GENERALIZED MULTIPLIERS ON LOCALLY COMPACT ABELIAN GROUPS

by

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Abstract

Generalized Multipliers on Locally Compact Abelian Groups

Let G be a locally compact Abelian group with dual Γ, $1 \leqslant p, q \leqslant 2$, and $C_p = \{f \in L^p(G) \mid \operatorname{supp}(\hat{f}) \text{ is compact}\}$. Then for $1 \leqslant r \leqslant s \leqslant 2$, $C_1 \subseteq C_r \subseteq C_s \subseteq C_2$, the containments are proper if G is noncompact, and C_1 is a dense, translation invariant subspace of $L^t(G)$ for $1 \leqslant t < \infty$. Let λ be a complex valued function defined on Γ, and $J_\lambda^{p,q} = \{f \in L^p(G) \,|\, \lambda\hat{f} \in \widehat{L^q(G)}\}$. Suppose $J_\lambda^{p,q} \supseteq C_p$. Define the operator, $T_\lambda : J_\lambda^{p,q} \to L^q(G)$ by the equation $\widehat{T_\lambda f} = \lambda\hat{f}$ for each $f \in J_\lambda^{p,q}$. Then $J_\lambda^{p,q}$ is a module over M(G), T_λ is a module homomorphism, and T_λ is (p, q) closed. We call T_λ a generalized (p, q) multiplier.

The main results include:

(1) Suppose T is an operator satisfying:

(a) The domain D(T) is a translation invariant subspace of $L^p(G)$, and the range $R(T) \subseteq L^q(G)$;

(b) $D(T) \supseteq C_p$;

(c) T is (p, q) closed, linear, and commutes with all translations;

(d) $C \times T(C)$ is dense in $C_p \times T(C_p)$.

Then $T = T_\lambda$ for some λ.

(2) The set of all generalized (p, q) multipliers, denoted $X_{p,q}$, is a linear space, and the set of all generalized (p, p) multipliers, denoted X_p, is an algebra containing X_1 and contained in X_2.

(3) If $T_\lambda \in X_{p,q}$, then λ is locally the transform of a bounded (p, q) multiplier.

Further sections include a deeper study of X_1, X_2, and special results obtainable for compact G.

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Introduction

Let G be a locally compact Abelian group with dual Γ, and A, B be two Banach spaces of measurable functions on G for which the Fourier transform is defined. Suppose λ is a measurable function on Γ such that $\lambda\hat{f}$ is the transform of a function in space B whenever f is a function in A. Then λ determines a bounded linear operator from A into B, and if the spaces are translation invariant the operator commutes with all translations. λ is called the transform of the operator, and such operators are called (A, B) multipliers. The problem of determining all (A, B) multipliers is called the "multiplier problem".

The multiplier problem has drawn the interest of many mathematicians since the turn of the century. The early investigators considered spaces of real 2π-periodic functions, and looked for real sequences which multiplied Fourier coefficients of these functions to produce Fourier coefficients from a second space. Bochner was the first to use the complex Fourier series of a function. This allows one to think of the functions as being defined on the circle group T, and provides the prototype for compact Abelian groups. Others, notably Hörmander, studied the multiplier problem on R^n, which provides the prototype for noncompact LCA groups.

Since World War II much of harmonic analysis has been done on locally compact Abelian groups. This includes the study of (p, q) multipliers for $1 \leq p, q \leq \infty$. A (p, q) multiplier is a bounded, linear operator from $L^p(G)$ to $L^q(G)$ which commutes with all the translation operators on G. An equivalent description is an operator defined by a function λ

on Γ which multiplies $\widehat{L^p}$ into $\widehat{L^q}$. Brainerd and Edwards [2], Gaudry [17], [18], and Larsen [24], provide excellent sources for the study of (p, q) multipliers.

An interesting multiplier problem arises if one considers functions λ on Γ with the property that $\lambda\hat{f} \in \widehat{L^q(G)}$ for f in a dense subspace of $L^p(G)$. The operator determined by λ is closed, but not necessarily bounded. $L^1(G)$ provides a guide for this development. A version of this problem on L^1 appears in Ford [16], where it is shown that these operators have a minimal domain, and they form a commutative algebra containing the bounded L^1 multipliers. We call these the generalized L^1 multipliers on G. One area of application is the study of multipliers on Segal algebras, as in Burnham [3], since every multiplier on a Segal algebra is a generalized L^1 multiplier.

Our goal is to characterize generalized (p, q) multipliers for $1 \leq p, q \leq 2$, using the L^1 case as a guide. Chapter I contains the preliminaries on density, convolution, and translation needed elsewhere in the development; the most important results are Lemma 1.7, which relates convolution of L^p functions by L^1 functions to sums of translates of the L^p functions, and Lemma 1.8 which shows that closed, linear operators which commute with all translations also commute with convolution by finite regular Borel measures on G.

Chapter II defines a generalized (p, q) multiplier as an operator determined by a function λ having certain properties. These operators are shown to be closed, linear, and commute with convolution by measures in M(G). It follows that they commute with all translations.

We show that λ is locally the transform of a bounded (p, q) multiplier, and from this obtain inclusion results similar to those which hold for bounded multipliers.

Chapter III extends the results on generalized L^1 multipliers contained in [16]. Chapter IV is a deeper study of generalized L^2 multipliers, and Chapter V gives the additional results obtainable when G is compact.

Throughout the paper, G is an LCA group with Haar measure m. The dual group Γ has Haar measure m normalized so that the Plancherel transform is an isometry. M(G) denotes the set of finite regular Borel measures on G. If $f \in L^p(G)$, $\hat{f}$ denotes the Fourier, Plancherel, or Hausdorff-Young transform for $p = 1$, 2, or $1 < p < 2$, respectively. If $f \in L^p$ and $a \in G$, then f_a is the translate of f by a, i.e., $f_a(x) = f(x-a)$ for $x \in G$. $\tilde{f}$ is the involution: $\tilde{f}(x) = \overline{f(-x)}$. If $f \in L^p$ and $g \in L^1$, then $f*g$ is the convolution of f with g. If E is a measurable subset of G or Γ, χ_E is the characteristic function of E. If $1 < p < \infty$, p' denotes the dual index: $1/p + 1/p' = 1$. If $p = 1$, then $p' = \infty$. If $f \in L^2(\Gamma)$, then $\check{f}$ is the inverse image of f under the Plancherel transform, i.e., $\hat{\check{f}} = f$. For $1 \leq p \leq 2$, $\mathscr{F}^p(\Gamma) = \{\hat{f} \in L^{p'}(\Gamma) \mid f \in L^p(G)\}$. If $x \in G$ and $\gamma \in \Gamma$, (x, γ) denotes the value $\gamma(x)$ (or dually, $x(\gamma)$). Finally, $C_0(G)$ are the continuous functions on G which "vanish at infinity", and $C_c(G)$ are the continuous functions on G having compact support.

