations, jump conditions, kinetic relation and boundary and initial conditions which describe the process generated by the perturbation is presented in Section 3.5. As in [9], both the inertial and inertia-free cases are included. The combined results of Sections 3.6 and 3.7 show that whenever it is static, regardless of the presence of inertial effects, the base process is linearly stable with respect to all perturbations of the type introduced in Section 3.2. Section 3.8 deals with the case where the base process involves an interface propagating at non-zero velocity. A *normal mode analysis* is performed which leads to a variety of conditions sufficient for the instability of the undisturbed process. These conditions depend on the monotonicity properties of the kinetic response function. Highlighted in Section 3.9 is one set of sufficient conditions which is of particular interest. The relevant conditions alter the conclusions reached in the purely mechanical context considered in [9] in two ways. First, in contrast to the results obtained in the latter setting, instability may arise even when the normal velocity of the phase boundary is a monotonically increasing function of driving traction as long as the temperature dependence in the kinetic response function is of an appropriate nature. Second, the instability that arises in these thermomechanical circumstances occurs only in the *long waves* of the the Fourier decomposition of the moving phase boundary, suggesting that the interface favors a highly wrinkled configuration. This conclusion is akin to that reached in similar linear stability analyses performed within the context of models for dendritic crystal growth where an otherwise thermally unstable process is stabilized for sufficiently large wave-numbers by the inclusion of surface tension at the interface in lieu of including mechanical effects.[1] The final topic addressed in Section 3.9 pertains to the physical suitability of kinetic response functions which are *mechanically stable* but *thermally unstable*.

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[1] See, for example, LANGER [13], MULLINS & SEKERKA [14] and STRAIN [16].

2. PRELIMINARIES

2.1. Notation, kinematics and balance principles. In the following $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. The intervals $(0, \infty)$ and $[0, \infty)$ are represented by $\mathbb{R}_+$ and $\bar{\mathbb{R}}_+$. The symbol $\mathbb{R}^n$, with n equal to 2 or 3, represents real n-dimensional space equipped with the standard Euclidean norm. If U is a set, then its closure, interior and boundary are designated by $\bar{U}$, $\overset{\circ}{U}$, and ∂U, respectively. The complement of a set V with respect to U is written as $U \setminus V$. Given a function $\psi : U \to W$ and a subset V of U, $\psi(V)$ stands for the image of V under the map ψ.

Vectors and linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ (referred to herein as *tensors*) are distinguished from scalars with the aid of boldface type—lower and upper case for vectors and tensors, respectively. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$, their inner product is then written as $\mathbf{a} \cdot \mathbf{b}$; the Euclidean norm of $\mathbf{a}$ is, further, written as $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. The set of unit vectors—that is, vectors with unit Euclidean norm—in $\mathbb{R}^3$ is designated by $\mathcal{N}$. The symbol $\mathcal{L}$ refers to the set of tensors, $\mathcal{L}_+$ denotes the set of all tensors with positive determinant, and $\overset{+}{\mathcal{S}}$ stands for the collection of all symmetric positive definite tensors. If $\mathbf{F}$ is in $\mathcal{L}$ then $\mathbf{F}^T$ represents its transpose; if, moreover, $\det \mathbf{F} \neq 0$, then the inverse of $\mathbf{F}$ and its transpose are written as $\mathbf{F}^{-1}$ and $\mathbf{F}^{-T}$, respectively. The notation $\mathbf{a} \otimes \mathbf{b}$ refers to the tensor $\mathbf{A}$, formed by the outer product of $\mathbf{a}$ with $\mathbf{b}$, defined such that $\mathbf{A}\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ for any vector $\mathbf{c}$ in $\mathbb{R}^3$. If $\mathbf{A}$ and $\mathbf{B}$ are tensors then their inner product is written as $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr} \mathbf{A}\mathbf{B}^T$.

When component notation is used, Greek indices range only over $\{1, 2\}$; summation of repeated indices over the appropriate range is implicit. A subscript preceded by a comma denotes partial differentiation with respect to the corresponding coordinate. Also, a superposed dot signifies partial differentiation with respect to time.

Consider now a body $\mathcal{B}$ which, in a reference configuration, occupies a region $\mathcal{R}$ contained in $\mathbb{R}^3$. A motion of $\mathcal{B}$ on a time interval $\mathcal{T} \subset \mathbb{R}$ is characterized by a one-parameter family of invertible mappings $\hat{\mathbf{y}}(\cdot,t) : \mathcal{R} \to \mathcal{R}_t$, with

$$\hat{\mathbf{y}}(\mathbf{x},t) = \mathbf{x} + \mathbf{u}(\mathbf{x},t) \quad \forall(\mathbf{x},t) \in \mathcal{M}, \tag{2.1.1}$$

where $\mathcal{M} = \mathcal{R} \times \mathcal{T}$ represents the *trajectory* of the motion. Assume that the *deformation* $\hat{\mathbf{y}}$, or equivalently the *displacement* $\mathbf{u}$, is continuous and possesses piecewise continuous first and second partial derivatives on $\mathcal{M}$. Let S_t be the set of points contained in $\mathcal{R}$ defined so that, at each instant t in $\mathcal{T}$, $\hat{\mathbf{y}}(\cdot,t)$ is twice continuously differentiable on the set $\mathcal{R} \setminus S_t$. Let the set $\varSigma$ be defined by

$$\varSigma = \{(\mathbf{x},t) | \, \mathbf{x} \in S_t, t \in \mathcal{T}\}. \tag{2.1.2}$$

Introduce the *deformation gradient tensor* $\mathbf{F} : \mathcal{M} \setminus \varSigma \to \mathcal{L}$ by

$$\mathbf{F}(\mathbf{x},t) = \nabla\hat{\mathbf{y}}(\mathbf{x},t) \quad \forall(\mathbf{x},t) \in \mathcal{M} \setminus \varSigma, \tag{2.1.3}$$

where the associated *Jacobian determinant*, $J : \mathcal{M} \setminus \varSigma \to I\!R$, of $\hat{\mathbf{y}}$ is restricted to be strictly positive on its domain of definition:

$$J(\mathbf{x},t) = \det \mathbf{F}(\mathbf{x},t) > 0 \quad \forall(\mathbf{x},t) \in \mathcal{M} \setminus \varSigma.$$

Hence, $\mathbf{F} : \mathcal{M} \setminus \varSigma \to \mathcal{L}_+$. The *left Cauchy-Green tensor* $\mathbf{G} : \mathcal{M} \setminus \varSigma \to \overset{+}{\mathcal{S}}$ corresponding to the deformation $\hat{\mathbf{y}}$ is given by

$$\mathbf{G}(\mathbf{x},t) = \mathbf{F}(\mathbf{x},t)\mathbf{F}^T(\mathbf{x},t) \quad \forall(\mathbf{x},t) \in \mathcal{M} \setminus \varSigma. \tag{2.1.4}$$

The *deformation invariants* associated with $\hat{\mathbf{y}}$ exist on $\mathcal{M} \setminus \varSigma$ and are supplied through the fundamental scalar invariants of $\mathbf{G}$:

$$I_1(\mathbf{G}) = \operatorname{tr}\mathbf{G}, \quad I_2(\mathbf{G}) = \tfrac{1}{2}\left((\operatorname{tr}\mathbf{G})^2 - \operatorname{tr}(\mathbf{G}^2)\right), \quad I_3(\mathbf{G}) = \det\mathbf{G}. \tag{2.1.5}$$

With the above kinematic antecedents in place introduce the *nominal mass density* $\rho : \mathcal{R} \to \mathbb{R}_+$, the *nominal body force per unit mass* $\mathbf{b} : \mathcal{M} \to \mathbb{R}^3$, and the *nominal stress tensor* $\mathbf{S} : \mathcal{M} \setminus \Sigma \to \mathcal{L}$, and suppose that ρ is constant on $\mathcal{R}$ and $\mathbf{b}$ is continuous on $\mathcal{M}$, while $\mathbf{S}$ is piecewise continuous on $\mathcal{M}$, continuous on $\mathcal{M} \setminus \Sigma$, and has a piecewise continuous gradient on $\mathcal{M}$. Let ρ_* be the mass density in the deformed configuration associated with $\hat{\mathbf{y}}$. Given a *regular* subregion $\mathcal{P}$ of $\mathcal{R}$, with $\partial\mathcal{P} \cap S_t$ a set of measure zero in $\partial\mathcal{P}$ for each t in $\mathcal{T}$, let $\mathbf{m} : \partial\mathcal{P} \to \mathcal{N}$ denote the unit outward normal to $\partial\mathcal{P}$. Then the global balance laws of mass, linear momentum, and angular momentum require that

$$\int_{\mathcal{P}} \rho \, dV = \int_{\hat{\mathbf{y}}(\mathcal{P})} \rho_* \, dV \quad \text{on} \quad \mathcal{T}, \tag{2.1.6}$$

$$\int_{\partial\mathcal{P}} \mathbf{S}\mathbf{m} \, dA + \int_{\mathcal{P}} \rho\mathbf{b} \, dV = \overline{\int_{\mathcal{P}} \rho\dot{\mathbf{u}} \, dV}^{\,\cdot} \quad \text{on} \quad \mathcal{T}, \tag{2.1.7}$$

and

$$\int_{\partial\mathcal{P}} \hat{\mathbf{y}} \wedge \mathbf{S}\mathbf{m} \, dA + \int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho\mathbf{b} \, dV = \overline{\int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho\dot{\mathbf{u}} \, dV}^{\,\cdot} \quad \text{on} \quad \mathcal{T}, \tag{2.1.8}$$

respectively, for every such regular subregion $\mathcal{P}$ contained in $\mathcal{R}$.

Next, introduce the *nominal internal energy per unit mass* $\varepsilon : \mathcal{M} \setminus \Sigma \to \mathbb{R}$, the *nominal heat flux* $\mathbf{q} : \mathcal{M} \setminus \Sigma \to \mathbb{R}^3$, and the *nominal heat supply per unit mass* $r : \mathcal{M} \setminus \Sigma \to \mathbb{R}$. Suppose that ε and $\mathbf{q}$ are piecewise continuous on $\mathcal{M}$, continuous on $\mathcal{M} \setminus \Sigma$, and have piecewise continuous partial derivatives on $\mathcal{M}$, and that r is continuous on $\mathcal{M}$. The first law of thermodynamics requires that

$$\int_{\partial\mathcal{P}} (\mathbf{S}\mathbf{m} \cdot \dot{\mathbf{u}} + \mathbf{q} \cdot \mathbf{m}) \, dA + \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \dot{\mathbf{u}} + r) \, dV = \overline{\int_{\mathcal{P}} \rho(\varepsilon + \tfrac{1}{2}|\dot{\mathbf{u}}|^2) \, dV}^{\,\cdot} \quad \text{on} \quad \mathcal{T}, \tag{2.1.9}$$

for every regular subregion $\mathcal{P}$ contained in $\mathcal{R}$ such that $\partial\mathcal{P} \cap S_t$ a set of measure zero in $\partial\mathcal{P}$ for each t in $\mathcal{T}$.

Finally, introduce the *nominal entropy per unit mass* $\eta : \mathcal{M} \setminus \Sigma \to I\!R$ and the *nominal absolute temperature* $\theta : \mathcal{M} \to I\!R$. Stipulate that η is piecewise continuous on $\mathcal{M}$, continuous on $\mathcal{M} \setminus \Sigma$, and has piecewise continuous first partial derivatives on $\mathcal{M}$, and that θ is continuous on $\mathcal{M}$ with piecewise continuous first partial derivatives on $\mathcal{M}$. The *Clausius-Duhem* version of the second law of thermodynamics requires that the *rate of entropy production* $\Gamma(\cdot;\mathcal{P}) : \mathcal{T} \to I\!R$ satisfies

$$\Gamma(\cdot;\mathcal{P}) = \overline{\int_{\mathcal{P}} \rho\eta \, dV}^{\,\cdot} - \int_{\partial\mathcal{P}} \frac{\mathbf{q}\cdot\mathbf{m}}{\theta} \, dA - \int_{\mathcal{P}} \frac{\rho r}{\theta} \, dV \geq 0 \quad \text{on} \quad \mathcal{T}, \tag{2.1.10}$$

for every regular subregion $\mathcal{P}$ contained in $\mathcal{R}$ such that $\partial\mathcal{P} \cap S_t$ a set of measure zero in $\partial\mathcal{P}$ for each t in $\mathcal{T}$.

Localization of the balance laws (2.1.6)–(2.1.9) and the imbalance law (2.1.10) at an arbitrary point contained in the interior of $\mathcal{M} \setminus \Sigma$ yields the following familiar field equations and field inequality:

$$\begin{aligned}
\rho = \rho_*(\hat{\mathbf{y}})J \quad &\text{on} \quad \mathcal{M} \setminus \Sigma, \\
\nabla \cdot \mathbf{S} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}} \quad &\text{on} \quad \mathcal{M} \setminus \Sigma, \\
\mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T \quad &\text{on} \quad \mathcal{M} \setminus \Sigma, \\
\mathbf{S} \cdot \dot{\mathbf{F}} + \nabla \cdot \mathbf{q} + \rho r = \rho\dot{\varepsilon} \quad &\text{on} \quad \mathcal{M} \setminus \Sigma, \\
\nabla \cdot \left(\frac{\mathbf{q}}{\theta}\right) + \frac{\rho r}{\theta} \leq \rho\dot{\eta} \quad &\text{on} \quad \mathcal{M} \setminus \Sigma.
\end{aligned} \tag{2.1.11}$$

Suppose, from now on, that the set S_t is a regular surface for every t in $\mathcal{T}$. The set Σ then represents the trajectory of a surface of discontinuity in $\mathbf{F}$, $\mathbf{S}$ and, perhaps, ε, $\mathbf{q}$ and η. Let $g(\cdot,t)$ denote a generic field quantity $g(\cdot,t) : S_t \to I\!R$ which is discontinuous across S_t at the instant t in $\mathcal{T}$. Define the *jump* $[\![g(\cdot,t)]\!]$ of $g(\cdot,t)$ across S_t by

$$[\![g(\mathbf{x},t)]\!] = \lim_{h\searrow 0} \big(g(\mathbf{x} + h\mathbf{n}(\mathbf{x},t),t) - g(\mathbf{x} - h\mathbf{n}(\mathbf{x},t),t)\big) \quad \forall(\mathbf{x},t) \in \Sigma, \tag{2.1.12}$$

where $\mathbf{n}(\cdot,t) : S_t \to \mathcal{N}$ is a unit normal to S_t at each t in $\mathcal{T}$. Then, localization of (2.1.6)–(2.1.10) at an arbitrary element of Σ yields the following jump conditions

$$\begin{aligned} [\![\rho_*(\hat{\mathbf{y}})J]\!] &= 0 \quad \text{on} \quad \Sigma, \\ [\![\mathbf{Sn}]\!] + \rho V_n [\![\dot{\mathbf{u}}]\!] &= \mathbf{0} \quad \text{on} \quad \Sigma, \\ [\![\mathbf{Sn}\cdot\dot{\mathbf{u}}]\!] + \rho V_n [\![\varepsilon + \tfrac{1}{2}|\dot{\mathbf{u}}|^2]\!] + [\![\mathbf{q}\cdot\mathbf{n}]\!] &= 0 \quad \text{on} \quad \Sigma, \\ \rho V_n [\![\eta]\!] + \frac{1}{\theta}[\![\mathbf{q}\cdot\mathbf{n}]\!] &\le 0 \quad \text{on} \quad \Sigma, \end{aligned} \tag{2.1.13}$$

where $V_n(\cdot,t) : S_t \to I\!R$ is the component of the velocity of the surface S_t in the direction of $\mathbf{n}(\cdot,t)$ at the instant t in $\mathcal{T}$.

Equations $(2.1.11)_1$ and $(2.1.13)_1$ are, evidently, completely decoupled from equations $(2.1.11)_{2,3,4,5}$ and $(2.1.13)_{2,3,4}$; that is, given a solution to, say, a boundary value problem involving $(2.1.11)_{2,3,4,5}$ and $(2.1.13)_{2,3,4}$, ρ_* can be calculated *a posteriori*. For this reason equations $(2.1.11)_1$ and $(2.1.13)_1$ will be disregarded in the subsequent analysis.

In this investigation an *inertia-free* motion is defined as one wherein the inertial terms on the right hand sides of the global balance equations (2.1.7) and (2.1.8) are replaced by the zero vector. In the context of an inertia-free motion the field equation $(2.1.11)_2$ simplifies to read

$$\nabla\cdot\mathbf{S} + \rho\mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathcal{M}\setminus\Sigma, \tag{2.1.14}$$

and the jump condition $(2.1.13)_2$ becomes

$$[\![\mathbf{Sn}]\!] = \mathbf{0} \quad \text{on} \quad \Sigma. \tag{2.1.15}$$

Equations $(2.1.11)_{1,3,4,5}$ and $(2.1.13)_{1,3,4}$ remain, of course, unaltered.

