Boundary Behavior of Cauchy Integrals and Rank One Perturbations of Operators

Thesis by

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To my wife Svetlana

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Abstract

We develop new methods based on Rohlin-type decompositions of Lebesgue measure on the unit circle and on the real line to study the boundary behavior of Cauchy integrals. We also apply these methods to investigate the notion of Krein spectral shift of a self-adjoint operator. Using this notion we study the spectral properties of rank one perturbations of operators.

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Introduction.

Let φ be an analytic function on the unit disk $\mathbb{D}$ such that $|\varphi| \leq 1$. Then for every $\alpha \in \mathbb{T} = \partial\mathbb{D}$ the function $(\alpha + \varphi)(\alpha - \varphi)^{-1}$ has positive real part. Therefore, there is a unique positive measure μ_α on $\mathbb{T}$ such that

$$\mathcal{P}\mu_\alpha = \operatorname{Re}\frac{\alpha + \varphi}{\alpha - \varphi},$$

where $\mathcal{P}\mu_\alpha$ denotes the Poisson integral:

$$\mathcal{P}\mu_\alpha = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} d\mu_\alpha(\xi).$$

The family $\mathcal{M}_\varphi = \{\mu_\alpha\}_{\alpha \in \mathbb{T}}$ has many interesting properties. In this paper we shall apply these properties in two closely related fields: boundary behavior of Cauchy integrals and spectral properties of rank one perturbations of operators.

Let μ be a finite positive Borel measure on $\mathbb{T}$. We can always find a family $\mathcal{M}_\varphi$ such that $\mu \in \mathcal{M}_\varphi$. After that, we can use the structure of the whole family to study the boundary behavior of the Cauchy integral

$$(\mathcal{K}\mu)(z) = \int_{\mathbb{T}} \frac{1}{1 - \bar{\xi} z} d\mu(\xi),$$

or the conjugate Poisson integral

$$(Q\mu)(z) = 2\operatorname{Im}(\mathcal{K}\mu)(z) = \int_{\mathbb{T}} \frac{2\operatorname{Im}(z\bar{\xi})}{|\xi - z|^2} d\mu(\xi).$$

In Part 1 we will use this approach to give new short proofs to some known facts and obtain new results on the distributions of boundary values of $\mathcal{K}\mu$ and $Q\mu$.

The first result in this area is probably due to G. Boole who discovered, in 1857, the following formula in the case when μ is a finite positive linear combination of point masses:

$$m(\{Q\mu > t\}) = m(\{Q\mu < -t\}) = \frac{1}{\pi} \arctan \frac{\|\mu\|}{t} \quad , t > 0 \tag{0.1}$$

(the functions $\mathcal{K}\mu$ and $Q\mu$ are defined almost everywhere on $\mathbb{T}$ by their nontangential boundary values).

Later on, this result was extended to the case of an arbitrary positive singular measure, see [T1], [T2], [D1]. We discuss the results of this type in Section 1.1.

In Section 1.2 we study the asymptotic behavior of $m(\{|Q\mu| > t\})$ as $t \to \infty$ for arbitrary measures. The classical result of Kolmogorov states that if $\mu \ll m$ (μ is absolutely continuous with respect to m) then

$$m(\{|Q\mu| > t\}) < C||\mu||/t$$

(the exact constant C was obtained by Davis in [D2]) and therefore

$$\mu \ll m \quad \Rightarrow \quad m(\{|Q\mu| > t\}) = o(\frac{1}{t}). \tag{0.2}$$

The case of an arbitrary measure was investigated by Vinogradov, Hruschev [V-H] and Goluzina [G]. In Section 1.2 we prove the Vinogradov-Hruschev result in the following refined form.

We denote by $\mathcal{M}$ the set of all finite complex Borel measures on $\mathbb{T}$.

Theorem 1.2.1. *Let $\mu \in \mathcal{M}$. Then*

(i)
$$\pi t \chi_{\{|\mathcal{K}\mu|>t\}} \cdot m \xrightarrow[t\to\infty]{*-weakly} |\mu^s|.$$

Moreover, if μ is a real measure then

(ii)
$$\pi t \chi_{\{Q\mu>t\}} \cdot m \xrightarrow[t\to\infty]{*-weakly} |\mu^s|$$

and

(iii)
$$\pi t \chi_{\{Q\mu<-t\}} \cdot m \xrightarrow[t\to\infty]{*-weakly} |\mu^s|.$$

Here $\chi_{\{Q\mu>t\}}$ is the characteristic function of the set $\{Q\mu > t\} \subset \mathbb{T}$ and $|\mu^s|$ is the variation of the singular component of μ.

In Section 1.3 we generalize (0.2) in another direction:

Theorem 1.3.2. *Let $\mu, \nu \in \mathcal{M}$. Then*

$$\nu^s \ll \mu^s \Leftrightarrow \lim_{c \to \infty} \lim_{t \to \infty} t \cdot m(\{|\mathcal{K}\nu| > t\} \setminus \{|\mathcal{K}\mu| > t/c\}) = 0.$$

In some problems on rank-one perturbations of operators it is important to understand how the resolvent function $\langle (A-z)^{-1}\phi, \phi \rangle$ of a cyclic self-adjoint operator A depends on the choice of the vector ϕ. In terms of Cauchy integrals this amounts to the problem of comparing the integrals $\mathcal{K}\mu$ and $\mathcal{K}\nu$ of two equivalent measures. Theorem 2 above gives a partial solution to this problem.

Our methods in Part 1 will be based on the following decomposition of the normalized Lebesgue measure m on $\mathbb{T}$ into the integral of the measures μ_α , see [A1]:

$$\int_{\mathbb{T}} \mu_\alpha dm(\alpha) = m. \tag{0.3}$$

(This formula should be understood in the sense that any Lebesgue measurable set $E \subset \mathbb{T}$ is μ_α-measurable for m-a.e. α and $\int_{\mathbb{T}} \mu_\alpha(E) dm(\alpha) = m(E)$.)

In Part 2 we will concentrate on the convergence of Cauchy integrals to their boundary values. Our main result is the following

Theorem 2.3.8. *Let $\mu \in \mathcal{M}$ and $f \in L^1(\mu)$. Denote by F the function*

$$\frac{\mathcal{K}(\mu f)}{\mathcal{K}\mu},$$

which is meromorphic in $\mathbb{D}$. Then for μ-almost all $\xi \in \mathbb{T}$ the limit

$$\lim_{\substack{z \to \xi \\ \angle}} F(z)$$

exists, and it equals $f(\xi)$ almost everywhere with respect to the singular component of the measure μ.

This result will be obtained as a corollary of the following theorem on the boundary behavior of pseudocontinuable functions.

Let θ be an inner function on $\mathbb{D}$ ($|\theta| = 1$ a. e. on $\mathbb{T}$). We will denote by $\theta^*(H^p)$ the invariant subspace of the backward shift operator in the Hardy space H^p corresponding to θ (for the exact definition see Section 2.1).

Theorem 2.2.4. *Let $f \in \theta^*(H^p)$, $p \geq 2$, $\sigma_\alpha \in \mathcal{M}_\theta$. Then for σ_α-almost all $\xi \in \mathbb{T}$*

$$f(z) \underset{\substack{z \to \xi \\ \not<}}{\longrightarrow} (U_\alpha f)(\xi).$$

We will also prove analogous results on the convergence in L^p and some estimates on the L^p norms of maximal functions.

In Section 2.4 we will apply Theorem 2.3.8 to solve some known problems on the multiplication of Cauchy integrals.

As was shown by Clark in [C], if φ is an inner function ($|\varphi| = 1$ a. e. on $\mathbb{T}$) then $\mathcal{M}_\varphi$ is the system of the spectral measures of all unitary one-dimensional perturbations of the model contraction with characteristic function φ. This connection with perturbation theory is even more transparent in the context of self-adjoint operators, see [Ar], [Do], [R-J-L-S], [S-W] and [S]. We discuss this connection in Section 3.1.

In Section 3.2 we discuss the notion of Krein spectral shift and its relations with the objects from Parts 1 and 2. In the rest of Part 3 we use this notion to study spectral properties of rank one perturbations of self-adjoint operators.

In Section 3.3 we give some sufficient conditions of the existence of an absolutely continuous perturbation of a given self-adjoint operator.

In Section 3.4 we give a sufficient condition for two operators to be equivalent modulo rank one perturbation. We supply each result with a number of examples.

In Section 3.5 we provide an example partially answering the question about the existence of a mixed spectrum.

Finally, in Section 3.6 we give a necessary and sufficient conditions for a given operator to have only diagonal rank one perturbations.

Part 1. On the distributions of boundary values of Cauchy integrals.

1.1. Metric properties of conjugate functions.

In addition to (0.3) the family $\mathcal{M}_\varphi$ has the following properties, see [A1]. If μ_α^s and $\mu_\alpha^{a.c.}$ are the singular and absolutely continuous components of μ_α and $\Sigma = \{\xi \in \mathbb{T} \mid |\varphi| = 1 \}$, then

(1.1.1) $$\chi_\Sigma \cdot m = \int_\mathbb{T} \mu_\alpha^s dm(\alpha)$$

and

(1.1.2) $$(1 - \chi_\Sigma) \cdot m = \int_\mathbb{T} \mu_\alpha^{a.c.} dm(\alpha).$$

An analogous result for the real line is contained in [S]. Instead of the measures μ_α Simon integrates the spectral measures of one-dimensional perturbations of a self-adjoint operator.

Remark. Formulas (0.1), (1.1.1) and (1.1.2) can be proved simply by integrating a Poisson kernel over the both parts of the equation, cf. [A1].

Let $w : \mathbb{R} \to \mathbb{T}$ be the mapping

$$w(\lambda) = \frac{\lambda + i}{\lambda - i}.$$

In this paper we will use (1.1.1) in the following form:

Lemma 1.1.1. *Let φ be an analytic function in $\mathbb{D}$, $|\varphi| \le 1$ and $\varphi(0) \in \mathbb{R}$. Let $\{\mu_\alpha\}_{\alpha \in \mathbb{T}} = \mathcal{M}_\varphi$ and $\Lambda = \left\{ \frac{d\mu_1}{dm} > 0 \right\}$. Then for any $t > 0$*

(1.1.3) $$\chi_{(\{Q\mu_1 > t\} \setminus \Lambda)} \cdot m = \int_{w((t;\infty))} \mu_\alpha^s dm(\alpha);$$

in particular,

(1.1.4) $$m(\{Q\mu_1 > t\} \setminus \Lambda) = \int_{w((t;\infty))} \|\mu_\alpha^s\| dm(\alpha).$$

Proof. For each $\alpha \in \mathbb{T}$, the measure μ_α^s is concentrated on the set

$$\Sigma_\alpha = \left\{ \xi \mid \lim_{\substack{z \to \xi \\ \not>}} \varphi = \alpha \right\}.$$

Since the set Σ from (1.1.1) coincides up to a set of Lebesgue measure 0 with the set $\mathbb{T} \setminus \Lambda$ and since

$$Q\mu_1 = \operatorname{Im} \frac{1+\varphi}{1-\varphi},$$

we have that up to a set of Lebesgue measure 0

$$\{Q\mu_1 > t\} \setminus \Lambda = \left\{ \xi \mid \lim_{\substack{z \to \xi \\ \not>}} \varphi \in w((t;\infty)) \right\} = \bigcup_{\alpha \in w((t;\infty))} \Sigma_\alpha.$$

That means that if we multiply both sides of formula (1.1.1) by $\chi_{\{Q\mu_1 > t\} \setminus \Lambda}$, we obtain (1.1.3). Integrating (1.1.3) over $\mathbb{T}$ we obtain (1.1.4). ▲

Remark. If we dropped the condition $\varphi(0) \in \mathbb{R}$ in the statement of the corollary, we would have to replace the set $w((t;\infty))$ in the formulas (1.1.2) and (1.1.3) with the set $w((t+c;\infty))$ where $c = \operatorname{Im} \frac{1+\varphi(0)}{1-\varphi(0)}$. Now we will give short proofs to some metric properties of conjugate functions of positive measures.

We denote by $\mathcal{M}_+$ the subset of $\mathcal{M}$ consisting of all nonnegative measures.

Theorem 1.1.2. *Let* $\mu \in \mathcal{M}_+$, $t > 0$ *and* $\Lambda = \left\{ \frac{d\mu}{dm} > 0 \right\}$. *Then*

(i) $$\frac{1}{\pi} \arctan \frac{\|\mu\|}{t} - m(\Lambda) \le m(\{Q\mu > t\} \setminus \Lambda) \le \frac{1}{\pi} \arctan \frac{\|\mu\|}{t}$$

(ii) $$\frac{1}{\pi} \arctan \frac{\|\mu\|}{t} - m(\Lambda) \le m(\{Q\mu < -t\} \setminus \Lambda) \le \frac{1}{\pi} \arctan \frac{\|\mu\|}{t}.$$

Proof. Without loss of generality we can assume that $\|\mu\| = 1$. Consider an analytic function φ such that $\varphi(0) = 0$ and

$$\mathcal{P}\mu = \operatorname{Re} \frac{1+\varphi}{1-\varphi}.$$

Since

$$\|\mu_\alpha\| = \frac{\alpha + \varphi(0)}{\alpha - \varphi(0)} = 1,$$

we have :

$$\int_{w([t;\infty))} \|\mu_\alpha^s\| dm(\alpha) \le m(w([t;\infty))) = \frac{1}{\pi}\int_t^\infty \frac{dt}{1+t^2} = \frac{1}{\pi}\arctan\frac{1}{t}. \tag{1.1.5}$$

We also have :

$$\begin{aligned}\int_{w([t;\infty))} \|\mu_\alpha^s\| dm(\alpha) &= \int_{w([t;\infty))} 1 - \|\mu_\alpha^{a.c.}\| dm(\alpha) = \\ &= \frac{1}{\pi}\arctan\frac{1}{t} - \int_{w([t;\infty))} \|\mu_\alpha^{a.c.}\| dm(\alpha) \ge \\ &\ge \frac{1}{\pi}\arctan\frac{1}{t} - \int_{\mathbb{T}} \|\mu_\alpha^{a.c.}\| dm(\alpha) = \frac{1}{\pi}\arctan\frac{1}{t} - m(\Lambda)\end{aligned} \tag{1.1.6}$$

because

$$\int_{\mathbb{T}} \|\mu_\alpha^{a.c.}\| dm(\alpha) = m(\Lambda)$$

by (1.1.2). Now, if we combine (1.1.4), (1.1.5) and (1.1.6), we obtain (i). Formula (ii) can be proven in the same way. ▲

Since for any two sets A and B

$$m(A) - m(B) \le m(A \setminus B) \le m(A),$$

Theorem 1.1.2 implies the following result of Tsereteli:

Corollary 1.1.3 ([T1], [T2]). *Let $\mu \in \mathcal{M}_+$, $t > 0$. Then*

(i)
$$\left|\frac{1}{\pi}\arctan\frac{\|\mu\|}{t} - m(\{Q\mu > t\})\right| \le m(\left\{\frac{d\mu}{dm} > 0\right\}).$$

(ii)
$$\left|\frac{1}{\pi}\arctan\frac{\|\mu\|}{t} - m(\{Q\mu < -t\})\right| \le m(\left\{\frac{d\mu}{dm} > 0\right\}).$$

In [T1] it is also shown that using this result one can prove the theorems of Riesz and Zygmund on metric properties of conjugate functions.

Both Theorem 1.1.2 and Corollary 1.1.3 imply Boole's formula (0.1). An analogous proof was obtained in [R-J-L-S]. Davis in [D1] proved (0.1) using the Brownian motion.

1.2. Reconstruction of the singular part of a measure.

Theorem 1.2.1. *Let $\mu \in \mathcal{M}$. Then*

(i) $$\pi t \chi_{\{|\mathcal{K}\mu|>t\}} \cdot m \xrightarrow[t\to\infty]{*-weakly} |\mu^s|.$$

Moreover, if μ is a real measure then

(ii) $$\pi t \chi_{\{Q\mu>t\}} \cdot m \xrightarrow[t\to\infty]{*-weakly} |\mu^s|$$

and

(iii) $$\pi t \chi_{\{Q\mu<-t\}} \cdot m \xrightarrow[t\to\infty]{*-weakly} |\mu^s|.$$

Proof. Let us first prove (ii) in three steps.

1) Let μ be a singular positive measure. Then for the measures from $\mathcal{M}_\varphi$, where $\mu = \mu_1 \in \mathcal{M}_\varphi$ we have

(1.2.1) $$\mu_\alpha \xrightarrow[\alpha\to 1]{*-weakly} \mu_1 = \mu.$$

Indeed, the definition of $\mathcal{M}_\varphi$ implies that

$$(\mathcal{K}\mu_\alpha)(z) \underset{\alpha\to 1}{\longrightarrow} (\mathcal{K}\mu_1)(z)$$

for any $z \in \mathbb{D}$. But linear combinations of Cauchy kernels and their complex conjugates are dense in the space of continuous functions on $\mathbb{T}$.

Now, formula (1.1.2) gives us

$$\pi t \chi_{\{Q\mu>t\}} \cdot m = \pi t \int_{w((t;\infty))} \mu_\alpha dm(\alpha).$$

Since $\pi t \sim \frac{1}{m(w((t;\infty)))}$, the right-hand side of the last equation has the same limit (if any) as

(1.2.2) $$\frac{1}{m(w((t;\infty)))} \int_{w((t;\infty))} \mu_\alpha dm(\alpha).$$

But since

$$w(t) \underset{t\to\infty}{\longrightarrow} w(\infty) = 1,$$

the expression (1.2.2) is just the average over the interval "tending to the point 1." Thus by (1.2.1) we obtain (ii).

2) Let μ be an arbitrary positive measure from $\mathcal{M}$ then the relation

$$(\{Q\mu^s > (1-\epsilon)t\} \cup \{Q\mu^{a.c.} > \epsilon t\}) \supset$$

$$\supset \{Q\mu > t\} \supset$$

$$\supset (\{Q\mu^s > (1-\epsilon)t\} \cap \{Q\mu^{a.c.} > \epsilon t\})$$

together with part 2 and formula (0.2) implies (ii).

(This part of the proof can also be obtained from the result of Vinogradov and Hruschev (see Corollary 1.2.2 below) or from the results of Tsereteli, mentioned in Section 1, via the following argument, suggested by the referee of my paper "On the distributions of boundary values of Cauchy integrals" in Proceedings of the AMS.

To prove (ii) for positive measures, let $P_z(e^{i\theta}) = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$ be the Poisson kernel. Start with the formula

$$\pi t \cdot m(\{Q\mu) > t\}) \underset{t\to\infty}{\longrightarrow} \mu^s(\mathbb{T}).$$

Using linear fractional transformations it follows that

$$\pi t \cdot \int_{\{Q\mu)>t\}} P_z \frac{d\theta}{2\pi} \underset{t\to\infty}{\longrightarrow} \int P_z d\mu^s$$

for all z. The result then follows since linear combinations of the functions P_z are dense in $C(\mathbb{T})$. I am thankful to the referee for this remark.)

3) Now, let $\mu = \mu_+ - \mu_-$ be a real measure, $\mu_\pm \in \mathcal{M}_+, \mu_+ \perp \mu_-$. Our proposition easily follows from the previous part if μ_+ and μ_- are concentrated on closed disjoint subsets of $\mathbb{T}$, i.e., if there are closed subsets F_+ and F_- of $\mathbb{T}$ such that $\|\mu_\pm\| = \mu_\pm(F_\pm), F_+ \cap F_- = \emptyset$.

If now $\mu = \mu_+ - \mu_-$ is an arbitrary real measure, consider disjoint closed subsets F_+ and F_- of $\mathbb{T}$ such that $\mu_\pm(F_\mp) = 0, \mu_\pm(F_\pm) = \|\mu_\pm\| - \epsilon^2$ where ϵ is a small positive constant. Let $\nu_\pm$ be the restriction of $\mu_\pm$ on $F_\pm$. Since

$$\{Q(\mu - \nu) > t\} \subset (\left\{Q(\mu_+ - \nu_+) > \frac{t}{2}\right\} \cup \left\{-Q(\mu_- - \nu_-) > \frac{t}{2}\right\}),$$

by (0.1) we have:

$$m(\{Q(\mu - \nu) > t\}) \leq \frac{4\epsilon^2}{\pi t}$$

(because measures $\mu_+ - \nu_+$ and $\mu_- - \nu_-$ are positive). Thus, from the relation

$$(\{Q\nu > (1-\epsilon)t\} \cup \{Q(\mu - \nu) > \epsilon t\}) \supset \tag{1.2.3}$$

$$\supset \{Q\mu > t\} \supset$$

$$\supset (\{Q\nu > (1-\epsilon)t\} \cap \{Q(\mu - \nu) > \epsilon t\})$$

we obtain:

$$\pi t \chi_{\{Q\mu > t\}} \cdot m = \pi t \chi_{\{Q\nu > t\}} \cdot m + \eta, \tag{1.2.4}$$

where $\eta \in \mathcal{M}, \|\eta\| \leq \frac{4\epsilon}{\pi t}\|\nu\| + \frac{4\epsilon}{\pi t} + o(\frac{1}{t})$. Since

$$\pi t \chi_{\{Q\nu > t\}} \cdot m \xrightarrow[t \to \infty]{*-weakly} |\nu|,$$

we obtain (ii).

4) To prove (iii) one should replace μ in (ii) by $-\mu$.

