THE CONSERVATION LAWS OF THREE DIMENSIONAL LINEARIZED ELASTICITY THEORY

Thesis by

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Abstract

For linearized isotropic elastodynamics and elastostatics, Noether's theorem on invariant variational principles is used to obtain all conservation laws arising from a reasonably general group of infinitesimal transformations. A theorem regarding the completeness of the derived laws is proved, and the conservation laws are then used to derive the wave speed equation for the Rayleigh problem on the surface of an anisotropic half space. An example of additional laws following from the same group but from a more general version of Noether's theorem is given in an appendix devoted to a discussion of limitations on the completeness theorem.

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Introduction

In a paper devoted to the analysis of stress concentrations near the tips of cracks and notches, Rice [1] introduced a "path-independent integral" implied by the field equations of elastostatics and demonstrated its utility in connection with the asymptotic analysis of singular stress fields. Although the path-independent integral used by Rice can be traced to the earlier work of Eshelby [2] on the theory of dislocations, much of the current interest in these integrals and their applications was stimulated by the results reported in [1][1].

Let (x_1, x_2) be rectangular cartesian coordinates, and suppose that $u_\alpha, \sigma_{\alpha\beta}$ are components of displacement and stress[2], respectively, associated with a two-dimensional infinitesimal deformation of a homogeneous elastic solid in the absence of body forces. Let $W(\underset{\sim}{x})$ be the strain-energy density at the point with position vector $\underset{\sim}{x}$. If C is any closed curve on and within which the equilibrium equations and stress-displacement relations of infinitesimal elasticity hold, the path-independent integral used in [1] can be stated as follows:

$$\oint_C \left(W n_\alpha - \sigma_{\beta\gamma} n_\gamma u_{\beta,\alpha} \right) ds = 0 \, . \tag{1.1}$$

Here $\underset{\sim}{n}$ is the unit outward normal vector on C and s denotes arc-length on C. The fact that (1.1) follows from the equilibrium equations, the

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[1] For applications to crack problems in elastostatics, see [3], [4], [5] and [6].

[2] Greek subscripts, here, have the range 1, 2; subscripts preceded by a comma indicate partial differentiation with respect to the corresponding Cartesian coordinate, and summation over repeated subscripts is implied.

stress-displacement relations, and the definition of W is easily demonstrated by means of the divergence theorem. Such a proof of (1.1) was given by Rice in [1]; it amounts to a verification and, as such, gives no indication of why (1.1) holds or whether other path-independent integrals exist.

In a recent paper by Knowles and Sternberg [7] it was shown that the path-independent integral — or conservation law — (1.1) and its three-dimensional counterpart follow from an application of Noether's theorem on invariant variational principles [8] to the principle of minimum potential energy in elastostatics. Roughly speaking, Noether's theorem states that if a given set of differential equations are the Euler-Lagrange equations corresponding to a variational principle which remains invariant under an n-parameter group of infinitesimal transformations, then there is an associated set of n conservation laws satisfied by all solutions of the original differential equations.

When derived in this way, (1.1) appears as a consequence of the invariance under a coordinate translation of the potential energy associated with an elastically homogeneous material. If the material is elastically isotropic as well as homogeneous, the potential energy is also invariant under a rotation of coordinates. This fact, together with Noether's theorem, is used in [7] to establish a second conservation law.

The conservation laws corresponding to invariance under translation and rotation are also shown in [7] to be valid in finite elasticity provided they are suitably interpreted.

A third conservation law for strictly linear elastostatics, corresponding to infinitesimal invariance of the potential energy under a family of coordinate scale changes, is also established in [7].

Finally it is shown in [7] that, within the context of linear isotropic, homogeneous elastostatics, the three conservation laws mentioned above are complete in the sense that they are the only ones furnished by Noether's theorem when applied to the restricted group of transformations considered in [7].

An earlier treatment of conservation laws in linear elastostatics — unknown to the authors of [7] until after the publication of that reference — is to be found in [9].

In this paper we extend the results of Knowles and Sternberg [7] to linear elastodynamics, and we establish the completeness of the corresponding conservation laws under a somewhat more general group of transformations than that employed in [7]. Our completeness result, when specialized to linear elastostatics, shows that the three conservation laws derived in [7] are the only ones furnished by Noether's theorem, even when the group of transformations under consideration is significantly larger than that admitted in [7].

In the next section we give a brief review of linear elastodynamics in order to establish the notation to be used in the remainder of the paper, and we state the version of Noether's theorem which we shall employ.

In Section 3 we state our principal results. Theorem 1 provides three conservation laws for elastodynamics, while Theorem 2 establishes the completeness of these results under certain conditions.

The proof of Theorem 2 is given in Section 4 by means of an argument similar to that used by S. Lie [10] and by Bateman [11].

In Section 5 we show how the conservation laws can be used to discuss two-dimensional linear Rayleigh waves on the surface of an anisotropic, homogeneous elastic half-space.

Finally we present in the Appendix a brief informal discussion of the application to linear elastodynamics of a more general version of Noether's theorem in which the notion of infinitesimal invariance is replaced by that of "weak invariance".

2. Preliminaries. Linear elastodynamics. Noether's theorem.

Let D be the closed, bounded, regular region in three-dimensional space occupied by a homogeneous elastic solid in its undeformed state. We consider a motion of the solid in which a particle, located at $\underset{\sim}{x}$ in the undeformed configuration, is found at time t at the point with position vector $\underset{\sim}{y}(\underset{\sim}{x},t)$. The displacement vector field $\underset{\sim}{u}$ associated with the motion is then defined by

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \underset{\sim}{y}(\underset{\sim}{x},t) - \underset{\sim}{x} \quad , \quad \underset{\sim}{x} \in D \ , \quad t \geq 0 \ . \tag{2.1}$$

The components of the infinitesimal strain tensor field $\underset{\sim}{\gamma}$ are

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{on} \quad D \times [0,\infty) \ . \tag{2.2}$$ [1]

If $c_{ijk\ell}$ are the components of the elasticity tensor of the material under consideration, the components σ_{ij} of the stress tensor $\underset{\sim}{\sigma}$ are given by

$$\sigma_{ij} = c_{ijk\ell}\gamma_{k\ell} \quad \text{on} \quad D \times [0,\infty) \ , \tag{2.3}$$

where the constants $c_{ijk\ell}$ satisfy the symmetry requirements

$$c_{ijk\ell} = c_{jik\ell} = c_{k\ell ij} \ . \tag{2.4}$$

In the absence of body forces, the momentum equations are

$$\sigma_{ij,j} = \rho\ddot{u}_i \quad \text{on} \quad D \times [0,\infty) \ , \tag{2.5}$$

where ρ is the mass density — assumed constant — and

$$\ddot{u}_i(\underset{\sim}{x},t) = \frac{\partial^2 u_i}{\partial t^2}(\underset{\sim}{x},t) \ . \tag{2.6}$$

From (2.2), (2.3), (2.4) and (2.5) follow the displacement equations

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[1] Latin subscripts have the range 1, 2, 3, unless otherwise stated.

of motion

$$c_{ijk\ell}u_{k,\ell j} = \rho\ddot{u}_i \quad \text{on } D \times [0,\infty) \, . \tag{2.7}$$

Throughout this paper we shall be concerned with solutions $\underset{\sim}{u}$ of (2.7) which are twice continuously differentiable on $D \times [0,\infty)$.

The elastic potential associated with the material at hand is defined by

$$\Gamma(\underset{\sim}{\gamma}) = \frac{1}{2} c_{ijk\ell}\gamma_{ij}\gamma_{k\ell} \tag{2.8}$$

for all symmetric two-tensors $\underset{\sim}{\gamma}$. If $\underset{\sim}{\sigma}$ and $\underset{\sim}{\gamma}$ are related by (2.3), it follows that

$$\sigma_{ij} = \Gamma_{,\gamma_{ij}}(\underset{\sim}{\gamma}) \quad , \quad \Gamma(\underset{\sim}{\gamma}) = \frac{1}{2}\sigma_{ij}\gamma_{ij} \, . \tag{2.9}$$ [1]

Furthermore, the symmetries (2.4) and the strain-displacement relations (2.2) can be used to write (2.8) in the form

$$\Gamma(\underset{\sim}{\gamma}) = \frac{1}{2} c_{ijk\ell}u_{i,j}u_{k,\ell} \, . \tag{2.10}$$

For an <u>isotropic</u> material, the constants $c_{ijk\ell}$ are given by

$$c_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \, , \tag{2.11}$$

where the constants λ and μ are the Lamé moduli. In this case, (2.8) becomes

$$\Gamma(\underset{\sim}{\gamma}) = \lambda\gamma_{ii}\gamma_{jj} + 2\mu\gamma_{ij}\gamma_{ij} \tag{2.12}$$

For any twice continuously differentiable vector field $\underset{\sim}{v}$ defined on $D \times [0,\infty)$, let

$$\mathcal{L}[\underset{\sim}{v}] = \int_0^T \int_D L(\nabla\underset{\sim}{v}, \dot{\underset{\sim}{v}}) \, d\underset{\sim}{x} \, dt \quad , \quad T > 0 \, , \tag{2.13}$$

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[1] Where, in the indicated differentiation, it is understood that we treat the elements γ_{ij} and γ_{ji} as mutually independent.

where the Lagrangian density L is given by

$$L(\nabla\underset{\sim}{v}, \dot{\underset{\sim}{v}}) = \Gamma(\mathrm{sym}\,\nabla\underset{\sim}{v}) - \frac{1}{2}\rho\,\dot{\underset{\sim}{v}}\cdot\dot{\underset{\sim}{v}}\,, \tag{2.14}$$

and sym $\nabla\underset{\sim}{v}$ stands for the symmetric part of $\nabla\underset{\sim}{v}$. Thus

$$L(\nabla\underset{\sim}{v}, \dot{\underset{\sim}{v}}) = \frac{1}{2} c_{ijk\ell} v_{i,j} v_{k,\ell} - \frac{1}{2}\rho\,\dot{v}_i\,\dot{v}_i\,. \tag{2.15}$$

The Lagrangian $\mathcal{L}$ of (2.13) occurs in the statement of Hamilton's principle for elastic solids[1]. As is easily verified, the formal Euler-Lagrange differential equations associated with the functional $\mathcal{L}$ of (2.13) are precisely the displacement equations of motion (2.7).

Our main tool in the present paper is a version of Noether's theorem which may be found in the textbook by Gelfand and Fomin [13]. In order to state this theorem in a form suitable for our purposes, we first introduce some additional notation. Let $\xi_1, \xi_2, \ldots \xi_n$ be rectangular cartesian coordinates in n-dimensional Euclidean space E^n, and let R be a bounded, closed, regular region in E^n. For any three-dimensional vector field $\underset{\sim}{w} = (w_1, w_2, w_3)$ defined and twice continuously differentiable on R, set

$$\mathfrak{F}[\underset{\sim}{w}] = \int_R F\Big(\underset{\sim}{\xi}, \underset{\sim}{w}(\underset{\sim}{\xi}), \nabla\underset{\sim}{w}(\underset{\sim}{\xi})\Big)\, d\underset{\sim}{\xi}\,, \tag{2.16}$$

where F is a given real valued function, defined and infinitely differentiable for all values of its arguments. We shall be interested in the behavior of the functional $\mathfrak{F}$ under transformations which carry the point $\underset{\sim}{\xi}$ and the vector field $\underset{\sim}{w}$ into a new point $\underset{\sim}{\xi}^*$ and a new vector

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[1] For a discussion of Hamilton's principle, as well as other variational principles in linear elastodynamics, see [12].

field $\underset{\sim}{w}^*$. More precisely, given the point $\underset{\sim}{\xi} \in R$ and the vector field $\underset{\sim}{w} \in C^2(R)$, define a family of transformations $(\underset{\sim}{\xi}, \underset{\sim}{w}) \rightarrow (\underset{\sim}{\xi}^*, \underset{\sim}{w}^*)$ by

$$\underset{\sim}{\xi}^* = \Phi\left(\underset{\sim}{\xi}, \underset{\sim}{w}(\underset{\sim}{\xi}), \underset{\sim}{\nabla}\underset{\sim}{w}(\underset{\sim}{\xi}); \epsilon\right) \quad , \quad \underset{\sim}{w}^*(\underset{\sim}{\xi}^*) = \Psi\left(\underset{\sim}{\xi}, \underset{\sim}{w}(\underset{\sim}{\xi}), \underset{\sim}{\nabla}\underset{\sim}{w}(\underset{\sim}{\xi}); \epsilon\right) \ . \qquad (2.17)$$

The given functions $\underset{\sim}{\Phi}$ and $\underset{\sim}{\Psi}$ are, respectively, $\underset{\sim}{n}$- and 3-dimensional vector valued functions of their arguments which depend on a parameter ϵ. They are assumed to be defined and twice continuously differentiable for sufficiently small values $|\epsilon|$, for all $\underset{\sim}{\xi} \in D$, and for all values of their remaining arguments. It is required that the transformation (2.17) shall reduce to the identity $\underset{\sim}{\xi}^* = \underset{\sim}{\xi}$, $\underset{\sim}{w}^*(\underset{\sim}{\xi}^*) = \underset{\sim}{w}(\underset{\sim}{\xi})$ when $\epsilon = 0$.

In general, (2.17) may be viewed as follows. Given a vector field $\underset{\sim}{w} \in C^2(R)$, the first of (2.17) provides a mapping of R onto a region R^* which depends on $\underset{\sim}{w}$, while the second of (2.17) defines a new vector field $\underset{\sim}{w}^*$ whose domain is R^*. Thus, in general, both the new domain R^* and the new vector field $\underset{\sim}{w}^*$ depend on the original vector field $\underset{\sim}{w}$.

The functional $\mathfrak{F}$ in (2.16) is said to be invariant at $\underset{\sim}{w}$ under the transformation (2.17) if

$$\int_{R^*} F\left(\underset{\sim}{\xi}^*, \underset{\sim}{w}^*(\underset{\sim}{\xi}^*), \underset{\sim}{\nabla}^* \underset{\sim}{w}^*(\underset{\sim}{\xi}^*)\right) d\underset{\sim}{\xi}^* = \int_R F\left(\underset{\sim}{\xi}, \underset{\sim}{w}(\underset{\sim}{\xi}), \underset{\sim}{\nabla}\underset{\sim}{w}(\underset{\sim}{\xi})\right) d\underset{\sim}{\xi} \qquad (2.18)^1$$

for all sufficiently small values of $|\epsilon|$. If, for a given $\underset{\sim}{w}$,

$$\left\{\frac{d}{d\epsilon} \int_{R^*} F\left(\underset{\sim}{\xi}^*, \underset{\sim}{w}^*(\underset{\sim}{\xi}^*), \underset{\sim}{\nabla}^* \underset{\sim}{w}^*(\underset{\sim}{\xi}^*)\right) d\underset{\sim}{\xi}^*\right\}_{\epsilon=0} = 0 \ , \qquad (2.19)$$

then $\mathfrak{F}$ is said to be infinitesimally invariant at $\underset{\sim}{w}$. Note that if $\mathfrak{F}$ is invariant at $\underset{\sim}{w}$, then $\mathfrak{F}$ is infinitesimally invariant at $\underset{\sim}{w}$.

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[1] $\underset{\sim}{\nabla}^* \underset{\sim}{w}^*$ is the $n \times 3$ matrix whose elements are $\partial w_i^* / \partial \xi_j^*$, $(i=1,2,3, j=1,\ldots,n)$.