In addition to the jump conditions given in (2.1.13) in the inertial case or $(2.1.13)_{1,3,4}$ and (2.1.15) in the inertia-free case, the stipulated continuity of $\hat{\mathbf{y}}$

and θ gives the following *kinematic* jump conditions

$$[\![\mathbf{u}]\!] = \mathbf{0} \quad \text{on} \quad \Sigma, \qquad [\![\theta]\!] = 0 \quad \text{on} \quad \Sigma. \tag{2.1.16}$$

2.2. Rate of entropy production and driving traction. Using the field equations (2.1.11), the jump conditions (2.1.13) , and the assumed smoothness of the deformation $\hat{\mathbf{y}}$, ABEYARATNE & KNOWLES [1] have demonstrated that for any continuum the rate of entropy production $\Gamma(\cdot;\mathcal{P})$ can, for any regular region $\mathcal{P}$ contained in $\mathcal{R}$, be represented in the form

$$\Gamma(t;\mathcal{P}) = \Gamma_{\text{loc}}(t;\mathcal{P}) + \Gamma_{\text{con}}(t;\mathcal{P}) + \Gamma_s(t;\mathcal{P}) \quad \forall t \in \mathcal{T}, \tag{2.2.1}$$

where $\Gamma_{\text{loc}}(\cdot;\mathcal{P})$, $\Gamma_{\text{con}}(\cdot;\mathcal{P})$, and $\Gamma_s(\cdot;\mathcal{P})$ are defined by

$$\begin{aligned}
\Gamma_{\text{loc}}(\cdot;\mathcal{P}) &= \int\limits_{\mathcal{P}\setminus S_t} \frac{1}{\theta}(\mathbf{S}\cdot\dot{\mathbf{F}} - \rho(\dot{\psi} + \dot{\theta}\eta))\,dV \quad \text{on} \quad \mathcal{T}, \\
\Gamma_{\text{con}}(\cdot;\mathcal{P}) &= \int\limits_{\mathcal{P}\setminus S_t} \frac{1}{\theta^2}\mathbf{q}\cdot\nabla\theta\,dV \quad \text{on} \quad \mathcal{T}, \\
\Gamma_s(\cdot;\mathcal{P}) &= \int\limits_{\mathcal{P}\cap S_t} \frac{1}{\theta}(\rho[\![\psi]\!] - \langle\!\langle \mathbf{S}\rangle\!\rangle\cdot[\![\mathbf{F}]\!])V_n\,dV \quad \text{on} \quad \mathcal{T},
\end{aligned} \tag{2.2.2}$$

with $\psi : \mathcal{M}\setminus\Sigma \to I\!R$ representing the *nominal Helmholtz free energy per unit mass* in defined in terms of ε, θ and η by

$$\psi = \varepsilon - \theta\eta \quad \text{on} \quad \mathcal{M}\setminus\Sigma, \tag{2.2.3}$$

and, where—given a generic field quantity $g(\cdot,t) : S_t \to I\!R$ which jumps across S_t at the instant t in $\mathcal{T}$—$\langle\!\langle g(\cdot,t)\rangle\!\rangle$ is defined through

$$\langle\!\langle g(\mathbf{x},t)\rangle\!\rangle = \lim_{h\searrow 0}\tfrac{1}{2}\left(g\big(\mathbf{x}+h\mathbf{n}(\mathbf{x},t),t\big) + g\big(\mathbf{x}-h\mathbf{n}(\mathbf{x},t),t\big)\right) \quad \forall(\mathbf{x},t)\in\Sigma. \tag{2.2.4}$$

The representation (2.2.1) additively decomposes the total rate of entropy production $\Gamma(\cdot;\mathcal{P})$, at the instant t in $\mathcal{T}$, in the regular region $\mathcal{P}$ contained in $\mathcal{R}$ into three parts. The first two terms in the decomposition $\Gamma_{\text{loc}}(\cdot;\mathcal{P})$ and $\Gamma_{\text{con}}(\cdot;\mathcal{P})$ are the contributions to the rate of entropy production due, respectively, to local mechanical dissipation and heat conduction away from the surface S_t, while the third term $\Gamma_s(\cdot;\mathcal{P})$ represents the entropy production rate due to the motion of the surface S_t.

Motivated by $(2.2.2)_3$ define the *driving traction* $f(\cdot,t) : S_t \to I\!R$ which acts on the surface S_t at the instant t in $\mathcal{T}$ by

$$f(\cdot,t) = \rho[\![\psi(\cdot,t)]\!] - \langle\!\langle \mathbf{S}(\cdot,t)\rangle\!\rangle \cdot [\![\mathbf{F}(\cdot,t)]\!] \quad \text{on} \quad S_t \quad \forall t \in \mathcal{T}. \tag{2.2.5}$$

In the absence of inertial effects it can be demonstrated that (2.2.5) reduces to

$$f(\cdot,t) = \rho[\![\psi(\cdot,t)]\!] - \overset{\pm}{\mathbf{S}}(\cdot,t) \cdot [\![\mathbf{F}(\cdot,t)]\!] \quad \text{on} \quad S_t \quad \forall t \in \mathcal{T}, \tag{2.2.6}$$

where $\overset{+}{\mathbf{S}}(\cdot,t)$ (*resp.*, $\overset{-}{\mathbf{S}}(\cdot,t)$) is the limiting value of the field $\mathbf{S}(\cdot,t)$ on the side of the interface into which the unit normal $\mathbf{n}(\cdot,t)$ is (*resp.*, is not) directed at the instant t in $\mathcal{T}$.

Now, from (2.2.1) and $(2.2.2)_3$, localization of the imbalance law (2.1.10) at an arbitrary element of Σ yields the following alternative to $(2.1.13)_4$:

$$fV_n \geq 0 \quad \text{on} \quad \Sigma, \tag{2.2.7}$$

with f given by (2.2.5) or (2.2.6) depending on whether inertial effects are included or not. Observe , from (2.2.2), that under isothermal conditions the total rate of entropy production $\Gamma(\cdot;\mathcal{P})$ in a region $\mathcal{P}$ takes the form of the rate of mechanical dissipation per unit temperature.

2.3. Finite thermoelasticity. Let $\mathcal{B}$ be composed of a homgeneous *thermoelastic* material. Then there exists a *Helmholtz free energy potential*

$\hat{\psi} : \mathcal{L}_+ \times I\!R_+ \to I\!R$ such that the nominal Helmholtz free energy per unit mass ψ, the nominal stress tensor $\mathbf{S}$, and the nominal entropy per unit mass are given in terms of $\hat{\psi}$ as follows:

$$\begin{aligned} \psi &= \hat{\psi}(\mathbf{F},\theta) \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \\ \mathbf{S} &= \rho\hat{\psi}_{\mathbf{F}}(\mathbf{F},\theta) \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \\ \eta &= -\hat{\psi}_{\theta}(\mathbf{F},\theta) \quad \text{on} \quad \mathcal{M} \setminus \Sigma. \end{aligned} \tag{2.3.1}$$

It is assumed that $\hat{\psi}$ is once continuously differentiable and piecewise twice continuously differentiable on $\mathcal{L}_+ \times I\!R_+$. The nominal heat flux $\mathbf{q}$ is, for a thermoelastic material, given by a *heat flux response function* $\hat{\mathbf{q}} : \mathcal{L}_+ \times I\!R_+ \times I\!R^3 \to I\!R^3$ so that

$$\mathbf{q} = \hat{\mathbf{q}}(\mathbf{F},\theta,\nabla\theta) \quad \text{on} \quad \mathcal{M} \setminus \Sigma. \tag{2.3.2}$$

It is assumed that $\hat{\mathbf{q}}$ is piecewise twice continuously differentiable on its domain of definition.

Observe that a thermoelastic material is defined in a manner such that the rate of entropy production $\Gamma_{\text{loc}}(\cdot;\mathcal{P})$ in a region $\mathcal{P}$ due to mechanical dissipation is identically zero on $\mathcal{T}$. Hence, the localization of the imbalance law (2.1.10) at a point contained in the interior of $\mathcal{M} \setminus \Sigma$ yields, with the aid of (2.3.2), the inequality

$$\hat{\mathbf{q}}(\cdot,\cdot,\mathbf{d}) \cdot \mathbf{d} \geq 0 \quad \text{on} \quad \mathcal{L}_+ \times I\!R_+ \quad \forall \mathbf{d} \in I\!R^3 \tag{2.3.3}$$

as a condition necessary for the satisfaction of the second law of thermodynamics. The response function $\hat{\mathbf{q}}$ is assumed to be specified so that (2.3.3) holds; then, inequality $(2.1.11)_5$ is automatically satisfied and can be ignored in the following.

For remarks regarding the consequences of *objectivity* on the properties of the potential $\hat{\psi}$ and response function $\hat{\mathbf{q}}$, see Jiang [11].

2.4. Constitutive specialization. To facilitate the ensuing analysis suppose, henceforth, that the homogeneous thermoelastic body $\mathcal{B}$ is thermomechanically isotropic. Then the Helmholtz free energy potential $\hat{\psi}$ and heat flux response function $\hat{\mathbf{q}}$ can depend on the deformation gradient $\mathbf{F}$ only through the deformation invariants $I_k(\mathbf{G})$ defined in (2.1.5). Assume henceforth that both $\hat{\psi}$ and $\hat{\mathbf{q}}$ are independent of the second deformation invariant $I_2(\mathbf{G})$. Suppose, moreover, that the Helmholtz free energy potential $\hat{\psi}$ can be represented in terms of three functions $\tilde{\psi} : I\!R_+ \times I\!R_+ \to I\!R$, $\tilde{g} : I\!R_+ \times I\!R_+ \to I\!R$ and $\tilde{\tilde{g}} : I\!R_+ \times I\!R_+ \to I\!R$ in the form

$$\hat{\psi}(\mathbf{F},\theta) = \tilde{\psi}(I_1(\mathbf{G}),\theta) + \tilde{\psi}_{I_1}(I_1(\mathbf{G}),\theta)\tilde{g}(I_3(\mathbf{G}),\theta) + \tilde{\tilde{g}}(I_3(\mathbf{G}),\theta)$$
$$\forall(\mathbf{F},\theta) \in \mathcal{L}_+ \times I\!R_+, \tag{2.4.1}$$

and that the heat flux response function $\hat{\mathbf{q}}$ can be expressed in terms of a function $\tilde{\phi} : I\!R_+ \times I\!R_+ \times I\!R_+ \to I\!R$ via

$$\hat{\mathbf{q}}(\mathbf{F},\theta,\mathbf{d}) = \tilde{\phi}(I_1(\mathbf{G}), I_3(\mathbf{G}),\theta)\mathbf{d} \quad \forall(\mathbf{F},\theta,\mathbf{d}) \in \mathcal{L}_+ \times I\!R_+ \times I\!R^3. \tag{2.4.2}$$

In (2.4.1) and (2.4.2) $\mathbf{G}$ is the left Cauchy-Green tensor defined in terms of the deformation gradient tensor $\mathbf{F}$ by (2.1.4). In accordance with the stipulated smoothness of $\hat{\psi}$ and $\hat{\mathbf{q}}$, the functions $\tilde{\psi}$, $\tilde{g}$ and $\tilde{\tilde{g}}$ are taken to be once continuously differentiable and piecewise twice continuously differentiable on $I\!R_+ \times I\!R_+$, while $\tilde{\phi}$ is taken to be continuous and piecewise twice continuously differentiable on $I\!R_+ \times I\!R_+ \times I\!R_+$. Assume, in addition, that the functions $\tilde{\psi}$, $\tilde{g}$ and $\tilde{\tilde{g}}$ comply, for each θ in $I\!R_+$, with the following *isochoric* restrictions:

$$\tilde{\psi}(3,\theta) = 0, \quad \tilde{g}(1,\theta) = 0, \quad \tilde{g}_{I_3}(1,\theta) = -1, \quad \tilde{\tilde{g}}(1,\theta) = \tilde{\tilde{g}}_{I_3}(1,\theta) = 0. \tag{2.4.3}$$

In what follows, attention will be restricted to homogeneous isotropic thermoelastic materials wherein the Helmholtz free energy potential $\hat{\psi}$ obeys (2.4.3). A particular material of this type was studied by JIANG [11].

The nominal stress response of $\mathcal{B}$ is then determined, with the aid of $(2.3.1)_2$ and (2.4.1), by

$$\mathbf{S} = 2\rho\left(\chi_1(I_1(\mathbf{G}), I_3(\mathbf{G}),\theta)\mathbf{F} + \chi_2(I_1(\mathbf{G}), I_3(\mathbf{G}),\theta)\mathbf{F}^{-T}\right) \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \tag{2.4.4}$$

where the functions $\chi_\alpha : I\!R_+ \times I\!R_+ \times I\!R_+ \to I\!R$ are defined as follows:

$$\begin{aligned} \chi_1(I_1(\mathbf{G}), I_3(\mathbf{G}), \theta) &= \tilde{\psi}_{I_1}(I_1, \theta) + \tilde{\psi}_{I_1 I_1}(I_1, \theta)\tilde{g}(I_3, \theta) \\ &\qquad \forall (I_1, I_3, \theta) \in I\!R_+ \times I\!R_+ \times I\!R_+, \\ \chi_2(I_1(\mathbf{G}), I_3(\mathbf{G}), \theta) &= \tilde{\psi}_{I_1}(I_1, \theta)\tilde{g}_{I_3}(I_3, \theta) + \tilde{\tilde{g}}_{I_3}(I_3, \theta) \\ &\qquad \forall (I_1, I_3, \theta) \in I\!R_+ \times I\!R_+ \times I\!R_+. \end{aligned} \tag{2.4.5}$$

Observe that, for an isochoric deformation—where $I_3 = 1$ on $\mathcal{M}\setminus\Sigma$, use of (2.4.3) in (2.4.5) reduces (2.4.4) to read

$$\mathbf{S} = 2\rho\tilde{\psi}_{I_1}(I_1, \theta)\left(\mathbf{F} - \mathbf{F}^{-T}\right) \quad \text{on} \quad \mathcal{M}\setminus\Sigma. \tag{2.4.6}$$

Following the work of JIANG & KNOWLES [12] in the purely mechanical setting, it can be readily shown that a special thermoelastic material of the type characterized by (2.4.1)–(2.4.3) satisfies the *Baker-Ericksen inequalities* at all absolute temperatures if and only if

$$\tilde{\psi}_{I_1}(I_1, \theta) + \tilde{\psi}_{I_1 I_1}(I_1, \theta)\tilde{g}(I_3, \theta) > 0 \quad \forall (I_1, I_3, \theta) \in \mathcal{U} \times I\!R_+, \tag{2.4.7}$$

where the set $\mathcal{U}$ is given by

$$\mathcal{U} = \{(I_1, I_3) \,|\, 0 < I_3 < (I_1/3)^3\}.$$

Choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and consider the response of the thermoelastic material at hand to a simple shear deformation $\hat{\mathbf{y}}$ given as follows

$$\hat{\mathbf{y}}(\mathbf{x}, t) = (\mathbf{1} + \gamma \mathbf{e}_3 \otimes \mathbf{e}_1)\mathbf{x} \quad \forall (\mathbf{x}, t) \in \mathcal{M}, \tag{2.4.8}$$

where the constant γ—assumed non-negative without loss of generality—denotes the amount of shear. Note that the foregoing deformation is isochoric. From

$(2.3.1)_2$, (2.4.1), (2.4.3) and (2.4.6) the nominal shear stress corresponding to the deformation $\hat{\mathbf{y}}$ is, therefore, for each γ in $\bar{\mathbb{R}}_+$, found to be

$$\mathbf{e}_3 \cdot \mathbf{S}\mathbf{e}_1 = 2\rho\gamma\tilde{\psi}_{I_1}(3+\gamma^2,\theta) =: \tau(\gamma,\theta). \tag{2.4.9}$$

where θ takes on some positive value. The function $\tau : \bar{\mathbb{R}}_+ \times \mathbb{R}_+ \to \mathbb{R}$ will be referred to as the *shear stress response function* of the special thermoelastic material at hand in simple shear. An immediate consequence of $(2.4.3)_1$ and (2.4.9) is that $\rho\tilde{\psi}$ can be expressed via

$$\rho\tilde{\psi}(I_1,\theta) = \int_0^{\sqrt{I_1-3}} \tau(\kappa,\theta)\,d\kappa \quad \forall (I_1,\theta) \in [3,\infty) \times \mathbb{R}_+, \tag{2.4.10}$$

so that the nominal stress response of such a material, in *all* three dimensional deformations and absolute temperatures, is completely characterized by specifying a shear stress response function τ along with the functions $\tilde{g}$ and $\tilde{\tilde{g}}$ introduced in (2.4.1). Now, define the *secant modulus in shear* $M : \bar{\mathbb{R}}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by

$$M(\gamma,\theta) = 2\rho\tilde{\psi}_{I_1}(3+\gamma^2,\theta) \quad \forall (\gamma,\theta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}_+. \tag{2.4.11}$$