To prove (i), let us notice that (ii) and (iii) imply (i) for real measures because for any such measure μ we have $m(\{\mathcal{P}\mu > t\}) = o(1/t)$. Let us also notice that if μ is a complex measure then for any $\epsilon > 0$ there exist real mutually singular measures $\mu_1, \mu_2, ..., \mu_n$ and real constants $\alpha_1, \alpha_2, ..., \alpha_n$ such that for $\nu = \sum_{k=1}^n e^{i\alpha_n}\mu_n$ we have $\|\mu - \nu\| < \epsilon$. Since the measures μ_k are real and mutually singular , we can prove (i) for ν using the same argument as in part 3 (we just have to deal with $\mu_1, \mu_2, ..., \mu_n$ instead of μ_+ and μ_-). Applying (ii) and (iii) to the real and imaginary parts of $\mu - \nu$ we obtain : $\pi t m(\{|\mathcal{K}(\mu - \nu)| > t\}) < 2\epsilon$. Now we can finish the proof using the estimates similar to the ones we used for μ and $\mu - \nu$ from part 3. ▲

Corollary 1.2.2 ([V-H]). *Let $\mu \in \mathcal{M}$. Then*

$$\pi t \cdot m(\{|\mathcal{K}\mu| > t\}) \underset{t\to\infty}{\longrightarrow} \|\mu^s\|.$$

Remark. One can prove the following localized versions of (i), (ii) and (iii), cf. [G1]. Let $0 \le \theta_1 < \theta_2 < 2\pi$, $a = e^{i\theta_1}$, $b = e^{i\theta_2}$, and let $I \subset \mathbb{T}$ be an open arc with the ends a and b : $I = \{e^{i\theta} \ : \ \theta_1 < \theta < \theta_2\}$. Let $\mu \in \mathcal{M}$. Then

(i′)
$$\pi t \chi_{(\{|\mathcal{K}\mu|>t\}\cap I)} \cdot m \xrightarrow[t\to\infty]{*-weakly} \chi_{(I\cup\{a,b\})} \cdot |\mu^s|.$$

If $\mu \in \mathcal{M}_+$ then

(ii′)
$$\pi t \chi_{(\{Q\mu>t\}\cap I)} \cdot m \xrightarrow[t\to\infty]{*-weakly} \chi_{(I\cup\{a\})} \cdot |\mu^s|$$

and

(iii′)
$$\pi t \chi_{(\{Q\mu<-t\}\cap I)} \cdot m \xrightarrow[t\to\infty]{*-weakly} \chi_{(I\cup\{b\})} \cdot |\mu^s|.$$

1.3. Relations between Cauchy integrals.

Let us consider the following corollary of Theorem 1.2.1.

Corollary 1.3.1 ([T2]). *Let $\mu \in \mathcal{M}$. If*

$$\lim_{t\to\infty} t \cdot \min(m(\{Q\mu > t\}), m(\{-Q\mu > t\})) = 0$$

then $\mu \ll m$.

This shows that the inverse of the statement (0.2) is also true. The situation is different when we replace m with an arbitrary measure from $\mathcal{M}_+$ (see the remark after the proof of Theorem 1.3.2 below). However, we still can obtain some necessary and sufficient condition for one measure to be absolutely continuous with respect to another measure.

Theorem 1.3.2. *Let $\mu, \nu \in \mathcal{M}$. Then*

$$\nu^s \ll \mu^s \Leftrightarrow \lim_{c\to\infty} \lim_{t\to\infty} t \cdot m(\{|\mathcal{K}\nu| > t\} \setminus \{|\mathcal{K}\mu| > t/c\}) = 0.$$

Proof. Put $\eta(c) = |\nu^s| - |c\mu^s|$. Let $\eta_+(c)$ and $\eta_-(c)$ be the positive and negative parts of $\eta(c)$ $(\eta(c) = \eta_+(c) - \eta_-(c),\ \eta_\pm(c) \in \mathcal{M}_+,\ \eta_+(c) \perp \eta_-(c))$.

By Theorem 1.2.1

$$\pi t \chi_{\{|\mathcal{K}\nu|>t\}} \cdot m - \pi t \chi_{\{|\mathcal{K}\mu|>\frac{t}{c}\}} \cdot m \xrightarrow[t\to\infty]{*-weakly} \eta(c).$$

But

$$\pi t \chi_{\{|\mathcal{K}\nu|>t\}} \cdot m - \pi t \chi_{\{|\mathcal{K}\mu|>\frac{t}{c}\}} \cdot m =$$

$$= \pi t \chi_{\{|\mathcal{K}\nu|>t\}\setminus\{|\mathcal{K}\mu|>\frac{t}{c}\}} \cdot m - \pi t \chi_{\{|\mathcal{K}\mu|>\frac{t}{c}\}\setminus\{|\mathcal{K}\nu|>t\}} \cdot m.$$

Thus

$$\pi t \chi_{\{|\mathcal{K}\nu|>t\}\setminus\{|\mathcal{K}\mu|>\frac{t}{c}\}} \cdot m \xrightarrow[t\to\infty]{*-weakly} \eta_+(c)$$

and, in particular,

$$\pi t m(\{|\mathcal{K}\nu| > t\} \setminus \{|\mathcal{K}\mu| > \frac{t}{c}\}) \underset{t\to\infty}{\longrightarrow} ||\eta_+(c)||.$$

It is left to notice that $||\eta_+(c)|| \to_{c\to\infty} 0$ iff $\nu^s \ll \mu^s$. ▲

Remark. The proof shows that if measures μ and ν were real then we could replace the sets $\{|\mathcal{K}\nu| > t\}$ and $\left\{|\mathcal{K}\mu| > \frac{t}{c}\right\}$ from the statement of the theorem with $\{Q\nu > t\}$ and $\left\{Q\mu > \frac{t}{c}\right\}$ or with $\{Q\nu < -t\}$ and $\left\{Q\mu < -\frac{t}{c}\right\}$ respectively.

Remark. The fact that $m(\{|\mathcal{K}\nu| > t\}) = o(\frac{1}{t})$ as $t \to \infty$ for any $\nu \ll m$ can also be generalized in the following way:

Let $\mu, \nu \in \mathcal{M}, \mu > 0$. Then

$$\nu^s \ll \mu^s \quad \Rightarrow \quad m(\left\{\left|\frac{\mathcal{K}\nu}{\mathcal{K}\mu}\right| > t\right\}) = o(\frac{1}{t}) \quad as \quad t \to \infty. \tag{1.3.1}$$

Proof. We can suppose that $||\mu|| = 1$. Theorem 1.3.2 suggests that for any $\epsilon > 0$ there are positive c and T such that for any $t > T$

$$m(\{|\mathcal{K}\nu| > t\} \setminus \left\{|\mathcal{K}\mu| > \frac{t}{c}\right\}) < \frac{\epsilon}{t}.$$

Thus, for $n = 1, 2, 3, ...$

$$m(\{2^{n+1}T > |\mathcal{K}\nu| > 2^n T\} \setminus \left\{|\mathcal{K}\mu| > \frac{2^n T}{c}\right\}) < \frac{\epsilon}{2^n T}.$$

Since $\mu \geq 0$ and $||\mu|| = 1$, $|\mathcal{K}\mu| > 1/2$ on $\mathbb{D}$. Thus,

$$\left\{\left|\frac{\mathcal{K}\nu}{\mathcal{K}\mu}\right| > t\right\} \subset \{|\mathcal{K}\nu| > t/2\}$$

for any $t > 0$. Hence, if n is big enough (such that $2^n T > c$),

$$\left\{\left|\frac{\mathcal{K}\nu}{\mathcal{K}\mu}\right| > 2^{n+1}T\right\} \subset \bigcup_{k=n}^{\infty}(\{2^{k+1}T > |\mathcal{K}\nu| > 2^k T\} \setminus \left\{|\mathcal{K}\mu| > \frac{2^k T}{c}\right\}).$$

Hence

$$m(\left\{\left|\frac{\mathcal{K}\nu}{\mathcal{K}\mu}\right| > 2^{n+1}T\right\}) \leq \sum_{k=n}^{\infty} \frac{\epsilon}{2^k T} \leq \frac{\epsilon}{2^{n-1}T}$$

and because ϵ is arbitrary we obtain (1.3.1). ▲

The inverse of (1.3.1) is false. Even the weaker statement that $m(\left\{|\frac{\mathcal{K}\nu}{\mathcal{K}\mu}| > t\right\}) = o(\frac{1}{t})$ implies $\operatorname{supp}\nu^s \subset \operatorname{supp}\mu^s$ is false, as is evident if

$$\mu = \frac{1}{|\xi - 1|^{\frac{1}{2}}} m, \quad \nu = \delta_1$$

where $delta_x$ denotes a point mass at x, $||\delta_x|| = 1$.

We can, however, prove the following:

Let $f \in L^1(\mathbb{T})$. Let

$$\Sigma_f = \bigcap_{t>0} \operatorname{Clos}\{|\mathcal{K}f| > t\}.$$

Let $\mu, \nu \in \mathcal{M}, \mu = fm + \mu^s$. If

$$m(\left\{\left|\frac{\mathcal{K}\nu}{\mathcal{K}\mu}\right| > t\right\}) = o(\frac{1}{t}) \quad as \quad t \to \infty$$

then

$$\operatorname{supp} \nu^s \subset (\Sigma_f \cup \operatorname{supp} \mu^s).$$

Proof. Suppose $m(\left\{|\frac{\mathcal{K}\nu}{\mathcal{K}\mu}| > t\right\}) = o(\frac{1}{t})$ but $|\nu^s|(F_r) < \|\nu^s\|$ for some r where $F_r = \operatorname{Clos}\{|\mathcal{K}f| > r\} \cup \operatorname{supp} \mu^s$. Then there exists an open set E such that $|\nu^s|(E) = \delta > 0$ and $|\mu| < T$ on E for some $T > 0$. But Theorem 1 implies that

$$m(E \cap \{|\mathcal{K}\nu| > t\}) \sim \frac{\delta}{t} \quad as \quad t \to \infty;$$

thus, $m(\left\{|\frac{\mathcal{K}\nu}{\mathcal{K}\mu}| > t\right\}) \geq \frac{\delta}{T \cdot t} + o(\frac{1}{t}) \neq o(\frac{1}{t})$. ▲

Part 2. On the convergence of Cauchy integrals.

2.1. Spaces of pseudocontinuable functions.

Let H^p $(0 < p < \infty)$ denote the Hardy class on the unit disk $\mathbb{D}$ i.e. H^p is the space of all functions f holomorphic in $\mathbb{D}$ for which

$$\|f\|_p = \sup_{o<r<1} \left(\int_{\mathbb{T}} |f(r\xi)|^p \, dm(\xi) \right)^{1/p} < +\infty,.$$

For $p = +\infty$ the Hardy class H^p consists of the functions f that are analytic and bounded in $\mathbb{D}$:

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < +\infty.$$

Since any H^p-function has boundary values almost everywhere on $\mathbb{T}$, one can treat H^p as a closed subspace of $L^p(m)$ (for more details see [K] or [Ga]). A function θ, $\theta \in H^\infty$, is called *inner* if $|\theta(\xi)| = 1$ for m-almost every ξ in $\mathbb{T}$. We set

$$H^p_0 = \{\, f \in H^p \mid f(0) = 0 \,\},$$
$$H^p_- = \{\, f \in L^p \mid \overline{f} \in H^p_0 \,\}.$$

With each inner function θ we associate the subspaces $\theta^*(H^p)$, $p \geq 1$:

$$\theta^*(H^p) = \{\, f \in H^p \mid f\overline{\theta} \in H^p_- \,\}.$$

Each function lying in the space $\theta^*(H^p)$ is known to possess the so-called "pseudocontinuation" (see, e.g., [N]). Furthermore, the spaces $\theta^*(H^p)$ are precisely the closed invariant subspaces of the backward shift operator $S^*\colon H^p \longrightarrow H^p$, $(S^*f)(z) = (f(z) - f(0))/z$.

Clark [C] investigated the connections between the measures $\sigma_\alpha \in \mathcal{M}_\theta$ and unitary rank one perturbations of the model contraction. In [C] it was shown that σ_α's are the spectral measures for these operators.

The following theorem was also proved.

Theorem 2.1.1 ([C]). *Consider the operator U_α originally defined on the linear hull of the functions k_λ with $|\lambda| < 1$ and mapping this linear hull into the space $L^0(\sigma_\alpha)$ of all σ_α-measurable functions* (mod 0) *according to the rule*

$$(U_\alpha k_\lambda)(z) = (1 - \overline{\theta(\lambda)}\alpha)/(1 - \overline{\lambda}z).$$

After an appropriate extension, U_α becomes a unitary operator mapping $\theta^*(H^2)$ onto $L^2(\sigma_\alpha)$.

Let λ_a be the conformal automorphism of the unit disk that sends a to 0. Now the operator given by $f \mapsto f \cdot \frac{\sqrt{1-|a|^2}}{1-\overline{a}\theta}$ takes $\theta^*(H^p)$ onto $(\lambda_a \circ \theta)^*(H^p)$; in the case $p = 2$ this operator is an isometry. Furthermore, letting $\{\sigma'_\alpha\}_{\alpha\in\mathbb{T}}$ be the family of measures corresponding to the inner function $\lambda_a \circ \theta$, we have $\sigma'_\beta = \frac{\sigma_\alpha}{|\lambda'_a(\alpha)|}$ for $\beta = \lambda_a(\alpha)$. This observation will enable us to confine ourselves to the case where $\theta(0) = 0$. Under this assumption, all the σ_α's are probability measures, and the formulas become much simpler. In particular, the operator $U^*_\alpha \colon L^2(\sigma_\alpha) \longrightarrow \theta^*(H^2)$, which is the adjoint of the unitary operator U_α mapping $\theta^*(H^2)$ onto $L^2(\sigma_\alpha)$, is now given by

$$U^*_\alpha f = \frac{\mathcal{K}(f\sigma_\alpha)}{\mathcal{K}(\sigma_\alpha)} = (1 - \overline{\alpha}\theta)\mathcal{K}(f\sigma_\alpha).$$

In [A2], A. B. Aleksandrov studied the behavior of the map $U_\alpha \colon \theta^*(H^p) \longrightarrow L^p(\sigma_\alpha)$ for $p \neq 2$. His results were obtained as corollaries to the following general theorem.

Theorem 2.1.2 ([A2]). *Let $\mu \in M_+(\mathbb{T})$. With this μ we associate the map*

$$V_\mu f = \mathcal{K}(f_\mu)/\mathcal{K}\mu \qquad (f \in L^1(\mu)).$$

The operator V_μ is of weak type $(1,1)$, *and it is a continuous map from $L^p(\mu)$ to $L^p(m)$* $(1 < p \leq 2)$. *Moreover, the norm of V_μ can be estimated in terms of p only.*

Further properties of the map V_μ will be considered in Section 2.3. Since for $1 < p \leq 2$ we have $U_\alpha^{-1} = V_{\sigma_\alpha}$ (recall that we assume $\theta(0) = 0$), Theorem 3 implies the following statements.

Corollary 2.1.3 ([A2]). *Let $1 < p \le 2$. Then*

$$U_\alpha^{-1}(L^p(\sigma_\alpha)) \subset \theta^*(H^p).$$

Corollary 2.1.4 ([A2]). *Let $2 < p \le +\infty$. Then*

$$U_\alpha(\theta^*(H^p)) \subset L^p(\sigma_\alpha).$$

In [A2], it is also shown that though $U_\alpha(\theta^*(H^p))$ and $L^p(\sigma_\alpha)$ coincide for $p = 2$ by Clark's Theorem, this is no longer valid for $p \neq 2$ except for some degenerated cases.

Theorem 2.1.5 ([A2]). *Let μ be a singular measure in $M_+(\mathbb{T})$, $p > 2$. Suppose that $V_\mu(C(\mathbb{T})) \subset L^p$. Then μ is discrete.*

Corollary 2.1.6 ([A2]). *Let θ be an inner function and let $\alpha \in \mathbb{T}$. Suppose that $U_\alpha^{-1}(L^p(\sigma_\alpha)) = \theta^*(H^p)$ for some $p \in (1,2)$ or $U_\alpha(\theta^*(H^p)) = L^p(\sigma_\alpha)$ for some $p \in (2, +\infty]$. Then σ_α is a discrete measure.*

Now the following question arises (see [Sa]): how does the operator U_α act on functions that do not belong to the linear hull of the family $\{k_\lambda\}_{\lambda\in\mathbb{D}}$? In Clark's paper [C] it was shown that if a function f in $\theta^*(H^2)$ extends analytically across an arc of the circle $\mathbb{T}$, then we have $f = U_\alpha f$ almost everywhere with respect to σ_α on that arc. In [A2] it was mentioned that the relation $f = U_\alpha f$ also holds a. e. with respect to σ_α in the case where $f \in C(\mathbb{T}) \cap \theta^*(H^2)$. In Section 2.2 we obtain some results showing that the relation $f = U_\alpha f$ remains true for the rest of the functions in $\theta^*(H^p)$; moreover, it can be understood "literally."

In [Sa], D. Sarason proved that, given a probability measure μ in $M_+(\mathbb{T})$, the map V_μ is an isometry of the space $H^2(\mu)$ (the latter is the closure in $L^2(\mu)$ of the set of all polynomials of z) onto the so-called de Branges space $\mathfrak{H}(f)$ (the notation is the same as in [Sa]), where $f = 1 - 1/\mathcal{K}\mu$. So, in a sense, the map V_μ is a generalization of the operator

U_α^*. In Section 2.3 we extend the results of Section 2.2 to the case of V_μ, where μ is an arbitrary measure from $M(\mathbb{T})$.

Finally, in §3 we apply the results obtained in the previous sections to some old problems concerning division and multiplication for Cauchy integrals.

2.2. Convergence of functions from $\theta^*(H^p)$ to their boundary values

We start with some propositions showing that for $p \geq 2$ the image $U_\alpha f$ of a function $f \in \theta^*(H^p)$ under Clark's operator U_α coincides σ_α-a. e. with the boundary values (introduced in a certain way) of f on $\mathbb{T}$.

Theorem 2.2.1. *Suppose that $2 \leq p < +\infty$, $f \in \theta^*(H^p)$, and let f have the expansion $f(z) = \sum_{n\geq 0} a_n z^n$ for z in $\mathbb{D}$. The following statements hold.*

1) *The partial sums of the series $\sum a_n z^n$ are bounded in the $L^p(\sigma_\alpha)$-norm by $C\|f\|_{H^p}$, where the constant C depends only on p.*

2) *The series $\sum a_n z^n$ converges in $L^p(\sigma_\alpha)$ to the function $U_\alpha f$.*

Proof. We note that the k-th partial sum of the series $\sum a_n z^n$ coincides σ_α-a.e. with $U_\alpha f - z^{k+1} U_\alpha S^{*k} f$. This fact is obvious if $f \in C(\mathbb{T})$, because the map U_α takes continuous functions to themselves. But if $f \notin C(\mathbb{T})$, one can choose a sequence $\{f_n\}_{n\geq 0}$, $f_n \in C(\mathbb{T}) \cap \theta^*(H^p)$, for which $f_n \to f$ in H^p and apply a limit argument. By Theorem 2.1.2 and Corollary 2.1.4, there is a constant $C_0 = C_0(p)$ such that

$$\|z^{k+1} U_\alpha S^{*k} f\|_{L^p(\sigma_\alpha)} = \|U_\alpha S^{*k} f\|_{L^p(\sigma_\alpha)} \leq C_0 \|S^{*k} f\|_{H^p} \to 0. \blacktriangle$$

Corollary 2.2.2. *Let $f \in \theta^*(H^p)$, $p \geq 2$, $f_r(z) = f(rz)$, $0 < r < 1$. Then*

1) *$f_r \to U_\alpha f$ in $L^p(\sigma_\alpha)$ as $r \to 1^-$.*

2) *$\|f_r\|_{L^p(\sigma_\alpha)} \leq C\|f\|_{H^p}$, where $C = C(p)$.*

Proof. It suffices to apply the Abel–Poisson summation method to the power series of f and then to use Theorem 2.2.1. ▲

Remark. In [A2] it was shown that the inclusion map of $\theta^*(H^p)$ into $\theta^*(H^q)$ is compact if $p > q \geq 1$. It can be seen from the proof of Theorem 2.2.1 that this statement yields an estimate, in terms of p and θ only, for the rate of convergence of the series $\sum a_n z^n = f(z)$ ($f \in \theta^*(H^p)$) and of the functions f_r in $L^p(\sigma_\alpha)$.

Remark. The restriction $p \geq 2$ cannot be dropped in the statements of the theorem and its corollary. This can be seen from the following example.

Let us take an inner function θ with the following properties:

1) $\frac{\theta(z)-\theta(1)}{z-1} \in H^2$ for,

2) $\frac{\theta(z)-\theta(1)}{z-1} \notin H^p$ for every $p > 2$.

Such a function can be constructed, e.g., with the help of Theorem 3.3 in [A2]. In the same paper it was pointed out that 1) implies $\sigma_\alpha(\{1\}) \neq 0$ for $\alpha = \theta(1)$, and 2) implies that the point evaluation at 1 is a discontinuous functional on $\theta^*(H^p)$ for $p < 2$. Therefore, for any $p < 2$ there is a function $f \in \theta^*(H^p)$ for which the limit

$$\lim_{r \to 1^-} f(r)$$

does not exist. So the functions f_r also do not converge as $r \to 1^-$. Nor does the power series of f converge in $L^p(\sigma_\alpha)$.

The above example also shows that for $p < 2$ the functions f_r do not necessarily converge σ_α–a. e. However, soon we will see that they do converge for $p \geq 2$. In order to treat the problem of convergence almost everywhere, we have to do some preparatory work.

We say that z tends to ξ, $\xi \in \mathbb{T}$, nontangentially (and write $z \underset{\not<}{\to} \xi$) if z tends to ξ from inside the region Δ_ξ^φ,

$$\Delta_\xi^\varphi = \mathbb{D} \cap \left\{ z \mid |\arg(1 - z\bar{\xi})| < \varphi \right\}, \quad \text{where } \varphi \in \left(0, \frac{\pi}{2}\right)$$

(from now on we denote by arg the principal branch of the argument, so its values lie in $(-\pi, \pi]$).

Let $\mu \in \mathcal{M}(\mathbb{T})$, $f \in L^1(\mu)$. A point $\xi \in \mathbb{T}$ will be called a Lebesgue point of f with respect to the measure μ if

$$\frac{1}{2h}\int_{I(\xi,h)} |f(\xi) - f(z)|\, d\mu(z) \to 0 \qquad \text{as } h \to 0,$$

where $I(\xi, h)$ is the subarc of $\mathbb{T}$ centered at ξ whose length is $2h$.