I. Preliminary Lemmas on Translation, Convolution and Density.

Definition: $C = \{f \in L^1(G) \mid \text{supp}(\hat{f}) \text{ is compact}\}$.

Lemma 1.1: For each $f \in L^1(G)$ and $\epsilon > 0$ there is a function $\nu \in C$ such that $\|f - f*\nu\|_1 < \epsilon$.

For a proof, see Rudin [30], Theorem [2.6.6]. An immediate consequence of this lemma is that C is a dense ideal of $L^1(G)$.

Lemma 1.2: $C = \bigcap \{I \mid I \text{ is a dense ideal of } L^1(G)\}$.

Proof: Let I be a dense ideal of L^1, and $f \in C$. Let $K = \text{supp}(\hat{f})$. For each $\gamma \in \Gamma$, let $M_\gamma = \{f \in L^1(G) \mid \hat{f}(\gamma) = 0\}$. Then M_γ is a maximal closed ideal of L^1, so $I \not\subseteq M_\gamma$. So choose $f_\gamma \in I$ such that $\hat{f}_\gamma(\gamma) \neq 0$. Let $g_\gamma = f_\gamma * \tilde{f}_\gamma$. Then $g_\gamma \in I$, $\hat{g}_\gamma(\gamma) > 0$. Since $\hat{g}_\gamma$ is continuous, there is a neighborhood V_γ of γ such that $\hat{g}_\gamma > 0$ on V_γ. Now $\{V_\gamma \mid \gamma \in K\}$ is an open cover of K, so there exist $\gamma_1, \cdots, \gamma_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} V_{\gamma_i}.$$

Let

$$g = \sum_{i=1}^{n} g_{\gamma_i}.$$

Then $g \in I$ and $\hat{g} > 0$ on K. By the Wiener-Lévy theorem there is a function $h \in L^1(G)$ such that $\hat{h} = 1/\hat{g}$ on K. This implies that $\hat{f} = \hat{f}\hat{g}\hat{h}$ on Γ, or $f = f*g*h$. Therefore $f \in I$, so $C \subseteq I$, and the Lemma follows from Lemma 1.1.

Lemma 1.3: Let $K \subseteq \Gamma$ be compact and $\epsilon > 0$. Then there is a function $g \in C$ such that $\hat{g}(\Gamma) \subseteq [0,1]$, $\hat{g} = 1$ on K, and $||g||_1 < 1 + \epsilon$.

The Lemma follows from Rudin [30], Theorems [2.6.8] and [2.6.1].

Lemma 1.4: Let $1 \leq p \leq \infty$, $f \in L^p(G)$, and $\mu \in M(G)$. Then $f*\mu \in L^p(G)$, and $||f*\mu||_p \leq ||f||_p||\mu||$.

This is Theorem [20.12] of Hewitt and Ross [22].

Lemma 1.5: Let $1 \leq p < \infty$, and $f \in L^p(G)$. Then there is a neighborhood V of 0 with $\overline{V}$ compact such that if u is any measurable function on G with $u \geq 0$, $\text{supp}(u) \subseteq \overline{V}$, and $\int_G u(x)dx = 1$, then $||f - f*u||_p < \epsilon$.

For a proof of this Lemma, see, for example, Loomis [25], Theorem 31E. Call such functions u V - blips.

Lemma 1.6: Let $1 \leq p, q < \infty$. Then C is a dense, translation invariant subspace of $L^p(G)$, and if $f \in L^p(G)$ and $g \in L^q(G)$ we can find a sequence $\{f_n\}_{n=1}^{\infty} \subseteq C$ such that both $||f - f*f_n||_p < 1/n$ and $||g - g*f_n||_q < 1/n$.

Proof: C is a subspace of $L^p(G)$ since $L^1 \cap L^p$ is a dense ideal of $L^1(G)$, and C is contained in every dense ideal. C is translation invariant since $\hat{f}_a(\gamma) = (-a, \gamma)\, \hat{f}(\gamma)$. To show C is dense in $L^p(G)$ let $f \in L^p(G)$ and $\epsilon > 0$. We can find $g \in L^1 \cap L^p$ such that $||f - g||_p < \epsilon/3$. By Lemma 1.5 we can find V a neighborhood of 0 such that if u is any V-blip then $||g - g*u||_p < \epsilon/3$. Also for a fixed V-blip u, Lemma 1.1 tells us we can find $v \in C$ such that

$||u-v||_1 < \epsilon/3(||g||_p+1)$. Now note that $g*v \in C$, and $||f-g*v||_p < \epsilon$. So C is dense in $L^p(G)$.

To prove the second part of the Lemma, for each n let V_n be a neighborhood of 0 small enough so that both $||f-f*u_n||_p < 1/2n$ and $||g-g*u_n||_q < 1/2n$ for any V_n-blip u_n. Also for fixed choices of such u_n, choose $f_n \in C$ such that

$$||u_n - f_n||_1 \leq \frac{1}{2n(||f||_p+1)(||g||_q+1)} .$$

Then $||f-f*f_n||_p < 1/n$, $||g-g*f_n||_q < 1/n$.