Observe that, in compliance with the stipulated smoothness of $\tilde{\psi}$, both τ and M must be continuous and piecewise continuously differentiable on $\bar{\mathbb{R}}_+ \times \mathbb{R}_+$. From (2.2.9) and (2.2.11) that the shear stress response function τ must also satisfy

$$\tau(0,\theta) = 0 \quad \forall \theta \in \mathbb{R}_+, \quad \tau_\gamma(0,\theta) = M(0,\theta) \quad \forall \theta \in \mathbb{R}_+. \tag{2.4.12}$$

Note, also, that for the simple shear deformation defined via (2.1.1) and (2.4.5), the *Baker-Ericksen inequality* (2.4.10) reduces, with the aid of (2.4.11) and $(2.4.3)_4$, to a relation which involves only M: *viz.*,

$$M(\gamma,\theta) > 0 \quad \forall (\gamma,\theta) \in \mathbb{R}_+^2. \tag{2.4.13}$$

Restrict attention in the sequel to these special materials for which the infinitesimal shear modulus is positive; *i.e.*, require that

$$M(0,\theta) > 0 \quad \forall \theta \in \mathbb{R}_+. \tag{2.4.14}$$

Despite the significant restrictions which have been placed upon the class of materials which will be considered in this investigation, no presuppositions have been made regarding the sign of the derivative with respect to its first argument—where it exists—of the shear stress response function corresponding to the thermoelastic material defined in compliance with (2.4.1)–(2.4.3) and (2.4.10). JIANG [11] has shown that the monotonicity of $\tau(\cdot,\theta)$ is, for fixed θ in $\mathbb{R}_+$, related directly to the ellipticity of the material which it characterizes. If, in particular, $\tau(\cdot,\theta)$ is not a monotonically increasing function on its domain of definition—for some range of θ—then the associated material is non-elliptic. With this in mind, let (θ_m,θ_M) be contained in $\mathbb{R}_+$ and define functions $\underset{*}{\gamma} : (\theta_m,\theta_M) \to \mathbb{R}_+$ and $\overset{*}{\gamma} : (\theta_m,\theta_M) \to \mathbb{R}_+$ such that

$$\underset{*}{\gamma}(\theta) < \overset{*}{\gamma}(\theta) \quad \forall \theta \in (\theta_m,\theta_M). \tag{2.4.15}$$

Next, define three plane open subsets A_1, A_2, and A_3 of the *shear strain-temperature* quadrant as follows:

$$\begin{aligned} A_1 &= \{(\gamma,\theta)\,|\, 0 < \gamma < \underset{*}{\gamma}(\theta), \theta \in (\theta_m,\theta_M)\}, \\ A_2 &= \{(\gamma,\theta)\,|\, \underset{*}{\gamma}(\theta) < \gamma < \overset{*}{\gamma}(\theta), \theta \in (\theta_m,\theta_M)\}, \\ A_3 &= \{(\gamma,\theta)\,|\, \overset{*}{\gamma}(\theta) < \gamma < \infty, \theta \in (\theta_m,\theta_M)\}. \end{aligned} \tag{2.4.16}$$

This investigation will make use of a particular subclass of non-elliptic thermoelastic materialsof the above special form wherein the relevant shear stress response function τ is taken to be continuous on $\overline{\mathbb{R}}_+ \times \mathbb{R}_+$ and continuously differentiable on $A_1 \cup A_2 \cup A_3$ and is required to obey the following monotonicity requirements

$$\begin{aligned} \tau_\gamma > 0 \quad &\text{on} \quad A_1 \cup A_3, \\ \tau_\gamma < 0 \quad &\text{on} \quad A_2. \end{aligned} \tag{2.4.17}$$

Assume, also, that $\tau(\cdot,\theta)$ is monotonically increasing on $\bar{\mathbb{R}}_+$ for all θ in $\mathbb{R}_+ \setminus [\theta_m, \theta_M]$. Let the *nominal conductivity in shear* $\tilde{k} : \bar{\mathbb{R}} \times \mathbb{R}_+ \to \mathbb{R}_+$ and the *nominal specific heat per unit mass in shear* $\tilde{c} : \bar{\mathbb{R}} \times \mathbb{R}_+ \to \mathbb{R}_+$ of the thermoelastic material at hand be defined as follows

$$\begin{aligned} \tilde{k}(\gamma,\theta) &= \tilde{\phi}(3+\gamma^2, 1, \theta) \quad \forall(\gamma,\theta) \in \bar{\mathbb{R}} \times \mathbb{R}_+, \\ \tilde{c}(\gamma,\theta) &= -\theta\tilde{\psi}_{\theta\theta}(3+\gamma^2,\theta) \quad \forall(\gamma,\theta) \in \bar{\mathbb{R}} \times \mathbb{R}_+. \end{aligned} \tag{2.4.18}$$

Suppose that $\tilde{k}$ and $\tilde{c}$ are both continuous on $A_1 \cup A_2 \cup A_3$ and piecewise continuous on $\bar{\mathbb{R}}_+ \times \mathbb{R}_+$. In compliance with (2.3.3) let $\tilde{k}$ be positive on its domain of definition. Suppose, in addition, that $\tilde{c}$ is positive on its domain of definition.

The sets A_1 and A_3 are referred to as the *high* and *low strain phases* of the thermoelastic material specified by (2.4.1)–(2.4.3) and (2.4.10). These, together with the set of shear strain-temperature pairs in $(\bar{\mathbb{R}}_+ \times \mathbb{R}_+) \setminus (\bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3)$ comprise the *elliptic phases* of such a material. A thermoelastic material of the type at hand which is defined so that τ, $\tilde{k}$ and $\tilde{c}$ have the properties set forth above will be referred to herein as a *three-phase* thermoelastic material. See Figure 1 for a graph of $\tau(\cdot,\theta_0)$ for fixed θ_0 in (θ_m,θ_M) typical of those which specify three-phase thermoelastic materials. Consult Figure 2 for a division of the shear strain-temperature quadrant into regions of monotonicity of $\tau(\cdot,\theta)$ for fixed θ.

2.5. Completion of constitutive assumptions via the kinetic relation. Let $\mathcal{B}$ be composed of a three-phase thermoelastic material and consider a motion of $\mathcal{B}$ which involves a moving surface of discontinuity S_t in one or all of the field quantities $\mathbf{F}(\cdot,t)$, $\dot{\mathbf{u}}(\cdot,t)$, $\mathbf{S}(\cdot,t)$, $\psi(\cdot,t)$, $\eta(\cdot,t)$, and $\mathbf{q}(\cdot,t)$ at each instant t in $\mathcal{T}$. Assume that S_t separates high and low strain elliptic phases in the material at hand. In the context of such a motion it is necessary (see [1–4]) to supplement, in some fashion, the constitutive information provided in Section 2.4. An approach to this taken by Abeyaratne & Knowles [1] entails the provision of a *kinetic relation*. Two basic cases motivated by [1] can be considered: in the first a constitutive response function $\tilde{V} : I\!R \times (\theta_m, \theta_M) \to I\!R$ is specified so that

$$V_n = \tilde{V}(\tfrac{f}{\theta}, \theta) \quad \forall(\tfrac{f}{\theta}, \theta) \in I\!R \times (\theta_m, \theta_M), \tag{2.5.1}$$

while, in the second a constitutive response function $\tilde{\varphi} : I\!R \times (\theta_m, \theta_M) \to I\!R$ is furnished so that

$$f = \theta\tilde{\varphi}(V_n, \theta) \quad \forall(V_n, \theta) \in I\!R \times (\theta_m, \theta_M). \tag{2.5.2}$$

The functions $\tilde{V}$ and $\tilde{\varphi}$ are referred to as *kinetic response functions*. Since the three-phase thermoelastic material can lose ellipticity only for absolute temperatures θ in (θ_m, θ_M), the kinetic response functions $\tilde{V}$ and $\tilde{\varphi}$ are defined only on $I\!R \times (\theta_m, \theta_M)$. Both varieties of kinetic response functions will be considered in this investigation. If $\tilde{V}$ is such that $\tilde{V}(\Phi, \theta)\Phi \geq 0$ for all (Φ, θ) in $I\!R \times (\theta_m, \theta_M)$ then (2.2.6) is automatically satisfied and $\tilde{V}$ is referred to as *admissible*. If $\tilde{\varphi}(V, \theta)V \geq 0$ for all (V, θ) in $I\!R \times (\theta_m, \theta_M)$, $\tilde{\varphi}$ is, similarly, referred to as admissible. If an admissible kinetic response function $\tilde{V}$ (or $\tilde{\varphi}$) is continuous on $I\!R \times (\theta_m, \theta_M)$, then it must satisfy $\tilde{V}(0, \theta) = 0$ (or $\tilde{\varphi}(0, \theta) = 0$) for all θ in (θ_m, θ_M). If, in addition, to being admissible, $\tilde{V}$ (or $\tilde{\varphi}$) is continuously differentiable on $I\!R \times (\theta_m, \theta_M)$, then $\tilde{V}_\Phi(0, \theta) \geq 0$ and $\tilde{V}_\Theta(0, \theta) = 0$ (or $\tilde{\varphi}_V(0, \theta) \geq 0$ and $\tilde{\varphi}_\theta(0, \theta) = 0$) for all θ in (θ_m, θ_M)—here $\tilde{V}_\Phi$ and $\tilde{\varphi}_V$ refer to the first partial derivatives of $\tilde{V}$ and $\tilde{\varphi}$ with respect to their first arguments while $\tilde{V}_\Theta$ and $\tilde{\varphi}_\theta$ refer to the first partial derivatives of $\tilde{V}$ and $\tilde{\varphi}$ with respect to their second arguments. Otherwise, admissiblility implies nothing with regard to the sign of the derivative of a smooth kinetic response function. All kinetic response functions considered herein are assumed to be admissible. See Figure 3 and Figure 4 for illustrative graphs of $\tilde{V}(\cdot, \theta_0)$ and $\tilde{\varphi}(\cdot, \theta_0)$ for fixed θ_0 in (θ_m, θ_M).

2.6. Thermoelastic antiplane shear motions of a special thermoelastic material. Suppose, from now on, that $\mathcal{R}$ is a cylindrical region and choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ so that the unit base vector $\mathbf{e}_3$

is parallel to the generatrix of $\mathcal{R}$. A dynamical process will be referred to as a *thermoelastic antiplane shear* normal to the plane spanned by the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ if the deformation $\hat{\mathbf{y}}$ is of the form

$$\hat{\mathbf{y}}(\mathbf{x},t) = \mathbf{x} + u(x_1,x_2,t)\mathbf{e}_3 \quad \forall(\mathbf{x},t) \in \mathcal{M}, \tag{2.6.1}$$

and the nominal Helmholtz free energy per unit mass ψ, nominal entropy per unit mass η, nominal absolute temperature θ, and nominal heat flux vector $\mathbf{q}$ are—like the displacement field associated with (2.6.1)—independent of the x_3–coordinate. The non-trivial component of displacement u in (2.6.1) will be referred to as the *out-of-plane* displacement field. Inspection of (2.6.1) reveals that any discontinuities in the gradient and, perhaps, time derivative of $\hat{\mathbf{y}}$ must result from discontinuities in the spatial derivatives out-of-plane displacement field and, hence, occur across surfaces which do not vary with the x_3–coordinate; similarly, because of their independence of the x_3–coordinate, any discontinuities in ψ, η or $\mathbf{q}$ must occur across surfaces which do not vary with the x_3–coordinate. Let S_t denote a surface across which at least one of the above field quantities jumps at the instant t in $\mathcal{T}$ and let Σ be defined as in (2.1.2).

Following the work of Jiang [11] in the inertia-free context, it is possible to demonstrate that, although not every homogeneous and isotropic thermoelastic material can sustain thermoelastic antiplane shear motions, all thermoelastic materials defined in compliance with (2.4.1)–(2.4.3) and (2.4.10) are capable of doing so. It is easily shown that for such materials the local balance equations $(2.1.11)_{2,3,4,5}$ reduce, in the absence of body forces and heat supplies, to

$$\begin{aligned} \big(M(\gamma,\theta)u_{,\alpha}\big)_{,\alpha} &= \rho\ddot{u} \quad \text{on} \quad \mathcal{X}\setminus\Gamma, \\ \big(\tilde{k}(\gamma,\theta)\theta_{,\alpha}\big)_{,\alpha} + M_\theta(\gamma,\theta)\theta u_{,\alpha}\dot{u}_{,\alpha} &= \rho\tilde{c}(\gamma,\theta)\dot{\theta} \quad \text{on} \quad \mathcal{X}\setminus\Gamma, \end{aligned} \tag{2.6.2}$$

where $\mathcal{X}$ is given by $\mathcal{D}\times\mathcal{T}$, $\mathcal{D}$ is a generic cross section of $\mathcal{R}$, and $\Gamma = \{(x_1,x_2,t)|\,(x_1,x_2)\in C_t, t\in\mathcal{T}\}$ with $C_t = \mathcal{D}\cap S_t$ at each t in $\mathcal{T}$. See Fosdick &

SERRIN [8] and FOSDICK & KAO [7] for a general discussion of the circumstances under which the local balance equations $(2.1.11)_{2,3}$ reduce to a single scalar equation. In (2.4.2) M is the secant modulus in shear as defined in (2.4.11) and $\gamma : \mathcal{X} \setminus \Gamma \to I\!R$ is the *shear strain field* given by

$$\gamma(x_1, x_2, t) = \sqrt{u_{,\alpha}(x_1, x_2, t) u_{,\alpha}(x_1, x_2, t)} \quad \forall (x_1, x_2, t) \in \mathcal{X} \setminus \Gamma. \tag{2.6.3}$$

The jump conditions $(2.1.13)_{2,3}$ reduce, for a thermoelastic material of the type at hand subjected to antiplane shear, to

$$\begin{aligned} [\![M(\gamma, \theta) u_{,\alpha} n_\alpha]\!] + \rho V_n [\![\dot{u}]\!] = 0 \quad &\text{on} \quad \Gamma, \\ [\![\tilde{k}(\gamma, \theta) \theta_{,\alpha} n_\alpha]\!] + \rho V_n \theta [\![\eta]\!] + f V_n = 0 \quad &\text{on} \quad \Gamma, \end{aligned} \tag{2.6.4}$$

where $\Gamma = \{(\mathbf{x}, t) | \, \mathbf{x} \in C_t, t \in \mathcal{T}\}$, $\mathbf{n}(\cdot, t) : C_t \to \mathcal{N}$ is a unit normal to C_t, the nominal entropy per unit mass $\eta : \mathcal{X} \setminus \Gamma \to I\!R$ is given, from $(2.3.1)_3$, (2.4.1) and (2.4.3), by

$$\eta = -\tilde{\psi}_\theta(3 + \gamma^2, \theta) = -\frac{1}{\rho} \int_0^\gamma \tau_\theta(\kappa, \theta) \, d\kappa \quad \text{on} \quad \mathcal{X} \setminus \Gamma, \tag{2.6.5}$$

and $f : \Gamma \to I\!R$ is the driving traction introduced in Section 2.3. The kinematic jump condition (2.1.16) becomes

$$[\![u]\!] = 0 \quad \text{on} \quad \Gamma, \quad [\![\theta]\!] = 0 \quad \text{on} \quad \Gamma. \tag{2.6.6}$$

It is also readily shown that the driving traction f for a thermoelastic material defined via (2.4.1)–(2.4.3) and (2.4.10) subjected to an antiplane shear deformation involving a discontinuity in the gradient of the out-of-plane displacement across a moving curve C_t is given by

$$f = \int_{\bar{\gamma}}^{\overset{+}{\gamma}} \tau(\kappa, \theta) \, d\kappa - \langle\!\langle M(\gamma, \theta) u_{,\alpha} \rangle\!\rangle [\![u_{,\alpha}]\!] \quad \text{on} \quad \Gamma. \tag{2.6.7}$$

Recall that the jump condition $(2.1.13)_4$, or, equivalently, (2.2.7) is satisfied constitutively by requiring that the kinetic response function be admissible.