Lemma 2.2.3. *Let $\mu \in \mathcal{M}(\mathbb{T})$, $f \in L^1(\mu)$. Then for $|\mu|$-almost every $\xi \in \mathbb{T}$*

$$P(f\mu)/P\mu \xrightarrow[z \underset{\not\angle}{\to} \xi]{} f(\xi).$$

Proof. First we consider the case $\mu \geq 0$. Let ξ be a Lebesgue point of f with respect to the measure μ. We may assume $f \geq 0$ and $f(\xi) = 0$. Standard arguments involving the definition of a Lebesgue point and the properties of the Poisson kernel show that in this case

$$\big(P(f\mu)/P\mu\big)(r\xi) \xrightarrow[r \to 1^-]{} 0.$$

The passage from a radius to a sector will require some estimates similar to those that occur in Harnack's lemma.

Now let μ be an arbitrary measure from $\mathcal{M}(\mathbb{T})$, and let g be a μ-measurable function such that $\mu = g|\mu|$. The statement of the lemma now follows from the relation

$$P(f\mu)/P\mu = \frac{P(fg|\mu|)}{P(|\mu|)} \cdot \frac{P|\mu|}{P(g|\mu|)},$$

since $|g| = 1 \neq 0$ $[\mu]$–a. e. ▲

Lemma 2.2.4. *Let μ and ν be measures in $\mathcal{M}(\mathbb{T})$, $\mu \perp \nu$. Then for ν-almost every $\xi \in \mathbb{T}$ we have*

$$(P\mu/P\nu)(z) \xrightarrow[z \underset{\not\angle}{\to} \xi]{} 0.$$

Proof. We consider the case where $\nu \geq 0$ and $\mu \geq 0$. Let f be a function defined on $\mathbb{T}$ such that $f = 0$ μ-a. e. and $f = 1$ ν-a. e. Now by Lemma 1.2 we have for ν-almost every

ξ

$$\lim_{z \underset{\not<}{\to} \xi} \frac{P(f(\mu+\nu))}{P(\mu+\nu)} = \lim_{z \underset{\not<}{\to} \xi} \frac{P\nu}{P(\mu+\nu)} = 1.$$

Consequently,

$$\lim_{z \underset{\not<}{\to} \xi} \frac{P\mu}{P\nu} = \lim_{z \underset{\not<}{\to} \xi} \frac{P(\mu+\nu)}{P\nu} - 1 = 0$$

for ν-almost every ξ.

The passage to the case of arbitrary μ and ν is similar to that in the previous proof. ▲

Consider the space $\theta^*(H_0^p) = \{ f \in \theta^*(H^p) \mid f(0) = 0 \}$. For $f \in \theta^*(H^p)$ we denote the function $\theta\overline{f}$ by $\widetilde{f}$. The map $f \mapsto \widetilde{f}$ is an involution on the space $\theta^*(H_0^p)$. In what follows a function f in $\theta^*(H^p)$ will be called a Hermitian element if $\widetilde{f} = f$.

Some of the proofs of the theorems below rely on the following nice property of Hermitian elements: if $f \in \theta^*(H^p)$, $p \geq 2$, and $f = \widetilde{f}$, then $\arg U_\alpha f = \frac{\arg \alpha}{2}$ σ_α-a. e. In particular, the function $U_1 f$ is real σ_1-a. e., and $U_{-1} f$ is purely imaginary σ_{-1}-a. e.

Theorem 2.2.5. *Let $f \in \theta^*(H^p)$, $p \geq 2$, $\alpha \in \mathbb{T}$. Then for σ_α-almost every $\xi \in \mathbb{T}$*

$$f(z) \underset{z \underset{\not<}{\to} \xi}{\longrightarrow} (U_\alpha f)(\xi).$$

Proof. Assuming that $f(0) = 0$, we represent f as the sum

$$f = \frac{f + \widetilde{f}}{2} + \frac{f - \widetilde{f}}{2}.$$

Since both $f + \widetilde{f}$ and $i(f - \widetilde{f})$ are Hermitian elements, it suffices to prove the theorem in the case of Hermitian f.

In what follows we write f instead of $U_\alpha f$ if this leads to no confusion.

Now let f be a Hermitian element, $\alpha = 1$, and let ξ be such that

(A) $P(f\sigma_1)/P\sigma_1 \to f(\xi)$ as $z \underset{\not<}{\to} \xi$,

(B) $(P(f^2\sigma_{-1})/P\sigma_1)(z) \to 0$ as $z \underset{\not<}{\to} \xi$,

(C) $(1 - \theta(z)) \to 0$ as $z \underset{\not<}{\to} \xi$.

(Lemmas 1.2, 1.3 and the definition of σ_1 show that σ_1-almost every ξ's enjoy the above properties.)

We note that

$$\begin{aligned} P(f\sigma_1) &= 2\operatorname{Re}\mathcal{K}(f\sigma_1) = 2\operatorname{Re}\Big(\frac{1+\theta}{1-\theta}\mathcal{K}(f\sigma_{-1})\Big) \\ &= 2\Big(\operatorname{Re}\Big(\frac{1+\theta}{1-\theta}\Big)\operatorname{Re}\big(\mathcal{K}(f\sigma_{-1})\big) - \operatorname{Im}\Big(\frac{1+\theta}{1-\theta}\Big)\operatorname{Im}\big(\mathcal{K}(f\sigma_{-1})\big)\Big) \\ &= i\big(P\sigma_1 Q(f\sigma_{-1}) - Q\sigma_1 P(f\sigma_{-1})\big), \end{aligned}$$

because f is a Hermitian element. Therefore, using condition (A) we get

$$i\Big(Q(f\sigma_{-1}) - \frac{Q\sigma_1\cdot P(f\sigma_{-1})}{P\sigma_1}\Big) \underset{z \underset{\not\,}{\to} \xi}{\longrightarrow} f(\xi). \tag{2.2.1}$$

Next, by Hölder's inequality we have

$$|P(f\sigma_{-1})| \le \sqrt{P(\sigma_{-1})|P(f^2\sigma_{-1})|}.$$

In view of condition (B) imposed on ξ, it follows that

$$|P(f\sigma_{-1})| = o\Big(\sqrt{P\sigma_{-1}P\sigma_1}\Big) = o\Big(\frac{1-|\theta|^2}{|1-\theta^2|}\Big). \tag{2.2.2}$$

Now we are able to estimate the second summand on the left-hand side of (2.2.1):

$$\frac{Q\sigma_1\cdot P(f\sigma_{-1})}{P\sigma_1} = o\Big(\operatorname{Im}\Big(\frac{1+\theta}{1-\theta}\Big)\cdot\frac{|1-\theta|^2}{|1-\theta^2|}\Big) = o\Big(\frac{\operatorname{Im}(1-\theta)}{|1-\theta^2|}\Big).$$

From condition (C), imposed on the point ξ at the beginning of the proof, we derive $\operatorname{Im}(1-\theta)/|1-\theta^2| = O(1)$ as $\underset{\not\,}{\to}\xi$, and so (2.2.1) yields

$$iQ(f\sigma_{-1})(z) \underset{z \underset{\not\,}{\to} \xi}{\longrightarrow} f(\xi).$$

Furthermore, (2.2.2) implies $P(f\sigma_{-1}) = o(1)$. It remains to notice that $\int f\,d\sigma_{-1} = f(0) = 0$, and so

$$\begin{aligned} f(z) &= (1+\theta(z))\cdot\mathcal{K}(f\sigma_{-1})(z) \\ &= (1+\theta(z))\Big(\frac{1}{2}P(f\sigma_{-1}) + \frac{i}{2}Q(f\sigma_{-1})\Big)(z) \underset{z \underset{\not\,}{\to} \xi}{\longrightarrow} f(\xi). \blacktriangle \end{aligned}$$

The rest of this section is devoted to the properties of maximal functions corresponding to elements of the space $\theta^*(H^p)$ with $p \geq 2$.

Let $f \in L^1(\mu)$, $\mu \in \mathcal{M}(\mathbb{T})$. By f_μ^M we denote the Hardy–Littlewood maximal function:

$$f_\mu^M(\xi) = \sup_{\varepsilon>0} \frac{1}{\mu(I(\xi,\varepsilon))} \int_{I(\xi,\varepsilon)} |f|\, d\mu,$$

where $\xi \in \mathbb{T}$ and $I(\xi,\varepsilon)$ is the subarc of $\mathbb{T}$ of length 2ε and centered at ξ.

In what follows we make use of the so-called Luzin–Privalov "Ice-cream cone construction" (cf. [K]). Let $\xi \in \mathbb{T}$ and $\varphi \in (0,\pi/2)$. Into the sector Δ_ξ^φ we inscribe a circle $\mathbb{T}_\varphi$ centered at the origin. The two points at which $\mathbb{T}_\varphi$ is tangent to the rays $\{\, z \mid \arg(1 - z\xi) = \pm\varphi \,\}$ divide the circle $\mathbb{T}_\varphi$ into two arcs. We denote by Γ_ξ^φ the open region bounded by the shorter arc and the rays $\{\, z \mid \arg(1 - z\overline{\xi}) = \pm\varphi \,\}$. In other words, of the three parts into which $\mathbb{T}_\varphi$ divides Δ_ξ^φ, Γ_ξ^φ is the one adjacent to the point ξ.

Let $E \subset \mathbb{T}$. By Γ_E^φ we denote the set

$$\bigcup_{\xi \in E} \Gamma_\xi^\varphi.$$

For a function $f \in H^p$ we introduce its nontangential maximal function. Letting $\varphi \in (0,\pi/2)$ we set

$$f_\varphi^*(\xi) = \sup_{z \in \Gamma_\xi^\varphi} |f(z)|.$$

Before proceeding with the "maximal Theorem" 2.2.7, we establish an auxiliary statement.

Lemma 2.2.6. *Let $\mu, \nu \in \mathcal{M}_+(\mathbb{T})$, $f \in L^1(\mu+\nu)$, $\varphi \in (0,\pi/2)$. Then there is a positive constant C such that for ν-almost every $\xi \in \mathbb{T}$ and for all z in Γ_ξ^φ we have*

1) $|P(f\nu)| \leq C f_\nu^M P\nu$,

2) $|P(f\mu)| \leq C f_{\mu+\nu}^M P(\mu+\nu)$.

Proof. We may assume that $f \geq 0$. A standard argument shows that inequality 1) holds for all points z on the radius terminating at a Lebesgue point of f with respect to ν. In order to extend inequality 1) from the radius to the sector Γ_ξ^φ, one should reproduce the argument of the proof of Harnack's Lemma.

To deduce 2) from 1), one should substitute $\mu+\nu$ for ν in 1) and recall that both f and ν are positive. ▲

Now everything is ready to prove the maximal Theorem.

Theorem 2.2.7. *Assume that $\alpha \in \mathbb{T}$, $\varphi \in (0, \pi/2)$, $f \in \theta^*(H^p)$, $p \geq 2$. If $p > 2$, then $f_\varphi^* \in L^p(\sigma_\alpha)$. If $p = 2$, then f_φ^* belongs to the "weak L^2", i.e.,*

$$\sigma_\alpha(\{\, \xi \in \mathbb{T} \mid f_\varphi^*(\xi) > \lambda \,\}) < \frac{c}{\lambda^2},$$

where c is a constant.

Proof. The crucial estimates will be similar to the estimates in the proof of the pointwise convergence.

Since $(h+g)_\varphi^* \leq h_\varphi^* + g_\varphi^*$, it follows that we can once again confine ourselves to the case where f is a Hermitian element.

Now, by Lemma 2.2.6

$$\left|P(f\sigma_1)/P\sigma_1\right| \leq C f_{\sigma_1}^M.$$

From this we derive (like we did in the proof of Theorem 2.2.5) that

$$\left|Q(f\sigma_{-1}) - Q\sigma_1 P(f\sigma_{-1})/P\sigma_1\right| \leq C f_{\sigma_1}^M. \tag{2.2.3}$$

Hölder's inequality gives

$$|P(f\sigma_{-1})| \leq \sqrt{P(\sigma_{-1})|P(f^2\sigma_{-1})|}.$$

Furthermore, by Lemma 2.2.6

$$|P(f^2\sigma_{-1})| \leq C(f^2)_{\sigma_1+\sigma_{-1}}^M \cdot P(\sigma_1 + \sigma_{-1})$$

whence

$$|P(f\sigma_{-1})| \le \sqrt{C(f^2)^M_{\sigma_1+\sigma_{-1}} \cdot P(\sigma_1+\sigma_{-1}) \cdot P(\sigma_{-1})} \le C_1 \frac{1-|\theta|^2}{|1-\theta^2||1+\theta|}\sqrt{(f^2)^M_{\sigma_1+\sigma_{-1}}}. \tag{2.2.4}$$

Next, we use (2.2.4) to estimate the second summand on the left-hand side of (2.2.3):

(2.2.5)

$$\left|Q\sigma_1 \cdot P(f\sigma_{-1})/P\sigma_1\right| \le C_1\left|\operatorname{Im}\left(\frac{1+\theta}{1-\theta}\right)\right| \cdot \frac{|1-\theta|^2}{|1-\theta^2||1+\theta|}\sqrt{(f^2)^M_{\sigma_1+\sigma_{-1}}} \le \frac{C_2}{|1+\theta|}\cdot\sqrt{(f^2)^M_{\sigma_1+\sigma_{-1}}}.$$

Since f is a Hermitian element, we have $f(0) = \int f\, d\sigma_{-1} = 0$, and so (2.2.3), (2.2.4), and (2.2.5) together yield

$$\left|f(z)\right| = \frac{1}{2}\left|(1+\theta(z))(P(f\sigma_{-1}) + iQ(f\sigma_{-1}))(z)\right| \le Cf^M_{\sigma_1} + \frac{C_1+C_2}{2}\sqrt{(f^2)^M_{\sigma_1+\sigma_{-1}}}.$$

The fact that f belongs to both $L^p(\sigma_1)$ and $L^p(\sigma_{-1})$, combined with the well-known Hardy–Littlewood maximal Theorem, implies that in the case $p > 2$ the two summands in the last expression belong to $L^p(\sigma_1)$, whereas in the case $p = 2$ the first summand is in $L^2(\sigma_1)$ and the second one is in the "weak $L^2(\sigma_1)$." ▲

Remark. From the proof one sees that the norm of the maximal function f^*_φ can be majorized by the quantity $C\|f\|_{H^p}$, where C depends only on p and φ.

2.3. Boundary properties of functions from the range of the operator V_μ

In this section we generalize the results of Section 2.2 to the case of an arbitrary measure $\mu \in \mathcal{M}(\mathbb{T})$ instead of the singular measure σ_α. The space $\theta^*(H^p)$ will now be replaced by the image of $L^p(\mu)$ under the map $V_\mu\colon L^p(\mu) \longrightarrow H^p(\mathbb{D})$.

Let R denote the operator from $L^p(\mu)$ to $L^p(\mu)$ defined by the formula $Rf = \bar{z}\cdot\big(f - \int_{\mathbb{T}} f\, d\frac{\mu}{\|\mu\|}\big)$.

Lemma 2.3.1. *Let $\mu \in \mathcal{M}_+(\mathbb{T})$, $f \in L^1(\mu)$, $F = V_\mu f$, and let $\sum_{n\geq 0} a_n z^n$ be the Fourier series of F. Then for each $k \in \mathbb{Z}$, $k \geq 0$ we have*

$$\sum_{n=0}^{k} a_n z^n = f - z^{k+1} R^{k+1} f \tag{2.3.1}$$

μ-a. e. on $\mathbb{T}$.

Proof. We may assume that μ is a probability measure. Set $g = f - \int f\, d\mu$. We note that $\int g\, d\mu = 0$. We also have

$$\begin{aligned}\mathcal{K}\Big(\frac{g(\xi)}{\xi}\cdot\mu\Big)(z) &= \int \frac{1}{1-z\bar{\xi}}\frac{g(\xi)}{\xi}\, d\mu(\xi)\\ &= \Big(\int \frac{1}{1-z\bar{\xi}} g(\xi)\, d\mu(\xi) - \int \frac{1-z\bar{\xi}}{1-z\bar{\xi}} g(\xi)\, d\mu(\xi)\Big)\Big/ z\\ &= \frac{g\mathcal{K}(\mu)(z)}{z}.\end{aligned}$$

Consequently,

$$V_\mu R f = \frac{\mathcal{K}((Rf)_\mu)}{\mathcal{K}\mu} = \frac{\mathcal{K}(\frac{g(\xi)}{\xi}\mu)}{\mathcal{K}\mu} = \big(V_\mu f - (V_\mu f)(0)\big)/z = S^* V_\mu f.$$

Thus, for all $k \in \mathbb{Z}$, $k \geq 0$ we have $V_\mu R^k f = S^{*k} F$. Now we prove the statement of the lemma by induction. For $k = 0$, (2.3.1) obviously holds, since $F(0) = \int_{\mathbb{T}} f\, d\mu$. Further, if

$$\sum_{n=0}^{k} a_n z^n = f - z^{k+1} R^{k+1} f \quad \mu\text{-a. e. on } \mathbb{T},$$

then

$$\begin{aligned}\sum_{n=0}^{k+1} a_n z^n - \sum_{n=0}^{k} a_n z^n + a_{k+1} z^{k+1} &= (f - z^{k+1} R^{k+1} f) + z^{k+1}((S^{*k} V_\mu f)(0)\\ &= f - z^{k+1}(R^{k+1} f - (V_\mu R^{k+1} f)(0))\\ &= f - z^{k+1}\Big(R^{k+1} f - \int R^{k+1} f\, d\mu\Big)\\ &= f - z^{k+2} R^{k+1} f. \blacktriangle\end{aligned}$$

Lemma 2.3.2. *Let $\mu \in \mathcal{M}_+(\mathbb{T})$, $f \in L^2(\mu)$, $F = V_\mu f$, and let $\sum_{n\geq 0} a_n z^n$ be the Fourier series of F. Then for each $k \in \mathbb{Z}$, $k \geq 0$, we have*

$$\left\| \sum_{n=0}^{k} a_n z^n \right\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mu)},$$

where C is an absolute constant.

Proof. It is sufficient to prove the lemma for an arbitrary continuous function f.

Consider a sequence of singular measures $\{\mu_n\}_{n>0}$, $\mu_n \in \mathcal{M}_+(\mathbb{T})$, converging to μ in the weak* topology of the space $\mathcal{M}(\mathbb{T})$. Since f is continuous, we may deal with the sequence of functions $F_n = V_{\mu_n} f$. Let S_k denote the k-th partial sum of the Fourier series of F, and let S_k^n stand for the similar sum associated with F_n. By Theorem 2.2.1,

$$\|S_k^n\|_{L^2(\mu_n)} \leq C\|f\|_{L^2(\mu_n)}. \tag{2.3.2}$$

We note that $F_n \to F$ uniformly on compact subsets of $\mathbb{D}$, and so $S_k^n \to S_k$ uniformly on $\mathbb{T}$. It remains to let $n \to \infty$ in (2.3.2). ▲

Remark. Clark's Theorem and the proof of Theorem 2.2.1 together imply that the constant C occurring in the statement of Lemma 2.3.2 can be chosen to equal 2.

Lemma 2.3.3. *Suppose that either*

(A) *$\mu = m + fm$, where $f \geq 0$, $f \in L^1(m)$* or

(B) *$\mu = m + \nu$, where ν is a singular probability measure in $\mathcal{M}_+(\mathbb{T})$.*

Then for each function g in $L^2(\mu)$ the Fourier series of $V_\mu g$ converges in $L^2(\mu)$.

Proof. In view of Lemma 2.3.2 and the Banach–Steinhaus Theorem, it suffices to find a dense subset of $L^2(\mu)$ such that the statement holds for its elements.

Let μ satisfy (A). We shall prove that the power series of $1/\mathcal{K}\mu$ converges in $L^2(\mu)$. Set

$\mu_r = \mathcal{K}\mu(rz)$, $0 < r < 1$. We have

(2.3.3)

$$\int \left|S^{*n}\frac{1}{\mu_r}\right|^2 \operatorname{Re}\mu_r\, dm = \frac{1}{2}\int S^{*n}\frac{1}{\mu_r}\cdot \overline{S}^{*n}\frac{1}{\mu_r}(\mu_r + \overline{\mu}_r)\, dm$$
$$= \frac{1}{2}\left(\int \frac{z^n\mu_r}{\overline{\mu}_r}S^{*n}\frac{1}{\mu_r}\, dm + \int \overline{\frac{z^n\mu_r}{\mu_r}S^{*n}\frac{1}{\mu_r}}\, dm\right)$$
$$= \operatorname{Re}\int S^{*n}\frac{1}{\mu_r}\cdot\frac{z^n\mu_r}{\overline{\mu}_r}\, dm.$$

Now, since $1/\mathcal{K}\mu \in H^\infty$, the functions $\left|S^{*n}\frac{1}{\mu_r}\right|^2$ are bounded by a constant independent of r, and they converge to $\left|S^{*n}\frac{1}{\mu_r}\right|^2$ as $r \to 1^-$ m-a. e. Thus, by the Lebesgue dominated convergence Theorem, $\left|S^{*n}\frac{1}{\mu_r}\right|^2$ tends to $\left|S^{*n}\frac{1}{\mathcal{K}\mu}\right|^2$ as $r \to 1^-$ in the weak* topology of $L^\infty(m)$. Besides, $\operatorname{Re}\mu_r \to \operatorname{Re}\mathcal{K}\mu$ in $L^1(m)$. Consequently,

$$\int \left|S^{*n}\frac{1}{\mu_r}\right|^2 \operatorname{Re}\mu_r\, dm \xrightarrow[r\to 1-]{} \int \left|S^{*n}\frac{1}{\mu_r}\right|^2 \operatorname{Re}\mu_r\, dm. \tag{2.3.4}$$

Since $\mu_r \xrightarrow[r\to 1^-]{} \mathcal{K}\mu$ m-a. e. and $1/\mathcal{K}\mu \in H^\infty$, we have

$$\operatorname{Re}\int S^{*n}\frac{1}{\mu_r}\cdot\frac{z^n\mu_r}{\overline{\mu}_r}\, dm \xrightarrow[r\to 1-]{} \operatorname{Re}\int S^{*n}\frac{1}{\mathcal{K}\mu}\cdot\frac{z^n\mathcal{K}\mu}{\overline{\mathcal{K}\mu}}\, dm. \tag{2.3.5}$$

Now (2.3.3-5) and the properties of S^* yield

$$\int \left|S^{*n}\frac{1}{\mathcal{K}\mu}\right|^2 \operatorname{Re}\mathcal{K}\mu\, dm = \operatorname{Re}\int S^{*n}\frac{1}{\mathcal{K}\mu}\frac{z^n\mathcal{K}\mu}{\overline{\mathcal{K}\mu}}\, dm \xrightarrow[n\to\infty]{} 0.$$

Therefore, the power series of $1/\mathcal{K}\mu$ converges in $L^2(\mu)$. We set $h = \mathcal{P}/f + 1$, where $\mathcal{P}$ is a polynomial in z and $\overline{z}$. Then $V_\mu h = \mathcal{P}_1 + \mathcal{P}_2/\mathcal{K}\mu$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are polynomials in z. Thus, the series of $V_\mu h$ converges in $L^2(\mu)$. In view of the remark made at the beginning of the proof, it remains to observe that the functions h of the above form are dense in $L^2(\mu)$.

