THE STABILITY OF RELATIVISTIC, SPHERICALLY SYMMETRIC STAR CLUSTERS

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ABSTRACT

It has been suggested that very dense star clusters might play important roles in quasi-stellar sources and in the nuclei of certain galaxies, where violent events occur. Such star clusters should become unstable against relativistic gravitational collapse when, in the course of evolution, they contract down to a certain critical density. In this thesis the study of the relativistic instability which triggers such collapse is initiated: The theory of the stability of a spherically symmetric star cluster against small radial perturbations is developed within the framework of general relativity. Collisions between stars in the cluster are neglected, since in realistic situations the time scale for collisions should be much greater than the time scale for the growth of the relativistic instability. The equation of motion governing the small radial perturbations of a spherical cluster is derived and is shown to be self-conjugate. Associated with the equation of motion is a dynamically conserved quantity, and a multidimensional variational principle for the normal modes of radial pulsation. The variational principle provides a necessary and sufficient criterion for the stability of the cluster. Also derived are much simpler, one-dimensional, sufficient (but not necessary) criteria for stability. The most important sufficient criterion is this: A relativistic, spherical cluster is stable against radial perturbations if the gas

sphere with the same distributions of density and pressure is stable against radial perturbations with adiabatic index $\Gamma_1 = (\rho + p)p^{-1}(dp/dr)(d\rho/dr)^{-1}$.

The stability criteria are used to diagnose numerically the stability of (i) clusters of identical stars with heavily-truncated Maxwell-Boltzmann velocity distributions, and (ii) clusters whose densities and isotropic pressures obey polytropic laws of index 2 or 3. The calculations show that a cluster of either type is unstable against collapse if the redshift of a photon emitted from its center and received at infinity is $z_c \gtrsim 0.5$. The cluster is stable if $z_c \lesssim 0.5$.

For purposes of motivation, two new theorems on the theory of the stability of highly relativistic stars (not star clusters!) are also presented in this thesis. The first theorem states that a highly relativistic, spherical star is stable if and only if its adiabatic index (assumed to be constant in the interior regions) is greater than a certain critical value, Γ_{crit}, which depends in a specified way on the high-density equation of state. Because of relativistic effects this critical value is somewhat larger than the Newtonian value $\Gamma_{crit} = 4/3$. The second theorem shows that, at high central densities, the curves of - (binding energy) versus radius for certain hot, isentropic sequences of stellar models must exhibit damped clockwise spirals. This spiraling reflects the onset of instability in one radial mode of pulsation after another as the central density increases along the sequence.

TABLE OF CONTENTS

1. A Brief Outline of the Thesis

In this thesis we devote ourselves to the study of the stability of relativistic, spherically symmetric star clusters against radial perturbations. A brief outline of the thesis follows: In Part 2 we attempt to provide motivation for our study. In Part 3 we present an extensive summary of the remaining parts of this treatise.

Parts 4, 5, 6, and 7 are unmodified versions of papers which have been prepared for publication.

Part 4 is included as further motivation for our study of the relativistic instabilities in star clusters. We present there two new theorems on the theory of the stability of highly relativistic stars (i.e. gas spheres). These theorems quantify the strengths of the relativistic instabilities in highly relativistic stars.

In Part 5 we begin our study of relativistic star clusters. Actually, we first discuss in §III of Part 5 the theory of stability for Newtonian, spherical star clusters; and then in §IV we derive the self-conjugate eigenequation governing the spherical perturbations of a relativistic, spherical cluster, and we discuss the properties associated with the eigenequation. One of these, namely, that there exists a multidimensional variational principle for the normal modes, provides a necessary and sufficient condition for stability.

In Part 6 we present one-dimensional, sufficient (but not necessary) criteria for the stability of relativistic, spherical clusters.

In Part 7 we use our stability criteria to diagnose numerically the stability against gravitational collapse of a variety of spherical clusters with isotropic velocity distributions.

2. Motivation

Recent developments in astronomy and astrophysics - - e. g. the discovery and study of the quasi-stellar sources (Schmidt 1963; Greenstein and Schmidt 1964), of explosions in galactic nuclei (Burbidge, Burbidge, and Sandage 1963), of strong, extrasolar X-ray sources (Giacconi, Gursky, Paolini, and Rossi 1962), and of pulsating radio sources (Hewish, Bell, Pilkington, Scott,and Collins 1968) - - have led to a reawakening of interest in the possible roles in nature of relativistic systems with strong gravitational fields. Thus far, this interest has concentrated largely on the roles which relativistic stars might possibly play in various astrophysical situations. Consequently, much effort has been directed to the task of understanding the specifically relativistic features of the structures and stabilities of compact white dwarfs, neutron stars, and supermassive stars (see Part 4 for references to reviews devoted to this subject); these features should influence strongly the possible roles of such stars in nature.

At present, it appears that theoretical analyses of hitherto unobserved relativistic stars finally have begun to pay off. Today rotating neutron stars seem to be the only reasonable explanation of the pulsating radio sources (pulsars).

One can ask if there are, besides relativistic stars, any other systems which might play roles in nature that are strongly influenced by relativistic gravitational effects. Hoyle and Fowler (1967) offered an answer to this question; they suggested that each quasi-stellar source might lie at the center of a massive relativistic star cluster, and might derive its redshift from the gravitational field of the cluster.

The Hoyle-Fowler cluster model for quasars makes use of relativistic star clusters which have central redshifts (redshift of a photon emitted from the center of a cluster and received at infinity) as large as ~ 2.4. Besides the gravitational redshift, another relativistic gravitational effect which could be of great importance for a relativistic star cluster is the onset of gravitational collapse.

Indeed, there are suggestions from studies of relativistic stars that relativistic star clusters with central redshifts as large as 2.4 might be unstable against gravitational collapse. (Studies of the relativistic instability in stars are discussed in the reviews referenced in Part 4; also, see the two new theorems presented in Part 4, which reflect the strengths of the instabilities in the ultra-relativistic regime.) It is known that the relativistic forces in stars can induce instabilities which do not arise in Newtonian theory. Studies reveal that a relativistic star becomes unstable against gravitational collapse when it contracts sufficiently far that its central redshift surpasses a value which is typically (though not always) about 1. (In this connection, see the first theorem of Part 4, which deals with limits on the central densities of stable, highly relativistic stars.) It is not unlikely that the maximum central redshifts of stable star clusters are also about 1, but we cannot know until the theory of stability for relativistic star clusters has been worked out and applied.

Under what circumstances might one expect relativistic gravitational effects, such as the onset of collapse, to be important for star clusters ? A rough measure of the importance of relativity for a star cluster is the parameter[1]

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[1]Throughout this thesis we adopt "gravitational" units, in which the speed of light, c, Newton's gravitation constant, G, and Boltzmann's constant, k, are equal to unity. In these units all quantities can be measured in terms of, say, a fundamental length. Thus, for example, in gravitational units the mass of the Sun is 1.476 km.

$$\alpha = \frac{2M}{R} \approx 0.01 \left(\frac{M/10^{11} m_{\odot}}{R/1\,pc} \right) .$$

Here M is the mass of the cluster, and R is some mean radius of the star distribution (e.g. the radius in which half the mass is contained).

If $\alpha \gtrsim 0.01$ relativistic effects may be important. Notice that this value of α corresponds to 10^9 stars in 0.01 pc, 10^{11} stars in 1.0 pc, etc. That such star densities are enormous is evident from the estimate that our galaxy has about 10^{11} stars in 10^4 pc($\alpha \approx 10^{-6}$).

Consequently, it appears that, of all known astronomical systems, relativistic star clusters could possibly be associated with only the quasi-stellar sources [as proposed by Hoyle and Fowler (1967)] and the nuclei of certain galaxies (Seyfert galaxies, N galaxies, and compact galaxies).

A seemingly strong argument to be heard against the existence of relativistic star clusters anywhere in nature is that, according to Newtonian estimates, too long a time is required to evolve a cluster up to star densities for which $\alpha \gtrsim 0.01$. For example, the relaxation time of our galaxy is estimated to be about 10^{14} years (Chandrasekhar 1942). However, most Newtonian analyses of evolution ignore collective interactions; they approximate N-body interactions as a sum of 2-body interactions. Certain recent Newtonian studies indicate that collective interactions might evolve some clusters up to relativistic densities in times much shorter than the evolution times associated with 2-body interactions. The studies in question involve (i) numerical experiments on the classical N-body problem (Aarseth 1963; Von Hoerner 1963; see also Henon 1961, 1965), and (ii) analytic analyses of maximum-entropy clusters confined to the interior of a spherical box (Antonov 1962, Lynden-Bell and Wood 1968). These studies suggest that, due to collective interactions, rapid evolution can occur in some clusters. In this evolution a cluster develops a dense core and a diffuse, extended envelope on a time scale which might be less than 10^{10} years for the nuclei of some galaxies.

If relativistic star clusters do exist in nature, their roles should be strongly influenced by relativistic effects such as the onset of gravitational collapse. Indeed, it is not inconceivable that the violent events in the nuclei of certain galaxies and the outbursts in the quasi-stellar sources are associated with the onset of collapse in a relativistic cluster, or with interactions between an already-collapsed core and the surrounding stars.

Before these speculations can be pursued further, the relationships between the structures and stabilities of relativistic star clusters must be understood. In this thesis we undertake the study of these relationships.

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3. Detailed Summary

In general relativity the gravitational field is described by the metric tensor, $g_{\mu\nu}$, of spacetime through the expression

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

for the fundamental line element.[1] The Newtonian Poisson equation is replaced by the Einstein field equations,

$$G_{\mu\nu} = 8\pi T_{\mu\nu} . \tag{2}$$

The Einstein field tensor, $G_{\mu\nu}$, involves certain combinations of the $g_{\mu\nu}$ and their derivatives. The stress-energy tensor, $T_{\mu\nu}$, describes the distribution of matter in spacetime. The general structure of the gravitational field equations is discussed, e.g., by Synge (1960), chapter 1.

In this thesis we are interested in studying the gravitational fields in spherically symmetric, relativistic star clusters. We seek to learn whether the gravitational forces in a spherical cluster can become so strong that the "pressure" forces are overwhelmed and the cluster becomes unstable against gravitational collapse.

Our initial efforts in this thesis attempt to provide motivation for this interest: In Part 4 we present two new theorems on the relativistic instabilities in highly relativistic stars. The first theorem results from attempts to analyze the strength of the

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[1]Greek indices run from 0 to 3; Latin indices, from 1 to 3. We use the superscript 0 to denote the time coordinate, $t = x^0$, of a general curvilinear coordinate system. In our convention the line element has signature (+ - - -).

relativistic instabilities in highly relativistic stars in terms of adiabatic indices. This theorem places a lower limit on the adiabatic index (defined by eqn. 1 of Part 4, and assumed to be constant in the interior regions) of a stable, highly relativistic, spherical star. In Newtonian theory this lower limit is 4/3 independently of the structure of the star. But in general relativity this lower limit is greater than 4/3 and its value depends upon the structure of the star. (See eqn. 5 of Part 4.) We arrive at the theorem by using known approximate solutions of the stellar eigenequation governing adiabatic, radial oscillations of a spherical star. The second theorem states that plots of minus the gravitational binding energy versus the radius for certain one-parameter sequences of stellar models with radially-invariant entropy per baryon must undergo high-density spirals. (See eqns. 20 and Figure 2 of Part 4.) This spiraling is due to the fact that more and more modes of radial oscillation become unstable along such a sequence as the models become increasingly relativistic. Thus both theorems of Part 4 reflect the strength of the relativistic instability in stars.

Sufficiently motivated by this work on stellar pulsations, we begin our study of relativistic star clusters. From the outset we should attempt to restrict ourselves to models which would not be expected to evolve too rapidly. For some star cluster models evolution would be so rapid as to be violent. The structure of such a model would be altered very rapidly by evolutionary processes, and so it would be useless to attempt to study its stability against gravitational collapse. In such a study we should restrict attention to clusters whose evolutions are sufficiently moderate that they can be considered to be quasistatic. The evolutionary path of such a cluster will pass through a sequence of near-equilibrium states.

The ways in which collective interactions affect the evolution of a star cluster are poorly understood. If, because of ignorance, we ignore collective interactions, we can attribute the evolution to four processes: (i) distant encounters between pairs of stars via the long-range gravitational force; (ii) direct collisions, in which two stars have grazing, or closer, contact; (iii) evaporation, in which stars gain enough energy through encounters to escape from the cluster; and (iv) gravitational radiation-reaction. Rough estimates of the characteristic times, t_{relax}, t_{coll}, t_{evap}, t_{rad} associated with these processes are (Chandrasekhar 1942; Spitzer and Härm 1958; Zel'dovich and Podurets 1965; Spitzer and Saslaw 1966)

$$10^9 (m/m_\odot)^{-2} (r/r_\odot)^2 \alpha^2 t_{coll} \approx t_{relax}$$

$$\approx 0.1\, \alpha^{-1/2} RN/\log_{10}(N) \approx 10^{-2} t_{evap} \approx [10^{-4} \alpha^{5/2}/\log_{10}(N)]\, t_{rad}. \quad (3)$$

Here m and r are the mass and radius of a typical star of the cluster; $\alpha = 2M/R$, where M and R are the mass and radius of the cluster; and N is the total number of stars in the cluster. These estimates show that, even if a typical star of a cluster is as small as a compact white dwarf ($m \approx m_\odot$, $r \approx 10^{-2} r_\odot$), direct collisions dominate the evolution if the cluster is relativistic (i.e. if $\alpha \gtrsim 0.01$).

If a relativistic cluster is to evolve quasistatically, t_{coll} must be greater than, say, the period ($\approx 2\pi M^{-1/2} R^{3/2}$) for circular orbits at the boundary of the system. For a given value of α, this condition places a lower limit on the radius, R, and hence the mass, M, of the system. For example, if α is 0.1 and if a typical star has $m \approx m_\odot$, $r \approx r_\odot$, then R must be greater than about 0.01 pc. If

$R = 1$ pc (and $\alpha = 0.1$), then $M \approx 10^{12} m_{\odot}$, and $t_{coll} \approx 8000$ years.

One expects that the characteristic time associated with the growth of the relativistic instability in a cluster will be of the order of the star travel time across the cluster, just as the analogous time scale for a star is of the order of the sound travel time across the star. For a cluster which is evolving quasistatically, the star transit time can be estimated crudely through use of the Newtonian virial theorem. The virial theorem states that in an equilibrium cluster the total kinetic energy is half the magnitude of the potential energy. This implies that the velocity of a typical star is of the order of $\alpha^{1/2}$. Hence the star transit time across a cluster is of the order of $\alpha^{-1/2}R$, which is short compared with t_{coll} for a relativistic cluster which is evolving quasistatically. (For example, the ratio of t_{coll} to the star transit time is about $10^{-11}\alpha^{-2}N$ for a cluster whose typical star resembles the Sun. This ratio is approximately 10^3 for the quasistatic cluster considered above, with $\alpha = 0.1$ and $R = 1$ pc.)

Consequently, the equilibrium states through which a quasistatic cluster evolves can be idealized as statistical distributions of point masses interacting through only the smoothed-out, self-consistent gravitational field of the entire cluster. The basic equations which are used to describe such a model are the Einstein field equations (2) and the relativistic Boltzmann-Liouville (or collisionless Boltzmann) equation. This latter equation states that the density of stars in phase space is conserved along the trajectory of a star. (See the Appendix to this Summary.)

The type of statistical treatment which we employ here to describe a relativistic star cluster has been used in the theory of the structure of Newtonian star clusters for about fifty years.

However, the stability of Newtonian clusters has been investigated only recently (Antonov 1960; Lynden-Bell 1966, 1967; Milder 1967; Lynden-Bell and Sanitt 1969; see also §III of Part 5 where we review and slightly extend the work of Antonov). In general relativity the theory of the structures of spherical star clusters has been developed recently by Zel'dovich and Podurets (1965) and by Fackerell (1966, 1968). This thesis and the papers which make up Parts 4-7 of it constitute the first attack ever made on the stability of relativistic clusters.

In our study of relativistic star clusters we work with the invariant distribution function (i.e. density of stars in phase space), $\mathcal{N}$. The distribution function is defined and discussed in some detail in the Appendix to this Summary. Briefly, in terms of an arbitrary curvilinear coordinate system, $\mathcal{N}$ is the number, dN, of stars per unit phase-space volume, $d\mathcal{V}_p\, d\mathcal{V}_x$

$$\mathcal{N} = dN/d\mathcal{V}_p d\mathcal{V}_x , \tag{4}$$

where

$$d\mathcal{V}_p = -dp_0\, dp_1\, dp_2\, dp_3 / \sqrt{(-g)} , \tag{5a}$$

$$d\mathcal{V}_x = (p_0/m\, g_{00}) \sqrt{(-g)}\, dx^1 dx^2 dx^3 \tag{5b}$$

are volume elements in momentum space and in physical space. Here p_α are the covariant components of the momentum of a star, g is the determinant of the metric tensor, and $m = (p_\alpha p^\alpha)^{1/2}$ is the rest mass of a star. The volume elements (5a) and (5b) are invariants; consequently, the distribution function is an invariant.

The distribution function is in general a function of the coordinates of an eight-dimensional phase space sometimes referred to as the tangent bundle. We will use general spacetime coordinates, x^α, as four of the independent coordinates in phase space. We must also select four momentum coordinates. We shall not restrict ourselves to one particular set of momentum coordinates. At times we shall find it convenient to employ the covariant components, p_α, of the momentum as coordinates; on other occasions we shall choose coordinates specially adapted to spherical symmetry; etc.

As is discussed in the Appendix, the stress-energy tensor, $T_{\mu\nu}$, is determined by the distribution function through the equations

$$T_{\mu\nu} = \int (\mathcal{N}/m)\, p_\mu p_\nu \, d\mathcal{V}_p \,. \tag{6}$$

Consequently, the distribution function generates the metric of spacetime through the field equations (2).

In the absence of collisions, $\mathcal{N}$ satisfies the Boltzmann-Liouville, or collisionless Boltzmann, equation

$$\mathcal{L}\mathcal{N} = 0 \,. \tag{7}$$

Here the Liouville operator, $\mathcal{L}$, is differentiation with respect to proper time along the path of a star in the tangent bundle. If (x^α, p_α) are used as independent coordinates, the Liouville operator takes the form (A16). A proof of equation (7) is given in the Appendix.

Equations (2), (6), and (7) are the basic equations for our study of relativistic star clusters.

In §IV of Part 5 we begin the development of the theory of small, radial perturbations of a spherically symmetric, relativistic star cluster. As is discussed, e.g., by Synge (1960), chapter 7, under the assumption of spherical symmetry coordinates can be chosen such that the line element (2) assumes the form

$$ds^2 = e^{\nu(t,r)} dt^2 - e^{\lambda(t,r)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) . \qquad (8)$$

The coordinate system (t, r, θ, φ) is called the Schwarzschild coordinate system. The radial coordinate, r, is defined uniquely by the demand that $2\pi r$ be the circumference of a circle about the center of symmetry (point where $r = 0$). The time coordinate, t, is defined by the demands that at infinity it reduces to proper time for an observer at rest with respect to the coordinate system and that everywhere the one-form dt be orthogonal to dr, $d\theta$, and $d\varphi$.

An equilibrium spherical cluster is one for which the metric functions ν and λ, and the distribution function, $\mathcal{N}$, are explicitly independent of the Schwarzschild time coordinate. In this case the dependence of the distribution function upon $(\underset{\sim}{x}, \underset{\sim}{p})$ is severely limited: Notice that equation (7) states mathematically that the distribution function is conserved along a star's path through phase space. This implies the relativistic version of Jeans's theorem, namely, that $\mathcal{N}$ is a function of the integrals of the motion of the stars. Ignore for the moment the fact that the distribution function actually generates the metric tensor, $g_{\mu\nu}$, through equations (3) and (6). Pretend that the $g_{\mu\nu}$ are given functions of the x^{α}. Then equation (7) can be viewed as a linear, homogeneous, partial differential equation in eight variables for η (cf. expression [A16a] for the Liouville operator). It follows from the theory of partial differential equations (see, e.g., Garabedian 1964, chap. 2) that there are in general seven integrals of the motion. For a spherical equilibrium configuration, in which the metric tensor and the distribution function are independent of the Schwarzschild time coordinate, the time coordinate can be eliminated and there are six explicitly time-independent integrals of the motion. Three of these integrals are the rest mass, m, the "energy at infinity," E, and the total angular momentum, J. In terms of Schwarzschild coordinates,

$$m = (p_\alpha p^\alpha)^{1/2}, \; E = p_0, \; J = [p_\theta{}^2 + (p_\varphi/\sin\theta)^2]^{1/2} . \qquad (9)$$

Two additional integrals specify the conserved plane of a star's orbit, and there is a final integral which specifies the "orientation" of a star's orbit in the orbital plane. However, the star's orbit in general will not be closed, but will fill up ergodically the physical region of the orbital plane in which stars with given values of m, E, and J are constrained to move. And in the equilibrium situation under consideration, equation (7) implies that the distribution function, for fixed m, E, J, has the same constant value at all physical locations (r, θ, φ) along a star's orbit. Consequently, for each set of values for m, E, J, and for each orbital plane, the distribution function in general must have the same constant value throughout the physical region in which stars with the given values of m, E, J are constrained to move. Thus the sixth integral referred to above is redundant (for obvious reasons, it is called a non-isolating integral), and we can say that the distribution function is an arbitrary function of the five remaining integrals. Next, we recall that the distribution function of a cluster actually generates the gravitational field through the Einstein field equations (2). If the distribution function depended on the orientation of the orbital plane, it would lead to a non-spherical stress-energy tensor and thence to a non-spherical gravitational field. Therefore, the distribution function for a spherical equilibrium cluster can depend on only m, E, and J; and we have

$$\eta = F(m, E, J) \text{ at equilibrium.} \tag{10}$$

If the distribution function of an equilibrium spherical cluster depends on the angular momentum, J, then the velocity distribution in the cluster is anisotropic. This means that the stresses (i. e. the fluxes of momentum) in the radial and tangential (i. e. non-radial) directions are not equal. However at each physical location the radial direction is an axis of symmetry for the velocity distribution.

The field equations (2) for an equilibrium spherical cluster are (Synge 1960, chap. 7)

$$\begin{aligned} d\nu_A/dr &= (e^{\lambda_A} - 1)/r - 8\pi r e^{\lambda_A} T_{Ar}{}^{r} , \\ dM_A/dr &= 4\pi r^2 T_{A0}{}^{0} , \\ e^{-\lambda_A} &= 1 - 2M_A/r . \end{aligned} \tag{11}$$

In these equations, and throughout this thesis, we use the subscript A to refer to quantities in an equilibrium cluster. The quantity $M_A(r)$ is the total mass-energy contained inside radius r. The structure of an equilibrium cluster is computed by using the equilibrium distribution function (10) to express the stress-energy tensor (6) in terms of the metric coefficients ν_A and λ_A. Equations (11) can then be integrated to obtain ν_A and λ_A as functions of r (see, e.g., Fackerell 1966, 1968).

Once someone has given us a spherical equilibrium cluster, we perturb the cluster slightly without destroying its spherical symmetry. We analyze the dynamics of a radially perturbed cluster by linearizing in the perturbation the basic equations (2), (6), and (7). In a perturbed spherical cluster the metric coefficients ν and λ of the line element (8) take the form

$$\nu(t, r) = \nu_A(r) + \nu_B(t, r), \ \lambda(t, r) = \lambda_A(r) + \lambda_B(t, r) . \qquad (12)$$

Here the subscript B refers to the small perturbation of a quantity away from its value in the equilibrium configuration.

To define the perturbation of the distribution function we must decide how to identify points in the perturbed cluster with points in the unperturbed cluster. The choice which we make is motivated by both physical considerations and the desire for simplicity of formalism. We identify points with the same Schwarzschild coordinates (t, r, θ, φ), and the same physical components of the momentum,

$$p_{(\alpha)} \equiv |g^{\alpha\alpha}|^{1/2} p_\alpha \qquad (13)$$

(see Table 1 of Part 5, §IVc). The physical component $p_{(0)}$ is the energy of a star, as measured locally by an observer at rest with respect to the Schwarzschild coordinate system. The components $p_{(i)}$ are the negatives of the locally measured spatial components of the momentum along the r, θ, and φ directions. Notice that equations (9), when reexpressed in terms of the $p_{(\alpha)}$, become

$$m = [p_{(0)}{}^2 - p_{(r)}{}^2 - p_{(\theta)}{}^2 - p_{(\varphi)}{}^2]^{1/2} ,$$
$$E = e^{\nu/2}\, p_{(0)}, \quad J = r[p_{(\theta)}{}^2 + p_{(\varphi)}{}^2]^{1/2} . \tag{14}$$

For our choice of coordinates in phase space, the distribution function of the perturbed cluster takes the form [cf. eqns. (10) and (13)]

$$\eta = \eta_A(x^\alpha, p_{(\alpha)}) + \eta_B(x^\alpha, p_{(\alpha)})$$
$$= F(m, E_A, J) + f(x^\alpha, p_{(\alpha)}) , \tag{15}$$

where $$E_A = e^{\nu_A/2}\, p_{(0)} .$$

When equation (15) is combined with equation (6) for the stress-energy tensor, the perturbation in the stress-energy tensor becomes

$$T_{B\alpha}{}^{\beta} = \int (f/m)\, p_\alpha p^\beta\, d\mathcal{V}_p . \tag{16}$$

Since the perturbation f is a first-order quantity, implicit is the rule that in going from contravariant to covariant momentum components in equation (16), or from covariant to physical components, etc., the unperturbed metric tensor, $g_{A\mu\nu}$, is to be used. Also, the factor $\sqrt{(-g)}$ which appears in the definition (5a) of $d\mathcal{V}_p$ is to be evaluated in equations (16) by use of the unperturbed metric. Equations (16) follow readily from the fact that the volume element, $d\mathcal{V}_p$, of equation (5a), when reexpressed in terms of the physical components of the momentum through equations (13), becomes

$$d\mathcal{V}_p = -dp_{(0)}\,dp_{(r)}\,dp_{(\theta)}\,dp_{(\varphi)}\,. \tag{17}$$

Hence the stress-energy tensor is evaluated by integration over the physical components of the momentum. Consequently, when a perturbation is introduced, that perturbation of the distribution function which enters into the perturbation of the stress-energy tensor is f, the perturbation at fixed physical components of the momentum. Further, it is evident that

$$p_\alpha p^\beta = \pm(g^{\beta\beta}/g^{\alpha\alpha})^{1/2}\, p_{(\alpha)}\, p_{(\beta)} \tag{18}$$

for our diagonal metric. If $\alpha = \beta$ the perturbation in $p_\alpha p^\beta$ at fixed $(x^\alpha, p_{(\alpha)})$ obviously vanishes, and in this case equation (16) follows. If $\alpha \neq \beta$, the perturbation in $p_\alpha p^\beta$ does not vanish. In this case, in addition to the integral in equation (16), one has for $T_{B\alpha}{}^{\beta}$ an integral similar in form to that in equation (16), except that f is replaced by F. However the integrand, $F\,p_\alpha p^\beta$, of this integral is an odd function of at least one of the spatial physical components of the momentum; and so the integral vanishes. Equation (16) then follows.

When the Boltzmann-Liouville equation (7) is linearized in the perturbed quantities ν_B, λ_B, and f it becomes equation (10; R) of Part 5. The derivation of that equation is an example of a calculation in which we exercise our freedom to use coordinates which make the calculation as simple as possible. The distribution function of a spherical cluster can depend on $p_{(\theta)}$ and $p_{(\varphi)}$ only through the angular momentum, J. As is the case in the derivation of equation (10; R) of Part 5, it is often convenient to use J as a coordinate in phase space in place of both $p_{(\theta)}$ and $p_{(\varphi)}$. A calculation can often be simplified by making use of the fact that J is conserved along a stellar orbit in phase space. The same is true of the rest mass, m.

The perturbed Einstein field equations (2) are needed to complete the basic set of perturbation equations. The perturbed field equations are written down as equations (14; R) of Part 5. There is a dynamical field equation in addition to these, which involves the second time derivative of the perturbation λ_B (see, e. g. ,Chandrasekhar 1964). However, if equation (10; R) of Part 5 is multiplied by $p_r d\mathcal{V}_p$, and if the product is integrated over momentum space, and if the resulting p_r-moment is combined with equation (14a; R) of Part 5, then the dynamical field equation referred to above will be reproduced. Actually, not even all of equations (10; R) and (14; R) are independent. The p_0-moment of equation (10; R), when combined with equation (14a; R), yields (14b; R).

The equation of motion (10; R) of Part 5 is of first order in the time and radial derivatives. But one expects that a perturbed cluster will pulsate if it is stable, and collapse or explode if it is unstable. And throughout other areas of physics such motions are described by hyperbolic second-order differential equations. In analogy with a procedure in Newtonian theory developed by Antonov (1960), in §IVd of Part 5 we obtain such a second-order differential equation by first splitting the perturbation of the distribution function into its even and odd parts, f_+ and f_-, as functions of the spatial momenta, $p_{(i)}$ (cf. definition 16; R). We then split equations (10; R) and (14; R) into their even and odd parts, combine suitably, and eventually arrive at the desired equation of motion (eqn. 19R),

$$(1/F_E)\, \partial^2 f_- / \partial t^2 = \mathcal{J} f_- , \qquad (19)$$

where

$$F_E \equiv (\partial F/\partial E)_{m, J} \qquad (20)$$

[cf. eqns. (9) and (10) of this Summary], and where the operator $\mathcal{J}$ is defined by equations (15c; R) and (20; R) of Part 5.

Now the even part, f_+, of the perturbation determines the perturbation in the star density, in the mass-energy density, and in the stresses. The odd part, f_-, determines the mass motions and the average "displacements" in a perturbed cluster. Hence equation (19) is the analogue of the pulsation equation for the displacement in a perturbed star (see, e.g. Chandrasekhar 1964).

The operator $\mathcal{J}$ is a complicated integro-differential operator in phase space. Consequently it would appear that the analysis of the equation of motion (19) involves a hopelessly difficult problem. However, in §IVe of Part 5 we show that $\mathcal{J}$ is a self-conjugate operator in the sense that

$$\int h \mathcal{J} k \, d\mathcal{V}_p \, d\mathcal{V}_x = \int k \mathcal{J} h \, d\mathcal{V}_p \, d\mathcal{V}_x , \tag{21}$$

if h and k are functions which are bounded in phase space. From the self-conjugate nature of $\mathcal{J}$ follow many useful results, which are listed in §IVe of Part 5. One result is that there exists a dynamically conserved quantity (i.e. a quantity whose value for a perturbed cluster is independent of time)

$$H = \int \left[\frac{(\partial f_- / \partial t)^2}{-F_E} + f_- \mathcal{J} f_- \right] d\mathcal{V}_p \, d\mathcal{V}_x = \text{constant} . \tag{22}$$

Of special importance is the result that there exists a variational principle for the normal modes of radial pulsation of a spherical cluster: If f_- is split up into normal modes,

$$f_- = \mathfrak{f}(x^j, p_\alpha) e^{i\omega t} , \tag{23}$$

$\mathfrak{f}$ satisfies a self-conjugate eigenequation, obtained by combining equations (19) and (23), for which there is a variational principle (eqn. [27; R] of Part 5)

$$\omega^2 = \frac{\int \mathfrak{f} \mathcal{J} \mathfrak{f} \, d\mathcal{V}_p \, d\mathcal{V}_x}{\int (-1/F_E) \mathfrak{f}^2 \, d\mathcal{V}_p \, d\mathcal{V}_x} . \tag{24}$$

If $F_E \leq 0$ throughout phase space, each normal frequency, ω, is either real (stable mode) or imaginary (unstable mode). Consequently, since the variational principle is a minimal principle, a spherical equilibrium cluster with $F_E \leq 0$ throughout phase space is stable against spherical perturbations if and only if $\mathcal{J}$ is a positive-definite operator for odd spherical functions bounded in phase space:

$$\int h \, \mathcal{J} \, h \, d\mathcal{V}_p \, d\mathcal{V}_x > 0 \, . \tag{25}$$

The variational principle (24) is difficult to apply because it involves a multidimensional problem. It would seem worthwhile to search for stability criteria which involve analyses in only one dimension. One-dimensional <u>sufficient</u> (but not necessary) criteria for stability have been developed in Newtonian theory by Lynden-Bell (1966, 1967) and by Lynden-Bell and Sanitt (1969). In Part 6 of this treatise we develop relativistic analogues of their criteria.

In §II of Part 6 we derive a sufficient condition for stability which involves the positive-definiteness of a one-dimensional, second-order differential operator. We begin the derivation of this stability criterion by introducing a new perturbation function, q, closely related to f:

$$q(x^\alpha, p_{(\alpha)}) \equiv f(x^\alpha, p_{(\alpha)}) - F_E \, p_0 \, \nu_B/2 \, . \tag{26}$$

By combining equations (15) and (26) we obtain (to first order) the expression

$$\eta = F(m, E, J) + q \tag{27}$$

for the distribution function of the perturbed cluster, since

$$E \equiv p_0 = e^{(\nu_A + \nu_B)/2} p_{(0)} = e^{\nu_B/2} E_A \,. \tag{28}$$

Expression (27) states that q is the perturbation in the distribution function at fixed x^α, m, E, and J, while f is the perturbation at fixed x^α, $p_{(\alpha)}$.

In §IIc of Part 6 we reexpress the conserved quantity, H, of equation (22) in terms of q, and combine with the perturbed field equations (3). After a moderate amount of manipulation, we discover that we can rewrite expression (22) in the form [eqn. (37) of Part 6]

$$H = H_1 + (1/4) \int \partial\nu_B/\partial t \; S \; \partial\nu_B/\partial t \; d^3V_x \,. \tag{29}$$

Here S -- which is an ordinary second-order differential operator in one variable, the Schwarzschild radial coordinate, r -- and the volume element $d^3V_x \propto dr \, d\theta \, d\varphi$ are defined by equations (28) and (29b) of Part 6. For almost all physically interesting star clusters, $H_1 \geq 0$ for all perturbations (see Part 6). In such a case, if the second term in equation (29) is also positive for all perturbations, then the perturbations cannot grow in time faster than linearly. Otherwise, H could not possibly be constant in time because it would be the sum of two positive terms, each of which becomes arbitrarily large in time. As is discussed in §IVf of Part 5, linear time growth is associated with a mode which is "marginally stable," i.e. a mode which is in neutral equilibrium. (Such a mode is called a dynamical zero-frequency mode.) These considerations thus result in a sufficient condition for stability: A spherical cluster for which $F_E \leq 0$, and for which condition (39b) (see also eqn. 38b) of Part 6 is satisfied, is stable, or at least marginally stable, against small radial perturbations if the operator S (eqn. [28] of Part 6) is positive-definite over all physically acceptable perturbation functions, $\partial\nu_B/\partial t$, i.e. if

$$\int \partial\nu_B/\partial t \; S \; \partial\nu_B/\partial t \, d^3 V_x \geq 0 . \tag{30}$$

(This integral is actually one-dimensional rather than three-dimensional, since the integrand is independent of θ and φ.)

The boundary conditions on the acceptable $\partial\nu_B/\partial t$ are given by equations (42a, b) of Part 6: at the center of symmetry the field equations and the smoothness of the spacetime geometry guarantee that the acceptable $\partial\nu_B/\partial t$ are power series in r^2; at the surface of the cluster both they and their radial derivatives vanish. We could attempt to apply the criterion (30) by inserting various acceptable, radial trial functions into the integral. Alternatively, we can make use of the fact that, as follows from definition (28) of Part 6 for S, S is self-conjugate for bounded, radial functions [see eqn. (29a) of Part 6]. As we show in §IIc of Part 6, if we combine the fact that S is self-conjugate with the fact that it is a Sturm-Liouville operator, then we obtain the following method of applying condition (30): Integrate the differential equation

$$S\psi = 0 \tag{31}$$

from $r = 0$, where the boundary conditions $\psi = 1$, $d\psi/dr = 0$ are imposed, to $r = R$. If the resultant function, ψ, has no nodes, then condition (30) is satisfied for all acceptable perturbation functions, and the cluster -- if it satisfies the subsidiary conditions (39) of Part 6 -- is stable against small radial perturbations.

In §IId of Part 6 we turn our attention momentarily to a gas sphere (star) which has been perturbed radially away from its equilibrium state. We derive there a sufficient condition for the stability of the gas sphere which involves the positive-definiteness of the same operator S which enters into criterion (30) above. We obtain this condition for stability by manipulating a quantity which is conserved during the small, radial motions of a perturbed gas sphere. That quantity is [eqns. (59) - (61) of Part 6]

$$H^* = K^* + P^* , \tag{32}$$

where K^* and P^* are the kinetic and potential energies associated with the perturbation of the gas sphere. We find that, for perturbations with adiabatic index (54) of Part 6, we can reexpress H^* in the form [see eqns. (59), (60), and (72) of Part 6]

$$H^* = H_1^* + \frac{1}{8} \int \nu_B S \nu_B d^3 V_x , \tag{33}$$

which is similar to equation (29) above. For most interesting gas spheres, it turns out that $H_1^* \geq 0$. Hence by a line of reasoning similar to that which leads to criterion (30) for a cluster, we arrive at the sufficient condition for gas sphere stability stated in §IId of Part 6,which involves the positive-definiteness of the operator S over the set of acceptable perturbation functions ν_B.

In §III of Part 6 we prove the most important of our sufficient conditions for the stability of a spherical cluster. This condition establishes a relationship between the stability of a spherical cluster with isotropic velocity distribution (i.e. a cluster whose equilibrium distribution function, F, is independent of the angular momentum), and the stability of the corresponding gas sphere (which is by definition the gas sphere which has the same equilibrium radial distributions of pressure and of total density of mass-energy as has the cluster -- see eqns. [86] of Part 6). We obtain the stability criterion of §III of Part 6 by comparing the conserved quantity, H, of equation (22) for a cluster with the conserved quantity, H^*, of equation (32) for the corresponding gas sphere. We manipulate the quantity, H, for a cluster and rewrite it in the form

$$H = 2K + 2P, \tag{34}$$

where 2K is just the first (positive-definite) term in equation (22) above, and where P is defined by equation (83) of Part 6. We proceed to show that if the potential energy function, P^*, for a gas sphere is positive-definite for all physically-acceptable perturbations of the gas sphere with adiabatic index (54) of Part 6, then the function P for the corresponding cluster is positive-definite for all physically-acceptable perturbations of the cluster (cf. eqn. [85]of Part 6). But the theory of gas sphere stability guarantees that a gas sphere is stable against a given set of radial perturbations if and only if the potential energy function, P^*, is positive-definite for that set of perturbations. (If P^* is positive-definite then the gas sphere sits at the bottom of a potential well.) Thus, if the gas sphere is stable, the function P is positive-definite, and by equation (34) H is the sum of two positive terms. Hence perturbations cannot grow in time and the cluster is stable. This stability criterion can be stated as follows: A bounded, relativistic, spherical cluster with isotropic velocity distribution is stable against small radial perturbations, if the gas sphere with the same distributions of pressure and of density of total mass-energy (cf. eqns. [86] of Part 6) is stable against radial perturbations with adiabatic index (87) of Part 6.

In Part 7 we employ the stability criteria of Parts 5 and 6 to diagnose numerically the stability of specific star-cluster models with isotropic velocity distributions.

In §III of Part 7 we reduce the variational principle (24) from the rather general form in which it is given in Part 5 to a form more suitable for numerical calculations. We begin in §IIIa of Part 7 by obtaining a simplified expression for the quantity $\mathcal{L}_A h$, where $\mathcal{L}_A$ is the Liouville operator of an unperturbed cluster, and h is a spherical function in phase space. We choose (x^α, p_α) as our coordinates in phase space. Then the unperturbed Liouville operator takes the form given in equation (9d) of Part 7. Notice that it involves derivatives with respect to the six coordinates (x^i, p_i). However, when the unperturbed Liouville operator acts on a spherical function, h, only those terms involving derivatives of h with respect to r and p_r survive. This is because a spherical function is one that depends on θ, φ, p_θ, and p_φ only through the angular momentum, J, which is itself independent of r and p_r. And $\mathcal{L}_A J = 0$ since J is an integral of the motion along a stellar orbit in any spherical cluster. Consequently, the terms in $\mathcal{L}_A h$ which involve derivatives with respect to θ, φ, p_θ, p_φ sum up to zero, and we arrive at expression (15) of Part 7 for the action of the unperturbed Liouville operator on a spherical function.

One uses the variational principle (24) to study the stability of a specific cluster model by inserting various odd-parity, spherical trial functions, $\bar{f}$, into expression (24), and by searching for a minimum value of ω^2. It follows from the necessary and sufficient criterion (25) above that if a negative value of ω^2 is associated with any acceptable trial function, the cluster is unstable. If a trial function is well chosen in that it approximates the eigenfunction of the fundamental radial mode "to first order" in some sense, then the value obtained for ω^2 will agree "to second order" with the actual value of the squared frequency of oscillation of the fundamental mode.

In §IIIb of Part 7 we discuss our choice of trail functions. Our trial functions must have odd parity and spherical symmetry because they correspond to the odd-parity part of the perturbation in the distribution function. Hence, since p_θ and p_φ can enter only through J [see eqn. (9) above], any trial function must be odd in p_r. Another constraint on our choice of trial functions arises in the following way: A bounded cluster possesses an equilibrium distribution function, F, which vanishes for energies, $E = p_0$, greater than some limiting value (a star with $E \geq$ its rest mass, m, can escape to infinity; i.e. it is not bound to the cluster). Unless F drops smoothly to zero at this limiting energy, the perturbation in the distribution function <u>must</u> have a term proportional to a delta function at the limiting energy. This is the proper mathematical statement of the fact that the location in phase space of the cluster's sharp "surface" varies. (Compare this with the fact that if the equilibrium density is discontinuous across the surface of a pulsating star, then the Eulerian perturbation in the density includes a term proportional to a delta function at the star's surface.) We can take account of this singular behavior by choosing the energy dependence of our trial functions to be F_E (cf. eqn. 20). Other reasons for this choice are discussed in §IIIb of Part 7.

Our choice for the dependences upon r of our trial functions is motivated by results from studies of the radial pulsations of gas spheres. It is often true that the displacement associated with the fundamental radial mode of a gas sphere is nearly $\propto r$ if the gas sphere is not highly relativistic. If the gas sphere is highly relativistic, the fundamental displacement tends to be much larger in the inner regions. For our star clusters we choose the radial dependences of the trial functions in such a manner that the associated "mean stellar displacements" have forms similar to these fundamental radial modes of gas spheres.

In summary, using the above considerations as guides, we choose our trial functions to be of the form

$$\bar{f} = C'(r)\, r\, e^{-\mu r}\, F_E\, p_r \, ; \qquad (35)$$

we either set $C'(r)$ equal to a constant or choose it such that the mean stellar displacement has a given form [condition (23) of Part 7]. The constant μ is a "peaking parameter" with respect to which minimization can be performed.

In §IIIc of Part 7 we substitute the trial function (35) into the variational principle (24) and obtain an explicit, reduced form of the variational principle. From this form [eqns. (28) and (29) of Part 7] one notices that the application of the variational principle involves non-trivial integrations over r and over at least one momentum coordinate.

Our method of evaluating the multidimensional phase-space integrals which enter into the application of the variational principle is this: First we perform the required integrations over momentum space (in some instances we can do so analytically). Then we replace the remaining integrals over the radial coordinate by their equivalent differential equations. We integrate these differential equations right along with the equations of structure for the equilibrium configuration. In this way we can simultaneously compute structure and diagnose stability.