With reference to (2.1.14), (2.1.15) and (2.2.6) it is easily demonstrated that, in the absence of inertial effects, (2.4.2) is replaced by

$$\big(M(\gamma,\theta)u_{,\alpha}\big)_{,\alpha} = 0 \quad \text{on} \quad \mathcal{X}\setminus\Gamma, \tag{2.6.8}$$

while (2.4.4) becomes

$$[\![M(\gamma,\theta)u_{,\alpha}\, n_\alpha]\!] = 0 \quad \text{on} \quad \Gamma, \tag{2.6.9}$$

and (2.6.7) reduces to

$$f = \int_{\bar{\gamma}}^{\overset{+}{\gamma}} \tau(\kappa,\theta)\, d\kappa - M(\overset{\pm}{\gamma},\theta)\overset{\pm}{u}_{,\alpha}\, [\![u_{,\alpha}]\!] \quad \text{on} \quad \Gamma. \tag{2.6.10}$$

Observe that, within the context of a thermoelastic antiplane shear deformation of the type described above, no generality is lost by focussing exclusively upon the motion on a cross-section $\mathcal{D}$ of the cylinder $\mathcal{R}$ and the dynamics of the curve $C_t = \mathcal{D}\cap S_t$. In the following, curves C_t across which the gradient of the out-of-plane displacement field $u(\cdot,\cdot,t)$ and, perhaps, the out-of-plane velocity field $\dot{u}(\cdot,\cdot,t)$, the entropy field $\eta(\cdot,\cdot,t)$, and the gradient of the absolute temperature field $\theta(\cdot,\cdot,t)$ jumps, at some instant t in $\mathcal{T}$, and which segregate the high and low strain phases of the material at hand will, therefore, be referred to as *phase boundaries.*

3. LINEAR STABILITY OF A PROCESS INVOLVING A STEADILY MOVING PLANAR PHASE BOUNDARY IN A THREE-PHASE THERMOELASTIC MATERIAL

3.1. Description of the base process. Suppose that $\mathcal{B}$ is composed of a three-phase thermoelastic material and that the cylinder $\mathcal{R}$ degenerates so as to occupy all of $I\!R^3$. Let the rectangular Cartesian frame X be as in Section 2.4. Consider a thermoelastic antiplane shear motion on the time interval $(-\infty, 0)$ with an out-of-plane displacement field $u_0(\cdot, t) : I\!R \to I\!R$ given by

$$u_0(x_1, t) = \begin{cases} \gamma_l x_1 + v_l t & \text{if} \quad x_1 < v_0 t, \\ \gamma_r x_1 + v_r t & \text{if} \quad x_1 > v_0 t, \end{cases} \tag{3.1.1}$$

for each t in $(-\infty, 0)$, and an absolute temperature field θ_0 which is constant on $\mathcal{R} \times (-\infty, 0)$ and satisfies

$$\theta_0 \in (\theta_m, \theta_M), \tag{3.1.2}$$

where the shear strain-temperature pairs (γ_l, θ_0) and (γ_r, θ_0) satisfy one of the following

$$((\gamma_l, \theta_0), (\gamma_r, \theta_0)) \in A_3 \times A_1, \quad ((\gamma_l, \theta_0), (\gamma_r, \theta_0)) \in A_1 \times A_3. \tag{3.1.3}$$

Observe that the process described by (3.1.1)–(3.1.3) is *isothermal*.

Since one of (3.1.3) must hold, there is no loss in generality incurred by assuming that the base interface normal velocity v_0 is non-negative; that is,

$$v_0 \geq 0. \tag{3.1.4}$$

It is clear that u_0 and θ_0 satisfy the differential equations in (2.6.2) on the set $(I\!R^2 \times (-\infty, 0)) \setminus \Gamma_0$ with Γ_0 given by $\{(x_1, x_2, t) | \, (x_1, x_2) \in A_t, t \in (-\infty, 0)\}$ and $A_t = \{(x_1, x_2) | \, x_1 = v_0 t, x_2 \in I\!R\}$ for each t in $(-\infty, 0)$. The moving line A_t is, for each t in $(-\infty, o)$, a phase boundary.

Assume, in order to comply with the jump conditions in (2.6.4) and (2.6.6) on Γ_0, that the constants γ_l, γ_r, v_l, v_r, and v_0 associated with (3.1.1)–(3.1.3) are restricted to satisfy the following equations:

$$\begin{gathered} v_r - v_l + v_0(\gamma_r - \gamma_l) = 0, \\ \tau(\gamma_r, \theta_0) - \tau(\gamma_l, \theta_0) + \rho v_0(v_r - v_l) = 0, \\ v_0(f_0 + \rho\theta_0(\eta_r - \eta_l)) = 0. \end{gathered} \tag{3.1.5}$$

In (3.1.5) the base driving traction f_0 is given, with the aid of (2.6.7), by

$$f_0 = \int_{\gamma_l}^{\gamma_r} \tau(\gamma, \theta_0)\, d\gamma - \tfrac{1}{2}(\tau(\gamma_r, \theta_0) + \tau(\gamma_l, \theta_0))(\gamma_r - \gamma_l), \tag{3.1.6}$$

and the constants η_r and η_l are given in terms of γ_r, γ_l and θ_0 via the shear stress response function τ as follows

$$\eta_r = -\frac{1}{\rho}\int_0^{\gamma_r} \tau_\theta(\kappa, \theta_0)\, d\kappa, \quad \eta_l = -\frac{1}{\rho}\int_0^{\gamma_l} \tau_\theta(\kappa, \theta_0)\, d\kappa. \tag{3.1.7}$$

Observe, as a consequence of (3.1.2) and (2.2.7), that f_0 must satisfy

$$f_0 \geq 0. \tag{3.1.8}$$

Assume that v_0 complies with the inequality

$$v_0 < \min\left\{\sqrt{\tau'(\gamma_l, \theta_0)/\rho}, \sqrt{\tau'(\gamma_r, \theta_0)/\rho}\right\}, \tag{3.1.9}$$

so that the normal velocity of the phase boundary in the base process is *locally subsonic*. It is then permissible[2] to impose a kinetic relation in the form (2.3.8)

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[2] See ABEYARATNE & KKNOWLES [2].

or (2.3.9) on Γ_0 and require that the parameters γ_l, γ_r, v_l, v_r, and v_0 satisfy one of

$$v_0 = \tilde{V}(\tfrac{f_0}{\theta_0}, \theta_0), \quad f_0 = \theta_0 \tilde{\varphi}(v_0, \theta_0), \tag{3.1.10}$$

depending, respectively, upon whether a kinetic relation is provided in the form (2.5.1) or (2.5.2).

In a coordinate frame moving with the phase boundary, the base process described involves a piecewise homogeneous shear strain field and a homogeneous temperature field. If (γ_l, θ_0) and (γ_r, θ_0) are consistent with $(3.1.3)_1$ then the base process is one wherein the high strain elliptic phase of the material at hand *grows* at the expense of the low strain elliptic phase at constant temperature; whereas, if (γ_l, θ_0) and (γ_r, θ_0) comply with $(3.1.3)_2$ then the base process is such that the low strain elliptic phase of the material at hand *grows* at the expense of the high strain elliptic phase at constant temperature. In either case the discontinuity involved is, for the duration of the motion, a *normal* phase boundary—that is, the angle between the limiting values of the gradient of the out-of-plane displacement field on either side of the phase boundary is zero at every point of the phase boundary over the time interval $(-\infty, 0)$.

The constant *latent heat of transformation*—ℓ_0—associated with the thermoelastic process described by (3.1.1)–(3.1.3) is defined by

$$\ell_0 = \rho\theta_0(\eta_l - \eta_r) - f_0 = \theta_0 \int_{\gamma_l}^{\gamma_r} \tau_\theta(\kappa, \theta_0)\, d\kappa - f_0. \tag{3.1.11}$$

From $(3.1.5)_3$ it is clear that ℓ_0 must be zero if $v_0 = 0$—which agrees with the intuitive notion that the heat given off in the process of transformation must be zero in the absence of *heat flux.* Recall from Section 2.5 that, under the present assumption that the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ which is provided is continuous, $v_0 = 0$ if and only if $f_0 = 0$. Hence, when $v_0 = 0$, the latent heat of transformation simplifies to

$$\ell_0 = \rho\theta_0(\eta_l - \eta_r) = \theta_0 \int_{\gamma_l}^{\gamma_r} \tau_\theta(\kappa, \theta_0)\, d\kappa. \qquad (3.1.12)$$

Observe, however, that $(3.1.5)_3$ is satisfied for any real value of ℓ_0 when $v_0 = 0$.

Suppose, in addition to all the above, that the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ is chosen so that its derivative is non-zero at the base driving traction f_0, that is, assume that one of the following, as is appropriate to the specification of a kinetic relation in the form of either (2.5.1) or (2.5.2), must hold:

$$\tilde{V}_\Phi(\tfrac{f_0}{\theta_0}, \theta_0) \neq 0, \quad \tilde{\varphi}_V(v_0, \theta_0) \neq 0, \qquad (3.1.13)$$

This assumption is made in order to preclude the necessity of going to higher order in the context of the forthcoming linear stability analysis. See Figure 3 and Figure 4 for schematic graphs of smooth admissible kinetic response functions $\tilde{V}(\cdot, \theta_0)$ and $\tilde{\varphi}(\cdot, \theta_0)$ which satisfy (3.1.13).

When inertial effects are ignored it is clear that u_0 as defined in (3.1.1) also satisfies the field equation in (2.6.9) on $(\mathbb{R}^2 \times (-\infty, 0)) \setminus \Gamma_0$. Equations $(3.1.5)_{1,3}$ are, in this context, still sufficient to satisfy (2.6.8) and $(2.6.2)_2$ on Γ_0. In place of $(3.1.5)_2$, the constants γ_l, γ_r, v_l, v_r, and v_0 must, however, satisfy

$$\tau(\gamma_r, \theta_0) - \tau(\gamma_l, \theta_0) = 0, \qquad (3.1.14)$$

in order for the jump condition in (2.6.9) to hold on Γ_0. Although the expression for the base driving traction f_0 given in (3.1.6) remains valid in the inertia-free setting, (3.1.14) can in this case be used so that it simplifies to read

$$f_0 = \int_{\gamma_l}^{\gamma_r} \tau(\gamma, \theta_0)\, d\gamma - \tau_*(\gamma_r - \gamma_l), \qquad (3.1.15)$$

where $\tau_* = \tau(\gamma_l, \theta_0) = \tau(\gamma_r, \theta_0)$.

Given a shear stress response function τ which describes a particular three-phase thermoelastic material and an arbitrary kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ which describes the dynamics of phase boundaries which may occur therein, there may not, in general, exist constants γ_l, γ_r, v_l, v_r, and v_0 which satisfy one of $(3.1.3)_1$ or $(3.1.3)_2$ and are consistent with the restrictions embodied by (3.1.5), (3.1.9), $(3.1.10)_1$ or $(3.1.10)_2$, and $(3.1.13)_1$ or $(3.1.13)_2$, or, in the inertia-free case, $(3.1.5)_1$, $(3.1.13)_1$ or $(3.1.13)_2$, (3.1.9), $(3.1.10)_1$ or $(3.1.10)_2$, (3.1.14) and (3.1.15). Within the context of this investigation it will be assumed, however, that $\tilde{V}$ or $\tilde{\varphi}$ is chosen so that a non-trivial base process exists.

3.2. Perturbation of the base process. Suppose that at the instant $t = 0$ the out-of-plane displacement and velocity fields, the absolute temperature field and the configuration of the phase boundary associated with the thermoelastic process specified in Section 3.1 are subjected to a perturbation. Let this perturbation be such that the phase boundary can be, at $t = 0+$, described by the graph C_0 of a continuous function $h : \mathbb{R} \to \mathbb{R}$ of the x_2–coordinate, and segregates elliptic phases of the three-phase material at hand in a sense consistent with that which was present for t in $(-\infty, 0)$. Let the out-of-plane displacement field, out-of-plane velocity field, and absolute temperature field linked to this perturbation be given, respectively, by a once continuously differentiable function $\eta : \mathbb{R}^2 \to \mathbb{R}$, a continuous function $\varpi : \mathbb{R}^2 \to \mathbb{R}$, and a continuous function $\phi : \mathbb{R}^2 \to \mathbb{R}$. Assume that h, η, ϖ and ϕ represent small deviations, in some appropriate sense, from their counterparts in the base process. In particular, suppose that h, η, $\eta_{,\alpha}$, ϖ, and ϕ are all square integrable on their domains of definition. Require, furthermore, that the components of the gradient of η allow the satisfaction of the decay condition

$$\lim_{x_1^2 + x_2^2 \to \infty} \eta_{,\alpha}(x_1, x_2)\eta_{,\alpha}(x_1, x_2) = 0, \tag{3.2.1}$$

while ϖ and ϕ comply with the following decay conditions

$$\lim_{x_1^2+x_2^2\to\infty} \varpi(x_1,x_2)=0, \qquad \lim_{x_1^2+x_2^2\to\infty} \phi(x_1,x_2)=0, \tag{3.2.2}$$

so that the disturbance is *localized* in a neighborhood of the phase boundary associated with the base state at $t=0$.

The perturbation at $t=0$ will initiate a new process involving an out-of-plane displacement field $u:\mathbb{R}^2\times\bar{\mathbb{R}}_+\to\mathbb{R}$ and an absolute temperature field $\theta:\mathbb{R}^2\times\bar{\mathbb{R}}_+\to\mathbb{R}$ which are, in general, *weak solutions* of the field equations (2.6.2) and satisfy the jump conditions in (2.6.4) and (2.6.6) at all discontinuities in their gradients, the kinetic relation (2.5.1) or (2.5.2) at all phase boundaries, and the initial conditions

$$\begin{aligned} &u(\cdot,\cdot,0+)=u_0(\cdot,0+)+\eta \quad \text{on} \quad \mathbb{R}^2,\\ &\dot{u}(\cdot,\cdot,0+)=\dot{u}_0(\cdot,0+)+\varpi \quad \text{on} \quad \mathbb{R}^2,\\ &\theta(\cdot,\cdot,0+)=\theta_0+\phi \quad \text{on} \quad \mathbb{R}^2. \end{aligned} \tag{3.2.3}$$

Since the perturbation is small, assume that, the subsequent process involves only a single phase boundary $C_t=\{(x_1,x_2,t)\,|\,x_1=\varsigma(x_2,t),x_2\in\mathbb{R}\}$ for each t in $\mathbb{R}_+$, with $\varsigma:\mathbb{R}\times\bar{\mathbb{R}}_+\to\mathbb{R}$ continuously differentiable on its domain of definition and defined so that it is in accord with the initial condition

$$\varsigma(\cdot,0+)=h \quad \text{on} \quad \mathbb{R}. \tag{3.2.4}$$

With the intent of linearizing the field equations in (2.6.2) about the base process, write, for each t in $\bar{\mathbb{R}}_+$,

$$\begin{aligned} &u(x_1,x_2,t)=u_0(x_1,t)+w(x_1,x_2,t) \quad \forall(x_1,x_2)\in\mathcal{D}\setminus C_t,\\ &\theta(x_1,x_2,t)=\theta_0+T(x_1,x_2,t) \quad \forall(x_1,x_2)\in\mathcal{D}\setminus C_t, \end{aligned} \tag{3.2.5}$$

where w and its derivatives and T are assumed to represent small departures from the relevent quantities in the base process. Assume that the components of the gradient of w satisfy the following limits

$$\begin{aligned} &\lim_{x_1 \to \pm\infty} w_{,1}(x_1, \cdot, \cdot) = 0 \quad \text{on} \quad I\!\!R \times \bar{I\!\!R}_+, \\ &\lim_{x_2 \to \pm\infty} w_{,2}(\cdot, x_2, \cdot) = 0 \quad \text{on} \quad I\!\!R \times \bar{I\!\!R}_+, \end{aligned} \tag{3.2.6}$$

and also that T complies with the limit

$$\lim_{x_1^2 + x_2^2 \to \infty} T(x_1, x_2, \cdot) = 0 \quad \text{on} \quad \bar{I\!\!R}_+. \tag{3.2.7}$$

From (3.2.3) and (3.2.5) it is clear, moreover, that—when inertial effects are not ignored—the increment w to the out-of-plane displacement field must satisfy the following initial conditions:

$$\begin{aligned} w(\cdot, \cdot, 0+) &= \eta \quad \text{on} \quad I\!\!R^2, \\ \dot{w}(\cdot, \cdot, 0+) &= \varpi \quad \text{on} \quad I\!\!R^2. \end{aligned} \tag{3.2.8}$$

It is important to emphasize that these can not be imposed in the inertia-free setting.