Noether's theorem is concerned with the consequences of infinitesimal invariance of $\mathfrak{F}$ at $\underset{\sim}{w}$ under the special circumstances in which $\underset{\sim}{w}$ satisfies the Euler-Lagrange differential equations associated with $\mathfrak{F}$. The following version of the theorem is a slight variation of that to be found in [13]. We shall omit the proof, since it is virtually identical with the one given in that reference.

Theorem. Let $\hat{R}$ be a domain in E^n, and suppose the vector field $\underset{\sim}{w}$ satisfies the Euler-Lagrange equations

$$F_{,w_i}(X) - \frac{\partial}{\partial \xi_\alpha}\left[F_{,w_{i,\alpha}}(X)\right] = 0 \, , \qquad (2.20)[1]$$

where X stands for

$$X \equiv \left(\underset{\sim}{\xi}, \underset{\sim}{w}(\underset{\sim}{\xi}), \underset{\sim}{\nabla} \underset{\sim}{w}(\underset{\sim}{\xi})\right) \, , \; \underset{\sim}{\xi} \in \hat{R} \, . \qquad (2.21)$$

Then $\mathfrak{F}$ in (2.16) is infinitesimally invariant at $\underset{\sim}{w}$ under the family of transformations (2.17) for every bounded, regular subregion R of $\hat{R}$ if and only if $\underset{\sim}{w}$ also satisfies

$$\frac{\partial}{\partial \xi_\alpha}\left\{F_{,w_{i,\alpha}}(X)\,\overline{\psi}_i(X) + F(X)\,\varphi_\alpha(X)\right\} = 0 \, , \qquad (2.22)$$

where

$$\left.\begin{aligned} \varphi_\alpha(X) &= \frac{\partial}{\partial \epsilon}\,\Phi_\alpha(X,\epsilon)\Big|_{\epsilon=0} \, , \\ \psi_i(X) &= \frac{\partial}{\partial \epsilon}\,\Psi_i(X,\epsilon)\Big|_{\epsilon=0} \, , \\ \overline{\psi}_i(X) &= \psi_i(X) - w_{i,\alpha}(\underset{\sim}{\xi})\,\varphi_\alpha(X) \end{aligned}\right\} \qquad (2.23)$$

If R is a bounded regular subregion of $\hat{R}$, (2.22) and the divergence theorem immediately yield

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1. Latin subscripts have range 1,2,3, while Greek indices run from 1 to n. Repeated subscripts are to be summed over the appropriate range.

$$\oint_{\partial R} \left[F_{,w_{i,\alpha}}(X)\,\overline{\Psi}_i(X) + F(X)\,\varphi_\alpha(X) \right] n_\alpha(\underset{\sim}{\xi})\, dS = 0 \ , \tag{2.24}$$

where ∂R is the boundary of R, and n_α is the ξ_α-component of the unit outward normal on ∂R. Thus the flux out of R of the n-dimensional vector field whose ξ_α-component appears within the braces in (2. 24) vanishes. In the special case n=2, (2. 24) represents a path-independent line integral. In general we shall speak of (2. 24) — or its equivalent differential form (2. 22) — as a <u>conservation law.</u>

The statement of the theorem given above is more general than that used in [7] in that the transformation $\underset{\sim}{\xi} \rightarrow \underset{\sim}{\xi}^*$ represented by the first of (2. 17) is here allowed to depend on $\underset{\sim}{w}$ and $\underset{\sim}{\nabla} \underset{\sim}{w}$. Moreover the transformation $w \rightarrow w^*$ in (2. 17) is here permitted to depend on $\underset{\sim}{\xi}$ and $\underset{\sim}{\nabla} \underset{\sim}{w}$. In the version of Noether's theorem given in [7], (2. 17) would be replaced by

$$\underset{\sim}{\xi}^* = \Phi(\underset{\sim}{\xi}, \epsilon) \quad , \quad \underset{\sim}{w}^*(\underset{\sim}{\xi}^*) = \Psi\Big(\underset{\sim}{w}(\underset{\sim}{\xi}), \epsilon\Big) \ .$$

On the other hand, Noether's original theorem [8] is substantially more general than that stated here.

3. The conservation laws.

In this section we shall obtain the conservation laws which follow from an application of Noether's theorem to the Lagrangian functional $\mathcal{L}$ of (2.13).

Theorem 1. *Let* D *be a domain in three-dimensional space, and suppose that on* $D \times [0, T]$ *the displacement field* $\underset{\sim}{u}$ *satisfies the displacement equations of motion* (2.7) *for a linear, homogeneous* (but not necessarily isotropic) *elastic material. Then each of the following five conservation laws for* $\underset{\sim}{u}$ *also holds on* $D \times [0, T]$:

$$\text{(i)} \quad \frac{\partial}{\partial t}\left[\Gamma(\underset{\sim}{\nabla}\underset{\sim}{u}) + \frac{1}{2}\rho\dot{u}_j\dot{u}_j\right] + \frac{\partial}{\partial x_k}(-\dot{u}_j\sigma_{jk}) = 0 \,, \tag{3.1}$$

[1]

$$\text{(ii)} \quad \frac{\partial}{\partial t}(-\rho\dot{u}_i) + \frac{\partial\sigma_{ik}}{\partial x_k} = 0 \,, \tag{3.2}$$

$$\text{(iii)} \quad \frac{\partial}{\partial t}(\rho\epsilon_{ijk}x_j\dot{u}_k) + \frac{\partial}{\partial x_k}(-\epsilon_{ijm}x_j\sigma_{mk}) = 0 \,, \tag{3.3}$$

$$\text{(iv)} \quad \frac{\partial}{\partial t}(\rho\dot{u}_j u_{j,i}) + \frac{\partial}{\partial x_k}(-u_{j,i}\sigma_{jk} + L\delta_{ik}) = 0 \,, \tag{3.4}$$

$$\text{(v)} \quad \frac{\partial}{\partial t}\left[\rho\dot{u}_j(u_j + x_m u_{j,m} + t\dot{u}_j) + tL\right] + \frac{\partial}{\partial x_k}\left[-\sigma_{jk}(u_j + x_m u_{j,m} + t\dot{u}_j) + x_k L\right] = 0 \,. \tag{3.5}$$

If the elastic material is also isotropic, then, in addition to (i)-(v), *the following conservation law holds on* $D \times [0, T]$:

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1. Latin subscripts continue to have the range 1,2,3.

$$\text{(vi)}\quad \frac{\partial}{\partial t}(\rho\,\epsilon_{ijk}u_k\dot{u}_j + \rho\,\epsilon_{ijk}x_j\dot{u}_m u_{m,k})$$

$$+\frac{\partial}{\partial x_k}(\epsilon_{imj}u_m\sigma_{jk} - \epsilon_{imj}x_j u_{\ell,m}\sigma_{\ell k} + \epsilon_{imk}x_m L) = 0\ . \tag{3.6}$$

In (3.1) - (3.6), σ_{ij} are the components of stress and are to be regarded as defined in terms of $\underset{\sim}{u}$ by (2.3); ϵ_{ijk} are the components of the three-dimensional alternating tensor, δ_{ik} the Kronecker delta, and L is the Lagrangian density defined by (2.15).

Before proceeding to the proof of the theorem, we give the integral forms of the conservation laws corresponding to (3.1) - (3.6), and add a few additional remarks. If D_0 is any bounded, regular subregion of D, the divergence theorem applied to (3.1) - (3.6) immediately yields, for $0 \le t \le T$,

$$\text{(i}'\text{)}\quad \frac{d}{dt}\int_{D_0}[\Gamma(\underset{\sim}{\nabla}\underset{\sim}{u}) + \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i]\,d\underset{\sim}{x} - \oint_{\partial D_0}\sigma_{jk}n_k\dot{u}_j\,dS = 0\ , \tag{3.7}$$

$$\text{(ii}'\text{)}\quad \frac{d}{dt}\int_{D_0}\rho\dot{u}_i\,d\underset{\sim}{x} - \oint_{\partial D_0}\sigma_{ik}n_k\,dS = 0\ , \tag{3.8}$$

$$\text{(iii}'\text{)}\quad \frac{d}{dt}\int_{D_0}\rho\,\epsilon_{ijk}x_j\dot{u}_k\,d\underset{\sim}{x} - \oint_{\partial D_0}\epsilon_{ijm}x_j\sigma_{mk}n_k\,dS = 0\ , \tag{3.9}$$

$$\text{(iv}'\text{)}\quad \frac{d}{dt}\int_{D_0}\rho\dot{u}_j u_{j,i}\,d\underset{\sim}{x} + \oint_{\partial D_0}(Ln_i - u_{j,i}\sigma_{jk}n_k)\,dS = 0\ , \tag{3.10}$$

$$(v') \quad \frac{d}{dt}\int_{D_0} [\rho\dot{u}_j(u_j + x_m u_{j,m} + t\dot{u}_j) + tL]\, d\underset{\sim}{x}$$

$$- \oint_{\partial D_0} [\sigma_{jk} n_k (u_j + x_m u_{j,m} + t\dot{u}_j) - n_k x_k]\, dS = 0 \,, \qquad (3.11)$$

$$(vi') \quad \frac{d}{dt}\int_{D_0} (\rho\epsilon_{ijk} u_k \dot{u}_j + \rho\epsilon_{ijk} x_j \dot{u}_m u_{m,k})\, d\underset{\sim}{x}$$

$$+ \oint_{\partial D_0} (\epsilon_{imj} u_m \sigma_{jk} n_k - \epsilon_{imj} x_j u_{\ell,m} \sigma_{\ell k} n_k + \epsilon_{imk} n_k x_m L)\, dS = 0 \,. \qquad (3.12)$$

In (3.7) - (3.12), ∂D_0 is the boundary of D_0 and $\underset{\sim}{n}$ is the unit outward normal on ∂D_0.

The first three conservation laws (3.7) - (3.9) correspond respectively to conservation of energy, conservation of linear momentum, and conservation of angular momentum. They are easily verified in the usual way from the basic equations (2.7),(2.3) and are included here only for reasons which will become clearer in the proof of Theorem 1.

Equations (3.10) - (3.12) represent conservation laws for linear elastodynamics which are believed to be new. They reduce to the three conservation laws obtained in [7] for linear elastostatics when the displacement field $\underset{\sim}{u}$ in (3.10) - (3.12) and (2.7) is independent of the time[1]. In particular (3.10) is the dynamical counterpart of the static conservation law which, in two dimensions, reduces to Rice's

[1] The reduction to the time-independent case is somewhat more involved for (3.11) than for the others.

path-independent integral (1.1). In the elastostatic case a physical interpretation of the surface integrals occurring in the time-independent versions of (3.10) - (3.12) has been given by Rice and Budiansky [14].

We remark that (3.1) - (3.4) remain valid in finite elasticity if x_k are material coordinates, $\Gamma(\nabla \underset{\sim}{u}(\underset{\sim}{x}))$ is the strain energy density per unit undeformed volume, ρ the mass per unit undeformed volume, σ_{ij} the components of the Piola-Kirchhoff stress tensor, and L is defined as in (2.14). The conservation law (3.5) depends critically on the linearity of the field equations (2.7) and is not generally valid in finite elasticity[1]. Finally, (3.6) remains valid in finite elasticity for <u>isotropic</u> materials, provided x_k, $\Gamma(\nabla \underset{\sim}{u}(\underset{\sim}{x}))$, ρ, σ_{ij} and L are interpreted as above.

In order to write certain of the differential forms (3.1) - (3.6) of the conservation laws in such a way that they formally resemble analogous results in other fields, and also to facilitate the application of Noether's theorem, it is convenient to introduce some further notation. We set

$$x_0 \equiv t \quad , \quad \frac{\partial}{\partial t}(\ ) \equiv (\ )_{,0} \ . \tag{3.13}$$

We further adopt the convention that Greek subscripts have the range 0, 1, 2, 3, while Latin subscripts continue to take the values 1, 2, 3, and we continue to sum repeated subscripts over the appropriate range. The position vector with components x_α is denoted by $\underset{\sim}{\xi}$. We denote by R the four-dimensional region

$$R = [0,T] \times D \ , \tag{3.14}$$

and we define

[1]It would seem that (3.5) is reminiscent of certain results in classical mechanics which are related to the virial theorem.

$$\bar{c}_{i\alpha k\beta} = \begin{cases} c_{ijk\ell} & \text{if } \alpha = j\,,\ \beta = \ell \\ -\rho\delta_{ik} & \text{if } \alpha = \beta = 0\,, \\ 0 & \text{if } \alpha = 0\,,\ \beta \neq 0 \text{ or } \alpha \neq 0\,,\ \beta = 0\,. \end{cases} \tag{3.15}$$

Reference to (2.15),(3.15) then shows that the Lagrangian density can now be written in the more compact four-dimensional notation as follows:

$$L = L(u_{m,\alpha}) = \frac{1}{2}\,\bar{c}_{i\alpha k\beta}\,u_{i,\alpha}\,u_{k,\beta}\,. \tag{3.16}$$

We note that (2.4),(3.15) imply that

$$\bar{c}_{i\alpha k\beta} = \bar{c}_{k\beta i\alpha}\,. \tag{3.17}$$

As a final remark prior to proving Theorem 1, we introduce a formal four-dimensional energy-momentum tensor $\underset{\sim}{T}$, an angular momentum tensor $\underset{\sim}{A}$ and a spin tensor $\underset{\sim}{S}$ as follows. Let

$$T_{\alpha\beta} = u_{k,\alpha}\,L_{,u_{k,\beta}} - L\delta_{\alpha\beta}\,, \tag{3.18}$$

$$A_{k\alpha} = \epsilon_{km\ell}\,x_\ell\,T_{m\alpha}\,, \tag{3.19}$$

$$\left.\begin{aligned} S_{ki} &= \epsilon_{km\ell}\,u_m\,L_{,u_{i,\ell}} = \epsilon_{km\ell}\,u_m\,\sigma_{i\ell}\,, \\ S_{k0} &= -\rho\,\epsilon_{km\ell}\,u_m\,\dot{u}_\ell\,. \end{aligned}\right\} \tag{3.20}$$

It may be verified that the conservation laws (3.1) and (3.4) together are equivalent to

$$T_{\alpha\beta,\beta} = 0\,, \tag{3.21}$$

while (3.12) reduces to

$$(S_{k\beta} + A_{k\beta})_{,\beta} = 0\,. \tag{3.22}$$

Equation (3.6) suggests that the conservation law (3.4) is associated with the balance of "wave momentum". In general (3.21),(3.22) resemble certain results in the quantum theory of fields [15]. See also [2],[9].

We turn now to the

Proof of Theorem 1. All of the results (3. 1) - (3. 6) can be verified directly from the field equations of Section 2. Such a verification, however, fails to suggest the source of (3. 4) - (3. 6), and a proof based on Noether's theorem is more instructive.

Since the elastodynamic equations of motion (2. 7) are the Euler-Lagrange equations associated with the Lagrangian functional (2. 13), (2. 15), we apply Noether's theorem of Section 2 to the functional

$$\mathcal{L}[\underset{\sim}{w}] = \int_R L\left(w_{i,\alpha}(\underset{\sim}{\xi})\right) d\underset{\sim}{\xi} \; , \tag{3.23}$$

where, according to (3. 16)

$$L\left(w_{i,\alpha}(\underset{\sim}{\xi})\right) = \frac{1}{2}\bar{c}_{i\alpha k\beta} w_{i,\alpha}(\underset{\sim}{\xi}) w_{k,\beta}(\underset{\sim}{\xi}) \; , \tag{3.24}$$

We shall explicitly exhibit six transformations of the form (2. 17) under which the functional $\mathcal{L}$ is infinitesimally invariant. The conservation laws (i) - (vi) of Theorem 1 then emerge from Noether's theorem as specializations of the general result (2. 22) appropriate to the functional $\mathcal{L}$ and the particular transformations considered.