<u>Lemma 1.7</u>: Let $1 \leq p < \infty$, $f \in L^p(G)$, $g \in L^1(G)$, and $\epsilon > 0$. Then there are complex numbers $\alpha_1, \cdots, \alpha_n$, and $a_1, \cdots, a_n \in G$ such that

$$||f*g - \sum_{k=1}^{n} \alpha_k f_{a_k}||_p < \epsilon.$$

Proof: Without loss of generality, $g \neq 0$. Assume for the moment that $g \in C_c(G)$ with support K. Let V be a symmetric, precompact neighborhood of 0 in G small enough that $y \in V$ implies

$$||f - f_y||_p < \epsilon / ||g||_\infty m(K).$$

Now

$$K \subseteq \bigcup_{a \in K} (V+a),$$

so there exist $a_1, \cdots, a_n \in K$ such that

$$K \subseteq \bigcup_{k=1}^{n} (V + a_k).$$

Let $A_1 = K \cap (V+a_1)$, and

$$A_k = [K \cap (V+a_k)] - [\bigcup_{i=1}^{k-1} A_i] \text{ for } k = 2, \cdots, n.$$

Then

$$K = \bigcup_{k=1}^{n} A_k,$$

each A_k is measurable, and the union is pairwise disjoint. Let

$$\alpha_k = \int_{A_k} g(y)\, dy.$$

Then

$$||f*g - \sum_{k=1}^{n} \alpha_k f_{a_k}||_p = \left(\int_G |f*g(x) - \sum_{k=1}^{n} \alpha_k f_{a_k}(x)|^p dx\right)^{1/p}$$

$$= \left(\int_G | \int_G f(x-y)g(y)dy - \sum_{k=1}^{n} \alpha_k f_{a_k}(x)|^p dx\right)^{1/p}$$

$$= \left(\int_G | \int_K f(x-y)g(y)dy - \sum_{k=1}^{n} \alpha_k f_{a_k}(x)|^p dx\right)^{1/p}$$

$$= \left(\int_G | \sum_{k=1}^{n} \int_{A_k} [f(x-y) - f(x-a_k)] g(y)dy |^p dx\right)^{1/p}$$

$$\leqslant \sum_{k=1}^{n} \left(\int_G \left(\int_{A_k} |f(x-y) - f(x-a_k)| \, |g(y)| dy\right)^p dx\right)^{1/p}$$

where the inequality is obtained by first using Minkowski's inequality and then moving the absolute value signs inside the interior integrals. For each of those interior integrals we can use Hölder's inequality and get:

$$\left(\int_{A_k} |f(x-y) - f(x-a_k)| \, |g(y)| dy\right)^p$$

$$\leqslant m(A_k)^{p/p'} \int_{A_k} |f(x-y) - f(x-a_k)|^p \, |g(y)|^p dy$$

which implies

$$||f*g - \sum_{k=1}^{n} \alpha_k f_{a_k}||_p \leqslant$$

$$\sum_{k=1}^{n} m(A_k)^{1/p'} \left(\int_G \int_{A_k} |f(x-y) - f(x-a_k)|^p |g(y)|^p dydx \right)^{1/p}$$

$$= \sum_{k=1}^{n} m(A_k)^{1/p'} \left(\int_{A_k} \int_G |f(x-y) - f(x-a_k)|^p |g(y)|^p dxdy \right)^{1/p}$$

Now $(x-a_k) - (x-y) = y - a_k$, and if $y \in A_k$, then $y \in V + a_k$, or $y - a_k \in V$. Consequently the translation invariance of Haar measure gives us

$$\int_G |f(x-y) - f(x-a_k)|^p dx = \int_G |f(x-(y-a_k))- f(x)|^p dx$$

$$< \left(\epsilon / ||g||_\infty m(K) \right)^p .$$

Therefore

$$||f*g - \sum_{k=1}^{n} \alpha_k f_{a_k}||_p <$$

$$\sum_{k=1}^{n} m(A_k)^{1/p'} \frac{\epsilon}{||g||_\infty m(K)} \left(\int_{A_k} |g(y)|^p dy\right)^{1/p}$$

$$\leq \sum_{k=1}^{n} m(A_k)^{1/p'} \frac{\epsilon ||g||_\infty}{||g||_\infty m(K)} \left(\int_{A_k} 1\, dy\right)^{1/p}$$

$$= \sum_{k=1}^{n} m(A_k)^{1/p' + 1/p} \frac{\epsilon}{m(K)} = \sum_{k=1}^{n} m(A_k) \frac{\epsilon}{m(K)} = \epsilon$$

The lemma now follows from the density of $C_c(G)$ in $L^1(G)$.

Remark: Note that if we have $1 \leq p, q < \infty$, $f \in L^p(G)$, $h \in L^q(G)$, and $g \in L^1(G)$, and $\epsilon > 0$ we can find the complex numbers $\alpha_1, \cdots, \alpha_n$ and group elements $a_1, \cdots, a_n$ such that <u>both</u>

$$||f*g - \sum_{k=1}^{n} \alpha_k f_{a_k}||_p < \epsilon \text{ and } ||h*g - \sum_{k=1}^{n} \alpha_k h_{a_k}||_q < \epsilon \text{ hold.}$$

To do this we need only choose the neighborhood V small enough that translating either f or h by $y \in V$ leaves $||f - f_y||_p$ and $||h - h_y||_q$ small. After that step the construction depends only on the function g.

The Lemma is actually stronger than we need. For example, if $f \in L^p(G)$ and $S(f)$ is the closed subspace generated by the translates of f, then the Lemma implies that $S(f) = \overline{f*L^1(G)}$. When $p = 1$ we get the Proposition below and from it Wiener's Tauberian theorems, as done in Loomis [25], Chapter VII.

Proposition: Let $f \in L^1(G)$ such that $\hat{f}$ is never 0. Then $S(f) = L^1(G)$.

Proof: Let $g \in C$, with $\text{supp}(\hat{g}) = K$. $\hat{f}$ is bounded away from 0 on K, so there exists an $h \in L^1(G)$ such that $\hat{f}\hat{h} = 1$ on K. Then $\hat{g} = \hat{f}\hat{h}\hat{g}$ on Γ, or $g = f*h*g$. Let $\epsilon > 0$. By the Lemma we can get

$$||g - \sum_{k=1}^{n} \alpha_k f_{a_k}||_1 < \epsilon \quad ,$$

so $g \in S(f)$. The proposition follows from the density of C in $L^1(G)$.

Lemma 1.8: Let $1 \leq p, q < \infty$, and T a linear operator with the following properties:

(1) the domain $D(T) \subseteq L^p(G)$, the range $R(T) \subseteq L^q(G)$,

(2) T is closed,

(3) T commutes with all the translations on G.

Then D(T) is a module over $L^1(G)$ and if $f \in D(T)$ and $g \in L^1(G)$, we have $T(f*g) = Tf*g$.