In §IV of Part 7 we use the necessary and sufficient variational principle, and also the sufficient criteria derived in Part 6, to study the stability of clusters of identical stars with truncated, isotropic Maxwell-Boltzmann velocity distributions. Such an isothermal cluster has a distribution function $F \propto e^{-p_0/T}$ for p_0 less than some cutoff value, where T is a constant; F is zero above the cutoff value [see eqn. (30) of Part 7]. At the beginning of §IV b of Part 7 we spell out our integration scheme for studying the structure and stability of isothermal clusters. It turns out (see Appendix A of Part 7) that ten different momentum-space integrals enter into the analysis. In Appendix A of Part 7 we devise a method for evaluating seven of these in terms of the remaining three. Unfortunately, the remaining three integrals, two of which also enter into the computation of the structure of the model, must be computed numerically.

In §V of Part 7 we study polytropic star clusters. By definition, these are clusters whose isotropic pressures and total densities of mass-energy are related by the relativistic polytropic law (40) of Part 7. In §V b of Part 7 we use slight generalizations of methods due to Fackerell to obtain the distribution functions, F, which give rise to the polytropic models. At the beginning of §V c of Part 7 we summarize the integration scheme which we use to study the structure and stability of polytropic clusters. Fortunately, we discover in Appendix C of Part 7 that all the momentum-space integrals which enter into the stability analysis can be evaluated analytically.

In §IV b of Part 7 we carry out our stability analysis for isothermal clusters, and in §V c of Part 7 we study polytropic clusters. In both studies we collect the equilibrium configurations into smooth sequences parameterized by the central redshift, z_c (redshift of a photon emitted at the center of the cluster and received at infinity), of a model. As z_c increases along such a sequence, the models become more relativistic.

Our analyses indicate not only that the onset of collapse occurs along each of a wide variety of sequences of isotropic clusters, but also that the onset of collapse is largely independent of the nature of the sequence. As z_c increases from zero along each sequence, the contrast between the densities of stars in the interior and outer regions of a cluster increases, and the fractional binding energy (the gravitational binding energy divided by the total rest mass) increases. When z_c reaches a value of the order of 0.5, the fractional binding energy reaches a maximum; and thereafter it oscillates. To within the accuracy of our calculations, the variational principle indicates that instability against gravitational collapse sets in at the peak of the fractional binding energy, which is always near $z_c = 0.5$. (See Tables 1-6 and Figure 2 of Part 7.)

As we discuss in §IV b and §V c of Part 7, in most situations the sufficient conditions for stability derived in Part 6 are much less powerful than the variational principle. In fact, it turns out that the criterion associated with equation (31) above yields vacuous results for all isothermal and polytropic models. Further, the criterion of

§III of Part 6 which relates cluster stability to gas-sphere stability is useless for studying the isothermal models. The gas spheres which correspond to the isothermal models of Part 7 are all unstable. Turning to the polytropic models, we find that all of the gas spheres which correspond to the clusters of Tables 5 and 6 of Part 7 are unstable. And the gas spheres which correspond to the polytropes of Table 4 of Part 7 (the "adiabatic" polytropes of index 2) become unstable when the central redshift reaches the value $z_c = 0.315$.

Our analyses suggest an idealized story of the evolution of a spherical cluster: It might evolve along some one-parameter sequence of equilibrium configurations, by means of stellar collisions and evaporation of stars. When two stars collide and coalesce, they increase the cluster's rest mass and hence its fractional binding energy. When a star gains enough energy through encounters to escape from the cluster, it carries away excess kinetic energy, thereby leaving the cluster more tightly bound. Consequently the cluster should evolve along its sequence toward states with larger and larger fractional binding energy. When the cluster reaches the point, along its sequence, of maximum binding, quasistatic evolution must stop because collisions and evaporation could only increase further the fractional binding energy of the cluster. Something else must replace quasistatic contraction -- and indeed it does: The cluster undergoes relativistic gravitational collapse.

In Part 7 we avoid consideration of isotropic clusters which have high-density cores surrounded by extended, diffuse envelopes (i.e. clusters with ratios of mean density to central density $\leq 10^{-3}$). Our experience indicates that an accurate study of the stability of such a "core-halo" cluster cannot be achieved with the simple trial functions (35). However, we know of no sequence of isotropic clusters for which the fractional binding energy peaks at a central redshift, z_c, significantly different from 0.5. Hence, it appears likely that all isotropic clusters with $z_c \gtrsim 0.5$ are unstable. One could attempt to make this result more definite by searching for a theorem which relates the onset of instability to the behavior of the binding-energy curves associated with properly constructed sequences of clusters. Such a theorem would be analogous to the theorem which we use for gas spheres in Part 4, and which enables us to diagnose the stability of the stars along isentropic stellar sequences by simply examining the associated binding-energy curves.

In any event, our analyses suggest that it will be very difficult, perhaps even impossible, to construct stable clusters with central redshifts as large as ~ 2.4. Such clusters are needed in the Hoyle and Fowler (1967) star-cluster model for the quasi-stellar sources. Hopefully, future research will decide this issue conclusively, and will cast light on speculations that violent events in the nuclei of galaxies and in quasars might be associated with gravitational collapse.

Appendix to Part 3

In this appendix we discuss certain aspects of relativistic kinetic theory which are relevant to our analyses.

We want to discuss measurements made by local observers. Associated with a local observer is the unit vector, λ^α, tangent to his world-line through spacetime. This tangent vector is just the 4-velocity of the observer; i.e. the 4-momentum of a particle which is at rest with respect to the observer is $p^\alpha = m\lambda^\alpha$, where m is the rest mass of the particle.

Suppose that two spacetime events near a local observer are separated by an infinitesimal displacement dx^α. The physical length, $d\ell$, between the two events, as measured by the observer, is the magnitude of the projection of dx^α onto the 3-space orthogonal to the observer's unit tangent vector, λ^α. The operation of projection is most easily performed through use of the projection operator

$$P_{\alpha\beta} = g_{\alpha\beta} - \lambda_\alpha \lambda_\beta \quad , \tag{A1}$$

where $g_{\alpha\beta}$ is the metric tensor of spacetime. The projection of dx^α is

$$d\ell^\alpha \equiv P^\alpha{}_\beta \, dx^\beta = dx^\alpha - \lambda^\alpha \lambda_\beta \, dx^\beta \, . \tag{A2}$$

(Of course, indices are raised and lowered by contraction with the metric tensor.) Hence the physical length, $d\ell$, is

$$d\ell \equiv \surd(-d\ell^\alpha d\ell_\alpha) = \surd\{-[dx^\alpha dx_\alpha - (\lambda^\alpha dx_\alpha)^2]\} \, . \tag{A3}$$

(The overall minus sign under the radical is due to our choice of convention that the fundamental line element, equation (1) of this Summary, has signature [+ - - -].) Notice from equation (A2) that, as viewed by the observer, the spacetime displacement, dx^α, is a pure physical length (i.e. the two events separated by the displacement dx^α are simultaneous with respect to the observer) if and only if $\lambda_\beta dx^\beta$ vanishes. In a curvilinear coordinate system with respect to which the observer is at rest (in such a coordinate system the spatial coordinates, x^i, are constant along the observer's world line), the only non-vanishing component of the observer's 4-velocity is $\lambda^0 = 1/\sqrt{g_{00}}$. Hence dx^α is a pure physical length if and only if, in a rest system for the observer,

$$g_{\alpha\beta}\,\lambda^\alpha dx^\beta = g_{0\beta}\,dx^\beta/\sqrt{g_{00}} = dx_0/\sqrt{g_{00}} = 0 . \tag{A4}$$

In a rest system for the observer, who comoves with the coordinate system, expression (A3) reduces to

$$d\ell = \sqrt{[-(g_{ij} - g_{0i}\,g_{0j}/g_{00})dx^i dx^j]} \equiv \sqrt{(-{}^{(3)}g_{ij}\,dx^i dx^j)}. \tag{A5}$$

(A different derivation of this expression is given in Landau and Lifshitz 1962, § 84.) This implies that the 3-space orthogonal to the observer's 4-velocity is described by the 3-dimensional "metric" tensor ${}^{(3)}g_{ij}$. It follows that the physical 3-volume elements, dS, measured by an observer are given in terms of the spatial-coordinate elements, dx^i, of a rest system for the observer by the expression

$$dS = \sqrt{(-{}^{(3)}g)}\,dx^1 dx^2 dx^3, \tag{A6}$$

where ${}^{(3)}g$ is the determinant of ${}^{(3)}g_{ij}$. (Synge and Schild 1949, chap. 7, discuss in some detail the concept of volume for general curvilinear coordinate systems.)

A quantity which plays a fundamental role in our statistical treatment of relativistic star clusters is the distribution function (or density of stars in phase space), $\mathcal{N}$. Let attention be focused upon a group of stars localized near a spacetime event, $\underset{\sim}{x}$, in a star cluster, and with 4-momenta in the range

$$d\mathcal{V}_p = \surd(-g)\, dp^0 dp^1 dp^2 dp^3 = -dp_0\, dp_1\, dp_2\, dp_3 / \surd(-g) \qquad \text{(A7)}$$

centered about $\underset{\sim}{p}$. Here g is the determinant of the metric tensor, $g_{\mu\nu}$, of spacetime, and p^α and p_α are the contravariant and covariant components of the 4-momentum of a star in some curvilinear coordinate system. The expressions in definition (A7) are invariants (see e.g. chapters 1 and 4 of Synge 1960). Denote by λ'^α the 4-velocity of the observer at $\underset{\sim}{x}$ who moves with these stars. As mentioned earlier, $\lambda'^\alpha = p^\alpha/m$. Suppose that this comoving observer finds that dN of these stars occupy a physical 3-volume dS'. Then the distribution function, $\mathcal{N}$, is defined by

$$\mathcal{N} \equiv dN/(d\mathcal{V}_p\, dS') \ . \qquad \text{(A8)}$$

$\mathcal{N}$ is an invariant by construction. If another observer at the spacetime event, $\underset{\sim}{x}$, has a 4-velocity $\lambda^\alpha \neq \lambda'^\alpha$, then, because of the Lorentz contraction, he finds that the dN stars in question occupy a physical 3-volume

$$dS = dS'/(\lambda^\alpha \lambda'_\alpha) = (m/\lambda^\alpha p_\alpha)\, dS' \ . \qquad \text{(A9)}$$

In terms of the curvilinear coordinates of a rest system for this observer, it follows that

$$
\begin{aligned}
dS' &= (p_0/m\sqrt{g_{00}})\sqrt{(-{}^{(3)}g)}\, dx^1\, dx^2\, dx^3 \\
&= (p_0/m\, g_{00})\sqrt{(-g)}\, dx^1\, dx^2\, dx^3 \equiv d\mathscr{V}_x \ .
\end{aligned}
\tag{A10}
$$

(The first equality results from combining equations [A6] and [A9], and the fact that $\lambda^0 = 1/\sqrt{g_{00}}$ is the only non-vanishing component of an observer's 4-velocity in a rest system. The second equality results from the relation $g = g_{00}\,{}^{(3)}g$, which is easily proved by expressing the determinant g in terms of an expansion along the zeroth row of the matrix $g_{\alpha\beta}$.) $d\mathscr{V}_x$ is an invariant by construction. Combining equations (A7), (A8), (A10), and the expression,

$$
m^2 = p_\alpha p^\alpha = g^{\alpha\beta} p_\alpha p_\beta \ , \tag{A11}
$$

for the rest mass of a particle in terms of its 4-momentum, we obtain

$$
\eta \equiv dN/d\mathscr{V}_p d\mathscr{V}_x = dN[-(p_0/p^0 g_{00})dx^1 dx^2 dx^3 dm dp_1 dp_2 dp_3]^{-1}, \tag{A12}
$$

in an arbitrary curvilinear coordinate system. Equations (A10) and (A12) should replace the first of equations (1; R) and equation (2; R) of Part 5, which are valid actually only in coordinate systems for which $g_{0i} = 0$.

Notice from equations (A8) and (A11) that the product $(\eta/m)d\mathscr{V}_p$ is an invariant. Hence

$$T_{\alpha\beta} \equiv \int (\eta/m)\, p_\alpha p_\beta \, d\mathcal{V}_p \qquad \text{(A12)}$$

is a tensor of rank 2, the stress-energy tensor. It follows from equations (A8) and (A9) that $T_{\alpha\beta}\lambda^\beta$ is just the amount of covariant 4-momentum, p_α, per unit physical 3-volume as measured by the observer with 4-velocity λ^α. More particularly, $T_{\alpha\beta}\lambda^\alpha\lambda^\beta$ is the local density of energy as measured by the observer, and the projection of the vector $T_{\alpha\beta}\lambda^\beta$ on the 3-space orthogonal to the 4-velocity, λ^α, measures the local spatial flux of energy per unit area per unit time. The projection of the tensor $T_{\alpha\beta}$ on the 3-space orthogonal to λ^α measures the stress (i.e. the flux of spatial momentum in different spatial directions) due to the motions of the stars in a cluster (for a detailed discussion see Synge 1960, chap. 4).

In the absence of collisions between the stars which compose a cluster (the justification for our using such an idealization in our investigations is discussed at the beginning of this Summary), a star travels along a geodesic through spacetime. If proper time, s, as measured by an observer moving with a star is used as a parameter along the star's geodesic, then

$$dx^\alpha/ds = p^\alpha/m = g^{\alpha\beta} p_\beta/m \,, \qquad \text{(A13a)}$$

$$dp_\alpha/ds = -(1/2m)\,\partial g^{\beta\gamma}/\partial x^\alpha \, p_\beta p_\gamma \,, \qquad \text{(A13b)}$$

along the geodesic (see, e.g. Synge 1960, chap. 3). Here $\underset{\sim}{x}$ is the star's position in spacetime, and $\underset{\sim}{p}$ and m are its 4-momentum and rest mass. In a rest frame for the star $ds = \sqrt{g_{00}}\, dx^0$ (compare with equation 2).

Just as in Newtonian theory, so also in general relativity, in the absence of collisions the distribution function, $\mathcal{N}$, is conserved along the path of a star,

$$\mathcal{L}\mathcal{N} \equiv d\mathcal{N}/ds = 0 \tag{A14}$$

(Walker 1936). Here the operator $\mathcal{L}$, the Liouville operator, is the derivative with respect to proper time along a star's geodesic. Equation (A14) is called the Boltzmann-Liouville, or collisionless Boltzmann, equation. We can prove equation (A14) by generalizing a familiar proof of the corresponding Newtonian equation:

The motion of a particular star in a cluster can be described as a path through an 8-dimensional phase space, the tangent bundle, four of whose coordinates are the coordinates, x^α, of some spacetime coordinate system. The remaining coordinates are four momentum coordinates. At present it is convenient to choose these latter coordinates to be the covariant components, p_α, of the momentum of a star with respect to the chosen spacetime coordinate system. The element of volume in the tangent bundle is taken to be

$$d^8V = -dp_0 \ldots dp_3 \, dx^0 \ldots dx^3 , \tag{A15}$$

and is an invariant under spacetime-coordinate transformations. For our choice of independent coordinates in the tangent bundle, the invariant Liouville operator, $\mathcal{L}$, takes the form

$$\begin{aligned} \mathcal{L}h = \frac{dh}{ds} &= \frac{dx^\alpha}{ds}\frac{\partial h}{\partial x^\alpha} + \frac{dp_\alpha}{ds}\frac{\partial h}{\partial p_\alpha} \\ &= \frac{p^\alpha}{m}\frac{\partial h}{\partial x^\alpha} - \frac{1}{2m}\frac{\partial g^{\beta\gamma}}{\partial x^\alpha} p_\beta p_\gamma \frac{\partial h}{\partial p_\alpha} , \end{aligned} \tag{A16a}$$

when operating on some function, h, in the tangent bundle. The last equality follows from equations (A13). Notice from equations (A13) that equation (A16a) can be written in the equivalent form

$$\mathcal{L} h = \frac{\partial}{\partial x^\alpha}\left(\frac{p^\alpha}{m} h\right) + \frac{\partial}{\partial p_\alpha}\left(- \frac{1}{2m} \frac{\partial g^{\beta\gamma}}{\partial x^\alpha} p_\beta p_\gamma h\right) . \qquad \text{(A16b)}$$

We are interested in integrating expressions similar to (A16b) over small domains of the tangent bundle. To simplify the notation, we introduce what may appear to be a complication. We denote by X^A ($A = 0, \ldots, 7$) the coordinate, x^α and p_α, of the tangent bundle with the identifications

$$X^0, \ldots, X^3, X^4, \ldots, X^7 \equiv p_0, \ldots, p_3, x^0, \ldots, x^3 . \qquad \text{(A17)}$$

Expression (A16b) can then be written as

$$\mathcal{L} h = \partial h^A / \partial X^A , \qquad \text{(A16c)}$$

with an obvious definition for the tangent-bundle vector h^A. It follows quite generally from the generalized Stokes theorem (Synge and Schild 1949, chap. 7) that

$$\int_D \mathcal{L}(h)\, d^8V \equiv \int_D \partial h^A/\partial X^A (-) dX^0 \ldots dX^7 = \int_{\partial D} h^A\, d^7\Sigma_A , \qquad \text{(A18)}$$

where ∂D is the 7-dimensional surface which bounds the tangent-bundle domain D, and where the surface elements $d^7\Sigma_A$ are given by

$$d^7\Sigma_A = \pm \epsilon_{A A_0 \ldots A_6} \frac{\partial X^{A_0}}{\partial y^0} \cdots \frac{\partial X^{A_6}}{\partial y^6} \, dy^0 \ldots dy^6 . \qquad \text{(A19)}$$

In definition (A19) $\epsilon_{A_0 \ldots A_7}$ is the totally antisymmetric permutation symbol ($\epsilon_{0 \ldots 7} = 1$); $y^0, \ldots, y^6$ are parametric coordinates on the boundary ∂D; and the plus or minus sign is chosen so as to properly orient the surface elements.

We single out a small number, dN, of stars whose paths fill up a thin tube in the tangent bundle. Of course, we can do this actually only in the absence of collisions; otherwise stars would in general enter and leave the tube. At some position along this tube we choose an observer and denote by $d_1\Sigma$ that cross-sectional slice through the tube which lies on the observer's 7-surface of simultaneity. The paths of the stars intersect $d_1\Sigma$ at a value $s = s_1$, say, of proper time along the tube. At $s = s_2 = s_1 + ds$ we choose another observer and denote by $d_2\Sigma$ the corresponding cross-sectional slice for him. We denote by dD the cylindrical section of the tube bounded by $d_1\Sigma$, $d_2\Sigma$, and that portion, $d_3\Sigma$, of the tube's wall between $d_1\Sigma$ and $d_2\Sigma$. We set $h = 1$ in equation (A18) and obtain

$$0 = \int_{d_1\Sigma + d_2\Sigma} 1^A \, d^7\Sigma_A . \qquad \text{(A20)}$$

(The surface integral over $d_3\Sigma$ vanishes because at each point of $d_3\Sigma$ the vector 1^A is tangent to $d_3\Sigma$ by construction.) We evaluate the integral over $d_1\Sigma$ by choosing parametric coordinates in equation (A19) such that

$$y^0, \ldots, y^3, y^4, \ldots, y^6 = p_0, \ldots, p_3, x^1, \ldots, x^3, \qquad \text{(A21)}$$

where x^α are the coordinates of a rest system for the observer at $s = s_1$ along the tube. Since $d_1\Sigma$ is a surface of simultaneity, x^0 varies over the surface according to equation (A4). Combining equations (A4), (A19), and (A21) we obtain, for the proper orientation of $d_1\Sigma$,

$$\begin{aligned} d_1^7\Sigma_0 &= \ldots = d_1^7\Sigma_3 = 0, \\ g_{00}d_1^7\Sigma_4 &= g_{00}\,d^7y, \ldots, g_{00}\,d_1^7\Sigma_7 = g_{03}\,d^7y, \qquad \text{(A22)} \\ d^7y &= dp_0 \ldots dp_3\,dx^1 \ldots dx^3 . \end{aligned}$$

Equations (A16b, c) and (A22) yield

$$1^A\,d_1^7\Sigma_A = (p_0/m\,g_{00})\,dp_0 \ldots dp_3\,dx^1 \ldots dx^3, \qquad \text{(A23)}$$

which, except for a minus sign, is just the invariant, 7-dimensional, phase-space element, $d\mathcal{V}_p d\mathcal{V}_x$, used to define the distribution function, $\mathcal{N}$ [cf. eqns. (A7), (A10), and (A12)]. The form of the expression for the integral over $d_2\Sigma$ obviously differs from equation (A23) by only a minus sign. Thus the phase-space volume, $d\mathcal{V}_p d\mathcal{V}_x$, occupied by the dN stars is conserved along a star's path. Consequently, the distribution function

$$\eta \equiv dN/d\mathcal{V}_p \, d\mathcal{V}_x$$

is conserved, and the mathematical statement of this conservation law is the Boltzmann-Liouville equation (eqn. A14). Q.E.D.

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4. On the Stability of Ultrarelativistic Stars (Submitted for publication in Astrophysics and Space Science)

1. Introduction and Summary

In recent years relativistic astrophysicists have developed an extensive body of theoretical knowledge concerning the structure and stability of relativistic stars -- i.e., high-density white dwarfs, neutron stars, supermassive stars, and configurations which are unstable against gravitational collapse, and therefore which should not be realizable in nature. [For reviews see HARRISON, THORNE, WAKANO, and WHEELER (1965), denoted henceforth as HTWW; WHEELER (1966); THORNE (1967); and ZEL'DOVICH and NOVIKOV (1967).]

One part of this body of knowledge concerns itself with the manner in which general relativity catalyzes instabilities in stellar models. It is well known that in general relativity the gravitational forces at work within a stellar model are more sensitive to perturbations of that model than they are in Newtonian theory, and hence that relativistic forces induce instabilities which do not arise in Newtonian theory. However, there appears to remain some uncertainty concerning the precise strength of the relativistic instabilities in highly relativistic stars. For example, there is occasionally debate as to whether one can construct stable stellar models whose structures are as highly relativistic as may be desired.

In this paper we seek to partially dispose of this uncertainty. Specifically, in Section 2 we extend our understanding of the manner in which general relativity catalyzes instabilities by presenting an analysis, in terms of adiabatic indices, of the strength of the relativistic instabilities in highly relativistic, spherical stars.

In Section 3 we study how the presence of these instabilities is reflected in the structures of hot stellar models with radially invariant entropy per baryon. We show that, because of the instabilities, plots of binding energy versus radius for certain one-parameter sequences of such models must undergo damped high-ρ_c spirals (where ρ_c, the central density of a model, parameterizes each sequence). This behavior is analogous to the behavior of mass-versus-radius curves for certain sequences of cold stellar models at the endpoint of thermonuclear evolution (see, e.g., HTWW, chap. 5); and it has been observed (though not explained) in a number of numerical studies of hot, relativistic stellar models (e.g., BARDEEN, 1965; TOOPER, 1965).

Throughout this paper we use units in which the speed of light, c, and Newton's gravitation constant, G, are equal to unity.

2. Relativistic Instabilities in Ultrarelativistic Stars

CHANDRASEKHAR (1964a,b) and FOWLER (1964) have developed independently a beautifully simple method for measuring -- in the post-Newtonian regime -- the effects of general relativity upon the stability of stars. Their analysis, in slightly modified form, is the following: Consider a particular spherical stellar model with mass M, radius R, and adiabatic index

$$\Gamma_1 \equiv [(\rho + p)/p]\,(\partial p/\partial \rho)_{\text{constant entropy}} \quad , \qquad (1)$$

which is assumed to be constant throughout the star. Here ρ is the total density of mass-energy and p is the pressure of the star's matter. The stability of the star will depend upon the value of Γ_1, and upon the star's structure. Ignore for the moment the actual value of Γ_1. Pretend that the star possesses a different adiabatic index, Γ_{crit}, which is also constant throughout the star, and which is just the right magnitude to make the fundamental mode of radial pulsation of the star neutrally stable. Then a comparison of the actual value of the adiabatic index with the "critical value", Γ_{crit}, will reveal whether the star is stable or unstable. If Γ_1 exceeds Γ_{crit}, the star is stable; if Γ_1 is less than Γ_{crit}, the star is unstable.

In Newtonian theory Γ_{crit} is equal to 4/3 independently of the structure of the stellar model. However, in general relativity Γ_{crit} exceeds 4/3. (This is the manner in which relativistic instabilities manifest themselves.) <u>In the post-Newtonian regime</u> Γ_{crit} differs from 4/3 by an amount of the order of the relativistic effects on the star's structure:

$$\Gamma_{crit} - 4/3 = C(2M/R) \ll 1 , \qquad (2)$$

where C is a constant, usually between 0.5 and 1.5, which depends only on the Newtonian structure of the star.

In this section we examine the critical values of the adiabatic index for highly relativistic stellar models, where the post-Newtonian analysis is invalid. More specifically, we seek to derive the following approximate criterion for stability:

<u>Consider a relativistic, spherical stellar model whose distributions of total density of mass-energy, ρ, and of pressure, p, are related by an equation of state which approaches the "gamma-law" form</u>

$$p = (\Gamma_4 - 1)\,\rho \qquad , \qquad (3)$$

<u>where Γ_4 is a constant, at densities greater than some limiting value,</u> ρ_{lim}. <u>Assume that the adiabatic index,</u>

$$\Gamma_1 \equiv [(\rho + p)/p]\,(\partial p/\partial \rho)_{\text{constant entropy}} \qquad , \qquad (4)$$

of the model is constant throughout the region $\rho > \rho_{lim}$. If the density, ρ_c, at the center of symmetry is sufficiently greater than ρ_{lim}, then the critical adiabatic index, Γ_{crit}, below which the model is unstable against adiabatic radial perturbations,

$$\Gamma_1 < \Gamma_{crit} \Rightarrow \text{instability,} \qquad (5a)$$

is given (approximately) by the solution of the transcendental equation

$$b_{crit} \equiv \left[\frac{8}{\Gamma_{crit}} - \frac{3}{\Gamma_4}\left(\frac{3}{\Gamma_4} - 1\right) - \frac{1}{4}\right]^{\frac{1}{2}} = \frac{\pi/2 + \tan^{-1}\left[(1/b_{crit})(3/2 - 1/\Gamma_4)\right]}{\ln\,(R/r_{core})}, \qquad (5b)$$

$$r_{core} = \left[\frac{2(\Gamma_4 - 1)}{(\Gamma_4{}^2 + 4\Gamma_4 - 4)4\pi\rho_c}\right]^{\frac{1}{2}} , \qquad (5c)$$

with R the radius of the model $\equiv$ (surface area of model$/4\pi)^{\frac{1}{2}}$.

We should note that almost all physically interesting equations of state thus far proposed for cold matter at the endpoint of thermonuclear evolution or for hot non-degenerate matter satisfy the gamma law (3) at large densities [see, e.g., THORNE (1967) for references and examples of such equations of state].

We begin the derivation of our stability criterion by focusing attention upon a spherical equilibrium configuration with central density $\rho_c \gg \rho_{lim}$. If we employ Schwarzschild coordinates to describe the geometry of spacetime, the metric assumes the familiar form

$$ds^2 = e^{\nu(r)}\,dt^2 - e^{\lambda(r)}\,dr^2 - r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) \qquad , \qquad (6)$$

and the non-linear effects of general relativity lead to the following peculiar structure (HTWW, chap. 5): In the central region of the configuration is a high-density core of radius $\approx r_{core}$, where

$$r_{core} \equiv \left[\frac{2\,(\Gamma_4 - 1)}{(\Gamma_4{}^2 + 4\Gamma_4 - 4)\,4\pi\rho_c}\right]^{\frac{1}{2}} \quad ; \qquad (7)$$

throughout the core the density, the pressure, and the metric component e^{ν} are approximately constant and equal to their values at the origin, $r = 0$. Surrounding the core is a mantle in which

$$p(r) \approx (\Gamma_4 - 1)\,\rho(r) \approx \frac{2\,(\Gamma_4 - 1)^2}{(\Gamma_4{}^2 + 4\Gamma_4 - 4)\,4\pi r^2}\left[1 + \frac{\mathcal{P}\,(\Gamma_4, r\sqrt{\rho_c})}{(r^2\rho_c)^a}\right] \quad , \qquad (8a)$$

$$m(r) \approx \frac{2\,(\Gamma_4 - 1)r}{(\Gamma_4{}^2 + 4\Gamma_4 - 4)}\left[1 + \frac{\mathcal{M}\,(\Gamma_4, r\sqrt{\rho_c})}{(r^2\rho_c)^a}\right] \quad , \qquad (8b)$$

$$e^{\nu(r)} \approx \text{constant} \times \rho^{-2(1 - 1/\Gamma_4)} \quad , \qquad (8c)$$

$$a = 3/4 - 1/(2\Gamma_4) \quad , \qquad (8d)$$

where $\mathcal{P}$ and $\mathcal{M}$ are certain oscillatory functions of $r\sqrt{\rho_c}$ (cf. HTWW), and where $m(r)$ is the total mass-energy within coordinate radius r,

$$m(r) \equiv (r/2)\left[1 - e^{-\lambda(r)}\right] = \int_0^r \rho(r)\,4\pi r^2 dr \quad . \qquad (9)$$

Extending from the outer edge of the mantle (that point at which $\rho = \rho_{lim}$) to the surface of the star is an envelope in which the structure is complicated, since the simple limiting form (3) for the equation of state is not valid there.

We seek to analyze the stability of the above configuration by studying the solutions for it of the stellar eigenequation governing adiabatic radial perturbations [CHANDRASEKHAR (1964a), Eq. (12); see also BARDEEN, THORNE, and MELTZER (1966), Eq. (3)]. MELTZER and THORNE (1966) have obtained approximate solutions of the stellar eigenequation in the core and mantle under the assumption that $\Gamma_1 = \Gamma_4 = 4/3$ there. It is a straightforward task to generalize their analysis to situations where $\Gamma_1 \neq \Gamma_4 \neq 4/3$. Such a generalization yields the following solution of the stellar eigenequation for the special case of zero pulsation frequency:

$$\frac{\delta r_0(r)}{r} \approx \begin{cases} [3/(kr)^2]\,[\cosh kr - \sinh kr/(kr)] & \text{in core} \\ A r^{-2a} \cos\,[b\, \ln\,(r/r_{core}) + B] & \text{in mantle} \\ \text{something much more complicated} & \text{in envelope.} \end{cases} \qquad (10)$$

Here A and B are constants, and

$$k \equiv [8\pi\Gamma_4\rho_c/\Gamma_1]^{\frac{1}{2}}, \qquad b \equiv [(8/\Gamma_1) - (3/\Gamma_4)(3/\Gamma_4 - 1) - 1/4]^{\frac{1}{2}}. \qquad (11)$$

The solution δr_0, if it satisfies the correct boundary conditions at the surface, $r = R$ (in general it does not), corresponds physically to the displacement associated with a radial mode which is in neutral equilibrium (i.e., a zero-frequency mode).

We are especially interested in the solution δr_0 because it is known that the number of unstable normal modes of radial oscillation is equal to the number of nodes which this solution possesses.* Notice from Equation (10) that, to our level of approximation, the condition

$$b \; \ell n \; (r/r_{core}) + B \leq \pi/2 \tag{12}$$

must be satisfied if δr_0 is to have no nodes in the mantle or core at values of the radial coordinate less than r. We determine the phase factor B by demanding that the logarithmic derivatives of the core and mantle solutions (10) match at the common boundary, $r = r_{core}$, of the core and mantle. Thus we obtain

$$B = \tan^{-1}\left\{\frac{1}{b}\left[\frac{3}{2} + \frac{1}{\Gamma_4} - \frac{kr_{core} \sinh kr_{core}}{\cosh kr_{core} - \sinh kr_{core}/(kr_{core})}\right]\right\} \quad . \tag{13a}$$

For $\Gamma_1 \gtrsim \Gamma_4$ (the region of interest to us since Γ_{crit} turns out to be greater than Γ_4) the value of the last term in braces is always sufficiently close to 3 that we can say

$$B \approx \tan^{-1} \left[- (1/b)(3/2 - 1/\Gamma_4)\right] \quad . \tag{13b}$$

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*See BARDEEN (1965) [reviewed by BARDEEN, THORNE, and MELTZER (1966)], who points out that this statement is true actually only when the surface, r = R, is a singular point of the stellar eigenequation. If R is a regular point, the number of unstable modes is either n or n + 1, where n is the number of nodes in the solution δr_0. However, we shall ignore this distinction because, as will become evident, it does not introduce any significant error into our already-approximate analysis.

It is evident from Equations (10)-(13) that there are no nodes of δr_0 in the core, and that any node in the mantle moves outward to larger values of the radial coordinate as the value of Γ_1 in the core and mantle is increased. Once such a node reaches the common boundary of the mantle and envelope, it moves outward much more rapidly through the tenuous envelope as the core-mantle value of Γ_1 is increased further. The node eventually disappears at the surface for some core-mantle value of Γ_1, which is to a large extent <u>independent</u> of the precise manner in which Γ_1 varies in the envelope. Therefore, since we shall assume that condition (12) (with b constant and equal to its core-mantle value) is approximately valid out to the surface, we shall tend to overestimate the critical core-mantle value, Γ_{crit}, of Γ_1 for which δr_0 possesses one node only, and for which that node is located at the surface. However, the error thus introduced will be slight in most cases, since the radial coordinate enters only logarithmically in condition (12). According to Bardeen's node-counting theorem, the stellar model is stable if and only if the core-mantle value of Γ_1 exceeds this critical value (however, in this connection recall the preceeding footnote). Consequently, we can combine Equations (12) and (13), and thereby complete the derivation of our approximate criterion for stability. <u>QED</u>.

In the ultra-relativistic limit, Equations (5) imply that

$$\lim_{\rho_c \to \infty} \Gamma_{crit} = 8 \left[\frac{3}{\Gamma_4}\left(\frac{3}{\Gamma_4} - 1\right) + \frac{1}{4}\right]^{-1} \quad . \qquad (14)$$

Since the speed of sound, β_{sound}, is given by the relation (CURTIS, 1950)

$$\beta^2_{sound} = (\partial p/\partial \rho)_{constant\ entropy} = \Gamma_1 p/(\rho + p) \quad , \qquad (15)$$

it follows from Equation (3) and the limit (14) that

$$\beta^2_{\text{sound}} > \frac{8(\Gamma_4 - 1)}{3(3/\Gamma_4 - 1) + \Gamma_4/4} \tag{16}$$

in the core and mantle if a model is stable in the limit $\rho_c \to \infty$. If $\Gamma_4 \gtrsim 1.45$ the inequality (16) yields a speed of sound greater than the speed of light (= 1 in our units), which is unphysical. Hence if $\Gamma_4 \gtrsim 1.45$ the principle of causality alone places an upper limit upon the central density of a stable model. Actually, for any value of $\Gamma_4 \geq 1$, the ratio Γ_{crit}/Γ_4 approaches a limit sufficiently greater than unity such that it is perhaps difficult to imagine a realistic situation where $\Gamma_1 \geq \Gamma_{crit}$ at ultrahigh densities.

In order to examine the accuracy of our approximate criterion for stability, we have applied it to the cold Harrison-Wakano-Wheeler stellar configurations which are governed by the Harrison-Wheeler equation of state (HTWW, chap. 10). We exhibit the results in Figure 1, where we have collected the configurations in the usual way into a sequence parameterized by the central density, ρ_c. For comparison with our approximate formula for Γ_{crit}, we have calculated the exact value of Γ_{crit} for each configuration by setting Γ_1 equal to a constant throughout the configuration, and by numerically integrating the stellar eigenequation for zero pulsation frequency. For densities $\rho \gtrsim 10^{16}$ g/cm^3, the Harrison-Wheeler equation of state approximates the gamma law (3) with $\Gamma_4 = 4/3$. Although the peculiar core-mantle structure used in our derivation is not fully developed until $\rho_c \gtrsim 10^{18}$ g/cm^3, Figure 1 shows that, for ρ_c as small as 10^{17} g/cm^3, the stability criterion (5) predicts values for Γ_{crit} which agree to within

$\sim$ 5% with the exact values of Γ_{crit} obtained numerically. This agreement, which improves rapidly with increasing ρ_c, is modified hardly at all if Γ_1 is made to vary in some arbitrary way in the envelope.

3. High-Density Spiraling of Binding-Energy Versus Radius Curves for Isentropic Sequences

It has been known for some time that the strength of the relativistic instability is reflected in a peculiar behavior of mass versus radius curves for certain sequences of cold stars. This peculiar behavior was discovered independently by DMITRIEV and HOLIN (1963), HARRISON (1965), and WHEELER (HTWW, chap. 5) and may be summarized as follows: Consider a family of spherical stellar models governed by a unique equation of state $\rho(p)$ which approaches the form (3) at large densities. The members of such a family can be arranged into a sequence parameterized by the central density, ρ_c. (An example of such a sequence is the family of cold Harrison-Wakano-Wheeler configurations associated with Fig. 1.) For such a sequence construct a curve of mass M versus radius R (M upward, R to the right), parameterized by central density, ρ_c. As ρ_c increases, this M(R) curve asymptotically undergoes a high-ρ_c, counterclockwise spiral into a limit point, which corresponds to the configuration with infinite central density. This asymptotic behavior is intimately connected with the stability of the highly relativistic members of the sequence in the following way (see, e.g., THORNE, 1967, Sections 4.2.1 and 4.2.2): As the central density increases through each extremal point (each maximum and minimum of M) along the counterclockwise spiral, a normal mode of radial oscillation with adiabatic index

$$\Gamma_1 = \frac{\rho + p}{p} \frac{dp/dr}{d\rho/dr} \tag{17}$$

becomes unstable. Hence more and more modes become unstable as ρ_c increases.

Numerical studies by BARDEEN (1965), TOOPER (1965), and others have revealed that the M(R) curves for certain sequences of hot stellar models exhibit analogous high-ρ_c spirals. However, the existence and form of the spiraling has been explained analytically only for sequences of cold models.

We wish to show that high-ρ_c spiraling must exist for certain hot sequences because of the presence of an increasing number of unstable modes. We consider a one-parameter sequence of spherical stellar models for which (i) the entropy per baryon, s, is radially invariant in each model (isentropic configuration); (ii) the models all have the same radially invariant fractional nuclear abundances and the same total number of baryons (and hence also the same total rest mass); (iii) the matter from which the configurations are made is described by equations of state for the density of mass-energy, ρ, and the density of rest mass, ρ_0,

$$\rho = \rho(p,s) \quad , \quad \rho_0 = \rho_0(p,s) \quad , \tag{18}$$

which assume the limiting forms

$$\rho = p/(\Gamma_4 - 1) \quad , \quad \rho_0 = K(s)p^{1/\Gamma_4} \tag{19}$$

for large ρ, where Γ_4 is a constant and K(s) is an unspecified function of the entropy per baryon, s. Such a family of models can be parameterized by s or, more conveniently, by the central density, ρ_c. Stellar configurations with the above properties have been studied extensively, especially in connection with the theory of supermassive stars [see FOWLER (1966), also

THORNE (1967) chaps. 5 and 6 for reviews and references].

We shall show that the M(R) curve for the above sequence of hot isentropic models undergoes the high-ρ_c spiral

$$-(E_B - E_{B\infty}) = M - M_\infty = C_M \rho_c^{-a} \sin\left[(b'/2)\ \ln \rho_c + \delta_M\right], \qquad (20a)$$

$$R - R_\infty = C_R \rho_c^{-a/\ell(s_\infty)} \sin\left[(b'/2)\ \ln \rho_c + \delta_R\right] \quad . \qquad (20b)$$

Here $E_{B\infty}$, M_∞, and R_∞ are the binding energy, mass, and radius of the member of the sequence with infinite central density, and s_∞ is its entropy per baryon; the constants C_M, C_R, δ_M, and δ_R satisfy

$$C_M C_R \sin(\delta_M - \delta_R) > 0 \quad , \qquad (20c)$$

so that the spiraling in a curve of M up,R to the right is clockwise; the constant a is defined by Equation (8d), b' is defined by

$$b' = (-9/\Gamma_4{}^2 + 11/\Gamma_4 - 1/4)^{\frac{1}{2}} \quad , \qquad (20d)$$

and $\ell(s) > 0$ is defined by the behavior

$$\rho(r) \xrightarrow[r \to R_-]{} (R - r)^{\ell(s)} \quad , \qquad (20e)$$

which the unspecified equations of state (18) impose upon the density near the surface.

We begin the derivation of Equations (20) by noticing that Equations (7) and (11) imply that the number of nodes of the zero-eigenvalue solution (10) in the mantle of a high-density configuration increases by 1 for approximately each increase of the central density, ρ_c, by the amount

$$\Delta \ln \rho_c = 2\pi(-9/\Gamma_4{}^2 + 11/\Gamma_4 - 1/4)^{-\frac{1}{2}} \quad , \qquad (21)$$

if $\Gamma_1 = \Gamma_4$. And Equation (19) yields $\Gamma_1 \approx \Gamma_4$ for our isentropic sequence in the high-density limit if it is assumed that the adiabatic index is given by Equation (17). Hence it follows from BARDEEN'S (1965) node-counting theorem that in the high-density limit an additional radial mode with adiabatic index (17) becomes unstable for approximately each increase in ρ_c by the amount (21). BARDEEN (1965) has also proved the following theorem for our isentropic sequence: If the M(R) curve bends in the clockwise direction as ρ_c increases through a maximum or minimum of M, a radial mode with adiabatic index (17) becomes unstable there. Otherwise a mode becomes stable. Further, a mode can change stability at no other point. Since modes become unstable at the rate (21), this theorem enables us to conclude that in the high-ρ_c limit, the M(R) curve bends continuously in a clockwise fashion, with successive maxima and minima separated by the amount (21).

To complete our derivation of Equations (20) we compare the structure of a high-ρ_c member of our isentropic sequence with that of the limiting member of the sequence, which has infinite central density, ρ_c. Notice that the structure of the infinite-ρ_c configuration, from the center of symmetry to the outer edge of the mantle, is described by the limiting forms of Equations (8) and of the equation

$$m_0(r) \approx \frac{K(s)(4\pi)^{1-1/\Gamma_4}\left[2(\Gamma_4 - 1)^2\right]^{1/\Gamma_4} r^{3-2/\Gamma_4}}{(\Gamma_4{}^2 + 4\Gamma_4 - 4)^{1/\Gamma_4 - \frac{1}{2}}(3\Gamma_4 - 2)}\left[1 + \frac{\mathcal{M}_0(\Gamma_4, r\sqrt{\rho_c})}{(r^2\rho_c)^a}\right] , \qquad (22)$$

for $\rho_c \to \infty$, $s \to s_\infty$. Here $m_0(r)$ is the rest mass within radius r, and $\mathcal{M}_0$ is an oscillatory function of r/ρ_c; we use the subscript "∞" to denote the value of a quantity for the infinite-ρ_c configuration. Equation (22), which is valid only in the mantle of a high-ρ_c configuration, can be derived from Equations (8a,b), (19) and the expression

$$m_0(r) = \int_0^r \rho_0(r) e^{\lambda/2} \, 4\pi r^2 dr \quad . \tag{23}$$

The infinite-ρ_c configuration must contain the same total rest mass, M_0, as does each of the other members of the sequence, and this requirement uniquely determines its entropy per baryon, s_∞, which enters into the equations of state (18).