(i) To obtain (3. 1), we consider a family of transformations (2. 17) of the form

$$\underset{\sim}{\xi}^* = \underset{\sim}{\xi} + \epsilon \underset{\sim}{\delta} \quad , \quad \underset{\sim}{w}^* = \underset{\sim}{w} \tag{3.25}$$

where the four-dimensional constant vector $\underset{\sim}{\delta}$ is given by

$$\delta_0 = 1 \quad , \quad \delta_i = 0 \; .$$

Thus the transformation $\underset{\sim}{\xi} \to \underset{\sim}{\xi}^*$ is merely a translation of amount ϵ in time, while the transformation $\underset{\sim}{w} \to \underset{\sim}{w}^*$ is the identity. It is readily confirmed that the functional $\mathcal{L}$ at (3.23) is invariant at every $\underset{\sim}{w}$ under the time-translation (3.25). When (2.23) is specialized to the transformation (3.25), it is found that

$$\varphi_0 = 1 \quad , \quad \varphi_i = 0 \quad , \quad \overline{\psi}_i = -\dot{u}_i \ . \tag{3.26}$$

If these values of φ_α and $\overline{\psi}_i$ are substituted into (2.22) with L of (3.24) in place of F, the general conservation law (2.22) is easily shown to reduce to (3.1). Thus (3.1) — which in the integral form (3.7) corresponds to conservation of energy — follows from the invariance of the Lagrangian under time translation.

(ii) To obtain (3.2) we introduce the family of transformations

$$\underset{\sim}{\xi}^* = \underset{\sim}{\xi} \quad , \quad \underset{\sim}{w}^* = \underset{\sim}{w} + \epsilon\, \underset{\sim}{a} \tag{3.27}$$

where the three-dimensional vector $\underset{\sim}{a}$ is given by

$$a_k = \delta_{ki} \qquad \underline{i \text{ fixed}} \ .$$

The transformations (3.27) represent <u>rigid body translations</u> of D, and the invariance of $\mathcal{L}$ is again easily established. The general conservation law (2.22) specializes immediately to (3.2). Thus conservation of linear momentum is associated with the invariance of the Lagrangian under rigid body translations.

(iii) The angular momentum conservation law (3.3) is obtained from the invariance of $\mathcal{L}$ under <u>rigid body rotations</u>. Thus the appropriate transformations of the form (2.17) are

$$\underset{\sim}{\xi}^* = \underset{\sim}{\xi} \quad , \quad \underset{\sim}{w}^* = \underset{\sim}{Q}(\epsilon)\underset{\sim}{w}^* \ , \tag{3.28}$$

where the 3×3 matrix $\underset{\sim}{Q}$ is proper orthogonal and satisfies $\underset{\sim}{Q}(0)=\underset{\sim}{1}$, and $\underset{\sim}{1}$ is the identity matrix. It is easily shown that $\mathcal{L}$ of (3.23),(3.24) is invariant under the transformations (3.28), and that the corresponding conservation law is (3.3).

The relationship between the conservation of energy, linear and angular momentum on the one hand and invariance of the Lagrangian under time translations, rigid body translations and rigid body rotations on the other hand, is well known in mechanics [16].

(iv) To establish (3.4), which is the dynamical counterpart of Rice's path independent integral in elastostatics, we introduce a family of coordinate translations by writing

$$\underset{\sim}{\xi}^* = \underset{\sim}{\xi} + \epsilon \underset{\sim}{b} \quad , \quad \underset{\sim}{w}^* = \underset{\sim}{w} \, . \tag{3.29}$$

The invariance of $\mathcal{L}$ under (3.29) follows easily, and, (2.22), when appropriately specialized, furnishes (3.4). It should be noted that (3.1) and (3.4) have in common the fact that both are associated with coordinate translations in four dimensions.

(v) The conservation law (3.5) is obtained upon consideration of the family of scale changes

$$\underset{\sim}{\xi}^* = (1+\epsilon)\underset{\sim}{\xi} \quad , \quad w^* = (1-\epsilon)\underset{\sim}{w} \, . \tag{3.30}$$

The functional $\mathcal{L}$ is in this case infinitesimally invariant (although not invariant) under (3.30), as is easily confirmed. When (2.22) is specialized with the aid of (3.30), the conservation law (3.5) follows.

(vi) To obtain (3.6), we first assume that the elastic material is isotropic, so that the $c_{ijk\ell}$ are given by (2.11), the elastic potential by (2.12). It can then be shown that $\mathcal{L}$ of (3.23) is invariant at any $\underset{\sim}{w}$ under the transformations

$$\left.\begin{array}{c} \xi_0^* = \xi_0 \quad , \quad \xi_i^* = Q_{ij}(\epsilon)\,\xi_j \ , \\ w_i^* = Q_{ij}(\epsilon) w_j \end{array}\right\} \tag{3.31}$$

where $\underset{\sim}{Q}(\epsilon)$ is a 3×3 proper orthogonal matrix with elements $Q_{ij}(\epsilon)$, $Q_{ij}(0) = \delta_{ij}$. For an isotropic material, it is readily shown that Γ, L and hence $\mathcal{L}$ are invariant under (3.31). The corresponding conservation law is (3.6).

This completes the proof of Theorem 1.

4. Completeness of the set of conservation laws.

In this section we show that, for isotropic materials, the only transformations under which the Lagrangian functional $\mathcal{L}$ is infinitesimally invariant are those which produce the conservation laws obtained in the preceding section.

Theorem 2. *Suppose the elastic material under consideration is isotropic, and let* $\mathcal{L}$ *be the Lagrangian*

$$\mathcal{L}[\mathbf{w}] = \int_0^T \int_D L(\nabla \mathbf{w}, \dot{\mathbf{w}})\, d\mathbf{x}\, dt \,, \tag{4.1}$$

where D *is a bounded regular region,* L *is given by* (2.15), *and* (2.11) *holds. Then* $\mathcal{L}[w]$ *is infinitesimally invariant at* $\mathbf{w}$ *under transformations of the form*

$$\left.\begin{aligned} \boldsymbol{\xi}^* &= \boldsymbol{\Phi}(\boldsymbol{\xi}, \mathbf{w}(\boldsymbol{\xi}); \epsilon) \\ \mathbf{w}^*(\boldsymbol{\xi}^*) &= \boldsymbol{\Psi}(\boldsymbol{\xi}, \mathbf{w}(\boldsymbol{\xi}); \epsilon) \end{aligned}\right\} \tag{4.2}$$

for every $\mathbf{w}$ *satisfying the displacement equations of motion* (2.7) *on* D *and for every* D, *if and only if the four-dimensional transformations satisfy*

$$\left.\begin{aligned} \boldsymbol{\Phi}(\boldsymbol{\xi}, \mathbf{w}(\boldsymbol{\xi}); \epsilon) &= \boldsymbol{\xi} + \epsilon \boldsymbol{\varphi}(\boldsymbol{\xi}) + o(\epsilon) \,, \\ \boldsymbol{\Psi}(\boldsymbol{\xi}, \mathbf{w}(\boldsymbol{\xi}); \epsilon) &= \mathbf{w} + \epsilon \boldsymbol{\psi}(\mathbf{x}, \mathbf{w}) + o(\epsilon) \,, \end{aligned}\right\} \tag{4.3}$$

where the components φ_α *and* ψ_i *of* $\boldsymbol{\varphi}$ *and* $\boldsymbol{\psi}$ *are given by*

$$\left.\begin{aligned} \varphi_0(\mathbf{x}, \mathbf{w}) &= \nu x_0 + c_0 \equiv \nu t + c_0 \,, \\ \varphi_i(\mathbf{x}, \mathbf{w}) &= \nu x_i + \epsilon_{ijk} b_j x_k + c_i \,, \\ \psi_i(\mathbf{x}, \mathbf{w}) &= -\nu w_i + \epsilon_{ijk} b_j w_k + \epsilon_{ijk} a_j x_k + d_i \,, \end{aligned}\right\} \tag{4.4}$$

while ν, c_α, b_i, a_i *and* d_i *are arbitrary real constants.*

We remark that the "infinitesimal part" of the transformations $\underset{\sim}{\xi} \to \underset{\sim}{\xi}^*$, $\underset{\sim}{w} \to \underset{\sim}{w}^*$ is all that is determined by (4.3), (4.4). As (2.22) shows, however, this is all that is required for the corresponding conservation law.

In (4.4) the terms associated with ν represent a <u>change of scale</u>, those associated with c_α represent a <u>coordinate translation</u>, those involving b_i correspond to a <u>rotation of coordinates</u>, and those involving a_i, d_i comprise a rigid body rotation and translation, respectively. These are precisely the types of transformations which produced the various conservation laws in the preceding section.

<u>Proof of Theorem 2.</u> In the following, we use the notation presented in Equations (3.15) and (3.16) for the elastodynamic Lagrangian. As in those equations we also write $\underset{\sim}{x}$ for $\underset{\sim}{\xi}$ and $\underset{\sim}{u}$ for $\underset{\sim}{w}$. We first establish the necessity of Theorem 2.

Under a transformation of the form given by (4.2), the region R goes into a region R^* and the Lagrangian $\mathcal{L}$ goes into a Lagrangian $\mathcal{L}^*$, where

$$\mathcal{L}^*[u^*] = \int_{R^*} L(\underset{\sim}{\nabla}^* \underset{\sim}{u}^*)\, d\underset{\sim}{x}^* \tag{4.5}$$

Let $J(\underset{\sim}{x}^*, \underset{\sim}{x})$ be the Jacobian determinant defined as

$$J(\underset{\sim}{x}^*, \underset{\sim}{x}) = \det\left(\frac{\partial x^*_\alpha}{\partial x_\beta}\right) \tag{4.6}$$

then we can write

$$\mathcal{L}^*[\underset{\sim}{u}^*] = \int_R L(\underset{\sim}{\nabla}^* \underset{\sim}{u}^*)\, J(\underset{\sim}{x}^*, \underset{\sim}{x})\, d\underset{\sim}{x} \tag{4.7}$$

where it is understood that the integrand is evaluated at $\underset{\sim}{x}, \underset{\sim}{u}(\underset{\sim}{x})$; and where $\underset{\sim}{x}, \underset{\sim}{u}(\underset{\sim}{x})$ are the antecedents of $\underset{\sim}{x}^*, \underset{\sim}{u}^*(\underset{\sim}{x}^*)$ under the mapping given in (4. 2).

For $\mathcal{L}$ to be infinitesimally invariant at $\underset{\sim}{u}$, under the transformation (4. 2), Equation (2. 19) requires

$$\left\{\frac{d}{d\epsilon} \int_{R^*} L(\underset{\sim}{\nabla}^* \underset{\sim}{u}^*)\, d\underset{\sim}{x}^*\right\}_{\epsilon=0} = 0 \quad .$$

And using (4. 7) we are able to write (2. 19) as

$$\left\{\frac{d}{d\epsilon} \int_{R} L(\underset{\sim}{\nabla}^* \underset{\sim}{u}^*)\, J(\underset{\sim}{x}^*, \underset{\sim}{x})\, d\underset{\sim}{x}\right\}_{\epsilon=0} = 0 \quad .$$

Now since the region of integration, R, is independent of the parameter ϵ we may take the derivative under the integral sign, obtaining

$$\int_R \left\{\frac{d}{d\epsilon}[L(\underset{\sim}{\nabla}^* \underset{\sim}{u}^*)\, J(\underset{\sim}{x}^*, \underset{\sim}{x})]\right\}_{\epsilon=0} d\underset{\sim}{x} = 0 \,.$$

Furthermore, since the region R is arbitrary we obtain the following necessary condition

$$\left\{\frac{d}{d\epsilon}[L(\underset{\sim}{\nabla}^* \underset{\sim}{u}^*) J(\underset{\sim}{x}^*, \underset{\sim}{x})]\right\}_{\epsilon=0} = 0 \qquad (4.\ 8)$$

Defining $M(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{\nabla} \underset{\sim}{u})$ to be the left-hand side of Equation (4. 8) and partially carrying out the indicated differentiation, we have

$$M(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{\nabla} \underset{\sim}{u}) = L_{u_{i,\alpha}}(\underset{\sim}{\nabla} \underset{\sim}{u})\left\{\frac{d}{d\epsilon}\, u^*_{i,\alpha^*}\right\}_{\epsilon=0} + L(\underset{\sim}{\nabla} \underset{\sim}{u})\left\{\frac{d}{d\epsilon} J(\underset{\sim}{x}^*, \underset{\sim}{x})\right\}_{\epsilon=0} \qquad (4.\ 9)$$

using the fact that $J(\underset{\sim}{x}, \underset{\sim}{x}) = 1$.

At this point we must obtain an explicit representation for the quantities enclosed in the braces of (4.9). Considering the first of the two braces we see that we must compute u^*_{i,α^*}. To do so we must obtain an expression for the operator $\frac{\partial}{\partial x^*_\alpha}$

$$\frac{\partial}{\partial x_\beta} = \frac{\partial}{\partial x^*_\alpha}\frac{\partial x^*_\alpha}{\partial x_\beta} = \left[\delta_{\alpha\beta} + \epsilon\frac{\partial\varphi_\alpha}{\partial x_\beta}\right]\frac{\partial}{\partial x^*_\alpha} .$$

The inverse of the infinitesimal matrix in brackets is given by

$$\left[\delta_{\alpha\beta} + \epsilon\frac{\partial\varphi_\alpha}{\partial x_\beta}\right]^{-1} = \left[\delta_{\beta\alpha} - \epsilon\frac{\partial\varphi_\beta}{\partial x_\alpha}\right] + o(\epsilon) .$$

Multiplying by this inverse, we obtain

$$\frac{\partial}{\partial x^*_\alpha} = \frac{\partial}{\partial x_\alpha} - \epsilon\frac{\partial\varphi_\beta}{\partial x_\alpha}\frac{\partial}{\partial x_\beta} . \tag{4.10}$$

Therefore we may write

$$u^*_{i,\alpha^*} = u_{i,\alpha} + \epsilon\left[\frac{\partial\psi_i}{\partial x_\alpha} - \frac{\partial\varphi_\beta}{\partial x_\alpha}u_{i,\beta}\right] .$$

It follows that

$$\left\{\frac{d}{d\epsilon}u^*_{i,\alpha^*}\right\}_{\epsilon=0} = \frac{\partial\psi_i}{\partial x_\alpha} - \frac{\partial\varphi_\beta}{\partial x_\alpha}u_{i,\beta} . \tag{4.11}$$

Now considering the second brace of Equation (4.9), we can write, by (4.6),

$$J(\underset{\sim}{x}^*,\underset{\sim}{x}) = \frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\sigma\rho}x^*_{\alpha,\mu}x^*_{\beta,\nu}x^*_{\gamma,\sigma}x^*_{\delta,\rho} .$$ [1]

And using (4.10) we obtain

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[1] Where by $\epsilon_{\alpha\beta\gamma\delta}$ we mean the 4-dimensional alternator.

$$J(\underset{\sim}{x}^*, \underset{\sim}{x}) = 1 + \epsilon \frac{\partial \varphi_\gamma}{\partial x_\gamma} + o(\epsilon) \ .$$

Therefore

$$\left\{ \frac{d}{d\epsilon} J(\underset{\sim}{x}^*, \underset{\sim}{x}) \right\}_{\epsilon=0} = \frac{\partial \varphi_\gamma}{\partial x_\gamma} \ . \tag{4. 12}$$