Let μ satisfy (B). By θ we denote the inner function for which $\mathcal{K}\nu = 1/(1-\theta)$. Let $\mathcal{P}_1$ be a polynomial in z, $\mathcal{P}_2$ a polynomial in $\overline{z}$, and h_0 a function in $L^2(\nu)$. Consider a function h in $L^2(\mu)$ defined by

$$h = \begin{cases} h_0 & \nu\text{-a. e.,} \\ (2-\theta)\mathcal{P}_1 + \mathcal{P}_2 + V_\nu h_0 & m\text{-a. e.} \end{cases}$$

We have

$$V_\mu h = \frac{(1-\theta)((2-\theta)\mathcal{P}_1 + V_\nu h_0 + V_\nu h_0/(1-\theta))}{2-\theta} = (1-\theta)\mathcal{P}_1 + V_\nu h_0.$$

By Theorem 2.2.1, the Fourier series of θ and $V_\nu h_0$ converge in $L^2(\nu)$; they also converge in $L^2(m)$ because both θ and $V_\nu h_0$ are in H^2. Finally, the functions h under consideration form a dense subset of $L^2(\mu)$. ▲

Lemma 2.3.4. *Let $\mu \in \mathcal{M}_+(\mathbb{T})$. Then the Fourier series of $1/\mathcal{K}\mu$ converges in $L^2(\mu)$.*

Proof. The measure μ can be written in the form $\mu = fm + \sigma$, where $f \geq 0$, $f \in L^1(m)$, and σ is a singular measure in $\mathcal{M}_+(\mathbb{T})$. We may assume that σ is a probability measure. Now we shall show that the function $1/\mathcal{K}\mu$ lies both in the range of $V_{(f+1)m}$ and in the range of $V_{\sigma+m}$.

Since $|\mathcal{K}\mu| \geq 1/2$, the function $1/\mathcal{K}\mu$ is bounded. By Theorem 3, $\mathcal{K}(fm)/\mathcal{K}\mu$ is in H^2. Consequently,

$$\frac{\mathcal{K}(fm)+1}{\mathcal{K}\mu(f+1)} \in L^2((f+1)m),$$

and so

$$V_{(f+1)m}\left(\frac{\mathcal{K}(fm)+1}{\mathcal{K}\mu(f+1)}\right) = \frac{\mathcal{K}((f+1)m)}{\mathcal{K}\mu}\Big/\mathcal{K}((f+1)m) = \frac{1}{\mathcal{K}\mu}.$$

Further, consider a function h in $L^2(\sigma+m)$ such that $h = 0$ σ-a. e. and $h = (\mathcal{K}\sigma+1)/\mathcal{K}\mu$ m-a. e. (We note that $h \in L^2(m)$ because $1/\mathcal{K}\mu$ is bounded and $\mathcal{K}\sigma/\mathcal{K}\mu \in H^2$ by Theorem 3). We have

$$V_{\sigma+m}(h) = \frac{(\mathcal{K}\sigma+1)}{\mathcal{K}\mu}\Big/(\mathcal{K}\sigma+1) = \frac{1}{\mathcal{K}\mu}.$$

Now the desired conclusion follows from Lemma 2.3.3. ▲

The next theorem is a generalization of Theorem 2.2.1 in the case $p = 2$.

Theorem 2.3.5. *Let $\mu \in \mathcal{M}_+(\mathbb{T})$ and $f \in L^2(\mu)$. Then the Fourier series of $V_\mu f$ converges in $L^2(\mu)$.*

Proof. By Lemma 2.3.2 and the Banach–Steinhaus Theorem, it again suffices to find a dense subset of $L^2(\mu)$ for which the statement is valid.

First we show that if the series of $V_\mu f$ converges in $L^2(\mu)$, then the same is true for $V_\mu(zf)$ and $V_\mu(\overline{z}f)$. Computations similar to those in the proof of Lemma 2.3.1 show that

$$\begin{aligned}(V_\mu(zf))(\xi) &= \int_{\mathbb{T}} zf(z)\,d\mu(z)/\mathcal{K}\mu(\xi) + \xi(V_\mu f)(\xi),\\ (V_\mu(\overline{z}f))(\xi) &= \overline{\xi}\Big((V_\mu f)(\xi) - \int_{\mathbb{T}} f\,d\mu/\mathcal{K}m(\xi)\Big).\end{aligned}$$

The convergence of the series of $V_\mu(zf)$ and $V_\mu(\overline{z}f)$ now follows by Lemma 2.3.4.

Since for $c \in \mathbb{C}$ one has $V_\mu c = c$, the polynomials in z and $\overline{z}$ form a dense subset with the required property. ▲

Corollary 2.3.6. *Under the assumptions of the theorem, we set*

$$F_r = (V_\mu f)(rz), \qquad r \in [0,1).$$

Then the F_r's converge in $L^2(\mu)$ as $r \to 1^-$.

Remark. Clearly, in the theorem (respectively, in its corollary) it is impossible to guarantee that the series converges to f (respectively, that the functions F_r converge to f). However, the sum of the series and the limit of F_r do coincide with f almost everywhere with respect to the singular component of the measure μ. This is a consequence of Theorem 2.3.8 below.

Remark. In the statement of the theorem, one cannot replace L^2 by L^p, $p \neq 2$. This can be seen from the following example.

Let σ be a singular probability measure in $\mathcal{M}_+(\mathbb{T})$, and let $p > 2$. If σ is not discrete, then the corollary to Theorem 4 implies the existence of a function f in $L^p(\sigma)$ such that $V_\sigma f \notin H^p$. Consider the measure $\mu = m + \sigma$ and the function $g \in L^p(\mu)$ which equals 0 m-a. e. and equals f σ-a. e. We denote by θ the inner function that equals $1 - 1/\mathcal{K}\sigma$. Then $V_\mu g = V_\sigma f \cdot (2-\theta) \notin H^p$. Consequently, the Fourier series of $V_\mu g$ does not converge in $L^p(m)$, so not in $L^p(\mu)$ either. Moreover, the $L^p(\mu)$-norms of the partial sums S_n of that series tend to infinity. By Lemma 2.3.1, $S_n = g - z^{n+1}R^n g$ μ-a. e. on $\mathbb{T}$. Therefore, $\|R^n\| \to \infty$ as $n \to \infty$.

Consider the adjoint operator $R^*\colon L^q(\mu) \to L^q(\mu)$, where $q = \frac{p}{p-1}$. From the definition of R one easily derives that R^* is given by the formula $R^* f = zf - \int zf\, d\frac{\mu}{\|\mu\|}$. Since $\|R^{*n}\| \to \infty$ as $n \to \infty$, the Banach–Steinhaus Theorem implies that for some function $h \in L^q(\mu)$ the norms $\|R^{*n}h\|_{L^q(\mu)}$ are not uniformly bounded. Now for the powers of R and R^* we have the relation $R^{*n}f = \overline{(zR^n(\overline{zf}))}$ holding for all $f \in L^q(\mu)$. It follows that the norms $\|R^n(\overline{zh})\|_{L^q(\mu)}$ are not bounded, and so the Fourier series of $V_\mu(\overline{zh})$ does not converge in $L^q(\mu)$.

Now we are going to focus on the problem concerning nontangential limits of functions lying in the range of the map V_μ. This problem was treated in Section 1 for singular measures $\mu \in \mathcal{M}_+(\mathbb{T})$. In the proof of Theorem 2.3.8 below we reduce the general case to that special setting with the help of the following statement.

Lemma 2.3.7. *Assume that I is an open subarc of the circle $\mathbb{T}$, $J = \mathbb{T}\setminus I$, $\mu \in \mathcal{M}_+(\mathbb{T})$, $\mu(J) = 0$, $f \in L^\infty(\mu)$, $\varphi \in (0, \pi/2)$. Then for each $\varepsilon > 0$ there exists a discrete measure μ_0 and a function $f_0 \in L^\infty(\mu_0)$ such that $\|\mu_0\| \le \|\mu\|$, $\|f_0\|_{L^\infty(\mu_0)} \le \|f\|_{L^\infty(\mu)}$, and for all z in Γ_J^φ one has*

$$(1) \qquad |\mathcal{K}\mu(z) - \mathcal{K}(\mu_0)(z)| < \varepsilon,$$

$$(2) \qquad |\mathcal{K}(f\mu)(z) - \mathcal{K}(f_0\mu_0)(z)| < \varepsilon.$$

Proof. We may assume that $\partial I = \{e^{i\alpha}, e^{-i\alpha}\}$, where $\alpha \in (0, \pi]$ (if $\alpha = 0$, then μ itself is discrete). We shall show how to partition the arc I into suitable subarcs I_k, $k \in \mathbb{Z}$, so that the desired measure μ_0 could be obtained by replacing μ with a point mass on each of these subarcs.

For each $\varepsilon_0 > 0$ one can find a sequence $\{\alpha_n\}_{n \ge 0}$ of numbers in $[0, \alpha)$ with the following properties:

1) $\alpha_0 = 0$,

2) $\alpha_{n+1} > \alpha_n$ $(n = 0, 1, 2, \dots)$,

3) $\mu(\{e^{i\alpha_n}\}) = 0$ $(n = 0, 1, 2, \dots)$,

4) $\alpha_n \to \alpha$ as $n \to \infty$,

5) for each $k \in \mathbb{N}$

$$\sup_{z \in \Gamma_J^\varphi} \left| \frac{1}{1 - ze^{-i\alpha_{k-1}}} - \frac{1}{1 - ze^{-i\alpha_k}} \right| < \varepsilon_0.$$

We construct such a sequence for a suitable sufficiently small ε_0 (the choice of ε_0 will be specified at the end of the proof). Denote by I_k, $k \in \mathbb{N}$, the arc between $e^{i\alpha_{k-1}}$ and $e^{i\alpha_k}$ contained in I. By I_{-k} we denote the arc symmetric to I_k with respect to the real axis: $I_{-k} = \{ \xi \mid \bar{\xi} \in I_k \}$. Now let μ_0 be the measure having the point mass $\mu(I_k)$ at each $e^{i\alpha_k}$ and the point mass $\mu(I_{-k})$ at each $e^{-i\alpha_k}$, $k \in \mathbb{N}$. We define the function f_0 by letting it equal $\frac{1}{\mu(I_k)} \int_{I_k} f\,d\mu$ at $e^{i\alpha_k}$ and $\frac{1}{\mu(I_{-k})} \int_{I_{-k}} f\,d\mu$ at $e^{-i\alpha_k}$ $(k \in \mathbb{N})$.

The conditions $\|\mu_0\| \le \|\mu\|$ and $\|f\| \le \|f_0\|$ are easily verified. Further, for $z \in \Gamma_J^\varphi$ we have

$$|\mathcal{K}\mu(z) - \mathcal{K}(\mu_0)(z)| \le \sum_{k \in \mathbb{Z} \setminus \{0\}} |\mu(I_k)| \cdot \sup_{\xi \in I_k} \left| \frac{1}{1 - z\bar{\xi}} - \frac{1}{1 - ze^{-i\alpha_k}} \right| \le C\|\mu\|\varepsilon_0,$$

where the constant C depends only on φ. Thus, taking ε_0 smaller than $\varepsilon / C\|\mu\|$, we get (1). Writing the same estimates for $f\mu$ and $f_0\mu_0$ instead of μ and μ_0, we arrive at (2). ▲

Theorem 2.3.8. *Let $\mu \in \mathcal{M}(\mathbb{T})$ and $f \in L^1(\mu)$. Denote by F the function $V_\mu f$, which is meromorphic in $\mathbb{D}$. Then for μ-almost every $\xi \in \mathbb{T}$ the limit*

$$\lim_{\substack{z \to \xi \\ \not\,\,}} F(z)$$

exists, and it equals $f(\xi)$ almost everywhere with respect to the singular component of the measure μ.

Proof. A) First we consider the case where $\mu \ge 0$ and $f \in L^\infty(\mu)$. We denote the singular component of μ by σ. Since $\mu \ge 0$, Theorem 3 yields $F \in H^2$, and so the nontangential limits of F exist m-a. e. It remains to prove that they also exist and equal f σ-a. e. Let a subset $E \subset \mathbb{T}$ be such that $\sigma(E) = \|\sigma\|$, $m(E) = 0$. Since σ is a regular measure, there exist closed sets $E_1, E_2, \dots$ with $\bigcup_{n\in\mathbb{N}} E_n = E$. Let n be a fixed positive integer. The set $\mathbb{T} \setminus E_n$ can be written as the union of a countable collection of pairwise disjoint open arcs $I_1, I_2, \dots$. By Lemma 2.3.7, for every $\varepsilon > 0$ and $\varphi \in (0, \pi/2)$ one can find a discrete measure μ_k^ε supported on I_k and a function $f_k^\varepsilon \in L^\infty(\mu_k^\varepsilon)$ such that $\|\mu_k^\varepsilon\| \le \mu(I_k)$, $\|f_k^\varepsilon\|_{L^\infty(\mu_k^\varepsilon)} \le \|f\|_{L^\infty(\mu)}$ and for every $z \in \Gamma^\varphi_{\mathbb{T}\setminus I_k}$ one has

$$\left| \int_{I_k} \frac{1}{1 - z\overline{\xi}}\, d\mu - \int_{I_k} \frac{1}{1 - z\overline{\xi}}\, d\mu_k^\varepsilon \right| < \frac{\varepsilon}{2^k}$$

and

$$\left| \int_{I_k} \frac{1}{1 - z\overline{\xi}} f\, d\mu - \int_{I_k} \frac{1}{1 - z\overline{\xi}} f_k^\varepsilon\, d\mu_k^\varepsilon \right| < \frac{\varepsilon}{2^k}.$$

Let χ_{E_n} denote the characteristic function of the set E_n. We set $\mu_\varepsilon = \sum_{k\in\mathbb{N}} \mu_k^\varepsilon + \chi_{E_n} \cdot \sigma$. Let a function f_ε be equal to f_k^ε μ_k^ε-a. e. for all $k \in \mathbb{N}$, and to f σ-a. e. on E_n. Then we have $\|\mu_\varepsilon\| \le \|\mu\|$, $\|f_\varepsilon\|_{L^\infty(\mu_\varepsilon)} \le \|f\|_{L^\infty(\mu)}$. Setting $F_\varepsilon = V_{\mu_\varepsilon} f_\varepsilon$, we see that the following relations hold on $\Gamma^\varphi_{E_n}$:

$$\begin{aligned}(2.3.6) \qquad |F - F_\varepsilon| &\le \left| \frac{\mathcal{K}(f\mu)}{\mathcal{K}\mu} - \frac{(f_\varepsilon \mu_\varepsilon)}{\mathcal{K}\mu} \right| + \left| \frac{\mathcal{K}(f_\varepsilon \mu_\varepsilon)}{\mathcal{K}\mu} - \frac{\mathcal{K}(f_\varepsilon \mu_\varepsilon)}{\mathcal{K}(\mu_\varepsilon)} \right| \\ &= \frac{1}{|\mathcal{K}\mu|} \left(|\mathcal{K}(f_\mu) - \mathcal{K}(f_\varepsilon \mu_\varepsilon)| + |F_\varepsilon| |\mathcal{K}(\mu_\varepsilon) - \mathcal{K}\mu| \right) \\ &\le \frac{2\varepsilon}{\|\mu\|} (1 + |F_\varepsilon|).\end{aligned}$$

By Theorem 2.2.7, the maximal function $(F_\varepsilon)^*_\varphi$ is finite μ_ε-a. e., and consequently also μ-a. e. on E_n. From (2.3.6) it follows that F^*_φ is also finite μ-a. e. on E_n. Let $\{\varepsilon_k\}_{k\ge 0}$ be a sequence of positive numbers tending to zero. For each ε_k, we construct a measure μ_{ε_k} and functions f_{ε_k} and F_{ε_k} in the same way as μ_ε, f_ε, and F_ε were constructed for ε. Let a point ξ in E_n be such that for all $k \in \mathbb{N}$

$$\lim_{\substack{z \to \xi \\ \not<}} F_{\varepsilon_k}(z) = f(\xi)$$

and $F_\varphi^* < +\infty$ (μ-almost every points in E_n enjoy these properties). Then for each $z \in \Gamma_\xi^\varphi$ we have

(2.3.7)

$$|F(z) - F_{\varepsilon_k}(z)| \le \left|\frac{\mathcal{K}(f\mu)}{\mathcal{K}\mu} - \frac{\mathcal{K}(f\mu)}{\mathcal{K}(\mu_{\varepsilon_k})}\right| + \left|\frac{\mathcal{K}(f\mu)}{\mathcal{K}(\mu_{\varepsilon_k})} - \frac{\mathcal{K}(f_{\varepsilon_k}\mu_{\varepsilon_k})}{\mathcal{K}(\mu_{\varepsilon_k})}\right|$$
$$= |F(z)|\left|\frac{1}{\mathcal{K}(\mu_{\varepsilon_k})}\right|\left|\mathcal{K}\mu - \mathcal{K}(\mu_{\varepsilon_k})\right| + \frac{1}{|\mathcal{K}(\mu_{\varepsilon_k})|}\left|\mathcal{K}(f\mu) - \mathcal{K}(f_{\varepsilon_k}\mu_{\varepsilon_k})\right|$$
$$\le F_\varphi^*(\xi)\frac{4\varepsilon_k}{\|\mu\|}.$$

Since $\varepsilon_k \to 0$ and all F_{ε_k} tend to $f(\xi)$ along Γ_ξ^φ, the Stokes–Seidel Theorem and (2.3.7) together imply that F also tends to $f(\xi)$ along the set Γ_ξ^φ. Thus,

$$\lim_{\substack{z \to \xi \\ \measuredangle}} F(z) = f(\xi)$$

for σ-almost every $\xi \in E_n$. Recalling that $\bigcup E_n = E$, we come to the desired conclusion.

B) Now let $\mu \in \mathcal{M}_+(\mathbb{T})$, $f \in L^1(\mu)$, and let σ be the singular part of μ. We may assume that $f \ge 1$. By A), the limit

$$\lim_{\substack{z \to \xi \\ \measuredangle}} (V_{f\mu} 1/f)(z) \tag{2.3.8}$$

exists $f\mu$-a. e. (and consequently μ-a. e.), its value being equal to $1/f(\xi)$ $f\sigma$-a. e. (and consequently σ-a. e.). Moreover, the function $V_{f\mu}1/f$ belongs to H^2. Therefore, the limit (2.3.8) is nonzero m-a. e. on $\mathbb{T}$. Since $f \in L^1(\mu)$, one has $1/f \ne 0$ σ-a. e. Consequently, the limit (2.3.8) is also nonzero σ-a. e., and it is equal to $1/f(\xi)$ for σ-almost every ξ. The statement of the theorem now follows from the relation

$$V_\mu f = \left(V_{f\mu} 1/f\right)^{-1}.$$

C) Finally, let $\mu \in \mathcal{M}(\mathbb{T})$ and $f \in L^1(\mu)$. Let a function g be such that $g|\mu| = \mu$. Then $|g| = 1$ μ-a. e.; the desired conclusion is now implied by B) and the fact that

$$V_\mu f = V_{|\mu|} fg / V_{|\mu|} g. \blacktriangle$$

Corollary 2.3.9. *Assume that* $\mu, \nu \in \mathcal{M}(\mathbb{T})$, ν *is singular,* $\mu \perp \nu$. *Then*

$$\mathcal{K}\mu = o(\mathcal{K}\nu) \quad \text{as } z \underset{\measuredangle}{\longrightarrow} \xi$$

for ν*-almost every* ξ.

Proof. Consider a function f in $L^\infty(\mu+\nu)$ such that $f = 1$ ν-a. e. and $f = 0$ μ-a. e. Now, by Theorem 2.3.8,

$$V_{\mu+\nu} f = \frac{\mathcal{K}\nu}{\mathcal{K}\mu + \mathcal{K}\nu} \underset{z \underset{\measuredangle}{\to} \xi}{\longrightarrow} 1$$

for ν-almost every ξ. ▲

Corollary 2.3.10. *Let* $f \in H^1$ *and let* θ *be an inner function. Then*

$$(1-\theta)f \longrightarrow 0 \quad \text{as } z \underset{\measuredangle}{\longrightarrow} \xi$$

for σ_1*-almost every* ξ.

Proof. Set $fm = \mu$, $\sigma_1 = \nu$ and apply Corollary 1. ▲

Thus, the set $\{\xi \in \mathbb{T} \mid (1-\theta)f \underset{z \underset{\measuredangle}{\to} \xi}{\longrightarrow} 0\}$ is nonempty, whenever $f \in H^1$ and θ is a nonconstant inner function.

Remark. Since Clark's operator $U_\alpha \colon \theta^*(H^p) \longrightarrow L^p(\sigma_\alpha)$ is unbounded for $p \in (1;2)$, the theorems of Section 1 deal with the case $p \geq 2$ only. Theorem 2.3.8 enables us to generalize Theorem 2.2.5 in the following way.

Let $f \in \theta^*(H^p)$ and $U_\alpha f \in L^1(\sigma_\alpha)$ (i.e., $f = V_{\sigma_\alpha} g$ for some $g \in L^1(\sigma_\alpha)$). Then for σ_α-almost every ξ one has

$$\lim_{z \underset{\measuredangle}{\to} \xi} f(z) = (U_\alpha f)(\xi).$$

Remark. As it was mentioned in the Introduction, for $\mu \in \mathcal{M}_+(\mathbb{T})$ the range of the map V_μ contains de Branges's space $\mathfrak{H}(f)$ with $f = 1 - \frac{\|\mu\|}{\mathcal{K}\mu}$. Thus, the theorems proved in this section also describe the boundary behavior of functions in $\mathfrak{H}(f)$ with respect to the measure μ.