Also, the increment T to the absolute temperature field must satisfy the following initial condition

$$T(\cdot, \cdot, 0+) = \phi \quad \text{on} \quad I\!\!R^2. \tag{3.2.9}$$

Next, define $s : I\!\!R \times \bar{I\!\!R}_+ \to I\!\!R$, the correction to the interface position due to the perturbation, via

$$\varsigma(\cdot, t) = v_0 t + s(\cdot, t) \quad \text{on} \quad I\!\!R \quad \forall t \in \bar{I\!\!R}_+. \tag{3.2.10}$$

Note, from (3.2.4) that the increment s to the phase boundary position must satisfy the initial condition

$$s(\cdot,0+) = h \quad \text{on} \quad I\!R. \tag{3.2.11}$$

Observe that the unit normal vectors $\mathbf{n}_\pm(\cdot,t) : I\!R \to \mathcal{N}$ to C_t are given by

$$\mathbf{n}_\pm(\cdot,t) = \pm\frac{\mathbf{e}_1 - s,_2(\cdot,t)\mathbf{e}_2}{\sqrt{1+s,_2^2(\cdot,t)}} \quad \text{on} \quad I\!R \quad \forall t \in I\!R_+. \tag{3.2.12}$$

For the remainder of this work, choose the unit normal vector associated with the plus sign in (3.2.12) and drop this sign when referring to it. The normal velocity $V_n(\cdot,t) : I\!R \to I\!R$ of C_t is, thus, given, for each t in $I\!R_+$, by

$$V_n(\cdot,t) = \frac{v_0 + \dot{s}(\cdot,t)}{\sqrt{1+s,_2^2(\cdot,t)}} \quad \text{on} \quad I\!R \quad \forall t \in I\!R_+. \tag{3.2.13}$$

3.3. Linearization of the field equations associated with the process initiated by the perturbation. Let $\mathcal{D}_t^l$ and $\mathcal{D}_t^r$ denote, for each t in $I\!R_+$, plane sets defined as shown below:

$$\mathcal{D}_t^l = \{(x_1,x_2)|\, x_1 \le \varsigma(x_2,t)\}, \quad \mathcal{D}_t^r = I\!R^2 \setminus \mathring{\mathcal{D}}_t^l. \tag{3.3.1}$$

Let $\mathcal{X}_l$ and $\mathcal{X}_r$ be given, in turn, by

$$\mathcal{X}_l = \{(x_1,x_2,t)|\, (x_1,x_2) \in \mathcal{D}_t^l, t \in I\!R_+\}, \tag{3.3.2}$$

and

$$\mathcal{X}_r = \{(x_1,x_2,t)|\, (x_1,x_2) \in \mathcal{D}_t^r, t \in I\!R_+\}. \tag{3.3.3}$$

The displacement equations of motion which hold on $\mathring{\mathcal{X}}_l$ and $\mathring{\mathcal{X}}_r$ can be obtained following Fried [9] and are given, in turn, by

$$\begin{aligned} a_l^2 w,_{11} + b_l^2 w,_{22} &= \ddot{w}, \\ a_r^2 w,_{11} + b_r^2 w,_{22} &= \ddot{w}, \end{aligned} \tag{3.3.4}$$

where the positive constants a_l, b_l, and a_r, b_r are defined as follows:

$$\begin{aligned} a_l &= \sqrt{\tau_\gamma(\gamma_l,\theta_0)/\rho}, \quad b_l = \sqrt{M(\gamma_l,\theta_0)/\rho}, \\ a_r &= \sqrt{\tau_\gamma(\gamma_r,\theta_0)/\rho}, \quad b_r = \sqrt{M(\gamma_r,\theta_0)/\rho}. \end{aligned} \tag{3.3.5}$$

In writing (3.3.5), the positivity of $\tau_\gamma(\gamma_l,\theta_0)$ and $\tau_\gamma(\gamma_r,\theta_0)$—which are results of whichever of $(3.1.3)_{1,2}$ is appropriate, and of $M(\gamma_l,\theta_0)$ and $M(\gamma_r,\theta_0)$—which follow from (2.4.13), have been used.

The energy equations which hold on $\mathring{\mathcal{X}}_l$ and $\mathring{\mathcal{X}}_r$ can be obtained by linearizing the partial differential equation $(2.6.2)_2$ about (γ_l,θ_0) and (γ_r,θ_0), respectively. Turn, now, to the derivation of the linearized energy equation which holds on $\mathring{\mathcal{X}}_l$. It is easy to show, following [9], that

$$\gamma \cong \gamma_l + w_{,1} \quad \text{on} \quad \mathring{\mathcal{X}}_l. \tag{3.3.6}$$

With the aid of $(3.2.5)_2$ and Taylor's theorem the relation (3.3.6) leads to the following expansions:

$$\begin{aligned} \tilde{k}(\gamma,\theta) &\cong \tilde{k}(\gamma_l,\theta_0) + \tilde{k}_\gamma(\gamma_l,\theta_0)w_{,1} + \tilde{k}_\theta(\gamma_l,\theta_0)T \quad \text{on} \quad \mathring{\mathcal{X}}_l, \\ \tilde{c}(\gamma,\theta) &\cong \tilde{c}(\gamma_l,\theta_0) + \tilde{c}_\gamma(\gamma_l,\theta_0)w_{,1} + \tilde{c}_\theta(\gamma_l,\theta_0)T \quad \text{on} \quad \mathring{\mathcal{X}}_l, \\ M_\theta(\gamma,\theta) &\cong M_\theta(\gamma_l,\theta_0) + M_{\gamma\theta}(\gamma_l,\theta_0)w_{,1} + M_{\theta\theta}(\gamma_l,\theta_0)T \quad \text{on} \quad \mathring{\mathcal{X}}_l. \end{aligned} \tag{3.3.7}$$

Next, using (3.2.5) and $(3.3.7)_{1,2}$ in the left hand side of the partial differential equation in $(2.6.2)_2$ gives

$$\begin{aligned} &(\tilde{k}(\gamma,\theta)\theta_{,\alpha})_{,\alpha} + M_\theta(\gamma,\theta)\theta u_{,\alpha}\dot{u}_{,\alpha} \cong \\ &\qquad \tilde{k}(\gamma_l,\theta_0)T_{,\alpha\alpha} + \theta_0\tau_\theta(\gamma_l,\theta_0)\dot{w}_{,1} \quad \text{on} \quad \mathring{\mathcal{X}}_l, \end{aligned} \tag{3.3.8}$$

while using (3.2.5) and $(3.2.7)_3$ in the right-hand-side of the same equation gives

$$\rho\tilde{c}(\gamma,\theta)\dot{\theta} \cong \rho\tilde{c}(\gamma_l,\theta_0)\dot{T} \quad \text{on} \quad \mathring{\mathcal{X}}_l. \tag{3.3.9}$$

The linearized energy equation which holds on $\mathring{\mathcal{X}}_l$ is, thus, from (3.2.5), (3.3.8) and (3.3.9), given by

$$\alpha_l T_{,\alpha\alpha} = \dot{T} + \beta_l \dot{w}_{,1}\,, \tag{3.3.10}$$

where the positive constant α_l and the real constant β_l are defined by

$$\alpha_l = \frac{\tilde{k}(\gamma_l, \theta_0)}{\rho\tilde{c}(\gamma_l, \theta_0)}, \quad \beta_l = -\frac{\theta_0 \tau_\theta(\gamma_l, \theta_0)}{\rho\tilde{c}(\gamma_l, \theta_0)}. \tag{3.3.11}$$

Similarly, the linearized energy equation which holds on $\mathring{\mathcal{X}}_r$ is

$$\alpha_r T_{,\alpha\alpha} = \dot{T} + \beta_r \dot{w}_{,1}\,, \tag{3.3.12}$$

where the positive constant α_r and the real constant β_r are defined by

$$\alpha_r = \frac{\tilde{k}(\gamma_r, \theta_0)}{\rho\tilde{c}(\gamma_r, \theta_0)}, \quad \beta_r = -\frac{\theta_0 \tau_\theta(\gamma_r, \theta_0)}{\rho\tilde{c}(\gamma_r, \theta_0)}. \tag{3.3.13}$$

From (2.6.9) it is clear that, in the inertia-free setting, the displacement equations of motion (3.3.4) are supplanted by

$$\begin{aligned} a_l^2 w_{,11} + b_l^2 w_{,22} &= 0, \\ a_r^2 w_{,11} + b_r^2 w_{,22} &= 0, \end{aligned} \tag{3.3.14}$$

which hold, repectively, on $\mathring{\mathcal{X}}_l$ and $\mathring{\mathcal{X}}_r$.

3.4. Linearization of the jump conditions and kinetic relation associated with the process initiated by the perturbation. Since the set $\Gamma = \{(x_1, x_2, t) |\, (x_1, x_2) \in C_t, t \in \mathbb{R}_+\}$ represents the post-disturbance trajectory of the phase boundary, the jump conditions in (2.6.4) and (2.6.6) and the kinetic relation in (2.5.1) or (2.5.2)—with V_n and f given, respectively, by (3.2.13) and (2.6.7)—must hold on it. Assume, henceforth, that the function s introduced via (3.2.10) and its derivatives are small in the same sense that w and T are small.

Note, first, that this assumption implies, using (3.2.12) and (3.2.13), the following approximations for $\mathbf{n}$ and V_n on Γ:

$$\begin{aligned} \mathbf{n} &\cong \mathbf{e}_1 - s_{,2}\,\mathbf{e}_2 \quad \text{on} \quad \Gamma, \\ V_n &\cong v_0 + \dot{s} \quad \text{on} \quad \Gamma. \end{aligned} \tag{3.4.1}$$

It is easy to show, following FRIED [9], that the linearized form of the jump condition $(2.6.4)_1$ is as follows

$$\begin{aligned} (a_r^2 - v_0^2)w_{,1}(v_0t+, x_2, t) - (a_l^2 - v_0^2)w_{,1}(v_0t-, x_2, t) \\ = 2v_0(\gamma_r - \gamma_l)\dot{s}(x_2, t) \quad \forall (x_2, t) \in I\!R \times I\!R_+, \end{aligned} \tag{3.4.2}$$

while that of $(2.6.6)_1$ is

$$w(v_0t+, x_2, t) - w(v_0t-, x_2, t) = (\gamma_l - \gamma_r)s(x_2, t) \quad \forall (x_2, t) \in I\!R \times I\!R_+. \tag{3.4.3}$$

Linearization of the jump condition $(2.6.6)_2$ gives, next, since no heat flux is present in the base process (3.1.1)–(3.1.3),

$$T(v_0t+, x_2, t) - T(v_0t-, x_2, t) = 0 \quad \forall (x_2, t) \in I\!R \times I\!R_+, \tag{3.4.4}$$

so that the increment T to the absolute temperature field is continuous across the phase boundary in the post perturbation process.

The driving traction f can be linearized in a manner analagous to that displayed in [9] to give, with the aid of (3.4.4),

$$\begin{aligned} f(x_2, t) \cong f_0 &+ \tfrac{1}{2}\rho(\gamma_l - \gamma_r)\big((a_r^2 - v_0^2)w_{,1}(v_0t+, x_2, t) + (a_l^2 - v_0^2)w_{,1}(v_0t-, x_2, t)\big) \\ &+ \big(\rho(\eta_l - \eta_r) + \tfrac{1}{2}(\gamma_l - \gamma_r)(\tau_\theta(\gamma_r, \theta_0) + \tau_\theta(\gamma_l, \theta_0))\big)\, T(v_0t, x_2, t) \\ &\forall (x_2, t) \in I\!R \times I\!R_+, \end{aligned} \tag{3.4.5}$$

where the base driving traction f_0 is given by (3.1.6).

Turn, now, to the linearization of the energy jump condition $(2.6.4)_2$. From $(3.3.7)_1$, $(3.4.1)_1$ and $(3.2.5)_2$ it is clear that the first term on the left-hand-side of the energy balance jump condition $(2.6.4)_2$ linearizes as follows:

$$
\begin{aligned}
&[\![\tilde{k}(\gamma(\zeta(x_2,t),x_2,t),\theta(\zeta(x_2,t),x_2,t))\theta,_\alpha(\zeta(x_2,t),x_2,t)n_\alpha(x_2,t)]\!] \\
&\qquad\qquad \cong \tilde{k}(\gamma_r,\theta_0)T,_1(v_0t+,x_2,t) - \tilde{k}(\gamma_l,\theta_0)T,_1(v_0t-,x_2,t) \\
&\qquad\qquad\qquad \forall (x_2,t) \in I\!R \times I\!R_+. \qquad (3.4.6)
\end{aligned}
$$

Furthermore, (2.6.5), $(2.6.6)_2$, (3.4.1), (3.4.4) and (3.4.5) yield the following linearization of the remaining two terms on the left-hand-side of the energy balance jump condition $(2.6.4)_2$:

$$
\begin{aligned}
&(\rho\theta(\zeta(x_2,t),x_2,t)[\![\eta(\zeta(x_2,t),x_2,t)]\!] + f(x_2,t))V_n(x_2,t) \\
&\quad\cong \rho v_0(\tilde{c}(\gamma_r,\theta_0) - \tilde{c}(\gamma_l,\theta_0))T(v_0t,x_2,t) \\
&\qquad + \tfrac{1}{2}v_0(\gamma_l-\gamma_r)(\tau_\theta(\gamma_r,\theta_0)+\tau_\theta(\gamma_l,\theta_0)))T(v_0t,x_2,t) \\
&\qquad + \tfrac{1}{2}\rho v_0(\gamma_l-\gamma_r)((a_r^2-v_0^2)w,_1(v_0t+,x_2,t)+(a_l^2-v_0^2)w,_1(v_0t-,x_2,t)) \\
&\qquad - v_0\theta_0(\tau_\theta(\gamma_r,\theta_0)w,_1(v_0t+,x_2,t) - \tau_\theta(\gamma_l,\theta_0)w,_1(v_0t-,x_2,t)) \\
&\qquad - \ell_0\dot{s}(x_2,t) \quad \forall (x_2,t) \in I\!R \times I\!R_+. \qquad (3.4.7)
\end{aligned}
$$

Together, (3.4.6) and (3.4.7) give the following expression for the linearization of the energy jump condition $(2.6.4)_2$:

$$
\begin{aligned}
0 =& \tilde{k}(\gamma_r,\theta_0)T,_1(v_0t+,x_2,t) - \tilde{k}(\gamma_l,\theta_0)T,_1(v_0t-,x_2,t) \\
&+ \rho v_0(\tilde{c}(\gamma_r,\theta_0) - \tilde{c}(\gamma_l,\theta_0))T(v_0t,x_2,t) \\
&+ \tfrac{1}{2}v_0(\gamma_l-\gamma_r)(\tau_\theta(\gamma_r,\theta_0)+\tau_\theta(\gamma_l,\theta_0)))T(v_0t,x_2,t) \\
&+ \tfrac{1}{2}\rho v_0(\gamma_l-\gamma_r)((a_r^2-v_0^2)w,_1(v_0t+,x_2,t)+(a_l^2-v_0^2)w,_1(v_0t-,x_2,t)) \\
&- v_0\theta_0(\tau_\theta(\gamma_r,\theta_0)w,_1(v_0t+,x_2,t) - \tau_\theta(\gamma_l,\theta_0)w,_1(v_0t-,x_2,t)) \\
&- \ell_0\dot{s}(x_2,t) \quad \forall (x_2,t) \in I\!R \times I\!R_+. \qquad (3.4.8)
\end{aligned}
$$

Linearization of the kinetic relation and use of whichever of (3.1.10) is appropriate in a manner completely analogous to that performed in [9] gives, with the aid of (3.4.4),

$$\begin{aligned}\dot{s}(x_2,t) =& \frac{\gamma_l-\gamma_r}{2v_*}((a_r^2-v_0^2)w_{,1}(v_0t+,x_2,t)+(a_l^2-v_0^2)w_{,1}(v_0t-,x_2,t)) \\ &+\frac{\gamma_l-\gamma_r}{2\rho v_*}(\tau_\theta(\gamma_r,\theta_0)+\tau_\theta(\gamma_l,\theta_0)))T(v_0t,x_2,t) \\ &+(v_\vartheta+\tfrac{\ell_0}{\rho v_*\theta_0})T(v_0t,x_2,t) \quad \forall(x_2,t)\in I\!R\times I\!R_+, \end{aligned} \tag{3.4.9}$$

where the constants v_* and v_ϑ are defined by either

$$v_* = \frac{\theta_0}{\rho\tilde{V}_\Phi(\frac{f_0}{\theta_0},\theta_0)}, \qquad v_\vartheta = \tilde{V}_\Theta(\tfrac{f_0}{\theta_0},\theta_0), \tag{3.4.10}$$

if the kinetic relation is furnished in the form (2.5.1), or

$$v_* = \frac{\theta_0\tilde{\varphi}_V(v_0,\theta_0)}{\rho}, \qquad v_\vartheta = -\frac{\tilde{\varphi}_\theta(v_0,\theta_0)}{\tilde{\varphi}_V(v_0,\theta_0)}, \tag{3.4.11}$$

if the kinetic relation is supplied in the form (2.5.2). Note, from (3.1.13), that v_* is a real—but nonzero—constant, while v_ϑ is a real—and possibly zero—constant.