And it follows that (4. 9) becomes

$$M(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{\nabla} \underset{\sim}{u}) = L_{,u_{i,\alpha}}(\underset{\sim}{\nabla} \underset{\sim}{u}) \left[\frac{\partial \psi_i}{\partial x_\alpha} - \frac{\partial \varphi_\beta}{\partial x_\alpha} u_{i,\beta} \right] + L(\underset{\sim}{\nabla} \underset{\sim}{u}) \frac{\partial \varphi_\gamma}{\partial x_\gamma} \ . \tag{4. 13}$$

Fully expanding the derivatives of (4. 13) and using (3. 15),(3. 16) we obtain finally

$$\begin{aligned} 2M(\underset{\sim}{x}, \underset{\sim}{u}, \underset{\sim}{\nabla} \underset{\sim}{u}) = {} & 2c_{i\alpha k\beta} u_{i,\alpha} \psi_{k,\beta} \\ & + c_{i\alpha k\beta} u_{i,\alpha} [\varphi_{\gamma,\gamma} u_{k,\beta} + 2\psi_{k,n} u_{n,\beta} - 2\varphi_{\rho,\beta} u_{k,\rho}] \\ & + c_{i\alpha k\beta} u_{i,\alpha} [\varphi_{\gamma,n} (u_{k,\beta} u_{n,\gamma} - 2u_{k,\gamma} u_{n,\beta})] \ . \end{aligned} \tag{4. 14}$$

The expression (4. 14) must vanish without restriction on $\underset{\sim}{\nabla} \underset{\sim}{u} \equiv (u_{i,\alpha})$ for the Lagrangian to be infinitesimally invariant under the mapping (4. 2). Since $\underset{\sim}{\varphi}, \underset{\sim}{\psi}$ are independent of $\underset{\sim}{\nabla} \underset{\sim}{u}$, it follows that the linear, quadratic, and cubic parts in $\underset{\sim}{\nabla} \underset{\sim}{u}$ of expression (4. 14) must vanish separately. We will treat each of these parts in the remainder of this chapter. It is therefore expedient to make the following definitions.

$$2M_1 \equiv 2c_{i\alpha k\beta} u_{i,\alpha} \psi_{k,\beta} \tag{4. 15}$$

$$2M_2 \equiv c_{i\alpha k\beta} u_{i,\alpha} [\varphi_{\gamma,\gamma} u_{k,\beta} + 2\psi_{k,n} u_{n,\beta} - 2\varphi_{\rho,\beta} u_{k,\rho}] \tag{4. 16}$$

$$2M_3 \equiv c_{i\alpha k\beta} u_{i,\alpha} [\varphi_{\gamma,s} (u_{k,\beta} u_{s,\gamma} - 2u_{k,\gamma} u_{s,\beta})] \tag{4. 17}$$

Linear Terms[1]

It is required that (4. 15) vanish without any restriction on $\underset{\sim\sim}{\nabla u}$.

$$M_1 = c_{i\alpha k\beta} u_{i,\alpha} \psi_{k,\beta} = 0 .$$

Using (3. 15) we have

$$M_1 = c_{i0k0} u_{i,0} \psi_{k,0} + c_{ijk\ell} u_{i,j} \psi_{k,\ell}$$

$$= -\rho u_{i,0} \psi_{i,0} + c_{ijk\ell} u_{i,j} \psi_{k,\ell} .$$

Therefore we must require

$$\psi_{i,x_0} = 0 \quad \text{all } i \tag{4.18}$$

$$c_{ijk\ell} u_{i,j} \psi_{k,\ell} = 0 \tag{4.19}$$

Using (2. 11) in Equation (4. 19) we require

$$\sum_{i,k} u_{i,i}(\lambda\psi_{k,k} + 2\mu\psi_{i,i}) + \mu \sum_{\substack{i,k \\ i\neq k}} u_{i,k}(\psi_{i,k} + \psi_{k,i}) = 0 .$$

By the independence of the displacement gradients, we are led to

$$\lambda\psi_{k,k} + 2\mu\psi_{i,i} = 0 \quad \text{all } i,k \ \ (\text{no sum}),$$

$$\psi_{i,k} + \psi_{k,i} = 0 \quad \text{all } i,k\,;\, i\neq k .$$

Choosing i=k in the first of these two equations, we see that we have

$$\psi_{k,k} = 0 \quad \text{all } k \ \ (\text{no sum}).$$

Therefore, using the second relation, we have

$$\psi_{i,x_k} + \psi_{k,x_i} = 0 \quad \text{all } i,k . \tag{4.20}$$

---

[1] We note explicitly that in our treatment of the linear terms $\psi_{k,j}$ invariably means ψ_{k,x_j} (j=1,2,3) whereas $\psi_{k,\alpha}$ means ψ_{k,x_α} ($\alpha = 0,1,2,3$).

We see therefore that infinitesimal invariance under the transformation (4. 2) imposes the requirements expressed by (4. 18) and (4. 20) on the linear terms of (4. 14).

We now consider the cubic terms.

<u>Cubic Terms</u>[1]

It is required that (4. 17) vanish without restricting the displacement gradients $\nabla \underset{\sim}{u}$.

$$2M_3 = c_{i\alpha k\beta} u_{i,\alpha} [\varphi_{\gamma,s} (u_{k,\beta} u_{s,\gamma} - 2u_{k,\gamma} u_{s,\beta})] = 0 \ .$$

Relabeling the indices, we may write

$$2M_3 = [\varphi_{\gamma,s} c_{i\alpha k\beta} - 2\varphi_{\beta,s} c_{i\alpha k\gamma}] \, u_{i,\alpha} u_{k,\beta} u_{s,\gamma} = 0 \ .$$

The term of this sum for which $\alpha = \beta = \gamma = 0$ is independent of the others and must therefore vanish separately

$$0 = -\varphi_{0,s} c_{i0k0} u_{i,0} u_{k,0} u_{s,0} = +\rho \varphi_{o,s} (u_{i,0})^2 u_{s,0} \ .$$

Therefore we obtain

$$\varphi_{0,u_s} = 0 \qquad s = 1, 2, 3 \ . \tag{4. 21}$$

Considering the remaining seven choices of zero or nonzero values to α, β, γ and using (4. 21) we obtain

$$0 = [\varphi_{\ell,s} c_{i0k0} - 2\varphi_{\ell,k} c_{i0s0}] u_{i,0} u_{k,0} u_{s,\ell}$$
$$+ [\varphi_{n,s} c_{ijk\ell} - 2\varphi_{\ell,s} c_{ijkn}] u_{i,j} u_{k,\ell} u_{s,n} \ . \tag{4. 22}$$

These two sums must vanish separately. Considering the first and using (2. 11) we obtain

[1] In our treatment of the cubic terms $\varphi_{\gamma,s}$, <u>for example</u>, means φ_{γ,u_s}, ($s=1, 2, 3$).

$$0 = -\sum_{\ell,s} \varphi_{\ell,s} u_{s,\ell} u_{s,0} u_{s,0} + \sum_{\substack{k,\ell,s \\ k \neq s}} (\varphi_{\ell,s} u_{k,0} - 2\varphi_{\ell,k} u_{s,0}) u_{k,0} u_{s,\ell} \,. \tag{4.23}$$

The two sums in (4. 23) are independent in $\underset{\sim\sim}{\nabla u}$ and we must therefore enforce

$$-\sum_{\ell,s} \varphi_{\ell,s} u_{s,\ell} u_{s,0} u_{s,0} = 0 \,.$$

But for this sum to vanish we need

$$\varphi_{\ell,u_s} = 0 \quad \text{all } \ell, s \,. \tag{4.24}$$

Now we note that (4. 24) is sufficient to ensure the vanishing of the remaining terms in (4. 23) and (4. 22). It follows therefore that (4. 24) and (4. 21) together ensure the vanishing of (4. 17).

We now write (4. 24) and (4. 21) together as

$$\varphi_{\alpha,u_k} = 0 \quad \text{all } \alpha, k \,. \tag{4.25}$$

This completes the treatment of the cubic terms for the elastodynamic case. It is important to note however that this proof requires a nontrivial modification for the elastostatic case. Namely, the expression (4. 17) reduces to just the second sum of (4. 22). Thus for the elastostatic case, we must show directly that

$$[\varphi_{n,s} c_{ijk\ell} - 2\varphi_{\ell,s} c_{ijkn}] u_{i,j} u_{k,\ell} u_{s,n} = 0 \tag{4.26}$$

leads to (4. 24).

The proof of this fact is given below:

The contribution due to the first term within the bracket of (4. 26) may be written (with a change of indices), using (2. 11), as

$$\varphi_{n,s}c_{ijk\ell}u_{i,j}u_{k,\ell}u_{s,n} = (\lambda+2\mu)\sum_{\substack{k,n\\k\neq n}}(u_{k,k})^2 u_{k,n}\varphi_{n,k}$$

$$+\mu\sum_{\substack{k,n\\k\neq n}}u_{k,k}u_{n,k}u_{k,n}\varphi_{k,k} + (\lambda+2\mu)\sum_{k}(u_{k,k})^3\varphi_{k,k}$$

$$+(\lambda+2\mu)\sum_{\substack{k,m,n\\m\neq n}}(u_{k,k})^2 u_{n,m}\varphi_{m,n} + \lambda\sum_{\substack{i,k,m,n\\i\neq k}}u_{k,k}u_{i,i}u_{n,m}\varphi_{m,n}$$

$$+\mu\sum_{\substack{k,n\\n\neq k}}u_{k,k}(u_{n,k})^2\varphi_{k,k} + \mu\sum_{\substack{k,m,n\\n\neq k,m\neq k}}u_{m,m}u_{n,k}(u_{k,n}+u_{n,k})\varphi_{m,m}$$

$$+\mu\sum_{\substack{i,k,m,n\\k\neq n,i\neq m}}u_{n,k}u_{i,m}(u_{k,n}+u_{n,k})\varphi_{m,i}\ .$$

The contribution due to the second term within the bracket may be written (with a change of indices), using (2. 11), as

$$-2\varphi_{\ell,s}c_{ijkn}u_{i,j}u_{k,\ell}u_{s,n} = -2(\lambda+2\mu)\sum_{k,m,n}u_{k,k}u_{n,k}u_{k,m}\varphi_{m,n}$$

$$-2\lambda\sum_{\substack{k,\ell,m,n\\\ell\neq k}}u_{k,k}u_{n,\ell}u_{\ell,m}\varphi_{m,n}$$

$$-2\mu\sum_{\substack{k,\ell,m,n\\\ell\neq k}}u_{n,k}u_{k,\ell}u_{\ell,m}\varphi_{m,n}$$

$$-2\mu\sum_{k,\ell,m,n}u_{\ell,k}u_{n,k}u_{\ell,m}\varphi_{m,n}\ .$$

Resolving the first three terms on the right-hand side of this expression into three parts each where m≠n, m=n≠k, m=n=k we obtain

$$-2\varphi_{\ell,s}c_{ijkn}u_{i,j}u_{k,\ell}u_{s,n} = -2(\lambda+2\mu)\sum_{\substack{k,m,n\\ m\neq n}} u_{k,k}u_{n,k}u_{k,m}\varphi_{m,n}$$

$$-2(\lambda+2\mu)\sum_{\substack{k,n\\ n\neq k}} u_{k,k}u_{n,k}u_{k,n}\varphi_{n,n} - 2(\lambda+2\mu)\sum_{k}(u_{k,k})^3\varphi_{k,k}$$

$$-2\lambda\sum_{\substack{k,\ell,m,n\\ \ell\neq k, m\neq n}} u_{k,k}u_{n,\ell}u_{\ell,m}\varphi_{m,n} - 2\lambda\sum_{\substack{k,\ell,n\\ \ell\neq k, n\neq k}} u_{k,k}u_{n,\ell}u_{\ell,n}\varphi_{n,n}$$

$$-2\lambda\sum_{\substack{k,n\\ n\neq k}} u_{k,k}u_{k,n}u_{n,k}\varphi_{k,k} - 2\mu\sum_{\substack{k,\ell,m,n\\ \ell\neq k, m\neq n}} u_{n,k}u_{k,\ell}u_{\ell,m}\varphi_{m,n}$$

$$-2\mu\sum_{\substack{k,\ell,n\\ \ell\neq k, n\neq k}} u_{n,k}u_{k,\ell}u_{\ell,n}\varphi_{n,n} - 2\mu\sum_{k,\ell,m,n} u_{\ell,k}u_{n,k}u_{\ell,m}\varphi_{m,n}\,.$$

Resolve the first term above into three parts m=n=k, m=n≠k, m≠n. In the eighth term, interchange k and n, and resolve term eight into two parts k=ℓ, k≠ℓ. Combine the k=ℓ part with terms 2, 6, 9. Finally add the entire result to the contribution due to the first bracket to obtain the expression given below for the total elastostatic cubic terms.

$$\left.\begin{aligned}&[\varphi_{n,s}c_{ijk\ell} - 2\varphi_{\ell,s}c_{ijkn}]u_{i,j}u_{k,\ell}u_{s,n} = \\ &\quad -(\lambda+2\mu)\sum_{k\neq n}(u_{k,k})^2u_{k,n}\varphi_{n,k} - 2(\lambda+2\mu)\sum_{\substack{k,m,n\\ k\neq m, m\neq n}} u_{k,k}u_{n,k}u_{k,m}\varphi_{m,n}\end{aligned}\right\} \quad (4.26)$$

$$
\left.
\begin{aligned}
&-2\sum_{\substack{k,n\\ k\neq n}} u_{k,k}u_{n,k}u_{k,n}\left[(\lambda+2\mu)\varphi_{n,n}+\left(\lambda+\tfrac{3}{2}\mu\right)\varphi_{k,k}\right]\\
&-(\lambda+2\mu)\sum_{k}(u_{k,k})^3\varphi_{k,k}+(\lambda+2\mu)\sum_{\substack{k,m,n\\ m\neq n}}(u_{k,k})^2u_{n,m}\varphi_{m,n}\\
&+\lambda\sum_{\substack{i,k,m,n\\ i\neq k}} u_{k,k}u_{i,i}u_{n,m}\varphi_{m,n}+\mu\sum_{\substack{n,k\\ n\neq k}} u_{k,k}(u_{n,k})^2\varphi_{k,k}\\
&+\mu\sum_{\substack{k,m,n\\ n\neq k,m\neq k}} u_{m,m}u_{n,k}(u_{k,n}+u_{n,k})\varphi_{m,n}\\
&+\mu\sum_{i,k,m,n} u_{n,k}u_{i,m}(u_{k,n}+u_{n,k})\varphi_{m,i}\\
&-2\lambda\sum_{\substack{k,\ell,m,n\\ \ell\neq k,m\neq n}} u_{k,k}u_{n,\ell}u_{\ell,m}\varphi_{m,n}-2\lambda\sum_{\substack{k,\ell,n\\ \ell\neq k,n\neq k}} u_{k,k}u_{n,\ell}u_{\ell,n}\varphi_{n,n}\\
&-2\mu\sum_{\substack{k,\ell,n\\ k\neq\ell\neq n\neq k}} u_{k,n}u_{n,\ell}u_{\ell,k}\varphi_{k,k}-2\mu\sum_{\substack{k,\ell,m,n\\ \ell\neq k}} u_{n,k}u_{\ell,k}u_{\ell,m}\varphi_{m,n}\\
&-2\mu\sum_{\substack{k,\ell,m,n\\ \ell\neq k,m\neq n}} u_{n,k}u_{k,\ell}u_{\ell,m}\varphi_{m,n}\ .
\end{aligned}
\right\}
\quad \begin{matrix}(4.26)\\ \text{cont.}\end{matrix}
$$

Only the first and third terms of (4.26) matter. Conditions necessary to ensure that they vanish are sufficient to ensure the vanishing of the remainder. The first and third terms are independent of each other in ∇u and are moreover independent of everything else.