Proof: Let $f \in D(T)$ and $g \in L^1(G)$, and $\epsilon > 0$. As we have already observed we can find $\alpha_1, \cdots, \alpha_n$ complex and $a_1, \cdots, a_n \in G$ such that

$$\left|\left|f*g - \sum_{k=1}^{n} \alpha_k f_{a_k}\right|\right|_p < \epsilon$$

and

$$\left|\left|Tf*g - \sum_{k=1}^{n} \alpha_k (Tf)_{a_k}\right|\right|_q < \epsilon$$

hold simultaneously. Note that

$$\sum_{k=1}^{n} \alpha_k f_{a_k} \in D(T)$$

since D(T) is a translation invariant subspace, and that

$$T\left(\sum_{k=1}^{n} \alpha_k f_{a_k}\right) = \sum_{k=1}^{n} \alpha_k (Tf)_{a_k}$$

since T is linear and commutes with translations. Since T has a closed graph we now have $f*g \in D(T)$ and $Tf*g = T(f*g)$.

II. Generalized Multipliers

The model for generalized multipliers is provided by the L^1 case. If T is a densely defined closed linear operator on $L^1(G)$, such that T commutes with all translations, then Lemma 1.8 implies the domain D(T) is a dense ideal of $L^1(G)$, and Lemma 1.2 implies that $D(T) \supseteq C$. If $f \in C$ with $\text{supp}(\hat{f}) = K$, Lemma 1.3 allows us to choose $g \in C$ such that $\hat{g} = 1$ on K. This gives $\widehat{Tf} = \widehat{T(f*g)} = \widehat{Tg*f} = \widehat{Tg}\,\hat{f}$, allowing us to define a function λ on Γ such that $\widehat{Tf} = \lambda\hat{f}$ for each $f \in D(T)$. In this manner we can identify the class of all densely defined, closed, linear operators which commute with all translations on $L^1(G)$ with a function algebra of multiplication operators on $\mathfrak{F}^1(\Gamma)$.

There are some minor complications in attempting to copy the same development for operators mapping from $L^p(G)$ into $L^q(G)$. First of all, for noncompact LCA groups we restrict our attention to $1 \leq p, q \leq 2$ in order to have the transform algebra available. Second, there is not usually a minimal dense submodule of $L^p(G)$ over $L^1(G)$ available for $1 < p \leq 2$. As an example, let G be noncompact, $p = 2$, and $\gamma \in \Gamma$. Let $M_\gamma = \{f \in L^1(G) \cap L^2(G) \,|\, \hat{f}(\gamma) = 0\}$. Then M_γ is a dense submodule of L^2 over L^1, but

$$\bigcap_{\gamma \in \Gamma} M_\gamma = \{0\}.$$

To avoid this difficulty we could restrict our attention to operators T with domain $D(T) \supseteq C$. This however would not be sufficient to guarantee that we could compose operators. For example, consider

$L^2(R)$, and the operator defined by $\widehat{Tf} = \chi_{[0,1]}\hat{f}$. This T does not map C into C, and if we could find an operator S with D(S) = C we could not make the composition $S \cdot T$. We avoid this difficulty by requiring the domain of the (p, q) - operator to contain $C_p = \{f \in L^p(G) \,|\, \text{supp}(\hat{f}) \text{ is compact}\}$. We will begin by making some observations on the Plancherel and Hausdorff-Young transforms.

Let $1 \leq p \leq 2$, and $1/p + 1/p' = 1$. Then for $f \in L^1(G) \cap L^p(G)$ we have $\hat{f} \in L^{p'}(\Gamma)$ and $\|\hat{f}\|_{p'} \leq \|f\|_p$. Equality holds when p = 2. We can extend the transform uniquely to all of $L^p(G)$, and the transform is well behaved in the sense that the following hold:

Lemma 2.1: Let $1 \leq p \leq 2$, $f \in L^p(G)$, and $g \in L^1(G)$. Then

i) $\hat{f} = 0$ iff $f = 0$

ii) $\widehat{f*g} = \hat{f}\hat{g}$

iii) If $f \in L^q(G)$ for $1 \leq q \leq 2$, $p \neq q$, then the transform of f as an L^p function and the transform of f as an L^q function agree a.e. on Γ.

For a proof, see [22], Theorems (31.32), (31.27), and (31.26).

Definition 2.1: Let $1 \leq p \leq 2$.
$C_p = C_p(G) = \{f \in L^p(G) \,|\, \text{supp}(\hat{f}) \text{ is compact}\}$.

Note: We agree to make the usual identification of functions that agree a.e., and under this equivalence find a representative function with compact support. Note also that C_1 is just C. We will continue to write C for

C_1. Finally, if G is compact then $C = C_1 = C_p = C_2 =$ {trigonometric polynomials} =

$$= \{f \in C(G) \,|\, f(x) = \sum_{k=1}^{n} \alpha_k(x,\gamma_k),\ \alpha_k \text{ complex},\ \gamma_k \in \Gamma\}.$$

<u>Theorem 2.1</u>: Let $1 \leq p < q \leq 2$. Then $C_p = L^p * C$, $C_1 \subseteq C_p \subseteq C_q \subseteq C_2$, and $C_p \neq C_q$ if G is noncompact.

Proof: If $f \in C_p$ with $K = \mathrm{supp}(\hat{f})$, choose $g \in C$ with $\hat{g} = 1$ on K. Then $\hat{f} = \hat{f}\hat{g} = \widehat{f*g}$, and $f = f*g \in L^p*C$. Clearly $L^p*C \subseteq C_p$, so $L^p*C = C_p$. Next note that $f \in C_0(G)$. If $p = 1$, then $f \in C \subseteq L^2(G)$ by Lemma 1.6; also $g \in L^2(G)$. If $p > 1$, then $g \in C \subseteq L^{p'}(G)$ by Lemma 1.6. In either case $f = f*g \in C_0(G)$ by Rudin [30], Theorem (1.1.6)d. Since $C_0(G) \cap L^p(G) \subseteq L^q(G)$, we have $f \in L^q(G)$, and Lemma 2.1 (iii) gives $f \in C_q$.