By virtue of the foregoing calculations it is crucial to note that, within the scope of the linearization, it is legitimate to enforce the partial differential equations in $(3.3.4)_1$ and (3.3.10) on the interiors of the set Ω_l defined by

$$\Omega_l = \{(x_1,x_2,t)|\,(x_1,x_2)\in\Pi_t^l, t\in I\!R_+\}, \tag{3.4.12}$$

with $\Pi_t^l = \{(x_1,x_2)|\,x_1\le v_ot, x_2\in I\!R\}$ for each t in $I\!R_+$, instead of the set $\mathring{\mathcal{X}}_l$, and the partial differential equations in $(3.3.4)_2$ and (3.3.12) on the interior of the set Ω_r defined by

$$\Omega_r = \{(x_1,x_2,t)|\,(x_1,x_2)\in\Pi_t^r, t\in I\!R_+\}, \tag{3.4.13}$$

with $\Pi_t^r = \{(x_1,x_2) | \, x_1 \geq v_o t, x_2 \in I\!R\}$ for each t in $I\!R_+$, instead of the set $\mathring{\mathcal{X}}_r$. For the purposes of the forthcoming analysis it is useful to define a set I as follows:

$$I = \{(x_1,x_2,t) | \, x_1 = v_0 t, x_2 \in I\!R, t \in I\!R_+\}. \tag{3.4.14}$$

In the inertia-free case it is readily shown that, while (3.4.3) and (3.4.4) continue to hold, (3.4.2) is replaced by

$$a_r^2 w,_1 (v_0 t+, x_2, t) - a_l^2 w,_1 (v_0 t-, x_2, t) = 0 \quad \forall (x_2,t) \in I\!R \times I\!R_+, \tag{3.4.15}$$

(3.4.8) reduces to

$$\begin{aligned}
0 =& \tilde{k}(\gamma_r,\theta_0) T,_1 (v_0 t+, x_2, t) - \tilde{k}(\gamma_l,\theta_0) T,_1 (v_0 t-, x_2, t) \\
&+ \rho v_0 (\tilde{c}(\gamma_r,\theta_0) - \tilde{c}(\gamma_l,\theta_0)) T(v_0 t, x_2, t) \\
&+ \tfrac{1}{2} v_0 (\gamma_l - \gamma_r)(\tau_\theta(\gamma_r,\theta_0) + \tau_\theta(\gamma_l,\theta_0))) T(v_0 t, x_2, t) \\
&+ \tfrac{1}{2} \rho v_0 (\gamma_l - \gamma_r)(a_r^2 w,_1 (v_0 t+, x_2, t) + a_l^2 w,_1 (v_0 t-, x_2, t)) \\
&- v_0 \theta_0 (\tau_\theta(\gamma_r,\theta_0) w,_1 (v_0 t+, x_2, t) - \tau_\theta(\gamma_l,\theta_0) w,_1 (v_0 t-, x_2, t)) \\
&- \ell_0 \dot{s}(x_2,t) \quad \forall (x_2,t) \in I\!R \times I\!R_+,
\end{aligned} \tag{3.4.16}$$

and (3.4.9) simplifies to read

$$\begin{aligned}
\dot{s}(x_2,t) =& \frac{\gamma_l - \gamma_r}{2 v_*} (a_r^2 w,_1 (v_0 t+, x_2, t) + a_l^2 w,_1 (v_0 t-, x_2, t)) \\
&+ \frac{\gamma_l - \gamma_r}{2 \rho v_*} (\tau_\theta(\gamma_r,\theta_0) + \tau_\theta(\gamma_l,\theta_0))) T(v_0 t, x_2, t) \\
&+ (v_\vartheta + \tfrac{\ell_0}{\rho v_* \theta_0}) T(v_0 t, x_2, t) \quad \forall (x_2,t) \in I\!R \times I\!R_+.
\end{aligned} \tag{3.4.17}$$

Finally, remarks analogous to those made regarding the enforcement of the partial differential equations in $(3.3.4)_1$ and $(3.3.4)_2$ on $\mathring{\Omega}_l$ and $\mathring{\Omega}_r$ apply also to those in $(3.3.14)_1$ and $(3.3.14)_2$.

3.5. Specialization of the base process and the associated linearized description of the post perturbation process. Suppose, henceforth, that the base process described in Section 3.1 is restricted so that

$$\tau_\theta(\gamma_l, \theta_0) = \tau_\theta(\gamma_r, \theta_0) = 0; \tag{3.5.1}$$

it is implicitly assumed that the shear stress response function τ allows for the possibility of (3.5.1). Observe that (3.5.1) requires that the *coefficients of thermoelastic coupling* in the low and high strain phases of the material at hand are, by $(3.3.11)_2$ and $(3.3.13)_2$, both identically zero. Although this assumption is made in order to simplify the forthcoming analysis, it is not inconsistent with the isochoric nature of the deformation under consideration. The linearized field equations, jump conditions, kinetic relation, initial conditions (where appropriate), and far field decay conditions satisfied by the increments w, T and s to the out-of-plane displacement field, absolute temperature field and the interface position are now listed in both the inertial and inertia-free cases.

In the inertial case, (3.3.4), (3.3.10) and (3.3.12) give the following linearized field equations

$$\begin{aligned} a_l^2 w,_{11} + b_l^2 w,_{22} &= \ddot{w} \quad \text{on} \quad \mathring{\Omega}_l, \\ a_r^2 w,_{11} + b_r^2 w,_{22} &= \ddot{w} \quad \text{on} \quad \mathring{\Omega}_r, \\ \alpha_l T,_{\alpha\alpha} &= \dot{T} \quad \text{on} \quad \mathring{\Omega}_l, \\ \alpha_r T,_{\alpha\alpha} &= \dot{T} \quad \text{on} \quad \mathring{\Omega}_r. \end{aligned} \tag{3.5.2}$$

In addition, from (3.4.2), (3.4.8), (3.4.3) and (3.4.4) the following jump conditions hold

$$\begin{aligned} [\![(a^2 - v_0^2)w,_1]\!] &= 2v_0(\gamma_r - \gamma_l)\dot{s} \quad \text{on} \quad I, \\ [\![kT,_1]\!] + \rho v_0 [\![c]\!] T &= \rho v_0 (\gamma_r - \gamma_l) \langle\!\langle (a^2 - v_0^2) w,_1 \rangle\!\rangle + \ell_0 \dot{s} \quad \text{on} \quad I, \\ [\![w]\!] &= (\gamma_l - \gamma_r)s \quad \text{on} \quad I, \\ [\![\theta]\!] &= 0 \quad \text{on} \quad I, \end{aligned} \tag{3.5.3}$$

where the constants a_+^2 and a_-^2 are given by

$$a_+^2 = a_r^2, \quad a_-^2 = a_l^2, \tag{3.5.4}$$

and $k_+ = k_r$, $k_- = k_l$, $c_+ = c_r$, and $c_- = c_l$ are defined via

$$\begin{aligned} k_+ &= \tilde{k}(\gamma_r, \theta_0), \quad k_- = \tilde{k}(\gamma_l, \theta_0), \\ c_+ &= \tilde{c}(\gamma_r, \theta_0), \quad c_- = \tilde{c}(\gamma_l, \theta_0). \end{aligned} \tag{3.5.5}$$

Next, from (3.4.9) and (3.5.1) the following linearized kinetic relation holds:

$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle (a^2 - v_0^2) w_{,1} \rangle\!\rangle + (v_\vartheta + \tfrac{\ell_0}{\rho v_* \theta_0}) T \quad \text{on} \quad I. \tag{3.5.6}$$

Observe that, despite the restrictions imposed on the coefficients of thermoelastic coupling by (3.5.1), the corrections to the out-of-plane displacement and absolute temperature fields reamain coupled through $(3.5.3)_2$ and (3.5.6).

The initial conditions satisfied by w and s are, from (3.2.8), (3.2.9) and (3.2.11),

$$\begin{aligned} w(\cdot,\cdot,0+) &= \eta \quad \text{on} \quad \mathbb{R}^2, \\ \dot{w}(\cdot,\cdot,0+) &= \varpi \quad \text{on} \quad \mathbb{R}^2, \\ T(\cdot,\cdot,0+) &= \phi \quad \text{on} \quad \mathbb{R}^2, \\ s(\cdot,0+) &= h \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{3.5.7}$$

Finally, from (3.2.6) and (3.2.7), it is assumed that, for each t in $\mathbb{R}_+$, the following far field decay conditions hold

$$\begin{aligned} &\lim_{x_1 \to \pm\infty} w_{,1}(x_1, \cdot, t) = 0 \quad \text{on} \quad \mathbb{R}, \\ &\lim_{x_2 \to \pm\infty} w_{,2}(\cdot, x_2, t) = 0 \quad \text{on} \quad \mathbb{R}, \\ &\lim_{x_1^2 + x_2^2 \to \pm\infty} T(x_1, x_2, t) = 0. \end{aligned} \tag{3.5.8}$$

In the inertia-free case, $(3.5.2)_{1,2}$ are replaced by

$$
\begin{aligned}
a_l^2 w_{,11} + b_l^2 w_{,22} &= 0 \quad \text{on} \quad \mathring{\Omega}_l, \\
a_r^2 w_{,11} + b_r^2 w_{,22} &= 0 \quad \text{on} \quad \mathring{\Omega}_r.
\end{aligned}
\tag{3.5.9}
$$

Furthermore, the jump condition $(3.5.3)_1$ is, by virtue of (3.4.15), replaced by

$$
[\![a^2 w_{,1}]\!] = 0 \quad \text{on} \quad I, \tag{3.5.10}
$$

and $(3.5.3)_2$ is, from (3.4.16) and (3.5.1), supplanted by

$$
[\![kT_{,1}]\!] + \rho v_0 [\![c]\!] T = \rho v_0 (\gamma_r - \gamma_l) \langle\!\langle a^2 w_{,1} \rangle\!\rangle + \ell_0 \dot{s} \quad \text{on} \quad I. \tag{3.5.11}
$$

while $(3.5.3)_{3,4}$ continue to hold. Finally, the linearized kinetic relation (3.5.6) is, upon referring to (3.4.17) and (3.5.1), superceded by

$$
\dot{s} = \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle a^2 w_{,1} \rangle\!\rangle + (v_\vartheta + \tfrac{\ell_0}{\rho v_* \theta_0}) T \quad \text{on} \quad I. \tag{3.5.12}
$$

In the absence of inertial effects initial conditions cannot be given for the increments to the out-of-plane displacement and velocity fields w and $\dot{w}$; the initial condition $(3.5.7)_{3,4}$ pertaining to T and s still, however, continue to be applicable. The decay conditions (3.5.8) also still hold.

3.6. Normal mode analysis for a base process involving a static interface in the absence of inertia. Suppose that v_0 in (3.1.1) is zero. Then the base process described by (3.1.1)–(3.1.3) is a piecewise homogeneous isothermal two-phase state involving a static planar interface. Recall, from Section 2.5, that when $v_0 = 0$ and the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ is continuously differentiable on its domain of definition then $v_* \geq 0$. Since $v_* = 0$ is ruled out by whichever of (3.1.13) is appropriate and the corresponding expression $(3.4.10)_1$ or $(3.4.11)_1$, it is clear that—in the present context—$v_* > 0$. Consider, now, the initial boundary value problem composed by (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$, (3.5.12), $(3.5.7)_{3,4}$ and (3.5.8). Note, since $v_0 = 0$, that (3.5.11) and (3.5.12) reduce with the aid of (3.1.12) to

$$\begin{aligned} [\![kT_{,1}]\!] &= \rho\theta_0(\eta_l - \eta_r)\dot{s} \quad \text{on} \quad I, \\ \dot{s} &= \frac{\gamma_l - \gamma_r}{v_*}\langle\!\langle a^2 w_{,1} \rangle\!\rangle + \frac{\eta_l - \eta_r}{v_*} T \quad \text{on} \quad I. \end{aligned} \tag{3.6.1}$$

Observe that, by virtue of the linearization, the relevant partial differential equations, jump conditions and kinetic relation are all linear with constant coefficients; note, also, that the domains $\mathring{\Pi}_0^l$ and $\mathring{\Pi}_0^r$ are rectangular. It is therefore possible to find a solution to the linearized partial differential equations, jump conditions and kinetic relation in the form

$$\begin{aligned} w(x_1, x_2, t) &= \begin{cases} W_l e^{+\xi_l x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2, t) \in \mathring{\Pi}_0^l \times I\!R_+, \\ W_r e^{-\xi_r x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2, t) \in \mathring{\Pi}_0^r \times I\!R_+, \end{cases} \\ T(x_1, x_2, t) &= \begin{cases} \Theta_l e^{+\zeta_l x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2, t) \in \mathring{\Pi}_0^l \times I\!R_+, \\ \Theta_r e^{-\zeta_r x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2, t) \in \mathring{\Pi}_0^r \times I\!R_+, \end{cases} \\ s(x_2, t) &= S e^{i\kappa x_2} e^{pt} \quad \forall (x_2, t) \in I\!R \times I\!R_+, \end{aligned} \tag{3.6.2}$$

where the *amplitudes* W_l, W_r, Θ_l, Θ_r and S, *wave-numbers* ξ_l, ξ_r, ζ_l, ζ_r and κ, and *growth-rate* p are all constants. To comply with the decay conditions $(3.5.8)_{1,3}$ it is clear that $\Re(\xi_l)$, $\Re(\xi_r)$, $\Re(\zeta_l)$ and $\Re(\zeta_r)$ must all be positive. Although (3.6.2) are not, in general, consistent with neither the initial conditions $(3.5.7)_{3,4}$ which hold in the absence of inertial effects nor the decay conditions $(3.5.8)_{2,3}$, since ϕ and h are stipulated to be square integrable on $I\!R$, and hence can be represented as Fourier integrals—

$$\begin{aligned} \phi(x_1, x_2) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\phi}(x_1, \kappa) e^{i\kappa x_2}\, d\kappa \quad \forall (x_1, x_2) \in I\!R^2, \\ h(x_2) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{h}(\kappa) e^{i\kappa x_2}\, d\kappa \quad \forall x_2 \in I\!R, \end{aligned} \tag{3.6.3}$$

it is reasonable to expect that stability results can be obtained by a *normal-mode analysis*; such an analysis entails substitution of (3.6.2) into (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$ and (3.5.12) to determine the growth-rate p as a function of the complex wave-numbers ξ_l, ξ_r, ζ_l, ζ_r and the real wave-number κ. If there exists a complex growth-rate p with positive real part which arises as a solution to the aforementioned problem then the base process will be referred to as *linearly unstable.* Otherwise, the base process will be called *linearly stable.*

Observe that the amplitudes Θ_l, Θ_r and S and wave-numbers ζ_l, ζ_r and κ must be viewed as given for the normal mode analysis to prove effective in determining necessary and sufficient conditions, via the analysis of a *dispersion relation* like that performed in [9], for the linear instability of the base process with respect to arbitrary disturbances contained in the class of perturbations described in Section 3.2. Before proceeding, note, from $(3.5.3)_4$, that $\Theta_l = \Theta_r =: \Theta$. Now, substitution of (3.6.2) into (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$ and (3.5.12) yields the following relations

$$
\begin{gathered}
W_l = -\frac{\nu^2|\kappa|}{a_l b_l(\gamma_l - \gamma_r)} S, \quad W_r = -\frac{\nu^2|\kappa|}{a_r b_r(\gamma_r - \gamma_l)} S, \quad \Theta = -\frac{pG_0(\kappa,p)}{\eta_l - \eta_r} S, \\
\xi_l = \frac{b_l}{a_l}|\kappa|, \quad \xi_r = \frac{b_r}{a_r}|\kappa|, \quad \zeta_l = \sqrt{\kappa^2 + p/\alpha_l}, \quad \zeta_r = \sqrt{\kappa^2 + p/\alpha_r}, \qquad (3.6.4) \\
p + \frac{1}{v_*}(\nu^2|\kappa| + pG_0(\kappa,p)) = 0,
\end{gathered}
$$

where $G_0 : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is given by

$$
G_0(\kappa,p) = \frac{\theta_0(\eta_l - \eta_r)^2}{c_l\sqrt{\alpha_l^2\kappa^2 + \alpha_l p} + c_r\sqrt{\alpha_r^2\kappa^2 + \alpha_r p}} \quad \forall(\kappa,p) \in \mathbb{R} \times \mathbb{C}, \qquad (3.6.5)
$$

and the constant ν^2 is defined as follows:

$$
\nu^2 = \frac{a_l b_l a_r b_r(\gamma_l - \gamma_r)^2}{a_l b_l + a_r b_r}. \qquad (3.6.6)
$$

It is clear from $(3.6.4)_{3,6,7}$ that, for (3.6.2) to represent a solution to (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$ and (3.5.12), the amplitude Θ and the wave-numbers ζ_l and ζ_r cannot be chosen independently of S and κ. Hence, the normal mode analysis is only of use in analyzing the linear stability of the base process at hand with respect to a certain class of perturbations; that is, it is only possible—via this analysis—to determine conditions necessary and sufficient for the instability of the base process with respect to a proper subset of the class of perturbations introduced in Section 3.2. To achieve such results it suffices to analyze the zero structure of the dispersion relation $(3.6.4)_8$ as a function of the growth-rate p for fixed values of the wave-number κ and the parameters γ_l, γ_r, v_0, θ_0, a_l, a_r, b_l, b_r, α_l, α_r, c_l, c_r, ρ and v_*. This is done below.