The condition necessary to ensure the vanishing of the first term above is

$$\varphi_{n,k} = 0 \quad \text{all} \quad n,k\,,\; n \neq k\,. \tag{4.27}$$

The condition necessary to ensure the vanishing of the third term above is

$$a\varphi_{n,n} + b\varphi_{k,k} = 0 \quad \text{all} \quad n,k\,,\; n \neq k\,,\; \text{no sum} \tag{4.28}$$

where $a = \lambda+2\mu$, $b = \lambda+\frac{3}{2}\mu$. Consider this second condition. Let $m \neq n$, $m \neq k$, then

$$a\varphi_{n,n} + b\varphi_{k,k} = 0 = a\varphi_{m,m} + b\varphi_{k,k}\,.$$

Thus we deduce $\varphi_{n,n} = \varphi_{m,m}$ all m,n. And the necessary condition (4.28) for the vanishing of the third term of expression (4.26) becomes $(a+b)\varphi_{n,n} = 0$, all n. Hence $\varphi_{n,n} = 0$ all n. And it follows from the first term necessary condition (4.27) that

$$\varphi_{n,k} = 0 \quad \underline{\text{all}} \quad n,k\,. \tag{4.29}$$

This condition is manifestly sufficient to ensure the vanishing of the remaining cubic terms, and is therefore a necessary (and sufficient) condition for the vanishing of the cubic elastostatic terms. Note, moreover, that it is precisely condition (4.25) restricted to the elastostatic case.

This completes the treatment of the cubic terms in $\underset{\sim}{\nabla}\underset{\sim}{u}$.

Quadratic Terms[1]

It is required that (4.16) vanish without restricting $\underset{\sim}{\nabla}\underset{\sim}{u}$.

[1] We note explicitly that in our treatment of the quadratic terms $\varphi_{\gamma,n}$ means φ_{γ,x_n} (n=1,2,3) whereas $\psi_{k,n}$ means ψ_{k,u_n} (n=1,2,3).

$$2M_2 \equiv c_{i\alpha k\beta} u_{i,\alpha}[\varphi_{\gamma,\gamma} u_{k,\beta} + 2\psi_{k,n} u_{n,\beta} - 2\varphi_{\rho,\beta} u_{k,\rho}] = 0 .$$

Relabeling the indices, we may write this expression as

$$[\varphi_{\gamma,\gamma} c_{i\alpha k\beta} - 2\varphi_{\beta,\gamma} c_{i\alpha k\gamma} + 2\psi_{n,k} c_{i\alpha n\beta}] u_{i,\alpha} u_{k,\beta} = 0 .$$

Expanding this expression somewhat, we have

$$\begin{aligned}
&[\varphi_{\gamma,\gamma} c_{i0k0} - 2\varphi_{0,\gamma} c_{i0k\gamma} + 2\psi_{n,k} c_{i0n0}] u_{i,0} u_{k,0} \\
&\quad + [\varphi_{\gamma,\gamma} c_{i0k\ell} - 2\varphi_{\ell,\gamma} c_{i0k\gamma} + 2\psi_{n,k} c_{i0n\ell}] u_{i,0} u_{k,\ell} \\
&\quad + [\varphi_{\gamma,\gamma} c_{i\ell k0} - 2\varphi_{0,\gamma} c_{i\ell k\gamma} + 2\psi_{n,k} c_{i\ell no}] u_{i,\ell} u_{k,0} \\
&\quad + [\varphi_{\gamma,\gamma} c_{ijk\ell} - 2\varphi_{\ell,\gamma} c_{ijk\gamma} + 2\psi_{n,k} c_{ijn\ell}] u_{i,j} u_{k,\ell} = 0 .
\end{aligned}$$

Using (3. 15) this expression becomes

$$\begin{aligned}
&[\varphi_{\gamma,\gamma} c_{i0k0} - 2\varphi_{0,0} c_{i0k0} + 2\psi_{n,k} c_{i0n0}] u_{i,0} u_{k,0} \\
&\quad - 2[\varphi_{\ell,0} c_{i0k0} + \varphi_{0,m} c_{k\ell im}] u_{i,0} u_{k,\ell} \\
&\quad + [\varphi_{\gamma,\gamma} c_{ijk\ell} - 2\varphi_{\ell,m} c_{ijkm} + 2\psi_{n,k} c_{ijn\ell}] u_{i,j} u_{k,\ell} = 0 . \qquad (4.30)
\end{aligned}$$

Each of these bracketed coefficients must vanish separately. Considering the first, and using (3. 15) we obtain

$$\rho[(\varphi_{\gamma,\gamma} - 2\varphi_{0,0}) u_{i,0} u_{i,0} + 2\psi_{i,k} u_{i,0} u_{k,0}] = 0 .$$

Rearranging, we have

$$\sum_i \left[\sum_\gamma \varphi_{\gamma,\gamma} - 2\varphi_{0,0} + 2\psi_{i,i}\right] u_{i,0} u_{i,0} + \sum_{\substack{i,k \\ i \neq k}} \left[\psi_{i,k} + \psi_{k,i}\right] u_{i,0} u_{k,0} = 0 .$$

Again each sum must vanish separately, and we therefore obtain

$$\sum_\gamma \varphi_{\gamma,\gamma} - 2\varphi_{0,0} + 2\psi_{i,i} = 0 \qquad \text{all} \quad i . \qquad (4.31)$$

$$\psi_{i,u_k}+\psi_{k,u_i}=0 \quad \text{all} \quad i,k \quad i\neq k\,. \tag{4.32}$$

Considering the second sum in (4.30) we have

$$[\varphi_{\ell,0}c_{i0k0}+\varphi_{0,m}c_{k\ell im}]u_{i,0}u_{k,\ell}=0\,.$$

Again by the independence of $u_{i,0}$ and $u_{k,\ell}$ we obtain

$$[\varphi_{\ell,0}c_{i0k0}+\varphi_{0,m}c_{k\ell im}]u_{k,\ell}=0 \quad \text{all} \quad i\,.$$

Using (3.15) and (2.11) we have

$$-\rho\varphi_{m,0}u_{i,m}+\lambda\varphi_{0,i}u_{m,m}+\mu\varphi_{0,m}[u_{i,m}+u_{m,i}]=0\,.$$

Rearranging this expression we have, for all i, (no sum on i)

$$[(\lambda-2\mu)\varphi_{0,i}+\rho\varphi_{i,0}]u_{i,i}+\sum_{\substack{m\\ m\neq i}}(-\rho\varphi_{m,0}+\mu\varphi_{0,m})u_{i,m}$$

$$+\mu\sum_{\substack{m\\ m\neq i}}\varphi_{0,m}u_{m,i}+\lambda\varphi_{0,i}\sum_{\substack{m\\ m\neq i}}u_{m,m}=0\,. \tag{4.33}$$

Again, each of these terms must vanish separately. In particular we therefore have

$$\lambda\varphi_{0,i}\sum_{\substack{m\\ m\neq i}}u_{m,m}=0 \quad \text{all} \quad i\,.$$

This leads immediately to

$$\varphi_{0,x_i}=0 \quad \text{all} \quad i\,. \tag{4.34}$$

Using (4.34) in the first term of (4.33) and equating that term to zero, we obtain

$$-\rho\varphi_{i,0}u_{i,i}=0 \quad \text{all} \quad i$$

which yields

$$\varphi_{i,x_0} = 0 \quad \text{all} \quad i\ .$$

Equations (4.31), (4.32), (4.34), (4.35) are necessary (and jointly sufficient) for the vanishing of the first and second terms of (4.30). It remains only to consider the third term of (4.30).

As a final necessary condition for the vanishing of (4.30) and thus the cubic terms in $\underset{\sim}{\nabla}\underset{\sim}{u}$, we must require that

$$[\varphi_{\gamma,\gamma}c_{ijk\ell} - 2\varphi_{\ell,n}c_{ijkn} + 2\psi_{n,k}c_{ijn\ell}]u_{i,j}u_{k,\ell} = 0\ . \tag{4.36}$$

The term $i=j=k=\ell$ must vanish independently of the others, therefore we obtain, by means of (2.11),

$$\sum_{\gamma}\varphi_{\gamma,\gamma} - 2\varphi_{i,i} + 2\psi_{i,i} = 0 \quad \text{all} \quad i \ (\text{no sum})\ . \tag{4.37}$$

Now, (4.37) together with (4.31) yields

$$\varphi_{\alpha,x_\alpha} = \beta(\underset{\sim}{x},\underset{\sim}{u}) \quad \text{all} \quad \alpha \ (\text{no sum}) \tag{4.38}$$

$$\psi_{i,u_i} = -\beta(\underset{\sim}{x},\underset{\sim}{u}) \quad \text{all} \quad i \ (\text{no sum}) \tag{4.39}$$

where $\beta(\underset{\sim}{x},\underset{\sim}{u})$ is a function to be determined later. Using (4.38), (2.11) in (4.36) and rearranging the sums, we obtain

$$0 = 4\beta(\underset{\sim}{x},\underset{\sim}{u})c_{ijk\ell}u_{i,j}u_{k,\ell} + 2\lambda\sum_{i,k}(\psi_{k,k} - \varphi_{k,k})u_{i,i}u_{k,k}$$

$$+ 2\mu\sum_{i,k}(\psi_{i,i} - \varphi_{k,k})u_{i,k}(u_{k,i} + u_{i,k}) + 2T_1 + 2T_2\ .$$

where

$$T_1 \equiv \lambda \sum_{\substack{i,k,m \\ m\neq k}} (\psi_{k,m}-\varphi_{k,m}) u_{m,k} u_{i,i} \tag{4.40}$$

$$T_2 \equiv \mu \sum_{\ell,k} (u_{\ell,k}+u_{k,\ell}) \left[\sum_{\substack{m \\ m\neq k}} \psi_{k,m} u_{m,\ell} - \sum_{\substack{m \\ m\neq \ell}} \varphi_{m,\ell} u_{k,m} \right]. \tag{4.41}$$

Again using (4.38), (4.39), (2.11) we obtain

$$0 = 4\beta(\underset{\sim}{x}, \underset{\sim}{u}) c_{ijk\ell} u_{i,j} u_{k,\ell} + 2\left(-2\beta(x,u)\right) c_{ijk\ell} u_{i,j} u_{k,\ell} + 2T_1 + 2T_2 .$$

Therefore it remains only to examine the requirement

$$T_1 + T_2 = 0 \tag{4.42}$$

In the expression (4.40) for T_1 we replace the dummy index i by ℓ, and resolve the sum into the two parts $m=\ell$ and $m\neq\ell$ to obtain

$$T_1 = \lambda \sum_{\substack{\ell,k \\ \ell\neq k}} (\psi_{k,\ell}-\varphi_{k,\ell}) u_{\ell,k} u_{\ell,\ell} + \lambda \sum_{\substack{\ell,k,m \\ m\neq k, m\neq\ell}} (\psi_{k,m}-\varphi_{k,m}) u_{m,k} u_{\ell,\ell} .$$

Resolve the second term above into two parts $k=\ell, k\neq\ell$ and collect to obtain

$$T_1 = \lambda \sum_{\substack{\ell,k \\ \ell\neq k}} u_{\ell,\ell} [u_{\ell,k}(\psi_{k,\ell}-\varphi_{k,\ell}) + u_{k,\ell}(\psi_{\ell,k}-\varphi_{\ell,k})$$

$$+ \lambda \sum_{\substack{\ell,k,m \\ m\neq k, m\neq\ell \\ \ell\neq k}} (\psi_{k,m}-\varphi_{k,m}) u_{m,k} u_{\ell,\ell} . \tag{4.43}$$

A straightforward manipulation of the expression (4.41) for T_2 yields

$$T_2 = \mu \sum_{\substack{\ell,k \\ \ell\neq k}} (u_{\ell,k}+u_{k,\ell}) u_{\ell,\ell} (\psi_{k,\ell}-\varphi_{\ell,k}) + \mu \sum_{\ell,k} (u_{\ell,k}+u_{k,\ell}) \sum_{\substack{\ell,k,m \\ m\neq k, m\neq\ell}} (\psi_{k,m} u_{m,\ell} - \varphi_{m,\ell} u_{k,m}) . \tag{4.44}$$

Resolve the second term in expression (4.44) into sums over $k=\ell$ and $k\neq\ell$. Then, in the $k=\ell$ part of this resolution, replace the index k by m. In the $k\neq\ell$ part, interchange k and ℓ and collect to obtain

$$T_2 = \mu \sum_{\substack{\ell,k \\ \ell\neq k}} u_{\ell,\ell}[u_{\ell,k}(\psi_{k,\ell}-\varphi_{\ell,k}-2\varphi_{k,\ell}) + u_{k,\ell}(\psi_{k,\ell}-\varphi_{\ell,k}+2\psi_{\ell,k})]$$

$$+\mu \sum_{\substack{\ell,k,m \\ m\neq k, m\neq\ell \\ k\neq\ell}} (u_{\ell,k}+u_{k,\ell})(\psi_{k,m}u_{m,\ell}-\varphi_{m,k}u_{\ell,m}) \,. \qquad (4.45)$$

Adding (4.43) and (4.45) we obtain

$$T_1 + T_2 = \sum_{\substack{\ell,k \\ \ell\neq k}} u_{\ell,\ell}\Big\{u_{\ell,k}[(\lambda+\mu)\psi_{k,\ell}-\mu\varphi_{\ell,k}-(\lambda+2\mu)\varphi_{k,\ell}] + u_{k,\ell}[\mu\psi_{k,\ell}-(\lambda+\mu)\varphi_{\ell,k}+(\lambda+2\mu)\psi_{\ell,k}]\Big\}$$

$$+\mu \sum_{\substack{\ell,k,m \\ k\neq\ell, m\neq k \\ m\neq\ell}} (u_{\ell,k}+u_{k,\ell})(\psi_{k,m}u_{m,\ell}-\varphi_{m,k}u_{\ell,m})$$

$$+\lambda \sum_{\substack{\ell,k,m \\ k\neq\ell, m\neq k \\ m\neq\ell}} u_{m,k}u_{\ell,\ell}(\psi_{k,m}-\varphi_{k,m}) \qquad (4.46)$$

The three summation terms in (4.46) are independent in the displacement gradients. Furthermore, within the first summation, the terms within the braces are independent. It is necessary therefore that the brackets within the first summation vanish separately.

$$(\lambda+\mu)\psi_{k,\ell}-\mu\varphi_{\ell,k}-(\lambda+2\mu)\varphi_{k,\ell} = 0 \quad \text{all} \quad k,\ell \;\; k\neq\ell \qquad (4.47)$$

$$\mu\psi_{k,\ell}-(\lambda+\mu)\varphi_{\ell,k}+(\lambda+2\mu)\psi_{\ell,k} = 0 \quad \text{all} \quad k,\ell \;\; k\neq\ell \qquad (4.48)$$

Adding (4.47) and (4.48) and using (4.32) we obtain

$$\varphi_{k,x_\ell} + \varphi_{\ell,x_k} = 0 \quad \text{all} \quad k,\ell \;\; k \neq \ell \; . \tag{4.49}$$

Using (4.49) in (4.47) we obtain

$$\psi_{k,u_\ell} - \varphi_{k,x_\ell} = 0 \quad \text{all} \quad k,\ell \;\; k \neq \ell \; . \tag{4.50}$$

Using (4.50) in the third term of (4.46) we see that the third term vanishes and (4.42) becomes

$$T_1 + T_2 = \mu \sum_{\substack{\ell,k,m \\ k\neq\ell, m\neq k \\ m\neq\ell}} (u_{\ell,k} + u_{k,\ell})(\psi_{k,m} u_{m,\ell} - \varphi_{m,k} u_{\ell,m}) = 0$$

as the remaining condition necessary for the vanishing of (4.36) and thus (4.30). But it is easy to see that (4.50), (4.49), and (4.32) ensure the vanishing of the expression above. Namely (4.49) and (4.50) give

$$T_1 + T_2 = \mu \sum_{\substack{\ell,k,m \\ k\neq\ell, m\neq k \\ m\neq\ell}} (u_{\ell,k} + u_{k,\ell}) \psi_{k,m} (u_{m,\ell} + u_{\ell,m}) \; . \tag{4.51}$$

Now define a pair of matrices $\underset{\sim}{A}$ and $\underset{\sim}{S}$ with elements

$$A_{km} \equiv \begin{cases} \psi_{k,m} & k \neq m \\ 0 & k = m \end{cases} ,$$

$$S_{\ell k} \equiv \begin{cases} u_{\ell,k} + u_{k,\ell} & k \neq \ell \\ 0 & k = \ell \end{cases} .$$

Then by (4.32) the matrix $\underset{\sim}{A}$ is antisymmetric, and the matrix $\underset{\sim}{S}$ is clearly symmetric. So the expression (4.51) may be written as

$$T_1 + T_2 = \mu \cdot \text{Trace}\,\{\underset{\sim}{S}\,\underset{\sim}{A}\,\underset{\sim}{S}\} \; .$$

But $\underset{\sim}{S}\underset{\sim}{A}\underset{\sim}{S}$ is antisymmetric, and its trace is therefore equal to zero. It follows that (4.30) is satisfied through the necessary restrictions (4.32), (4.34), (4.35), (4.38), (4.39), (4.49), (4.50).