Consider the differences $\Delta p(r) \equiv p(r) - p_\infty(r)$, etc., between the values of various quantities at corresponding radii of a particular high-ρ_c configuration and the infinite-ρ_c configuration. Equations (8) imply that

$$\Delta p(r) \propto \Delta\rho(r) \propto \Delta m(r) \propto \rho_c^{-a}$$

in the mantle. That these quantities $\propto \rho_c^{-a}$ also in the envelope, and that $\Delta\rho_0(r) \propto \Delta m_0(r) \propto \rho_c^{-a}$ in the mantle and envelope, are guaranteed by the following self-consistent argument: First assume (incorrectly) that $\Delta s \equiv s - s_\infty = 0$, which implies through Equations (19) and (22) that $\Delta\rho_0(r) \propto \Delta m_0(r) \propto \rho_c^{-a}$ in the mantle. For $\Delta s = 0$, the equations of structure,

$$\begin{aligned} dp/dr &= -(\rho + p)(m + 4\pi r^3 p)/[r(r - 2m)] \quad , \\ dm/dr &= 4\pi r^2 \rho \quad , \\ dm_0/dr &= 4\pi r^2 e^{\lambda/2} \rho_0 \quad , \end{aligned} \tag{24}$$

may be combined with the equations of state (18) to obtain a closed set of first order "perturbation" equations in Δp, $\Delta\rho$, $\Delta\rho_0$, Δm, and Δm_0 which guarantee that these differences are of order ρ_c^{-a} in the envelope if they are of order ρ_c^{-a} in the mantle.* (The proof is essentially identical to that given in Chapter 5 of HTWW for certain one-parameter sequences of zero-temperature stars, so we shall not reproduce it here.) For the hot stellar models of physical interest, ρ (and hence also ρ_0) drops to zero at the surface, R,

$$\rho \xrightarrow[r \to R_-]{} (R - r)^{\ell(s)} \quad , \qquad (25)$$

where the exponent, $\ell(s)$, is determined by the behavior of the equations of state (18) at low densities. Since $\Delta\rho \propto \rho_c^{-a}$ in the envelope if $\Delta s = 0$, the limit (25) implies that $\Delta R \equiv R - R_\infty \propto \rho_c^{-a/\ell(s_\infty)}$ in such a case. It then follows from Equation (23) that the difference between the rest masses of the two configurations is

$$\Delta M_0 \equiv m_0(R) - m_{0\infty}(R_\infty)$$
$$= m_0(R) - m_{0\infty}(R) - \int_R^{R_\infty} \rho_{0\infty}(r) e^{\lambda_\infty(r)/2} \, 4\pi r^2 dr \propto \rho_c{}^{-a} \quad . \qquad (26)$$

---

*For simplicity, the analyses of this section tacitly assume that the functional relationships (18) are well behaved. However, one can show that the final result, namely, the behavior described by Equations (20), follows from less stringent conditions. For example, one can show that Equations (20) remain valid if the equations of state (18) allow a phase transition, i.e., if ρ experiences a finite jump as p passes some critical value. One need only demand that the magnitude of this jump and the critical value of p be smooth functions of s.

[The first two terms in the final expression give $\Delta m_0(R)$, which is guaranteed to be $\propto \rho_c^{-a}$; and the remaining integral is $\propto \rho_c^{-a(1+1/\ell)}$.] Thus if $\Delta s = 0$, ΔM_0 is of order ρ_c^{-a} but is in general not equal to zero. However, we are demanding that all the members of our isentropic sequence possess the same total rest mass, M_0, i.e., that $\Delta M_0 = 0$. Now one ineluctably deduces from Equations (18), (19), and (24) that if the value of s for the high-ρ_c configuration is varied by some small amount ϵ, the values of p, ρ, and m in the envelope, and the values of ρ_0 and m_0 in all regions, are changed by amounts $\propto \epsilon$. Hence M_0 is changed by an amount $\propto \epsilon$. Consequently, since $\Delta M_0 \propto \rho_c^{-a}$ if $\Delta s = 0$, the correct value of Δs (the value for which $\Delta M_0 = 0$) $\propto \rho_c^{-a}$. And the values of Δp, $\Delta \rho$, $\Delta \rho_0$, Δm, and Δm_0 in the mantle and envelope for the correct value of Δs cannot differ from their values for $\Delta s = 0$ by amounts of order $> \rho_c^{-a}$. But the values for $\Delta s = 0$ are of order ρ_c^{-a}. Thus we have succeeded in showing that

$$\Delta p(r) \propto \Delta \rho(r) \propto \Delta \rho_0(r) \propto \Delta m(r) \propto \Delta m_0(r) \propto \rho_c^{-a} \tag{27}$$

in the mantle and envelope of a high-ρ_c member of our isentropic sequence.

We can now combine Equations (25) and (27) in a straightforward fashion to show that

$$\Delta R \equiv R - R_\infty \propto \rho_c^{-a/\ell(s_\infty)} \quad . \tag{28a}$$

Finally, Equations (9), (27), and (28a) imply that the difference between the masses of a high-ρ_c configuration and the infinite-ρ_c configuration is

$$\Delta M \equiv M - M_\infty = \left[\Delta m(R) - \int_R^{R_\infty} \rho_\infty(r)\, 4\pi r^2 dr\right] \propto \rho_c^{-a} \quad . \tag{28b}$$

Combining Equations (28) with the previously-obtained result that in the high-ρ_c limit the M(R) curve for our isentropic sequence bends continuously in a clockwise fashion with extremal points separated by the amount (21), we conclude that the M(R) curve undergoes the high-ρ_c spiral described by Equations (20). Since all of the members of the sequence possess the same rest mass, a plot of minus the binding energy ($-E_B = M - M_0$) versus radius exhibits the same behavior.

This spiraling is evident in Figure 2, where we exhibit the $-E_B(R)$ curve for the adiabatic polytropes of index 2, which are governed by the equations of state

$$\rho_0 = K(s)p^{2/3}, \qquad \rho = \rho_0 + 2p \ . \tag{29}$$

Here K depends upon only the radially invariant entropy per baryon, s. At high densities Equations (29) approach the limiting forms (19) with $\Gamma_4 = 3/2$. By combining the first of Equations (24) with Equations (29), one can show that $\ell = 2$ in Equation (20e) for all of these models. In Figure 2 we follow BARDEEN (1965) (reviewed by THORNE, 1967, Section 5.1) and employ M_0, the rest mass of a model, as the unit of length in terms of which all other structural quantities are measured.

4. Conclusion

In this paper we have added two items to the current body of knowledge concerning the structure and stability of relativistic stars. These items reflect the strength of the relativistic instabilities in ultrarelativistic stars; and they indicate the difficulty involved in constructing stable stellar models with arbitrarily large central densities.

I express my thanks to Professor Kip S. Thorne for suggestions and advice concerning the work reported here, much of which was performed while I was a National Science Foundation Predoctoral Fellow.

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Figure Captions

Fig. 1. The critical adiabatic index, Γ_{crit}, for the Harrison-Wakano-Wheeler stellar configurations. The configurations are parameterized by the central density, ρ_c. Two curves are given for Γ_{crit} (ρ_c): The exact, correct (under the assumption that Γ_1 is constant throughout each configuration) curve as obtained by numerical integrations (curve labeled "exact"); and the curve given by our approximate formula (5) (curve labeled "approximate"). The approximations which went into the derivation of formula (5) are valid, for H-W-W configurations, only at central densities $\rho_c \gtrsim 10^{18}$ g/cm^3; and this figure shows the formula itself to be very accurate in this range.

Fig. 2. The curve of minus the binding energy versus radius for the adiabatic polytropes of index 2. The curve is parameterized by the central density of total mass-energy in units of M_0^{-2}, where M_0 is the rest mass of a model. Plotted horizontally is the radius $R \equiv$ (surface area of model/$4\pi)^{\frac{1}{2}}$ in units of $2M_0$ ($\approx$ gravitational radius of model). Plotted vertically is minus the binding energy, $- E_B$, in units of M_0. At each peak or valley of the curve a normal radial mode becomes unstable. The curve undergoes a high-density spiral which is described by Equations (20) with $\Gamma_4 = 3/2$ and $\ell = 2$.

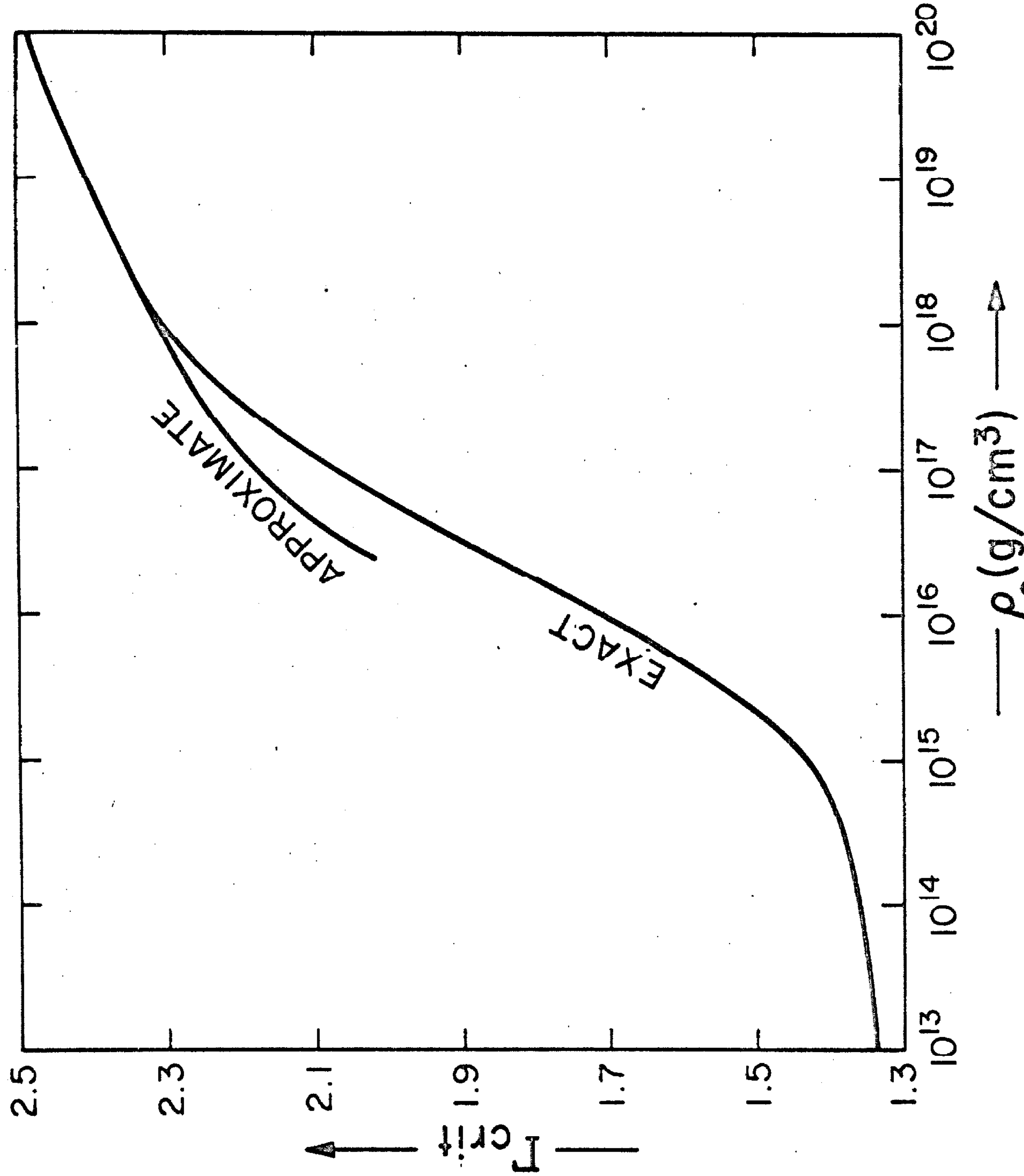
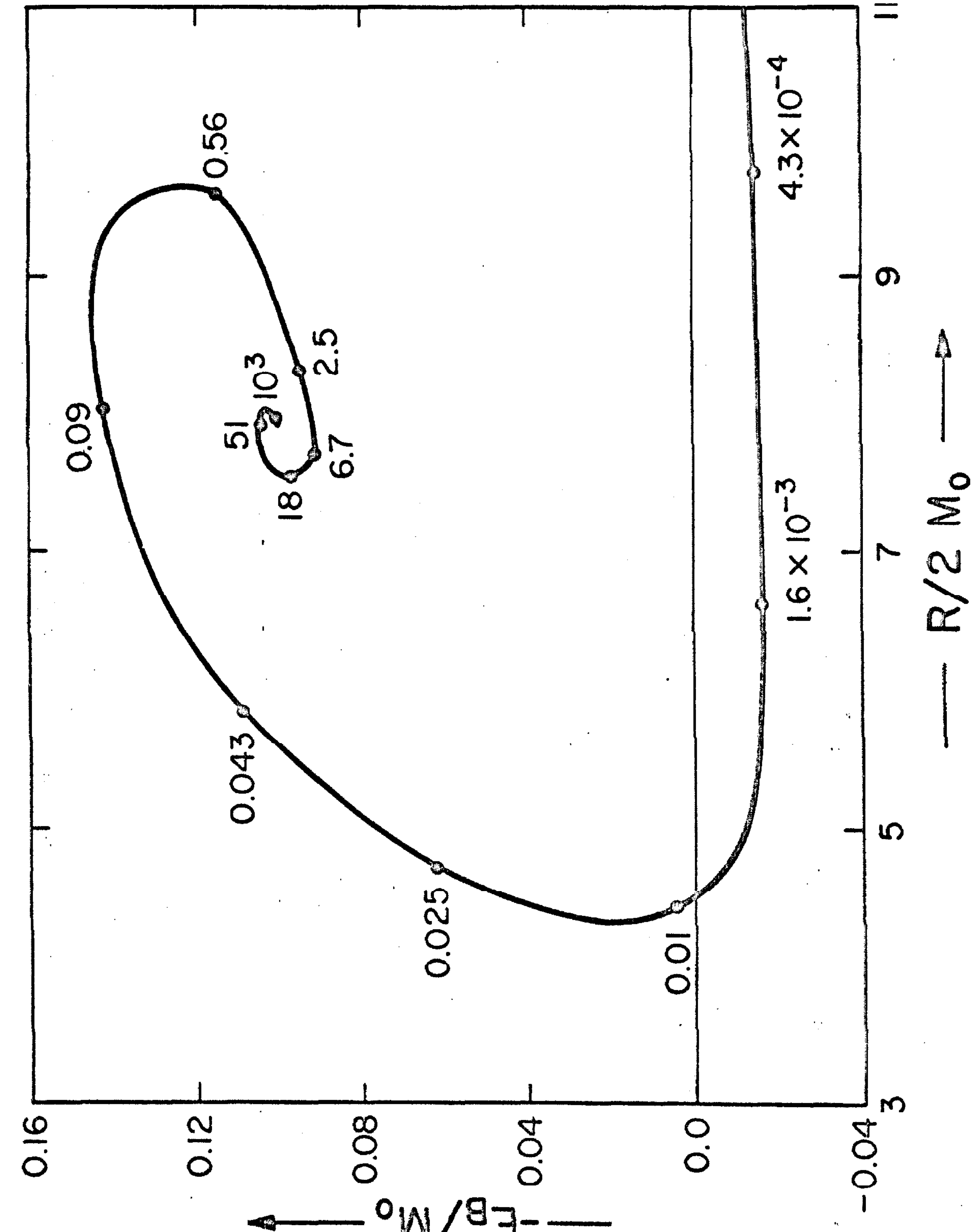

APPROXIMATE
EXACT
2.5
2.3
2.1
1.9
1.7
1.5
1.3
Γ_{crit}
10^{13}
10^{14}
10^{15}
10^{16}
10^{17}
10^{18}
10^{19}
10^{20}
$\rho_c (g/cm^3)$

Fig. 1



0.16
0.12
0.08
0.04
0.0
−0.04
3
5
7
9
11
— R/2 M$_0$ →
— $-E_B/M_0$ →
0.01
0.025
0.043
0.09
0.56
2.5
10^3
51
18
6.7
1.6×10^{-3}
4.3×10^{-4}

Fig. 2

5. Relativistic, Spherically Symmetric Star Clusters.
I. Stability Theory for Radial Perturbations

(Co-authored by Kip S. Thorne; published in Astrophysical Journal, 154, 251 [1968])

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RELATIVISTIC, SPHERICALLY SYMMETRIC STAR CLUSTERS
I. STABILITY THEORY FOR RADIAL PERTURBATIONS*

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ABSTRACT

There is some indication that very dense star clusters might play important roles in quasi stellar sources and in the nuclei of certain galaxies. The roles of such star clusters should be strongly influenced by a relativistic instability, which sets in when a cluster surpasses a certain critical density. In this paper the groundwork is laid for the study of that instability: the theory of small, radial perturbations of a spherically symmetric star cluster is developed within the framework of general relativity. The cluster is idealized as a solution to the collisionless Boltzmann-Liouville equation (an idealization which should be valid on the short time scale associated with the relativistic instability). The equation of motion governing small radial perturbations is derived and is shown to be self-conjugate. From the equation of motion follows a variational principle for the normal modes, which provides a necessary and sufficient condition for the stability of the cluster. Also presented are (1) the corresponding Newtonian analysis, much of which has been developed previously by Antonov and by Lynden-Bell, (2) the relationship between the Newtonian and relativistic analyses, and (3) necessary and sufficient conditions for the existence of a zero-frequency mode of radial motion.

I. MOTIVATION

Between 1964 and 1967 it was generally believed that the redshifts of quasi-stellar sources (QSSs) could not possibly be gravitational in origin. (One of us—K. S. T.—was a particularly firm proponent of this view.) Not only are there difficulties with the sharpness of the spectral lines in a gravitational redshift model (Greenstein and Schmidt 1964); there is also an absolute upper limit of $z < 0.63$ on the redshift of light from the surface of any non-rotating equilibrium configuration of perfect fluid with reasonable equation of state and density distribution (Bondi 1964), and this limit probably cannot be changed much by angular velocities which are compatible with the sharpness of the emission lines. (Large angular velocities are not permitted because of Doppler broadening.)

However, in early 1967 Hoyle and Fowler (1967) revived the gravitational redshift hypothesis by introducing a new model which may circumvent both of the above difficulties: they suggested that each QSS might rest at the center of a very massive relativistic star cluster and might derive its redshift from the gravitational field of the cluster. One can, indeed, construct star-cluster models in which sharp spectral lines and large gravitational redshifts are produced. However, one does not know today whether such star clusters are stable against gravitational collapse.

There is good reason to fear that star clusters with central redshifts as large $z \approx 2$ might be unstable against collapse. In Newtonian theory the stabilities of collisionless, spherical star clusters and of gas spheres are somewhat related (Lynden-Bell 1966; see also § III*g* of this paper), and a similar relationship seems likely in general relativity. Chandrasekhar (1964) showed that when a gas sphere (star) of given mass contracts beyond a certain critical point, it becomes unstable against gravitational collapse. This

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instability, which is catalyzed by general-relativistic effects, has been studied in great detail since it was first discovered (see Thorne 1967 for a review). An examination of a variety of relativistic gas spheres as computed, e.g., by Tooper (1965 and private communication) and by us (unpublished) reveals this: a contracting gas sphere becomes unstable against collapse when the redshift from its center exceeds a limit which is typically $z_{\text{central}} \sim 1$. It is not unlikely that the critical redshifts for spherical star clusters will be similar in magnitude, but we cannot know until the theory of pulsating star clusters has been developed fully.

There is another motivation for studying the relativistic instability in star clusters: two independent lines of investigation have suggested recently that, when a *Newtonian* star cluster contracts beyond a certain critical density—one far less than the density for relativistic instabilities—it may become unstable against a "thermal runaway." In this thermal runaway the cluster gradually develops a dense core and a diffuse envelope. For some clusters of astrophysical interest (e.g., the compact nuclei of certain galaxies), the core evolves toward ever higher densities on a time scale which may be short compared with 10^{10} years but which is very long compared with the time scale for the relativistic instability ($\lesssim 1$ year). The evidence which suggests that a thermal runaway may occur comes (1) from dynamical computer experiments on the many-body gravitational problem (Arseth 1963; see also Hénon 1961, 1965) and (2) from analytic studies of the configurations of maximal entropy for a star cluster inclosed in a spherical cavity (Antonov 1962; Lynden-Bell and Wood 1968).

One is invited to speculate that the star densities in the nuclei of some galaxies (and in potential QSSs) may exceed the critical density for thermal runaway, that runaway may occur, and that the nuclei may thereby evolve in times $t < 10^{10}$ years to such high densities that relativistic effects become important and collapse sets in. Indeed, the outbursts which occur in the nuclei of galaxies might conceivably be associated with the onset of collapse or with encounters between an already collapsed nucleus and surrounding stars.

The above discussion of models for QSSs and of outbursts in galaxies is necessarily very speculative. Before these speculations can be analyzed with confidence, we must understand, among many other things, the onset of the relativistic instability in star clusters. This paper is the first of several in which we shall attempt to delineate the theory of the stability of relativistic star clusters and thereby contribute to the tools needed for studying dense galactic nuclei and QSS models.

II. SUMMARY

In relativistic gas spheres the time scale for the growth of the relativistic instability is roughly the sound travel time across the sphere. Similarly, in star clusters one expects the time scale to be roughly the star travel time across the cluster, which—for the clusters that interest us—is short compared with the mean time between close stellar encounters. Consequently, in discussing cluster stability, we shall idealize the cluster as a statistical distribution of mass points, which interact only through the smoothed-out gravitational field of the entire cluster. The mathematical formalism used in such a treatment is relativistic kinetic theory (Synge 1934; Walker 1936; Tauber and Weinberg 1961; Lindquist 1966). The cluster is described by a density in phase space and by a metric for the curvature of spacetime. The density in phase space determines a stress-energy tensor, which generates the metric through the Einstein field equations; the metric in turn determines the density in phase space via the collisionless Boltzmann-Liouville equation.

This type of statistical treatment of star clusters has been used in Newtonian theory for about fifty years. However, only very recently (Antonov 1960; Lynden-Bell 1966; Milder 1967) has the collisionless stability of Newtonian clusters been investigated, and those Newtonian investigations of stability have all been of a formal nature: no applications have been made yet to specific models for clusters. In general relativity the structures of spherical star clusters have been investigated recently by Zel'dovich and Podurets (1965) and by Fackerell (1966, 1968*a*–*c*), but no treatment of their stability has been attempted.[1]

In this paper we shall treat the stability of star clusters by means of a relativistic analysis which is patterned after the Newtonian analysis of Antonov (1960). However, Antonov made two restrictions in his Newtonian analysis which we do not wish to make: he assumed that the stars in his cluster all had identical masses, and he assumed that the number density in phase space for the equilibrium configuration depends only on energy. Before presenting our relativistic treatment of stability, we shall redo the Newtonian treatment, dropping Antonov's restrictions but imposing in their place the demand that both the equilibrium and the perturbed configurations be spherically symmetric. We shall also extend the Newtonian analysis somewhat beyond that of Antonov: in addition to obtaining his stability criterion, we shall derive a variational principle (action principle) for the pulsation of the cluster; we shall obtain from our variational principle a conserved quantity for arbitrary radial pulsations; and we shall derive an elegant, new criterion for the existence of a zero-frequency mode of motion. All of this Newtonian discussion is found in § III.

In § IV we shall use the Newtonian analysis as a guide in developing the corresponding relativistic analysis. All the Newtonian results will be generalized to relativity theory except Lynden-Bell's relationship between the stabilities of star clusters and gas spheres: we shall obtain (1) a self-conjugate equation of motion for the small spherical pulsations of a spherical cluster, (2) an action principle for the pulsations, (3) a variational principle for the normal modes, which is also a necessary and sufficient condition for stability, (4) a conserved quantity analogous to pulsational energy, and (5) an elegant criterion for the existence of a zero-frequency mode.

Throughout this paper we adopt the mathematical conventions of Thorne (1967), including the use of "geometrized units" in which the speed of light, c, Newton's gravitational constant, G, and Boltzmann's constant, k, are equal to unity. Also, we number the equations in a manner designed to bring out the close relationship between the Newtonian and relativistic analyses; for example, the relativistic equation (12;R) has as its Newtonian limit equation (12;N).

III. NEWTONIAN THEORY OF STABILITY

a) *Equations of Stellar Dynamics*

In Newtonian theory the density of stars in phase space, which we denote by $\mathfrak{N}$, is defined as follows: At a particular time t an observer concentrates his attention on a particular volume $d\mathcal{V}_x$ in physical space and a particular volume $d\mathcal{V}_p$ in momentum space. In a Cartesian coordinate system these volumes are

$$d\mathcal{V}_x = dxdydz\,, \quad d\mathcal{V}_p = dp^x dp^y dp^z dm\,, \tag{1;N}$$

where m is the rest mass of a star and $p^j = m dx^j/dt$. If the observer sees dN stars in the volume $d\mathcal{V}_x d\mathcal{V}_p$ at time t, then the number density in phase space ("distribution function") is

$$\mathfrak{N} \equiv dN/d\mathcal{V}_x d\mathcal{V}_p = dN/(dxdydzdp^x dp^y dp^z dm)\,. \tag{2;N}$$

The density $\mathfrak{N}$ is a function of time, t, and of location (x^j, p^j) in the seven-dimensional phase space.

[1] Zel'dovich and Podurets (1965) and Zel'dovich and Novikov (1967, § 11.19) argued without proof that one should be able to diagnose the stability of isothermal, relativistic star clusters from binding-energy considerations; but the discussion presented in § IV*f* of this paper makes that seem highly improbable.

The smoothed-out gravitational field of the star cluster is described by the Newtonian gravitational potential, $\Phi(t,x,y,z)$. The distribution function determines Φ through the source equation

$$\nabla^2\Phi = 4\pi\rho\,, \quad \rho = \int m\mathfrak{N}d\mathcal{V}_p\,, \tag{3a;N}$$

which has the solution

$$\Phi(t,x) = -\int \frac{m'\mathfrak{N}(t,x',p',m')}{|x - x'|}\, d\mathcal{V}_{x'}d\mathcal{V}_{p'}\,. \tag{3b;N}$$

The gravitational field determines the distribution function through the collisionless Boltzmann-Liouville equation (or simply "Liouville equation")

$$\mathfrak{D}\mathfrak{N} = 0\,. \tag{4;N}$$

Here $\mathfrak{D}$, the Liouville operator, is differentiation with respect to time along the path of a star in phase space. In a Cartesian coordinate system, $\mathfrak{D}$ is given by

$$\mathfrak{D} = \frac{\partial}{\partial t} + \frac{dx^i}{dt}\frac{\partial}{dx^i} + \frac{dp^i}{dt}\frac{\partial}{\partial p^i} + \frac{dm}{dt}\frac{\partial}{\partial m} = \frac{\partial}{\partial t} + \frac{p^i}{m}\frac{\partial}{\partial x^i} - m\frac{\partial\Phi}{\partial x^i}\frac{\partial}{\partial p^i}\,. \tag{5;N}$$

(We sum over repeated indices unless otherwise indicated.)

Equations (3;N) and (4;N), which couple Φ and $\mathfrak{N}$, are the fundamental equations of Newtonian stellar dynamics.

b) Spherical Equilibrium Configurations

In stellar dynamics an equilibrium configuration is one for which the distribution function and the gravitational field are independent of time. From the Liouville equation (4;N) one readily verifies that a Newtonian star cluster is in equilibrium if and only if $\mathfrak{N}$ is a function of the integrals of the motion of the stars ("Jeans's theorem"; see, e.g., Ogorodnikov 1965). For spherically symmetric equilibrium configurations there are five independent integrals of the motion: the stellar mass m, the energy E, the total angular momentum J, and the two angles which determine the (conserved) plane of the orbit. Of these, $\mathfrak{N}$ can depend only on m, E, and J, since a dependence on the plane of the orbit would lead to a non-spherical mass density and thence to a non-spherical gravitational field (cf. eq. [3a;N]). Consequently, the distribution function and the gravitational potential have the form

$$\mathfrak{N} = F(m,E,J)\,, \quad \Phi = \Phi(r) = \Phi[(x^2 + y^2 + z^2)^{1/2}]\,, \tag{6a;N}$$

where

$$E = (p^i)^2/2m + m\Phi(r)\,, \quad J = |x \times p|\,. \tag{6b;N}$$

When F is independent of J, the cluster has an isotropic velocity distribution at each point in space.

Equilibrium configurations for spherical star clusters have been studied extensively during the last fifty years. (See Ogorodnikov 1965 for references.) However, none of the models constructed have ever been tested for collisionless stability.

c) Equations of Motion for a Perturbed Spherical Cluster

Consider a particular spherically symmetric equilibrium configuration described by the distribution function $\mathfrak{N} = F(m,E,J)$ and by the gravitational potential $\Phi = \Phi_A(r)$. Perturb the equilibrium configuration slightly without destroying its spherical symmetry. The perturbed configuration can be described by a gravitational potential and a distribution function of the forms

$$\Phi(t,x^j) = \Phi_A(r) + \Phi_B(t,r) , \qquad (7;N)$$

$$\mathfrak{N}(t,x^j,p^j,m) = F(m,E_A,J) + f(t,x^j,p^j,m) , \qquad (8;N)$$

where

$$E_A = (p^j)^2/2m + m\Phi_A , \quad J = |x \times p| . \qquad (9;N)$$

Notice that f is the perturbation in the distribution function at a fixed point in space x, for mixed momentum p, and for fixed rest mass m; i.e., it is an "Eulerian perturbation" in phase space.

Throughout this paper, as above, the subscript A will refer to quantities in the unperturbed cluster, and a subscript B will refer to perturbations in those quantities accurate to first order in the amplitude of the motion. Our treatment of stability will not be carried beyond the first order.

The distribution function, $\mathfrak{N}$, for the perturbed cluster must satisfy the Liouville equation (4;N). When the Liouville equation is linearized in the perturbation functions Φ_B and f, it takes the form

$$\partial f/\partial t + \mathfrak{D}_A f - F_E p^r \partial\Phi_B/\partial r = 0 . \qquad (10;N)$$

Here F_E stands for

$$F_E \equiv (\partial F/\partial E_A)_{m,J} , \qquad (11;N)$$

and $\mathfrak{D}_A$ is the Liouville operator of the unperturbed cluster,

$$\mathfrak{D}_A = \frac{p^j}{m}\frac{\partial}{\partial x^j} - m\frac{\partial\Phi_A}{\partial x^j}\frac{\partial}{\partial p^j} . \qquad (12;N)$$

The derivation of the perturbed Liouville equation (10;N) follows.

The full Liouville equation states:

$$\mathfrak{D}\mathfrak{N} = [\,\partial/\partial t + \mathfrak{D}_A - m\,(\partial\Phi_B/\partial x^j)\,(\partial/\partial p^j)\,][F+f] = 0 . \qquad (13a;N)$$

Linearizing in f and Φ_B, and subtracting the zero-order Liouville equation, we obtain

$$\partial f/\partial t + \mathfrak{D}_A f - m(\partial F/\partial p^j)_{x^j,p^j,m}(\partial\Phi_B/\partial x^j) = 0 . \qquad (13b;N)$$

Since Φ_B depends only on t and $r = |x|$, we have

$$\begin{aligned} m(\partial F/\partial p^j)_{x^j,p^j,m}\partial\Phi_B/\partial x^j &= m(\partial F/\partial p^r)_{x^j,p^j,m}\partial\Phi_B/\partial r \\ &= mF_E(p^r/m)\partial\Phi_B/\partial r . \end{aligned} \qquad (13c;N)$$

By combining equations (13b,c;N), we obtain equation (10;N). Q.E.D.

The perturbed Liouville equation (10;N) must be supplemented by an equation for Φ_B in terms of f. From the linearity of equations (3;N) one readily sees that the required relation is

$$\nabla^2\Phi_B = 4\pi\int mf d\mathcal{V}_p , \qquad (14a;N)$$

which has the solution

$$\Phi_B(t,x) = -\int \frac{m'f(t,x',p',m')}{|x - x'|}\, d\mathcal{V}_{x'}d\mathcal{V}_{p'} . \qquad (14b;N)$$

Equations (10;N) and (14b;N)—or their analogues for his version of this analysis—are taken by Antonov (1960) to be the equations of motion of the perturbed cluster.[2] Unfortunately, one cannot readily obtain an analogue of equation (14b;N) in general relativity. In order to produce a Newtonian analysis which parallels so far as possible the relativistic analysis, we shall use in place of equation (14b;N) the relation

$$\frac{\partial^2\Phi_B}{\partial t\partial r} = \frac{1}{r^2}\frac{\partial}{\partial t}\begin{pmatrix}\text{mass inside}\\ \text{radius } r\end{pmatrix} = \frac{1}{r^2}(-4\pi r^2)\begin{pmatrix}\text{mass flux in}\\ \text{radial direction}\end{pmatrix}$$
$$= -4\pi\int p^r f d\mathcal{V}_p\,,$$

so that our version of the equations of motion is

$$\partial f/\partial t + \mathfrak{D}_A f - F_E p^r \partial\Phi_B/\partial r = 0\,, \tag{15a;N}$$

$$\partial^2\Phi_B/\partial t\partial r = -4\pi\int p^r f d\mathcal{V}_p\,. \tag{15b;N}$$

d) Equation of Motion for the Odd Part of f

It may seem surprising that the equation of motion (15a;N) is of first order rather than second order. Physical intuition suggests that a perturbed cluster should pulsate, collapse, or explode, and such motions are usually described by hyperbolic second-order differential equations. Actually, a hyperbolic second-order differential equation is hidden in equation (15a;N) and can be extracted by a method due to Antonov (1960):

i) Split the function $f(t,x,p,m)$ into "even" and "odd" parts:

$$\begin{aligned} f_+(t,x,p,m) &= \tfrac{1}{2}[f(t,x,p,m) + f(t,x,-p,m)]\,,\\ f_-(t,x,p,m) &= \tfrac{1}{2}[f(t,x,p,m) - f(t,x,-p,m)]\,. \end{aligned} \tag{16;N}$$

The even part, f_+, is that part which is unaffected by reflections in momentum space ("even parity" in momentum space); the odd part, f_-, is that which changes sign under reflections in momentum space ("odd parity" in momentum space):

$$\begin{gathered} f_+(t,x,-p,m) = f_+(t,x,p,m)\,, \quad f_-(t,x,-p,m) = -f_-(t,x,p,m)\,;\\ f = f_+ + f_-\,. \end{gathered} \tag{17;N}$$

Notice that the even part of f, f_+, determines the star density, the mass density, and the stresses inside the star cluster (these are even moments of f in phase space), while the odd part, f_-, determines the flow of stars, the flow of mass, and the flow of energy (odd moments of f).

ii) Similarly, split equations (15;N) into even and odd parts, noticing in the process that $\mathfrak{D}_A$ is an odd operator (it changes the parity of a function) and that only the odd part of f contributes to the integral in equation (15b;N)

$$\partial f_+/\partial t + \mathfrak{D}_A f_- = 0\,, \tag{18a;N}$$

$$\partial f_-/\partial t + \mathfrak{D}_A f_+ - F_E p^r \partial\Phi_B/\partial r = 0\,, \tag{18b;N}$$

$$\partial^2\Phi_B/\partial t\partial r = -4\pi\int p^r f_- d\mathcal{V}_p\,. \tag{18c;N}$$

[2] One can readily verify that Antonov's equations of motion and all other results of his analysis are valid, not only when F depends on E alone (the case he considered) and not only for spherical perturbations of spherical clusters (the case presented here), but also for those perturbations of any cluster which do not destroy the space symmetries of the equilibrium configuration. For example, his results are valid for all axially symmetric perturbations of a rotating, axially symmetric equilibrium configuration. We do not present the more general treatment here because our motivation is to obtain a Newtonian guide for the relativistic analysis, and in relativity only spherical motions of spherical clusters are free of the difficulties of gravitational radiation.

iii) Differentiate equation (18b;N) with respect to t and combine with equations (18a,c;N), to get

$$(1/F_E)(\partial^2 f_-/\partial t^2) = \mathfrak{J} f_- \,, \tag{19;N}$$

where $\mathfrak{J}$ is the operator

$$\mathfrak{J} f_- = \frac{\mathfrak{D}_A \mathfrak{D}_A f_-}{F_E} - 4\pi p^r \int p^r f_- d\mathcal{V}_p \,. \tag{20;N}$$

Equation (19;N) is the fundamental dynamical equation which governs the pulsation of Newtonian star clusters. Once equation (19;N) has been integrated to give the odd part of f, equation (18a;N) can be solved for the even part, and equation (14b;N) or (18c;N) can be integrated to give Φ_B.

e) Properties of the Equation of Motion; Variational Principles

The dynamical equation (19;N) has a key property which simplifies considerably the study of its solutions: *the operator $\mathfrak{J}$ is self-conjugate for functions which are bounded in phase space.* That is, if h and k are functions which are zero outside some finite region of phase space, then they satisfy

$$\begin{aligned}\int h \mathfrak{J} k d\mathcal{V}_p d\mathcal{V}_x = \int k \mathfrak{J} h d\mathcal{V}_p d\mathcal{V}_x = & \int \frac{(\mathfrak{D}_A h)(\mathfrak{D}_A k)}{-F_E} d\mathcal{V}_x d\mathcal{V}_p \\ & - 4\pi \int \left(\int p^r h d\mathcal{V}_p\right)\left(\int p^r k d\mathcal{V}_p\right) d\mathcal{V}_x \,.\end{aligned} \tag{21;N}$$

Proof of equation (21;N): The second term on the right-hand side of equation (21;N) follows trivially from equation (20;N). The first term follows from the fact that $\mathfrak{D}_A$ is anti-self-conjugate for bounded functions u and v

$$\int u \mathfrak{D}_A v d\mathcal{V}_x d\mathcal{V}_p = -\int v \mathfrak{D}_A u d\mathcal{V}_x d\mathcal{V}_p \tag{22;N}$$

and from the fact that F_E is a function of the integrals of motion of the equilibrium configuration, so that $\mathfrak{D}_A F_E = 0$. That $\mathfrak{D}_A$ is anti-self-conjugate for bounded functions (eq. [22;N]) follows from simple integrations by parts (cf. eq. [12;N]). Q.E.D.

Since $\mathfrak{J}$ is self-conjugate for bounded functions, the dynamical equation (19;N) has a number of well-known and useful properties, *provided only that the star cluster is bounded.*

Property 1: The dynamical equation (19;N) follows from the action principle

$$\delta \int \left[\frac{(\partial f_-/\partial t)^2}{-F_E} - f_- \mathfrak{J} f_-\right] d\mathcal{V}_p d\mathcal{V}_x dt = 0 \,. \tag{23;N}$$

Property 2: Associated with the action principle (23;N) there is a dynamically conserved quantity analogous to pulsational energy:[3]

$$H = \int \left[\frac{(\partial f_-/\partial t)^2}{-F_E} + f_- \mathfrak{J} f_-\right] d\mathcal{V}_p d\mathcal{V}_x = \text{constant} \,. \tag{24a;N}$$

[3] Lynden-Bell (1966, eq. [17]) has previously discussed a conserved quantity similar to expressions (24;N). Rewritten in our notation, his conserved quantity is

$$\epsilon = \tfrac{1}{2} \int \frac{f^2}{-F_E} d\mathcal{V}_p d\mathcal{V}_x - \tfrac{1}{2} \int \int \frac{m m' f f'}{|x - x'|} d\mathcal{V}_p d\mathcal{V}_x d\mathcal{V}_{p'} d\mathcal{V}_{x'} \,.$$

His conserved quantity, ϵ, can be obtained from ours, H, as follows: Re-express the second term of expression (24b;N) for H in terms of $\partial f/\partial t$ by using eqs. (15b;N) and (14;N). Then simply replace $\partial f/\partial t$ in H by f and divide by 2. The resultant quantity is ϵ. By splitting f into its normal modes and using their

With the help of equations (17;N), (18a;N), and (21;N), we can rewrite this conserved quantity in terms of the full perturbation $f = f_+ + f_-$:

$$H = \int(-1/F_E)(\partial f/\partial t)^2 d\mathcal{V}_p d\mathcal{V}_x - 4\pi \int(\int p^r f d\mathcal{V}_p)^2 d\mathcal{V}_x . \qquad (24b;N)$$

Property 3: If f_- is split up into normal modes

$$f_- = \mathfrak{f}(x^j,p^j,m)e^{i\omega t} , \quad f_+ = (i/\omega)\mathfrak{D}_A \mathfrak{f} e^{i\omega t} , \qquad (25;N)$$

then the eigenfunctions $\mathfrak{f}$ satisfy the self-conjugate eigenequation

$$(-\omega^2/F_E)\mathfrak{f} = \mathfrak{J}\mathfrak{f} , \qquad (26;N)$$

for which there is a variational principle

$$\omega^2 = \frac{\int \mathfrak{f}\, \mathfrak{J}\mathfrak{f} d\mathcal{V}_p d\mathcal{V}_x}{\int(-1/F_E)\mathfrak{f}^2 d\mathcal{V}_p d\mathcal{V}_x} . \qquad (27;N)$$

The stationary values of the right-hand side of this equation are the squared eigenfrequencies, ω^2; and the functions $\mathfrak{f}$ which produce those stationary values are the corresponding eigenfunctions.

Property 4: If F_E is negative or zero throughout the phase space of the equilibrium configuration, then the squared eigenfrequencies, ω^2, are all real; i.e , each eigenfrequency is real (stable mode) or imaginary (unstable mode).

Property 5: The eigenfunctions belonging to different eigenfrequencies satisfy the orthogonality relation

$$\int(-1/F_E)\mathfrak{f}_m \mathfrak{f}_n d\mathcal{V}_p d\mathcal{V}_x = 0 . \qquad (28;N)$$

Property 6: If F_E is negative or zero throughout the phase space of the equilibrium configuration, then that configuration is stable against spherical perturbations if and only if $\mathfrak{J}$ is a positive-definite operator for spherical functions bounded in phase space—i.e., if and only if

$$\int h\, \mathfrak{J} h d\mathcal{V}_p d\mathcal{V}_x > 0 \qquad (29;N)$$

for all non-zero, bounded h. (*Note:* The condition $F_E \leq 0$ will be satisfied by most if not all equilibrium configurations of physical interest, since it states that there are fewer high-energy stars than low-energy stars.)