2.4 Applications to multiplication and division problems for Cauchy integrals

In this section we apply the above results to some well-known problems. The questions treated here were discussed earlier in [G1, G2, G-K-V].

Let θ be an inner function. It is well known that θ possesses a natural extension to $\mathbb{D}_- = \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$ as a meromorphic function. Given $f \in H^\infty$, we say that f has a pseudocontinuation equal to θg, where θ is inner and g is analytic and bounded on $\mathbb{D}_-$ with $g(\infty) = 0$, provided that for m-almost every ξ one has

$$\lim_{r \to 1^-} f(r\xi) = \lim_{r \to 1^+} (\theta g)(r\xi).$$

We define the Hardy class $H^p(\mathbb{D}_-)$ of the region $\mathbb{D}_-$ as the set of all functions f analytic in $\mathbb{D}_-$ and satisfying

$$\sup_{r>1} \int_{\mathbb{T}} |f(r\xi)|^p \, dm(\xi) < +\infty.$$

The following theorem is due to A. B. Aleksandrov.

Theorem 2.4.1 ([A3, A4]). *Assume that $f \in H^\infty(\mathbb{D})$ and f has a pseudocontinuation equal to θg. Let $\mu \in \mathcal{M}(\mathbb{T})$ and $\theta \mathcal{K}\mu \in H^p(\mathbb{D}_-)$ for some $p > 0$. Then there is a measure ν such that*

$$f \mathcal{K}\mu = \mathcal{K}\nu$$

on $\mathbb{C} \setminus \mathbb{T}$ (it is meant that $f = \theta g$ on $\mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$).

Now the following question arises: what is the relationship between the measures μ and ν? More precisely, when is it possible to let ν equal $f\mu$? Some results on this matter can be found in [G1], [G2], and [G-K-V]. Theorem 2.3.8 enables us to prove the following proposition.

Theorem 2.4.2. *Under the assumptions of the preceding theorem, for μ-almost every ξ the limit*

$$\lim_{\substack{z \to \xi \\ \not\angle}} f(z) = f(\xi)$$

exists. The measure ν can be chosen to equal $f\mu$:

$$f\mathcal{K}\mu = \mathcal{K}(f\mu).$$

Proof. From I. I. Privalov's results [P] it follows that the absolutely continuous component ν_a of the measure ν equals $f\mu_a$ (μ_a being the absolutely continuous component of the measure μ). It was shown in M. G. Goluzina's paper [G1] that ν is absolutely continuous with respect to μ. Thus, ν equals $g\mu$, where $g \in L^1(\mu)$ and $g = f$ m-a. e. on $\mathbb{T}$.

It remains to prove that f has angular limits almost everywhere with respect to μ_s (which is the singular component of μ) and these limits equal the values of g μ_s-a. e.

By Theorem 2.3.8, the function $\mathcal{K}\nu/\mathcal{K}\mu$ has angular limits μ-a. e.; moreover, for μ_s-almost every ξ

$$\left(\frac{\mathcal{K}\nu}{\mathcal{K}\mu}\right)(z) = f(z) \underset{\substack{z\to\xi \\ \measuredangle}}{\longrightarrow} g(\xi). \blacktriangle$$

In particular, Theorem 2.4.2 can be applied to the following corollary of Theorem 2.4.1, due to S. A. Vinogradov.

Corollary 2.4.3 ([V]). *Let $\mu \in \mathcal{M}(\mathbb{T})$, let θ be an inner function, and let $\mathcal{K}\mu/\theta \in H^p$ for some $p > 0$. Then there is a measure $\nu \in \mathcal{M}(\mathbb{T})$ such that $\mathcal{K}\mu/\theta = \mathcal{K}\nu$.*

Once again, the question of relationship between μ and ν arises. One of the ways to construct the measure ν for given μ and θ was pointed out in [G2]. Applying Theorem 2.3.8 we get the following result.

Corollary 2.4.4. *Under the assumptions of the preceding theorem, the function θ has nontangential boundary values of modulus one μ-a. e. on the circle $\mathbb{T}$. The measure ν can be chosen to equal $\mu\bar{\theta}$.*

Part 3. Rank one perturbations of self adjoint operators

3.1 Families $\mathcal{M}_\varphi$ and rank one perturbations of self-adjoint operators.

We will denote by $\mathcal{M}(\mathbb{R})$ the space of Borel complex measures on $\mathbb{R}$ with the norm

$$||\mu|| = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\mu(t)}{1+t^2}.$$

We will also use the notation $\mathcal{M}_+(\mathbb{R})$ for the subset consisting of positive measures.

Let φ be an analytic function on the upper half plane $\mathbb{C}_+$ such that $|\varphi| \le 1$. Then for any $\alpha \in \mathbb{T}$ we again can consider a measure $\mu_\alpha \in \mathcal{M}_+(\mathbb{R})$ such that its Poisson integral satisfies

$$(\mathcal{P}\mu_\alpha)(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\,d\mu_\alpha(t)}{(x-t)^2+y^2} = \operatorname{Re}\frac{\alpha+\varphi}{\alpha-\varphi}.$$

The family $\mathcal{M}_\varphi(\mathbb{R}) = \{\mu_\alpha\}_{\alpha\in\mathbb{T}}$ preserves most of the properties of an analogous family on the circle. All the results from Parts 1 and 2 concerning families $\mathcal{M}_\varphi$ can be easily adopted to the case of the real line.

As was shown by Clark in [C], if φ is an inner function ($|\varphi| = 1$ a. e. on $\mathbb{T}$) then $\mathcal{M}_\varphi(\mathbb{T})$ is the system of the spectral measures of all unitary one-dimensional perturbations of the model contraction with the characteristic function φ. A similar connection can be made between rank one perturbations of self-adjoint operators and families $\mathcal{M}_\varphi(\mathbb{R})$, see [Ar], [Do], [S-W], [S] and [R-J-L-S].

If A_0 is a cyclic self-adjoint operator, acting in a separable Hilbert space, and φ is its cyclic vector, then we can consider the family of one-dimensional perturbations

$$A_\lambda = A_0 + \lambda(*,\varphi)\varphi,$$

$\lambda \in \mathbb{R}$. If we denote by ν_λ the spectral measure of φ for A_λ then the relation for the resolvents

$$(A_0-z)^{-1} - (A_\lambda - z)^{-1} = \left[\lambda(*,\varphi)\left((A_\lambda - z)^{-1}\varphi\right)\right](A_0-z)^{-1},$$

for any $z \in \mathbb{C}_+$, gives us

$$F_\lambda(z) = \frac{F_0(z)}{1 + i\pi\lambda F_0(z)} \tag{3.1.1}$$

where

$$F_\lambda(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\nu_\lambda(t)}{z-t} = \frac{i}{\pi}((z - A_\lambda)^{-1}\varphi, \varphi) \tag{3.1.2}$$

(see [Ar]).

Since ν_0 is a positive Borel measure on $\mathbb{R}$, F_0 is an analytic function with a positive real part on $\mathbb{C}_+$. Thus

$$F_0 = \frac{1+\varphi}{1-\varphi}$$

for some $\varphi \in \mathcal{H}^\infty(\mathbb{C}_+)$, $\|\varphi\|_\infty \leq 1$.

Hence by (3.1.1)

$$(\mathcal{P}\nu_\lambda)(x+iy) = \operatorname{Re} F_0(x+iy) = \operatorname{Re} \frac{\frac{1-\varphi}{1+\varphi}}{1 + i\pi\lambda\frac{1-\varphi}{1+\varphi}} = c \operatorname{Re} \frac{\beta - \varphi}{\beta + \varphi} \tag{3.1.3}$$

where $\beta = \frac{1+i\pi\lambda}{1-i\pi\lambda}$, $c = \operatorname{Re} \frac{1}{1+i\pi\lambda}$. Thus

$$\mathcal{P}\nu_\lambda = c\mathcal{P}\mu_\beta$$

where $\{\mu_\beta\}_{\beta\in\mathbb{T}} = \mathcal{M}_\varphi(\mathbb{R})$.

Remark. The same argument works for the families of unitary operators (in particular the ones considered by Clark [C]).

Let $\mathcal{U}_1$ be a unitary cyclic operator in a separable Hilbert space. Let probability measure $\mu_1 \in \mathcal{M}_+(\mathbb{T})$ be the spectral measure of some cyclic vector v for $\mathcal{U}_1$. Then we can consider the family of one-dimensional unitary perturbations of $\mathcal{U}_1$:

$$\mathcal{U}_\alpha = \mathcal{U}_1 + (\alpha - 1)(*, \mathcal{U}_1^{-1}v)v.$$

For the resolvents we have

$$(\mathcal{U}_1 - z)^{-1} - (\mathcal{U}_\alpha - z)^{-1} = (\mathcal{U}_1 - z)^{-1}[(*, \mathcal{U}_1^{-1}v)v](\mathcal{U}_\alpha - z)^{-1},$$

and since $\mathcal{K}\mu_\alpha = ((\mathcal{U}_\alpha - z)^{-1}v, v)$ for $z \in \mathbb{D}$, where μ_α is the spectral measure of v for $\mathcal{U}_\alpha$, we have that

(3.1.4) $$\mathcal{K}\mu_\alpha = \frac{\alpha\mathcal{K}\mu_\alpha}{1+(\alpha-1)\mathcal{K}\mu_1}.$$

If we consider $\varphi \in \mathcal{H}^\infty$ such that $\|\varphi\|_\infty \le 1, \varphi(0) = 0$ and

$$(\mathcal{P}\mu_1)(z) = \operatorname{Re}\frac{1+\varphi(z)}{1-\varphi(z)}$$

for each $z \in \mathbb{D}$, then

$$\mathcal{K}\mu_1 = \frac{1}{1-\varphi}$$

and by (3.1.4)

$$\mathcal{P}\mu_\alpha = 2\operatorname{Re}\mathcal{K}\mu_\alpha - 1 = 2\operatorname{Re}\left[\frac{\frac{\alpha}{1-\varphi}}{1+\frac{\alpha-1}{1-\varphi}} - \frac{1}{2}\right] = \operatorname{Re}\frac{\alpha+\varphi}{\alpha-\varphi}.$$

Thus $\{\mu_\alpha\}_{\alpha\in\mathbb{T}} = \mathcal{M}_\varphi$ for some $\varphi \in \mathcal{H}^\infty$.

As we mentioned before, this result was obtained by Clark [C] for inner φ and for one-dimensional unitary perturbations of the model contraction $T_\varphi = \mathcal{S}P_\varphi$, where $\mathcal{S} : f \mapsto zf$ is a shift operator in $\mathcal{H}^2$ and P_φ is the orthogonal projector from $\mathcal{H}^2$ onto the model space $\varphi^*(\mathcal{H}^2) = \mathcal{H}^2 \ominus \varphi\mathcal{H}^2$.

Clark's result shows that if in addition to our conditions the operator $\mathcal{U}_1$ is singular, then $\mathcal{U}_1 - (*, \mathcal{U}_1^{-1}v)v$ is a C_0 completely nonunitary contraction with the characteristic function φ.

3.2 Krein spectral shift.

Let $u \in L^\infty(\mathbb{R})$, $||u||_\infty \le \pi/2$ and let $\mathcal{H}u$ be its Herglotz integral:

$$(\mathcal{H}u)(z) = \frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{z-t} + \frac{t}{t^2+1}\right)u\,dt.$$

Then the function $\exp(-i\mathcal{H}u)$ has positive real part on $\mathbb{C}_+$. Thus there exists a unique measure $\mu \in \mathcal{M}_+(\mathbb{R})$ such that

(3.2.1) $$\operatorname{Re}\left[\exp(-i\mathcal{H}u)\right] = \mathcal{P}\mu.$$

Definition. If u satisfies (3.2.1) for some $\mu \in \mathcal{M}_+(\mathbb{R})$ we will say that u is a shift-function of μ.

Since

$$||\mu|| = \mathrm{Re}\left[\exp(-i\mathcal{H}u(i))\right] = \cos\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x)dx}{1+x^2}\right), \tag{3.2.2}$$

we have that $||\mu|| \leq 1$. Conversely, let μ be a Borel positive measure on $\mathbb{R}$ such that $||\mu|| = c \leq 1$. Put $d = \sqrt{1-c^2}$. Then

$$\mathcal{H}\mu \pm id = \exp(-i\mathcal{H}u_\pm)$$

for some real functions u_+ and u_- on $\mathbb{R}$ such that $||u_\pm||_\infty \leq \pi/2$ and

$$\sin\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u_\pm(x)dx}{1+x^2}\right) = \pm d.$$

The functions u_+ and u_- are *all* shift-functions of μ. If $||\mu|| = 1$ then μ has a unique shift-function.

One can also see that if u is a shift function of μ then $-u$ is equal to the argument of the nontangential boundary values of

$$\mathcal{H}\mu - i\sin\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x)dx}{1+x^2}\right)$$

almost everywhere on $\mathbb{R}$.

When μ is a spectral measure of a self adjoint operator A, then u is sometimes called a Krein spectral shift of A.

First, let us point out the following important connections between u and μ.

Let φ be the function from $H^\infty(\mathbb{C}_+)$ such that $||\varphi||_\infty \leq 1$, $\varphi(0) \in \mathbb{R}$ and

$$\mathcal{H}\mu = \frac{1+\varphi}{1-\varphi}.$$

We will denote by Qu the conjugate Poisson integral of u:

$$Qu(x+iy) = \operatorname{Im} \mathcal{H}u(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{x-t}{(x-t)^2+y^2}+\frac{t}{t^2+1}\right)u(t)dt.$$

3.2.1. The absolutely continuous component of μ is concentrated on the set $\{|u| < \pi/2\}$. The measure is singular if and only if $|u| = \pi/2$ a. e.

Proof. As we discussed in Parts 1 and 2, $\mu_{a.c.}$ is concentrated on the set

$$\{|\varphi|<1\} = \{\operatorname{Re}\frac{1+\varphi}{1-\varphi} > 0\} = \{|u| = \left|\arg((1+\varphi)(1-\varphi)^{-1})\right| < \pi/2\}. \ \blacktriangle$$

3.2.2. For μ^s-a.e ξ

$$Qu(z) \xrightarrow[\substack{z\to\xi \\ \measuredangle}]{} +\infty.$$

Proof. Follows from the definition of u and Fatou Theorem. ▲

3.2.3. Consider $M_\varphi = \{\mu_\alpha\}$ $(\mu = \mu_1 \in \mathcal{M}_\varphi)$. Put

$$c = \sin\left(\frac{1}{\pi}\int_{\mathbb{R}}\frac{u(x)dx}{1+x^2}\right), \quad \gamma = \frac{ic-1}{ic+1}.$$

Then

$$Qu(z) \xrightarrow[\substack{z\to\xi \\ \measuredangle}]{} -\infty$$

for μ^s_γ-a.e. ξ.

Proof. Notice that the condition

$$Qu(z) \xrightarrow[\substack{z\to\xi \\ \measuredangle}]{} -\infty$$

is equivalent to

$$\mathcal{H}\mu - i\sin\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x)dx}{1+x^2}\right) = \mathcal{H}\mu - ic \xrightarrow[\substack{z\to\xi \\ \measuredangle}]{} 0$$

The last condition, in its turn, holds if and only if

$$\frac{1+\varphi}{1-\varphi} \underset{\substack{z \longrightarrow \xi \\ \not\succ}}{\longrightarrow} ic$$

i.e. if and only if

$$\varphi(z) \underset{\substack{z \longrightarrow \xi \\ \not\succ}}{\longrightarrow} \gamma.$$

By the definition of the measures μ_α, this is true for μ^s_γ-a.e. ξ. ▲

3.2.4. Suppose that μ and ν $(\neq \mu)$ belong to the same family M_φ. Then there exists a unique function u such that u is a shift-function of $c_1\mu$ and $-u$ is a shift function of $c_2\nu$ for some positive constants c_1 and c_2.

Definition. We will call such u the shift-function of the pair $(\mu; \nu)$.

Proof 1. Let $\mu = \mu_1 \in M_\varphi$, $\nu = \mu_\gamma \in M_\varphi$.

1)Existence. Put

$$ic = \frac{1+\gamma}{1-\gamma}.$$

Then the function

$$F = \frac{\mathcal{H}\mu - ic}{|\ ||\mu|| - ic|}$$

has positive real part. Thus we can consider u such that $||u||_\infty \leq \pi/2$ and

$$\exp(-i\mathcal{H}u) = F.$$

By the definition of a shift-function, u is a shift-function of $c_1\mu$ where $c_1 = 1/|\ ||\mu|| - ic|$. Also, since $\mathcal{H}\mu = \frac{1+\varphi}{1-\varphi}$ (recall that $\varphi(0)$ is real),

$$\operatorname{Re}\exp(-i\mathcal{H}(-u)) = \operatorname{Re} F^{-1} = \operatorname{Re}\left[|\ ||\mu|| - ic|\,(1 - \operatorname{Re}\gamma)\frac{\gamma+\varphi}{\gamma-\varphi} - |\ ||\mu|| - ic|\,1\operatorname{Im}\gamma\right]$$

$$= |\ ||\mu|| - ic|\,(1 - \operatorname{Re}\gamma)\mathcal{P}\nu.$$

2)Uniqueness. Let us prove it in the case when $||\mu|| = 1$ ($\varphi(0) = 0$) and $\gamma = -1$. Suppose u is a shift-function of $c_1\mu$ and $-u$ is a shift-function of $c_2\nu$ for some positive c_1 and c_2. Note that since $||\mu|| = 1$,

$$c_1 = \cos\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x)dx}{1+x^2}\right) \leq 1.$$

Put

$$a = \frac{1-c_1}{1+c_1}$$

and let $\lambda_a(z)$ be a Möbius transform of the unit disk $\mathbb{D}$ such that $\lambda_a(0) = a$. Then $\lambda_a(1) = 1$, $\lambda_a(-1) = -1$ and $\lambda_a'(1) = (1+a)(1-a)^{-1}$. As we discussed in Section 2.1, that means that

$$\mathcal{H}(c_1\mu) = \frac{1+\lambda_a\circ\varphi}{1-\lambda_a\circ\varphi}.$$

Thus

$$\frac{1-a}{1+a} = c_1 = \frac{1+\lambda_a\circ\varphi}{1-\lambda_a\circ\varphi}(0) = \frac{1+a}{1-a},$$

which implies $a = 0$ and $c_1 = 1$. Hence u is the shift-function of μ which is unique because $||\mu|| = 1$. ▲

Ideas of Proof 2. Let $\mu = \mu_1 \in M_\varphi$, $\nu = \mu_\gamma \in M_\varphi$.

Let u be the shift-function of $\mu/||\mu||$. Let $\mu/||\mu|| = \mu_1 \in \mathcal{M}_\phi$. Then $c\nu = \mu_{\gamma_0} \in \mathcal{M}_\phi$ for some $\gamma_0 \in \mathbb{T}$ and $c > 0$.

If u_c is a shift-function of $c\mu$ then

$$\operatorname{Re}\exp(-i\mathcal{H}u_c) = \operatorname{Re}\frac{1+\lambda_a\circ\phi}{1-\lambda_a\circ\phi}$$

where $\lambda_a : \mathbb{D} \to \mathbb{D}$ is a Möbius transform such that $\lambda_a(1) = 1$.

If u_c is the shift function of the pair $(\mu;\nu)$ then λ_a must satisfy $\lambda_a(1) = 1$ and $\lambda_a(\gamma_0) = \gamma$ which determines λ_a uniquely. ▲

3.2.5. Let u be the shift-function of a pair $(\mu;\nu)$. Then

$$Qu(z) = -\log|z-x| + \log c + o(1) \qquad (3.2.3)$$

for some $c > 0$ as $z \underset{\not>}{\rightarrow} x$ if and only if μ has a point mass c at x and

$$Qu(z) = \log|z - x| + \log c + o(1) \tag{3.2.4}$$

as $z \underset{\not>}{\rightarrow} x$ if and only if ν has a point mass c at x.

Condition (3.2.3) is equivalent to

$$\int_{-1}^{1} \frac{(\pi/2\,\mathrm{sign}(x-y) - u(y))dy}{x-y} < \infty \tag{3.2.5}$$

where $\mathrm{sign}\, t = t/|t|$ for $t \neq 0$.

Proof. It is well-known that μ has a point mass c at x if and only if

$$|\mathcal{H}\mu(z)| \sim \frac{c}{|z-x|}$$

as $z \underset{\not>}{\rightarrow} x$ which is equivalent to (3.2.3).

Since for $u = \pi/2\,\mathrm{sign}(x-y)$

$$Qu(z) = -\log|z-x| + O(1)$$

as $z \underset{\not>}{\rightarrow} x$ and $Qu(x+iy) \sim C\int_{-1}^{1} \frac{(x-t)u(t)dt}{(x-t)^2+y^2}$, (3.2.3) is equivalent to (3.2.5). ▲

Remark. Suppose μ and ν belong to the same family M_φ for some analytic function φ. If μ and ν are linear combinations of point masses at points $a_1, a_2, .., a_n$ and $b_1, b_2, .., b_n$ respectively, $a_1 < b_1 < a_2 < b_2 < ... < a_n < b_n$, then the shift-function u of the pair $(\mu;\nu)$ depends only on the sequences $\{a_n\}$ and $\{b_n\}$: $u = \pi/2$ on $(-\infty; a_1) \cup (b_1; a_2) \cup ... \cup (b_{n-1}; a_n) \cup (b_n; \infty)$ and $u = -\pi/2$ elsewhere.

That means that for each pair of measures (μ', ν') such that $\mu' \sim \mu$, $\nu' \sim \nu$ and μ', $\nu' \in M_\phi$ for some analytic function ϕ, we have $\mu = \mu'$, $\nu = \nu'$ and $\varphi = \phi$. However in general there may exist two equivalent pairs belonging to different families.