This completes the treatment of the quadratic terms for the elastodynamic case. Again it is important to note that the proof requires a slight modification for the elastostatic case. Namely, expressions (4.32), (4.49), (4.51) though still necessary must be shown to follow by some other means. To do so, we note that (4.36) becomes

$$[\varphi_{s,s}c_{ijk\ell}-2\varphi_{\ell,n}c_{ijkn}+2\psi_{n,k}c_{ijn\ell}]u_{i,j}u_{k,\ell}=0$$

for the elastostatic case. Thus we obtain

$$\sum_{s}\varphi_{s,s}-2\varphi_{i,i}+2\psi_{i,i}=0 \quad \text{all } i \text{ (no sum)}$$

which leads to

$$\varphi_{i,x_i}=\beta(\underset{\sim}{x},\underset{\sim}{u}) \quad \text{all } i \text{ (no sum)}$$

and therefore

$$\psi_{i,u_i}=-\frac{1}{2}\beta(\underset{\sim}{x},\underset{\sim}{u}) \quad \text{all } i \text{ (no sum)}.$$

Thus we are still led to (4.47), (4.48). But these form a system of 12 linear homogeneous equations in 12 unknowns. Moreover the coefficient determinant vanishes. Any solution of this system must be of the form

$$\psi_{k,\ell}=f_{k\ell}(\underset{\sim}{x},\underset{\sim}{u}) \qquad k>\ell$$

$$\psi_{k,\ell}=-f_{\ell k}(\underset{\sim}{x},\underset{\sim}{u}) \qquad k<\ell$$

$$\varphi_{k,\ell}=f_{k\ell}(\underset{\sim}{x},\underset{\sim}{u}) \qquad k>\ell$$

$$\varphi_{k,\ell} = -f_{\ell k}(\underset{\sim}{x}, \underset{\sim}{u}) \quad k<\ell$$

where the three functions $f_{k\ell}$ $(k>\ell)$ are arbitrary. These, however, lead immediately to (4.32), (4.49), (4.51).

Deductions from the Necessary Conditions[1]

In the preceding portion of this chapter, we have deduced a number of restrictions on the transformation (4.2) characterized infinitesimally by the functions $\underset{\sim}{\varphi}, \underset{\sim}{\psi}$ of expression (4.3). We summarize those restrictions here for convenience

$$\psi_{i,x_0} = 0 \qquad (4.18)$$

$$\psi_{i,x_k} + \psi_{k,x_i} = 0 \qquad (i \neq k) \qquad (4.20)$$

$$\varphi_{\alpha,u_i} = 0 \qquad (4.25)$$

$$\psi_{i,u_k} + \psi_{k,u_i} = 0 \qquad (i \neq k) \qquad (4.32)$$

$$\varphi_{0,x_i} = 0 \qquad (4.34)$$

$$\varphi_{i,x_0} = 0 \qquad (4.35)$$

$$\varphi_{\alpha,x_\alpha} = \beta(\underset{\sim}{x},\underset{\sim}{u}) \qquad (4.38)$$

$$\psi_{i,u_i} = -\beta(\underset{\sim}{x},\underset{\sim}{u}) \qquad (4.39)$$

$$\varphi_{k,x_\ell} + \varphi_{\ell,x_k} = 0 \qquad (k \neq \ell) \qquad (4.49)$$

---

[1] In the remainder of this chapter, we suspend the summation convention, and to avoid ambiguity we explicitly write out the variables of differentiation, viz we write φ_{i,x_j} rather than $\varphi_{i,j}$.

$$\psi_{k,u_\ell} - \varphi_{k,x_\ell} = 0 \qquad (k \neq \ell) \tag{4.50}$$

Now $\underset{\sim}{\varphi}, \underset{\sim}{\psi}$ are C^2 functions on their domains (cf. pg. 8) and the above conditions taken jointly impose severe restrictions upon their form.

Define two sets of three functions $f_{ik}(\underset{\sim}{x}, \underset{\sim}{u})$ and $g_{ik}(\underset{\sim}{x}, \underset{\sim}{u})$ for $i>k$ by the expressions

$$\begin{aligned} \psi_{i,u_k} &= f_{ik}(\underset{\sim}{x}, \underset{\sim}{u}) \quad i>k \\ \psi_{i,x_k} &= g_{ik}(\underset{\sim}{x}, \underset{\sim}{u}) \quad i>k \ . \end{aligned} \tag{4.52}$$

Then (4.20), (4.32) together imply

$$\begin{aligned} \psi_{k,u_i} &= -f_{ik}(\underset{\sim}{x}, \underset{\sim}{u}) \quad i>k \\ \psi_{k,x_i} &= -g_{ik}(\underset{\sim}{x}, \underset{\sim}{u}) \quad i>k \ . \end{aligned} \tag{4.53}$$

Using (4.50) we have

$$\begin{aligned} \varphi_{i,x_k} &= f_{ik} \qquad i>k \\ \varphi_{k,x_i} &= -f_{ik} \qquad i>k \ . \end{aligned} \tag{4.54}$$

Using (4.25) we have, for $i>j$

$$0 = \left(\varphi_{i,u_k}\right)_{,x_j} = \left(\varphi_{i,x_j}\right)_{,u_k} = f_{ij}(\underset{\sim}{x}, \underset{\sim}{u})_{,u_k} \ .$$

Therefore, since k is arbitrary

$$f_{ij}(\underset{\sim}{x}, \underset{\sim}{u}) = f_{ij}(\underset{\sim}{x}) \tag{4.55}$$

Again, using (4.25) we have

$$0 = \left(\varphi_{\alpha,u_k}\right)_{,x_\alpha} = \left(\varphi_{\alpha,x_\alpha}\right)_{,u_k} = \beta(\underset{\sim}{x}, \underset{\sim}{u})_{,u_k} \ .$$

Therefore, since k is arbitrary

$$\beta(\underset{\sim}{x}, \underset{\sim}{u}) = \beta(\underset{\sim}{x}) . \tag{4.56}$$

Using (4. 20) and (4. 56) we have

$$0 = \left(\psi_{k,x_k}\right)_{,u_k} = \left(\psi_{k,u_k}\right)_{,x_k} = -\beta(\underset{\sim}{x})_{,x_k} .$$

And, again, since k is arbitrary we obtain

$$\beta(\underset{\sim}{x}, \underset{\sim}{u}) = \nu \equiv \text{constant} . \tag{4.57}$$

Taking $k > \ell$ and using (4. 57), (4,38) we have

$$0 = \nu_{x_\ell} = \left(\varphi_{k,x_k}\right)_{,x\ell} = \left(\varphi_{k,x_\ell}\right)_{,x_k} = \left(f_{k\ell}(\underset{\sim}{x})\right)_{,x_k} .$$

Similarly we can obtain

$$0 = \left(f_{k\ell}(\underset{\sim}{x})\right)_{,x_\ell}$$

and by (4. 35)

$$0 = \left(f_{k\ell}(\underset{\sim}{x})\right)_{,x_0} .$$

We may thus write

$$f_{31}(\underset{\sim}{x}) = f_{31}(x_2)$$

$$f_{32}(\underset{\sim}{x}) = f_{32}(x_1)$$

$$f_{21}(\underset{\sim}{x}) = f_{21}(x_3) .$$

Using these relations we can write

$$f_{31,x_2} = \varphi_{3,x_1,x_2} = \varphi_{3,x_2,x_1} = f_{32,x_1}$$

$$f_{21,x_3} = \varphi_{2,x_1,x_3} = \varphi_{2,x_3,x_1} = -f_{32,x_1}$$

$$f_{32,x_1} = \varphi_{3,x_2,x_1} = \varphi_{3,x_1,x_2} = f_{31,x_2} .$$

Therefore $f_{31,x_2} = f_{32,x_1} = -f_{21,x_3} = \text{constant} \equiv a$. We conclude therefore that

$$\begin{aligned} f_{21} &= -ax_3 + b_3 \\ f_{32} &= ax_1 + b_1 \\ f_{31} &= ax_2 + b_2 . \end{aligned} \tag{4. 58}$$

By exactly similar reasoning, we obtain

$$\begin{aligned} g_{21} &= -\alpha(\underset{\sim}{u})x_3 + \beta_3(\underset{\sim}{u}) \\ g_{31} &= \alpha(\underset{\sim}{u})x_2 + \beta_2(\underset{\sim}{u}) \\ g_{32} &= \alpha(\underset{\sim}{u})x_1 + \beta_1(\underset{\sim}{u}) \end{aligned}$$

where $\alpha, \underset{\sim}{\beta}$ are at this point unknown functions of $\underset{\sim}{u}$. To obtain the functional form of $\alpha(\underset{\sim}{u}), \underset{\sim}{\beta}(\underset{\sim}{u})$ we carry the analysis of g_{ij} further through the use of restrictions (4. 39) and (4. 57). We have

$$0 = \left(\psi_{1,u_1}\right)_{,x_2} = \left(\psi_{1,x_2}\right)_{,u_1} = -g_{21,u_1} = \left(\alpha_{,u_1}\right) x_3 + \beta_{3,u_1} .$$

We thus deduce $\alpha_{,u_1} = \beta_{3,u_1} = 0$. By similar reasoning we can deduce $\alpha_{,u_1} = \alpha_{,u_2} = \alpha_{,u_3} = 0$ and also $\beta_{1,u_2} = \beta_{1,u_3} = 0$; $\beta_{2,u_1} = \beta_{2,u_3} = 0$; $\beta_{3,u_1} = \beta_{3,u_2} = 0$. Therefore

$$\begin{aligned} g_{21} &= -\alpha x_3 + b_3(u_3) \\ g_{31} &= \alpha x_2 + b_2(u_2) \\ g_{32} &= \alpha x_1 + b_1(u_1) \end{aligned}$$

where α is a constant.

Proceeding still further, we have

$$b_3' = g_{21,u_3} = \left(\psi_{2,x_1}\right)_{,u_3} = \left(\psi_{2,u_3}\right)_{,x_1} = -f_{32,x_1} = -a \ .$$

Thus, $b_3(u_3) = -au_3 + a_3$. Similar reasoning applied to b_2' and b_1' yields

$$\begin{aligned} g_{21} &= -\alpha x_3 - au_3 + a_3 \\ g_{31} &= \alpha x_2 - au_2 + a_2 \\ g_{32} &= \alpha x_1 + au_1 + a_1 \end{aligned} \tag{4.59}$$

The equations (4.52), (4.53), (4.54) with the right-hand sides given by (4.58), (4.59) form an overdetermined system of equations for φ_i, ψ_i. Direct integration yields $a=\alpha=0$ as a necessary condition for the existence of a solution. When $a=\alpha=0$ is enforced, the result is exactly the set of equations (4.4) for φ_i, ψ_i where c_i, d_i are the constants of integration. It remains only to consider φ_0. But it is easy to see from (4.34), (4.38), (4.57) that φ_0 must have the form given in (4.4).

This completes the proof of the necessity.

To prove sufficiency we may, if we wish, directly verify the infinitesimal invariance of (4.1) under (4.3), (4.4). Alternatively, we may observe that the preceding proof of necessity obtained conditions that taken jointly are manifestly sufficient to ensure the vanishing of the linear, quadratic, and cubic terms in $\nabla \underset{\sim}{u}$ of (4.8), and therefore to ensure infinitesimal invariance of (4.1).

This completes the proof of Theorem 2.

5. An illustration: surface waves on a half-space.

All of the applications of the conservation laws of elasticity have been concerned with problems in elastostatics [3] - [6], [17]. In the present section we consider the problem of periodic free surface waves on an elastic half-space, and we show how one of the conservation laws can be used to provide a simple derivation of the secular equation governing the speed of propagation of such waves.

We consider an elastic half-space composed of an anisotropic material with one plane of elastic symmetry. Rectangular cartesian coordinates are chosen so that the half-space occupies the region $x_2 \geq 0$, and planes perpendicular to the x_3-axis are planes of elastic symmetry. We seek solutions of the displacement equations of motion (2.7) corresponding to a "plane deformation", so that $u_3 \equiv 0$ and u_1, u_2 are independent of x_3. The equations (2.7) can thus be written in the form

$$c_{\alpha\beta\gamma\delta}\, u_{\gamma,\delta\beta} = \rho \ddot{u}_\alpha \tag{5.1}$$

where here and throughout the remainder of this section Greek subscripts have the range 1,2. The stresses $\sigma_{\alpha\beta}$ are found from (2.3), (2.2) as

$$\sigma_{\alpha\beta} = c_{\alpha\beta\gamma\delta}\, u_{\gamma,\delta} \, . \qquad (5.2)^1$$

It is assumed that $c_{\alpha\beta\gamma\delta}$ is positive definite.