To show that the containment is proper for noncompact G, let $h \in C$ with h positive real, $h(0) > 1$, and $h(x) = h(-x)\ \forall\, x \in G$. Let V be a symmetric, precompact neighborhood of 0 in G with $|h(x)| > 1$ for $x \in V+V$. Since G is noncompact there is a sequence $\{a_n\}_{n=1}^{\infty}$ of points in G such that $(V + a_n) \cap (V + a_m) = \phi$ for $n \neq m$. Let

$$f = \sum_{n=1}^{\infty} 1/n^{1/p}\, \chi_{V+a_n}.$$

Then for $1 \leq r < \infty$,

$$|f|^r = f^r = \sum_{n=1}^{\infty} 1/n^{r/p}\, \chi_{V+a_n},$$

and

$$\int_G |f(x)|^r\, dx = \sum_{n=1}^{\infty} 1/n^{r/p}\, \mu(V).$$

Hence $f \in L^q(G) - L^p(G)$, and

$$h*f = \sum_{n=1}^{\infty} 1/n^{1/p}\, h*\chi_{V+a_n}$$

$$= \sum_{n=1}^{\infty} 1/n^{1/p}\, (h*\chi_V)_{a_n}.$$

Also

$$h*\chi_V(x) = \int_G h(x-y)\chi_V(y)dy$$

$$= \int_V h(x-y)dy = \int_V h(y-x)dy = \int_V h_x(y)dy$$

We claim $h*f \notin L^p(G)$, i.e.,

$$\int_G |h*f(x)|^p dx = \infty.$$

To see this, note

$$\int_G |h*f(x)|^p dx = \int_G \Big| \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} (h*\chi_V)_{a_n}(x) \Big|^p dx$$

$$\geq \int_G \sum_{n=1}^{\infty} \frac{1}{n} [(h*\chi_V)_{a_n}(x)]^p \, dx$$

$$\geq \int_G \sum_{n=1}^{N} \frac{1}{n} [(h*\chi_V)_{a_n}(x)]^p \, dx$$

$$= \sum_{n=1}^{N} \int_G \frac{1}{n} [(h*\chi_V)(x)]^p \, dx$$

$$\geq \sum_{n=1}^{N} \frac{1}{n} \int_V (h*\chi_V(x))^p \, dx$$

$$= \sum_{n=1}^{N} \frac{1}{n} \int_V \left(\int_V h_x(y)dy \right)^p dx$$

$$\geq \sum_{n=1}^{N} \frac{1}{n} \int_V 1 \circ \mu(V)^p \, dx = \sum_{n=1}^{N} \frac{1}{n} \mu(V)^{1+p}.$$

This holds for any $N \geq 1$, so

$$\int_G |h*f(x)|^p \, dx = \infty.$$

Definition 2.2: Let $1 \leq p, q \leq 2$.

$\hat{X}_{p,q} = \{\lambda$ measurable on $\Gamma \mid \lambda\hat{f} \in \mathfrak{F}^q(\Gamma)$ for each $f \in C_p\}$. If $p = q$, we denote $\hat{X}_{p,q}$ by $\hat{X}_p$.

Theorem 2.2: Let $1 \leq r \leq p \leq 2$, $1 \leq q \leq s \leq 2$. Then $\hat{X}_{p,q} \subseteq \hat{X}_{r,s}$.

Proof: Let $\lambda \in \hat{X}_{p,q}$, and $f \in C_r$. Then $f \in C_p$, and $\lambda\hat{f} \in \mathfrak{F}^q(\Gamma)$. But $K = \operatorname{supp}(\hat{f})$ is compact, and $\operatorname{supp}(\lambda\hat{f}) \subseteq K$. So $\lambda\hat{f} \in \hat{C}_q$, and by Lemma 2.1 (iii) and Theorem 2.1, $\lambda\hat{f} \in \hat{C}_s \subseteq \mathfrak{F}^s(\Gamma)$. Therefore $\lambda \in \hat{X}_{r,s}$.

Definition 2.3: Let $1 \leq p, q \leq 2$, and $\lambda \in \hat{X}_{p,q}$.

$J_\lambda^{p,q} = \{f \in L^p(G) \mid \lambda\hat{f} \in \mathfrak{F}^q(\Gamma)\}$. Again we will write J_λ^p for $J_\lambda^{p,p}$. Note that $J_\lambda^{p,q} \supseteq C_p$.

Definition 2.4: Let $1 \leq p, q \leq 2$, and $\lambda \in \hat{X}_{p,q}$. $T_\lambda = T_\lambda^{p,q}$ is the operator with domain $J_\lambda^{p,q}$, range contained in $L^q(G)$, defined by the equation $\widehat{T_\lambda f} = \lambda\hat{f}$ for each $f \in J_\lambda^{p,q}$. $X_{p,q} = \{T_\lambda^{p,q} \mid \lambda \in \hat{X}_{p,q}\}$.

Theorem 2.3: Let $1 \leq p, q \leq 2$ and $T_\lambda \in X_{p,q}$. Then T_λ is (p, q)-closed, linear, and commutes with convolution by elements of $M(G)$.

Proof: If $f, g \in J_\lambda^{p,q}$, and α, β are complex numbers, then $\widehat{T_\lambda(\alpha f + \beta g)} = \lambda\widehat{(\alpha f + \beta g)} = \alpha\lambda\hat{f} + \beta\lambda\hat{g} = \widehat{\alpha T_\lambda f + \beta T_\lambda g}$, so T_λ is linear. If $\mu \in M(G)$, then $\widehat{T_\lambda f * \mu} = \lambda\hat{f}\hat{\mu} = \lambda\widehat{f*\mu}$, so $f * \mu \in J_\lambda^{p,q}$ and $T_\lambda f * \mu = T_\lambda(f * \mu)$.

Suppose $\{f_n\} \subseteq J_\lambda^{p,q}$, $f \in L^p(G)$, and $g \in L^q(G)$ such that $f_n \xrightarrow{p} f$ and $T_\lambda f_n \xrightarrow{q} g$. Let $h \in C$ with $K = \operatorname{supp}(\hat{h})$. Then $f_n * h$,

$f*h \in C_p \subseteq J_\lambda^{p,q}$, and $||\lambda\hat{f}\hat{h} - \hat{g}\hat{h}||_{q'} \leq ||\lambda\hat{h}(\hat{f} - \hat{f}_n)||_{q'} + ||\hat{h}(\lambda\hat{f}_n - \hat{g})||_{q'}$
$\leq ||\lambda\hat{h}(\hat{f} - \hat{f}_n)||_{q'} + ||h||_1 ||T_\lambda f_n - g||_q$.