Example 3.2.6. Let $\mathcal{A} = \{a_n\}$ be some enumeration of the set

$$\{x \in (0;1) | x = \frac{2k+1}{2^{2n}}; k, n \in \mathbb{N}\}$$

and $\mathcal{B} = \{b_n\}$ be some enumeration of the set

$$\{x \in (0;1) | x = \frac{2k+1}{2^{2n+1}}; k, n \in \mathbb{N}\}.$$

Then Alexandrov's Theorem (see Section 3.4, Theorem 3.4.3′) implies that there exist sequences of positive real numbers $\{\alpha_n\}$ and $\{\beta_n\}$ such that the measures $\mu = \sum \alpha_n \delta_{a_n}$ and $\nu = \sum \beta_n \delta_{b_n}$ belong to the same family M_φ. Let u be the shift-function of the pair $(\mu; \nu)$. For every $n \in \mathbb{N}$ define $v_n(x) = u(2^{2n}x - 2^{2n-1})$ on $[0;1]$ and $v_n(x) = \pi/2$ elsewhere on $\mathbb{R}$; define $w_n(x) = -u(2^{2n}(1-x) - 2^{2n-1})$ on $[0;1]$ and $w_n(x) = \pi/2$ elsewhere on $\mathbb{R}$.

Since $\mathcal{A}$ and $\mathcal{B}$ are symmetric with respect to the point $1/2$ and $1/2 - x \in \mathcal{A}(\in \mathcal{B})$ implies $1/2 - x/4 \in \mathcal{A}(\in \mathcal{B})$, all the functions v_i and w_i are the shift-functions of equivalent pairs of measures. Suppose that

$$v_1 \equiv v_2 \equiv v_3 \equiv \ldots \equiv w_1 \equiv w_2 \equiv \ldots$$

Denote

$$c = \int_0^1 v_1 dx = \int_0^1 v_2 dx = \ldots = \int_0^1 w_1 dx = \int_0^1 w_2 dx = \ldots$$

Then by the construction of v_i and w_i for any $n \in \mathbb{N}$

$$2^{2n} \int_{1/2-1/2^{2n}}^{1/2} u dx = c$$

and

$$2^{2n} \int_{1/2}^{1/2+1/2^{2n}} u dx = -c.$$

Since $1/2 \in \mathcal{B}$, by formula (3.2.5) $c = -\pi/2$. Since $|u| \leq \pi/2$, this implies that $u \equiv -\pi/2$ a. e. on $[1/4; 1/2]$ which is impossible. That means that the functions v_i and w_i can not be all the same. (In fact there must be infinitely many different functions among v_i and among w_i.)

The proofs of Theorems 3.4.3′ and 3.4.4 give some further ideas on how to construct different shift-functions for equivalent pairs of measures.

3.3 Existence of an absolutely continuous perturbation

Let A be a cyclic self adjoint operator, μ be its spectral measure. In this section we will give some sufficient conditions for A to have an absolutely continuous self-adjoint rank one perturbation.

We say that operator B is absolutely continuous on $E \subset \mathbb{R}$ if its singular spectrum $\sigma_s(B)$ does not intersect E.

Definition. Let E be a closed subset of $\mathbb{R}$. We will say that point x is *deep* inside E if and only if there exists a positive function $\psi(t)$ on $\mathbb{R}_+$ monotonically decreasing to 0 as $t \to 0+$ such that

$$\int_0^1 \frac{\psi(t)}{t^2} dt < \infty$$

and

$$\text{dist}(y, E) \leq \psi(|x - y|) \tag{3.3.1}$$

for any $y \in \mathbb{R}$.

We will prove the following

Theorem 3.3.1. *Let A be a cyclic self-adjoint operator, μ be its spectral measure. Suppose that μ^s-a. e. x is deep in $\sigma_{a.c.}(A) \cap [0;1]$. Then there exists a cyclic vector ϕ of A such that $A + (*, \phi)\phi$ is absolutely continuous on $[0;1]$.*

Before we prove Theorem 3.3.1 let us consider some cases when such an absolutely continuous perturbation *does not* exist.

Example 3.3.2. If $\sigma_s(A) \not\subset \sigma_{a.c.}(A)$ then A has an absolutely continuous rank one perturbation only in some degenerated cases:

Claim. *Suppose A has an absolutely continuous rank one perturbation and I is an open interval such that $\sigma_{a.c.}(A) \cap I = \emptyset$. Then $\sigma_s(A) \cap I$ contains at most 1 point.*

Proof. Let ϕ be a cyclic vector of A such that $A + (*;\phi)\phi$ is absolutely continuous, μ be the spectral measure of ϕ for A. If I contains points from the essential spectrum of A then an absolutely continuous perturbation does not exist because of the stability of the absolutely continuous and essential spectrum. Thus I can only contain isolated eigenvalues of A. Let $a, b \in I, a \neq b$ be eigenvalues of A, $(a;b) \cap \sigma(A) = \emptyset$. Then μ has point masses at a and b. Hence F_0 (defined as in (3.1.2)) is analytic and takes all real values on $(a;b)$. Thus by (3.1.1) each μ_λ has a point mass between a and b and we have a contradiction.

Example 3.3.3. Condition $\sigma_s(A) \subset \sigma_{a.c.}(A)$ does not imply the existence of an absolutely continuous perturbation. The easiest example is $\mu^{a.c.} = \chi_E m$, where $E = [0;1] \setminus [1/3;2/3]$, $\mu^s = \delta_{1/3} + \delta_{2/3}$.

It may seem however that if $\sigma_s(A) \subset \sigma_{a.c.}(A)$ and for any two eigenvalues $a, b \in \mathbb{R}$ of A $\sigma(A)_{a.c.} \cap (a;b) \neq \emptyset$ then an absolutely continuous perturbation must exist. The following example shows that it is not true.

If $\mu \in M(\mathbb{R})$ we will denote by A_μ the operator of multiplication by the independent variable in $L^2(\mu)$:

$$A_\mu : f \mapsto xf.$$

Example 3.3.4. Let us construct the standard Cantor null set C on the unit interval:

Let

$$C_0 = I_0^0 = [0;1],\ C_1 = I_1^1 \cup I_2^1, \ldots,\ C_n = I_1^n \cup \ldots \cup I_{2^n}^n, \ldots$$

where

$$I_{2k}^n \cup I_{2k-1}^n = I_k^{n-1} \setminus \Delta_k^{n-1}$$

and $\Delta_k^n = (a_k^n; b_k^n)$ is the open interval placed in the center of the interval I_k^n such that $m(\Delta_k^n) = 1/3m(I_k^n)$. Put $C = \bigcap_{n=0}^\infty C_n$.

Let

$$\mu^s = \sum_{n,k=1}^{\infty} 1/2^{n+k} \delta_{b_k^n}.$$

Put

$$d_k^n = \frac{2}{3}a_k^n + \frac{1}{3}b_k^n, \; E = \bigcup (a_k^n; d_k^n).$$

Consider $\mu = \chi_E m + \mu^s$, $A = A_\mu$. Suppose A has an absolutely continuous rank one perturbation $A' = A + (*;\phi)\phi$. Denote by ν and ν' the spectral measures of ϕ for A and A' respectively. Let v be the shift function of the pair $(\nu;\nu')$. Since ν' is absolutely continuous, v must be constant on each $(d_k^n; b_k^n)$. Since ν has a point mass at each b_k^n, $v = \pi/2$ on each $(d_k^n; b_k^n)$. But then condition (3.2.5) does not hold at any point of C and we have a contradiction.

Our next example shows that even when $\sigma_s(A_\mu) \subset \sigma_{a.c.}(A_\mu)$ and μ^s is continuous, an absolutely continuous rank one perturbation may not exist.

Example 3.3.5. Let C, a_k^n, b_k^n, d_k^n be as in the previous example. Put

$$c_k^n = (a_k^n + b_k^n)/2, \; e_k^n = \frac{1}{3}a_k^n + \frac{2}{3}b_k^n.$$

Define $u = -\pi/2$ on $\bigcup (a_k^n; c_k^n)$ and $u = \pi/2$ elsewhere. Let μ^s be the measure for which u is a shift-function. Then μ^s is concentrated on C. Since condition (3.2.5) does not hold at any point of C, μ^s is continuous. Put

$$E = \bigcup \left((a_k^n; b_k^n) \setminus (d_k^n; e_k^n)\right).$$

Consider

$$\mu = \mu^s + \chi_E m, \; A = A_\mu.$$

Suppose A has an absolutely continuous rank one perturbation $A' = A + (*;\phi)\phi$. Denote by ν and ν' the spectral measures of ϕ for A and A' respectively. Let v be the shift function of the pair $(\nu;\nu')$. Since ν' is absolutely continuous, v must be constant ($-\pi/2$ or $\pi/2$) on each interval $(d_k^n; e_k^n)$. Thus on each $(d_k^n; e_k^n)$ function $u - v$ is either $-\pi$ on $(d_k^n; c_k^n)$ and 0 on $(c_k^n; e_k^n)$ or π on $(c_k^n; e_k^n)$ and 0 on $(d_k^n; c_k^n)$. Hence its Poisson integral $\mathcal{P}(u - v)$ does not have a nontangential boundary limit at *any* point of C. But $\mu \sim \nu$ and Theorem 2.3.8 implies that the ratio

$$\frac{\mathcal{H}\mu}{\mathcal{H}\nu} = \exp(i\mathcal{H}(u - v))$$

must have nonzero nontangential limits μ-a. e. So we have a contradiction.

Remark. Suppose that for μ^s-a. e. x there exists c, $0 < c < 1$ such that for any $y \in \mathbb{R}$ we have

$$\operatorname{dist}(y, \sigma_{a.c.}(A)) \leq c|x - y|.$$

Then A_μ still may not have an absolutely continuous rank one perturbation. Indeed, in Example 3.3.5 for μ^s-a. e. x and for any $y \in \mathbb{R}$ we have

$$\operatorname{dist}(y, \sigma_{a.c.}(A)) \leq \frac{2}{3}|x - y|.$$

To prove Theorem 3.3.1 we will need the following fact proven in [C-P].

Lemma 3.3.2. *Let μ be a singular Borel measure on $[0;1]$ such that $m(\operatorname{supp}\mu) = 0$. Then there exists $f \in L^1(\mu)$ such that*

$$\frac{d(f\mu)}{dm}(x) = \infty$$

for every $x \in \operatorname{supp}\mu$.

The following Lemma shows that if A is absolutely continuous then there exists its cyclic vector ϕ and $C > 0$ such that $A + \lambda(*, \phi)\phi$ is absolutely continuous for all $\lambda \in (-C; C)$.

Lemma 3.3.3. *For each set $E \subset \mathbb{R}$ there exists $f \in L^\infty$ such that $f \geq 0, f > 0 = E$ and $|\mathcal{H}f| < 1$ in $\mathbb{C}_+$.*

Proof. Without loss of generality we can assume that $m(E) > 0$ and $m(\mathbb{R} \setminus E) > 0$. Put

$$c = \frac{1}{\pi} \int_E \frac{dx}{1 + x^2}.$$

Define

$$g : \mathbb{T} \to \mathbb{R}$$

as

$$g(e^{i\xi}) = \frac{\operatorname{dist}(\xi; \mathbb{R} \setminus (0; c))}{2\pi}$$

for any $\xi \in (0; 2\pi]$. Then $|\mathcal{H}g(z)| < 1$ for any $z \in \mathbb{D}$ (where Hg denotes the Herglotz integral of g in the unit disk).

Consider $\phi \in H^\infty(\mathbb{D})$ and $\varphi \in H^\infty(\mathbb{C}_+)$ such that

$$\frac{1+\varphi}{1-\varphi} = \mathcal{H}g, \quad \frac{1+\phi}{1-\phi} = \exp(i\mathcal{H}u),$$

where $u = \pi\chi_E - \pi/2$. Define

$$F = \frac{1+\varphi\circ\phi}{1-\varphi\circ\phi}.$$

One can show that then $F = \mathcal{H}\mu$ for some absolutely continuous measure $\mu \in \mathcal{M}_+(\mathbb{R})$. Also $|F| < 1$ on $\mathbb{C}_+$ and $\{\operatorname{Re} F > 0\} = E$ up to a set of Lebesgue measure 0. Thus f can be obtained from the equation $fm = \mu$. ▲

Definition. Let f be a function defined in $\mathbb{C}_+$. We will say that f is less than C (greater than C) at some point $x \in \mathbb{R}$ if

$$\limsup_{\substack{z\to x\\ \not\to}} f(z) < C \qquad (\liminf_{\substack{z\to x\\ \not\to}} f(z) > -C).$$

Lemma 3.3.4. *Consider $E \subset (0; 1)$. Suppose for some $x \in (0; 1)$ and for any $\epsilon > 0$*

$$\frac{m((x-\epsilon; x+\epsilon)\cap E)}{2\epsilon} < \psi(\epsilon)$$

where ψ is a monotonic function on $\mathbb{R}_+$ such that

$$\int_0^1 \frac{\psi(t)}{t}\,dt < \infty.$$

Then

$$\int_E \frac{dy}{|x-y|} < \infty$$

Proof. Put $E_n = E \cap (x - 1/2^n; x + 1/2^n)$. Then

$$\int_E \frac{dy}{|x-y|} \leq \sum 2^{n+1} m(E_n \setminus E_{n+1}) \leq$$

$$\sum 2^{n+1} \frac{\psi(1/2^n)}{2^{n-1}} \leq 2 \int_0^1 \frac{\psi(t)}{t} dt. \blacktriangle$$

We will also need the following corollary of John-Nirenberg Theorem.

Lemma 3.3.5. *Consider $u \in L^\infty(\mathbb{R})$, $||u||_\infty \leq \pi/2$. Then for any $t > 0$*

$$m(\ |Qu| > \ln t\ ) < \frac{C \cos\left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{u(x)dx}{1+x^2}\right)}{t}$$

for some absolute constant C.

Proof. Since u is a shift-function of some measure μ such that

$$||\mu|| = \cos\left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{u(x)dx}{1+x^2}\right),$$

the statement follows from Corollary 1.2.2. $\blacktriangle$

Lemma 3.3.6. *Let $u \in L^\infty$, $||u||_\infty \leq \pi/2$. Let v be the shift-function of a pair of measures $(\eta; \sigma)$. Let $I \subset \mathbb{R}$ be an open interval. Suppose that $u > v$ and $m(\{u - v \neq 0\}) < \epsilon$. Then there exists some absolute constant $C > 0$ such that:*

1) If $Qu > C_1$ for some $C_1 \in \mathbb{R}$ on I, then $\sigma^s(I) \leq Ce^{-C_1}\epsilon$.

2) If $Qu < C_2$ for some $C_2 \in \mathbb{R}$ on I, then $\sigma^s(I) \leq Ce^{C_2}\epsilon$.

Proof.

1) Since $0 < u - v < \pi$ by Lemma 3.3.5

$$m(\{Q(v-u) < -\ln t\}) < \frac{C\epsilon}{t}.$$

Thus

$$m(\{Qv < -\ln t\} \cap I) \leq m(\{Q(v-u) < -\ln t + C_1\} \leq C\frac{e^{-C_1}\epsilon}{t}.$$

Hence

$$m(\{|H\sigma| > t\} \cap I) \leq C\frac{e^{-C_1}\epsilon}{t}.$$

By the Remark after the proof of Theorem 1.2.1 , that means that $||\sigma^s \chi_I|| \leq C\exp(-C_1)\epsilon$.

Part 2) can be proven in the same way. ▲

Definition. Let $E \subset (0;1)$. We will denote by $\operatorname{essClos} E$ the closure of the set of all x such that

$$\frac{m((x-\epsilon; x+\epsilon) \cap E)}{2\epsilon} \to 1$$

as $\epsilon \to 0$.

Remark. Note that if $\mu = hm + \mu^s$ is a spectral measure of A then $\sigma_{a.c.}(A) = \operatorname{essClos}\{h > 0\}$.

Lemma 3.3.7. *Let $E, F \subset (0;1)$, $m(F) = 0$ and F is a set of the type F_σ. Then there exists $G \subset E$ such that $\operatorname{essClos} G = \operatorname{essClos} E$ and*

$$\int_G \frac{dy}{|x-y|} < \infty$$

for any $x \in F$.

Proof. The statement is easy to prove when F is closed. In the general case, $F = \bigcup F_n$ where $F_1, F_2, ..$ are closed disjoint sets. For each F_n we can obtain $G_n \subset E$ such that $\operatorname{essClos} G_n = \operatorname{essClos} E$ and

$$\int_{G_n} \frac{dy}{|x-y|} < \infty$$

for any $x \in F_n$. After that we can consider open sets $H_n \supset F_n$ such that $m(H_n) < 1/2^n$ and put $G = E \setminus \bigcup [I_n \cap H_n]$ where $I_n = (0;1) \setminus G_n$. ▲

Proof of Theorem 3.3.1. Let $\mu = hm + \mu^s$. Without loss of generality we can assume that $\operatorname{supp} \mu \subset (0;1)$.

We can always find disjoint closed sets

$$F^1, F^2, ..., F^n, ...$$

such that $m(F^i) = 0$, $\mu^s(\bigcup F^i) = ||\mu^s||$ and for each n condition (3.3.1) holds for μ^s-a. e. $x \in F^n$ uniformly, i.e. there exists a positive function $\psi^n(t)$ on $\mathbb{R}_+$ monotonically decreasing to 0 as $t \to 0+$ such that

$$\int_0^1 \frac{\psi^n(t)}{t^2} dt < \infty$$

and for μ^s-a. e. $x \in F^n$ and any $y \in \mathbb{R}$

$$\text{dist}(y, E) \leq \psi^n(|x - y|).$$

Denote by μ^s_n the restriction of μ^s on F^n, by $I^n_1 = (a^n_1; b^n_1), I^n_2 = (a^n_2; b^n_2), ...,$ the disjoint intervals constituting the complement of $\bigcup_1^n F^k$. By Lemma 3.3.2, without loss of generality, we can assume that

$$\frac{d\mu^s}{dm} = \infty \tag{3.3.2}$$

everywhere on $\bigcup F^n$.

Let us first prove the theorem in the case when

$$\int_{\{h>0\}} \frac{1}{|x - y|} dy < \infty \tag{3.3.3}$$

for any $x \in \bigcup F^n$.

Step 1.

Consider a shift-function u^1 of μ^s_1. On each interval $I^1_n = (a^1_n; b^1_n)$ function u is either constant (equal to $\pi/2$ or $-\pi/2$) or $u^1 = -\pi/2$ on $(a^1_n; c^1_n)$ and $u^1 = \pi/2$ on $(c^1_n; b^1_n)$ for some $c^1_n \in I^1_n$. Let $\{I^1_{n_k}\}_{k=1}^{\infty}$ be the sequence of intervals on which u^1 has "jumps." Let $\{c^1_{n_k}\}_k = 1^{\infty}, c^1_{n_k} \in I^1_{n_k}$ be the sequence of points in which u^1 has "jumps."

By Lemma 3.3.3 there exists $f \in L^\infty$ such that $1 \geq f \geq 0$, $\{f > 0\} = \{h > 0\}$ and $\mathcal{H}f < 1$ in $\mathbb{C}_+$.

In each interval $I^1_{n_k}$ let us choose a point $d^1_{n_k}$ such that $d^1_{n_k} > c_{n_k}$, $d^1_{n_k}$ is a Lebesgue point of f, $f(d^1_{n_k}) > 0$ and

$$|c^1_{n_k} - d^1_{n_k}| < 2\psi^1\left(\min(\operatorname{dist}(c^1_{n_k}; F^1); \operatorname{dist}(d^1_{n_k}; F^1))\right). \tag{3.3.4}$$

Now let us define function v^1 on $\mathbb{R}$ as following: $v^1 = u^1$ on $\mathbb{R} \setminus \bigcup [c^1_{n_k}; d^1_{n_k}]$, $v^1 = -u^1$ on $\bigcup [c^1_{n_k}; d^1_{n_k}]$ (we "shift" each "jump" of u^1 from $c^1_{n_k}$ to $d^1_{n_k}$).

Now let us define function $w^1 = v^1 + g^1 f$, where $g^1(x) = -\operatorname{sign} v^1(x)$ on $(0; 1)$ and $g^1 = 0$ elsewhere.

On each I^1_n where v^1 is constant $|\mathcal{H}w^1(x)| < \infty$ for any x by the choice of f. On the intervals $I^1_{n_k}$ the integral $\mathcal{H}w^1(x)$ is finite for any x in $I^1_{n_k} \setminus \{d^1_{n_k}\}$. Since each point $d^1_{n_k}$ is a Lebesgue point of f and $0 < f(d^1_{n_k}) \leq 1$ we have

$$Qw^1(d^1_{n_k} + iy) \sim (1 - 2f(d^1_{n_k})/\pi) \log y + o(\log y) \tag{3.3.5}$$

as $y \to 0$.

Since $|w^1| \leq \pi/2$ we can consider a measure ν_1 such that w^1 is its shift-function. Since $\{|w^1| < \pi/2\} = \{|u^1| < \pi/2\}$ by property 3.2.1 of shift-functions $\nu_1^{a.c.}$ is equivalent to $\mu^{a.c.}$. Since $Qw^1 < +\infty$ on $[0; 1] \setminus \operatorname{supp} \mu_1^s$, by property 3.2.2 of shift-functions $\operatorname{supp} \nu_1^s \subset \operatorname{supp} \mu_1^s$. Also let us notice that for each $x \in F^1$

$$\left(\mathcal{H}(u^1 - w^1)\right)(x) = \mathcal{H}(\sum_k \pi \operatorname{sign}(d^1_{n_k} - c^1_{n_k}) \chi_{[c^1_{n_k}; d^1_{n_k}]}) + \mathcal{H}(fg^1) \leq$$

$$C\left(\int_{\bigcup [c^1_{n_k}; d^1_{n_k}]} \frac{dy}{|x-y|} + \int_{\{h>0\}} \frac{||f||_\infty dy}{|x-y|}\right).$$

The first integral is finite for any $x \in \operatorname{supp} \mu_1^s$ by Lemma 3.3.4 because

$$\frac{m\left((x-t; x+t) \cap \bigcup [c^1_{n_k}; d^1_{n_k}]\right)}{2t} \leq \frac{\psi^1(t)}{t}$$

(here we use the fact that ψ^1 is monotonic). The second integral is finite by (3.3.3). Hence $\mathcal{H}\mu_1/\mathcal{H}\nu_1$ is finite on $\operatorname{supp}\mu_1^s$. By Theorem 2.3.8, together with the fact that $\operatorname{supp}\nu_1^s \subset \operatorname{supp}\mu_1^s$, that implies that ν_1^s is equivalent to μ_1^s. So ν_1 is equivalent to μ_1.