We now investigate solutions u_α of (5.1) which correspond to a wave propagating in the positive x_1-direction with speed c, so that

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[1] Plane deformations are possible in materials with one plane of elastic symmetry. There are six independent constants among the $c_{\alpha\beta\gamma\delta}$.

$$u_\alpha(x_1,x_2,t) = u_\alpha(x_1 - ct, x_2) \; . \tag{5.3}$$

We furthermore require that u_α and all of its derivatives tend to zero as $x_2 \to +\infty$. Finally the surface $x_2 = 0$ is required to remain traction free, so that

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = 0 \quad \text{at} \quad x_2 = 0 \; . \tag{5.4}$$

The third of these conditions is automatically satisfied for plane deformations of the anisotropic materials under consideration here, so that the free surface conditions reduce, by (5.2),(5.4), to

$$c_{21\gamma\delta} u_{\gamma,\delta} = c_{22\gamma\delta} u_{\gamma,\delta} = 0 \quad \text{at} \quad x_2 = 0 \; . \tag{5.5}$$

We wish to determine the possible values of the propagation speed c corresponding to nontrivial solutions u_α which are periodic in x_1 and t. Referring to (5.3) we thus require that

$$u_\alpha\left(z + \frac{\ell}{2\pi}, x_2\right) = u_\alpha(z, x_2) \tag{5.6}$$

for all real z, and for $x_2 \geq 0$. The wavelength of the motion is $\ell/2\pi$, while the period is $2\pi\ell/c$.

We seek a solution of the form

$$u_\alpha(x_1 - ct, x_2) = U_\alpha(x_2) \exp[i(x_1 - ct)/\ell] \; . \tag{5.7}$$

Substitution of (5.7) into the differential equations (5.1) provides two ordinary differential equations for U_α. These may be written as follows:

$$\ell^2 A_{\alpha\gamma} \ddot{U}_\gamma + i\ell B_{\alpha\gamma} \dot{U}_\gamma + \left(C_{\alpha\gamma} + \rho c^2 \delta_{\alpha\gamma}\right) U_\gamma = 0 \quad , \quad x_2 \geq 0 \; , \tag{5.8}$$

where the superposed dot indicates differentiation with respect to x_2, and

$$A_{\alpha\beta} = c_{\alpha 2\gamma 2} \;,\quad B_{\alpha\beta} = c_{\alpha 1\gamma 2} + c_{\alpha 2\gamma 1} \;,\quad C_{\alpha\gamma} = -c_{\alpha 1\gamma 1} \;. \tag{5.9}$$

Note that the matrices $\underset{\sim}{A} = (A_{\alpha\gamma})$, $\underset{\sim}{B} = (B_{\alpha\gamma})$ and $\underset{\sim}{C} = (C_{\alpha\gamma})$ are real and symmetric.

If (5.7) is substituted into (5.5) there follow the boundary conditions at $x_2=0$ for the system of ordinary differential equations (5.8)

$$\ell A_{\alpha\beta}\dot{U}_\beta + iD_{\alpha\beta}U_\beta = 0 \quad \text{at} \quad x_2 = 0 \;, \tag{5.10}$$

where

$$D_{\alpha\beta} = c_{\alpha 2\beta 1} \;. \tag{5.11}$$

Finally, we require

$$U_\alpha \to 0 \;,\quad \dot{U}_\alpha \to 0 \quad \text{as} \quad x_2 \to \infty \;. \tag{5.12}$$

Equations (5.8), (5.10) and (5.12) comprise an eigenvalue problem on the interval $[0,\infty)$, with c^2 as the eigenvalue parameter. We now show that the conservation law (3.4) can be used to by-pass the process of solving this eigenvalue problem in detail, insofar as the determination of c is concerned. When specialized to the plane deformation under consideration at present, (3.4) with i=2 becomes

$$\frac{\partial}{\partial t}\left(\rho u_{\beta,2}\frac{\partial u}{\partial t}\beta\right) + \frac{\partial}{\partial x\beta}\left(-u_{\gamma,2}\sigma_{\gamma\beta} + L\delta_{2\beta}\right) = 0 \;, \tag{5.13}$$

where L is the Lagrangian density. If we now introduce into (5.13) the assumption (5.7), we find that (5.13) reduces to

$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} = 0 \qquad -\infty < x_1 < \infty \,,\, x_2 \geq 0 \;, \tag{5.14}$$

where

$$\left.\begin{aligned} Q_1 &= -\rho c u_{\beta,1} u_{\beta,2} - \sigma_{\beta,1} u_{\beta,2} \,, \\ Q_2 &= -\sigma_{\beta,2} u_{\beta,2} + \frac{1}{2} c_{\alpha\beta\gamma\delta} u_{\alpha,\beta} u_{\gamma,\gamma} - \frac{1}{2} \rho \frac{\partial u_\alpha}{\partial t} \frac{\partial u_\alpha}{\partial t} \,, \end{aligned}\right\} \tag{5.15}$$

and u_α is given by (5.7).

We note that, in virtue of (5.7) and (5.15), Q_1 and Q_2 in (5.14) are functions of $x_1 - ct$ and x_2. We now integrate (5.14) with respect to x_1 over one period and use the assumed periodicity of u_1, u_2 — and thus Q_1 — to get

$$\frac{d}{dx_2} \int_0^\ell Q_2(z, x_2)\, dz = 0 \,. \tag{5.16}$$

It follows from (5.16) that

$$\int_0^\ell Q_2(z, x_2)\, dz = \text{constant} \,, \quad 0 \le x_2 < \infty \,.$$

But (5.12), (5.7) and (5.15) show that $Q_2(z, x_2) \to 0$ as $x_2 \to +\infty$, so

$$\int_0^\ell Q_2(z, x_2)\, dz = 0 \,, \quad 0 \le x_2 < \infty \,. \tag{5.17}$$

From (5.7), (5.15) it is possible to compute the integral in (5.17) in terms of the U_α's, and hence to obtain (5.17) in the form

$$\ell^2 A_{\alpha\gamma} \dot{U}_\alpha(x_2) \dot{U}^*_\gamma(x_2) + \left(C_{\alpha\gamma} + \rho c^2 \delta_{\alpha\gamma}\right) U_\alpha(x_2) U^*_\gamma(x_2) = 0 \,, \quad 0 \le x_2 < \infty \tag{5.18}$$

where the asterisk indicates complex conjugate. It should be observed that (5.18) — which plays the role of a first integral for (5.8) — depends only on the field equations, the periodicity assumption (5.7), and the decay conditions (5.12), but <u>not</u> on the free surface conditions (5.10). Since the differential equations (5.1) have constant coefficients, the derivatives $u_{\alpha,2}$ of solutions u_α of (5. 1) also satisfy (5. 1). Moreover if the u_α have the general form (5. 7), so do $u_{\alpha,2}$, $u_{\alpha,2}$ also decay at $x_2=\infty$. It follows that (5. 18) also holds with U_α replaced by $\dot{U}_\alpha$, $\dot{U}_\alpha$ by $\ddot{U}_\alpha$, viz

$$\ell^2 A_{\alpha\gamma}\ddot{U}_\alpha(x_2)\ddot{U}^*_\gamma(x_2)+\left(C_{\alpha\gamma}+\rho c^2\delta_{\alpha\gamma}\right)\dot{U}_\alpha(x_2)\dot{U}_\gamma(x_2)=0\ , \quad 0\le x_2<\infty\ . \tag{5. 19}$$

We now consider the differential equations (5. 8) evaluated at $x_2=0$, the results (5. 18), (5. 19) of the conservation laws evaluated at $x_2=0$, and the free surface conditions (5. 10) as a system of six equations for the for the six unknowns $U_\alpha(0)$, $\dot{U}_\alpha(0)$, $\ddot{U}_\alpha(0)$. This system can be written as follows

$$\text{from (5. 8):}\quad \ell^2\underset{\sim}{A}\ddot{\underset{\sim}{U}}(0)+i\ell\underset{\sim}{B}\dot{\underset{\sim}{U}}(0)+\left(\underset{\sim}{C}+\rho c^2\underset{\sim}{1}\right)\underset{\sim}{U}(0)=0\ , \tag{5. 20}$$

$$\text{from (5. 18):}\quad \ell^2\dot{\underset{\sim}{U}}^T(0)\underset{\sim}{A}\dot{\underset{\sim}{U}}^*(0)+\underset{\sim}{U}^T(0)\left(\underset{\sim}{C}+\rho c^2\underset{\sim}{1}\right)\dot{\underset{\sim}{U}}^*(0)=0\ , \tag{5. 21}$$

$$\text{from (5. 19):}\quad \ell^2\ddot{\underset{\sim}{U}}^T(0)\underset{\sim}{A}\ddot{\underset{\sim}{U}}^*(0)+\dot{\underset{\sim}{U}}^T(0)\left(\underset{\sim}{C}+\rho c^2\underset{\sim}{1}\right)\dot{\underset{\sim}{U}}^*(0)=0\ , \tag{5. 22}$$

$$\text{from (5. 20):}\quad \ell\underset{\sim}{A}\dot{\underset{\sim}{U}}(0)+i\underset{\sim}{D}\underset{\sim}{U}(0)=0\ , \tag{5. 23}$$

where $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, $\underset{\sim}{D}$ are the real, symmetric 2×2 matrices whose elements are defined by (5. 9), (5. 11), $\underset{\sim}{U}$ is the two-dimensional column vector with components U_α, and a superposed T indicates matrix transpose. The matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ and $\underset{\sim}{D}$ in the system (5. 20) - (5. 28) depend only on the elastic constants $c_{\alpha\beta\gamma\delta}$; we wish to characterize the values of the propagation speed c for which the above system has a nontrivial solution $\underset{\sim}{U}(0)$, $\dot{\underset{\sim}{U}}(0)$, $\ddot{\underset{\sim}{U}}(0)$. We note that any such value of c will clearly be independent of ℓ.

We now discuss the analysis of (5. 21) - (5. 23). From (5. 20), (5. 23)

$$\left.\begin{aligned} \ell\dot{\underset{\sim}{U}}(0) &= -i\underset{\sim}{A}^{-1}\underset{\sim}{D}\underset{\sim}{U}(0)\ , \\ \ell^2\ddot{\underset{\sim}{U}}(0) &= \left(-\underset{\sim}{A}^{-1}\underset{\sim}{B}\underset{\sim}{A}^{-1}\underset{\sim}{D} - \underset{\sim}{A}^{-1}\underset{\sim}{C} - \rho c^2\underset{\sim}{A}^{-1}\right)\underset{\sim}{U}(0)\ . \end{aligned}\right\} \tag{5.24}$$

(The inverse $\underset{\sim}{A}^{-1}$ of $\underset{\sim}{A}$ can be shown to exist as a consequence of the assumed positive definiteness of $c_{\alpha\beta\gamma\delta}$.) Substitution of (5.24) into (5.21), (5.22) then shows that the column vector $\underset{\sim}{U}(0)$ must satisfy the pair of quadratic equations.

$$\underset{\sim}{U}^T(0)\left[\underset{\sim}{D}^T\underset{\sim}{A}^{-1}\underset{\sim}{D} + \underset{\sim}{C} + \rho c^2\underset{\sim}{1}\right]\underset{\sim}{U}^*(0) = 0 \tag{5.25}$$

$$\begin{aligned} \underset{\sim}{U}^T(0)\Big[&\underset{\sim}{D}^T\underset{\sim}{A}^{-1}\underset{\sim}{B}\underset{\sim}{A}^{-1}\underset{\sim}{B}\underset{\sim}{A}^{-1}\underset{\sim}{D} + \underset{\sim}{D}^T\underset{\sim}{A}^{-1}\underset{\sim}{B}\underset{\sim}{A}^{-1}\underset{\sim}{C} + \underset{\sim}{C}\underset{\sim}{A}^{-1}\underset{\sim}{B}\underset{\sim}{A}^{-1}\underset{\sim}{D} + \underset{\sim}{C}\underset{\sim}{A}^{-1}\underset{\sim}{C} + \underset{\sim}{D}^T\underset{\sim}{A}^{-1}\underset{\sim}{C}\underset{\sim}{A}^{-1}\underset{\sim}{D} \\ &+ \rho c^2\left(\underset{\sim}{D}^T\underset{\sim}{A}^{-1}\underset{\sim}{B}\underset{\sim}{A}^{-1} + \underset{\sim}{A}^{-1}\underset{\sim}{B}\underset{\sim}{A}^{-1}\underset{\sim}{D} + \underset{\sim}{D}^T\underset{\sim}{A}^{-1}\underset{\sim}{A}^{-1}\underset{\sim}{D} + \underset{\sim}{A}^{-1}\underset{\sim}{C} + \underset{\sim}{C}\underset{\sim}{A}^{-1}\right) \\ &+ \rho^2 c^4\underset{\sim}{A}^{-1}\Big]\underset{\sim}{U}^*(0) = 0\ . \end{aligned} \tag{5.26}$$

A general discussion of the pair of quadratic equations (5.25), (5.26) appears to be very difficult. In general, the matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, $\underset{\sim}{D}$ are given by (5.9), (5.11). For a material with a plane of elastic symmetry, there are six independent elastic constants[1], which may be conveniently chosen as c_{1111}, c_{2222}, c_{1122}, c_{1112}, c_{1222} and c_{1212}. In terms of these,

$$\left.\begin{aligned} &\underset{\sim}{A} = \begin{pmatrix} c_{1212} & c_{1222} \\ c_{1222} & c_{2222} \end{pmatrix}\ , \quad \underset{\sim}{B} = \begin{pmatrix} 2c_{1112} & c_{1122}+c_{1212} \\ c_{1122}+c_{1212} & 2c_{1222} \end{pmatrix}, \\ &\underset{\sim}{C} = -\begin{pmatrix} c_{1111} & c_{1112} \\ c_{1112} & c_{1212} \end{pmatrix}\ , \quad \underset{\sim}{D} = \begin{pmatrix} c_{1112} & c_{1212} \\ c_{1122} & c_{1222} \end{pmatrix}\ . \end{aligned}\right\} \tag{5.27}$$

[1] See [18].

For the special case of a transversely isotropic material for which the x_2-axis is an axis of symmetry, we have [18]

$$c_{1112} = c_{1222} = 0 , \tag{5.28}$$

and the condition that (5.25) and (5.26) shall possess a nontrivial solution can be shown to

$$\begin{aligned} & c_{1212}^2 c_{2222}\left(c_{2222} - c_{1212}\right)\left(\frac{c}{c_s}\right)^6 \\ & - \left[c_{1212}^2 c_{2222}\left(c_{2222} - c_{1111}\right) + 2c_{1212}c_{2222}\left(c_{1111}c_{2222} - c_{1122}^2\right)\right]\left(\frac{c}{c_s}\right)^4 \\ & + \left(c_{1111}c_{2222} - c_{1122}^2\right)\left(2c_{1212}c_{2222} + c_{1111}c_{2222} - c_{1122}^2\right)\left(\frac{c}{c_s}\right)^2 \\ & - \left(c_{1111}c_{2222} - c_{1122}^2\right)^2 = 0 , \end{aligned} \tag{5.29}$$

where

$$c_s = \sqrt{\frac{c_{1212}}{\rho}} \tag{5.30}$$

is a "shear wave" speed. Equation (5.29) is a bicubic for the surface wave speed c.

If we further specialize to an isotropic material, the $c_{\alpha\beta\gamma\delta}$ satisfy (5.28) and also

$$\left.\begin{aligned} c_{1111} = c_{2222} = \lambda + 2\mu , \\ c_{1122} = c_{1111} - 2c_{1212} = \lambda , \end{aligned}\right\} \tag{5.31}$$

where λ, μ are Lamé's moduli. When (5.29) is specialized in accordance with (5.31), there follows the usual bicubic for the Rayleigh surface wave speed [19].

$$\left(\frac{c}{c_s}\right)^6 - 8\left(\frac{c}{c_s}\right)^4 + \left[24 - 16\left(\frac{c_s}{c_d}\right)^2\right]\left(\frac{c}{c_s}\right)^2 + 16\left[\left(\frac{c_s}{c_d}\right)^2 - 1\right] = 0 . \qquad (5.32)$$

Finally we remark that whenever c is such that (5.25), (5.26) have a nontrivial solution, c must indeed represent the speed of propagation of any surface wave which is an arbitrary periodic function of x_1-ct (not merely the special periodic function assumed in (5.7)). This follows from the fact that any such wave can be represented as a Fourier superposition of waves of the form (5.7) with different wavelengths ℓ, and the property that c is independent of ℓ.