Most of these properties have been discussed previously by Antonov (1960) for clusters with F a function of E only. However, he did not mention properties 1 and 2 or the variational principle (27;N).

f) Criterion for the Existence of a Zero-Frequency Mode

From the equations of motion in the form (18;N) one can derive an elegant criterion for the existence of a zero-frequency mode: *In a spherically symmetric Newtonian star*

orthogonality (eq. [28;N]), one can show that the conservation of H implies the conservation of ϵ. Neither Lynden-Bell's conserved quantity nor ours appears to be the pulsational energy of the cluster. Lynden-Bell claims that there is an intimate relation between pulsational energy and his conserved quantity, but his analysis proves only the trivial result that his conserved quantity differs from pulsational energy by a constant. Milder (1967) has also discussed the relation between pulsational energy and the conserved quantity, ϵ, but the physical meaning of his formal mathematical result is unclear to us.

cluster for which $F_E \leq 0$,[4] there exists a zero-frequency mode of spherical, collisionless motion if and only if the following holds: there exists another, slightly different equilibrium configuration such that the difference in distribution functions between the two configurations,

$$\Delta\mathfrak{N}(x^j,p^j,m) = \mathfrak{N}_2(x^j,p^j,m) - \mathfrak{N}_1(x^j,p^j,m) , \tag{30;N}$$

satisfies the relation

$$\Delta\mathfrak{N} = \mathfrak{D}_A G \tag{31;N}$$

for some function, G, in phase space. Equivalently, it is necessary and sufficient that

i) *When $\Delta\mathfrak{N}$ is integrated around any closed stellar orbit, $\mathfrak{C}$, in the phase space of the equilibrium configuration, the result is zero:*

$$\int_{\mathfrak{C}} \Delta\mathfrak{N} dt = 0 ; \tag{32a;N}$$

and

ii) *When $\Delta\mathfrak{N}$ is integrated along any possible stellar orbit in phase space which originates outside the cluster and terminates outside the cluster, the result is also zero:*

$$\int_{\mathfrak{C}'} \Delta\mathfrak{N} dt = 0 . \tag{32b;N}$$

Moreover, when a zero-frequency mode is present, it has the form

$$f_+ = (t/\tau)\Delta\mathfrak{N} , \quad f_- = -G/\tau , \quad \Phi_B' = (t/\tau)\Delta\Phi , \tag{33;N}$$

where τ is a constant. Hence the zero-frequency mode carries the cluster from one of its two equilibrium configurations to the other during the lapse of time τ.

The significance of this theorem will be discussed in the relativistic section (§ IV*f*).

Proof of the theorem: We first determine the general form for a zero-frequency mode. Any zero-frequency mode must be a finite power series in time, t, for which f_+ vanishes at time $t = 0$:

$$\begin{aligned} f_- &= a_-^{(0)} + a_-^{(1)}t + \ldots + a_-^{(n)}t^n , \\ f_+ &= \qquad a_+^{(1)}t + \ldots + a_+^{(n)}t^n . \end{aligned} \tag{34a;N}$$

The exponent n must be 1 for the following reason. The equations of motion (19;N) and (18a;N) demand that

$$\mathfrak{J}a_-^{(n)} = \mathfrak{J}a_-^{(n-1)} = 0 , \tag{34b;N}$$

$$n(n-1)a_-^{(n)}/F_E = \mathfrak{J}a_-^{(n-2)} , \quad (n-1)(n-2)a_-^{(n-1)}/F_E = \mathfrak{J}a_-^{(n-3)} , \tag{34c;N}$$

$$na_+^{(n)} = -\mathfrak{D}_A a_-^{(n-1)} . \tag{34d;N}$$

Multiplying equations (34c;N) by $a_-^{(n)}$ and $a_-^{(n-1)}$, integrating over phase space, and using equations (21;N) and (34b;N), we obtain

$$n(n-1)\int \frac{[a_-^{(n)}]^2}{F_E} d\mathcal{V}_x d\mathcal{V}_p = (n-1)(n-2)\int \frac{[a_-^{(n-1)}]^2}{F_E} d\mathcal{V}_x d\mathcal{V}_p = 0 . \tag{34e;N}$$

[4] If we define a zero-frequency mode to be one for which f has the form

$$f_+ = \beta(x^j,p^j)t ; \quad f_- = \gamma(x^j,p^j) ,$$

then we can drop from the theorem the demand that $F_E \leq 0$.

Suppose $n > 2$. Then equations (34e;N) together with the condition $F_E \leq 0$ and equation (34d;N) tell us that

$$a_+^{(n)} = a_-^{(n)} = a_-^{(n-1)} = 0 . \qquad \text{(34f;N)}$$

Hence n must be ≤ 2. When $n = 2$, the above argument tells us only that $a_-^{(2)} = 0$, but equations (34a;N) and (18b;N) allow us to conclude that $a_-^{(1)} = 0$ as well; and equation (34d;N) then reveals that $a_+^{(2)} = 0$. Consequently, n can only be equal to 1; and the general zero-frequency mode is of the form (33;N).[5]

Next we verify that expression (33;N) represents a zero-frequency motion if and only if $\Delta\mathcal{N}$ and G satisfy conditions (30;N) and (31;N). Equation (30;N) is equivalent to the statement that $\Delta\mathcal{N}$ satisfies the perturbed Liouville equation

$$\mathcal{D}_A \Delta\mathcal{N} = F_E p^r \partial\Delta\Phi/\partial r , \quad \nabla^2(\Delta\Phi) = 4\pi \int m \Delta\mathcal{N} d\mathcal{V}_p \qquad \text{(34g;N)}$$

(cf. eq. [10;N] or eqs. [4;N] and [5;N]). Hence equations (30;N) and (31;N) are equivalent to equations (34g;N) and (31;N). On the other hand, expression (33;N) represents a zero-frequency mode if and only if it satisfies the equations of motion (18a,b;N) and (14a;N), which become identical with equations (34g;N) and (31;N) upon manipulation. Q.E.D.

Only condition (32;N) remains to be verified. Equations (32;N) are nothing more than the integrability conditions for the existence of the potential function, G, of equation (31;N). This is because $\mathcal{D}_A$ is the derivative with respect to time along the unique stellar orbit that goes through a given point in the phase space of the equilibrium configuration. Q.E.D.

g) Relation between Stabilities of Clusters and of Gas Spheres

The variational principles and stability criterion derived in § III*e* will be much more difficult to apply than the corresponding results in the theory of gas spheres For a gas sphere the variational principles and eigenequations involve only one coordinate, r, whereas for clusters the radius r, radial momentum p^r, angular momentum J, and mass m all enter non-trivally. In certain circumstances one may be able to handle the effects of J and m analytically (recall that J and m are conserved along a stellar orbit in the pulsating cluster), but typically one may have to analyze numerically a two-dimensional problem in (r,p^r).

Recently Lynden-Bell (1966) has partially saved us from the pain of two-dimensional numerical analyses by devising a simple one-dimensional criterion for the stability of certain star clusters. Lynden-Bell's criterion has one drawback: it is a sufficient condition for stability but not (so far as we know) a necessary condition. Nevertheless, it should prove extremely useful for many problems.

Lynden-Bell's criterion for the special case of spherical clusters with isotropic velocity distributions (F independent of J)[6] says this: *Consider a bounded, spherically symmetric Newtonian cluster with isotropic velocity distribution and with $F_E \leq 0$. Such a cluster is stable against collisionless, spherical perturbations if the gas sphere with the same radial distributions of density,*

$$\rho = \int mF d\mathcal{V}_p = 4\pi \int m^2 [2m(E_A - m\Phi_A)]^{1/2} F dE_A dm , \qquad \text{(35a;N)}$$

and of pressure,

$$\begin{aligned} P &= \int (p^r p^r/m) F d\mathcal{V}_p = \tfrac{1}{2} \int (J^2/mr^2) F d\mathcal{V}_p \\ &= (4\pi/3) \int [2m(E_A - m\Phi_A)]^{3/2} F dE_A dm , \end{aligned} \qquad \text{(35b;N)}$$

[5] Antonov (1960) concluded incorrectly that zero-frequency modes with $n = 2$ are possible.

[6] Lynden-Bell proves his theorem in a somewhat more general context, but here we are concerned only with spherical clusters.

is stable against radial perturbations for which the "adiabatic index" is

$$\Gamma_1 = \frac{\rho}{P}\frac{dP/dr}{d\rho/dr}\,. \qquad (35c;N)$$

Since it is a simple one-dimensional problem to determine whether a gas sphere is stable, this theorem gives us a simple, one-dimensional, sufficient criterion for the stability of a spherical cluster.

Paragraph added July 3, 1968.—Recent discussions between Donald Lynden-Bell and James R. Ipser, motivated in part by remarks of Edward Lee, have revealed that Lynden-Bell's (1966) proof of this theorem was incorrect.[7] However, a new, corrected proof of the theorem has been devised by Lynden-Bell (paper in preparation), and a relativistic version of the theorem has been proved by Ipser (to be published in Paper II of this series).

IV RELATIVISTIC THEORY OF STABILITY

We now develop the relativistic generalization of our Newtonian discussion of stability. Our treatment follows as closely as possible the corresponding Newtonian treatment, with the corresponding equations being given similar numbers (e.g., eq. [1;R] corresponds to eq. [1;N]).

a) Equations of Stellar Dynamics

In general relativity the density of stars in phase space, which we denote by $\mathfrak{N}$, is defined as follows: we concentrate attention on those stars near a particular event, x, in spacetime with 4-momenta near a particular value, p. As seen in the rest frame of these stars, they occupy a particular three-dimensional volume, $d\mathcal{V}_x$, in physical space and a particular four-dimensional volume, $d\mathcal{V}_p$, in momentum space In terms of a general curvilinear coordinate system, $d\mathcal{V}_x$ and $d\mathcal{V}_p$ are given by

$$d\mathcal{V}_x = (p^0/m)\sqrt{(-g)}dx^1dx^2dx^3\,; \quad d\mathcal{V}_p = -dp_0dp_1dp_2dp_3/\sqrt{(-g)}\,. \quad (1;R)$$

Here p^α and p_α are the contravariant and covariant components of the 4-momentum, g is the determinant of the metric tensor, and $m = (p_\alpha p^\alpha)^{1/2}$ is the rest mass of a star with 4-momentum p. If there are dN stars in the volume $d\mathcal{V}_x d\mathcal{V}_p$, then the number density in phase space ("distribution function") is given by

$$\mathfrak{N} \equiv dN/d\mathcal{V}_x d\mathcal{V}_p = dN/(-\,dx^1dx^2dx^3dp_1dp_2dp_3dm)\,. \qquad (2;R)$$

The density $\mathfrak{N}$ is a function of location (x,p) in eight-dimensional phase space. Through part of our discussion we shall use as coordinates in phase space general curvilinear spacetime coordinates, x^α, and the "conjugate" covariant components of the 4-momentum, p_α. However, we shall sometimes employ other sets of coordinates, for example, (x^α,p_j,m)[8] and coordinates specially adapted to spherical symmetry.

The smoothed-out gravitational field of the star cluster is described by the metric

[7] The error lies in the argument showing that positive-definiteness of the Lynden-Bell operator

$$S \equiv -\frac{\nabla^2}{4\pi} - \rho\frac{d\rho/dr}{dP/dr}$$

is a *necessary* condition for stability of the gas sphere. It is *not* necessary for stability.

[8] Greek indices run from 0 to 3; Latin indices, from 1 to 3.

tensor, $g_{\alpha\beta}(x)$. The distribution function determines a smoothed-out stress-energy tensor through the equations

$$T_\alpha{}^\beta = \int p_\alpha p^\beta (\mathfrak{N}/m) d\mathcal{V}_p \,, \tag{3a;R}$$

and that stress-energy tensor determines the metric, $g_{\alpha\beta}$, through Einstein's equations,

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \,. \tag{3b;R}$$

The gravitational field in turn determines the distribution function through the collisionless Boltzmann-Liouville equation (or "Liouville equation")

$$\mathfrak{D}\mathfrak{N} = 0 \,. \tag{4;R}$$

Here $\mathfrak{D}$, the Liouville operator, is differentiation with respect to proper time along the path of a star in phase space:

$$\mathfrak{D} = \frac{dx^\alpha}{ds}\frac{\partial}{\partial x^\alpha} + \frac{dp_\alpha}{ds}\frac{\partial}{\partial p_\alpha} = \frac{p^\alpha}{m}\frac{\partial}{\partial x^\alpha} - \frac{1}{2m}\frac{\partial g^{\mu\nu}}{\partial x^\alpha} p_\mu p_\nu \frac{\partial}{\partial p_\alpha} \,. \tag{5;R}$$

Equations (3;R) and (4;R), which couple $g_{\alpha\beta}$ and $\mathfrak{N}$, are the fundamental equations of relativistic stellar dynamics.

b) Spherical Equilibrium Configurations

In general relativity, as in Newtonian theory, the distribution function for an equilibrium configuration depends only on the integrals of the motion. For spherical symmetry the relevant integrals of the motion are the rest mass m, the "energy at infinity," $E \equiv p_0$, and the total angular momentum J; hence we have

$$\mathfrak{N} = F(m,E,J) \,. \tag{6a;R}$$

When F is independent of J, the cluster has an isotropic velocity distribution at each point in space We shall use the "Schwarzschild coordinate system" (t,r,θ,ϕ) to describe spherical equilibrium configurations. In this coordinate system the gravitational field is described by

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \,, \tag{6b;R}$$

where ν and λ are functions of r, and the angular momentum and "energy at infinity" are given by

$$J = [p_\theta{}^2 + (p_\phi/\sin\theta)^2]^{1/2} \,, \quad E = p_0 \,. \tag{6c;R}$$

The theory of spherically symmetric equilibrium configurations has been developed in great detail by Fackerell (1966; 1968*a–c*). Independently Zel'dovich and Podurets (1965) have treated the restricted problem of a cluster of identical stars with a truncated, isotropic Maxwell-Boltzmann velocity distribution—i.e., a cluster with

$$\begin{aligned} F &= Ae^{-E/T}\delta(m - m_0) \quad && \text{if } E < E_0 \\ &= 0 && \text{if } E > E_0 \,. \end{aligned}$$

c) Equation of Motion for a Perturbed Spherical Star Cluster

If a spherically symmetric equilibrium configuration is perturbed in a spherical manner, and if Schwarzschild coordinates are adopted for the perturbed configuration as for the unperturbed configuration, then the perturbed gravitational field is described by

$$ds^2 = e^{\nu_A(r)+\nu_B(t,r)}dt^2 - e^{\lambda_A(r)+\lambda_B(t,r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \,. \tag{7;R}$$

As in the Newtonian case, so also here, we work only to first order in the perturbation quantities ν_B and λ_B.

The radial coordinate, r, in equation (7;R) is defined uniquely by the demand that $4\pi r^2$ be the area of an invariant sphere about the center of symmetry. The time coordinate, t, is also defined uniquely if we insist that the perturbed metric (7;R) become the static Schwarzschild metric outside the cluster (Birkhoff's theorem). Consequently, there is no coordinate arbitrariness in the functions ν_A, ν_B, λ_A, and λ_B.

In defining the perturbation of the distribution function, we must decide how to identify points in the phase space of the perturbed cluster with points in the phase space of the unperturbed cluster. There is a variety of possibilities: We could identify points with the same Schwarzschild coordinates, x^a, and with the same covariant components of the momentum, p_a, so that

$$\mathfrak{N} = \mathfrak{N}_A(x^a, p_a) + \mathfrak{N}_B(x^a, p_a) .$$

Alternatively, we could use contravariant components of the momentum, p^a, in making the identification:

$$\mathfrak{N} = \mathfrak{N}_A(x^a, p^a) + \mathfrak{N}_B(x^a, p^a) .$$

Either of these choices is reasonable on mathematical grounds, but from a physical standpoint it is preferable to identify points with the same Schwarzschild coordinates, x^a, and the same *physical components* of the momentum, $p_{(a)} = |g^{aa}|^{1/2} p_a$. (See Table 1.)

TABLE 1

PHYSICAL COMPONENTS OF THE 4-MOMENTUM*

Component	Value in Equilibrium Configuration	Value in Perturbed Configuration
$p_{(0)}$	$p_0 \exp(-\nu_A/2) = p^0 \exp(\nu_A/2)$	$p_0 \exp[-(\nu_A+\nu_B)/2] = p^0 \exp[(\nu_A+\nu_B)/2]$
$p_{(r)}$	$p_r \exp(-\lambda_A/2) = -p^r \exp(\lambda_A/2)$	$p_r \exp[-(\lambda_A+\lambda_B)/2] = p^r \exp[(\lambda_A+\lambda_B)/2]$
$p_{(\theta)}$	$p_\theta r^{-1} = -p^\theta r$	$p_\theta r^{-1} = -p^\theta r$
$p_{(\phi)}$	$p_\phi (r \sin\theta)^{-1} = -p^\phi (r \sin\theta)$	$p_\phi (r \sin\theta)^{-1} = -p^\phi (r \sin\theta)$

* The physical components are the projections of p on an orthonormal tetrad with legs in the t, r, θ, and ϕ directions. (See, e.g., chap. ii of Thorne 1967.)

This is the type of identification which observers using proper reference frames or locally inertial reference frames would make, and it leads to a formalism which is considerably simpler than the other choices. With this choice of identification of points in phase space, the distribution function of the perturbed cluster takes the form

$$\mathfrak{N} = F(m, E_A, J) + f(x^a, p_{(a)}) , \tag{8;R}$$

where

$$m = [p_{(0)}{}^2 - p_{(r)}{}^2 - p_{(\theta)}{}^2 - p_{(\phi)}{}^2]^{1/2} , \quad E_A = p_{(0)} e^{\nu_A/2}, \quad J = r[p_{(\theta)}{}^2 + p_{(\phi)}{}^2]^{1/2} . \tag{9;R}$$

The Liouville equation which this distribution function obeys takes the following form when linearized in the perturbations ν_B, λ_B, and f:

$$\frac{p^0}{m}\frac{\partial f}{\partial t} + \mathfrak{D}_A f - \tfrac{1}{2}\frac{p_0}{m} F_E p^r \frac{\partial \nu_B}{\partial r} + \frac{1}{2m} F_E p_r p^r \frac{\partial \lambda_B}{\partial t} = 0 . \tag{10;R}$$

Here F_E stands for

$$F_E \equiv (\partial F/\partial E_A)_{m,J} , \tag{11;R}$$

and $\mathfrak{D}_A$ is the Liouville operator of the unperturbed cluster, which has the form

$$\mathfrak{D}_A = \frac{p^j}{m}\frac{\partial}{\partial x^j} - \frac{1}{2m}\frac{\partial g_A{}^{\mu\nu}}{\partial x^j} p_\mu p_\nu \frac{\partial}{\partial p_j} \tag{12;R}$$

when (x^α, p_α) are used as coordinates in phase space. (Note that we must use care in the choice of coordinates only while defining f. Now that f has been defined explicitly, we are free to use whatever coordinates we wish in manipulating it, except that *we shall demand that our coordinates leave the equilibrium configuration explicitly static.*)

The derivation of the perturbed Liouville equation (10;R) follows.

For the purpose of the derivation we shall use as coordinates in phase space the Schwarzschild space coordinates (t,r,θ,ϕ), the physical zero component of the momentum, $p_{(0)}$, the angular momentum, J, and the rest mass of a star, $m = [p_{(0)}{}^2 - (J/r)^2 - p_{(r)}{}^2]^{1/2}$. The rest mass, m, is used in place of $p_{(r)}$; and J is used in place of *both* $p_{(\theta)}$ and $p_{(\phi)}$. (This is possible because spherical symmetry guarantees that $\mathfrak{N}$ can depend on $p_{(\theta)}$ and $p_{(\phi)}$ only through J.) The full Liouville equation for a dynamical, spherically symmetric cluster is

$$\frac{p^\alpha}{m}\frac{\partial \mathfrak{N}}{\partial x^\alpha} + \frac{dp_{(0)}}{ds}\frac{\partial \mathfrak{N}}{\partial p_{(0)}} + \frac{dJ}{ds}\frac{\partial \mathfrak{N}}{\partial J} + \frac{dm}{ds}\frac{\partial \mathfrak{N}}{\partial m} = 0\,. \tag{13a;R}$$

Because J and m are integrals of the motion in a dynamical, spherical cluster, we have

$$dJ/ds = dm/ds = 0\,. \tag{13b;R}$$

The change in $p_{(0)}$ along a star's world line, as calculated from the geodesic equation, is

$$dp_{(0)}/ds = (1/2m)e^{-\nu/2}[p_r p^r(\partial\lambda/\partial t) - p_0 p^r(\partial\nu/\partial r)]\,. \tag{13c;R}$$

Consequently, the Liouville equation (13a;R) reads

$$\frac{p_{(0)}e^{-\nu/2}}{m}\frac{\partial \mathfrak{N}}{\partial t} - \frac{p_{(r)}e^{-\lambda/2}}{m}\frac{\partial \mathfrak{N}}{\partial r} + \frac{1}{2m}\left[-e^{-\nu/2}p_{(r)}p_{(r)}\frac{\partial \lambda}{\partial t} + e^{-\lambda/2}p_{(0)}p_{(r)}\frac{\partial \nu}{\partial r}\right]\frac{\partial \mathfrak{N}}{\partial p_{(0)}} = 0\,. \tag{13d;R}$$

When we split into unperturbed and perturbed parts,

$$\mathfrak{N} = F + f\,, \quad \nu = \nu_A + \nu_B\,, \quad \lambda = \lambda_A + \lambda_B\,, \tag{13e;R}$$

and linearize in the perturbation, this becomes

$$\frac{p^0}{m}\frac{\partial f}{\partial t} + \mathfrak{D}_A f - \frac{p^r}{m}\frac{\lambda_B}{2}\left(\frac{\partial F}{\partial r}\right)_{t,p_{(0)},J,m} + \frac{e^{-\nu_A/2}}{2m}\left[p_r p^r \frac{\partial \lambda_B}{\partial t} + p_0 p^r\left(\tfrac{1}{2}\frac{\partial \nu_A}{\partial r}\lambda_B - \frac{\partial \nu_B}{\partial r}\right)\right]\left(\frac{\partial F}{\partial p_{(0)}}\right)_{t,r,J,m} = 0\,. \tag{13f;R}$$

The value of F depends on $p_{(0)}$ and r only through $E_A = p_{(0)}e^{\nu_A/2}$ when J and m are held fixed. Consequently,

$$\left(\frac{\partial F}{\partial r}\right)_{t,p_{(0)},J,m} = \tfrac{1}{2}p_0\frac{\partial \nu_A}{\partial r}F_E\,, \quad \left[\frac{\partial F}{\partial p_{(0)}}\right]_{t,r,J,m} = e^{\nu_A/2}F_E\,. \tag{13g;R}$$

When equations (13g;R) are combined with equation (13f;R), the perturbed Liouville equation (10;R) results. Q.E.D.

The perturbed Liouville equation (10;R) must be supplemented by equations for ν_A and ν_B in terms of f. The required relations are the perturbations of Einstein's field equations (3;R). The perturbation in the stress-energy tensor, which enters into the field equations, is

$$T_{B\alpha}{}^{\beta} = \int p_\alpha p^\beta (f/m) d\mathcal{V}_p \,,$$

and the perturbation in the Einstein tensor is the same as that used by Chandrasekhar (1964) in studying the radial pulsation of gas spheres. By combining the perturbed Einstein and stress-energy tensors, one obtains three useful field equations:

$$\partial\lambda_B/\partial t = -8\pi r e^{\lambda_A} \int p_0 p^r (f/m) d\mathcal{V}_p \,, \qquad (14a;R)$$

$$(\partial/\partial r)(r e^{-\lambda_A}\lambda_B) = 8\pi r^2 \int p_0 p^0 (f/m) d\mathcal{V}_p \,, \qquad (14b;R)$$

$$\partial\nu_B/\partial r = (\partial\nu_A/\partial r + 1/r)\lambda_B - 8\pi r e^{\lambda_A} \int p_r p^r (f/m) d\mathcal{V}_p \,. \qquad (14c;R)$$

Equations (10;R) and (14;R) are the equations of motion for the perturbed cluster. These four equations for f, ν_B, and λ_B are not all independent. The Liouville equation (10;R), when combined with equation (14a;R), can be made to yield (14b;R); when combined with (14b;R), it yields (14a;R).

Equations (10;R) and (14;R) are not the most useful forms for the equations of motion. Rather, it is convenient to remove ν_B and $\partial\lambda_B/\partial t$ from equation (10;R) by use of (14a,c;R), and to take the resultant equation along with (14a;R) as coupled equations for f and λ_B:

$$\frac{\partial f}{\partial t} + \mathcal{B} f - \frac{p_0}{p^0}\left(1 + r\frac{\partial\nu_A}{\partial r}\right) F_E p^r \frac{\lambda_B}{2r} = 0 \,, \qquad (15a;R)$$

$$\frac{\partial}{\partial t}\left(\frac{\lambda_B}{2r}\right) = -4\pi e^{\lambda_A} \int \frac{p_0}{m} p^r f d\mathcal{V}_p \,. \qquad (15b;R)$$

Here $\mathcal{B}$ is the operator in phase space,

$$\mathcal{B}\psi \equiv \frac{m}{p^0}\mathfrak{D}_A\psi + 4\pi r e^{\lambda_A}\frac{m}{p^0} F_E\left(\frac{p_0 p^r}{m}\int \frac{p_r p^r}{m}\psi d\mathcal{V}_p - \frac{p_r p^r}{m}\int \frac{p_0 p^r}{m}\psi d\mathcal{V}_p\right). \qquad (15c;R)$$

d) Equation of Motion for the Odd Part of f

In order to convert the equations of motion (15;R) into self-adjoint, hyperbolic, second-order form, we follow the Newtonian procedure of splitting them into even and odd parts. Such a split in general is not Lorentz-invariant in momentum space, because the parity is defined in terms of inversions of the *space* part of the 4-momentum ($p_{(0)}$ is not inverted); and the space part of p is not a Lorentz-invariant entity. Fortunately, this need not disturb us. The static nature of the unperturbed geometry provides us with preferred time directions in both physical space and momentum space. In the pulsating cluster, the preferred time directions are well defined to zero order in the perturbations—which is sufficiently well defined for our purposes—and they are automatically embodied in the coordinate system $(t,r,\theta,\phi,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)})$ which we are using.

Consequently, without any loss of generality, and without any introduction of arbitrariness into the analysis, we can define the even and odd parts of f as

$$\begin{aligned} f_+(x,p) &= \tfrac{1}{2}[f(x,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)}) + f(x,p_{(0)},-p_{(r)},-p_{(\theta)},-p_{(\phi)})] \,, \\ f_-(x,p) &= \tfrac{1}{2}[f(x,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)}) - f(x,p_{(0)},-p_{(r)},-p_{(\theta)},-p_{(\phi)})] \,. \end{aligned} \qquad (16;R)$$

As in Newtonian theory, f_+ and f_- have even and odd parities in the spatial part of momentum space, and their sum is f:

$$f_+(x,p_{(0)},-p_{(r)},-p_{(\theta)},-p_{(\phi)}) = f_+(x,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)}) ,$$
$$f_-(x,p_{(0)},-p_{(r)},-p_{(\theta)},-p_{(\phi)}) = -f_-(x,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)}) , \qquad (17;R)$$
$$f = f_+ + f_- .$$

If we split equations (15;R) into even and odd parts, noticing in the process that $\mathfrak{B}$ is an odd operator and that only the odd part of f contributes to the integral in equation (15b;R), we obtain the equations

$$(\partial f_+/\partial t) + \mathfrak{B}f_- = 0 , \qquad (18a;R)$$

$$\frac{\partial f_-}{\partial t} + \mathfrak{B}f_+ - \frac{p_0}{p^0}\left(1 + r\frac{\partial \nu_A}{\partial r}\right)F_E p^r \frac{\lambda_B}{2r} = 0 , \qquad (18b;R)$$

$$\frac{\partial}{\partial t}\left(\frac{\lambda_B}{2r}\right) = -4\pi e^{\lambda_A}\int \frac{p_0}{m} p^r f_- d\mathfrak{V}_p . \qquad (18c;R)$$

Finally, if we differentiate equation (18b;R) with respect to t and combine it with equations (18a,c;R), we obtain the desired hyperbolic second-order differential equation

$$(1/F_E)(\partial^2 f_-/\partial t^2) = \mathfrak{J}f_- . \qquad (19;R)$$

Here $\mathfrak{J}$ is the operator

$$\mathfrak{J}f_- = \frac{1}{F_E}\mathfrak{B}\mathfrak{B}f_- - 4\pi\left(1 + r\frac{\partial \nu_A}{\partial r}\right)e^{\lambda_A}\frac{m}{p^0}\frac{p_0}{m}p^r\int \frac{p_0}{m}p^r f_- d\mathfrak{V}_p . \qquad (20;R)$$

Equation (19;R) is the fundamental dynamical equation which governs the pulsation of relativistic star clusters. Once it has been integrated to give f_-, equation (18a;R) can be solved for f_+, and the field equations (14;R) can be solved for λ_B and ν_B.

e) *Properties of the Equation of Motion; Variational Principles*

The operator $\mathfrak{J}$, *like its Newtonian counterpart, is self-conjugate for functions which are bounded in phase space.* That is, if h and k are spherically symmetric functions which are zero outside some finite region of phase space, then they satisfy

$$\int h\,\mathfrak{J}k\,d\mathfrak{V}_p d\mathfrak{V}_x = \int k\,\mathfrak{J}h\,d\mathfrak{V}_p d\mathfrak{V}_x = \int \frac{(\mathfrak{B}h)(\mathfrak{B}k)}{(-F_E)}d\mathfrak{V}_x d\mathfrak{V}_p$$
$$- 4\pi\int\left(1 + r\frac{\partial \nu_A}{\partial r}\right)e^{\lambda_A}\left(\int \frac{p_0}{m}p^r h\,d\mathfrak{V}_p\right)\left(\int \frac{p_0}{m}p^r k\,d\mathfrak{V}_p\right)\frac{m}{p^0}d\mathfrak{V}_x . \qquad (21;R)$$

Proof of equation (21;R): the second term of the right-hand side follows directly from definition (20;R) of $\mathfrak{J}$ and from expression (1;R) for $d\mathfrak{V}_x$ and $d\mathfrak{V}_p$. The first term follows, once we have verified that $\mathfrak{B}$ is anti-self-conjugate with the weighting function $1/F_E$, i.e., once we have shown that, for bounded, spherical u and v,

$$\int (1/F_E)u\mathfrak{B}v\,d\mathfrak{V}_p d\mathfrak{V}_x = -\int (1/F_E)v\mathfrak{B}u\,d\mathfrak{V}_p d\mathfrak{V}_x . \qquad (22a;R)$$

Equation (22a;R) is readily verified from definition (15c;R) of $\mathfrak{B}$, once it is recognized that $\mathfrak{D}_A$ is also anti-self-conjugate, but with the weighting function (m/p^0),

$$\int \frac{m}{p^0}u\mathfrak{D}_A v\,d\mathfrak{V}_p d\mathfrak{V}_x = -\int \frac{m}{p^0}v\mathfrak{D}_A u\,d\mathfrak{V}_p d\mathfrak{V}_x , \qquad (22b;R)$$

and that $\mathfrak{D}_A F_E = 0$. Relation (22b;R) follows from integration by parts plus simple manipulations, if (x^α, p_α) are used as the coordinates in phase space. Note that with this choice of coordinates $\mathfrak{D}_A$ has the form (12;R), $d\mathcal{V}_x$ and $d\mathcal{V}_p$ have the form (1;R), and the relation

$$\frac{\partial}{\partial x^i}\left(\frac{p^i}{m}\right) - \frac{\partial}{\partial p_j}\left(\frac{1}{2m}\frac{\partial g_A{}^{\mu\nu}}{\partial x^j}\, p_\mu p_\nu\right) = 0 \qquad \text{(22c;R)}$$

is satisfied. Q.E.D.

Since $\mathfrak{J}$ is self-conjugate for bounded functions, the dynamical equation (19;R) has the same types of well-known and useful properties as its Newtonian analogue (19;N), ***provided only that the star cluster is bounded.***

Property 1: The dynamical equation (19;R) follows from the action principle

$$\delta \int \left[\frac{(\partial f_-/\partial t)^2}{-F_E} - f_- \mathfrak{J} f_-\right] d\mathcal{V}_p d\mathcal{V}_x dt = 0\,. \qquad \text{(23;R)}$$

Property 2: Associated with the action principle (23;R) there is a dynamically conserved quantity

$$H = \int \left[\frac{(\partial f_-/\partial t)^2}{-F_E} + f_- \mathfrak{J} f_-\right] d\mathcal{V}_p d\mathcal{V}_x = \text{constant}\,. \qquad \text{(24a;R)}$$

With the help of equations (17;R), (18a;R), and (21;R), we can rewrite this conserved quantity in terms of the full perturbation $f = f_+ + f_-$:

$$H = \int \frac{(\partial f/\partial t)^2}{-F_E} d\mathcal{V}_p d\mathcal{V}_x - 4\pi \int \left(1 + r\frac{\partial \nu_A}{\partial r}\right) e^{\lambda_A} \left(\int \frac{p_0}{m} p^r f d\mathcal{V}_p\right)^2 \frac{m}{p^0} d\mathcal{V}_x\,. \qquad \text{(24b;R)}$$

Property 3: If f_- is split up into normal modes,

$$f_- = \mathfrak{f}(x^j, p_a) e^{i\omega t}\,, \qquad f_+ = (i/\omega)\mathfrak{B}\mathfrak{f} e^{i\omega t}\,, \qquad \text{(25;R)}$$

then the eigenfunctions $\mathfrak{f}$ satisfy the self-conjugate eigenequation

$$(-\omega^2/F_E)\mathfrak{f} = \mathfrak{J}\mathfrak{f}\,, \qquad \text{(26;R)}$$

for which there is a variational principle

$$\omega^2 = \frac{\int \mathfrak{f}\, \mathfrak{J}\mathfrak{f} d\mathcal{V}_p d\mathcal{V}_x}{\int (-1/F_E)\mathfrak{f}^2 d\mathcal{V}_p d\mathcal{V}_x} \qquad \text{(27;R)}$$

analogous to the Newtonian variational principle (27;N).

Property 4: If F_E is negative or zero throughout the phase space of the equilibrium configuration, then the squared eigenfrequencies, ω^2, are all real; i.e., each eigenfrequency is real (stable mode) or imaginary (unstable mode).

Property 5: The eigenfunctions belonging to different eigenfrequencies satisfy the orthogonality relation

$$\int (-1/F_E)\mathfrak{f}_m \mathfrak{f}_n d\mathcal{V}_p d\mathcal{V}_x = 0\,. \qquad \text{(28;R)}$$

Property 6: If F_E is negative or zero throughout the phase space of the equilibrium configuration, then that configuration is stable against spherical perturbations if and only if $\mathfrak{J}$ is a positive-definite operator for spherical functions bounded in phase space:

$$\int h\, \mathfrak{J} h d\mathcal{V}_p d\mathcal{V}_x > 0\,. \qquad \text{(29;R)}$$

f) Criterion for the Existence of a Zero-Frequency Mode

Zel'dovich and Podurets (1965) and Zel'dovich and Novikov (1967) have argued that one should be able to diagnose the stability of isothermal star clusters from binding-energy curves, in much the same way as one diagnoses the stability of isentropic stellar models from such curves (Fowler 1964; Bardeen 1965; Thorne 1967, § 4.1.4). This seems highly unlikely to us, because in isothermal clusters one must, in some arbitrary manner, introduce a cutoff at high energies in the distribution function, and this cutoff must be chosen uniquely for each central density in order to produce a one-parameter binding-energy curve (cf. end of § IV*b*). Only a very special choice of the cutoff—which choice is not yet known—could lead to a binding-energy criterion for stability, and perhaps no choice will work.

That the situation in star clusters is much more complicated than that in stars is indicated also by the following theorem, which is the direct generalization of our Newtonian theorem of § III*f*:

In a spherically symmetric, relativistic star cluster for which $F_E \leq 0$ (see n. 4, p. 259), *there exists a zero-frequency mode of spherical, collisionless motion if and only if the following holds: there exists another, slightly different equilibrium configuration such that the difference in distribution functions* (at fixed *physical* components of the momentum) *between the two configurations,*

$$\Delta\mathfrak{N} = \mathfrak{N}_2(r,\theta,\phi,p_{(r)},p_{(\theta)},p_{(\phi)},p_{(0)}) - \mathfrak{N}_1(r,\theta,\phi,p_{(r)},p_{(\theta)},p_{(\phi)},p_{(0)}) \,, \qquad (30;R)$$

satisfies the relation

$$\Delta\mathfrak{N} = \mathfrak{B}G \qquad (31;R)$$

for some function, G, in phase space. Equivalently (*integrability condition for eq.* [*31;R*]), *it is necessary and sufficient that*

i) *When one integrates the following quantity around any closed stellar orbit,* $\mathcal{C}$, *in the phase space of the equilibrium configuration, the result is zero:*

$$\int_{\mathcal{C}} [\Delta\mathfrak{N} + \tfrac{1}{2}(F_E/p^0)p_r p^r \Delta\lambda] dt = 0 \,; \qquad (32a;R)$$

and

ii) *When the same quantity is integrated along any possible stellar orbit in phase space which originates outside the cluster and terminates outside the cluster, the result is also zero:*

$$\int_{\mathcal{C}'} [\Delta\mathfrak{N} + \tfrac{1}{2}(F_E/p^0)p_r p^r \Delta\lambda] dt = 0 \,. \qquad (32b;R)$$

Moreover, when a zero-frequency mode is present, it has the form

$$f_+ = (t/\tau)\Delta\mathfrak{N} \,, \quad f_- = -G/\tau \,, \quad \nu_B = (t/\tau)\Delta\nu \,, \quad \lambda_B = (t/\tau)\Delta\lambda \,, \qquad (33;R)$$

where τ *is a constant. Hence the zero-frequency mode carries the cluster from one of its two equilibrium configurations to the other during the lapse of time* τ.

Proof of the theorem: We first verify that any zero-frequency mode must have the general form (i.e., time dependence) of expressions (33;R). This is done by precisely the same procedure as was used in the Newtonian analysis (eqs. [34a–f;N]).

Next we verify that expression (33;R) represents a zero-frequency motion if and only if $\Delta\mathfrak{N}$ and G satisfy conditions (30;R) and (31;R). Equation (30;R) is equivalent to the statement that $\Delta\mathfrak{N}$ satisfies the perturbed Liouville equation

$$\mathfrak{B}\Delta\mathfrak{N} - \frac{p_0}{p^0}\left(1 + r\frac{\partial\nu}{\partial r}\right)F_E p^r \frac{\Delta\lambda}{2r} = 0 \,. \qquad (34a;R)$$

(cf. eq. [15a;R]). Hence equations (30;R) and (31;R) are equivalent to equations (34a;R) and (31;R). On the other hand, expression (33;R) represents a zero-frequency mode if and only if it satisfies the equations of motion (18a,b;R) and (14b,c;R), which upon manipulation become identical with equations (34a;R), (31;R), and the perturbed field equations for $\Delta\lambda$ and $\Delta\nu$. Q.E.D.

Condition (32;R) remains to be verified. Equation (31;R), when combined with definition (15c;R) of $\mathcal{B}$, with the form (33;R) of the zero-frequency mode, and with the field equation (15b;R), becomes

$$(m/p^0)\mathfrak{D}_A G = \Delta\mathfrak{N} + \tfrac{1}{2}(F_E/p^0)p_r p^r \Delta\lambda \,. \qquad (34b;R)$$

The operator $(m/p^0)\mathfrak{D}_A$ is the derivative with respect to coordinate time along the unique stellar orbit that goes through a given point in the phase space of the equilibrium configuration. Consequently, equations (32) are the integrability conditions for the potential function G. Q.E.D.

The criteria for zero-frequency modes provided by this theorem are quite elegant conceptually, but without some sort of extension they are useless for numerical calculations. This theorem can be compared to the statement that a hot stellar model possesses a zero-frequency mode if and only if there exists another, slightly different model with identically the same chemical composition, rest mass, and binding energy and with the same distribution of entropy. In the stellar case, the demand for identical entropy distributions provides an infinity of constraints analogous to the constraints (31;R) or (32;R) for clusters. In the stellar case, we know how to simplify the stability criterion by looking only at isentropic configurations (Bardeen 1965; Thorne 1967, § 4.1.4). Perhaps future thought will reveal an analogous simplification for star clusters.

V. CONCLUSION

In this paper we have reviewed and extended the tools available for analyzing the collisionless stability of Newtonian star clusters, and we have derived a number of analogous tools for studying relativistic star clusters. One of the authors (J. R. I.) is now using these tools to study numerically the onset of the relativistic instability in spherical star clusters. It is hoped that the numerical analyses (which will be reported in a sequel to this paper) will yield improved understanding of possible processes in the nuclei of galaxies and of the Fowler-Hoyle star-cluster model for QSSs.

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6. Relativistic, Spherically Symmetric Star Clusters.
II. Sufficient Conditions for Stability against Radial Perturbations

(To be published in May 1969 issue of Astrophysical Journal)

I. INTRODUCTION AND SUMMARY

In a recent paper (Ipser and Thorne 1968; henceforth this paper will be referred to as I) the theory of the stability of spherically symmetric star clusters for small radial perturbations was developed within the framework of general relativity. Among the results derived therein were a self-conjugate equation governing the small, collisionless, radial pulsations of a relativistic cluster, and a variational principle for the normal modes of radial motion. In practice, the study of cluster stability through the use of these tools involves the analysis of a numerical problem which is at least two-dimensional. In this paper we shall derive sufficient (but not necessary) criteria for stability, that necessitate only a one-dimensional analysis, and that, consequently, are much easier to use than the necessary and sufficient, multidimensional criteria of Paper I. In subsequent papers we shall apply both sets of criteria to determine the stability of specific star-cluster models.

One-dimensional, sufficient conditions for stability have been derived in Newtonian theory by Lynden-Bell (1966, 1967, 1968). Of Lynden-Bell's results, those most important to us here are: (i) a proof that a spherical, Newtonian cluster is stable if (but not only if) a certain one-dimensional differential operator is positive-definite (Lynden-Bell 1966, 1967); and (ii) the derivation of a relationship between the stability of a collisionless, Newtonian cluster with isotropic velocities, and the stability of the corresponding gas sphere (i.e., the gas sphere which has the same radial distribution of pressure and density as has the cluster). More specifically, Lynden-Bell (1968) has shown that a bounded, spherical cluster, with fewer high-energy stars than low-energy stars, is stable against collisionless, radial perturbations if the corresponding gas sphere is stable against radial perturbations for which the adiabatic index is[1] $\Gamma_1(r) = (\rho/p)\,(dp/dr)\,(d\rho/dr)^{-1}$.

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[1]Unfortunately, there exists an important error in Lynden-Bell's (1966, 1967) analyses. This error was discovered during discussions between Professor Lynden-Bell and the author, concerning the boundary conditions satisfied by physically acceptable perturbation functions, and concerning the resulting restrictions upon the differential equations which such functions must satisfy. It is sufficient for us to note here that those proofs in Lynden-Bell (1966, 1967) which purport to establish a necessary "Schroedinger equation" condition for the stability of a gas sphere are invalid. In Lynden-Bell's (1968) paper, this matter is clarified, and the theorem relating cluster stability to gas sphere stability is reproved, correctly.

The purpose of this paper is to develop relativistic generalizations of Lynden-Bell's results. In §II we shall present a sufficient condition for the stability of a spherical, relativistic, collisionless star cluster with, in general, an anisotropic velocity distribution. This condition, like Lynden-Bell's corresponding Newtonian condition, involves the positive-definiteness of a one-dimensional, second-order differential operator. We shall also discuss a sufficient condition for the stability of a relativistic gas sphere in terms of the same differential operator. In §III we shall establish, within general relativity, the relationship between the stabilities of an isotropic, spherical cluster and the associated gas sphere. We shall find that the radial stability of a gas sphere, with adiabatic index $\Gamma_1 = (\rho + p)\, p^{-1}\, (dp/dr)\, (d\rho/dr)^{-1}$, implies the stability of the associated cluster against collisionless, radial perturbations.

Throughout this paper we will employ the notation and conventions of Paper I, including the use of "geometrized units" in which the speed of light, c, the gravitation constant, G, and the Boltzmann constant, k, are set equal to unity.