Now $\hat{f}_n \xrightarrow{p'} \hat{f}$, so there exists a subsequence f_{n_k} such that $\hat{f}_{n_k} \to \hat{f}$ a.e. on Γ. Also $h \in C \subseteq C_p$, so $\lambda\hat{h} \in L^{q'}(\Gamma)$, and $\lambda\hat{h}$ is finite a.e. on Γ. Therefore $\lambda\hat{h}\hat{f}_{n_k} \to \lambda\hat{h}\hat{f}$ a.e. on Γ. But $\lambda\hat{h}\hat{f}_{n_k} \xrightarrow{q'} \hat{h}\hat{g}$, so there is a subsequence $f_{n_{k_m}}$ such that $\lambda\hat{h}\hat{f}_{n_{k_m}} \to \hat{h}\hat{g}$ a.e. on Γ. Therefore $\hat{h}\hat{g} = \lambda\hat{h}\hat{f}$ a.e. on Γ. Since $h \in C$ was arbitrary, $\hat{g} = \lambda\hat{f}$ a.e. on Γ, so $||\hat{g} - \lambda\hat{f}||_{q'} = 0$. Therefore $f \in J_\lambda^{p,q}$, $T_\lambda f = g$, and T_λ is closed.

<u>Corollary 1</u>: $J_\lambda^{p,q}$ is a translation invariant subspace of $L^p(G)$ and T_λ commutes with all translations.

<u>Corollary 2</u>: T_λ is bounded iff $J_\lambda^{p,q} = L^p(G)$.

Proof: Closed graph theorem.

<u>Corollary 3</u>: T_λ maps C_p into C_q.

<u>Theorem 2.4</u>: Let $\lambda \in \hat{X}_{p,q}$ and $h \in C$. Then $\lambda\hat{h} \in \hat{X}_{p,q}$ and $T_{\lambda\hat{h}}$ is (p, q) bounded.

Proof: Let $f \in L^p(G)$. Then $h*f \in C_p$. Since $\lambda \in \hat{X}_{p,q'}$ we know $\widehat{\lambda h*f} = \lambda\hat{h}\hat{f} \in \widehat{C_q}$. Therefore $J_{\lambda\hat{h}}^{p,q} = L^p(G) \supseteq C_p$, so $\lambda\hat{h} \in \hat{X}_{p,q}$. Furthermore, by Theorem 2.3 $T_{\lambda\hat{h}}$ is (p, q) closed, so the Closed Graph Theorem implies $T_{\lambda\hat{h}}$ is (p, q) bounded.

<u>Definition 2.5</u>: Let λ be a function on Γ. <u>λ has property P locally on Γ</u> if $\lambda|_K$ has property P for each compact $K \subseteq \Gamma$.

Examples of local properties we shall consider are locally measurable, locally L^∞, and locally the transform of a bounded (p, q)-multiplier.

Theorem 2.5: $\hat{X}_{p,q}$ is the set of all functions on Γ which are locally the transform of a (p, q)-multiplier.

Proof: Suppose that λ is locally the transform of a (p, q)-multiplier on Γ. Let $f \in C_p$ with $K = \operatorname{supp}(\hat{f})$. Then there is a (p, q)-multiplier T such that $\lambda = \hat{T}$ on K. Then $\lambda\hat{f} = \widehat{Tf}$ on Γ, and $Tf \in \widehat{L^q(G)}$. Therefore, $J_\lambda^{p,q} \supseteq C_p$, and $\lambda \in \hat{X}_{p,q}$.

If $\lambda \in \hat{X}_{p,q}$ and $K \subseteq \Gamma$ is compact, choose $h \in C$ such that $\hat{h} = 1$ on K. Then $\lambda = \lambda\hat{h} = \widehat{T_{\lambda\hat{h}}}$ on K, so on K λ is the transform of a (p, q) multiplier.

Corollary 1: Let $1 \leq q < p \leq 2$, and G noncompact. Then $\hat{X}_{p,q} = \{0\}$.

Proof. From Gaudry [17], Sec. 5, we know that the only (p, q) multiplier for $q < p$ is 0 when G is noncompact.

Corollary 2: Let $1 \leq p \leq q \leq 2$. Then $\hat{X}_p \subseteq \hat{X}_q$.

Proof: This follows from the fact that an L^p-multiplier is also an L^q-multiplier for $1 \leq p \leq q \leq 2$. (See Larsen [24], Corollary 4.1.3, page 97).

Corollary 3: Let T be a (p, q)-multiplier. Then $T \in X_{p,q}$.

Proof: $\hat{T}$ is locally the transform of T, which is a (p, q) multiplier.

We call $X_{p,q}$ <u>the set of generalized (p, q)-multipliers</u>. Note that every multiplier is also a generalized multiplier by Corollary 3 above.

<u>Theorem 2.6</u>: Let $1 \leq p, q \leq 2$, and T be an operator with the following properties:

(1) the domain D(T) satisfies $C_p \subseteq D(T) \subseteq L^p$

(2) the range $R(T) \subseteq L^q$

(3) T is (p, q) closed, linear, and commutes with translations by elements of G

(4) $C \times T(C)$ is dense in $C_p \times T(C_p)$.

Then $T \in X_{p,q}$.

Proof: By Lemma 1.8 D(T) is a module over $L^1(G)$ and $T(f*g) = Tf*g$ for each $f \in D(T)$ and $g \in L^1(G)$. Let $K \subseteq \Gamma$ be compact and $h \in C$ such that $\hat{h} = 1$ on K. Let $\lambda = \widehat{Th}$ on K. If $g \in C$ with $\hat{g} = 1$ on K, then $\widehat{Th} = \widehat{Th}\hat{g} = \widehat{Tg}\hat{h} = \widehat{Tg}$, a.e. on K by Lemma 2.1. So λ is well defined as a function in $L^{p'}(K)$, and we can extend λ to a well defined, locally $L^{p'}$ function on Γ. λ is measurable since it is locally measurable (see Hewitt and Ross I [22], Theorem (11.42)). If $f \in C$, and $h \in C$ with $\hat{h} = 1$ on $\mathrm{supp}(\hat{f})$, then $\widehat{Tf} = \widehat{T(f*h)} = \widehat{Th}\,\hat{f} = \lambda\hat{f}$. If $f \in C_{p'}$, then we can find $f_n \in C$ such that $f_n \xrightarrow{p} f$ and $Tf_n \xrightarrow{q} Tf$, so $\widehat{Tf_n} = \lambda\hat{f}_n \xrightarrow{q'} \widehat{Tf}$. But there is a subsequence f_{n_k} so that $\hat{f}_{n_k} \to \hat{f}$, a.e., so $\lambda\hat{f}_{n_k} \to \lambda\hat{f}$, a.e., or $\widehat{Tf} = \lambda\hat{f}$. Therefore $J_\lambda^{p,q} \supseteq C_{p'}$, and $T = T_\lambda$ on C_p. Now we can use Theorem 2.4 to conclude $T = T_{\lambda'}$, since for each $h \in C$ $T_{\lambda\hat{h}}$ is (p, q) bounded, and $Tf*h = T_{\lambda\hat{h}}f$ for each $f \in D(T)$ or $J_\lambda^{p,q}$.