Appendix

A Note on Weak Invariance

It is our purpose in this appendix to emphasize, by example, that the completeness theorem of Section 4 is restricted to that version of Noether's theorem requiring infinitesimal invariance under the group (4. 2).

In a verbal communication, noted in [20], E.Noether extended her theorem to include those cases wherein the variational principle is merely "weakly invariant" under a given group of infinitesimal transformations. In the following, we present a short, informal, discussion of the altered requirement of merely weak invariance on the Lagrangian (2. 13) under the group (4. 2). We establish a set of equations whose solutions will yield transformations leaving the Lagrangian weakly invariant, and we observe that the class of solutions to the displacement equations of motion (2. 7) will satisfy these equations as well. It follows then that there exists a large number of conservation laws derivable in a consistent manner from the transformation group (4. 2) and yet independent of those laws presented in Theorem 1 and discussed in Theorem 2.

We indicate here, in an abbreviated manner, the generalization, mentioned above, of Noether's theorem regarding "weak invariance" of the variational principle (see also [20]).

The functional $\mathfrak{F}$ (2. 16) is said to be infinitesimally weakly invariant under the transformation (2. 17) if there is a vector valued function $\underset{\sim}{E}(\underset{\sim}{\xi}, \underset{\sim}{w}, \underset{\sim}{\nabla}\underset{\sim}{w})$ such that

$$\frac{d}{d\epsilon}\left[F\left(\underset{\sim}{\xi}^*(\underset{\sim}{\xi}), \underset{\sim}{w}^*(\underset{\sim}{\xi}), \underset{\sim}{\nabla}^*\underset{\sim}{w}^*(\underset{\sim}{\xi})\right) \cdot J(\underset{\sim}{\xi}^*, \underset{\sim}{\xi})\right]_{\epsilon=0} = \underset{\sim}{\nabla} \cdot \underset{\sim}{E}(\underset{\sim}{\xi}, \underset{\sim}{w}, \underset{\sim}{\nabla}\underset{\sim}{w}) \qquad (A.1)$$

where F is the density (2.16).

If this is the case, and $\underset{\sim}{w}(\underset{\sim}{\xi})$ is a solution of the Euler-Lagrange equations (2.20), then Noether's theorem asserts that the following conservation law holds

$$\frac{\partial}{\partial \xi_\alpha}\left\{F_{,w_{i,\alpha}}\overline{\psi}_i + F\varphi_\alpha - E_\alpha\right\} = 0 \qquad (A.2)$$

where all functions are evaluated at $\underset{\sim}{\xi}$, $\underset{\sim}{w}(\underset{\sim}{\xi})$, $\underset{\sim}{\nabla}\underset{\sim}{w}(\underset{\sim}{\xi})$ and where $\underset{\sim}{\varphi}, \overline{\underset{\sim}{\psi}}$ are defined as in (2.23).

To be consistent with Section 4 we use the notation $\underset{\sim}{x}$, $\underset{\sim}{u}$, $\underset{\sim}{\nabla}\underset{\sim}{u}$ for $\underset{\sim}{\xi}$, $\underset{\sim}{w}$, $\underset{\sim}{\nabla}\underset{\sim}{w}$ for our elastodynamic application of this theorem, and in this notation we note that (A.1) corresponds to (4.8).

In our analysis of expression (4.8) we found that a transformation of the form (4.2) led to terms which were linear, quadratic, and cubic expressions in the displacement gradients $\underset{\sim}{\nabla}\underset{\sim}{u}$, viz. (4.15), (4.16), (4.17). These expressions were required to vanish to maintain infinitesimal invariance. For weak invariance, they need only be divergence expressions.

Defining M as in (4.9) we may write (4.14) as

$$2M = A_{i\alpha}(\underset{\sim}{x}, \underset{\sim}{u})u_{i,\alpha} + B_{i\alpha k\beta}(\underset{\sim}{x}, \underset{\sim}{u})u_{i,\alpha}u_{k,\beta} + D_{i\alpha k\beta s\gamma}(\underset{\sim}{x}, \underset{\sim}{u})u_{i,\alpha}u_{k,\beta}u_{s,\gamma} \qquad (A.3)$$

where $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{D}$ are defined as follows: Let

$$\left.\begin{aligned} \hat{A}_{i\alpha} &= 2c_{i\alpha k\beta}\psi_{k,\beta} \\ \hat{B}_{i\alpha k\beta} &= \varphi_{\gamma,\gamma}c_{i\alpha k\beta} - 2\varphi_{\beta,\gamma}c_{i\alpha k\gamma} + 2\psi_{n,k}c_{i\alpha n\beta} \end{aligned}\right\} \qquad (A.4)$$

$$\left.\hat{D}_{i\alpha k\beta s\gamma} = \varphi_{\gamma,s} c_{i\alpha k\beta} - 2\varphi_{\beta,s} c_{i\alpha k\gamma}\right\} \qquad \text{(A. 4) cont.}$$

then we define $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{D}$ as

$$\left.\begin{aligned} A_{i\alpha} &= \hat{A}_{i\alpha} \\ B_{i\alpha k\beta} &= \frac{1}{2}\left[\hat{B}_{i\alpha k\beta} + \hat{B}_{k\beta i\alpha}\right] \\ D_{i\alpha k\beta s\gamma} &= \frac{1}{6}\Big[\hat{D}_{i\alpha k\beta s\gamma} + \hat{D}_{k\beta s\gamma i\alpha} + \hat{D}_{s\gamma i\alpha k\beta} \\ &\qquad + \hat{D}_{i\alpha s\gamma k\beta} + \hat{D}_{s\gamma k\beta i\alpha} + \hat{D}_{k\beta i\alpha s\gamma}\Big] . \end{aligned}\right\} \qquad \text{(A. 5)}$$

In this notation, (A. 3) may be written as

$$\begin{aligned} 2M = \frac{\partial}{\partial x_\alpha}&\left[\left(A_{i\alpha} + B_{i\alpha k\beta} u_{k,\beta} + D_{i\alpha k\beta s\gamma} u_{k,\beta} u_{s,\gamma}\right) u_i\right] \\ &- u_i\left\{\frac{\partial}{\partial x_\alpha}(A_{i\alpha}) + \frac{\partial}{\partial x_\alpha}(B_{i\alpha k\beta}) u_{k,\beta} + \frac{\partial}{\partial x_\alpha}(D_{i\alpha k\beta s\gamma}) u_{k,\beta} u_{s,\gamma}\right. \\ &\qquad \left. + u_{k,\alpha,\beta}\left[B_{i\alpha k\beta} + 2D_{i\alpha k\beta s\gamma} u_{s,\gamma}\right]\right\} . \end{aligned} \qquad \text{(A. 6)}$$

And it follows that a sufficient condition for (2. 13) to be weakly invariant under (4. 2) is that

$$\begin{aligned} 0 = u_i &\left\{\frac{\partial}{\partial x_\alpha}(A_{i\alpha}) + \frac{\partial}{\partial x_\alpha}(B_{i\alpha k\beta}) u_{k,\beta} + \frac{\partial}{\partial x_\alpha}(D_{i\alpha k\beta s\gamma}) u_{k,\beta} u_{s,\gamma}\right. \\ &\qquad \left. + u_{k,\alpha,\beta}\left[B_{i\alpha k\beta} + 2D_{i\alpha k\beta s\gamma}\right]\right\} . \end{aligned}$$

But $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{D}$ are functions of $\underset{\sim}{x}, \underset{\sim}{u}$ only, therefore the coefficient of the second derivatives must vanish separately.

$$u_i\left[B_{i\alpha k\beta} + B_{i\beta k\alpha} + 2(D_{i\alpha k\beta s\gamma} + D_{i\beta k\alpha s\gamma}) u_{s,\gamma}\right] = 0 .$$

And within this expression the coefficient of the first order derivatives must separately vanish. Therefore we obtain the following conditions for weak invariance of (2. 13).

$$u_i\left[\frac{\partial}{\partial x_\alpha}(A_{i\alpha}) + \frac{\partial}{\partial x_\alpha}(B_{i\alpha k\beta})u_{k,\beta} + \frac{\partial}{\partial x_\alpha}(D_{i\alpha k\beta s\gamma})u_{k,\beta}u_{s,\gamma}\right] = 0 \qquad (A.\,7)$$

$$u_i[B_{i\alpha k\beta} + B_{i\beta k\alpha}] = 0 \qquad (A.\,8)$$

$$u_i[D_{i\alpha k\beta s\gamma} + D_{i\beta k\alpha s\gamma}] = 0\,. \qquad (A.\,9)$$

Rather than examine these equations more closely in this generality, we will consider the special case $\underset{\sim}{B} \equiv \underset{\sim}{D} \equiv 0$. Note that we are implicitly enforcing the restrictions of Section 4 on the quadratic and cubic terms.

We therefore obtain from (A. 7) the restriction

$$\frac{\partial}{\partial x_\alpha}(A_{i\alpha})u_i = 0 \qquad (A.\,10)$$

where $A_{i\alpha}(\underset{\sim}{x}, \underset{\sim}{u}) = c_{i\alpha k\beta}\psi_{k,x_\beta}$. Expanding the derivatives we obtain

$$u_i c_{i\alpha k\beta}\psi_{k,x_\alpha x_\beta} + u_i c_{i\alpha k\beta}\psi_{k,x_\beta,u_j}u_{j,x_\alpha} = 0\,.$$

And again, since $\psi_k = \psi_k(\underset{\sim}{x}, \underset{\sim}{u})$ the derivative coefficient must vanish separately, therefore

$$u_i c_{i\alpha k\beta}\psi_{k,x_\beta,u_j} = 0 \qquad (A.\,11)$$

$$u_i c_{i\alpha k\beta}\psi_{k,x_\alpha,x_\beta} = 0\,. \qquad (A.\,12)$$

Any solution $\underset{\sim}{\psi}(\underset{\sim}{x}, \underset{\sim}{u})$ of these 13 equations which is consistent with Equations (4. 32), (4. 39), (4. 49), (4. 50) will leave the Lagrangian (2. 13) weakly invariant. In particular, note that if $\underset{\sim}{\psi}(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{\psi}(\underset{\sim}{x})$ (a function of $\underset{\sim}{x}$ alone), such that 4. 49 is satisfied, and $\underset{\sim}{\varphi} \equiv 0$ then Equations (A. 11), (A. 12) and all remaining necessary equations of

Section 4, (namely: 4.25, 4.34, 4.35, 4.38, 4.39, 4.50) are satisfied. Thus, in particular, if the functions $\underset{\sim}{\varphi}, \underset{\sim}{\psi}$ of (4.2) are chosen so that

$$\begin{aligned} \underset{\sim}{\varphi}(\underset{\sim}{\xi}, \underset{\sim}{w}) &= 0 \\ \underset{\sim}{\psi}(\underset{\sim}{\xi}, \underset{\sim}{w}) &= \underset{\sim}{\psi}(\underset{\sim}{\xi}) \end{aligned} \tag{A. 13}$$

where $\underset{\sim}{\psi}(\underset{\sim}{\xi})$ satisfies the displacement equations of motion (2.7) then the Lagrangian (2.13) is weakly invariant under (4.2) and the conservation law is given by

$$\frac{\partial}{\partial \xi_\alpha}[c_{j\alpha k\beta} u_{k,\beta} \psi_j - c_{j\alpha k\beta} \psi_{k,\beta} u_j] = 0 \ . \tag{A. 14}$$

We explicitly note that the $\psi(x)$ need only satisfy the field equations and (A. 12), and that it may violate the boundary conditions of any problem to which the resulting conservation law is applied. As an elastostatic example we note that since the field equations are second order, $u_i(\underset{\sim}{x}) = x_i$ is a solution. Thus taking $\psi_i(\underset{\sim}{x}) = x_i$ we see that $\mathcal{L}$ (2.13) is invariant to within $E = \frac{\partial}{\partial x_j}(c_{ijkk} u_i)$, and therefore

$$\frac{\partial}{\partial x_j}[c_{ijk\ell} u_{k,\ell} x_i - c_{ijkk} u_i] = 0$$

is a conservation law, which is independent of those covered by Theorems 1, 2.

As a further remark we note that our result of Section 5 required the use of a conservation law not directly derived from Noether's original theorem. We see that in the applications of the conservation laws we must frequently consider generalizations of Noether's theorem.

References

[1] J. R. Rice, A Path Independent Integral and the Approximate Analysis of Strain Concentrations by Notches and Cracks, Journal of Applied Mechanics, 35 (1968), 2, p. 379.

[2] J. D. Eshelby, The Continuum Theory of Lattice Defects, Solid State Physics, edited by F. Seitz and D. Turnbull, Volume 3, Academic Press, New York, 1956.

[3] J. R. Rice and G. F. Rosengren, Plane Strain Deformation Near a Crack Tip in a Power-Law Hardening Material, Journal of the Mechanics and Physics of Solids, 16 (1968), p. 1.

[4] J. W. Hutchinson, Singular Behavior at the End of a Tensile Crack in a Hardening Material, Journal of the Mechanics and Physics of Solids, 16 (1968), 3, p. 504.

[5] J. W. Hutchinson, Plastic Stress and Strain Fields at a Crack Tip, Journal of the Mechanics and Physics of Solids, 16 (1968), p. 13.

[6] J. K. Knowles and E. Sternberg, a paper to appear in Journal of Elasticity.

[7] J. K. Knowles and E. Sternberg, On a Class of Conservation Laws in Linearized and Finite Elasticity, Archive for Rational Mechanics and Analysis, 44 (1972), p. 187.

[8] E. Noether, Invariante Variationsprobleme, Göttinger Nachrichten Mathematisch-Physicalishe Klasse, 2 (1918), p. 235.

[9] W. Günther, Über einige Randintegrale der Elastomechanik, Abhandlungen, Braunschweiger Wissenschaftliche Gesellschaft, 14 (1962), p. 53.

[10] S. Lie, Transformation-gruppen, Göttinger Nachrichten, 3 (1871), p. 351.

[11] H. Bateman, The Transformation of the Electrodynamical Equations, Proc. of the London Mathematical Society, (2), 8, p. 228.

[12] M. E. Gurtin, The Linear Theory of Elasticity, Handbuch der Physik, Band VIa/2, p. 1-295.

[13] I. M. Gelfand and S. V. Fomin, Calculus of Variations, English translation by R. A. Silverman, Prentice-Hall, 1963.

[14] B. Budiansky and J. R. Rice, Conservation Laws and Energy Release Rates, Journal of Applied Mechanics, 40 (1973), p. 201.

[15] N. N. Bogliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, Interscience, New York, 1959.

[16] L. D. Landau,and E. M. Lifshitz, Mechanics, 2nd edition, Addison-Wesley, 1969.

[17] A. E. Green, On Some General Formulae in Finite Elastostatics, Archive for Rational Mechanics and Analysis, 50 (1972), p. 73.

[18] S. G. Lekhnitskii, Theory of Elasticity of an Anisotropic Elastic Body, Holden-Day, Inc., San Francisco, 1963.

[19] A. E. H. Love, The Mathematical Theory of Elasticity, 4th ed., Dover Publications, New York, 1944.

[20] E. Bessel-Hagen, Über die Erhaltungsätze der Electrodynamik, Mathematisch Annalen, 84 (1921), p. 259.