II. A SUFFICIENT CONDITION FOR STABILITY

In this section we shall show that a cluster is stable if (but not only if) a certain one-dimensional, second-order differential operator is positive-definite. Our approach will be this: In subsection a we shall reiterate some of the fundamental equations of relativistic, spherical stellar dynamics that were derived in Paper I; in subsection b we shall derive useful formulae for certain integrals over the momentum space of a spherical equilibrium cluster; in subsection c we shall derive the relevant operator and prove its relationship to cluster stability; and in subsection d we shall show that the positive-definiteness of this same operator guarantees the stability of a gas sphere. Our notation will be identical to that of Paper I.

a) Some Fundamental Equations of Relativistic Stellar Dynamics

In relativistic stellar dynamics one works with the number density in phase space or distribution function, $\mathcal{N}$, which is defined to be the number of stars, dN, per unit invariant phase space volume, $d\mathcal{V}_p\, d\mathcal{V}_x$:

$$\mathcal{N} = dN/(d\mathcal{V}_p\, d\mathcal{V}_x) \qquad , \qquad (1)$$

where

$$d\mathcal{V}_x = (p^0/m)\sqrt{(-g)}\, dx^1\, dx^2\, dx^3 \qquad ;$$

$$d\mathcal{V}_p = -\, dp_0\, dp_1\, dp_2\, dp_3/\sqrt{(-g)} \qquad . \tag{2}$$

Here p^α are the covariant components of the four-momentum, g is the determinant of the metric tensor, and m is the rest mass. The stress-energy tensor is determined by $\mathcal{N}$ through the equations

$$T_\alpha{}^\beta = \int (\mathcal{N}/m)\, p_\alpha\, p^\beta\, d\mathcal{V}_p \qquad , \tag{3}$$

and the metric $g_{\alpha\beta}$ is determined by Einstein's equations

$$G_{\alpha\beta} = 8\pi\, T_{\alpha\beta} \qquad . \tag{4}$$

We focus attention upon a spherical cluster undergoing small radial motions about its equilibrium state. Adopting Schwarzschild coordinates to describe the geometry of spacetime, we have

$$ds^2 = e^{\nu_A(r)+\nu_B(t,r)}\, dt^2 - e^{\lambda_A(r)+\lambda_B(t,r)}\, dr^2$$

$$- r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad . \tag{5}$$

The subscript A denotes the equilibrium values of various quantities, and B denotes the small departures of these quantities away from their equilibrium values. We will carry our analysis only to first order in the perturbation.

As discussed in §IVc of Paper I, the distribution function, $\mathcal{N}$, is split into an equilibrium part, F, and a perturbation, f:

$$\mathcal{N} = F(m, E_A, J) + f(x^\alpha, p_{(\alpha)}) \qquad . \tag{6}$$

Here a star's rest mass, m, its energy at infinity, E, and its angular momentum, J, are given by

$$m = (p_\alpha p^\alpha)^{1/2} \quad , \quad E = p_0 \quad , \quad J = [p_\theta{}^2 + (p_\phi/\sin\theta)^2]^{1/2} \quad ; \qquad (7)$$

and $p_{(\alpha)}$ are the physical components of the momentum,

$$p_{(\alpha)} = |g^{\alpha\alpha}|^{1/2} p_\alpha \quad ; \quad p^{(\alpha)} = |g_{\alpha\alpha}|^{1/2} p^\alpha \quad . \qquad (8)$$

In terms of the physical components of the momentum, we have

$$\begin{aligned} m &= [p_{(0)}{}^2 - p_{(r)}{}^2 - p_{(\theta)}{}^2 - p_{(\phi)}{}^2]^{1/2} \quad , \\ E_A &= e^{\nu_A/2} \; p_{(0)} \quad , \\ J &= r\,[p_{(\theta)}{}^2 + p_{(\phi)}{}^2]^{1/2} \quad . \end{aligned} \qquad (9)$$

The unperturbed and perturbed parts of the stress-energy tensor are given by

$$T_{A\alpha}{}^\beta = \int (F/m)\, p_\alpha p^\beta \, d\mathcal{V}_p \quad , \qquad (10)$$

$$T_{B\alpha}{}^\beta = \int (f/m)\, p_\alpha p^\beta \, d\mathcal{V}_p \quad . \qquad (11)$$

In both of these equations $p_\alpha p^\beta d\mathcal{V}_p$ are to be evaluated in the unperturbed cluster -- i.e., they have implicit subscripts A.

For the equilibrium configuration, some of the field equations (4) take the form

$$(d/dr)\,(r\,e^{-\lambda_A}) = 1 - 8\pi\, r^2\, T_{A0}{}^0 \quad , \qquad (12a)$$

$$r\,e^{-\lambda_A}\, d\nu_A/dr = 1 - e^{-\lambda_A} - 8\pi\, r^2\, T_{Ar}{}^r \quad , \qquad (12b)$$

while some of the field equations for the perturbed cluster read

$$(\partial/\partial r)\ (r\ e^{-\lambda_A}\ \lambda_B) = 8\pi\ r^2\ T_{BO}{}^{0} \quad , \tag{13a}$$

$$\partial\nu_B/\partial r = (d\nu_A/dr + 1/r)\ \lambda_B - 8\pi\ r\ e^{\lambda_A}\ T_{Br}{}^{r} \quad , \tag{13b}$$

$$\partial\lambda_B/\partial t = -\ 8\pi\ r\ e^{\lambda_A}\ T_{BO}{}^{r} \quad . \tag{13c}$$

To determine fully the dynamical evolution of the cluster, one must use, in addition to these field equations, the perturbed Boltzmann-Liouville equation for f (equation [15a; R] of Paper I); but since it will not be needed in our analysis, we shall not write it down.

b) A Useful Identity

In this subsection we shall establish a relationship among certain integrals taken over the momentum space of a bounded, spherical, equilibrium cluster. We shall use this relationship in our stability analysis of §IIc. The identity which we seek to prove is this:

$$\int (F_E/m)\ (p_0)^a\ (p_r)^b\ d\mathcal{V}_p$$

$$= -\ a \int (F/m)\ (p_0)^{a-1}\ (p_r)^b\ d\mathcal{V}_p - (b-1)\ e^{\lambda_A - \nu_A} \int (F/m)\ (p_0)^{a+1}\ (p_r)^{b-2}\ d\mathcal{V}_p \ . \tag{14}$$

Here a and b are constants, and F_E is defined by

$$F_E \equiv (\partial F/\partial E_A)_{m,J} \quad . \tag{15}$$

Proof of the identity: Notice that r, θ, and ∅ are treated as constants in equation (14), and that m and J (and hence also $p_{(\theta)}$ and $p_{(\phi)}$) are held constant in obtaining F_E from F by differentiation. Because of this we can simplify our proof by using m, $p_{(r)}$, $p_{(\theta)}$, $p_{(\phi)}$

as the independent variables in our integrals. With this choice of independent variables the left-hand side of equation (14) becomes

$$\int (F_E/m)\,(p_0)^a\,(p_r)^b\,d\mathcal{V}_p$$
$$= e^{(\lambda_A-\nu_A)/2} \int (\partial F/\partial p_{(r)})_{m,p_{(\theta)},p_{(\phi)}}\,(p_0)^a\,(p_r)^{b-1}\,[-\,dm\,dp_{(r)}\,dp_{(\theta)}\,dp_{(\phi)}]\ . \tag{16}$$

To arrive at this expression first use equations (8) and (9) to reexpress F_E in the form

$$F_E = e^{(\lambda_A/2-\nu_A)}\,(p_0/p_r)\,(\partial F/\partial p_{(r)})_{m,p_{(\theta)},p_{(\phi)}}\quad ; \tag{17}$$

Then use equations (2), (8), and (9) to express $d\mathcal{V}_p$ in the form

$$d\mathcal{V}_p = -\,dp_{(0)}\,dp_{(r)}\,dp_{(\theta)}\,dp_{(\phi)}$$
$$= -\,e^{\nu_A/2}\,(m/p_0)\,dm\,dp_{(r)}\,dp_{(\theta)}\,dp_{(\phi)}\quad ; \tag{18}$$

and then combine expressions (17) and (18) to obtain (16). Next perform an integration by parts in the integral on the right side of equation (16), obtaining

$$\int (F_E/m)\,(p_0)^a\,(p_r)^b\,d\mathcal{V}_p$$
$$= -\,e^{(\lambda_A-\nu_A)/2} \int F(\partial/\partial p_{(r)})_{m,p_{(\theta)},p_{(\phi)}}\,[(p_0)^a\,(p_r)^{b-1}]$$
$$\times\,[-\,dm\,dp_{(r)}\,dp_{(\theta)}\,dp_{(\phi)}]\ . \tag{19}$$

Finally, in equation (19) employ equation (18) and the relations

$$(\partial p_0/\partial p_{(r)})_{m, p_{(\theta)}, p_{(\phi)}} = e^{(\nu_A - \lambda_A/2)} (p_r/p_0) \quad ;$$
$$(\partial p_r/\partial p_{(r)})_{m, p_{(\theta)}, p_{(\phi)}} = e^{\lambda_A/2} \quad ; \tag{20}$$

-- which are readily derived from equations (5), (8), and (9) -- to arrive at the desired result, equation (14). QED.

c) Stability of Spherical Clusters

In deriving our operator criterion for stability we use an approach analogous to the Newtonian approach of Lynden-Bell (1966, 1967). The first step consists of changing variables from f to

$$q(x^\alpha, p_{(\alpha)}) \equiv f(x^\alpha, p_{(\alpha)}) - F_E\, p_0\, \nu_B/2$$
$$\approx f - F_E\, m\, \Phi_B \quad \text{in Newtonian limit} \quad . \tag{21}$$

Here Φ_B is the perturbation in the Newtonian potential. From its definition one verifies that q is the perturbation in the distribution function at fixed $(t, r, \theta, \phi, m, p_{(\theta)}, p_{(\phi)}, E)$, while f is the perturbation at fixed $(t, r, \theta, \phi, m, p_{(\theta)}, p_{(\phi)}, p_{(r)})$. Equivalently, q compares the number density of particles, before and after the perturbation is turned on, at the same value of the angular momentum, J, and energy at infinity, $p_0 \equiv E$, but at different values of the radial momentum, $p_{(r)}$; while f measures the change in the number density for fixed physical components, $p_{(\alpha)}$, of four-momentum but for different energies at infinity, $E \equiv p_0 = e^{\nu/2}\, p_{(0)}$.

Our derivation of the stability criterion makes use of the two perturbation functions q and ν_B rather than the three functions f, ν_B, λ_B of equations (13). From the perturbed field equations (13) we obtain a "source equation" for ν_B in terms of q by the following procedure:

First define the quantities

$$Q \equiv \int (q/m)\, p_0\, p^0\, d\mathcal{V}_p \approx \int q\, m\, d\mathcal{V}_p \quad \text{in Newtonian limit} \quad ,$$

$$W \equiv \int (q/m)\, p_r\, p^r\, d\mathcal{V}_p \approx 0 \quad \text{in Newtonian limit} \quad . \tag{22}$$

Then use equations (11), (21), and (22) to reexpress the perturbed field equations (13) in terms of q, Q, W, ν_B, λ_B rather than f, ν_B, λ_B

$$(\partial/\partial r)\,(r\, e^{-\lambda_A}\, \lambda_B) = 8\pi\, r^2\, Q + 4\pi\, r^2\, \nu_B \int (F_E/m)\, p_0\, p_0\, p^0\, d\mathcal{V}_p \quad , \tag{23a}$$

$$\partial\nu_B/\partial r = (d\nu_A/dr + 1/r)\,\lambda_B - 8\pi\, r\, e^{\lambda_A}\, W$$
$$- 4\pi\, r\, e^{\lambda_A}\, \nu_B \int (F_E/m)\, p_0\, p_r\, p^r\, d\mathcal{V}_p \quad , \tag{23b}$$

$$\partial\lambda_B/\partial t = -\, 8\pi\, r\, e^{\lambda_A} \int (q/m)\, p_0\, p^r\, d\mathcal{V}_p \quad . \tag{23c}$$

(There is no term proportional to ν_B in equation (23c) because F_E/m is an even function in three-momentum space and thus has vanishing odd moments.) From equations (10) and (14) and the relations $p^0 = e^{-\nu_A}\, p_0$, $p^r = -\, e^{-\lambda_A}\, p_r$ find that

$$\int (F_E/m)\, p_0\, p_r\, p^r\, d\mathcal{V}_p = T_{A0}{}^{0} - T_{Ar}{}^{r} \quad . \tag{24}$$

Next use the unperturbed field equations (12a) and (12b) to show that

$$(d/dr)\,(\nu_A + \lambda_A) = 8\pi\, r\, e^{\lambda_A}\,(T_{A0}{}^{0} - T_{Ar}{}^{r}) \quad . \tag{25}$$

Combine equations (23b), (24), and (25) to obtain

$$e^{-(\nu_A+\lambda_A)/2} (\partial/\partial r) [e^{(\nu_A+\lambda_A)/2} \nu_B] = (d\nu_A/dr + 1/r) \lambda_B - 8\pi r e^{\lambda_A} W \quad . \qquad (26)$$

Finally, use equation (23a) to eliminate λ_B from equation (26), obtaining

$$S \nu_B = -2Q + \frac{2}{r^2} \frac{\partial}{\partial r} \left(\frac{r^2 W}{d\nu_A/dr + 1/r}\right) \quad , \qquad (27)$$

where S is the operator

$$S \nu_B \equiv -\frac{1}{4\pi r^2} \frac{\partial}{\partial r} \left[\frac{r e^{-(\nu_A+3\lambda_A)/2}}{d\nu_A/dr + 1/r} \frac{\partial}{\partial r} (e^{(\nu_A+\lambda_A)/2} \nu_B)\right]$$

$$+ \nu_B \int \frac{F_E}{m} p_0 p_0 p^0 d\mathcal{V}_p \quad . \qquad (28)$$

This is the desired source equation for ν_B in terms of q, which enters through the integrals Q and W of equation (22).

Notice that the operator S is defined in coordinate space rather than in phase space. In fact, it is a one-dimensional operator; it involves only the radial coordinate, r. An important property of the operator S is that it is self-conjugate for bounded functions h, k:

$$\int h S k d^3 V_x = \int k S h d^3 V_x \quad , \qquad (29a)$$

where

$$d^3 V_x = (m/p^0) d\mathcal{V}_x = \sqrt{(-g)} dx^1 dx^2 dx^3$$

$$= e^{(\nu_A+\lambda_A)/2} r^2 \sin\theta \, dr \, d\theta \, d\phi \quad . \qquad (29b)$$

For the special case of an equilibrium cluster with isotropic velocity distribution (F independent of J), it is useful to define the total density of mass-energy, ρ_A, and the isotropic pressure, p_A, by

$$\rho_A = T_{A0}{}^0 = \int (F/m)\, p_0\, p^0\, d\mathcal{V}_p \qquad ,$$

$$p_A = -T_{Ar}{}^r = -T_{A\theta}{}^\theta = -T_{A\phi}{}^\phi = -\int (F/m)\, p_r\, p^r\, d\mathcal{V}_p$$

$$= (1/2) \int (F/m)\, (J/r)^2\, d\mathcal{V}_p \qquad . \qquad (30)$$

The integral appearing in expression (28) for S is related to the density and pressure by

$$\int \frac{F_E}{m}\, p_0\, p_0\, p^0\, d\mathcal{V}_p = \frac{2\, d\rho_A/dr}{d\nu_A/dr} = -(\rho_A + p_A)\, \frac{d\rho_A/dr}{dp_A/dr} \qquad . \qquad (31)$$

(The first equality results from taking the radial derivative of expression (30) for ρ_A -- for which purpose it is most convenient to use $p_{(0)}$, $p_{(r)}$, $p_{(\theta)}$, and $p_{(\phi)}$ as the coordinates in momentum space and to employ equations (8), (9), and (18); the second equality results from the relation $T_{Ar}{}^{\mu}{}_{;\mu} = 0$.) Consequently, for isotropic clusters S takes the form

$$S\,\nu_B = -\frac{1}{4\pi\, r^2}\, \frac{\partial}{\partial r} \left[\frac{r\, e^{-(\nu_A + 3\lambda_A)/2}}{d\nu_A/dr + 1/r}\, \frac{\partial}{\partial r} \left(e^{(\nu_A + \lambda_A)/2}\, \nu_B \right) \right]$$

$$- (\rho_A + p_A)\, \frac{d\rho_A/dr}{dp_A/dr}\, \nu_B \qquad . \qquad (32)$$

As the next step in our derivation of the stability criterion we take the quantity

$$H = \int \frac{(\partial f/\partial t)^2}{-F_E}\, d\mathcal{V}_p\, d\mathcal{V}_x - 4\pi \int \left(1 + r\, \frac{d\nu_A}{dr}\right) e^{\lambda_A} \left[\int \frac{f}{m}\, p_0\, p^r\, d\mathcal{V}_p\right]^2 d^3V_x \quad ,$$

(33)

which is conserved during any small, radial motion of a cluster (cf. equation [24b; R] of Paper I), and reexpress it in terms of q and ν_B by the following procedures:

First rewrite H in terms of q and Q by simply substituting expressions (21) and (22) into (33) and by remembering that F_E/m is an even function in momentum space

$$H = \int \frac{(\partial q/\partial t)^2}{-F_E} \, d\mathcal{V}_p \, d\mathcal{V}_x - \int \frac{\partial Q}{\partial t} \frac{\partial \nu_B}{\partial t} \, d^3V_x$$

$$- \frac{1}{4} \int \left[\int \frac{F_E}{m} \, p_0 \, p_0 \, p^0 \, d\mathcal{V}_p\right] \left(\frac{\partial \nu_B}{\partial t}\right)^2 \, d^3V_x$$

$$- 4\pi \int \left(1 + r \frac{d\nu_A}{dr}\right) e^{\lambda_A} \left[\int \frac{q}{m} \, p_0 \, p^r \, d\mathcal{V}_p\right]^2 \, d^3V_x \quad . \tag{34}$$

By using equations (23a), (23c), (26), (29b) and an integration by parts to manipulate the last term in equation (34), convert H into the form

$$H = \int \frac{(\partial q/\partial t)^2}{-F_E} \, d\mathcal{V}_p \, d\mathcal{V}_x - \frac{1}{2} \int \frac{\partial Q}{\partial t} \frac{\partial \nu_B}{\partial t} \, d^3V_x$$

$$- \frac{1}{2} \int \frac{\partial W}{\partial t} \frac{\partial \lambda_B}{\partial t} \, d^3V_x \quad . \tag{35}$$

Next transform the second term in this expression by using equations (28) and (29b), by integrating by parts, and by then employing equation (26), to obtain

$$H = \int \frac{(\partial q/\partial t)^2}{-F_E} \, d\mathcal{V}_p \, d\mathcal{V}_x + \frac{1}{4} \int \frac{\partial \nu_B}{\partial t} \, S \, \frac{\partial \nu_B}{\partial t} \, d^3V_x$$

$$- \int \frac{4\pi \, r \, e^{\lambda_A}}{d\nu_A/dr + 1/r} \left(\frac{\partial W}{\partial t}\right)^2 \, d^3V_x \quad . \tag{36}$$

Use equations (22) and (29b) to rewrite this expression in the form

$$H = \int \frac{1}{-F_E} \left(\frac{\partial q}{\partial t} + F_E \frac{p_r\, p^r}{p^0} \sigma \frac{\partial W}{\partial t} \right)^2 d\mathcal{V}_p\, d\mathcal{V}_x$$

$$+ \frac{1}{4} \int \frac{\partial \nu_B}{\partial t} S \frac{\partial \nu_B}{\partial t} d^3V_x \qquad , \qquad (37)$$

where σ is given by

$$\sigma = \frac{1}{\mu} \left[1 \pm \left(1 - \frac{4\pi\, r\, e^{\lambda_A} \mu}{d\nu_A/dr + 1/r} \right)^{1/2} \right] \qquad , \qquad (38a)$$

and where μ is defined by

$$\mu = \int \frac{-F_E}{m} \frac{(p_r\, p^r)^2}{p^0} d\mathcal{V}_p \qquad . \qquad (38b)$$

Expression (37) is the form of the conserved quantity, H, which we were seeking; and it provides the basis for our stability criterion: If F_E is nowhere positive throughout the phase space of the equilibrium cluster, and if σ is real, the first term in expression (37) is positive. Hence, if the second term is also positive -- i.e., if S is a positive-definite operator -- then the constancy of H prevents the perturbation from growing in time faster than linearly. (Linear time growth corresponds to marginal stability; i.e., zero-frequency motion.) Therefore we have the following theorem:

<u>A spherically symmetric, relativistic star cluster, for which</u>

$$F_E \leq 0 \qquad (39a)$$

<u>and</u>

$$\frac{4\pi\, r\, e^{\lambda_A} \mu}{d\nu_A/dr + 1/r} \leq 1 \qquad (39b)$$

throughout phase space, is stable, or at least marginally stable, against small radial perturbations if the operator S is positive-definite over the set of all physically acceptable perturbation functions, $\partial\nu_B/\partial t$:

$$\int \frac{\partial\nu_B}{\partial t}\; S\; \frac{\partial\nu_B}{\partial t}\; d^3V_x \geq 0 \qquad . \tag{40}$$

The conditions (39) are reasonable, in that one expects them to be satisfied by almost all physically interesting clusters. Condition (39a) says that there are fewer high-energy stars than low-energy stars. Notice that in the Newtonian limit μ becomes zero; S becomes the Lynden-Bell (1966, 1967, 1968) operator

$$S \approx -\frac{\nabla^2}{4\pi} + \int m^2\, F_E\, d\mathcal{V}_p \qquad \text{in Newtonian limit} \quad ; \tag{41}$$

ν_B becomes $2\Phi_B$; and the stability criterion (40) reduces to that of Lynden-Bell.

Our theorem is incomplete until we have delineated the physically acceptable perturbations, $\partial\nu_B/\partial t$, of $\partial(\ell n\, g_{00})/\partial t$. At the center of the cluster the field equations (13), and also the smoothness of the spacetime geometry, force ν_B -- and hence also $\partial\nu_B/\partial t$ -- to be a power series in r^2:

$$\partial\nu_B/\partial t = a(t) + b(t)\, r^2 + \ldots \quad \text{at} \quad r \approx 0 \qquad . \tag{42a}$$

At the cluster's surface the interior line element (5) must join onto the static Schwarzschild geometry

$$ds^2 = e^{\nu_A+\nu_B}\, dt^2 - e^{\lambda_A+\lambda_B}\, dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= \left[1 - \frac{2(M_A+M_B)}{r}\right] dt^2 - \frac{dr^2}{1 - 2(M_A+M_B)/r} - r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \quad , \qquad (43)$$

in which M_A and M_B are constants.[2] The join must be smooth in the sense

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[2]For any physical perturbation the rest mass of the cluster (i.e., the sum of the rest masses of the stars) is the same in the perturbed state as in the equilibrium state. However, the total mass-energy, $M = M_A+M_B$, need not be the same; i.e., M_B need not vanish. For example, if, at a particular instant of time, every star is given a one percent increase in $p_{(0)}$ with no change in direction of motion and no change in position (i.e., in surface area of the sphere on which it sits), then $T_0^{\ 0}$ increases by one percent everywhere, and M_B is one percent of M_A. However, M_B is constant in time during the subsequent motion of the cluster (conservation of total mass-energy; Birkhoff's theorem). Any acceptable perturbation can be expressed as a linear combination of normal modes and zero-frequency modes. Paper I discusses normal modes, and those zero-frequency modes which correspond to dynamical, uniform motion of a cluster from one configuration to another. Such a zero-frequency mode -- which occurs only in a set of measure zero of all clusters -- corresponds to the onset of instability of some normal mode. The perturbation, M_B, in M vanishes for all normal modes and for these dynamical zero-frequency modes. However, as noted by Professor K. S. Thorne (private communication), there exist other, more general zero-frequency modes which are possible in any cluster and which may induce a perturbation, M_B, in the conserved mass-energy, M. These are zero-frequency modes for which the perturbation

function f is completely static. These modes take the cluster mathematically, but not dynamically, between two slightly different equilibrium configurations which may or may not have the same total mass-energy, M. General dynamical perturbations of a cluster will typically contain zero-frequency components of this sort, and as a consequence they will have $M_B \neq 0$. In fact, the example given above must contain such a zero-frequency mode.

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that the first and second fundamental forms of the cluster's surface, $r = R + \xi(t)$, as induced by the interior four-geometry (5), must be the same as the first and second fundamental forms induced by the exterior four-geometry (43). Straightforward computations (see Appendix A) show that the first fundamental forms agree if and only if ν_B is continuous at the surface for all t, and the second fundamental forms then agree if and only if λ_B and $\partial\nu_B/\partial r$ are continuous at the surface for all t. Since $\partial\nu_B/\partial t$ and $(\partial/\partial r)(\partial\nu_B/\partial t)$ are zero outside the cluster, they must approach zero as one approaches the surface of the cluster from the interior

$$\partial\nu_B/\partial t \to 0 \quad \text{and} \quad (\partial/\partial r)(\partial\nu_B/\partial t) \to 0 \quad \text{as} \quad r \to R_- \quad . \tag{42b}$$

The function space over which S acts in the stability criterion (40) is the space of all functions satisfying the conditions (42a,b). In practice, testing cluster stability by inserting various trial functions into the integral (40) will be a risky business. One can never be completely certain, from an examination of a limited set of trial functions, that all trial functions will give a positive value of the integral. Fortunately, an alternative stability criterion involving the operator S can be derived, one which is slightly weaker than positive-definiteness, but which is much more easily applied.

Integrate the differential equation

$$S \psi = 0 \tag{44}$$

from $r = 0$, where the boundary conditions $\psi = 1$, $d\psi/dr = 0$ are imposed, to $r = R$. If the resultant function, ψ, has no nodes, then S is positive-definite on the space of trial functions satisfying conditions (42), and the cluster -- if it satisfies conditions (39) -- is stable against small radial perturbations.

Proof of the theorem: Consider the Sturm-Liouville eigenvalue problem

$$\begin{aligned} &S \psi_n = - \lambda_n V(r) \psi_n \quad , \\ &\psi_n = 1 \text{ and } d\psi_n/dr = 0 \text{ at } r = 0 \quad , \\ &\psi_n = 0 \text{ at } r = R \quad , \end{aligned} \tag{45a}$$

where

$$V(r) = \int (F_E/m) p_0 p_0 p^0 d\mathcal{V}_p \quad . \tag{45b}$$

We assume without proof that the eigenfunctions ψ_n form a complete set on the interval [0, R]. Since both the acceptable $\partial v_B/\partial t$ and their radial derivatives vanish at the surface, $r = R$, the $\partial v_B/\partial t$, for fixed t, cannot be solutions of a second-order differential equation in r. However, any trial function, $\partial v_B/\partial t$, can be expressed as a linear combination of two or more eigenfunctions, ψ_n, of equations (45):

$$\partial v_B/\partial t = \sum_n a_n \psi_n \quad \text{for fixed } t \quad ;$$
$$\sum_n a_n \, d\psi_n/dr = 0 \quad \text{at } r = R \quad . \tag{46}$$

Through use of the expansion (46) and the orthonormality relations,

$$\int \psi_m S \psi_n d^3V_x = \lambda_n \int (-V) \psi_m \psi_n d^3V_x = \lambda_n \delta_{mn} \quad , \tag{47}$$

for the Sturm-Liouville problem (45), it follows that

$$\int (\partial v_B/\partial t) S (\partial v_B/\partial t) = \sum_n a_n^2 \lambda_n \tag{48}$$

for an acceptable trial function, $\partial v_B/\partial t$. If there are no negative eigenvalues for the problem (45), equation (48) implies that S is positive-definite over the acceptable $\partial v_B/\partial t$ and that the cluster is stable. If there are at least two negative eigenvalues, λ_0 and λ_1, the trial function $\partial v_B/\partial t = a_0 \psi_0 + a_1 \psi_1$ produces a negative value when inserted into expression (48), and S is not positive-definite. If there is only one negative eigenvalue, one does not know if there is an acceptable trial function for which expression (48) is negative. Consequently, we conclude that: (i) <u>if S has no negative eigenvalues, then it is positive-definite over the</u> $\partial v_B/\partial t$, <u>and the spherical cluster satisfying conditions (39) is stable against small radial perturbations</u>; (ii) <u>if S has at least two negative eigenvalues, S is not positive-definite over the</u> $\partial v_B/\partial t$; (iii) <u>if S has only one negative eigenvalue, one does not know whether S is positive-definite over the</u> $\partial v_B/\partial t$. The first of these conclusions is all we need to prove our theorem: Because (45) is a Sturm-Liouville problem, the Sturm oscillation theorem guarantees that the n-th eigenfunction, ψ_n, has exactly n nodes -- excluding the node at $r = R$ -- and that ψ_n has one node in the region between any two adjacent nodes of ψ_{n-1}. If the differential equation $(S + \lambda V)\,\psi = 0$ is integrated from $r = 0$, where $\psi = 1$ and $d\psi/dr = 0$, to $r = R$ for every value of λ, it follows that the nodes of ψ will move inward toward $r = 0$ continuously with increasing λ (recall from definition [45b] that V is everywhere negative if $F_E \leq 0$). Consequently, since ψ_0 has a node at only $r = R$, there will be no nodes in the range [0, R] for $\lambda < \lambda_0$. For $\lambda_0 < \lambda < \lambda_1$ there will be one node in the range [0, R] -- which node moves inward with increasing λ until another node appears at $r = R$ when $\lambda = \lambda_1$, etc. Hence, if there are no nodes for $\lambda = 0$ (equation [44]), it follows that $\lambda_0 > 0$, which in turn implies that all eigenvalues of equation (45) are positive and that (conclusion (i) above) the cluster is stable. QED.

We conclude this section by emphasizing that the stability criteria developed above should be applied only to clusters with anisotropic velocity distributions. The stability criterion for isotropic clusters developed in §III of this paper is more powerful and is even easier to apply.

d) Stability of Gas Spheres

Let us now turn our attention from star clusters to gas spheres exhibiting small radial motions about their equilibrium states. In discussing the stability of gas spheres we shall employ a method analogous to that which we pursued in our discussion of the stability of spherical clusters -- i.e., we shall manipulate a quantity conserved during the motion of a perturbed, gas sphere; and we shall thereby derive a sufficient (but not necessary) condition for stability against radial perturbations for which the adiabatic index is given by $\Gamma_1(r) = (\rho+p)\, p^{-1}\, (dp/dr)\, (d\rho/dr)^{-1}$. This stability criterion will involve the positive-definiteness of a second-order differential operator, which is similar to the operator S that we encountered in §IIc. Before proceeding to our stability criterion we will first review some of the fundamental equations which govern the radial motion of a perturbed gas sphere.

If we employ Schwarzschild coordinates to describe the geometry of spacetime (cf. equation [5]) the equilibrium field equations analogous to the cluster equations (12) become (see, for example, Chandrasekhar 1964),

$$(d/dr)\,(r\, e^{-\lambda_A}) = 1 - 8\pi\, r^2\, \rho_A \quad , \tag{49a}$$

$$r\, e^{-\lambda_A}\, d\nu_A/dr = 1 - e^{-\lambda_A} + 8\pi\, r^2\, p_A \quad , \tag{49b}$$

while the perturbed field equations analogous to (13) take the form

$$(\partial/\partial r)\,(r\, e^{-\lambda_A}\, \lambda_B) = 8\pi\, r^2\, \rho_B \quad , \tag{50a}$$

$$\partial\nu_B/\partial r = (d\nu_A/dr + 1/r)\, \lambda_B + 8\pi\, r\, e^{\lambda_A}\, p_B \quad , \tag{50b}$$

$$\lambda_B = -\, 8\pi\, r\, e^{\lambda_A}\, (\rho_A + p_A)\, \xi \quad . \tag{50c}$$

Here ξ is the radial displacement of the fluid (or "gas"), and ρ and p are the density of total mass-energy and the pressure as measured by an observer comoving with the fluid. We are assuming a perfect fluid (no viscosity or heat transfer; isotropic pressure in the comoving frame). From the perturbed field equations (50a) and (50c) it follows that

$$\rho_B = -(1/r^2)\,(\partial/\partial r)\,[r^2\,(\rho_A + p_A)\,\xi] \quad . \tag{51}$$

If we make use of the relation

$$\frac{d\nu_A}{dr} = -2\,\frac{dp_A/dr}{\rho_A + p_A} \quad , \tag{52}$$

which follows from the equation of motion $T_r{}^{\alpha}{}_{;\alpha} = 0$, we can rewrite equation (51) in the alternate form

$$\rho_B = -\frac{d\rho_A}{dr}\,\xi - \frac{(\rho_A + p_A)}{r^2}\,e^{\nu_A/2}\,\frac{\partial}{\partial r}\,(r^2\,e^{-\nu_A/2}\,\xi) \quad . \tag{53}$$

We will restrict attention to those perturbations of a gas sphere for which the adiabatic index is given by

$$\Gamma_1 \equiv \frac{(\rho_A + p_A)}{p_A}\,\left(\frac{\partial p}{\partial \rho}\right)_{\text{constant entropy}} = \frac{(\rho_A + p_A)}{p_A}\,\frac{dp_A/dr}{d\rho_A/dr} \quad . \tag{54}$$

It then follows that

$$p_B = \frac{dp_A/dr}{d\rho_A/dr}\,\rho_B = \frac{dp_A}{d\rho_A}\,\rho_B \quad . \tag{55}$$

(Henceforth we write $(dp_A/dr)\,(d\rho_A/dr)^{-1}$ symbolically as $dp_A/d\rho_A$.)

In arriving at an operator criterion for the stability of gas spheres we employ a series of steps patterned after the analysis of §IIc. The initial step thus consists in obtaining a source equation for ν_B analogous to equation (27):

First introduce the quantities

$$Q^* \equiv \rho_B - \frac{d\rho_A/dr}{d\nu_A/dr}\,\nu_B = \rho_B + \frac{(\rho_A + p_A)}{dp_A/d\rho_A}\,\frac{\nu_B}{2} \quad , \tag{56a}$$

$$W^* = -p_B + \frac{dp_A/dr}{d\nu_A/dr}\ \nu_B = -p_B - (\rho_A + p_A)\ \frac{\nu_B}{2} = -\frac{dp_A}{d\rho_A}\ Q^* \quad , \qquad (56b)$$

which are changes in ρ and -p at fixed potential, ν. These are the analogues of the quantities Q and W for a cluster (equations [22]) in the following sense: If a cluster with isotropic velocity distribution is perturbed radially, and if we define its equilibrium density and pressure by equations (30), then its perturbation functions Q and W of equations (22) take the form

$$Q = T_{BO}{}^{O} - \frac{d\rho_A/dr}{d\nu_A/dr}\ \nu_B = T_{BO}{}^{O} + \frac{(\rho_A + p_A)}{dp_A/d\rho_A}\ \frac{\nu_B}{2} \quad , \qquad (22a')$$

$$W = T_{Br}{}^{r} + \frac{dp_A/dr}{d\nu_A/dr}\ \nu_B = T_{Br}{}^{r} - (\rho_A + p_A)\ \frac{\nu_B}{2}$$

$$\neq (dp_A/d\rho_A)\ Q \text{ in general} \quad . \qquad (22b')$$

(Equation [22a'] follows directly from equations [21], [22], and [31]; while equation [22b'] follows from equations [14], [21], and [22].) In the derivation of a source equation for ν_B, next rewrite equations (50a,b) in terms of Q^*, W^*, ν_B, and λ_B, thereby eliminating ρ_B and p_B:

$$\frac{\partial}{\partial r}\ (r\ e^{-\lambda_A}\ \lambda_B) = 8\pi\ r^2\ Q^* - 4\pi\ r^2\ \frac{(\rho_A + p_A)}{dp_A/d\rho_A}\ \nu_B \quad , \qquad (57a)$$

$$e^{-(\nu_A+\lambda_A)/2}\ (\partial/\partial r)\ [e^{(\nu_A+\lambda_A)/2}\ \nu_B] = (d\nu_A/dr + 1/r)\ \lambda_B - 8\pi\ r\ e^{\lambda_A}\ W^* \ . \qquad (57b)$$

(In deriving equation [57b] employ equation [25], with the identifications $\rho_A = T_{AO}{}^{O}$ and $p_A = -T_{Ar}{}^{r}$, which follow directly from the field equations [50a,b] for the gas sphere.) Next combine equations (57a) and (57b) and thereby obtain the desired source equation for ν_B in terms of Q^* and W^*,

$$S\ \nu_B = -2\,Q^* + \frac{2}{r^2}\frac{\partial}{\partial r}\left[\frac{r^2\,W^*}{d\nu_A/dr + 1/r}\right] \quad . \qquad (58)$$

Here S is precisely the same operator (equation [32]) as appears in the analogous source equation (27) for a cluster with isotropic velocity distribution. Consequently, as for clusters, so also for gas spheres, the operator S is self-conjugate over the space of bounded functions (equation [29a]).

Continuing our derivation of the stability criterion along the lines of §IIc, we next consider a quantity which is conserved during the small, radial, adiabatic motions of a gas sphere. This quantity is

$$H^* = K^* + P^* \quad , \qquad (59)$$

where

$$K^* = (1/2) \int e^{(\lambda_A - \nu_A)} (\rho_A + p_A)\,(\partial\xi/\partial t)^2\ d^3V_x \quad , \qquad (60)$$

and where

$$\begin{aligned} P^* = {} & 2 \int (dp_A/dr)\,(\xi^2/r)\ d^3V_x \\ & + (1/2) \int e^{\nu_A}\,\Gamma_1\,(p_A/r^4)\,[(\partial/\partial r)\,(r^2 e^{-\nu_A/2}\,\xi)]^2\ d^3V_x \\ & + 4\pi \int e^{\lambda_A}\,p_A\,(\rho_A + p_A)\,\xi^2\ d^3V_x \\ & - (1/2) \int (\rho_A + p_A)^{-1}\,(dp_A/dr)^2\,\xi^2\ d^3V_x \quad . \end{aligned} \qquad (61)$$

The quantities K^* and P^* are the kinetic and potential energies (to second order in the radial displacement, ξ) associated with the perturbation, and H^* is the second-order change in the total mass-energy from its value at equilibrium (see equation [B26] of Harrison, Thorne, Wakano, and Wheeler [1965] with an obvious error corrected).

Our present aim is to express P^* in terms of Q^* and ν_B. To this end, we first rewrite P^* in terms of ρ_B and λ_B by the following manipulations:

Substitute equations (53) and (54) into the second integral in the definition of P^* and obtain

$$P^* = \frac{1}{2} \int \frac{dp_A/d\rho_A}{\rho_A + p_A} (\rho_B)^2 d^3V_x$$
$$+ \frac{1}{2} \int \frac{dp_A/dr}{\rho_A + p_A} \left\{ 2 \rho_B \xi + \left[\frac{d\rho_A}{dr} + \frac{4}{r} (\rho_A + p_A) - \frac{dp_A}{dr} \right] \xi^2 \right\} d^3V_x$$
$$+ 4\pi \int e^{\lambda_A} p_A (\rho_A + p_A) \xi^2 d^3V_x \quad . \tag{62}$$

Next use equation (51) in the second integral of the above expression to find

$$P^* = \frac{1}{2} \int \frac{dp_A/d\rho_A}{\rho_A + p_A} (\rho_B)^2 d^3V_x$$
$$+ \frac{1}{2} \int \frac{e^{\lambda_A}}{r^2} (m_A + 4\pi r^3 p_A) \left\{ \frac{\partial}{\partial r} [(\rho_A + p_A) \xi^2] + 2 \frac{dp_A}{dr} \xi^2 \right\} d^3V_x$$
$$+ 4\pi \int e^{\lambda_A} p_A (\rho_A + p_A) \xi^2 d^3V_x \quad , \tag{63}$$

where m_A is the total gravitating mass within coordinate radius r (cf. equation [49a])

$$m_A(r) = (1/2)\, r\, (1 - e^{-\lambda_A}) = \int_0^r \rho_A \, 4\pi r^2 \, dr \quad . \tag{64}$$

(In arriving at equation [63] employ the Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium,

$$dp_A/dr = - e^{\lambda_A} (\rho_A + p_A) (m_A + 4\pi r^3 p_A)/r^2 \quad , \tag{65}$$

which is derivable from the equilibrium equations [40a,b], [52], and [64].) Perform an integration by parts on the first term in the second integral in equation (63) and use equations (25), (29b), (52), (64), and (65) to obtain

$$P^* = \frac{1}{2} \int \frac{dp_A/d\rho_A}{\rho_A + p_A} (\rho_B)^2 d^3V_x$$

$$- 2\pi \int \frac{e^{\lambda_A}}{r^2} [2r e^{\lambda_A} (m_A + 4\pi r^3 p_A) + r^2] (\rho_A + p_A)^2 \xi^2 d^3V_x \quad . \tag{66}$$

Next use equation (65) to rewrite equation (52) in the form

$$2r e^{\lambda_A} (m_A + 4\pi r^3 p_A) + r^2 = r^2 (1 + r\, d\nu_A/dr) \quad ; \tag{67}$$

and combine this relation with equations (50c) and (66) to obtain the desired expression for P^* in terms of ρ_B and λ_B:

$$P^*(\rho_B, \lambda_B) = \frac{1}{2} \int \frac{dp_A/d\rho_A}{\rho_A + p_A} (\rho_B)^2 d^3V_x$$

$$- \frac{1}{32\pi} \int \frac{e^{-\lambda_A}}{r^2} \left(1 + r \frac{d\nu_A}{dr}\right) (\lambda_B)^2 d^3V_x \quad . \tag{68}$$

We will use this expression for P^* in §III to establish a relationship between the stabilities of isotropic clusters and gas spheres; but we presently wish to continue our manipulations so as to reexpress P^* in terms of Q^* and ν_B: Rewrite the first integral in expression (68) in terms of Q^* and ν_B by using definition (56a), and transform the second integral via equation (57b) to obtain

$$P^* = \frac{1}{2} \int \frac{dp_A/d\rho_A}{\rho_A + p_A} \, Q^{*2} \, d^3V_x - \frac{1}{2} \int Q^* \, \nu_B \, d^3V_x$$

$$+ \frac{1}{8} \int \frac{\rho_A + p_A}{dp_A/d\rho_A} \, (\nu_B)^2 \, d^3V_x$$

$$- \frac{1}{32\pi} \int \frac{e^{-\lambda_A}}{r^2} \, \lambda_B \, [r \, e^{-(\nu_A+\lambda_A)/2} \, \frac{\partial}{\partial r} (e^{(\nu_A+\lambda_A)/2} \, \nu_B)$$

$$+ 8\pi \, r^2 \, e^{\lambda_A} \, W^*] \, d^3V_x \quad . \qquad (69)$$

Next perform an integration by parts on the last integral in the above expression, and use equation (57a), to find

$$P^* = \frac{1}{2} \int \frac{dp_A/d\rho_A}{\rho_A + p_A} \, Q^{*2} \, d^3V_x - \frac{1}{4} \int Q^* \, \nu_B \, d^3V_x - \frac{1}{4} \int W^* \, \lambda_B \, d^3V_x \quad . \qquad (70)$$

By using equations (57b) and (58) and an integration by parts, show that

$$\int Q^* \, \nu_B \, d^3V_x = - \frac{1}{2} \int \nu_B \, S \quad \nu_B \, d^3V_x - \int W^* \, \lambda_B \, d^3V_x$$

$$+ \int \frac{8\pi \, r \, e^{\lambda_A}}{d\nu_A/dr + 1/r} \, (W^*)^2 \, d^3V_x \quad . \qquad (71)$$

Finally, combine equations (56b), (70), and (71) to obtain

$$P^* = \frac{1}{2} \int \frac{dp_A/d\rho_A}{\rho_A + p_A} \left[1 - \frac{4\pi \, r \, e^{\lambda_A} \, (\rho_A+p_A)(dp_A/d\rho_A)}{d\nu_A/dr + 1/r} \right] Q^{*2} \, d^3V_x$$

$$+ \frac{1}{8} \int \nu_B \, S \quad \nu_B \, d^3V_x \quad . \qquad (72)$$

This is the desired expression for P^* in terms of Q^* and ν_B, and it enables us to arrive at a stability criterion for gas spheres which is analogous to the stability criterion of §IIc for spherical clusters:

The kinetic energy of motion, K^*, of a perturbed gas sphere is positive-definite (cf. expression [60]). If the coefficient of Q^{*2} in the first integral in expression (72) is non-negative throughout the equilibrium configuration of the gas sphere, and if S is a positive-definite operator, then P^* is also positive-definite -- and the perturbation cannot grow in time since H^* (cf. equation [59]) is a constant of the motion. Hence our analysis results in the following theorem:

A relativistic gas sphere, for which

$$\frac{4\pi\, r\, e^{\lambda_A} (\rho_A + p_A)(dp_A/d\rho_A)}{d\nu_A/dr + 1/r} \leq 1 \tag{73}$$

throughout space, is stable against small, radial perturbations for which the adiabatic index is given by

$$\Gamma_1 = \frac{\rho_A + p_A}{p_A} \left(\frac{\partial p}{\partial \rho}\right)_{\text{constant entropy}} = \frac{\rho_A + p_A}{p_A} \frac{dp_A/dr}{d\rho_A/dr} \geq 0 \quad , \tag{74}$$

if the operator S is positive-definite over the set of all physically acceptable perturbation functions, ν_B:

$$\int \nu_B\, S\, \nu_B\, d^3V_x \geq 0 \quad . \tag{75}$$

For a gas sphere -- but not for a star cluster -- the perturbed configuration has identically the same total mass-energy as the unperturbed configuration, to first order in the perturbation (see, e.g., Harrison et al. 1965). Consequently, ν_B and $\partial\nu_B/\partial r$ must vanish identically at the boundary of the gas sphere, just as $\partial\nu_B/\partial t$ and $(\partial/\partial r)\,(\partial\nu_B/\partial t)$ do for a cluster; and the boundary conditions on acceptable perturbation functions, ν_B, for a gas sphere are identical to those (equations [42]) on acceptable perturbation functions, $\partial\nu_B/\partial t$, for a cluster.