<u>Theorem 2.7:</u> Let $1 \leq p, q, r, s \leq 2$, and $q \leq r$. Let $\lambda_1 \in \hat{X}_{p,q}$ and $\lambda_2 \in \hat{X}_{r,s}$. Then $\lambda_2\lambda_1 \in \hat{X}_{p,s}$, so $T_{\lambda_2}T_{\lambda_1} = T_{\lambda_2\lambda_1} \in X_{p,s}$.

Proof: Since $C_q \subseteq C_r$, we have for each $f \in C_p$, $\lambda_1\hat{f} \in C_r$, so $\lambda_2\lambda_1\hat{f} \in C_s$ or $J^{p,s}_{\lambda_2\lambda_1} \supseteq C_p$.

<u>Theorem 2.8:</u> Let $1 \leq p \leq 2$. Then X_p is a commutative algebra.

Proof: By Theorem 2.7, if $\lambda_1, \lambda_2 \in \hat{X}_p$ then $\lambda_1\lambda_2 = \lambda_2\lambda_1 \in \hat{X}_p$, so $T_{\lambda_2\lambda_1} = T_{\lambda_1}T_{\lambda_2} = T_{\lambda_1\lambda_2} \in X_p$.

III. Generalized L^1 Multipliers

This chapter contains a deeper study of X_1, the generalized multipliers on $L^1(G)$. This algebra is the prototype of generalized multipliers, and because of the structure available in $L^1(G)$ we can avoid many of the difficulties encountered in X_p for $1 < p \leq 2$. Also from Corollary 2 of Theorem 2.5 we have $X_1 \subseteq X_p$.

One of the areas of current research on multipliers is in Segal algebras, which are subalgebras of $L^1(G)$ which are Banach algebras under their own norms and also Banach modules (i.e., $f \in A$, $g \in L^1(G)$, then $f*g \in A$ and $||f*g||_A \leq ||f||_A ||g||_1$). These algebras are usually taken to be dense. The virtue of studying X_1 is that multipliers on (dense) Segal algebras are all in X_1, and inherit all of its structure. We proceed forthwith.

Theorem 3.1: Let $T_\lambda \in X_1$. Then λ is continuous.

Proof: Let $\gamma \in \Gamma$. Since Γ is locally compact, there is a neighborhood V_γ of γ such that $\overline{V_\gamma}$ is compact. Choose $g \in C$ such that $\hat{g} = 1$ on $\overline{V_\gamma}$. $T_\lambda g \in L^1(G)$, $T_\lambda f = \lambda\hat{f}$ for each $f \in J_\lambda^{1,1}$, so $\widehat{T_\lambda g} = \lambda$. So λ is continuous on $\overline{V_\gamma}$ and λ is continuous at γ. Since $\gamma \in \Gamma$ is arbitrary, λ is continuous.

Theorem 3.2: Let $T_\lambda \in X_1$ and $f \in C$. Then $T_\lambda f(x) = \int_\Gamma \lambda(\gamma)\hat{f}(\gamma)(x,\gamma)d\gamma$.

Proof: By Corollary 3 to Theorem 2.3, $T_\lambda f \in C$. Since λ is continuous and $\hat{f}$ has compact support, $\lambda\hat{f} \in L^1(\Gamma)$. The inversion theorem can be applied to give the above formula.

Theorem 3.3: Let T be an operator with domain D(T) a dense ideal of $L^1(G)$, range $R(T) \subseteq L^1(G)$, such that $f \in D(T)$ and $g \in L^1(G)$ implies $f*g \in D(T)$ and $T(f*g) = Tf*g$. Then T is a restriction of some element of X_1.

Proof: Since D(T) is dense, $D(T) \supseteq C$, and we can apply the construction in the proof of Theorem 2.6 to get the function λ on Γ with the property that $\widehat{Tf} = \lambda\hat{f}$ for each $f \in C$. If $g \in D(T)$ and $K \subseteq \Gamma$ is compact, let $h \in C$ such that $\hat{h} = 1$ on K. Then on K, $\widehat{Tg} = \widehat{Tg}\hat{h} = \widehat{Th}\hat{g} = \lambda\hat{g} = \widehat{T_\lambda g}$. Therefore $Tg = T_\lambda g$ and T is a restriction of T_λ.

Definition 3.1: (Generalized Strong Operator Topology). Let $\mathcal{T}$ be the set of all operators T with domain D(T) a dense ideal of $L^1(G)$, range $R(T) \subseteq L^1(G)$, such that T is closed. A net $\{T_\alpha \mid \alpha \in A\} \subseteq \mathcal{T}$ converges to $T \in \mathcal{T}$ if $\|T_\alpha f - Tf\|_1 \to 0$ for each $f \in C$. The topology determined by this convergence is the generalized strong operator topology on L^1.

Remark: The generalized strong operator topology extends the strong operator topology to unbounded operators.

Theorem 3.4: X_1 is closed in $\mathcal{T}$ under the generalized strong operator topology, and $Bdd(L^1(G)) \cap X_1$ is closed in $Bdd(L^1(G))$ under the strong operator topology.

Proof: Let $T \in \mathcal{T}$, $\{T_\alpha\} \subseteq X_1$, $T_\alpha \to T$ in the generalized strong operator topology. Let $f \in C$, and $g \in L^1(G)$. Note that $C \subseteq D(T)$ and that f and $f*g \in C \subseteq D(T)$. Let $\epsilon > 0$. Then there is an $\alpha_1 \in A$ such that $\alpha \geq \alpha_1$ implies that $\|T_\alpha f - Tf\|_1 < \epsilon/2\,\|g\|_1$. Also there is an α_2 such that $\alpha \geq \alpha_2$ implies $\|T_\alpha(f*g) - T(f*g)\|_1 < \epsilon/2$.