To comply with the restriction that $\Re(\zeta_l)$ and $\Re(\zeta_r)$ are both positive, the square roots which appear in the definition of G_0 are defined so that for p in $I\!R$,

$$\begin{aligned} \alpha_l^2\kappa^2 + \alpha_l p > 0 \quad &\Longrightarrow \quad \sqrt{\alpha_l^2\kappa^2 + \alpha_l p} > 0 \quad \forall \kappa \in I\!R, \\ \alpha_r^2\kappa^2 + \alpha_r p > 0 \quad &\Longrightarrow \quad \sqrt{\alpha_r^2\kappa^2 + \alpha_r p} > 0 \quad \forall \kappa \in I\!R, \end{aligned} \tag{3.6.7}$$

from which it is clear that for p in $\mathbb{C}$,

$$\begin{aligned} \Re(\alpha_l^2\kappa^2 + \alpha_l p) > 0 \quad &\Longrightarrow \quad \Re\left(\sqrt{\alpha_l^2\kappa^2 + \alpha_l p}\right) > 0 \quad \forall \kappa \in I\!R, \\ \Re(\alpha_r^2\kappa^2 + \alpha_r p) > 0 \quad &\Longrightarrow \quad \Re\left(\sqrt{\alpha_r^2\kappa^2 + \alpha_r p}\right) > 0 \quad \forall \kappa \in I\!R. \end{aligned} \tag{3.6.8}$$

Furthermore, it is evident from (3.6.8) that

$$\Re(p) > 0 \quad \Longleftrightarrow \quad \Re(G_0(\kappa, p)) > 0 \quad \forall \kappa \in I\!R. \tag{3.6.9}$$

This result shows that there cannot exist a root p in $\mathbb{C}$ with $\Re(p) > 0$ to $(3.6.4)_8$ unless $v_* < 0$. Since $v_* > 0$ it is clear that, at present, there cannot exist a p in $\mathbb{C}$ to $(3.6.4)_8$ with $\Re(p) > 0$ for any κ in $I\!R \setminus \{0\}$. Hence, when $v_0 = 0$ and inertial effects are disregarded the base process described in Section 3.1 is linearly stable with respect to all perturbations within the narrowed set under consideration.

If, in place of the foregoing normal mode analysis, a full-fledged Fourier-Laplace transform analysis of (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$ and

(3.5.12) is performed, then the narrowing of the class of initial data necessitated by the normal mode analysis does not occur. Furthermore, in this case it transpires that the Fourier-Laplace transform of s can be expressed in the form

$$S(\kappa, p) = \frac{\hat{h}(\kappa) + \frac{\gamma_l - \gamma_r}{v_*} H(\kappa, p)}{p + \frac{1}{v_*} F(\kappa, p)} \quad \forall(\kappa, p) \in I\!R \times \mathbb{C}, \tag{3.6.10}$$

where $\hat{h}$ is the Fourier transform of h and, for each (κ, p) in $I\!R \times \mathbb{C}$, $H(\kappa, p)$ is a functional of the initial data η, ϖ and ϕ. From the foregoing discussion it is apparent that, since $v_* > 0$ at present, there exist no unstable zeros of the denominator of the expression on the right-hand-side of (3.6.10). Hence, when the base process is one wherein the associated phase boundary is static prior to the instant at which the perturbation is imposed and inertial effects are ignored, it is linearly stable with respect to all perturbations within the class introduced in Section 3.2.

3.7. Energy analysis for a base process involving a static interface with inertial effects present. Suppose, as in Section 3.6, that v_0 in (3.1.1) is zero; the parameter v_* is, as such, positive. Consider, now, the inertial initial boundary value problem formed by (3.5.2), (3.5.3) and (3.5.6)–(3.5.8). Observe that, since $v_0 = 0$, $(3.5.3)_2$ and (3.5.6) are replaced by $(3.6.1)_1$ and $(3.6.1)_2$, respectively. Furthermore, $(3.5.3)_1$ simplifies to its inertia-free counterpart (3.5.10). In place of a normal mode analysis like that performed in Section 3.6 an energy analysis will be used in this section to show that, when inertial effects are accounted for but $v_0 = 0$, the base process described by (3.1.1)–(3.1.3) is linearly stable with respect to all perturbations of the type put forth in Section 3.2. Preliminary to doing so define the *total energy* $\mathcal{E} : [0, t_*) \to \bar{I\!R}_+$ by

$$\mathcal{E}(t) = E_K(t) + E_W(t) + E_T(t) \quad \forall t \in [0, t_*), \tag{3.7.1}$$

where $E_K : [0, t_*) \to \bar{I\!R}_+$ is the *kinetic energy* given by

$$E_K(t) = \frac{\rho\ell}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dot{w}(x_1, x_2, t)\, dx_1 dx_2 \quad \forall t \in [0, t_*), \tag{3.7.2}$$

$E_W : [0, t_*) \to \bar{I\!R}_+$ is the *elastic energy* defined via

$$\begin{aligned} E_W(t) =& \frac{\rho\ell}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{0} \left(a_l^2 w_{,1}^2 (x_1, x_2, t) + b_l^2 w_{,2}^2 (x_1, x_2, t)\right) dx_1 dx_2 \\ &+ \frac{\rho\ell}{2} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \left(a_r^2 w_{,1}^2 (x_1, x_2, t) + b_r^2 w_{,2}^2 (x_1, x_2, t)\right) dx_1 dx_2 \\ & \qquad \forall t \in [0, t_*), \end{aligned} \tag{3.7.3}$$

and $E_T : [0, t_*) \to \bar{I\!R}_+$ is the *thermal energy* given by

$$\begin{aligned} E_T(t) =& \frac{\rho\ell}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{0} c_l T^2(x_1, x_2, t) dx_1 dx_2 + \frac{\rho\ell}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{0} c_r T^2(x_1, x_2, t) dx_1 dx_2 \\ &+ \rho\ell \int_0^t \int_{-\infty}^{+\infty} \int_0^{+\infty} k_l T_{,\alpha} (x_1, x_2, \tau) T_{,\alpha} (x_1, x_2, \tau) dx_1 dx_2 d\tau \\ &+ \rho\ell \int_0^t \int_{-\infty}^{+\infty} \int_0^{+\infty} k_r T_{,\alpha} (x_1, x_2, \tau) T_{,\alpha} (x_1, x_2, \tau) dx_1 dx_2 d\tau \\ & \qquad \forall t \in [0, t_*). \end{aligned} \tag{3.7.4}$$

The constant ℓ which appears in (3.7.2)–(3.7.4) is assumed to be positive and carries units of length. It is clear from $(3.5.7)_{1,2,3}$ and the stipulated square integrability of $\eta_{,\alpha}$, ϖ and ϕ that $\mathcal{E}(0)$ exists. In writing (3.7.2)–(3.7.4) it is assumed, however, that there exists a positive time t_*, which may possibly be very small, such that the relevant integrals exist on $[0, t_*)$. A reasonable definition of linear stability is, at present, that $\mathcal{E}$ remain bounded on $\bar{\mathbb{R}}_+$. A straightforward but long calculation which makes use of (3.5.8), (3.6.1) and, recalling the foregoing remarks regarding the coincidence of $(3.5.3)_1$ with (3.5.10) when $v_0 = 0$, show that the *power* $\dot{\mathcal{E}}$ is given by

$$\dot{\mathcal{E}}(t) = -\rho\ell v_* \int_{-\infty}^{+\infty} \dot{s}^2(x_2, t)\, dx_2 \quad \forall t \in [0, t_*). \tag{3.7.5}$$

Since $v_0 = 0$ at present, $\dot{\mathcal{E}}(t) \leq 0$ for all t in $[0, t_*)$; under these circumstances the interval over which $\mathcal{E}$ is defined can be extended incrementally to $\bar{\mathbb{R}}_+$ leading to the following inequality:

$$\dot{\mathcal{E}}(t) \leq 0 \quad \forall t \in \bar{\mathbb{R}}_+. \tag{3.7.6}$$

Evidently, then, by the definition of linear stability given above, the base process at hand is stable with respect to all perturbations under consideration if the associated phase boundary is static prior to the instant at which the perturbation is imposed and inertial effects are accounted for.

Note that a normal mode analysis akin to that performed in Section 3.6 produces the following dispersion relation

$$p + \frac{1}{v_*}(F_0(\kappa, p) + pG_0(\kappa, p)) = 0, \tag{3.7.7}$$

where $F_0 : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is defined by

$$F_0(\kappa, p) = \frac{a_l b_l a_r b_r (\gamma_l - \gamma_r)^2 \sqrt{\kappa^2 + p^2/b_l^2}\sqrt{\kappa^2 + p^2/b_r^2}}{a_l b_l \sqrt{\kappa^2 + p^2/b_l^2} + a_r b_r \sqrt{\kappa^2 + p^2/b_r^2}} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C}, \tag{3.7.8}$$

and G_0 is as defined in (3.6.5). A study of this dispersion relation allows the recovery of the results obtained by the foregoing energy analysis.

The combined results of this and the preceeding section are consistent with those presented by FRIED [9] in the purely mechanical analogue of the problem considered here. Hence, when $v_0 = 0$, the inclusion of thermal effects does not alter the linear stability of the base state (3.1.1)–(3.1.3) from its obvious mechanical analogue.

3.8. Normal mode analysis for a base process involving a moving interface with or without inertial effects. Suppose that v_0 in (3.1.1) is positive. Consider now both the inertia-free intial value problem consisting of (3.5.9), $(3.5.2)_{3,4}$, (3.5.10), (3.5.11), $(3.5.3)_{3,4}$, (3.5.11), $(3.5.7)_{3,4}$ and (3.5.8) and the inertial initial value problem comprised by (3.5.2), (3.5.3) and (3.5.6)–(3.5.8). Note that, in both of these cases, $(3.1.5)_3$, (3.1.11) and the assumed positivity of v_0 imply that $\ell_0 = 0$. Hence, in the inertia-free case, (3.5.11) and (3.5.12) simplify as shown below

$$
\begin{aligned}
[\![kT_{,1}]\!] + \rho v_0 [\![c]\!] T &= \rho v_0 (\gamma_r - \gamma_l) \langle\!\langle a^2 w_{,1} \rangle\!\rangle \quad \text{on} \quad I, \\
\dot{s} &= \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle a^2 w_{,1} \rangle\!\rangle + v_\vartheta T \quad \text{on} \quad I,
\end{aligned}
\tag{3.8.1}
$$

while, in the inertial case, $(3.5.3)_2$ and (3.5.6) become

$$
\begin{aligned}
[\![kT_{,1}]\!] + \rho v_0 [\![c]\!] T &= \rho v_0 (\gamma_r - \gamma_l) \langle\!\langle (a^2 - v_0^2) w_{,1} \rangle\!\rangle \quad \text{on} \quad I, \\
\dot{s} &= \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle (a^2 - v_0^2) w_{,1} \rangle\!\rangle + v_\vartheta T \quad \text{on} \quad I.
\end{aligned}
\tag{3.8.2}
$$

Next, a normal mode analysis analogous to those undertaken in Section 3.6 and [9] will be performed based upon the following representation of a solution to the relevant partial differential equations, jump conditions and kinetic relation:

$$
\begin{aligned}
w(x_1, x_2, t) &= \begin{cases} W_l e^{+(\xi_l - v_0 t) x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}_t^l, \quad t \in I\!R_+, \\ W_r e^{-(\xi_r - v_0 t) x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}_t^r, \quad t \in I\!R_+, \end{cases} \\
T(x_1, x_2, t) &= \begin{cases} \Theta_l e^{+(\zeta_l - v_0 t) x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}_t^l, \quad t \in I\!R_+, \\ \Theta_r e^{-(\zeta_r - v_0 t) x_1} e^{i\kappa x_2} e^{pt} & \forall (x_1, x_2) \in \mathring{\Pi}_t^r, \quad t \in I\!R_+, \end{cases} \\
s(x_2, t) &= S e^{i\kappa x_2} e^{pt} \quad \forall (x_2, t) \in I\!R \times I\!R_+.
\end{aligned}
\tag{3.8.3}
$$

Substitution of (3.8.3) into the equations appropriate to the equations appropriate to the inertia-free and inertial cases gives, respectively, the following dispersion relations

$$
\begin{aligned}
p + \frac{\nu^2|\kappa|}{v_*}(1 + v_\vartheta G(\kappa, p)) &= 0, \\
p + \frac{F(\kappa, p)}{v_*}(1 + v_\vartheta G(\kappa, p)) &= 0,
\end{aligned}
\tag{3.8.4}
$$

where $G : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ and $F : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ are given by

$$
\begin{aligned}
G(\kappa, p) &= \frac{2v_*}{c_l - c_r + c_l g_l(\kappa, p) + c_r g_r(\kappa, p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C}, \\
F(\kappa, p) &= \frac{(\gamma_l - \gamma_r)^2 (f_l(\kappa, p) f_r(\kappa, p) + v_0^2 p^2)}{f_l(\kappa, p) + f_r(\kappa, p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C},
\end{aligned}
\tag{3.8.5}
$$

where $g_l : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ and $g_r : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ are given by

$$
\begin{aligned}
g_l(\kappa, p) &= \sqrt{1 + \tfrac{4}{v_0^2}(\alpha_l^2 \kappa^2 + \alpha_l p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C}, \\
g_r(\kappa, p) &= \sqrt{1 + \tfrac{4}{v_0^2}(\alpha_r^2 \kappa^2 + \alpha_r p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C},
\end{aligned}
\tag{3.8.6}
$$

and $f_l : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ and $f_r : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ are defined via

$$
\begin{aligned}
f_l(\kappa, p) &= \sqrt{(a_l^2 - v_0^2) b_l^2 \kappa^2 + a_l^2 p^2} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C}, \\
f_r(\kappa, p) &= \sqrt{(a_r^2 - v_0^2) b_r^2 \kappa^2 + a_r^2 p^2} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C}.
\end{aligned}
\tag{3.8.7}
$$

When $v_\vartheta = 0$ the dispersion relations in (3.8.4) reduce to

$$
\begin{aligned}
p + \frac{\nu^2|\kappa|}{v_*} &= 0, \\
p + \frac{F(\kappa, p)}{v_*} &= 0,
\end{aligned}
\tag{3.8.8}
$$

Observe that $(3.8.8)_1$ and $(3.8.8)_2$ are structurally identical to the inertia-free and inertial dispersion relations obtained by FRIED [9] in the purely mechanical analogue of the investigation at hand. Hence, if the kinetic response function $\tilde{V}$

or $\tilde{\varphi}$ is chosen so that $v_\vartheta = 0$ then the linear stability of the base process at hand remains unaltered from that of its purely mechanical analogue by the presence of thermal effects; specifically, when $v_\vartheta = 0$, the linear stability of the base process (3.1.1)–(3.1.3) is determined entirely by the sign of v_*. That is, $v_* < 0$ is a necessary condition for the base process to be linearly unstable with respect to any perturbation of the type introduced in Section 3.2 and, further, $v_* < 0$ is a sufficient condition for the base process to be linearly unstable with respect to all but a small class of very special initial disturbances contained within the full set under consideration. Note, in particular, that $v_\vartheta = 0$ if either $\tilde{V}$ depends only on θ through the ratio f/θ or $\tilde{\varphi}$ is independent of θ. Assume, henceforth, that $v_\vartheta \neq 0$.

The branches of the square roots which define g_l and g_r are chosen so that, for p in $\mathbb{R}$,

$$\begin{aligned} 1 + \tfrac{4}{v_0^2}(\alpha_l^2\kappa^2 + \alpha_l p) > 0 \quad &\Longrightarrow \quad g_l(\kappa, p) > 0 \quad \forall \kappa \in \mathbb{R}, \\ 1 + \tfrac{4}{v_0^2}(\alpha_r^2\kappa^2 + \alpha_r p) > 0 \quad &\Longrightarrow \quad g_r(\kappa, p) > 0 \quad \forall \kappa \in \mathbb{R}, \end{aligned} \tag{3.8.9}$$

from which it is clear that, for p in $\mathbb{C}$,

$$\begin{aligned} \Re(1 + \tfrac{4}{v_0^2}(\alpha_l^2\kappa^2 + \alpha_l p)) > 0 \quad &\Longrightarrow \quad \Re(g_l(\kappa, p)) > 0 \quad \forall \kappa \in \mathbb{R}, \\ \Re(1 + \tfrac{4}{v_0^2}(\alpha_r^2\kappa^2 + \alpha_r p)) > 0 \quad &\Longrightarrow \quad \Re(g_r(\kappa, p)) > 0 \quad \forall \kappa \in \mathbb{R}. \end{aligned} \tag{3.8.10}$$

Evidently, then, (3.8.10) and $(3.8.5)_1$ yield the following result:

$$\Re(p) > 0 \quad \Longleftrightarrow \quad \Re(v_* G(\kappa, p)) > 0 \quad \forall \kappa \in \mathbb{R}. \tag{3.8.11}$$

The square roots which appear in f_l and f_r are defined via the principal branch of the complex logarithm. It is, therefore, clear that

$$\Re(p) > 0 \quad \Longleftrightarrow \quad \Re(F(\kappa, p)) > 0 \quad \forall \kappa \in \mathbb{R}. \tag{3.8.12}$$

An immediate consequence of (3.8.11) and (3.8.12) is that one or both of the parameters v_* or v_ϑ must be negative in order for a root p in $\mathbb{C}$ of either $(3.8.4)_1$ or $(3.8.4)_2$ to have a positive real part. It is also obvious, from (3.8.11) and (3.8.12), that if both $v_* < 0$ and $v_\vartheta < 0$ then there exists a root p in $\mathbb{C}$ with $\Re(p) > 0$ to both of (3.8.4) regardless of the value of the wave number κ in $\mathbb{R} \setminus \{0\}$. A more subtle condition sufficient for the existence of a root p in $\mathbb{C}$ to either of (3.8.4) occurs under the assumption that $v_* > 0$, $v_\vartheta < 0$ and $v_*|v_\vartheta|/c_l > 1$. Specifically, when $v_* > 0$, $v_\vartheta < 0$ and $v_*|v_\vartheta|/c_l > 1$ it is, then, possible to show that there exists a root p in $\mathbb{C}$ to both of (3.8.4) provided the wave-number κ in $\mathbb{R} \setminus \{0\}$ is sufficiently small so that the inequality

$$\frac{c_l - c_r}{c_l + c_r} + \frac{f_l(\kappa, p)}{1 + \frac{c_r}{c_l}} + \frac{f_r(\kappa, p)}{1 + \frac{c_l}{c_r}} < \frac{2v_*|v_\vartheta|}{c_l + c_r} \tag{3.8.13}$$

holds. A similar condition which guarantess the existence of a root p in $\mathbb{C}$ to either of (3.8.4) occurs under the assumption that $v_* < 0$, $v_\vartheta > 0$ and $|v_*|v_\vartheta/c_l < 1$. In this case there always exists such a root to either of (3.8.4) as long as the wave-number κ is sufficiently large so that the following inequality is satisfied:

$$\frac{c_l - c_r}{c_l + c_r} + \frac{f_l(\kappa, p)}{1 + \frac{c_r}{c_l}} + \frac{f_r(\kappa, p)}{1 + \frac{c_l}{c_r}} > \frac{2|v_*|v_\vartheta}{c_l + c_r} \tag{3.8.14}$$

The foregoing discussion shows that, unlike the purely mechanical process investigated in [9], the present context is not, when $v_0 > 0$, amenable to the statement of necessary and sufficient conditions for the linear instability of the base process at hand. The sufficient conditions which have been presented above are, however, of interest.