Condition (73) for a gas sphere is in general not the same as condition (39b) for an isotropic cluster. Notice that the equilibrium equations (49) can be used to rewrite condition (73) in the equivalent form

$$\frac{1}{2}\ \frac{d\nu_A/dr + d\lambda_A/dr}{d\nu_A/dr + 1/r}\ \frac{dp_A}{d\rho_A} \leq 1 \quad . \tag{73'}$$

For a gas sphere whose structure is not highly relativistic, $dp_A/d\rho_A$ is much less than unity and $1/r$ is large compared to either $d\nu_A/dr$ or $d\lambda_A/dr$. Hence one expects that condition (73') will be satisfied in this case. On the other hand, for a gas sphere which is highly relativistic, one frequently finds that there exists a region in the sphere throughout which $p_A \approx (\gamma-1)\ \rho_A$ with $1 \leq \gamma \leq 2$. Thus $dp_A/d\rho_A = (\gamma-1)$ can be as large as unity in such a region. However, one can show by using the equilibrium equations (49), (64), and (65) that condition (73) is satisfied even in this latter case. Hence (73) is likely to be satisfied for most physically interesting gas spheres.

III. RELATIONSHIP BETWEEN THE STABILITIES OF ISOTROPIC CLUSTERS AND GAS SPHERES

In this section we shall complete our discussion of the connection between the stabilities of spherical clusters and gas spheres. We shall show that an isotropic cluster is stable against small radial perturbations if the gas sphere which has the same radial distribution of density and pressure is stable against small, radial perturbations for which the adiabatic index is given by $\Gamma_1 = (\rho+p)\ p^{-1}\ (dp/dr)\ (d\rho/dr)^{-1}$. We will establish this connection by comparing the conserved quantity H of equation (33) for a perturbed cluster with the potential energy function P^*, as expressed in equation (68), for the associated gas sphere.

We begin by transforming the conserved quantity H for a perturbed, isotropic cluster into a form closely related to that of expression (68) for the potential energy, P^*, of the corresponding gas sphere:

First use equations (11) and (13c) to rewrite expression (33) as

$$H = \int \frac{(\partial f/\partial t)^2}{-F_E}\, d\mathcal{V}_p\, d\mathcal{V}_x - \frac{1}{16\pi}\int \frac{e^{-\lambda_A}}{r^2}\left(1+r\frac{d\nu_A}{dr}\right)\left(\frac{\partial\lambda_B}{\partial t}\right)^2 d^3V_x \quad . \qquad (76a)$$

Next split f into a part f_+ which is even, and a part f_- which is odd under reflections of the spatial momenta ($p_{(r)} \to -p_{(r)}$, $p_{(\theta)} \to -p_{(\theta)}$, $p_{(\phi)} \to -p_{(\phi)}$, $p_{(0)} \to +p_{(0)}$). (Cf. equation [16; R] of Paper I, and associated discussion.) When H is rewritten in terms of the even and odd parts of f, it takes the form

$$H = \int \frac{(\partial f_-/\partial t)^2}{-F_E}\, d\mathcal{V}_p\, d\mathcal{V}_x + \int \frac{(\partial f_+/\partial t)^2}{-F_E}\, d\mathcal{V}_p\, d\mathcal{V}_x$$

$$- \frac{1}{16\pi}\int \frac{e^{-\lambda_A}}{r^2}\left(1+r\frac{d\nu_A}{dr}\right)\left(\frac{\partial\lambda_B}{\partial t}\right)^2 d^3V_x \quad . \qquad (76b)$$

This is the desired expression for H.

For an isotropic cluster, in which $F_E \leq 0$, the following inequality is valid for the second integral in this expression:

$$\int \frac{(\partial f_+/\partial t)^2}{-F_E}\, d\mathcal{V}_p\, d\mathcal{V}_x \geq \int \frac{dp_A/d\rho_A}{\rho_A + p_A}\left(\frac{d\rho_B}{dt}\right)^2 d^3V_x \quad , \qquad (77)$$

where the identifications $\rho_A = T_{A0}{}^{0}$, $p_A = -T_{Ar}{}^{r} = -T_{A\theta}{}^{\theta} = -T_{A\phi}{}^{\phi}$, and $\rho_B = T_{B0}{}^{0}$ have been made. Proof of inequality (77): The proof involves an application of the familiar Schwarz inequality,

$$(h, h)\ (k, k)\ \geq\ (h, k)^2 \qquad , \qquad (78a)$$

where, for our purposes, we take the inner product to be defined by

$$(h, k)\ \equiv\ \int (hk/m)\ p_0\ p^0\ d\mathcal{V}_p \qquad . \qquad (78b)$$

For the functions

$$h\ =\ (\partial f_+/\partial t)/\sqrt{(-F_E\ p_0)} \quad , \quad k\ =\ \sqrt{(-F_E\ p_0)} \quad ,$$

the Schwarz inequality (78a) reads

$$\left[\int \frac{(\partial f_+/\partial t)^2}{-F_E}\ \frac{p^0}{m}\ d\mathcal{V}_p\right]\left[\int \frac{-F_E}{m}\ p_0\ p_0\ p^0\ d\mathcal{V}_p\right]\ \geq\ \left[\int \frac{\partial f_+/\partial t}{m}\ \ p_0\ p^0\ d\mathcal{V}_p\right]^2 . \quad (79)$$

Divide by the second term on the left and integrate over d^3V_x to obtain

$$\int \frac{(\partial f_+/\partial t)^2}{-F_E}\ d\mathcal{V}_p\ d\mathcal{V}_x\ \geq\ \int\ \frac{\left\{\int [(\partial f_+/\partial t)/m]\ p_0\ p^0\ d\mathcal{V}_p\right\}^2}{\int\ (-F_E/m)\ p_0\ p_0\ p^0\ d\mathcal{V}_p}\ \ d^3V_x \quad . \quad (80)$$

Finally use equations (31) and

$$\rho_B\ \equiv\ T_{B0}{}^0\ =\ \int\ (f/m)\ p_0\ p^0\ d\mathcal{V}_p\ =\ \int\ (f_+/m)\ p_0\ p^0\ d\mathcal{V}_p \qquad (81)$$

to manipulate the inequality (80) into the desired form (77). QED.

Turn attention next to expression (68) for the potential energy P^* of a perturbed gas sphere. As follows from the variational principle governing the normal modes of radial pulsation of gas spheres (Chandrasekhar 1964), a necessary and sufficient condition for the stability of a gas sphere against small, radial perturbations for which the adiabatic index is given by $\Gamma_1 = (\rho_A+p_A)\ p_A{}^{-1}\ (dp_A/dr)\ (d\rho_A/dr)^{-1}$ is that expression (68) is positive-definite for all acceptable perturbation functions. Recall that the perturbation functions, $\rho_{B\ gas}$ and

$\lambda_{B\ gas}$, for a gas sphere are related through equation (50a), which fixes $\lambda_{B\ gas}$ uniquely once $\rho_{B\ gas}$ is specified as a function of t and r. As follows from the field equations and the smoothness of the spacetime geometry at the center of symmetry, the acceptable perturbations $\rho_{B\ gas}$ behave as a power series in r^2 near the origin:

$$\rho_{B\ gas} = a(t) + b(t)\, r^2 + \ldots \qquad \text{near} \quad r = 0 \quad . \tag{82a}$$

At the surface, r = R, of the gas sphere the Lagrangian change in pressure must vanish. For any equilibrium gas sphere with the same density and pressure distributions, $\rho_A(r)$ and $p_A(r)$, as an isotropic star cluster, $\rho_A(r)$ will approach zero as r approaches R (no discontinuity in density at the surface; see Appendix B). In this case a vanishing Lagrangian change in pressure is equivalent to a vanishing Eulerian change in density

$$\rho_{B\ gas} \to 0 \qquad \text{as} \qquad r \to R_- \quad . \tag{82b}$$

A final restriction on $\rho_{B\ gas}$ is that the change in total gravitating mass, M, from its equilibrium value vanishes to first order in the perturbation

$$M_B = \int_0^R \rho_{B\ gas}\ 4\pi\, r^2\, dr = 0 \quad . \tag{82c}$$

If expression (68) is positive-definite, i.e., if $P^*(\rho_{B\ gas}, \lambda_{B\ gas}) \geq 0$ for all perturbation functions $\rho_{B\ gas}$ which satisfy the conditions (82) (with $\lambda_{B\ gas}$ given in terms of $\rho_{B\ gas}$ by [50a]), then the gas sphere is stable against all small, radial perturbations associated with the special form of the adiabatic index, $\Gamma_1(r)$, which we are considering. Conversely, if P^* in expression (68) is not positive-definite, the gas sphere is unstable.

We are now in a position to prove the following statement: _The expression P^* of equation (68) is positive-definite over the set of allowable, radial, perturbation functions,_ $(\partial/\partial t)\,(\rho_{B\ \mathrm{cluster}})$ _and_ $(\partial/\partial t)\,[\lambda_B\,(\rho_B)_{\mathrm{cluster}}]$, _for an isotropic cluster, i.e.,_ $P^*\,[(\partial/\partial t)\,(\rho_{B\ \mathrm{cluster}}),\ \ (\partial/\partial t)\,(\lambda_{B\ \mathrm{cluster}})] \geq 0$, _if the corresponding gas sphere (the gas sphere which has the same radial distribution of density and pressure as has the cluster) is stable against radial perturbations for which the adiabatic index is_ $\Gamma_1(r) = (\rho_A + p_A)\,p_A^{-1}\,(dp_A/dr)\,(d\rho_A/dr)^{-1}$. To establish the validity of this statement we need merely note that $(\partial/\partial t)\,(\rho_{B\ \mathrm{cluster}})$ and $(\partial/\partial t)\,(\lambda_{B\ \mathrm{cluster}})$ are related by an equation of the same form as equation (50a), that $(\partial/\partial t)\,(\rho_{B\ \mathrm{cluster}})$ satisfies the conditions of equations (82), that $(\partial/\partial t)\,(\rho_{B\ \mathrm{cluster}})$ is therefore an acceptable $\rho_{B\ \mathrm{gas}}$, and that P^* is positive-definite over the acceptable sets $[\rho_{B\ \mathrm{gas}},\ \lambda_B\,(\rho_B)_{\mathrm{gas}}]$ if the gas sphere is stable. (That $(\partial/\partial t)\,(\rho_{B\ \mathrm{cluster}})$ satisfies condition (82a) follows from the smoothness of the spacetime geometry at the center of symmetry. That it satisfies condition (82b) is proved in Appendix B. That it satisfies condition (82c) follows from the fact that $(\partial/\partial t)\,(M_B) = 0$ for a perturbed cluster.)

We can now compare the quantity, H, conserved during the motion of a perturbed, isotropic cluster with the quantity P^*. We define a quantity

$$P\left[\frac{\partial f_+}{\partial t}, \frac{\partial}{\partial t}(\lambda_{B\ \mathrm{cluster}})\right] \equiv \frac{1}{2}\int \frac{(\partial f_+/\partial t)^2}{-F_E}\, d\mathcal{V}_p\, d\mathcal{V}_x$$

$$-\frac{1}{32\pi}\int \frac{e^{-\lambda_A}}{r^2}(1+r\, d\nu_A/dr)\left[\frac{\partial}{\partial t}(\lambda_{B\ \mathrm{cluster}})\right]^2 d^3V_x \quad , \qquad (83)$$

so that

$$H = \int \frac{(\partial f_-/\partial t)^2}{-F_E}\, d\mathcal{V}_p\, d\mathcal{V}_x + 2P\left[\frac{\partial f_+}{\partial t}, \frac{\partial}{\partial t}(\lambda_{B\ \mathrm{cluster}})\right] \qquad (84)$$

for a perturbed cluster. For an isotropic cluster with $F_E \leq 0$ it follows from equations (68), (77), and (83) that

$$P[\partial f_+/\partial t, (\partial/\partial t)(\lambda_{B\ \mathrm{cluster}})]$$
$$\geq P^*[(\partial/\partial t)(\rho_{B\ \mathrm{cluster}}), (\partial/\partial t)(\lambda_{B\ \mathrm{cluster}})] \quad . \qquad (85)$$

This inequality enables us to establish the relationship which we were seeking between the stability of a spherical cluster with isotropic pressure and the stability of the corresponding gas sphere: Suppose that the corresponding gas sphere is stable against radial perturbations for which $\Gamma_1 = (\rho+p)\, p^{-1}(dp/dr)(d\rho/dr)^{-1}$. Then our previous discussion shows that $P^*[(\partial/\partial t)(\rho_{B\ \mathrm{cluster}}), (\partial/\partial t)(\lambda_{B\ \mathrm{cluster}})] \geq 0$ at all times throughout the motion of the cluster. Coupled with inequality (85), this result implies in turn that H is the sum of two positive-definite terms (cf. equation [84]), and that the constancy of H therefore prevents the perturbations from growing in time. We have proved the following theorem:

Consider a bounded, spherical, relativistic cluster with isotropic pressure and with $F_E \leq 0$. Such a cluster is stable against collisionless, radial perturbations if the gas sphere which has the same radial distribution of density of total mass-energy,

$$\rho = T_0{}^0 = \int (F/m)\, p_0\, p^0\, d\mathcal{V}_p \quad , \tag{86a}$$

and of pressure,

$$p = -T_1{}^1 = -T_2{}^2 = -T_3{}^3 = -\int (F/m)\, p_r\, p^r\, d\mathcal{V}_p \quad , \tag{86b}$$

is stable against radial perturbations for which the adiabatic index is given by

$$\Gamma_1 \equiv \frac{\rho + p}{p} \left(\frac{\partial p}{\partial \rho}\right)_{\text{constant entropy}} = \frac{\rho + p}{p} \frac{dp/dr}{d\rho/dr} \quad . \tag{87}$$

Note that this theorem is not limited by any restriction such as that of condition (39b).

IV. CONCLUSION

In this paper we have developed methods which should be useful in studying the collisionless stability of relativistic star clusters. Their usefulness lies in the fact that their application requires only the solution of one-dimensional problems. The price paid for this relative simplification is that these tools only provide sufficient conditions for stability. For clusters in which they fail to prove stability, one must turn for more conclusive results to the necessary and sufficient, multidimensional variational principle of Paper I. These methods are now being used by the author to study the stability of specific relativistic star clusters. The results will be reported in a subsequent paper in this series.

I wish to express my gratitude to Professor Donald Lynden-Bell for helpful suggestions; I extend my special thanks to Professor Kip S. Thorne for his encouragement and useful advice throughout the course of this research.

APPENDIX A

JUNCTION CONDITIONS AT THE SURFACE OF A PULSATING CLUSTER

The proper junction conditions to be imposed on the metric at the surface of a pulsating spherical cluster are that the first and second fundamental forms of the cluster's timelike three-surface, as computed from the interior four-geometry, agree with the first and second fundamental forms as computed from the exterior four-geometry.

The equation of the surface of a perturbed spherical cluster is

$$r - R - \xi(t) = 0 \qquad , \qquad \text{(A1)}$$

where R is the value of the radial coordinate at the equilibrium position of the surface, and where $\xi(t)$ is the displacement of the surface from its equilibrium position. In the Schwarzschild coordinate system which we are employing to describe both the interior and exterior four-geometries, it follows that displacements in the radial and time directions on the surface are related by the equation

$$dr = (d\xi/dt)\, dt \qquad . \qquad \text{(A2)}$$

Hence the first fundamental form, Φ_1, of the surface is

$$\Phi_1 = [e^{\nu} - e^{\lambda}\,(d\xi/dt)^2]\, dt^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad . \qquad \text{(A3)}$$

The displacement, ξ, is a first-order quantity. Consequently, to first order, Φ_1 is continuous across the surface if and only if ν_A and ν_B are continuous there.

The second fundamental form, Φ_2, of the surface is given by

$$\Phi_2 = (-\, n_{\alpha;\beta}\, dx^{\alpha}\, dx^{\beta})_s \qquad , \qquad \text{(A4)}$$

where n^{α} is the unit normal to the surface. The subscript "s" denotes that relation (A2) must be employed in the definition (A4). The only non-vanishing components of the unit normal are

$$n^0 = e^{-\nu} \frac{d\xi/dt}{N} \quad ; \quad n^r = \frac{e^{-\lambda}}{N} \quad ; \quad N = \sqrt{[e^{-\lambda} - e^{-\nu} (\frac{d\xi}{dt})^2]} \quad . \qquad \text{(A5)}$$

A simple computation, correct to first order, then reveals that

$$\Phi_2 = K_{tt} \, dt^2 + K_{\theta\theta} \, (d\theta^2 + \sin^2\theta \, d\phi^2) \qquad , \qquad \text{(A6a)}$$

where

$$K_{tt} = - e^{\lambda_A/2} \frac{d^2\xi}{dt^2} - \frac{1}{2} e^{\nu_A - \lambda_A/2} \left[\frac{d\nu_A}{dr}\left(1 + \nu_B - \frac{\lambda_B}{2}\right) + \frac{\partial \nu_B}{\partial r}\right] \quad ,$$

$$K_{\theta\theta} = r \, e^{-\lambda_A/2} \, (1 - \lambda_B/2) \quad . \qquad \text{(A6b)}$$

Continuity of Φ_2 for a displacement in the (θ, ϕ) plane demands that $K_{\theta\theta}$, and hence λ_A and λ_B, are continuous across the surface. Continuity of K_{tt} then demands that $d\nu_A/dr$ and $\partial\nu_B/\partial r$ are continuous across the surface. Hence λ_B, ν_B, and $\partial\nu_B/\partial r$ must be continuous at the surface of a perturbed spherical cluster.

APPENDIX B

BOUNDARY CONDITIONS ON THE DENSITY AT THE SURFACE OF AN ISOTROPIC CLUSTER

For an equilibrium cluster with isotropic velocity distribution the distribution function is independent of angular momentum; $F = F(p_o, m)$. In this case equations (2), (8), (9), and (10) enable one to write the cluster's pressure and density as

$$p_A = -T_{Ar}{}^{r} = -T_{A\theta}{}^{\theta} = -T_{A\phi}{}^{\phi}$$

$$= (4\pi/3) \int_1^\infty \left[\int F(e^{\nu_A(r)/2}\, m\, x,\, m)\, m^4\, dm\right] (x^2-1)^{3/2}\, dx \quad , \qquad \text{(B1a)}$$

$$\rho = T_{Ao}{}^{0} = 4\pi \int_1^\infty \left[\int F(e^{\nu_A(r)/2}\, m\, x,\, m)\, m^4\, dm\right] x^2 \sqrt{(x^2-1)}\, dx \quad , \qquad \text{(B1b)}$$

(see, e.g., Fackerell 1968). At the surface, $r = R$, of the cluster the radial stress, $T_{Ar}{}^{r} = -p_A(r)$, must be zero. This means that $F(e^{\nu_A(R)/2}\, m\, x,\, m)$ must vanish for all $x > 1$; i.e., it means that

$$F(p_o, m) = 0 \quad \text{for all} \quad p_o > m\, e^{\nu_A(R)/2} \quad . \qquad \text{(B2)}$$

Conditions (B1) and (B2) guarantee not only that $\lim_{r \to R_-} p_A(r) = 0$, but also that

$$\lim_{r \to R_-} \rho_A(r) = 0 \quad , \qquad \text{(B3a)}$$

unless F has a strong delta-function singularity at $p_o = m\, e^{\nu_A(R)/2}$. However, delta-function singularities in p_o are excluded since we are concerned only with clusters for which $F_E \leq 0$. Not only does ρ_A vanish at the cluster's surface; $d\rho_A/dr$ also vanishes

$$\lim_{r \to R_-} d\rho_A/dr = 0 \quad . \tag{B3b}$$

One discovers this from equation (31), and the fact that $d\nu_A/dr$ is finite and non-zero at the cluster's surface (cf. Appendix A).

Turn attention now to perturbations of an isotropic cluster. For all acceptable perturbations except a set of measure zero, considerations similar to the above enable one to conclude that $T_0{}^0$ vanishes at the surface, $r = R + \xi(t)$, of the perturbed cluster. That special class of perturbations which leads to non-vanishing $T_0{}^0$ at the surface consists of those perturbations which attach a shell of stars with circular orbits to the cluster's surface. Hence, except for this special set of very physically unreasonable perturbations, the Lagrangian change in $T_0{}^0$ vanishes at $r = R$:

$$\rho_B + (d\rho_A/dr)\,\xi = 0 \quad \text{at} \quad r = R \quad . \tag{B4}$$

Equations (B3b) and (B4) may then be combined to show that ρ_B and $\partial\rho_B/\partial t$ vanish at the cluster's surface

$$\lim_{r \to R_-} \rho_B = \lim_{r \to R_-} \partial\rho_B/\partial t = 0 \quad . \tag{B5}$$

In summary, the above analysis shows that in all physically reasonable cases ρ_A, ρ_B, and $\partial\rho_B/\partial t$ approach zero as $r \to R_-$. These limits play an important role in §III.

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7. Relativistic, Spherically Symmetric Star Clusters.
III. Stability of Compact Isotropic Models

(To be published in Astrophysical Journal)

I. INTRODUCTION

In the two preceding papers of this series [Ipser and Thorne 1968 (Paper I), Ipser 1969 (Paper II)] the theory of the stability against spherical perturbations of collisionless, spherically symmetric star clusters was developed within the framework of general relativity. Among the results derived in Paper I was a variational principle for the normal modes of radial oscillation. This variational principle provides a necessary and sufficient condition for the stability of the cluster. In Paper II were presented sufficient (but not necessary) criteria for stability. These sufficient criteria are much easier to apply than the variational principle. In applying them one has only to determine whether a certain ordinary differential operator is positive definite over a set of one-dimensional perturbation functions (cf. §IIc and §III of Paper II), whereas, to apply the variational principle of Paper I, one must attack a problem which is at least two-dimensional.

In this paper we shall use the stability criteria of Papers I and II to study numerically the stability of some compact models (i.e., models for which the ratio of mean density of total mass-energy, $\langle \rho \rangle$, to central density, ρ_c, is $\gtrsim 10^{-3}$) with isotropic velocity distributions. More specifically, we will employ both sets of criteria to study the stability of (i) clusters of identical stars with heavily-truncated, isotropic, Maxwell-Boltzmann velocity distributions (heavily-truncated "isothermal" clusters); and (ii) isotropic clusters whose density and pressure distributions obey a polytropic law of index 2 or 3. Our main conclusion will be that neither of these classes of models contains a stable equilibrium configuration with a central redshift (redshift of a photon emitted from the center of the cluster and received at infinity) significantly larger than 0.5.

The outline of this paper is as follows: In §II we shall restate those basic definitions and results of Papers I and II which will be needed in this paper. In §III we shall reduce the variational principle of Paper I from its rather general, original form to a form which is more useful for numerical calculations. In §IV we shall use the variational principle and also the sufficient criteria of Paper II to study the stability of isothermal clusters; and in §V we shall apply these criteria to polytropic clusters.

Our notation and conventions will conform to those of Papers I and II. Thus we shall set equal to unity the gravitation constant, G, the speed of light, c, and the Boltzmann constant, k.

II. REVIEW OF STABILITY THEORY

In this section we shall briefly review certain aspects of the theory of the stability of spherically symmetric, relativistic star clusters. For a detailed discussion of the material presented here, the reader is referred to the relevant sections of Papers I and II.

We begin our review with the definition of the invariant distribution function, $\mathcal{N}$, in phase space. $\mathcal{N}$ is defined to be the number of stars, dN, per unit invariant volume, $d\mathcal{V}_p d\mathcal{V}_x$, in phase space

$$\mathcal{N} = dN/d\mathcal{V}_p d\mathcal{V}_x \quad , \tag{1}$$

where the volume elements in momentum space and physical space are given by

$$d\mathcal{V}_p = -dp_0 dp_1 dp_2 dp_3/\sqrt{(-g)} \; ; \quad d\mathcal{V}_x = (p^0/m)\sqrt{(-g)}dx^1 dx^2 dx^3 . \tag{2}$$

Here p^α and p_α are the contravariant and covariant components of the 4-momentum of a star, g is the determinant of the metric tensor, and m is the rest mass of a star.

In analyzing the dynamics of a spherically perturbed cluster we employ Schwarzschild coordinates, so that the metric takes the form

$$ds^2 = e^{\nu_A(r)+\nu_B(t,r)}dt^2 - e^{\lambda_A(r)+\lambda_B(t,r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \tag{3}$$

As in Papers I and II, the subscripts A and B denote the equilibrium values of various quantities and the small departures of these quantities away from their equilibrium values, respectively.

Following §IVc of Paper I we split the distribution function, $\hbar$, into an unperturbed part, F, and a perturbed part, f:

$$\hbar = F(m, E_A, J) + f(x^\alpha, p_{(\alpha)}) \; . \tag{4}$$

Here the $p_{(\alpha)}$ are the physical components of the momentum,

$$p_{(\alpha)} = |g^{\alpha\alpha}|^{\frac{1}{2}} p_\alpha \; ; \quad p^{(\alpha)} = |g_{\alpha\alpha}|^{\frac{1}{2}} p^\alpha \quad , \tag{5}$$

and a star's rest mass, m, its energy at infinity, E_A, and its angular momentum, J, may be expressed in terms of the $p_{(\alpha)}$ through the relations

$$m = (p_{(\alpha)}p^{(\alpha)})^{\frac{1}{2}} \; ; \; E_A = e^{\nu_A/2} p_{(0)} \; ; \; J = r(p_{(\theta)}^2 + p_{(\phi)}^2)^{\frac{1}{2}}. \tag{6}$$

As described in §IVd of Paper I we write the perturbation, f, in the distribution function as the sum of that part, f_+, which is even, and that part, f_-, which is odd under reflections of the spatial momenta $(p_{(0)} \to +p_{(0)},\ p_{(r)} \to -p_{(r)},\ p_{(\theta)} \to -p_{(\theta)},\ p_{(\phi)} \to -p_{(\phi)})$. If f_- is split into normal modes,

$$f_- = f(x^j, p_\alpha)e^{i\omega t} \quad , \tag{7}$$

we have the <u>variational principle</u>

$$\omega^2 = \frac{\int f \mathcal{J} f \, d\mathcal{V}_p d\mathcal{V}_x}{\int (-1/F_E) f^2 d\mathcal{V}_p d\mathcal{V}_x} \tag{8}$$

for the normal frequencies of oscillation. Here

$$F_E \equiv (\partial F/\partial E_A)_{m,J} \quad , \tag{9a}$$

$$\mathcal{J} f \equiv (1/F_E)\mathcal{B}\mathcal{B} f - 4\pi\left(1 + r\frac{d\nu_A}{dr}\right)e^{\lambda_A}\frac{p_0}{p^0}p^r\int\frac{p_0}{m}p^r f\, d\mathcal{V}_p , \tag{9b}$$

and

$$\mathcal{B} f \equiv (m/p^0)\mathcal{D}_A f + 4\pi r e^{\lambda_A}\frac{m}{p^0}F_E\left(\frac{p_0 p^r}{m}\int\frac{p_r p^r}{m} f\, d\mathcal{V}_p - \frac{p_r p^r}{m}\int\frac{p_0 p^r}{m} f\, d\mathcal{V}_p\right). \tag{9c}$$

In definition (9c), $\mathcal{D}_A$ is the Liouville operator for the unperturbed cluster; when (x^α, p_α) are used as coordinates in phase space, $\mathcal{D}_A$ take the form

$$\mathcal{D}_A f = \frac{p^j}{m}\frac{\partial f}{\partial x^j} - \frac{1}{2m}\frac{\partial g_A^{\alpha\beta}}{\partial x^j}p_\alpha p_\beta\frac{\partial f}{\partial p_j} \quad . \tag{9d}$$

The variational principle (8) provides a necessary and sufficient condition for stability: <u>If F_E is nowhere positive throughout the phase space of a bounded, collisionless, spherical equilibrium cluster, then that cluster is stable against radial perturbations if and only if $\mathcal{J}$ is a positive-</u>

definite operator for odd-parity spherical functions, h, bounded in phase space:

$$\int h \mathcal{J} h \, d\mathcal{V}_p \, d\mathcal{V}_x > 0 \quad . \tag{10}$$

In Paper II sufficient (but not necessary) criteria for the stability of relativistic star clusters were discussed. Among the criteria for stability presented there the most important one is the theorem of §III of that reference, which relates cluster stability to gas-sphere stability: A bounded, spherical, relativistic cluster with isotropic pressure and with $F_E \leq 0$ is stable against collisionless, radial perturbations if the gas sphere which has the same radial distribution of density of total mass-energy,

$$\rho = \int (F/m) p_0 p^0 \, d\mathcal{V}_p \; , \tag{11a}$$

and of pressure,

$$p = -\int (F/m) p_r p^r \, d\mathcal{V}_p \; , \tag{11b}$$

is stable against radial perturbations for which the adiabatic index is given by

$$\Gamma_1 \equiv \frac{\rho + p}{p} \left(\frac{\partial p}{\partial \rho}\right)_{\text{constant entropy}} = \frac{\rho + p}{p} \frac{dp/dr}{d\rho/dr} \quad . \tag{12}$$

In the following sections we shall use the above criteria to study the stability of some isothermal models and polytropic models. Our first task will be to reduce the variational principle of equation (8) to a more useful form. This is done in the next section.

III. REDUCTION OF THE VARIATIONAL PRINCIPLE

a) A Reduced Form of the Liouville Operator

In applying the variational principle (8) we shall use (x^{α}, p_{α}) as our coordinates in phase space. Then the unperturbed Liouville operator, $\mathcal{L}_A$, takes the form given by equation (9d). However, when $\mathcal{L}_A$ operates on a spherical function, h, in phase space, only those terms involving derivatives of h with respect to r and p_r survive. To see this, recall that a spherical function, h, can depend only on t, r, p_0, p_r, and J--which is given by

$$J = [p_{\theta}^2 + (p_{\phi}/\sin\theta)^2]^{\frac{1}{2}} \tag{13}$$

(cf. eqs. [5] and [6]). Consequently, for a spherical function, h, equation (9d) takes the form

$$\mathcal{L}_A h = -e^{-\lambda_A} \frac{p_r}{m}\left(\frac{\partial h}{\partial r}\right)_{x^{\mu}, p_{\mu}} - \frac{1}{2m}\frac{\partial g_A^{\alpha\beta}}{\partial r} p_{\alpha}p_{\beta}\left(\frac{\partial h}{\partial p_r}\right)_{x^{\mu}, p_{\mu}} + (\mathcal{L}_A J)\left(\frac{\partial h}{\partial J}\right)_{t, r, p_0, p_r} . \tag{14a}$$

But $\mathcal{L}_A J = 0$ since J is an integral of the motion along a stellar orbit in any spherical cluster. Hence

$$\mathcal{L}_A h = -e^{-\lambda_A} \frac{p_r}{m}\left(\frac{\partial h}{\partial r}\right)_{x^{\mu}, p_{\mu}} - \frac{1}{2m}\frac{\partial g_A^{\alpha\beta}}{\partial r} p_{\alpha}p_{\beta}\left(\frac{\partial h}{\partial p_r}\right)_{x^{\mu}, p_{\mu}} . \tag{14b}$$

All quantities in this equation can be re-expressed in terms of m, p_0, p_r, ν_A, and λ_A by the use of equations (5) and (6):

$$\mathcal{L}_A h = -e^{-\lambda_A} \frac{p_r}{m}\frac{\partial h}{\partial r} + \left[\left(\frac{m}{2}\frac{d\nu_A}{dr} - \frac{m}{r}\right)e^{-\nu_A}\left(\frac{p_0}{m}\right)^2 - \left(\frac{m}{2}\frac{d\lambda_A}{dr} - \frac{m}{r}\right)e^{-\lambda_A}\left(\frac{p_r}{m}\right)^2 + \frac{m}{r}\right]\frac{\partial h}{\partial p_r} , \tag{15}$$

when h is a spherical function in phase space.

b) Choice of a Trial Function

In applying the variational principle to study the stability of a specific cluster model, one inserts various odd-parity, spherical trial functions, f, into expression (8) and searches for a minimum value of ω^2. Hopefully, this minimum value of ω^2 will be close to the true value of the squared frequency for the fundamental mode of radial oscillation. Whether it is close or not, if it is negative, then the cluster is unstable. If a "good choice" of trial function is made, the method is powerful; this is because a trial function which is in some sense "good to first order" yields a value for the squared frequency which is "good to second order."

Any spherical trial function, f, must be an even function of p_θ and p_ϕ, since they can enter only through J. Hence, if f is to have odd-parity in the spatial momenta, it must be odd in p_r; i.e. it must have the form

$$f = \sum_{n=0}^{\infty} C_{2n+1}(r, p_0, m) p_r^{2n+1} \quad . \qquad (16)$$

The angular momentum, J, does not appear explicitly in the coefficients C_{2n+1} because J can be expressed in terms of r, p_0, p_r, and m by using equations (5) and (6)

$$J = r(e^{-\nu} A p_0^2 - e^{-\lambda} A p_r^2 - m^2)^{\frac{1}{2}} \, . \qquad (17)$$

In this paper we will confine ourselves to trial functions which are linear in p_r--i.e., for which only the first term in equation (16) is non-vanishing

$$f = C_1(r, p_0, m) p_r \quad . \qquad (18)$$

The obvious motivation for this is simplicity. Additional justification is this: As will later become evident, each extra factor of $(p_r)^2$ in a momentum-space integral reduces the value of that integral by a factor $\sim 1 - e^{-(\nu_{As}-\nu_{Ac})}$. Here ν_{Ac} and ν_{As} are the values of the potential, ν_A, at the center-of-symmetry and at the surface, respectively. Hence, at least for clusters which are not very relativistic in their structures [i.e., for clusters where $e^{-(\nu_{As}-\nu_{Ac})}$ is sufficiently close to unity] the importance of successive terms in the expansion (16) should decrease rapidly with increasing n.

We shall also restrict our trial functions (18) to have a simple dependence on p_0 and m:

$$f = C(r)F_E(m, p_0)p_r \; . \tag{19}$$

(F_E depends only on $p_0 = E$ and m, since we consider here only clusters with isotropic velocity distributions--i.e., clusters for which F and F_E are independent of J.) The reasons for this choice of dependence on p_0 and m are as follows: (i) As will later become evident, such a choice simplifies the analysis. (ii) For most of the clusters under consideration here, such a choice satisfies the physically reasonable condition that the perturbation, f, should be large where F is large and small where F is small. Why not take $f \propto F$ rather than $f \propto F_E$? Because (iii) the dynamical and zero-frequency equations of motion suggest taking $f \propto F_E$. In particular, the dynamical equation of motion (eq. [19;R] of Paper I) is of the form

$$(m/p^0)\mathcal{D}_A[(m/p^0)\mathcal{D}_A f_-] - \partial^2 f_-/\partial t^2$$

$$= F_E \times \begin{pmatrix}\text{terms where } f_- \text{ appears only in the} \\ \text{integrands of momentum-space integrals}\end{pmatrix} . \tag{20}$$

Since $\mathcal{D}_A F_E = 0$, $f_- \propto F_E$ produces on the left side of this equation a dependence upon F_E which counterbalances the F_E-dependence of the right side. Similar remarks are applicable to the equation

$$\mathcal{D}_A f_- = -F_E (p_0 p^0/2m)\partial\nu_B/\partial t + 4\pi F_E p_r p^r r e^{\lambda_A}\int (f_-/m) p_0 p^r d\mathcal{V}_p , \qquad (21)$$

which governs zero-frequency motion. (This equation is readily derived from the theorem on zero-frequency motion in §IVf of Paper I.) (iv) The choice $f \propto F_E$ is particularly crucial because many bounded cluster models have a term in F_E which is proportional to a delta function in $E = p_0$. This arises from the fact that a bounded cluster possesses a distribution function, F, which vanishes for energies, E, greater than a certain limiting value (see, e.g., Appendix B of Paper II). Unless F drops smoothly to zero at this limiting energy, F_E has a delta function in the energy. In such a case, the perturbation in the distribution function must contain a delta-function term like that which appears in F_E. Such a term is required in order to provide for the motion of the cluster's sharp "surface" in phase space (cf. Fig. 1 later in this paper). This requirement has an analogue in the pulsation theory of stars. There, if the equilibrium density is discontinuous across the surface of a star, then the pulsating configuration possesses an Eulerian change in density, $\rho_B = -\nabla\cdot[(\rho_A + p_A)\underset{\sim}{\xi}]$, which includes a term proportional to a delta function at the star's surface.

The radial dependence, $C(r)$, of the trial function (19) remains to be specified. A reasonable form for $C(r)$ is suggested by analogy with the radial pulsations of gas spheres (see, e.g., Meltzer and Thorne 1966). There, it is often true that the displacement, ξ, associated with the fundamental mode of radial oscillation is largely homologous; i.e., that $\xi \sim r$. Actually, this is usually the case only if the structure of the gas sphere is not too relativistic. Highly relativistic spheres have high-density cores to which the fundamental eigenfunction, $\xi(r)$, couples strongly; i.e., they have ξ/r much larger in the central regions than in the outer regions.

Using these familiar results for gas spheres as a guide, we will choose $C(r)$ to be of a form such that

$$\bar{f} = C(r)F_E p_r = C'(r) r e^{-\mu r} F_E p_r \ ; \tag{22}$$

and we will either set $C'(r)$ equal to a constant or choose $C(r)$ such that the displacement, ξ, associated with the trial function, $\bar{f}$, is given by

$$\frac{\partial \xi(r,t)}{\partial t} \equiv \frac{S^r}{S^0} \equiv \frac{\int (f_-/m) p^r d\mathcal{V}_p}{\int (F/m) p^0 d\mathcal{V}_p} = \frac{\int (\bar{f}/m) p^r d\mathcal{V}_p}{\int (F/m) p^0 d\mathcal{V}_p} e^{i\omega t} = (r/\ell) e^{-\mu r} e^{i\omega t} \ . \tag{23}$$

Here $S^\alpha \equiv \int (n/m) p^\alpha d\mathcal{V}_p$ is the number-flux vector, and the trial function $\bar{f}$ is related to the odd part of the distribution function by

$$f_- = \bar{f} e^{i\omega t} \ . \tag{24}$$

The arbitrary constant ℓ in equation (23) has dimensions of length. The constant μ in expressions (22) and (23) provides for a peaking of the fractional displacement, ξ/r, about the cluster's center. We shall later choose this "peaking parameter" to have a value which minimizes the squared frequency, ω^2.

Any physically reasonable perturbation should have the property that the perturbed cluster has the same distribution of rest masses of stars as has the equilibrium cluster. Also, the total rest masses of the perturbed and static configurations should be the same. These requirements place no constraints upon our choice of trial functions. In fact, once a trial function, f, has been chosen, the above requirements can be met by adding to the perturbation a static zero-frequency mode, i.e., a mode for which $f_- = 0$ and $f_+ = f_+(r, p_0, p_r, m)$. Such a static mode in no way affects the stability analysis because it has a vanishing odd-parity part, f_-.

c) Reduced Form of the Variational Principle

In this subsection we shall obtain an explicit, reduced expression for the variational principle (8) which facilitates its application. We arrive at the desired expression by the following procedures:

First employ definition (9b) and the anti-self-conjugate property of the operator $\mathcal{B}$ (cf. eq. [22a;R] of Paper I) in order to rewrite the numerator of the variational principle (8) in the form

$$\int f \mathcal{J} f \, d\mathcal{V}_p d\mathcal{V}_x = -\int \frac{1}{F_E} (\mathcal{B} f)^2 \frac{p^0}{m} d\mathcal{V}_p \, d^3V_x - 4\pi \int e^{\lambda_A} \left(1 + r\frac{d\nu_A}{dr}\right) \left[\int \frac{f}{m} p_0 p^r \, d\mathcal{V}_p\right]^2 d^3V_x \,, \tag{25}$$

where

$$d^3V_x \equiv (m/p^0) d\mathcal{V}_x = \sqrt{(-g)}\, dx^1 dx^2 dx^3 = e^{(\nu_A + \lambda_A)/2} r^2 \sin\theta \, dr \, d\theta \, d\phi \quad . \tag{26}$$

Next, in expression (15) replace h by the trial function (22) and find that

$$\mathcal{S}_A f = mF_E[a_0(r)(p_0/m)^2 + a_1(r)(p_r/m)^2 + C(r)/r] , \qquad (27a)$$

where

$$a_0(r) \equiv C(r)e^{-\nu_A}\left(\frac{1}{2}\frac{d\nu_A}{dr} - \frac{1}{r}\right) ; \quad a_1(r) \equiv e^{-\lambda_A}\left[C(r)\left(\frac{1}{r} - \frac{1}{2}\frac{d\lambda_A}{dr}\right) - \frac{dC(r)}{dr}\right] . \qquad (27b)$$

In a straightforward fashion, combine equations (5), (8), (9), (22), (25), (26), (27) to finally arrive at the desired reduced form of the variational principle,

$$\begin{aligned}\omega^2 = &\left(\int dr\, e^{(\lambda_A - \nu_A)/2} r^2 C^2 I'_{1,2}\right)^{-1} \\ &\times \int dr\, e^{3\nu_A/2}\Big\{ e^{\lambda_A/2}\Big[(ra_0)^2 I'_{3,0} + 2r^2 a_0 a_1 I'_{1,2} + 2ra_0 C I'_{1,0} \\ &\qquad + (ra_1)^2 I'_{-1,4} + 2ra_1 C I'_{-1,2} + C^2 I'_{-1,0}\Big] \\ &\quad - 8\pi e^{-\lambda_A/2} r^2 C I'_{1,2}\Big[ra_0 I'_{1,2} + ra_1 I'_{-1,4} + C I'_{-1,2}\Big] \\ &\quad + 16\pi^2 e^{-3\lambda_A/2} r^4 C^2 I'_{-1,4}(I'_{1,2})^2 \\ &\quad + 4\pi e^{-(\nu_A + \lambda_A/2)} r^2 (1 + r d\nu_A/dr) C^2 (I'_{1,2})^2\Big\} , \end{aligned} \qquad (28)$$

where

$$I'_{a,b}(r) \equiv \int F_E m^2 (p_0/m)^a (p_r/m)^b \, d\mathcal{V}_p \quad . \qquad (29)$$

Notice from equations (28) and (29) that the application of the variational principle entails the evaluation of multidimensional phase-space integrals. In this paper we confine ourselves to the study of clusters with isotropic velocity distributions. For such a cluster, equations (28) and (29) involve non-trivial integrations over the radial coordinate and over one momentum coordinate, if the cluster is composed of stars with identical rest masses. (This is evident from the analysis of §IV.) If an isotropic cluster is composed of stars with different rest masses, integrations over three phase-space coordinates are required. Such is the case for the polytropic models studied in §V. (However, we will find that two of the integrations can be handled analytically there.)