Let η_1 be a measure such that w^1 is the shift function of the pair $(\nu_1;\eta_1)$. Let us show that η_1 is absolutely continuous on $[0;1]$.

By our assumption $\frac{d\mu_1^s}{dm} = \infty$ on $\operatorname{supp}\mu_1^s$, so by Fatou Theorem $Qu = +\infty$ on F^1. Since $|\mathcal{H}(u^1 - w^1)| < \infty$ on F^1, $Qw^1 > -\infty$ on F. Thus by properties 3.2.3. and 3.2.4 of shift functions η_1 is absolutely continuous on F^1. Also $Qw^1 > -\infty$ everywhere on $[0;1]\setminus F$ except the points $d^1_{n_k}$. But by (3.2.4) $\eta_1(\{d^1_{n_k}\}) = 0$ because

$$(1 - 2f(d^1_{n_k})/\pi)\log y + o(\log y) \neq \log y + O(1)$$

as $y \to 0$. Thus η_1 is absolutely continuous on $[0;1]$.

Step 2.

Without loss of generality we can assume that $\mu^s(\{d^1_{n_k}\}) = 0$ and that $F^n \cap \{d^1_{n_k}\} = \emptyset$ for any n. Then inside each I^1_n there exist open intervals Δ^2_n and Σ^2_n such that $\operatorname{Clos}\Delta^2_n \subset (a^1_n; d^1_n)$, $\operatorname{Clos}\Sigma^2_n \subset (d^1_n; b^1_n)$ and $F^2 \cap I^1_n \subset (\Delta^2_n \cup \Sigma^2_n)$. Let μ'_n and μ''_n be the restrictions of μ_2 onto Δ^2_n and Σ^2_n respectively. Then by (3.2.2) for any ϵ^2_n and ε^2_n there exist positive c'_n and c''_n such that for a shift-function u'_n of $c'_n\mu'_n$ we have $u'_n = -\pi/2$ on $\mathbb{R}\setminus\Delta^2_n$ and

$$1/\pi \int_{\mathbb{R}} \frac{u'(x)dx}{1+x^2} = -\pi/2 + \epsilon^2_n \tag{3.3.6}$$

and for a shift-function u''_n of $c''_n\mu''_n$ we have $u''_n = \pi/2$ on $\mathbb{R}\setminus\Sigma^2_n$ and

$$1/\pi \int_{\mathbb{R}} \frac{u''(x)dx}{1+x^2} = \pi/2 - \varepsilon^2_n. \tag{3.3.7}$$

On each I^1_n define u^2 as u'_n on Δ^2_n, as u''_n on Σ^2_n and as v^1 elsewhere. Now we can "shift" the "jumps" $c^2_{n_k} \in I^2_{n_k}$ of u^2 inside each complementary interval $I^2_{n_k}$ into the Lebesgue points $\{d^2_{n_k}\}$ of f to obtain the function v^2. If we choose $\{d^2_{n_k}\}$ in such a way that

$d^2_{n_k} > c^2_{n_k}$ if $c^2_{n_k} \in \Delta^2_n$ and $d^2_{n_k} < c^2_{n_k}$ if $c^2_{n_k} \in \Sigma^2_n$, then $\{u^2 - v^2 \neq 0\} \subset \{u'_n \neq -\pi/2\}$ on each Δ^2_n and $\{u^2 - v^2 \neq 0\} \subset \{u''_n \neq \pi/2\}$ on each Σ^2_n. Thus by (3.3.6) and (3.3.7) $m(\{u^2 - v^2 \neq 0\} \cap \Delta^2_n) < 2\epsilon^2_n$ and $m(\{u^2 - v^2 \neq 0\} \cap \Sigma^2_n) < 2\varepsilon^2_n$. After that we define $w^2 = v^2 + g^2 f$ where $g^2 = -\operatorname{sign} v^2$.

Then on each I^1_n $\{g^2 \neq g^1\} \subset (\Delta^2_n \cup \Sigma^2_n)$ and

$$m(\{g^2 \neq g^1\} \subset (\{u'_n \neq -\pi/2\} \cup \{u''_n \neq \pi/2\}) < 2\epsilon^2_n + 2\varepsilon^2_n.$$

On each I^1_n $\operatorname{supp}(w^1 - w^2) \subset (\Delta^2_n \cup \Sigma^2_n)$ and

$$m(\operatorname{supp}(w^1 - w^2) \cap (\Delta^2_n \cup \Sigma^2_n)) < 2\epsilon^2_n + 2\varepsilon^2_n.$$

If we choose the constants ϵ^2_n and ε^2_n small enough, we can provide

$$|\mathcal{H}(w^1 - w^2)| < 1/4$$

on F^1. Hence, using the same argument as in step 1, we can prove that w^2 is the shift-function of a pair $(\nu^2; \eta^2)$ where ν^2 is equivalent to $\mu^s_1 + \mu^s_2 + \mu^{a.c.}$ and η^2 is absolutely continuous.

Step k.

We repeat the construction from step 2 operating with w^{k-1} rather than w^1. We choose the constants ϵ^k_n and ε^k_n in the following way:

$$\epsilon^k_n < \frac{1}{2^{k+1}} \operatorname{dist}(\Delta^k_n; \mathbb{R} \setminus (a^k_n; d^k_n)),$$

$$\varepsilon^k_n < \frac{1}{2^{k+1}} \operatorname{dist}(\Sigma^k_n; \mathbb{R} \setminus (d^k_n; b^k_n)),$$

$$\epsilon^k_n < \frac{1}{2^{k+1}} \exp(\inf_{\Delta^k_n} Qw^{k-1})$$

and

$$\varepsilon^k_n < \frac{1}{2^{k+1}} \exp(\inf_{\Sigma^k_n} Qw^{k-1}).$$

Then in particular we will have

$$m(\mathrm{supp}(w^{k-1} - w^k)) < 1/2^k \tag{3.3.8}$$

and

$$|\mathcal{H}(w^{k-1} - w^k)| < \delta^k < 1/2^k \tag{3.3.9}$$

in some neighborhood of $\bigcup_1^{k-1} F^n$.

Conclusion.

After each step we obtain a shift-function w^k of a pair $(\nu^k; \eta^k)$ where measure ν^k is equivalent to $\sum_1^k \mu_n^s + \mu^{a.c.}$ and η^k is absolutely continuous. Formula (3.3.8) implies that the sequence w^k converges in measure to some function w. Denote by ν and η the measures such that w is the shift function of the pair $(\nu; \eta)$.

Let us first show that η is absolutely continuous.

Consider an interval $I_n^k = (a_n^k; b_n^k) \subset (0;1) \setminus \bigcup_{i=1}^k F^i$. Then by our construction $w^k < 0$ on $(a_n^k; d_n^k)$ and $w^k > 0$ on $(d_n^k; b_n^k)$ for some d_n^k, $a_n^k \le d_n^k \le b_n^k$. Conditions (3.3.2) and (3.3.4) imply that $Qv^k \to \infty$ as $x \to a_n^k+$ for any n, k. By (3.3.9) and the choice of f that means that $Qw^k \to \infty$ as $x \to a_n^k+$ for any n, k and that Qw^k is bounded from below on each $(a_n^k; c] \subset (a_n^k; d_n^k)$.

Let $\{e_l\}_{l=k+1}^{\infty}$ be the sequence of points from $[a_n^k; d_n^k)$ increasing to d_n^k such that $e_k = a_n^k$, $\eta(e_l) = 0$ and and $[\mathrm{supp}(w^k - w^l) \cap (a_n^k; d_n^k)] \subset (e_k; e_l)$ for any $l > k$. Then for any $l \ge k$

$$\inf_{(e_l; e_{l+1})} Qw^k > C_l > -\infty.$$

For any $l \ge k$ define the function ω^l as $\omega^l = w^l$ on I_n^k and $\omega^l = w$ elsewhere. Then by (3.3.9) $Q\omega^k > C_l + \frac{1}{2^{k-1}}$ on $(e_k; e_l)$. Since for any $l > k$ $\omega^l > \omega^k$ on $(e_k; e_l)$,

$$\inf_{(e_k; e_l))} Q\omega^l = D_l > C_l + \frac{1}{2^{k-1}} \tag{3.3.10}$$

on $(e_l; e_{l+1})$.

By the choice of the constants ϵ_j^i and ε_j^i we have

$$m(\mathrm{supp}(\omega^l - w)) < \exp(D_l)\frac{d_n^k - a_n^k}{2^l} \tag{3.3.11}$$

and

$$m(\mathrm{supp}(\omega^l - \omega^{l+1}) \cap (e_l; e_{l+1})) < \frac{d_n^k - e_{l+1}}{2^l}. \tag{3.3.12}$$

Since $\eta(e_l) = 0$, by (3.3.10), (3.3.11) and part 2) of Lemma 3.3.6

$$\eta^s((e_k; e_l) \leq \frac{C(d_n^k - a_n^k)}{2^l}$$

for all $l > k$. Thus

$$\eta^s((a_n^k; d_n^k)) \leq \frac{C(d_n^k - a_n^k)}{2^{k-1}}.$$

In the same way we can show, that

$$\eta^s((d_n^k; b_n^k)) \leq \frac{C(b_n^k - d_n^k)}{2^{k-1}}.$$

Also (3.3.12) imply that $Q(w - w^k) > -\infty$ at d_n^k. Thus $\eta(d_n^k) = \eta_k(d_n^k) = 0$ and we have that

$$\eta^s(I_n^k) \leq \frac{Cm(I_n^k)}{2^{k-1}}.$$

Since by our construction $\eta_k^s(\bigcup_{i=1}^k F^i) = 0$ and by (3.3.9) $|Q(w - w^k)| < 1$ at each point from $\bigcup_{i=1}^k F^i$, $\eta^s(\bigcup_{i=1}^k F^i) = 0$. Hence

$$||\eta^s|| \leq \frac{C}{2^{k-1}}$$

for any k.

In a similar way, using Lemma 3.3.6, we can prove that by the choice of ϵ_j^i and ε_j^i we have $\nu^s((a_n^k; b_n^k)) \leq C(b_n^k - a_n^k)/2^{k-1}$. Thus $\nu^s((0;1) \setminus \bigcup_1^\infty F^i) = 0$. Also (3.3.9) implies that

$$\nu^s \chi_{(\bigcup_1^\infty F^i)} \sim \mu^s.$$

Thus $\nu^s \sim \mu^s$.

Since for each k $\{|w^k| \neq \pi/2\} = \{h > 0\}$, (3.3.8) implies that $\{|w| \neq \pi/2\} = \{h > 0\}$ up to a set of measure 0. Thus $\nu^{a.c.} \sim \mu^{a.c.}$.

If $\{h > 0\}$ does not satisfy (3.3.3), then by Lemma 3.3.7 there exists $G \subset \{h > 0\}$ such that $\operatorname{essClos} G = \operatorname{essClos}\{h > 0\}$ and

$$\int_G \frac{1}{|x-y|} dx < \infty$$

for any $x \in \bigcup F^n$. Hence we can obtain the shift-function w' of a pair $(\nu'; \eta')$ where $\nu' \sim \mu' = \mu^s + \chi_G \mu^{a.c.}$ and η is absolutely continuous. Then $|w'| = \pi/2$ a.e. on $H = \{h > 0\} \backslash G$. Denote $H^+ = H \cap \{w' = \pi/2\}$ and $H^- = H \cup \{w' = -\pi/2\}$. By Lemma 3.3.3 there exist f^+ and f^- such that $0 \leq f^\pm \leq 1$, $\{f^\pm > 0\} = H^\pm$ and $|Qf^\pm| \leq 1$. Put $w = w' + f^- - f^+$. Let w be the shift function of a pair $(\nu; \eta)$. Since $\{|w| \neq \pi/2\} = \{h > 0\}$, $\nu^{a.c.} \sim \mu^{a.c.}$. Since $|Q(w - w')| < \infty$, $\mu^s \sim \nu^s$ and η is absolutely continuous. ▲

3.4. The problem of two spectra

In this section we will give a partial answer to the following question.

The problem of two spectra. Let μ and ν be two finite Borel measures on $\mathbb{R}$. When does there exist a cyclic self-adjoint operator A and its cyclic vector ϕ such that A and $A^\phi = A + (*, \phi)\phi$ are unitarily equivalent to A_μ and A_ν respectively?

Definition. If such A and ϕ exist, we will say that μ and ν are equivalent modulo rank one perturbation.

In terms of the families M_φ, we can say that μ and ν are equivalent modulo rank one perturbationif and only if they are equivalent to two measures from the same family.

In this section we will discuss this problem in the case of pure point measures μ and ν.

The first result in this direction is the following theorem, proved by Gelfand and Levitan.

Definition. We will say that two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ are well-mixed if

1)For any $i \neq j$ there exists k such that $a_i < b_k < a_j$ and there exists l such that $b_i < a_l < b_j$;

2)The supremum and the infimum of the set $\{a_1, b_1, a_2, b_2, ...\}$ do not belong to the same sequence.

Theorem 3.4.1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two disjoint sequences of real numbers, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$ where c does not belong to $\{a_n\}$ or $\{b_n\}$. Then there exists a cyclic self-adjoint operator A and its cyclic vector ϕ such that $\{a_n\}$ are eigenvalues of A and $\{b_n\}$ are eigenvalues of $A + (*, \phi)\phi$ if and only if the sequences $\{a_n\}$ and $\{b_n\}$ are well-mixed.*

Proof. To prove this result we can again use the notion of Krein spectral shift.

Define function u on $\mathbb{R}$ to be continuous and equal to $\pi/2$ by absolute value everywhere on $\mathbb{R} \setminus (\{a_n\} \cup \{b_n\})$. At the points $\{a_n\}$ u must "jump" from $\pi/2$ to $-\pi/2$; at the points $\{b_n\}$ u must "jump" from $-\pi/2$ to $\pi/2$. Then we can put $A = A_\mu$ where u is a shift-function of μ. By properties 3.2.3 and 3.2.4 of shift-functions, the operators A and $A^c = A + (*, c)c$ will satisfy the condition of the theorem for some real constant c. ▲

Remark. In the conditions of Theorem 3.4.1 denote $\mu = \sum \frac{1}{2^n}\delta_{a_n}$ and $\nu = \sum \frac{1}{2^n}\delta_{b_n}$. Note that Theorem 3.4.1 provides us with operators whose spectral measures are only *absolutely continuous with respect to* μ and ν. Therefore it does not imply that μ and ν are equivalent modulo rank one perturbation. Moreover, the following example shows that such μ and ν *are not* generally equivalent modulo rank one perturbation.

Example 3.4.2. *Put $a_n = (-1)^n/2^n$ for $n = 1, 2, 3, ...$, $b_1 = -1$, $b_n = a_{n-1} + (-1)^n/4^n$ for $n = 2, 3, ...$ Then the sequences $\{a_n\}$ and $\{b_n\}$ are well-mixed.*

Let μ and ν be defined as in the last remark. Suppose that μ and ν are equivalent modulo rank one perturbation. Let u be a shift-function of the pair $(\mu'; \nu')$ where $\mu \sim \mu'$

and $\nu \sim \nu'$. Such u must be continuous on $\mathbb{R}\backslash(\{a_n\}\cup\{b_n\})$; u must jump from $\pi/2$ to $-\pi/2$ at any a_n and from $-\pi/2$ to $\pi/2$ at any b_n. Thus $u = \pi/2$ on $(1/2^{2n} - 1/4^{2n+1}; 1/2^{2n})$ and on $(-1/2^{2n-1} + 1/4^{2n}; -1/2^{2n+1})$ for $n = 1, 2, \ldots$ and $u = -\pi/2$ on the rest of $\mathbb{R}$.

But then $Qu(iy) = -\log y + \log c + o(1)$ for some $c > 0$ as $y \to 0$. By property 3.2.4 of shift-functions this implies $\mu(0) > 0$ and we have a contradiction.

If the sequences $\{a_n\}$ and $\{b_n\}$ are finite disjoint and well-mixed then μ and ν obviously are equivalent modulo rank one perturbation. More interesting example is provided by the following theorem.

We will denote by C_E the set of all cluster points of a set E.

Theorem 3.4.3 (Aleksandrov, private communications). *Let $\mathcal{A} = \{a_n\}$ and $\mathcal{B} = \{b_n\}$ be two disjoint sequences on the unit circle $\mathbb{T}$. Suppose that $C_{\mathcal{A}} = C_{\mathcal{B}} = \mathbb{T}$. Then there exist sequences of positive real numbers $\{\alpha_n\}$ and $\{\beta_n\}$ such that the measures $\sum \alpha_n \delta_{a_n}$ and $\sum \beta_n \delta_{b_n}$ belong to the same family M_φ*

Using Aleksandrov's ideas we can prove the following statement, which we will use in the proof of Theorem 3.4.4 below

Let $0 < \theta_1 < \theta_2$, $\theta_2 - \theta_1 < 2\pi$. We will denote by $I(e^{i\theta_1}; e^{i\theta_2})$ the open arc $\{e^{i\theta} | \theta_1 < \theta < \theta_2\}$.

Theorem 3.4.3$'$. *Let $I = I(e^{i\varphi}; e^{i\phi})$, $0 \leq \phi < \varphi < 2\pi$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of points from I, both of them dense in I. Then for any $0 < \epsilon < |\phi - \varphi|$ there exist sequences of positive real numbers $\{\alpha_n\}$ and $\{\beta_n\}$ such that the measures $\mu = \sum \alpha_n \delta_{a_n}$ and $\nu = \sum \beta_n \delta_{b_n}$ belong to the same family M_φ and the shift-function u of the pair $(\mu; \nu)$ satisfies*

$$m(I(e^{i\phi}; e^{i(\phi+\epsilon)}) \cap \{u \neq -\pi/2\}) < \epsilon^2 \tag{3.4.1}$$

$$m(I(e^{i(\varphi-\epsilon)}; e^{i\varphi}) \cap \{u \neq -\pi/2\}) < \epsilon^2 \tag{3.4.2}$$

and $u = -\pi/2$ on $\mathbb{T} \setminus I$.

Proof. Put $a_n = e^{i\phi_n}, b_n = e^{i\varphi_n}$ where $0 \le \phi_k < 2\pi$ and $0 \le \varphi_k < 2\pi$ for any k. For any two points $a, b \in I$ we will denote by $J(a; b)$ the subarc of I with the ends a and b.

Let us reenumerate $\{a_n\}$ and $\{b_m\}$ in the following way.

Step 1.

Put $n_1 = 1$. Choose m_1 in such a way that

$$\varphi_{n_1} > \phi_{n_1} \tag{3.4.3}$$

and

$$|\varphi_{n_1} - \phi_{n_1}| < \frac{(\min(|\phi_{n_1} - \phi|; |\phi_{n_1} - \varphi|))^2}{2}.$$

Put $m_2 = 1$ if $m_1 \neq 1$ or $m_2 = 2$ if $m_1 = 1$. Choose n_2 in such a way that

1)$J(a_{n_2}; b_{m_2})$ contains neither a_{n_1} nor b_{m_1} and

$$|\varphi_{m_2} - \phi n_2| < \frac{(\min(|\varphi_{m_2} - \phi|; |\varphi_{m_2} - \varphi|))^2}{4},$$

2) the function $F_2(z) = \frac{z - a_{n_2}}{z - b_{m_2}}$ satisfies $|F_2 - 1| < 1/4$ at a_{n_1} and b_{m_1},

3) the sequences ϕ_{n_1}, ϕ_{n_2} and $\varphi_{m_1}, \varphi_{m_2}$ are well mixed.

Step k.

Put $n_{2k-1} = \min\{i | i \neq n_1, n_2, ..., n_{2k-2}\}$. Choose m_{2k-1} in such a way that

1')$J(a_{n_{2k-1}}; b_{m_{2k-1}})$ does contain any elements of the set

$$\{a_{n_1}, a_{n_2}, ..., a_{n_{2k-2}}, b_{n_1}, b_{n_2}, ..., b_{n_{2k-2}}\}$$

and

$$|\varphi_{m_{2k-1}} - \phi n_{2k-1}| < \frac{(\min(|\phi_{n_{2k-1}} - \phi|; |\phi_{n_{2k-1}} - \varphi|))^2}{2^{2k-1}},$$

2') the function $F_{2k-1}(z) = \frac{z-a_{n_{2k-1}}}{z-b_{m_{2k-1}}}$ satisfies $|F_{2k-1}-1| < 1/2^{2k-1}$ at each point from

$$\{a_{n_1}, a_{n_2}, ..., a_{n_{2k-2}}, b_{n_1}, b_{n_2}, ..., b_{n_{2k-2}}\},$$

3') the sequences $\phi_{n_1}, ..., \phi_{n_{2k-1}}$ and $\varphi_{m_1}, ..., \varphi_{m_{2k-1}}$ are well mixed.

Put $m_{2k} = \min\{i | i \neq m_1, m_2, ..., m_{2k-1}\}$. Choose n_{2k} in such a way that

1")$J(a_{n_{2k}}; b_{m_{2k}})$ does not intersect

$$\{a_{n_1}, a_{n_2}, ..., a_{n_{2k-1}}, b_{n_1}, b_{n_2}, ..., b_{n_{2k-1}}\},$$

and

$$|\varphi_{m_{2k}} - \phi n_{2k}| < \frac{(\min(|\varphi_{m_{2k}} - \phi|; |\varphi_{m_{2k}} - \varphi|))^2}{2^{2k}},$$

2") the function $F_{2k}(z) = \frac{z-a_{n_{2k}}}{z-b_{m_{2k}}}$ satisfies $|F_{2k}-1| < 1/2^{2k}$ on

$$\{a_{n_1}, a_{n_2}, ..., a_{n_{2k-1}}, b_{n_1}, b_{n_2}, ..., b_{n_{2k-1}}\},$$

3") the sequences $\phi_{n_1}, ..., \phi_{n_{2k}}$ and $\varphi_{m_1}, ..., \varphi_{m_{2k}}$ are well mixed.