3.9 Conclusion. In [9] it is demonstrated that when the purely mechanical analogue of the parameter v_* is positive the appropriate purely mechanical version of the base process considered here is linearly stable with respect to all perturbations which are considered in that context. The last of the conditions sufficient for the linear instability of the thermoelastic base process (3.1.1)–(3.1.3), *viz.*,

$$v_* > 0, \quad v_\vartheta < 0, \quad \frac{v_* v_\vartheta}{c_l} > 1, \quad \frac{c_l - c_r}{c_l + c_r} + \frac{f_l(\kappa, p)}{1 + \frac{c_r}{c_l}} + \frac{f_r(\kappa, p)}{1 + \frac{c_l}{c_r}} < \frac{2v_*|v_\vartheta|}{c_l + c_r}, \tag{3.9.1}$$

where the parameters v_*, v_ϑ, c_l and c_r are as defined in $(3.4.10)_1$ or $(3.4.11)_1$, $(3.4.10)_2$ or $(3.4.11)_2$, $(3.5.5)_4$ and $(3.5.5)_3$, respectively, and the functions f_l and f_r are as defined in (3.8.7), is, hence, arguably the most interesting of the three which are presented. It exposes what might be described as a competition between mechanically *stabilizing* and thermally *destabilizing* effects and an explicit dependence of growth-rate upon wave-number. Significantly in these circumstances, it is the low wave-numbers (that is, *long waves*) with respect to which the base process is linearly unstable. Under conditions consistent with (3.9.1), a moving planar phase boundary, therefore, tends to prefer a highly wrinkled—*i.e.*, plate-like or dendritic—morphology. Instability of this variety is also found in models for dendritic crystal growth and solidification (see [13–14] and [16]).

In analogy to [9] where the physical plausibility of a purely mechanical kinetic response function for which the parameter analogous to v_* can be negative is addressed, it is now natural to consider the question of whether it is physically reasonable for a kinetic response function to depend monotonically on its first argument—so that v_* is always positive and the related purely mechanical process is linearly stable—but non-monotonically on its second argument, in which case v_ϑ may be negtive. The experimental work of Clapp & Yu [5] which studies, in part, the dependence upon temperature of transformation kinetics in a particular alloy capable of sustaining displacive solid-solid phase transformations indicates that the role of temperature in such kinetics is very complicated. In fact, despite what appears to be a very careful experimental procedure and analysis, Clapp & Yu [5] observe a severe scatter in the data which measure the dependence of phase boundary velocity upon temperature. This scatter indicates there may not be a simple functional dependence of interface normal velocity upon temperature. With regard to the issue at hand, these experimental results seem to indicate that, if a kinetic relation of the form (2.5.1) or (2.5.2) is insisted upon, monotonicity of a kinetic response function $\tilde{V}(\Phi, \cdot)$ for fixed Φ in $\mathbb{R}$ or $\tilde{\varphi}(V, \cdot)$ for fixed V in $\mathbb{R}$ may be the exception rather than the rule.

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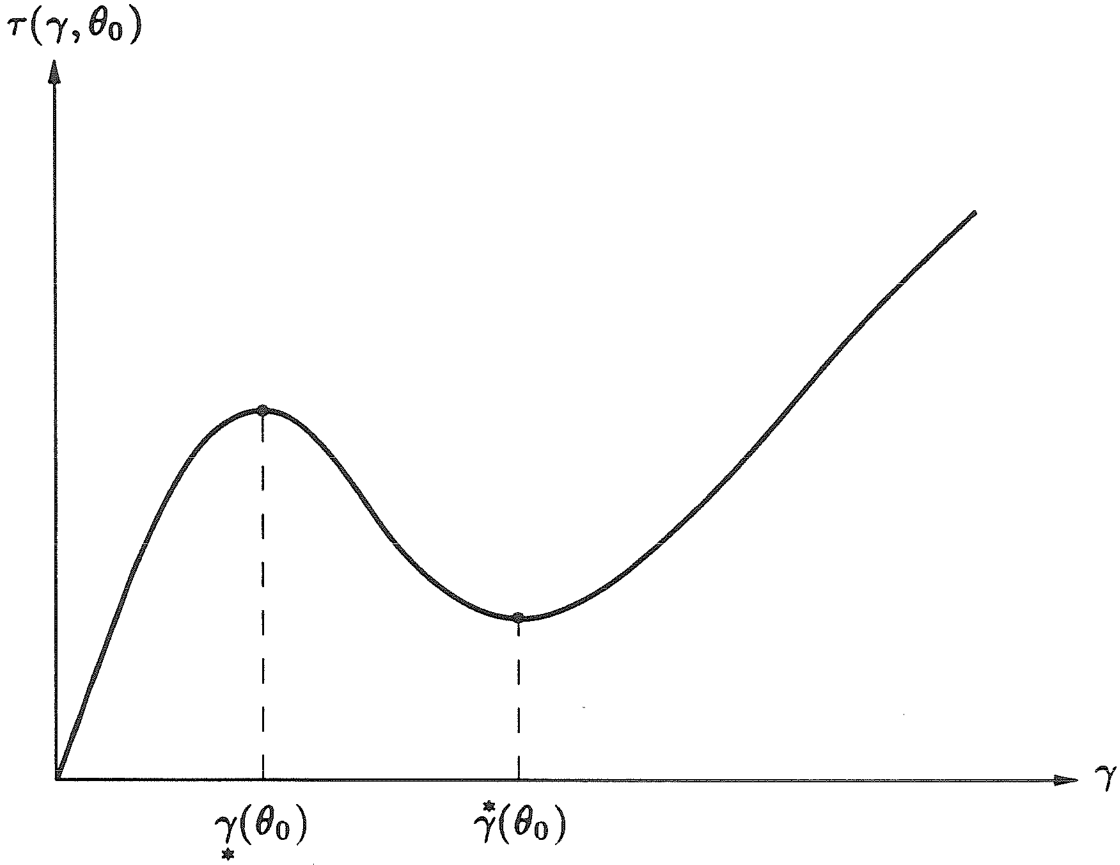

Figure 1: Graph of the shear stress response function $\tau(\cdot, \theta_0)$ for fixed θ_0 in (θ_m, θ_M).

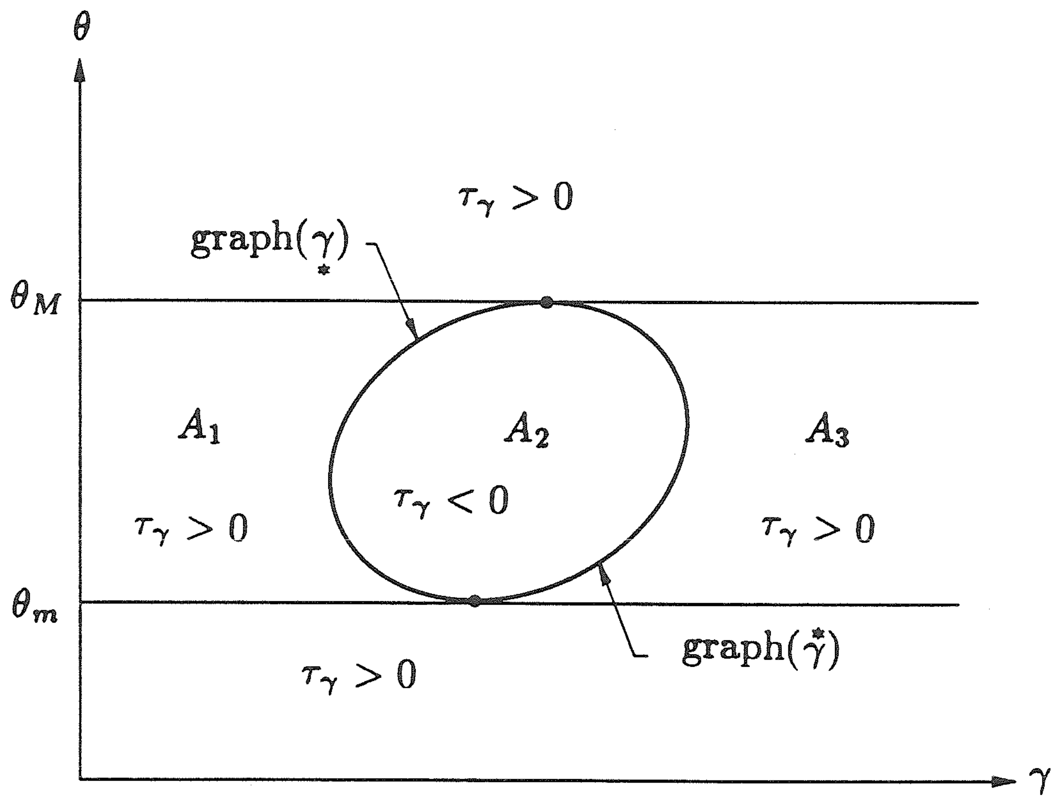

Figure 2: Plot of the shear strain-temperature quadrant.

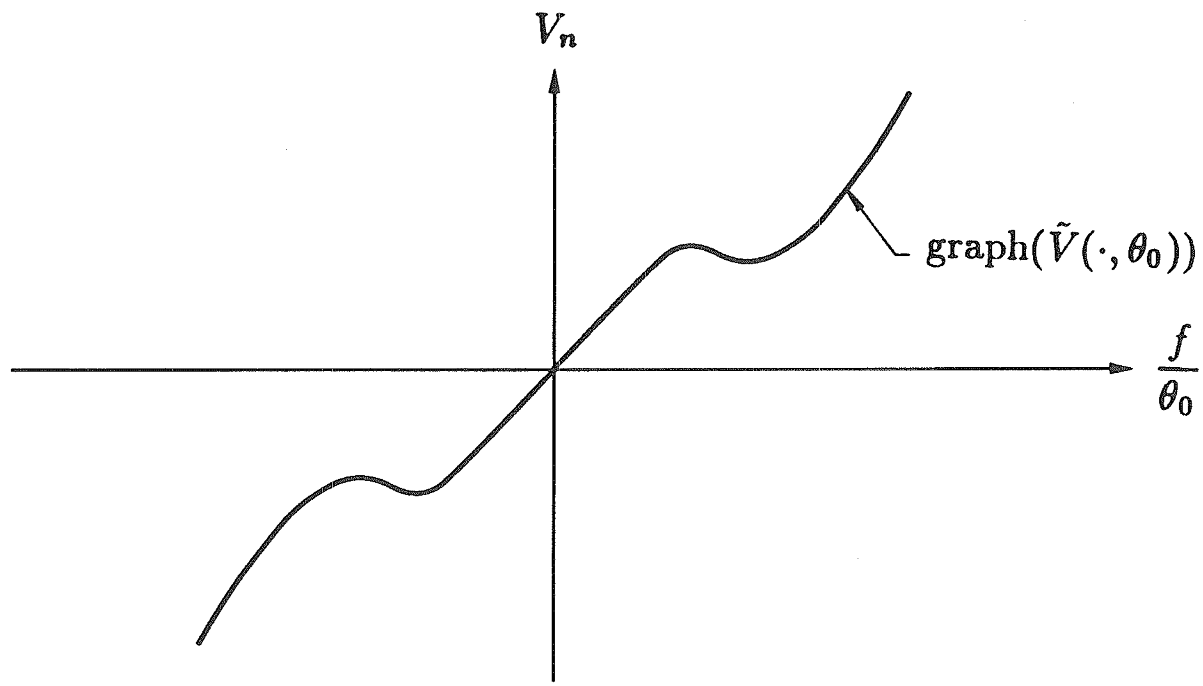

Figure 3: Graph of the kinetic response function $\tilde{V}(\cdot, \theta_0)$ for fixed θ_0 in (θ_m, θ_M).

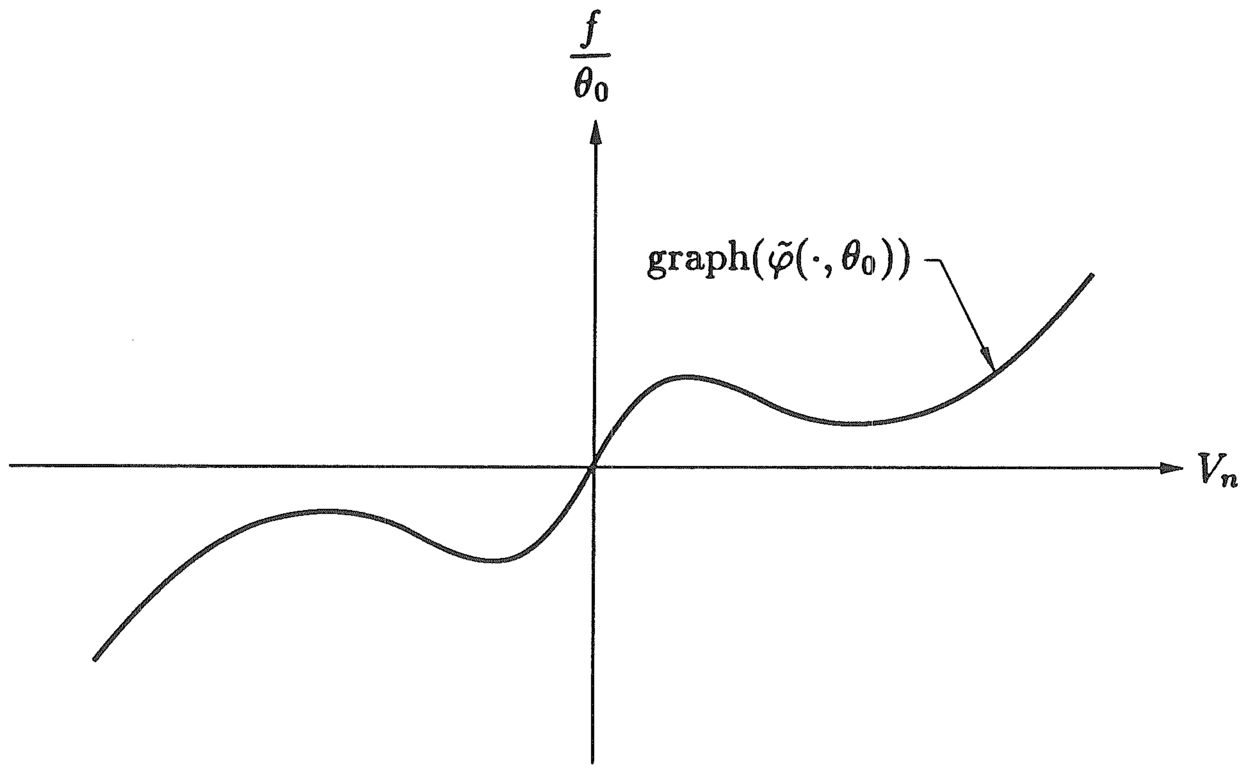

Figure 4: Graph of the kinetic response function $\tilde{\varphi}(\cdot,\theta_0)$ for fixed θ_0 in (θ_m,θ_M).