In this paper we shall evaluate the necessary phase-space integrals by a process which appears to introduce the least complication into the numerical analysis. We shall perform the required integrations over momentum space (in some instances we will be able to do so analytically), and we shall replace the remaining integrals over the radial coordinate by their equivalent differential equations. This will enable us to integrate these differential equations right along with the equations which govern the structure of the equilibrium configuration. Hence we will be able to simultaneously compute structure and diagnose stability.

We now proceed to detailed applications of the computational method outlined above.

IV. ISOTHERMAL CLUSTERS

In this section we shall study the stability of clusters of identical stars with truncated, isotropic Maxwell-Boltzmann velocity distributions (isothermal clusters). Our procedure will be as follows: In subsection a we shall review the theory of the equilibrium structures of isothermal clusters. In Appendix A we shall discuss methods for evaluating the momentum-space integrals which enter into the application of the variational principle. And in subsection b we shall outline the computational procedure and present the results of the computations of the cluster pulsation frequencies and stability.

a) Equilibrium Configurations

The structures of isothermal, relativistic clusters have been studied independently by Zel'dovich and Podurets (1965) and by Fackerell (1966). Such models have distribution functions, F, given by

$$F(m, p_0) = Ke^{-p_0/T}\delta(m-m_0)H(m\sqrt{\beta} - p_0) \quad . \tag{30}$$

Here K is a constant, T is the temperature of the configuration as measured by an infinitely-removed observer, m_0 is the rest mass of a star, H is the unit step function

$$H(x) = 1 \text{ if } x > 0$$
$$= 0 \text{ if } x < 0 \quad ,$$

and β is the value of e^{ν_A} at the surface, r = R, of the cluster: $\beta \equiv e^{\nu_A(R)}$. The step function in relation (30) guarantees that no star can be found at $r > R$ (see, e.g., Appendix B of Paper II). The metric coefficients ν_A and λ_A are determined in terms of the stress-energy tensor, $T_{A\alpha}{}^{\beta}$, (cf. equation [3a;R] of Paper I) by the equations

$$d\nu_A/dr = 2(M_A + 4\pi r^3 p_A)e^{\lambda_A}/r^2 \quad , \tag{31a}$$

$$dM/dr = 4\pi r^2 \rho_A \quad , \tag{31b}$$

$$e^{-\lambda_A} = 1 - 2M_A/r \quad , \tag{31c}$$

where

$$\rho_A(r) = T_{A0}{}^{0} = \int (F/m) p_0 p^0 \, d\mathcal{V}_p \ ,$$

$$p_A(r) = -T_{Ar}{}^{r} = -T_{A\theta}{}^{\theta} = -T_{A\phi}{}^{\phi} = -\int (F/m) p_r p^r \, d\mathcal{V}_p \qquad (31d)$$

are the density of mass-energy and pressure, and where $M_A(r)$ is the total mass-energy within coordinate radius r.

To reduce the equations of structure to a manageable, dimensionless form, we follow Fackerell's (1966) procedure: First introduce a scaling parameter, L, with dimensions of length

$$L \equiv \sqrt{(p_{Ac}/4\pi \rho_{Ac}^2)} \ . \qquad (32)$$

Here the subscript c denotes the value of a quantity at the center of symmetry, $r = 0$. Next define the dimensionless radius, x, mass, v, and redshift factor, y, by

$$x \equiv r/L \ ; \quad v \equiv M_A \rho_{Ac}/(L p_{Ac}) \ ; \quad y \equiv e^{\nu_A}/\beta \ . \qquad (33)$$

(Note that $[y(r)^{-\frac{1}{2}} - 1]$ is the redshift of a photon emitted at radius r and received at the cluster's surface, $r = R$.) Combine equations (31), (32) and (33) to obtain the desired dimensionless form of the equations of structure

$$\frac{dy}{dx} = 2 \frac{p_{Ac}}{\rho_{Ac}} y \frac{e^{\lambda_A}}{x^2} \left(v + \frac{p_{Ac}}{\rho_{Ac}} x^3 \frac{p_A}{p_{Ac}} \right) \ , \qquad (34a)$$

$$dv/dx = x^2 (\rho_A/\rho_{Ac}) \ , \qquad (34b)$$

$$e^{-\lambda_A} = 1 - 2 p_{Ac} v/(\rho_{Ac} x) \ . \qquad (34c)$$

This system of equations is complete once we have explicit expressions for p_{Ac}/ρ_{Ac}, p_A/p_{Ac}, and ρ_A/ρ_{Ac}. As Fackerell (1966) shows, it follows from equations (30) and (31d) that

$$\rho_A = 4\pi K m_0^4 \int_1^{1/\sqrt{y}} e^{-E'\sqrt{y}/T'} E'^2 (E'^2 - 1)^{\frac{1}{2}}\, dE' \quad , \tag{35a}$$

$$p_A = (4\pi/3) K m_0^4 \int_1^{1/\sqrt{y}} e^{-E'\sqrt{y}/T'} (E'^2 - 1)^{3/2} dE' \quad , \tag{35b}$$

where the variable of integration is $E' \equiv p_{(0)}/m_0$, and where

$$T' \equiv T/(m_0\sqrt{\beta}) \tag{36}$$

is the temperature, as measured by an observer at the surface of the configuration, divided by a star's rest mass. We shall also wish to compute the rest mass, $M_{0A}(r)$, within coordinate radius r. The rest mass, $M_{0A}(r)$, is computed by first defining the dimensionless rest mass

$$v_0 \equiv M_{0A}\rho_{Ac}/(Lp_{Ac}) \tag{37a}$$

and by then integrating the equation

$$dv_0/dx = e^{\lambda_A/2}(\rho_{0A}/\rho_{Ac})x^2 \tag{37b}$$

along with equations (34) and (35). Here the density of rest mass, ρ_{0A}, is given by

$$\rho_{0A} = \int F p^{(0)} d\mathcal{V}_p = 4\pi K m_0^4 \int_1^{1/\sqrt{y}} e^{-E'\sqrt{y}/T'} E'(E'^2 - 1)^{\frac{1}{2}}\, dE' \; . \tag{37c}$$

(A detailed discussion of momentum space integrals similar to those which enter into equations [35] and [37c] is presented in Appendix A.) Notice that the constant K drops out of equations (34) and (37b).

Hence, once values are chosen for the surface temperature, T', and the central-to-surface redshift factor, $y_c < 1$, equations (34), (35) and (37) can be integrated outward from $x = 0$, where $v_c = v_{0c} = 0$. The surface of the configuration is reached when p_A drops to zero, i.e., when $y = 1$ (cf. expressions 35).

We shall discuss the structures of the isothermal clusters in relation to their stability in the following subsection.

b) Analysis of the Stability of Compact Isothermal Clusters

Our scheme of integration for computing the structure of an isothermal cluster, and for studying its stability through use of the variational principle, now becomes evident. It consists of the following procedures, all performed simultaneously for initially-chosen values of the surface temperature, T', the center-to-surface redshift parameter, $y_c = e^{\nu_{Ac}}/\beta < 1$, and the trial-function peaking parameter, μ: (i) integrate the structure equations (34) and (37b); (ii) evaluate the momentum-space integrals which enter into the analysis by using the techniques discussed in Appendix A; (iii) chose the function $C(r)$ in the trial function (22) such that the displacement, ξ, is given by expression (23)--or alternatively choose $C'(r) = \text{constant}$ in equation (22); (iv) evaluate the integrals appearing in the numerator and denominator of expression (28) by integrating their equivalent ordinary differential equations in suitable dimensionless forms (see Appendix B). Perform the radial integrations outward from the origin, $x = 0$, until the pressure drops to zero, identifying the point where $p_A = 0$ as the cluster's surface.

The outstanding remaining question concerns itself with the relationship, if any particular one is desired, between the choices of the parameters T' and y_c. Each set of choices corresponds to picking a certain cluster temperature and a certain way of cutting off the distribution function above the energy $p_0 = m\sqrt{\beta}$. Zel'dovich and Podurets (1965) chose to write the cutoff in the form

$$\sqrt{\beta} = 1 - \epsilon T'\sqrt{\beta} = 1 - \epsilon T/m \quad , \tag{38}$$

and to study all those clusters for which ϵ is a fixed constant. In this way they obtained a one-parameter sequence of models. It is the members of such sequences whose stability we shall study here. Once we have singled out for study one of these sequences by specifying the cutoff parameter, ϵ, we obtain each of its members by first choosing a value for the surface temperature, T', and then searching for a value of $y_c = e^{\nu_A(0)}/\beta$ which produces the cutoff given by equation (38). Of course, there is no guarantee that a model with the desired cutoff exists for the chosen value of T'. Notice that the value of β for a particular model is not known until the structure of the model has been fully computed. Consequently, the task of finding the sets of parameters (T', y_c) which belong to those models with cutoffs given by equation (38) for a prescribed value of the cutoff parameter, ϵ, is somewhat time-consuming.[1]

We have performed a numerical analysis of the isothermal sequence with cutoff parameter $\epsilon = 0.5$. The structures of the models in this sequence have also been studied by Zel'dovich and Podurets (1965) (see their Table 1, where the entries in the second column should

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[1] The parameters (T', y_c) for the sequences of models studied in this section were calculated for the author by B. A. Zimmerman.

be multiplied by 0.5). The results of our analysis of this sequence are given in Table 1. These results exhibit several phenomena, which turn out to be common to all sequences with the cutoff (38): The dimensionless temperature, T/m_0, as measured by an observer at infinity, reaches a maximum value along the sequence; and hence different models can have the same temperature. The product $\mathfrak{M}_0^2\rho_{Ac}$, where $\mathfrak{M}_0$ is the total rest mass of a model, increases monotonically along the sequence; but the fractional binding energy, $\mathcal{E}/\mathfrak{M}_0$, oscillates. Here

$$\mathcal{E}/\mathfrak{M}_0 \equiv 1 - \mathfrak{M}/\mathfrak{M}_0 \qquad (39)$$

where $\mathfrak{M}$ is the total mass-energy of a model.

For central redshifts $z_c \leq 0.64$, the entries in Table 1 for the squared frequencies of oscillation, ω^2, were obtained by setting equal to zero the peaking parameter, μ, in equation (23). Thus the associated displacements are homologous: ξ/r = constant. For $z_c > 0.64$ "upper limits" are given for ω^2. Actually, those entries are not minimum upper limits even with respect to our trial functions (22), since the study of a model with $z_c > 0.64$ was discontinued once a value of μ which yielded a negative value of ω^2 was found. Because of this, and also because our simple trial functions (22) cannot be expected to be "good" for highly relativistic models, no significance should be attached to the way in which the given upper limits vary along the sequence of Table 1.

We have found that, for each model in Table 1 which has a central redshift $z_c \leq 0.64$, the value of ω^2 obtained by minimization with respect to the peaking parameter, μ, of equation (23) differs by less than ~ one-tenth of a per cent from the value of ω^2 obtained by letting $\mu = 0$ in equation (23)! But the most favorable value of μ increases as the central redshift becomes larger along the sequence. For example, for the model with $z_c = 1.08$ the ratio of the value of the fractional displacement, ξ/r, at the surface to its value near the origin is ≈ 0.6, for that value of μ which minimizes ω^2. We have also found that it seems to matter little whether the parameter C' of equation (22) is set equal to a constant or is chosen such that equation (23) is satisfied.

For the model of Table 1 with $z_c = 0.392$, we exhibit in Figure 1 the momentum-space dependences of the equilibrium distribution function, F, and of the perturbation trial function, f, at two values of the radial coordinate, r. The odd-parity part, f_-, of the perturbation is taken to be that given by the trial function (22) with $C' = 1.28$ and $\mu = 0$. The perturbed distribution function, $\mathfrak{N} = F + f$, is shown at a moment of time when the even-parity part, f_+, of the perturbation is zero, i.e., when $\mathfrak{N} = F + f_-$. (It is evident from equation [25;R] of Paper I that the even and odd parts of a normal mode oscillate with a phase difference of $\pi/2$.)

Since a model is unstable if the squared frequency of oscillation of its fundamental radial mode is negative, Table 1 indicates that an instability occurs along the isothermal sequence with cutoff parameter $\epsilon = \frac{1}{2}$ when the central redshift, z_c, reaches a value which is about 0.5. We can be certain that the point of onset of instability occurs at a central redshift < 0.52. This is because the variational principle is a minimal principle.

Table 1 exhibits another interesting phenomenon, namely, that quite strongly correlated are the positions along the sequence of that point where the fractional binding energy, $\mathcal{E}/\mathfrak{M}_0$, attains its first maximum, and that point at which the variational analysis indicates that an instability sets in.[2] This behavior is clearly indicated also in Figure 2, where we plot ω^2/ρ_{Ac} and $-\mathcal{E}/\mathfrak{M}_0$ as functions of central redshift, z_c. This correlation between binding energy and stability is highly reminiscent of a similar correlation which exists for certain sequences of hot, isentropic stellar models (see, e.g., Chapters 4 and 6 of Thorne 1967): there each extremal point of the associated binding energy curve signals the change of stability of a normal mode of radial oscillation.

In Tables 2 and 3 we present the general properties of the isothermal sequences for which the cutoff parameter, ϵ, is equal to 0.1 and 0.01. It is evident that these sequences possess properties quite similar to those of the sequence with $\epsilon = 0.5$, which were discussed above.

The isothermal sequences of Tables 1 - 3 have heavily-truncated distribution functions; i.e., they have cutoffs (38) such that, even in the central region of a model, the "spectral density,"

$$dS^{(0)}/dE' \equiv 4\pi\left[\int F(m, e^{\nu_A/2} mE')m^3\, dm\right]E'(E'^2 - 1)^{\frac{1}{2}}$$

$$= -d\left(\begin{matrix}\text{proper density of stars with locally measured}\\ \text{energy per unit mass greater than } E'\end{matrix}\right)/dE',$$

attains its maximum at a value, $E'_m(r)$, of $E' \equiv p_{(0)}/m$ which is either greater than or only slightly less than the cutoff value, $E'_{cut}(r) \equiv e^{-\nu_A/2}\sqrt{\beta}$,

[2]The possibility of such a correlation was conjectured in 1967 by Ya. B. Zel'dovich and I. D. Novikov (private communication to K. S. Thorne); but Ipser and Thorne (Paper I) thought it quite unlikely.

above which the equilibrium distribution, F, vanishes. In general, for fixed central-to-surface redshift factor, y_c, the smaller is the value of the cutoff parameter, ϵ, the larger is $E'_m(r)$, whereas $E'_{cut}(r)$ is largely insensitive to the value of ϵ. As may be seen from Tables 1 - 3, values of $\epsilon \lesssim 0.5$ give rise to isothermal models for which--except in ultrarelativistic situations-- the ratio, $\langle \rho_A \rangle \rho_{Ac}^{-1} \equiv 3\mathfrak{M}(4\pi R^3 \rho_{Ac})^{-1}$, of mean density to central density is $\gtrsim 10^{-3}$. From this comes our loosely-applied statement that such heavily-truncated models are "compact." Tables 1 - 3 indicate that such a compact, isothermal cluster is unstable against radial perturbations if the redshift from its center is significantly larger than 0.5.[3]

One could argue that compact isothermal clusters are unrealistic, since they have heavily-truncated distribution functions, and since interactions between stars near the center of a cluster should tend to populate a significant portion of the tail of the Maxwellian distribution. However, Zel'dovich and Podurets (1965) have pointed out that inelastic collisions become more important than other evolutionary phenomena when the stars which compose a relativistic cluster have velocities a few per cent of the velocity of light or larger. The process whereby two stars collide and stick will tend to reduce the fraction of stars with large energies at a rate which grows larger as the cluster becomes more relativistic. Hence, at least some of the isothermal models examined here may be semi-realistic.

The analyses of this section suggest a highly idealized history of

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[3] In this connection, it should be noticed that the cutoff for any isothermal cluster can be written in the form (38) for some choice of the cutoff parameter, ϵ. However, only for $\epsilon \lesssim 0.5$ will the cluster be compact.

the evolution of a spherical cluster along one of the isothermal sequences studied here.[4] A proto-cluster might be expected to relax towards an isothermal distribution, after which it might evolve along one of the sequences considered here, by means of the ejection of stars and collisions between stars. (When a star is ejected from a cluster, it carries away non-zero kinetic energy as measured by an observer at infinity, and thereby increases the fractional binding energy of the cluster; when two stars collide and stick, they increase the cluster rest mass and hence also its binding energy.) Consequently, the cluster evolves along the sequence down the binding energy curve ($-\mathcal{E}/\mathfrak{m}_0$ decreasing) in the direction of increasing temperature. When the cluster reaches the minimum of the binding energy curve, evolution along the equilibrium sequence must stop because subsequent star ejection and collisions can only increase further the fractional binding energy of the cluster (i.e., decrease $-\mathcal{E}/\mathfrak{m}_0$). Something else must happen. What? The analyses presented in this section give us an answer: The cluster undergoes catastrophic gravitational collapse at this point. The fact that a compact isothermal cluster may move from one sequence to another as it evolves will not enable it to escape this catastrophe. This is because the analyses of this section indicate that the key features of the sequences are largely cutoff-independent; i.e., instabilities always occur when a compact isothermal cluster has evolved sufficiently far

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[4]For a related discussion, see Zel'dovich and Podurets (1965). However, it should be remembered, in contrast with their comments, that there appears to be little relationship between the temperature of a model and its stability.

that the redshift of a photon emitted from its center and received at infinity is of the order of 0.5.

We have argued that collisions and evaporation bring an isothermal cluster to the verge of collapse. But we have ignored these processes in performing our stability analyses. The justification given in §II of Paper I for doing so is that the time-scale associated with the growth of the instability in a cluster should be roughly the star-travel time across the cluster--a time much smaller than the interval between collisions for a typical star. Our analysis has verified this justification; it has shown that the time-scale for onset of the instability is $1/|\omega| \sim \rho_{Ac}^{-\frac{1}{2}}$ (cf. Figure 2), a time of the order of the star transit time.

In addition to the variational principle, we have applied to the isothermal models the sufficient condition for stability quoted in §II, which relates cluster stability to gas-sphere stability. Unfortunately, that stability criterion is useless for studying the isothermal models. The corresponding gas sphere was unstable for every model examined! We have also applied the sufficient condition for stability associated with equation (44) of Paper II. That criterion also is of no use for studying the isothermal models. It was not satisfied for any model examined.

In this section we have avoided consideration of isothermal models which possess high-density cores surrounded by extended, diffuse envelopes (i.e., models with the cutoff $\epsilon >> 0.5$). Our experience suggests that an accurate study of the stability of such a "core-halo" model cannot be achieved with the simple trial functions employed in this paper. However, we have as yet been unable to construct a sequence of core-halo isothermal models for which the model with maximum fractional binding energy, $\mathcal{E}/\mathfrak{m}_0$, possesses a central redshift significantly different from 0.5. Consequently, at present it appears likely but remains unproved that any isothermal cluster is unstable if it possesses a central redshift $\gtrsim 0.5$.

V. POLYTROPIC CLUSTERS

In this section we shall study compact, isotropic clusters with densities, $\rho_A \equiv T_{A0}{}^{0}$, and pressures, $p_A \equiv -T_{Ar}{}^{r} = -T_{A\theta}{}^{\theta} = -T_{A\phi}{}^{\phi}$, which are related by a relativistic polytropic law (eq. [40] below). The gas spheres which correspond to these clusters (i.e., the gas spheres which have the same radial distributions of density and pressure) have been studied extensively by various authors (see Thorne [1967] for references). Our approach to the study of the polytropic cluster models will be this: in subsection a we shall discuss the equations of structure and the dimensionless form in which they are usually written; in subsection b we shall discuss Fackerell's (1966, 1968a, 1968b) method for the determination of the distribution function; in Appendix C we shall develop a method for evaluating the momentum-space integrals which enter into the variational principle; and in subsection c we shall study the structures and stabilities of the polytropic models.

a) Equations of Structure

In this subsection we write the equilibrium equations of structure for polytropic cluster models in a form identical to that used by Tooper (1965) for polytropic stellar models.

By definition, a relativistic star cluster which is polytropic is one whose density, ρ_A, and isotropic pressure, p_A, obey the relations

$$p_A = \alpha\tau\Theta^{n+1} , \tag{40a}$$

$$\rho_A = \tau\Theta^n + p_A/(\Gamma_4 - 1) \quad . \tag{40b}$$

In these equations $\Theta(r)$ is a dimensionless function which is equal to unity at $r = 0$ and zero at $r = R$; α, τ, n, and Γ_4 are constants, with τ having the dimensions of an inverse length squared. The positive constant n is called the polytropic index. Study of equations (40) reveals that the larger is the value of the relativity parameter, α, the more relativistic is a cluster. By combining equation (40a) and the relation

$$dp_A/dr = -(1/2)(\rho_A + p_A)\, d\nu_A/dr \quad , \tag{41}$$

which follows from the equation of motion, $T_{Ar}{}^{\mu}{}_{;\mu} = 0$, we obtain the useful equation

$$y(r) \equiv e^{\nu_A(r) - \nu_A(R)} \equiv e^{\nu_A(r)}/\beta = [1 + \alpha\gamma_4\Theta(r)]^{-2(n+1)/\gamma_4} , \tag{42}$$

where we have introduced the parameter

$$\gamma_4 \equiv \Gamma_4/(\Gamma_4 - 1) \quad . \tag{43}$$

To reexpress the equations of structure in dimensionless form, we introduce the dimensionless radius, x, and mass, v, through the definitions

$$x = r/L \; ; \; v(x) = M_A(r)/(4\pi L^3\tau) \; ; \; L = \sqrt{[(n+1)\alpha/4\pi\tau]} \quad . \tag{44}$$

(In this section we shall thus reinterpret the symbols x, v, and L, which were employed in the previous section.) With the help of equations (40) and (41), we may then rewrite the equilibrium field equations (31) in the form

$$\frac{d\Theta}{dx} = \frac{-(1+\alpha\gamma_4\Theta)(v+\alpha x^3\Theta^{n+1})}{x^2[1-2(n+1)\alpha v/x]} , \tag{45a}$$

$$dv/dx = x^2\Theta^n[1+\alpha(\gamma_4-1)\Theta] , \tag{45b}$$

$$e^{-\lambda_A} = [1 - 2(n+1)\alpha v/x] . \tag{45c}$$

To compute the structure of a polytropic model, we first choose values for n, Γ_4 (or γ_4), and α. We then integrate equations (45) outward from $x = 0$, where $\Theta = 1$ and $v = 0$, until Θ drops to zero. This happens at the cluster's surface. The pressure and density throughout the configuration are obtained from Θ through equations (40). Notice that all quantities will be scaled by the parameter τ, which is related to the central density, ρ_{Ac}, by equations (40):

$$\rho_{Ac} = \tau[1 + \alpha/(\Gamma_4 - 1)] . \tag{46}$$

b) The Distribution Function

The distribution function, F, which gives rise to the polytropic models can be obtained by a method due to Fackerell (1966, 1968a). This method has been employed by Fackerell (1968b) to obtain expressions for F in those cases where $\Gamma_4 = 1 + 1/n$ (the "adiabatic" polytropes). A straightforward generalization of this method will allow us to obtain F when no such restriction is placed upon Γ_4.

We begin by combining equations (40a) and (42) so as to obtain an expression for the pressure, p_A, in terms of the redshift factor

$y = e^{\nu_A}/\beta$:

$$p_A = [\alpha\tau/(\alpha\gamma_4)^{n+1}]\, y^{-\gamma_4/2}\left(1 - y^{\gamma_4/[2(n+1)]}\right)^{n+1} . \qquad (47)$$

The pressure is given in terms of the equilibrium distribution function, F, by the expression

$$p_A = -\int F(m,p_0)(p_{(r)}p^{(r)}/m)\, d\mathcal{V}_p$$

$$= (2\pi/3)(\beta y)^{-2}\int_{\beta y}^{\infty}\left[\int_0^{\infty} F(m, m\sqrt{u})m^4\, dm\right](u-\beta y)^{3/2}u^{-1/2}\, du. \qquad (48)$$

In this integral the integration variable u corresponds to $(p_0/m)^2$. Expression (48) is easily derived by techniques similar to those employed to obtain relation (A7) (also see Fackerell 1968a). The integration over u in expression (48) actually runs up to only $u = \beta$. This is because the distribution function, F, must be zero for values of $E_A = p_0 > m\sqrt{\beta}$, or else particles would be found at $r > R$, where R is the value of the radial coordinate at the surface. If we define the mass-weighted distribution function,

$$\mathfrak{F}(u) \equiv [2\pi/(3\sqrt{u})]\int_0^{\infty} F(m, m\sqrt{u})m^4\, dm \quad , \qquad (49)$$

and if we combine equations (47), (48), and (49), we obtain

$$\mathcal{G}(y) \equiv \int_{\beta y}^{\beta} \mathfrak{F}(u)(u-\beta y)^{3/2}\, du$$

$$= [\alpha\tau\beta^2/(\alpha\gamma_4)^{n+1}]\, y^{2-\gamma_4/2}\left(1 - y^{\gamma_4/[2(n+1)]}\right)^{n+1} . \qquad (50)$$

The stability criteria of Papers I and II cannot be applied to models with polytropic index $n < 3/2$, since F_E is positive over a certain region of phase space for those models.[5] Restricting ourselves to those polytropic models for which $n \geq 3/2$, Fackerell's (1968b) analysis tells us that the solution of the integral equation (50) for $\mathcal{F}$ is

$$\mathcal{F}(u) = -(4/3\pi)\beta^{-5/2}\int_{u/\beta}^{1} d^3[\mathcal{G}(y)]/dy^3 (y-u/\beta)^{-1/2} dy. \quad (51)$$

By straightforward differentiation of equation (50), we obtain for the integrand of equation (51)

$$d^3[\mathcal{G}(y)]/dy^3 = [\alpha\tau\beta^2/(\alpha\gamma_4)^{n+1}]\, y^{-(1+\gamma_4/2)}$$
$$\times \left[1-y^{\gamma_4/[2(n+1)]}\right]^{n-2}\left[A_0 + A_1 y^{\gamma_4/[2(n+1)]} + A_2 y^{\gamma_4/(n+1)}\right] . \quad (52a)$$

In this expression,

$$A_0 = -\gamma_4 A_0'/2 \ , \quad A_1 = [\gamma_4/(n+1)](A_0' - nA_1'/2) \ , \quad (52b)$$
$$A_2 = [\gamma_4/(n+1)][A_1'/2 - (n-1)] \ ,$$

where

$$A_0' = (2 - \gamma_4)(4 - \gamma_4)/4 \ , \quad (52c)$$
$$A_1' = -(4-\gamma_4)[2(n+1) - \gamma_4]/[4(n+1)] + n\gamma_4/(n+1) - 2 \ .$$

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[5]This is because equations (40) reduce to the Newtonian polytropic law near the surface of a relativistic polytrope. Since only stars with large energies, p_0, close to the cutoff energy, $m\sqrt{\beta}$, can reach the surface region of a cluster, it follows from the known form of the distribution function for a Newtonian polytrope (Eddington 1916) that $\mathcal{F} \sim (m\sqrt{\beta} - p_0)^{n-3/2}$ for the largest energies, p_0, available to the stars which compose a relativistic polytrope. Hence F_E (cf. definition 9a) is positive at large energies if $n < 3/2$.

Equations (51) and (52) may now be combined to obtain the mass-weighted distribution function, $\mathfrak{F}(u)$. It is evident that the expression obtained for $\mathfrak{F}(u)$ will be complicated, particularly when $\gamma_4 \neq 1 + 1/n$. And in order to evaluate any of the momentum-space integrals which enter into the analysis, one must perform a complicated integration over the variable u appearing in our complicated result for $\mathfrak{F}$. These features suggest that the task of studying the stability of the polytropic models might be formidable. However, in Appendix C we devise a technique for evaluating analytically, in terms of the metric coefficients ν_A and λ_A, the momentum-space integrals which enter into our stability analysis when n is an integer.

c) Analysis of the Stability of Polytropic Models

In this subsection we study the stability of polytropic clusters with indices $n = 2$ and 3. The method used to apply the reduced variational principle (28) is the following (all procedures performed simultaneously): (i) integrate the structure equations (45a,b); (ii) evaluate--by employing the technique devised in Appendix C--the momentum-space integrals, $I'_{a,b}(r)$, of definition (29), which occur in the reduced variational principle (28); (iii) employ the trial function (22) with $C' =$ constant, and evaluate the numerator and denominator of the reduced variational principle (28) by integrating their equivalent differential equations in suitable dimensionless form (see Appendix D). The value thus obtained for the squared frequency of oscillation, ω^2, of the fundamental radial mode can be minimized by varying the peaking parameter, μ, of equation (22). The integrations are performed outward from the origin $(r = x = 0)$, where $v = 0$ and $\Theta = 1$, to the point where the structure variable Θ drops to zero. This defines the position of the surface. The run of the density of total mass-energy and the pressure throughout the equilibrium configuration are determined from equations (40). All results are scaled by the parameter τ (cf. eq. [46]).

In this paper we obtain a one-parameter sequence of polytropic clusters by choosing the polytropic index, n, to be equal to either 2 or 3, and by choosing a value for $\Gamma_4 > 1$. We then collect into a sequence the models which belong to different values of the relativity parameter, α. As the value of α increases, the models become more relativistic (i.e., the redshift of a photon emitted from the center of the model and received at infinity increases).

In applying the stability criteria of Papers I and II to the members of such a sequence, we must always be certain that F_E (cf. definition 9a) is non-positive throughout the phase space of a model. Otherwise, the stability criteria will be inapplicable. In any event, we might question the reasonableness of a model for which $F_E > 0$ in some region of phase space--in such a region there would be fewer low-energy stars than high-energy stars. Notice from §Vb that the distributions of density, $\rho_A(r)$, and of pressure, $p_A(r)$, throughout an isotropic cluster determine uniquely only the mass-weighted distribution function, $\mathfrak{F}[(p_0/m)^2]$. If only $\mathfrak{F}$ is known, we must make some assumption about the explicit dependence of $F(m,p_0)$ upon the rest mass, m, and the energy, p_0, of a star in order to study the behavior of F_E. Fackerell (1969) has carried out a study of the sign of F_E for the polytropes of indices 2 and 3, under the assumption that $F(m,p_0)$ depends upon the energy, p_0, only through the ratio p_0/m. His investigations show that the larger is the value of Γ_4 for a given polytropic index, n, the earlier

F_E becomes somewhere positive along the associated sequence (n and Γ_4 fixed along the sequence; α varying). More specificially, the larger is the value of Γ_4, the smaller is the maximum of the central redshifts, z_c, for those models which have $F_E \leq 0$ throughout phase space. Further, for given values of n and Γ_4, the assumption that F has a dependence on p_0 which is more complicated than that considered by Fackerell tends to make matters only worse; i.e., it tends to lower the maximum of the central redshifts for those models which have $F_E \leq 0$.

In Tables 4 - 6 we exhibit the results of our stability analysis for three representative sequences of polytropic clusters with indices $n = 2$ and 3. The structures and stabilities of the gas spheres which correspond to the models of Tables 4 and 5 have been studied by Tooper (1965). The entries in Tables 4 - 6 for the fractional binding energies, $\mathcal{E}/\mathfrak{m}_0$, were computed by Fackerell (1969). Under the assumption that the distribution function, $F(m,p_0)$ depends on p_0 only through the ratio p_0/m, Fackerell (1969) has found that, for values of the relativity parameter, α, only slightly greater than the last entries in Tables 4 and 6, F_E has a turning point and the stability criteria become inapplicable. However, all members of the "adiabatic" sequence with $n = 3$ and $\Gamma_4 = 4/3$ (Table 5) have $F_E \leq 0$ throughout phase space.

There are two outstanding features of the results given in Tables 4 - 6, which are quite similar to the main results of §IV b for compact, isothermal clusters. First, Tables 4 - 6 indicate that a polytropic cluster of index 2 or 3 is unstable if it has a central redshift $z_c \gtrsim 0.5$, and stable if it has $z_c \lesssim 0.5$. Secondly, the fractional binding energy, $\mathcal{E}/\mathfrak{m}_0$, reaches its maximum value along each of the polytropic cluster sequences considered here at a point quite close to the point of onset of instability as diagnosed by our variational principle.

If one applies to the polytropic models the sufficient condition for stability quoted in §II, which relates cluster stability to gas-sphere stability, he finds that criterion to be fairly weak. In fact, the gas spheres which correspond to the polytropic clusters with $n = 2$ and $\Gamma_4 = 3/2$ become unstable against radial perturbations with adiabatic index (12) when the central redshift reaches the value $z_c = 0.315$. All of the relativistic gas spheres with $n = 3$ and $\Gamma_4 = 4/3$ are unstable; and all of the gas spheres with $n = 3$ and $\Gamma_4 = 2$ which correspond to the specific clusters of Table 6 are unstable. Further, one can obtain no information regarding the stability of the polytropic clusters through use of the stability criterion associated with equation (44) of Paper II. That criterion failed for every polytrope to which it was applied.

It is to be expected that the results presented in this section are typical of at least all polytropic clusters with $2 \lesssim n \lesssim 3$. For $n \lesssim 2$, our stability criteria are not even applicable when the central redshift is $z_c \gtrsim 0.55$, because the distribution function has $F_E > 0$ somewhere in phase space (Fackerell 1969). For $n \gtrsim 3$, the polytropes have core-halo structures, and the simple trial functions (22) yield inconclusive results.

VI. CONCLUDING REMARKS

In this paper we have employed the stability criteria developed in previous papers of this series to study the stability against radial perturbations of two representative classes of compact, relativistic, spherical star clusters with isotropic velocity distributions. It is perhaps somewhat surprising that our study indicates that neither heavily-truncated isothermal models nor polytropic models of indices 2 and 3 admit stable equilibrium configurations which have central redshifts significantly greater than 0.5.

These findings are of some relevance to the Hoyle and Fowler (1967) star-cluster model for quasi-stellar sources; the results suggest that it will be difficult--and perhaps even impossible--to construct a stable spherical star-cluster model which has an isotropic velocity distribution, and which has a central redshift as large as ~ 2.3. If further analyses of compact and also of core-halo isotropic models confirm that stable, isotropic, spherical clusters cannot have the required large central redshifts, one would logically turn next to the study of spherical models with anisotropic velocity distributions, and to the study of rotating models.

Our analyses have demonstrated the existence of a strong correlation between the behavior of binding-energy curves and the onset of instability along a variety of sequences of isotropic, spherical clusters. This suggests that, for certain properly-chosen sequences of cluster models, one will be able to diagnose stability by simply appealing to the behavior of the associated binding-energy curves. Since the task of applying the variational principle can be somewhat time-consuming, it would seem well worthwhile to devote considerable effort to the search for such a binding-energy theorem for stability. Of some use in such a search should be the theorem on zero-frequency motions in §IVf of Paper I. The importance of a binding-energy theorem for stability is further enhanced by our findings that the one-dimensional sufficient criteria for stability of Paper II seem to be much less powerful than the multidimensional, necessary and sufficient variational principle of Paper I.

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APPENDIX A

EVALUATION OF MOMENTUM-SPACE INTEGRALS FOR ISOTHERMAL CLUSTERS

It is evident from the reduced expression (28) that our application of the variational principle (8) entails the evaluation of at least six of the momentum-space integrals $I'_{a,b}(r)$ defined by equation (29). In addition to the six $I'_{a,b}(r)$ which appear in expression (28), the integral $I'_{0,2}(r)$ also enters into the analysis if the trial function (22) is chosen such that the displacement is given by equation (23). Further, the integrals $I_{2,0}$, $I_{1,0}$, and $I_{0,2}$, where

$$I_{a,b}(r) \equiv \int Fm(p_0/m)^a(p_r/m)^b \, d\mathcal{V}_p \quad , \tag{A1}$$

enter into the structure equations (34a,b) and (37b). Thus ten momentum-space integrals $I_{a,b}$ and $I'_{a,b}$ enter into our analysis of the structures and stabilities of the isothermal clusters. Some of the relevant momentum-space integrals can be evaluated in terms of the others, as we shall show here.

In this appendix we derive useful expressions for $I'_{a,b}$ and $I_{a,b}$ when the distribution function is that of an isothermal model: First use equations (2), (5), and (6) to change variables in expression (29) from p_0, p_r, p_θ, p_ϕ to m, $p^{(r)}$, $p^{(\theta)}$, $p^{(\phi)}$

$$d\mathcal{V}_p = -dp_0\, dp_r\, dp_\theta\, dp_\phi/\sqrt{(-g)} = e^{\nu_A/2}(m/p_0)\, dm\, dp^{(r)} dp^{(\theta)} dp^{(\phi)} . \tag{A2}$$

Expression (29) then becomes

$$I'_{a,b}(r) = e^{\nu_A/2} \int F_E m^2 (p_0/m)^{a-1}(p_r/m)^b dm\, dp^{(r)}\, dp^{(\theta)}\, dp^{(\phi)} \ . \tag{A3}$$

At each point in physical space we set up a Cartesian coordinate system in 3-momentum space with coordinates $p^{(r)}$, $p^{(\theta)}$, $p^{(\phi)}$. Consider a vector $(p^{(r)}, p^{(\theta)}, p^{(\phi)})$ in this space with magnitude $p \equiv (\sum p^{(j)} p^{(j)})^{\frac{1}{2}}$. Let ζ be the angle between the $p^{(r)}$ axis and this 3-vector; and let η be the azimuthal angle in the $(p^{(\theta)}, p^{(\phi)})$ plane. By construction,

$$p^{(r)} = p \cos\zeta \ , \quad p^{(\theta)} = p \sin\zeta \cos\eta \ , \quad p^{(\phi)} = p \sin\zeta \sin\eta \ , \tag{A4a}$$

$$dp^{(r)} dp^{(\theta)} dp^{(\phi)} = p^2 dp \, d(\cos\zeta) \, d\eta \ . \tag{A4b}$$

Combine equations (A3), (A4), and the relations

$$p_0 = e^{\nu_A/2} p^{(0)} = e^{\nu_A/2} (p^2 + m^2)^{\frac{1}{2}} \ , \tag{A5a}$$

$$p_r = -e^{\lambda_A/2} p^{(r)} = -e^{\lambda_A/2} p \cos\zeta \ , \tag{A5b}$$

which follow from equations (5) and (6), to obtain

$$I'_{a,b}(r) = (-1)^b 2\pi e^{(a\nu_A + b\lambda_A)/2}$$
$$\times \int F_E(m, e^{\nu_A/2} p^{(0)}) m^4 (p^{(0)}/m)^{a-1} (p/m)^{b+2} \cos^b\zeta \, dm \, dp \, d(\cos\zeta). \tag{A6}$$

In arriving at this expression we have performed the trivial integration over η. Next use equation (A5a) to change variables from m, p, ζ to m, $E' \equiv p^{(0)}/m$, ζ in expression (A6); and then integrate over ζ, obtaining

$$I'_{a,b}(r) = [4\pi/(b+1)] \, e^{(a\nu_A + b\lambda_A)/2}$$
$$\times \int F_E(m, e^{\nu_A/2} mE') m^5 E'^a (E'^2 - 1)^{(b+1)/2} dm \, dE' \ . \tag{A7}$$

This expression is valid only if b is an even integer--which it is for all cases of interest in this paper. For isothermal clusters it follows from equations (9a) and (30) that

$$F_E(m,p_0) = -Ke^{-p_0/T}\,\delta(m-m_0)[\,H(m\sqrt{\beta}-p_0)/T + \delta(p_0-m\sqrt{\beta})]$$

$$= -F/T - Ke^{-p_0/T}\,\delta(m-m_0)\,\delta(p_0-m\sqrt{\beta}) \quad , \tag{A8}$$

since

$$\partial[\,H(m\sqrt{\beta}-p_0)]\,/\partial p_0 = -\delta(p_0-m\sqrt{\beta}) \quad . \tag{A9}$$

Notice that when F_E is reexpressed in terms of the variable $E' = p^{(0)}/m$, the quantity $\beta = e^{\nu_A(R)}$, the redshift factor, $y = e^{\nu_A}/\beta$, and the surface temperature, T', it becomes

$$F_E(m,e^{\nu_A/2}mE') = -Ke^{-E'\sqrt{y}/T'}\,\delta(m-m_0)/(m\sqrt{\beta})$$

$$\times[(1/T')H(1/\sqrt{y}-E') + \delta(E'\sqrt{y}-1)] \quad . \tag{A10}$$

Insert this expression for F_E into equation (A7) and finally arrive at the desired relation for $I'_{a,b}(r)$:

$$I'_{a,b}(r) = -[4\pi/(b+1)]\,Km_0^4\beta^{(a-1)/2}e^{b\lambda_A/2}$$

$$\times\,[(y^{a/2}/T')\,\mathcal{J}_{a,b}(r) + (e^{-1/T'}/\sqrt{y})(1/y-1)^{(b+1)/2}] \quad , \tag{A11}$$

where

$$\mathcal{J}_{a,b}(r) \equiv \int_1^{1/\sqrt{y}} e^{-E'\sqrt{y}/T'}E'^a(E'^2-1)^{(b+1)/2}\,dE' \quad . \tag{A12}$$

It follows from equations (30), (A8), and (A11) that we can also write the integrals $I_{a,b}(r)$ of definition (A1) for isothermal clusters in terms of $\mathcal{J}_{a,b}(r)$:

$$I_{a,b}(r) = [\,4\pi/(b+1)]\,Km_0^4\,\beta^{a/2} e^{b\lambda_A/2} y^{a/2} \mathcal{J}_{a,b}(r) \quad . \tag{A13}$$

Notice from equations (A11) and (A13) that we have reduced our task of evaluating the required integrals $I'_{a,b}$ and $I_{a,b}$ to the problem of evaluating the corresponding integrals $\mathcal{J}_{a,b}$. Scrutiny of equations (23), (28), (35), (37c), (A1), and (A13) will reveal that eight different integrals $\mathcal{J}_{a,b}$ enter into the structure and stability analysis. The relevant combinations (a,b) are: (3,0), (2,0), (1,2), (1,0), (0,2), (-1,4), (-1,2), and (-1,0). We shall now show how five of these integrals may be evaluated in terms of the remaining three: First reexpress definition (A12) for $\mathcal{J}_{a+1,b}$ in the form

$$\mathcal{J}_{a+1,b} = [1/(b+3)]\int_1^{1/\sqrt{y}} e^{-E'\sqrt{y}/T'} E'^a (d/dE')[(E'^2-1)^{(b+3)/2}]\,dE' \,.$$

Perform an integration by parts and use definition (A12) to obtain

$$(b+3)\mathcal{J}_{a+1,b} = e^{-1/T'}(1/\sqrt{y})^a (1/y-1)^{(b+3)/2}$$

$$+(\sqrt{y}/T')\mathcal{J}_{a,b+2} - a\mathcal{J}_{a-1,b+2} \,. \tag{A14a}$$

Also, employ definition (A12) and simple subtraction to arrive at the trivial identity

$$\mathcal{J}_{a,b+2} = \mathcal{J}_{a+2,b} - \mathcal{J}_{a,b} \quad . \tag{A14b}$$

The "recursion relations" (A14) imply that if, say, the integrals $\mathcal{J}_{2,0}$

$\mathcal{J}_{0,2}$, and $\mathcal{J}_{-1,0}$ are known, we can evaluate in terms of them the five remaining $\mathcal{J}_{a,b}$ which enter into our present analysis. This fortunate state of affairs greatly simplifies the numerical problem.