Now for each l put $\theta_k = \frac{1}{2}\sum_{i=1}^{k}(\varphi_{m_i} - \phi_{n_i}) + \frac{\pi}{2}$. Consider functions $F_k = e^{i\theta_k}\prod_{i=1}^{k}\frac{z-a_{n_i}}{z-b_{m_i}}$ and $G_k = F_k^{-1}$. Since the sequences $\phi_{n_1}, ..., \phi_{n_k}$ and $\varphi_{m_1}, ..., \varphi_{m_k}$ are well mixed, for any k functions F_k and G_k have positive real parts on $\mathbb{D}$. Thus there exist positive measures μ_k and ν_k such that $\mathcal{H}\mu_k = F_k - \operatorname{Im} F_k(0)$ and $\mathcal{H}\nu_k = G_k - \operatorname{Im} G_k(0)$. It is easy to show that $\mu_k = \sum_{i=1}^{k}\alpha_i^k\delta_{a_{n_i}}$ and $\nu_k = \sum_{i=1}^{k}\beta_i^k\delta_{b_{m_i}}$ for some positive constants $\alpha_1^k, ..., \alpha_k^k, \beta_1^k, ..., \beta_k^k$. Since

$$\lim_{k\to\infty} F_k(0) = \lim_{k\to\infty}(G_k(0))^{-1} = \lim_{k\to\infty}\exp(\frac{i\pi}{2} - \frac{i}{2}\sum(\varphi_{m_i} - \phi_{n_i})) = C \in -i\mathbb{C}_+$$

we have that

$$\lim_{k\to\infty}||\mu_k|| = \lim_{k\to\infty}||\nu_k|| = \operatorname{Re} C > 0. \tag{3.4.4}$$

Conditions 2), 2') and 2") above imply that for any k and $i \leq k$

$$|\alpha_i^k - \alpha_i^{k+1}| < D/2^k \tag{3.4.5}$$

and

$$(3.4.6) \qquad |\beta_i^k - \beta_i^{k+1}| < D/2^k$$

for some absolute constant $D > 0$. Put $\alpha_i = \lim_{k\to\infty} \alpha_i^k$, $\beta_i = \lim_{k\to\infty} \beta_i^k$, $\mu = \sum_{i=1}^{\infty} \alpha_i \delta_{a_{n_i}}$ and $\nu = \sum_{i=1}^{\infty} \beta_i \delta_{b_{m_i}}$. Then (3.4.4), (3.4.5) and (3.4.6) imply that

$$\mu_n \xrightarrow[n\to\infty]{*-weakly} \mu$$

and

$$\nu_n \xrightarrow[n\to\infty]{*-weakly} \nu.$$

Thus

$$\mathcal{H}\mu = \prod_{i=1}^{\infty} \frac{z - a_{n_i}}{z - b_{n_i}} - \mathrm{Im} \prod_{i=1}^{\infty} \frac{a_{n_i}}{b_{n_i}} = \frac{1}{\mathcal{H}\nu}.$$

Hence μ and ν belong to the same family M_φ (if $\mu = \mu_1 \in M_\varphi$ then $\nu = \mu_{-1}$).

If u_n is the shift-function of the pair $(\mu_n; \nu_n)$ then conditions 1), 1'), and 1") imply that u_n satisfies (3.4.1) and (3.4.2). Since u_n converge pointwise to the shift-function u of the pair $(\mu; \nu)$ as $n \to \infty$, u also satisfies (3.4.1) and (3.4.2).▲

One can easily show that in general the condition $C_\mathcal{A} = C_\mathcal{B}$ does not imply that the corresponding measures are equivalent modulo rank one perturbation. The following example shows that even if in addition to the condition $C_\mathcal{A} = C_\mathcal{B}$ we have that $\mathcal{A}$ and $\mathcal{B}$ are well-mixed, the corresponding measures still do not have to are equivalent modulo rank one perturbation.

Example. *Let C be the standard Cantor null set on the unit interval $[0;1]$, $C = [0;1] \setminus \bigcup I_n$ where $I_n = (x_n; y_n)$ are disjoint open intervals. Let $\mathcal{A} = \{a_n\}$ and $\mathcal{B} = \{b_n\}$ be two disjoint sequences of points of C such that*

$$(3.4.7) \qquad 0, x_1, x_2, .. \in \mathcal{A} \text{ and } 1, y_1, y_2, .. \in \mathcal{B}.$$

Then $\mathcal{A}$ and $\mathcal{B}$ are well-mixed and $C_\mathcal{A} = C_\mathcal{B} = C$. Define $\mu = \sum \frac{1}{2^n} \delta_{a_n}$ and $\nu = \sum \frac{1}{2^n} \delta_{b_n}$. Suppose that μ and ν are equivalent modulo rank one perturbation. Then there exists the *shift-function u of the pair $(\mu';\nu')$ where $\mu \sim \mu'$ and $\nu \sim \nu'$. Function u must be constant on each I_n. Condition (3.4.7) imply that $u = \pi/2$ a. e. and we have a contradiction.*

We can however combine the conditions of the two previous theorems in the following form.

Theorem 3.4.4. *Let $\mathcal{A} = \{a_n\}$ and $\mathcal{B} = \{b_n\}$ be two disjoint sequences of real numbers. Let $E = \operatorname{Int} C_{\mathcal{A}} = \bigcup_{n=1}^{\infty}(x_n;y_n)$ where $(x_n;y_n)$ are disjoint intervals. Denote by F the set of all isolated points of $\mathcal{A}$: $F = \mathcal{A} \setminus C_{\mathcal{A}}$. Suppose that*

1)$\mathcal{A}$ and $\mathcal{B}$ are well-mixed,

2) $C_F \subset \{x_1, y_1, x_2, y_2, \ldots\}$

3)$\mathcal{A} \setminus (F \cup E) = \emptyset$

Then there exist a cyclic self-adjoint operator A and its cyclic vector ϕ such that

1) A and $A + (*,\phi)\phi$ are diagonal

2) $\mathcal{A}$ and $\mathcal{B}$ are the sets of **all** *eigenvalues of A and $A + (*,\phi)\phi$ respectively.*

Proof. Let us define the function u_1 on $\mathbb{R} \setminus E$ in the following way.

If $x \in \mathbb{R} \setminus E$ denote $m_x = \sup\{y | y < x, y \in \mathcal{A} \cup \mathcal{B}\}$ and $n_x = \inf\{y | y > x, y \in \mathcal{A} \cup \mathcal{B}\}$. We put $u_1(x) = -\pi/2$ if $m_x \in \mathcal{A}, n_x \in \mathcal{B}$ and $u_1(x) = \pi/2$ otherwise. We put $u_1 = 0$ on E.

If $I = (x;y) \subset C_{\mathcal{A}}$ is an open interval we will denote by u_I^+ the shift function of the pair

$$\Big(\sum_{a_n \in I} \alpha_n \delta_{a_n}; \sum_{b_n \in I} \beta_n \delta_{b_n}\Big)$$

for some positive constants α_i and β_i satisfying the conditions

$$m((x;x+\epsilon) \cap \{u \neq -\pi/2\}) < \epsilon^2 \qquad (3.4.8)$$

and

$$m(y-\epsilon;y) \cap \{u \neq -\pi/2\}) < \epsilon^2 \qquad (3.4.9)$$

(such u_I^+ exists by Theorem 3.4.3′).

Obviously, if we replace $-\pi$ with π in the statement of Theorem 3.4.3′ it will remain true. We will denote by u_I^- the analogous shift-function satisfying the conditions

$$m((x; x+\epsilon) \cap \{u \neq +\pi/2\}) < \epsilon^2 \tag{3.4.10}$$

and

$$m(y-\epsilon; y) \cap \{u \neq +\pi/2\}) < \epsilon^2. \tag{3.4.11}$$

Now let us define u_2 on E in the following way. For each $I_n = (x_n; y_n)$ we compute

$$\limsup_{\epsilon \to 0} \frac{m(I(x_n - \epsilon; x_n) \cap \{u_1 \neq -\pi/2\})}{\epsilon} = C_n^- \tag{3.4.12}$$

and

$$\limsup_{\epsilon \to 0} \frac{m(I(y_n; y_n + \epsilon) \cap \{u_1 \neq -\pi/2\})}{\epsilon} = C_n^+. \tag{3.4.13}$$

If $C_n^- = C_n^+ = 0$ we put $u_2 = u_{I_n}^-$ on I_n; if $C_n^- = C_n^+ = 1$ we put $u_2 = u_{I_n}^+$ on I_n; if $C_n^- = 0,\ C_n^+ = 1$ we choose $b \in \mathcal{B} \cap I_n$ and put $u_2 = u_{I_n'}^-$ on $I_n' = (x_n; a)$ and $u_2 = u_{I_n''}^+$ on $I_n'' = (a; y_n)$; if $C_n^- = 1, C_n^+ = 0$ we choose $a \in \mathcal{A} \cap I_n$ and put $u_2 = u_{I_n'}^+$ on $I_n' = (x_n; b)$ and $u_2 = u_{I_n''}^-$ on $I_n'' = (b; y_n)$. In all other cases we put $u_2 = u_{I_n}^-$ on I_n. On the set $E \setminus \mathbb{R}$ we define $u_2 = 0$.

Now consider the function $u = u_1 + u_2$. Let $x \notin E$. Then the definition of u_1 implies that $\mathcal{H}u(x) = \infty$ if $x \in \mathcal{A} \setminus E$ and $\mathcal{H}u(x) = -\infty$ if $x \in \mathcal{B} \setminus E$. If $x \notin \mathcal{A} \cup \mathcal{B} \cup \{x_1, y_1, x_2, y_2, ...\}$ then $x \notin C_{\mathcal{A} \cup \mathcal{B}}$ and $|\mathcal{H}u_1(x)| < \infty$. By the definition of u_2 and (3.4.9-13) $|\mathcal{H}u_2(x)|$ is also finite.

Also by the definition of u_2,

$$\mathcal{H}u(x+iy) = -\log y + \log c + o(1)$$

for some $c = c(x) > 0$ as $y \to 0$ for any $x \in \mathcal{A} \cap E$,

$$\mathcal{H}u(x + iy) = \log y + \log c + o(1)$$

for some $c = c(x) > 0$ as $y \to 0$ for any $x \in \mathcal{B} \cap E$ and $|\mathcal{H}u(x)| < \infty$ for any $x \in E \setminus (\mathcal{A} \cup \mathcal{B})$.

Thus u is the shift function of the pair $(\sum \alpha_n \delta_{a_n}; \sum \beta_n \delta_{b_n})$ for some positive constants α_i and β_i. ▲

Remark. Theorem 3.4.1 and the example after it show that if we drop either 1) or 2) in the statement of Theorem 3.4.4 then it will no longer be true.

3.5. Pure point and singular continuous spectra

Let A be a cyclic self-adjoint operator and let ϕ be its cyclic vector. Denote $A_\lambda = A + \lambda(*, \phi)\phi$ where $\lambda \in \mathbb{R}$. Let μ_λ be the spectral measure of ϕ for A_λ. Let us denote by P and C the sets of λ for which μ_λ has nontrivial pure point and nontrivial singular continuous part on $[0; 1]$ respectively. Then the set $P \cap C$ will consist of those λ for which the corresponding measures are "mixed" on the interval $[0; 1]$.

One of the natural questions which arise from the recent results on rank one perturbations (see [R-J-L-S]) is *whether the set $P \cap C$ can be empty (or almost empty) when the sets P and C are sufficiently big (topologically or in measure).*

The following example gives a partial answer to this question.

Example 3.5.1.

We will show that there exist a self-adjoint cyclic operator A such that for some cyclic vector ϕ the operators $A + \lambda(*; \phi)\phi$ are diagonal for all $\infty < \lambda < 0$ and singular continuous for all $\infty \geq \lambda \geq 0$ on $[-2; 2]$.

To do that we will construct its Krein spectral shift u. We will start by constructing a Cantor set on the interval $[-2; 2]$.

Let $\{a_n\}$ be a sequence of real numbers monotonically decreasing to 0 and such that

$$\prod_{n=1}^{\infty} (1 - a_n) = c \tag{3.5.1}$$

and

(3.5.2)
$$1-\prod_{k=n}^{\infty}(1-a_k)\geq\frac{1}{n}.$$

Let

$$C_0=I_0^0=[0;1],C_1=I_1^1\cup I_2^1,...,C_n=I_1^n\cup...\cup I_{2^n}^n,...$$

where

$$I_{2k}^n\cup I_{2k-1}^n=I_k^{n-1}\setminus\Delta_k^{n-1}$$

and Δ_k^n is the open interval placed in the center of the interval I_k^n and such that $m(\Delta_k^n)=a_n m(I_k^n)$. Put $C=\bigcap_{n=0}^{\infty}C_n$.

The Cantor set C has the following properties:

a)

$$m(C)=c$$

b)

$$\frac{m(I_n^k\cap C)}{m(I_n^k)}=\prod_{n}^{\infty}(1-a_n)\geq\frac{1}{n}\geq\frac{1}{|\ln m(I_n^k)|}.$$

Also we can choose c in such a way that

(3.5.3)
$$\frac{1}{\pi}\int_C\frac{dt}{1+t^2}=\frac{1}{2}.$$

Define $u=\pi/2$ on C and $u=-\pi/2$ elsewhere on C. Denote $U(z)=\mathcal{H}u(z)$ for $z\in\mathbb{C}_+$.

Claim. *U has a nontangential derivative $U'(x)=\lim_{z\underset{\not\to}{\longrightarrow}x}\frac{U(x)-U(z)}{x-z}$ at a point $x\in\mathbb{R}$ if and only if $x\in\mathbb{R}\setminus C$.*

Proof. Since u is locally constant on $\mathbb{R}\setminus C$, U' obviously exists there.

Let $x\in C$. Let $\{I_{n_k}^k\}_{k=1}^{\infty}$ be the sequence of intervals containing x. Denote by x_k the middle of the interval $I_{n_k}^k$; put $y_k=|x-x_k|$. Then condition b) above imply that

$$\left|\frac{\pi}{2}-\mathcal{P}u(x_n+iy_n)\right|\geq\frac{c}{|\ln y_n|}$$

for come $c > 0$. Thus $U'(x)$ does not exist.▲

Consider a real measure μ such that u is its shift-function. Put $A = A_\mu, A_\lambda = A + \lambda(*;1)1$. Let μ_λ be the spectral measure of 1 for A_λ.

By the results from part 2 and (3.5.3)

$$U(z) \xrightarrow[z \not\to \xi]{} \pi/2$$

μ_λ-a. e. for all $\lambda \in (0;\infty)$. Thus all $\mu_\lambda, \lambda \in (0;\infty)$ are concentrated on the set C. Since U' does not exist on C and

$$F(z) = \exp(-i\mathcal{H}u(z)) \xrightarrow[z \not\to \xi]{} c \neq 0$$

μ_λ-a. e., F' does not exist on C for all $\lambda \in (0;\infty)$. Hence by Simon-Wolf criterion [S-W] all $\mu_\lambda, \lambda \in (0;\infty)$ are singular continuous. Similarly all μ_λ for $\lambda \in (-\infty;0)$ are concentrated on $\mathbb{R} \setminus C$, where U' exists, and so they are pure point.

To prove that $\mu = \mu_0$ is continuous, let us notice that since $Qu(z) \xrightarrow[z \not\to \xi]{} \infty$ μ-a. e., μ is concentrated on C. Let $x \in C$ and let the sequence $\{x_n + iy_n\}$ be constructed as in the proof of the last claim. Put $z_n = x_n + \frac{iy_n}{2}$. Then

$$Qu(z_n) \sim \int_{\mathbb{R}\setminus(-y_n;y_n)} \frac{u(t)dt}{t-y} \neq -\log|z_n - x| + O(1).$$

Thus by (3.2.3) μ can not have a point mass at x. Hence μ is continuous.

The case $\lambda = \infty$ can be proven in the same way.

Remark. To obtain a similar example with $[-2;2] \subset \sigma(A)$ we can insert "small copies" of C into each complimentary interval I_n^k, then insert "smaller copies" of C into each new complimentary interval and so on. If the size of these "copies" of C decreases to 0 fast enough, proceeding in the same way as above we will obtain an example of A and ϕ such that $A + \lambda(*;\phi)\phi$ is continuous *for all* $\lambda \in [0;\infty]$ and pure point *for almost all* $\lambda \in (-\infty;0)$.

3.6 Stability of the absence of continuous spectrum

In this section we will give a necessary and sufficient condition for an operator to have *only* diagonal rank one perturbations.

Theorem 3.6.1. *Let A be a self-adjoint cyclic operator. Then the following two conditions are equivalent:*

1) All self-adjoint rank one perturbations of A are pure point,

2) $\sigma(A)$ is countable.

Remark. We assume that A itself is included in the set of all its rank one perturbations.

Proof. 2) $\Rightarrow$ 1). Follows from the stability of the essential spectrum.

1) $\Rightarrow$ 2). Suppose $\sigma(A)$ is uncountable. Without loss of generality we can assume that $\sigma(A) \subset (0;1)$. Denote by $\mathcal{A} = \{a_n\}_{n=1}^{\infty}$ the set of all eigenvalues of A. Then at least one of the sets

$$F_- = \{x | (x-\epsilon; x) \cap \mathcal{A} \neq \emptyset \text{ for any } \epsilon > 0\}$$

or

$$F_+ = \{x | (x; x+\epsilon) \cap \mathcal{A} \neq \emptyset \text{ for any } \epsilon > 0\}.$$

is uncountable. Suppose that F_- is uncountable. Then there exists a closed uncountable set $F \subset F_-$ such that $F \cap \mathcal{A} = \emptyset$, $m(F) = 0$ and F does not contain any isolated points ($F = C_F$). Let $I_1 = (x_1; y_1), I_2 = (x_2; y_2), \ldots$ be disjoint open intervals such that $F = (0;1) \setminus \bigcup I_n$. Inside each I_n we can choose $a_n \in \mathcal{A}$ such that

$$\frac{y_n - a_n}{(y_n - x_n)^2} < 1. \tag{3.6.1}$$

Define function u to be equal to $\pi/2$ on each $(x_n; a_n)$ and $-\pi/2$ on each $(a_n; y_n)$. Then u is the shift-function of some pair of measures $(\mu_0; \nu_0)$. By the construction of u, μ_0 has point masses at each a_n. Condition (3.6.1) implies that $Qu < C < \infty$ at each point of F. Thus $\mu_0 = \sum \alpha_n^0 \delta_{a_n}$ for some positive constants α_n^0. Also by the construction of u, $\operatorname{supp} \nu_0 \subset F$. Since condition (3.2.5) is not satisfied at any point of F, ν_0 is continuous.

Let $\{b_n\}_{n=1}^{\infty}$ be some enumeration of the set $\mathcal{A}\setminus\{a_n\}_{n=1}^{\infty}$. For each $b_n \in I_k$ let us choose $c_n \in I_k$ such that

i)

$$|b_n - c_n| < \frac{\operatorname{dist}(b_n; (\mathbb{R}\setminus I_k) \cup \{b_1, b_2, ..., b_{n-1}\})}{2^{n+1}}$$

and

ii) for each n and k the sequences $\mathcal{B}_n = \{a_i\}_{i=1}^{\infty} \cup \{b_1, b_2, ..., b_n\}$ and $\mathcal{C}_n = \{c_1, c_2, ..., c_n\}$ are "well-mixed" on I_k i.e. between each two points of one of these sequences lying on I_k there is at least one point from the other sequence.

For each k define the function u_k in the following way:

1) $|u_k| = \pi/2$ everywhere on $\mathbb{R}$;

2) u_k is continuous everywhere except $F \cup \mathcal{B}_k \cup \mathcal{C}_k$;

3) u_k "jumps" from $\pi/2$ to $-\pi/2$ at each point of $\mathcal{B}_k$ and from $-\pi/2$ to $\pi/2$ at each point of $\mathcal{C}_k$.

Let u_k be the shift-function of a pair $(\mu_k; \nu_k)$. Then

$$\mu_k = \sum_{n=1}^{\infty} \alpha_n^k \delta_{a_n} + \sum_{n=1}^{k} \beta_n^k \delta_{b_n}$$

for some positive constants α_i^k and β_i^k and

$$\nu_k = \sum_{n=1}^{k} \gamma_n^k \delta_{c_n} + f_k \nu_0$$

for some positive constants γ_1^k and some positive function $f_k \in L^1(\nu_0)$. Conditions *i)* and *ii)* imply that the sequence $\{u_k\}$ converges in measure to some function u. Let u be the shift-function of a pair $(\mu; \nu)$. Then $\mu_k \to \mu$ and $\nu_k \to \nu$ in the $*$-weak topology.

Since

$$\frac{\mathcal{P}\mu_k}{\mathcal{P}\mu_{k+1}} = \operatorname{Re}\exp(-i\mathcal{H}(u_k - u_{k+1})$$

condition *i)* implies that

$$1 - \frac{1}{2^k} < \frac{\mathcal{P}\mu_k}{\mathcal{P}\mu_{k+1}} < 1 + \frac{1}{2^k}$$

μ_k-a.e. and ν_0-a.e. By Lemma 2.2.4 that means that

$$\mu_{k+1} = g_k\mu_k + \beta_{k+1}^{k+1}\delta_{c_{k+1}} \text{ where } g_k \in L^1(\mu_k),\ 1 - \frac{1}{2^k} < g_k < 1 + \frac{1}{2^k} \tag{3.6.2}$$

μ_k-a. e. and that

$$1 - \frac{1}{2^k} < \frac{f_k}{f_{k+1}} < 1 + \frac{1}{2^k} \tag{3.6.3}$$

ν_0-a. e.

Since $||\mu_k|| \to ||\mu||$, $\beta_{k+1}^{k+1} \to 0$ as $k \to \infty$. Since each μ_k is pure point, that implies, together with (3.6.2), that μ is pure point and that

$$\mu = \sum_{n=1}^{\infty} \alpha_n\delta_{a_n} + \sum_{n=1}^{\infty} \beta_n\delta_{b_n}$$

for some positive constants α_i and β_i. Thus $A \sim A_\mu$. Also, since $\nu = \eta + \sigma$ where η is some positive measure and σ is a $*$-weak limit of the sequence $\{f_k\nu_0\}$, (3.6.3) implies that $\sigma = f\nu_0$ for some $f \in L^1(\nu_0)$, $f > 0$ ν_0-a.e. Thus $A_\mu + (*;1)1$ has a nontrivial continuous part. ▲

Remark. Note that the case $m(F) > 0$ could be proved much easier than the general case.

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