Fackerell (1966) has shown how to efficiently evaluate integrals equivalent to $\mathcal{J}_{2,0}$ and $\mathcal{J}_{0,2}$ by summing infinite series involving hypergeometric functions. Unfortunately, the integral $\mathcal{J}_{-1,0}$ seems to be expressible only as a doubly-infinite series involving hypergeometric functions, so that it is most efficiently evaluated by use of Simpson's rule or some other numerical technique.

APPENDIX B

DIMENSIONLESS FORM OF THE VARIATIONAL PRINCIPLE FOR ISOTHERMAL CLUSTERS

For the isothermal clusters, the reduced variational principle (28) can be reexpressed in a dimensionless form which facilitates the computation of a squared frequency, ω^2. We begin by defining the dimensionless analogues, $\mathcal{a}_0$ and $\mathcal{a}_1$, of the functions a_0 and a_1 of definitions (27b):

$$\mathcal{a}_0(x) \equiv \beta L a_0(r) = \frac{C(Lx)}{y}\left[\frac{1}{2y}\frac{dy}{dx} - \frac{1}{x}\right] \qquad \text{(B1a)}$$

$$\mathcal{a}_1(x) \equiv L a_1(r) = e^{-\lambda_A}\left[C(Lx)\left(\frac{1}{x} - \frac{1}{2}\frac{d\lambda_A}{dx}\right) - \frac{dC(Lx)}{dx}\right], \qquad \text{(B1b)}$$

where L is defined by equation (32). (Recall from expressions [22] that the function $C(r) = C(Lx)$ is itself dimensionless.) The last equalities in these relations follow from equations (27b), (32), and (33). Next we introduce the dimensionless function (cf. equation [A11])

$$\mathcal{J}'_{ab}(x) = (y^{a/2}/T')\mathcal{J}_{a,b}(Lx) + (e^{-1/T'}/\sqrt{y})(1/y - 1)^{(b+1)/2} \qquad \text{(B2)}$$

so that

$$I'_{a,b}(r) = I'_{a,b}(Lx) = -[4\pi/(b+1)]Km_0^4\beta^{(a-1)/2}e^{b\lambda_A/2}\mathcal{J}'_{ab}(x) \quad . \qquad \text{(B3)}$$

It follows from equations (A12), (B2), and (B3) that the constant K appears raised to different powers in the various terms of the reduced equation (28). In order to reexpress the dependence on K of equation (28), we combine equations (32), (35), and (A12) to obtain

$$Km_0^4 L^2 = \mathcal{J}^c_{0,2}/[\,48\pi^2(\mathcal{J}^c_{2,0})^2\,] \quad , \qquad \text{(B4)}$$

where the superscript c denotes the value of one of the integrals $\mathcal{J}_{a,b}(r)$ at the center of symmetry, $r = x = 0$.

If we now combine equations (B1) - (B4) with equations (28), (32), and (33) we obtain the desired reduced form of equation (28):

$$(L\omega)^2/\beta = \left(\int dx\, y^{-1/2} e^{3\lambda_A/2} x^2 C^2 \mathcal{J}'_{1,2}\right)^{-1}$$

$$\times \int dx\, y^{3/2} e^{3\lambda_A/2} \Big\{ 3e^{-\lambda_A}(x\mathcal{Q}_0)^2 \mathcal{J}'_{3,0} + 2x^2 \mathcal{Q}_0 \mathcal{Q}_1 \mathcal{J}'_{1,2} + 6e^{-\lambda_A} x\mathcal{Q}_0 C \mathcal{J}'_{1,0}$$

$$+ (3/5)e^{\lambda_A}(x\mathcal{Q}_1)^2 \mathcal{J}'_{-1,4} + 2x\mathcal{Q}_1 C\mathcal{J}'_{-1,2} + 3e^{-\lambda_A} C^2 \mathcal{J}'_{-1,0}$$

$$+ [\,\mathcal{J}^c_{0,2}/(\mathcal{J}^c_{2,0})^2]\, x^2 C \mathcal{J}'_{1,2} [(2/9)x\mathcal{Q}_0 \mathcal{J}'_{1,2} + (2/15)e^{\lambda_A} x\mathcal{Q}_1 \mathcal{J}'_{-1,4}$$

$$+ (2/9)C\mathcal{J}'_{-1,2} - (1/9)(1/y)(1 + xy^{-1} dy/dx) C\mathcal{J}'_{1,2}]$$

$$+ (1/135)[(\mathcal{J}^c_{0,2})^2/(\mathcal{J}^c_{2,0})^4]\, e^{\lambda_A} x^4 C^2 (\mathcal{J}'_{1,2})^2 (\mathcal{J}'_{-1,4}) \Big\} \quad , \qquad \text{(B5)}$$

where $C \equiv C(r) \equiv C(Lx)$ in this expression. As stated in §IVb, we evaluate expression (B5) by integrating the differential equations which are equivalent to its numerator and denominator.

APPENDIX C

EVALUATION OF MOMENTUM-SPACE INTEGRALS FOR POLYTROPIC CLUSTERS

In order to apply the reduced variational principle (28) to the study of the stability of a polytropic cluster, we must first be able to evaluate the six integrals I'_{ab} of definition (29), which appear in equation (28). In this appendix we devise a method for evaluating these integrals analytically in terms of ν_A and λ_A when the polytropic index, n, is an integer ≥ 2: First recall identity (14) of Paper II,

$$\int (F_E/m)(p_0)^a(p_r)^b\,d\mathcal{V}_p$$

$$= -a\int (F/m)(p_0)^{a-1}(p_r)^b d\mathcal{V}_p - (b-1)e^{\lambda_A-\nu_A}\int (F/m)(p_0)^{a+1}(p_r)^{b-2}d\mathcal{V}_p. \quad \text{(C1)}$$

Since the rest mass, m, and the angular momentum, J, of a star are held constant in obtaining F_E from F by differentiation (cf. definition 9a), we can replace F_E and F in identity (C1) by $F_E G(m,J)$ and $FG(m,J)$, where G is an arbitrary function of its arguments--the identity will remain valid. For $G(m,J) = m^{3-a-b}$ the identity becomes

$$I'_{a,b}(r) = -aI_{a-1,b}(r) - (b-1)e^{\lambda_A-\nu_A}I_{a+1,b-2}(r)\ , \quad \text{(C2)}$$

where $I'_{a,b}$ and $I_{a,b}$ are defined by equations (29) and (A1). By employing the same methods used to derive expression (48), and by employing definition (49), obtain the relation

$$I_{a,b}(r) = [3/(b+1)]e^{[b\lambda_A-(b+2)\nu_A]/2}\int_{e^{\nu_A}}^{\beta} \mathfrak{F}(u)u^{a/2}(u-e^{\nu_A})^{(b+1)/2}du, \quad \text{(C3)}$$

when b is an even integer. (In the remainder of this Appendix we assume that such is the case.) Next combine equations (C2) and (C3) to arrive at the identity

$$I'_{a,b}(r) = -3e^{[b\lambda_A-(b+2)\nu_A]/2}\left\{[a/(b+1)]K_{a-1,b}(r)+K_{a+1,b-2}(r)\right\}, \tag{C4}$$

where

$$K_{a,b}(r) \equiv \int_{e^{\nu_A}}^{\beta} \mathcal{F}(u)u^{a/2}(u-e^{\nu_A})^{(b+1)/2}\,du$$

$$= -[4/(3\pi)]\int_{e^{\nu_A}}^{\beta}\left[\left[u^{a/2}(u-e^{\nu_A})^{(b+1)/2}\right.\right.$$

$$\left.\left.\times\int_{u}^{\beta}\left\{d^3[\mathcal{G}(w/\beta)]/dw^3\right\}(w-u)^{-1/2}dw\right]\right]du \;. \tag{C5}$$

The last equality in this equation results from employing equation (51) and the relation $y = e^{\nu_A}/\beta$. Examination of equations (52) reveals that we can evaluate the needed $K_{a,b}(r)$ of equation (C5) most simply by reversing the order of integration in equation (C5). That equation then reads

$$K_{a,b}(r) = -[4/(3\pi)]\int_{e^{\nu_A}}^{\beta} d^3[\mathcal{G}(w/\beta)]/dw^3\,L_{a,b}(w,e^{\nu_A})\,dw \;, \tag{C6}$$

where

$$L_{a,b}(w,e^{\nu_A}) \equiv \int_{e^{\nu_A}}^{w} u^{a/2}(u-e^{\nu_A})^{(b+1)/2}(w-u)^{-1/2}du \;. \tag{C7}$$

Scrutiny of equations (28), (C4), (C6) and (C7) leads us to conclude that all of the integrals $L_{a,b}$ which we encounter in our stability analysis are among those for which a is an even integer. (We have already assumed that b is an even integer.) Consequently, the needed $L_{a,b}$ are easily evaluated in terms of the variables w and e^{ν_A} by use of any of the standard tables of integrals, and it seems unnecessary for us to reproduce expressions for them here. However, we note that each of the needed $L_{a,b}$ is expressible as a sum of terms; each term depends on the variable w through its proportionality to only one of the factors $w^{-1/2}$, 1, w, or w^2.

Once we have analytic expressions for the $L_{a,b}$ we can combine with equations (52) and perform the integrations over the variable w in equation (C6) analytically, thereby obtaining explicit algebraic expressions for the $K_{a,b}$ when the polytropic index, n, is an integer ≥ 2. The results are these: Define the quantities

$$B_{j,k}(r) \equiv (1/\sigma_{jk})(1 - y^{\sigma_{jk}}) \quad \text{if } \sigma_{jk} \neq 0$$

$$\equiv -\ln y \quad \text{if } \sigma_{jk} = 0 , \tag{C8a}$$

$$\sigma_{jk} \equiv (\gamma_4/2)[j/(n+1) - 1] + k , \tag{C8b}$$

$$D_{jk} \equiv (-)^j A_k (n-2)!/[(j)!(n-2-j)!] , \tag{C8c}$$

$$Q_k(r) \equiv -4\alpha\tau/[3(\alpha\gamma_4)^{n+1}] \sum_{i=0}^{n-2} \sum_{j=0}^{2} D_{ij} B_{i+j,k}(r) . \tag{C8d}$$

Here $y = e^{\nu_A}/\beta$, γ_4 is defined by equation (43), the A_k are defined by equations (52b,c), and τ is the scaling parameter of equation (46). In terms of these quantities the relevant $K_{a,b}(r)$ are given by

$$K_{4,-2} = (\beta^2/8)(3Q_2 + 2yQ_1 + 3y^2 Q_0) ,$$

$$K_{2,0} = (\beta^2/8)(3Q_2 - 2yQ_1 - y^2Q_0) \quad ,$$

$$K_{2,-2} = (\beta/2)(Q_1 + yQ_0) \quad ,$$

$$K_{0,2} = (3\beta^2/8)(Q_2 - 2yQ_1 + y^2Q_0) \quad ,$$

$$K_{0,0} = (\beta/2)(Q_1 - yQ_0) \quad ,$$

$$K_{0,-2} = Q_0 \quad ,$$

$$K_{-2,4} = (\beta^2/8)(3Q_2 - 10yQ_1 + 15y^2Q_0 - 8y^{5/2}Q_{-1/2}) \quad ,$$

$$K_{-2,2} = (\beta/2)(Q_1 - 3yQ_0 + 2y^{3/2}Q_{-1/2}) \quad ,$$

$$K_{-2,0} = Q_0 - y^{1/2}Q_{-1/2} \quad . \tag{C9}$$

Equations (C4), (C8), and (C9) provide us with analytic expressions for the integrals $I'_{ab}(r)$ which enter into our stability analysis for polytropes of integral index $n \geq 2$.

APPENDIX D

DIMENSIONLESS FORM OF THE VARIATIONAL PRINCIPLE FOR POLYTROPIC CLUSTERS

In this appendix we rewrite the reduced variational principle (28) in a dimensionless form suitable for application to polytropic clusters of integral index $n \geq 2$. The first step consists of introducing the quantity

$$\mathcal{K}'_{a,b}(x) \equiv [3(\alpha\gamma_4)^{n+1}/(4\alpha\tau)]\,\beta^{-(a+b+1)/2}$$

$$\times\left\{[a/(b+1)]K_{a-1,b}(Lx) + K_{a+1,b-2}(Lx)\right\} \quad , \qquad \text{(D1)}$$

where, in this appendix, the length parameter, L, is defined by equation (44). It follows from equations (C4) and (D1) that

$$I'_{a,b}(Lx) = -[4\alpha\tau/(\alpha\gamma_4)^{n+1}]\,y^{-(b+2)/2}e^{b\lambda_A/2}\beta^{(a-1)/2}\mathcal{K}'_{a,b}(x) \quad . \qquad \text{(D2)}$$

Notice from equations (C8), (C9), and (D1) that the $\mathcal{K}'_{a,b}(x)$ do not depend upon the scaling parameter, τ, and that they depend upon the surface redshift factor, β, only through the quantity $y = e^{\nu_A}\beta^{-1}$, which is given in terms of the structure parameter, Θ, by equation (42). Consequently, as we integrate the structure equations (45), we can simultaneously evaluate the $\mathcal{K}'_{a,b}(x)$ which enter into the reduced variational principle (28) through equation (D2). If we introduce the dimensionless quantities

$$\mathcal{A}_0(x) = \beta L a_o(r) \quad ; \quad \mathcal{A}_1(x) = L a_1(r) \qquad \text{(D3)}$$

(cf. equations [27b] and [44]), and if we combine equations (28), (44), (D2), and (D3), we finally arrive at the following dimensionless form of the reduced expression (28):

$$(L\omega)^2/\beta = \left(\int dx\, y^{-5/2} e^{3\lambda_A/2} x^2 C^2 \mathcal{K}'_{1,2} \right)^{-1}$$

$$\times \int dx \Big\{ \sqrt{y}\, e^{\lambda_A/2} \Big[(x\mathfrak{a}_0)^2 \mathcal{K}'_{3,0} + (2/y) e^{\lambda_A} x^2 \mathfrak{a}_0 \mathfrak{a}_1 \mathcal{K}'_{1,2} + 2x\mathfrak{a}_0 C \mathcal{K}'_{1,0}$$

$$+ y^{-2} e^{2\lambda_A} (x\mathfrak{a}_1)^2 \mathcal{K}'_{-1,4} + (2/y) e^{\lambda_A} x\mathfrak{a}_1 C \mathcal{K}'_{-1,2} + C^2 \mathcal{K}'_{-1,0} \Big]$$

$$+ \left[8(n+1)\alpha^2/(\alpha\gamma_4)^{n+1} \right] y^{-5/2} e^{3\lambda_A/2} x^2 C \mathcal{K}'_{1,2} \Big[x\mathfrak{a}_0 \mathcal{K}'_{1,2}$$

$$+ (1/y) e^{\lambda_A} x\mathfrak{a}_1 \mathcal{K}'_{-1,4} + C\mathcal{K}'_{-1,2} - (1/2)(1/y)(1 + xy^{-1} dy/dx) C \mathcal{K}'_{1,2} \Big]$$

$$+ \left[16(n+1)^2 \alpha^4/(\alpha\gamma_4)^{2n+2} \right] y^{-11/2} e^{5\lambda_A/2} x^4 C^2 (\mathcal{K}'_{1,2})^2 \mathcal{K}'_{-1,4} \Big\}. \quad \text{(D4)}$$

As we integrate the equations of structure (45a,b) for a polytropic model of integral index $n \geq 2$, we simultaneously evaluate the right side of equation (D4) by integrating the differential equations which are equivalent to the integrals over the coordinate x in the numerator and denominator.

TABLE 1

THE ISOTHERMAL SEQUENCE WITH $\epsilon = 1/2^*$

z_c	$100\ \mathfrak{m}_0^2\rho_{Ac}$	$\langle\rho\rangle/\rho_{Ac}$	T/m_0	$\mathcal{E}/\mathfrak{m}_0$	$R/2\mathfrak{m}_0$	$z_{surface}$	T'	y_c	ω^2/ρ_{Ac}
0.067	0.0029	0.124	0.05	0.0108	20.1	0.0256	0.0512	0.923	+1.17
0.149	0.0255	0.115	0.10	0.0206	10.1	0.0526	0.105	0.840	+0.879
0.392	0.261	0.0848	0.20	0.0347	5.07	0.111	0.222	0.638	+0.181
0.423	0.307	0.0815	0.21	0.0352	4.86	0.117	0.235	0.616	+0.123
0.459	0.364	0.0779	0.22	0.0356	4.66	0.123	0.247	0.592	+0.065
0.516	0.464	0.0715	0.23	0.0357	4.42	0.131	0.262	0.556	-0.016
0.560	0.544	0.0681	0.24	0.0353	4.25	0.136	0.272	0.531	-0.068
0.640	0.707	0.0598	0.25	0.0338	4.09	0.144	0.288	0.487	-0.144
0.839	1.16	0.0433	0.27	0.0265	3.87	0.156	0.312	0.395	< -0.257
1.08	1.81	0.0267	0.27	0.0133	3.92	0.156	0.312	0.309	< -0.295
1.34	2.73	0.0166	0.25	-0.0028	4.27	0.143	0.286	0.239	< -0.303
1.56	3.89	0.0066	0.22	-0.014	4.90	0.123	0.247	0.192	< -0.267
1.63	4.49	0.0045	0.20	-0.0150	5.30	0.112	0.223	0.179	< -0.243
1.74	7.26	0.0012	0.15	-0.0073	7.00	0.081	0.162	0.156	< -0.182

Legend for Table 1

*With all quantities expressed in units where $G = c = k = 1$, the various columns are: z_c, the gravitational redshift of a photon emitted at the center of the cluster and received at infinity; $\mathfrak{m}_0^2 \rho_{Ac}$, the central density of total mass-energy of the cluster in units of $\mathfrak{m}_0^{-2}$, the inverse of the square of the total rest mass of the cluster; $\langle \rho \rangle / \rho_{Ac}$, the ratio of mean density to central density--$\langle \rho \rangle \equiv (3/4\pi)(\text{total mass-energy})(\text{radius})^{-3}$; T/m_0, the ratio of the temperature, as measured by an infinitely-removed observer, to the rest mass of one of the stars from which the cluster is made; $\mathcal{E}/\mathfrak{m}_0$, the ratio of the binding energy to the total rest mass of the cluster; $R/2\mathfrak{m}_0$, one-half the ratio of the radius $\equiv$ (surface area of cluster$/4\pi)^{1/2}$ to the total rest mass of the cluster; $z_{surface}$, the gravitational redshift of a photon emitted at the surface and received at infinity; T', the ratio of the temperature, as measured by an observer located on the surface, to the rest mass of one of the stars from which the cluster is made; y_c, the central-to-surface redshift factor--$(y_c^{-1/2} - 1)$ is the redshift of a photon emitted at the cluster's center and received at its surface; ω^2/ρ_{Ac}, the estimate, in units of ρ_{Ac}, of the square of the frequency of oscillation of the fundamental radial mode obtained from the variational principle; later entries in the column give only "upper limits"; an entry which is negative implies that the model is unstable. The cutoff parameter ϵ is equal to the ratio $z_{surface}/T'$. The entries in this and the remaining tables are accurate to about 1 per cent.

TABLE 2

THE ISOTHERMAL SEQUENCE WITH $\epsilon = 1/10$*

z_c	$100\,\mathfrak{m}_0^2\rho_{Ac}$	$\langle\rho\rangle/\rho_{Ac}$	$T/\mathfrak{m}_0$	$\mathcal{E}/\mathfrak{m}_0$	$R/2\mathfrak{m}_0$	$z_{surface}$	T'	y_c	ω^2/ρ_{Ac}
0.349	0.188	0.115	1.0	0.0337	5.11	0.110	1.11	0.678	+0.291
0.459	0.339	0.102	1.2	0.0357	4.37	0.133	1.33	0.602	+0.058
0.521	0.437	0.0944	1.25	0.0357	4.11	0.143	1.43	0.565	-0.0398
0.678	0.720	0.0782	1.4	0.0324	3.72	0.163	1.63	0.480	-0.215
1.14	1.75	0.0402	1.57	0.0059	3.48	0.183	1.83	0.305	< -0.381

*The notation is the same as that in Table 1. The entries for ω^2/ρ_{Ac} were obtained by using a homologous trial function (22). Hence ξ/r = constant, where ξ is the displacement.

TABLE 3

THE ISOTHERMAL SEQUENCE WITH $\epsilon = 1/100$*

z_c	$100\ \mathfrak{m}_0^2 \rho_{Ac}$	$\langle \rho \rangle / \rho_{Ac}$	T/m_0	$\mathcal{E}/\mathfrak{m}_0$	$R/2\mathfrak{m}_0$	$z_{surface}$	T'	y_c	ω^2/ρ_{Ac}
0.343	0.180	0.121	10.0	0.0336	5.11	0.111	11.1	0.683	+0.309
0.471	0.352	0.105	12.0	0.0357	4.26	0.137	13.7	0.597	+0.0369
0.508	0.409	0.101	12.5	0.0357	4.11	0.143	14.3	0.574	-0.0238
0.636	0.633	0.0937	14.0	0.0336	3.75	0.161	16.1	0.503	-0.184
1.16	1.77	0.0418	16.0	0.0035	3.42	0.188	18.8	0.301	< -0.396

*The notation is the same as that in Table 1. The entries for ω^2/ρ_{Ac} were obtained by using a homologous trial function (22). Hence ξ/r = constant, where ξ is the displacement.

TABLE 4

THE POLYTROPIC SEQUENCE WITH $n = 2,\ \Gamma_4 = 3/2^*$

α	z_c	ρ_{Ac}/τ	$R\sqrt{\tau}$	$\mathfrak{M}\sqrt{\tau}$	$\langle\rho\rangle/\rho_{Ac}$	$\mathcal{E}/\mathfrak{M}_0$	$z_{surface}$	ω^2/τ
0.05	0.225	1.10	0.427	0.0253	0.0825	0.0269	0.0650	+0.638
0.071	0.315	1.14	0.493	0.0367	0.0643	0.0321	0.0837	+0.387
0.09	0.394	1.18	0.542	0.0459	0.0583	0.0346	0.0973	+0.208
0.10	0.435	1.20	0.565	0.0505	0.0543	0.0353	0.103	+0.127
0.11	0.475	1.22	0.587	0.0548	0.0531	0.0357	0.109	+0.0540
0.115	0.495	1.23	0.597	0.0569	0.0521	0.0357	0.111	+0.0201
0.12	0.515	1.24	0.608	0.0590	0.0504	0.0357	0.114	-0.0121
0.125	0.535	1.25	0.618	0.0609	0.0492	0.0356	0.116	-0.0428
0.13	0.554	1.26	0.628	0.0628	0.0483	0.0354	0.118	-0.0720

Legend for Table 4

*The entries in the columns are: α, the relativity parameter, which uniquely specifies a member of the sequence; z_c, the redshift of a photon emitted at the center of the cluster and received at infinity; ρ_{Ac}/τ, the ratio of the central density of total mass-energy to the scaling parameter, τ; $R\sqrt{\tau}$, the radius $\equiv$ (surface area of cluster$/4\pi)^{1/2}$ in units of $\tau^{-1/2}$; $\mathfrak{M}\sqrt{\tau}$, the total mass energy, in units of $\tau^{-1/2}$, as sensed gravitationally by a distant observer; $\langle\rho\rangle/\rho_{Ac}$, the ratio of mean density to central density--$\langle\rho\rangle \equiv (3/4\pi)\mathfrak{M}R^{-3}$; $\mathcal{E}/\mathfrak{M}_0$, the binding energy per unit rest mass--the entries in this column were computed by Fackerell (1969); $z_{surface}$, the redshift of a photon emitted at the surface of the cluster and received at infinity; ω^2/τ, the squared frequency, in units of τ, obtained for the fundamental mode of the cluster by employing a trial function (22) with C' = constant and $\mu = 0$. A negative value of ω^2/τ implies that the model is unstable.

TABLE 5

THE POLYTROPIC SEQUENCE WITH $n = 3$, $\Gamma_4 = 4/3$*

α	z_c	ρ_{Ac}/τ	$R\sqrt{\tau}$	$\mathfrak{M}\sqrt{\tau}$	$\langle\rho\rangle/\rho_{Ac}$	$\mathcal{E}/\mathfrak{M}_0$	$z_{surface}$	ω^2/τ
0.04	0.197	1.12	0.790	0.0242	1.05×10^{-2}	0.0237	0.0321	+0.454
0.06	0.290	1.18	0.999	0.0376	7.62×10^{-3}	0.0296	0.0399	+0.299
0.08	0.378	1.24	1.21	0.0500	5.43×10^{-3}	0.0331	0.0441	+0.175
0.10	0.464	1.3	1.44	0.0613	3.78×10^{-3}	0.0348	0.0457	+0.0817
0.11	0.506	1.33	1.56	0.0666	3.13×10^{-3}	0.0351	0.0457	+0.0451
0.12	0.547	1.36	1.69	0.0717	2.84×10^{-3}	0.0352	0.0452	+0.0129
0.125	0.567	1.37	1.77	0.0742	2.39×10^{-3}	0.0352	0.0449	-0.0019
0.130	0.588	1.39	1.84	0.0767	2.04×10^{-3}	0.0350	0.0444	-0.013
0.135	0.608	1.405	1.92	0.0791	1.90×10^{-3}	0.0349	0.0439	-0.024
0.14	0.628	1.42	2.00	0.0815	1.71×10^{-3}	0.0347	0.0433	-0.035
0.20	0.859	1.6	3.47	0.109	3.90×10^{-4}	0.0298	0.0330	< -0.095
0.30	1.23	1.9	12.4	0.166	1.09×10^{-5}	0.0188	0.0137	< -0.071
0.40	1.61	2.2	53.0	0.302	2.20×10^{-7}	0.0121	0.0058	< -0.066

*The notation is identical to that of Table 4. The entries for the fractional binding energy, $\mathcal{E}/\mathfrak{M}_0$, were computed by Fackerell (1969). For the models with $z_c \leq 0.567$ the entries for the squared frequency, ω^2/τ, were obtained by letting $C' =$ constant and by minimizing with respect to the peaking parameter, μ. For $z_c > 0.567$ the entries for ω^2/τ were obtained by letting $C' =$ constant and $\mu = 0$.

TABLE 6

THE POLYTROPIC SEQUENCE WITH $n = 3$, $\Gamma_4 = 2$*

α	z_c	ρ_{Ac}/τ	$R\sqrt{\tau}$	$\mathfrak{M}\sqrt{\tau}$	$100\langle\rho\rangle/\rho_{Ac}$	$\mathcal{E}/\mathfrak{M}_0$	$z_{surface}$	ω^2/τ
0.05	0.261	1.05	0.862	0.0341	1.21	0.0284	0.0422	+0.312
0.075	0.393	1.075	1.07	0.0533	0.968	0.0339	0.0536	+0.145
0.09	0.473	1.09	1.20	0.0640	0.810	0.0352	0.0581	+0.0619
0.10	0.527	1.10	1.28	0.0709	0.736	0.0355	0.0604	+0.0104
0.102	0.538	1.102	1.30	0.0723	0.715	0.03543	0.0608	-0.0019
0.105	0.554	1.105	1.32	0.0743	0.698	0.03537	0.0613	-0.014
0.11	0.581	1.11	1.37	0.0775	0.648	0.0352	0.0621	-0.035
0.12	0.635	1.12	1.45	0.0839	0.588	0.0345	0.0632	-0.074
0.14	0.744	1.14	1.64	0.0961	0.457	0.0318	0.0642	-0.138

*The notation is identical to that of Table 4. The entries for the fractional binding energy, $\mathcal{E}/\mathfrak{M}_0$, were computed by Fackerell (1969). All entries for the frequency of oscillation, ω^2/τ, were obtained by letting C' = constant and $\mu = 0$.

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FIGURE CAPTIONS

Fig. 1. Momentum-space dependences of the equilibrium and perturbed distribution functions for the isothermal cluster of Table 1 with central redshift $z_c = 0.392$. Figure (a) shows the momentum space at a fractional distance $r/R = 0.15$ from the cluster's center, where $e^{\nu_A/2} = 0.73$. Figure (b) is at $r/R = 0.78$, where $e^{\nu_A/2} = 0.87$. The left portion of each figure is a plot of the contours of constant F^*, where $F \equiv K\delta(m-m_0)F^*$ is the equilibrium distribution function (cf. equation [30]). The right portion of each figure is a plot of F^* and of the perturbed quantity $F^* + f^*$, where $f \equiv K\delta(m-m_0)f^*$ is the perturbation induced by the trial function (22) for the assignments $C' = 1.28$, $\mu = 0$. The perturbed distribution function is shown at a moment of time when the even-parity part of the perturbation vanishes. The perturbed distribution function has a delta-function singularity at the cutoff energy, $p_0 = 0.9\, m_0$, of the cluster. The delta function is a mathematical tool for taking account of the motion of the cluster's sharp surface in phase space. The dashed extensions of the curves indicate the actual locations of the perturbed cluster's surface, as calculated from the demand that the area under each dashed curve be equal to the area under its delta-function idealization.

Fig. 2. Relationship between binding energy and the onset of instability of the fundamental radial mode for the isothermal sequence with cutoff parameter $\epsilon = 0.5$. The central redshift (the redshift of a photon emitted at a cluster's center and received at infinity)

is plotted horizontally. The binding-energy curve is parametrized by T/m_0, where T is the temperature of a model as measured by an observer at infinity, and where m_0 is the rest mass of one of the stars which compose the model. The solid portion of the curve for the squared frequencies of oscillation, ω^2/ρ_{Ac}, was obtained by choosing C' in equation (22) such that equation (23) is satisfied, and by minimizing with respect to the peaking parameter, μ, of equations (22) and (23). The dashed portion was obtained without minimizing with respect to μ and is included so as to show only that the corresponding models are unstable. Thus its shape is of no significance.

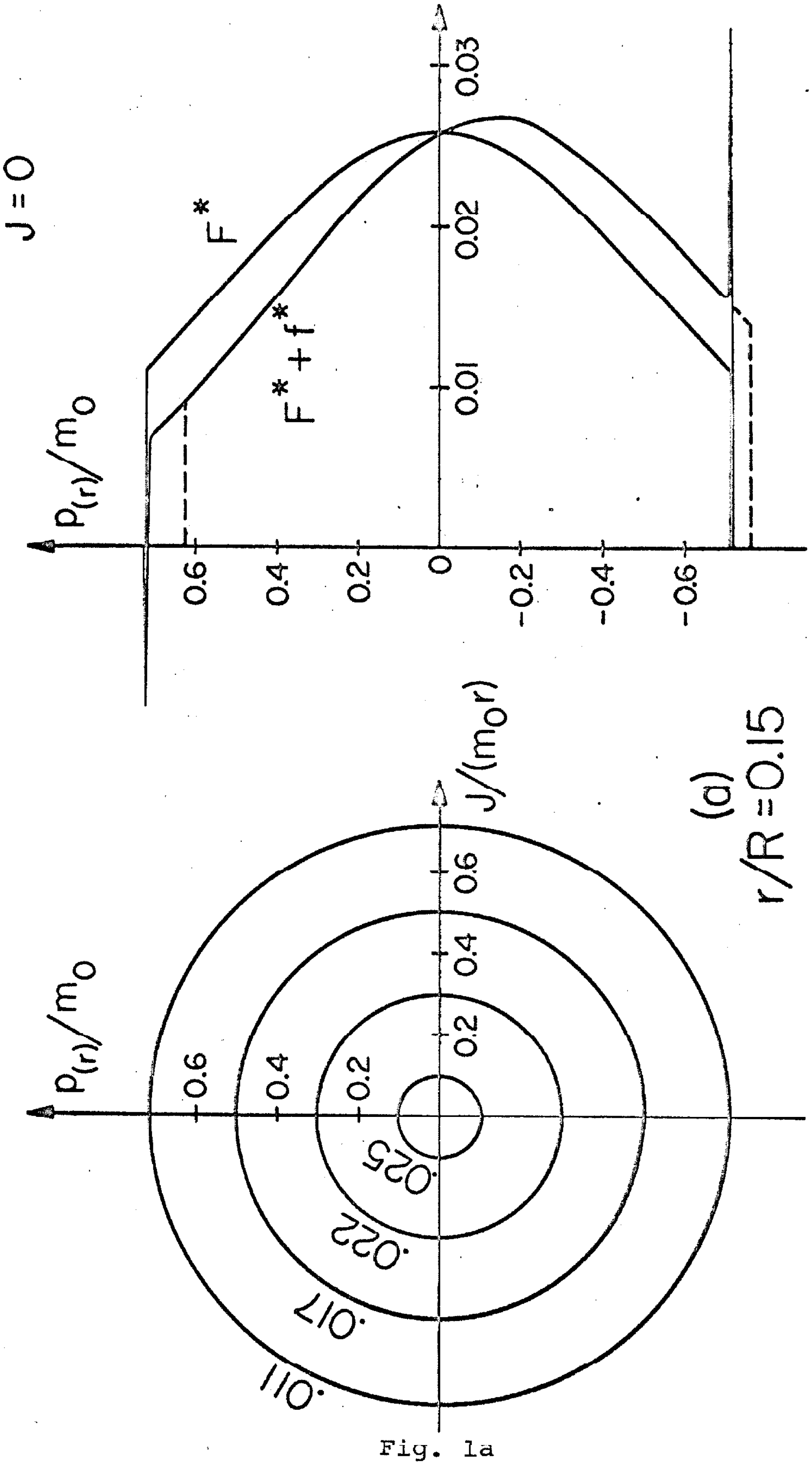
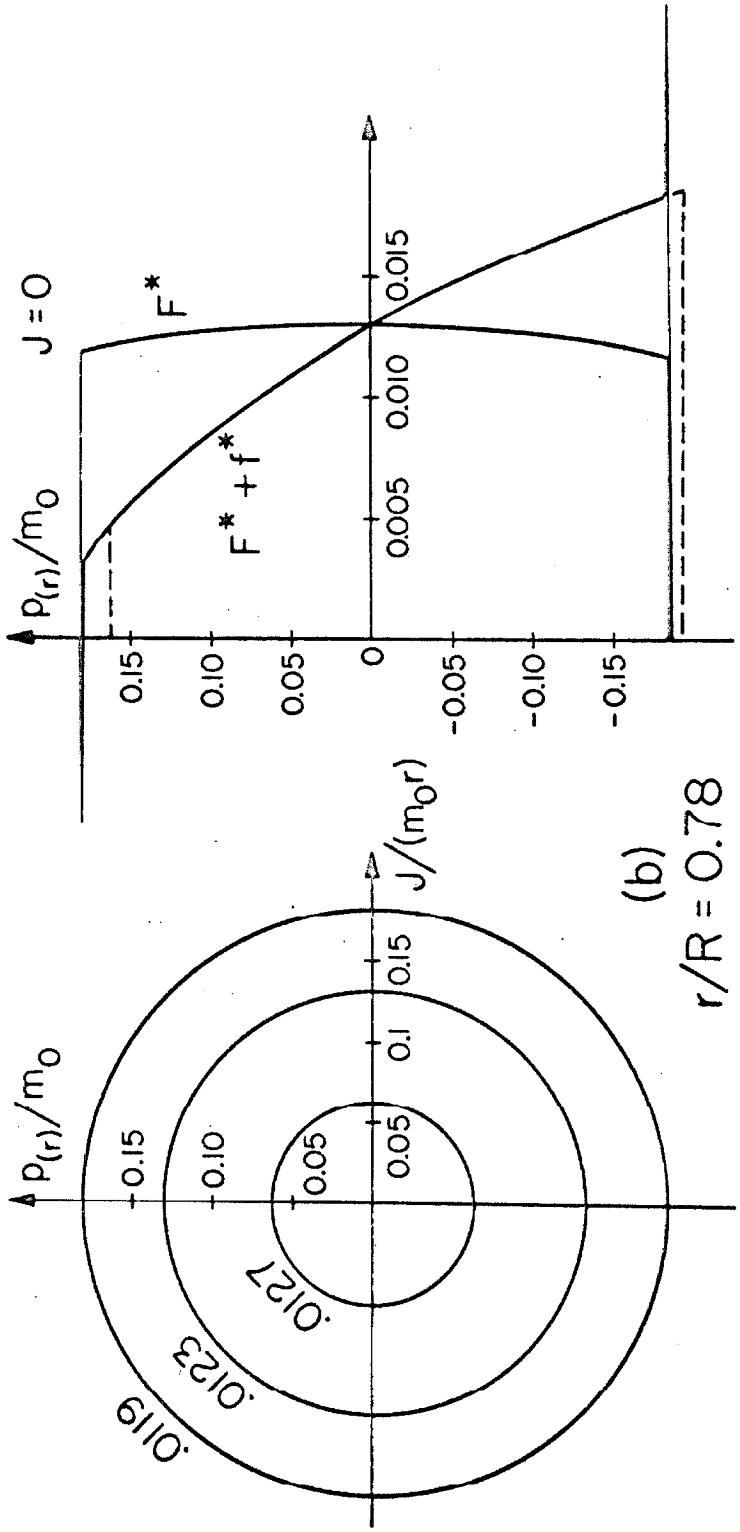
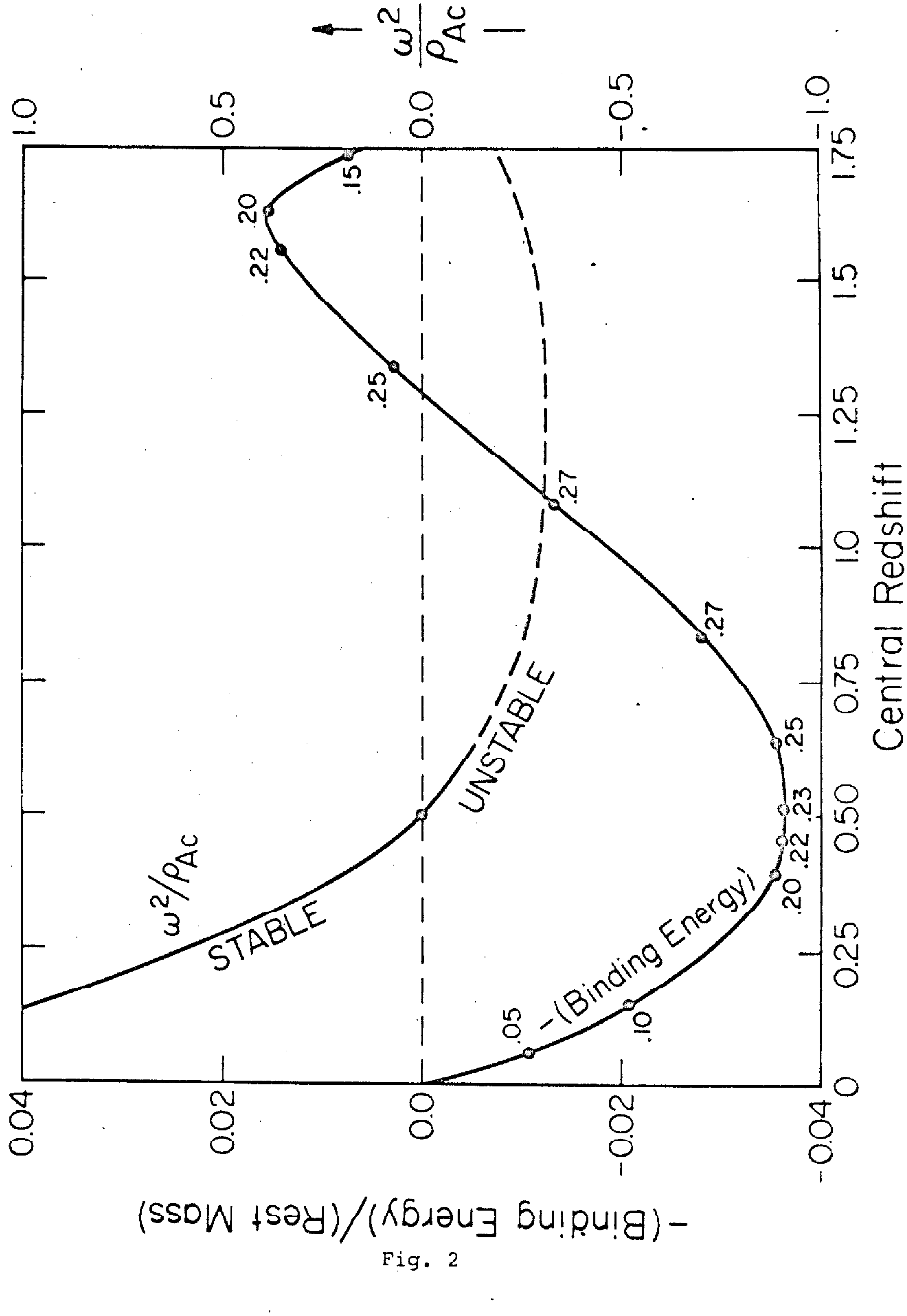

J = O
F*
F* + f*
p(r)/m0
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0.01
0.02
0.03
J/(m0r)
(a)
r/R = 0.15
.011
.017
.022
.025

Fig. 1a



$p_{(r)}/m_o$
J = 0
F^*
$F^* + f^*$
0.005
0.010
0.015
0.15
0.10
0.05
0
-0.05
-0.10
-0.15
$J/(m_o r)$
$p_{(r)}/m_o$
0.15
0.10
0.05
0.05
0.1
0.15
.0127
.0123
.0119
(b)
r/R = 0.78

Fig. 1b



Central Redshift
-(Binding Energy)/(Rest Mass)
ω^2/ρ_{Ac}
STABLE
UNSTABLE
-(Binding Energy)
0 0.25 0.50 0.75 1.0 1.25 1.5 1.75
0.04 0.02 0.0 -0.02 -0.04
1.0 0.5 0.0 -0.5 -1.0

Fig. 2