Then

$$\begin{aligned}
&\|T(f*g) - Tf*g\|_1 \\
&\qquad \leqslant \|T(f*g) - T_\alpha(f*g)\|_1 + \|T_\alpha f*g - Tf*g\|_1 \\
&\qquad \leqslant \|T(f*g) - T_\alpha(f*g)\|_1 + \|T_\alpha f - Tf\|_1 \ \|g\|_1 \\
&\qquad < \epsilon/2 + \epsilon/2 = \epsilon .
\end{aligned}$$

Therefore, $T(f*g) = Tf*g$.

By Theorem 3.3 T is the restriction of some $T_\lambda \in X_1$. Also $T = T_\lambda$ on C, and T is closed. Therefore by Theorem 2.6 $T \in X_1$, or $T = T_\lambda$.

If the T_α are all bounded, and $T \in \mathrm{Bdd}(L^1(G))$ such that for each $f \in L^1(G)$, $T_\alpha f \xrightarrow{1} Tf$, then T commutes with convolution on all $L^1(G)$, and $T = T_\lambda$ on all $L^1(G)$, so $\mathrm{Bdd}(L^1(G)) \cap X_1$ is closed under the strong operator topology in $\mathrm{Bdd}(L^1)$.

Remark: It is not true that $\mathrm{Bdd}(L^1(G)) \cap X_1$ is closed under the generalized strong operator topology. An example follows the next theorem.

<u>Theorem 3.5</u>: Let $\gamma \in \hat{X}_1$, and A analytic on a neighborhood of $\lambda(\Gamma)$. Then $A \circ \lambda \in \hat{X}_1$.

Proof: We must show $J^1_{A \circ \lambda} \supseteq C$. So let $f \in C$ with $K = \mathrm{supp}(\hat{f})$. Let $g \in C$ such that $\hat{g} = 1$ on K. Then $\lambda = \widehat{T_\lambda g}$ on K, and $A \circ \lambda = A(\widehat{T_\lambda g})$ on K. By the Wiener-Lévy theorem (see Reiter [28], Chapter 6) there exists $h \in L^1(G)$ such that $\hat{h} = A(\widehat{T_\lambda g})$ on K. So $\hat{h}\hat{f} = \widehat{h*f} = A(\widehat{T_\lambda g})\hat{f}$ on Γ. Therefore $A \circ \lambda \in \hat{X}_1$.

<u>Corollary 1</u>: $T_\lambda \in X$ is invertible iff λ is never 0. In this case $\widehat{T^{-1}} = 1/\hat{T}$.

<u>Corollary 2</u>: Let A be entire and T_λ bounded. Then $T_{A\circ\lambda}$ is bounded.

Proof: Suppose

$$A(z) = \sum_{k=0}^{\infty} \alpha_k z^k \qquad \text{for all } z \quad .$$

Let

$$T = \sum_{k=0}^{\infty} \alpha_k T_\lambda{}^k \quad .$$

Then $T \in \mathrm{Bdd}(L^1)$, T is the norm limit (hence strong operator limit) of elements of X_1, and by Theorem 3.4 $T \in X_1$. Also

$$\hat{T} = \sum_{k=0}^{\infty} \widehat{\alpha_k T_\lambda{}^k} = \sum_{k=0}^{\infty} \alpha_k \lambda^k = A \circ \lambda \quad .$$

So $T = T_{A\circ\lambda}$.

Remark: It might be conjectured that if T_λ is bounded and A analytic, with $\lambda(\Gamma)$ bounded away from the singularities of A, that $T_{A\circ\lambda}$ is bounded. However, it is known (see Rudin [30], Theorem 6.4.1) that for G nondiscrete, and z_0 any complex number, there is a $\mu \in M(G)$ with $\hat{\mu}(\Gamma) \subseteq [-1, 1]$ and z_0 in the spectrum of μ. Therefore $\hat{\mu} - z_0$ can be bounded arbitrarily far away from 0, but $J^1_{1/\hat{\mu}-z_0} \neq L^1(G)$.

Definition 3.2: Let λ be continuous on Γ, and $f \in C$. $F_{\lambda,f}$ is the function defined on G by

$$F_{\lambda,f}(x) = \int_{\Gamma} \lambda(\gamma)\hat{f}(\gamma)(x,\gamma)d\gamma \quad .$$

Theorem 3.6: $\lambda \in \hat{X}_1$ iff $F_{\lambda,f} \in L^1(G)$ for each $f \in C$.

Proof: If $\lambda \in \hat{X}_1$, then $F_{\lambda,f} = T_\lambda f \in L^1(G)$. If $F_{\lambda,f} \in L^1(G)$ for each $f \in C$, then $F_{\lambda,f} = \lambda\hat{f} \in \widehat{L^1(G)}$ for each $f \in C$. So $J_\lambda^1 \supseteq C$, and $\lambda \in \hat{X}_1$.

To close this chapter we characterize the bounded multipliers on $L^1(G)$. To do this we first need Bochner's Theorem, as stated in Rudin [30], Theorem 1.9.1:

Theorem 3.7: (Bochner's Theorem). The following are equivalent:

(1) $\lambda = \hat{\mu}$ where $\mu \in M(G)$ and $||\mu|| \leq M$.

(2) λ is continuous on Γ, and

$$\left|\sum_{k=1}^{m} c_k \lambda(\gamma_k)\right| \leq M \left|\left|\sum_{k=1}^{m} c_k (x,\gamma_k)\right|\right|_\infty$$

for each choice of $c_1, \ldots, c_m$ complex and $\gamma_1, \ldots, \gamma_m \in \Gamma$.

Theorem 3.8: Let λ be continuous on Γ. The following are equivalent:

(1) $\lambda = \hat{\mu}$, where $\mu \in M(G)$ and $||\mu|| \leq M$.

(2) $T_\lambda \in X_1$, and $||T_\lambda|| \leq M$.

(3) $F_{\lambda,f} \in L^1(G)$ for each $f \in C$, and $||F_{\lambda,f}||_1 \leq M||f||_1$.

Proof: (1) $\Rightarrow$ (2): If $\lambda = \hat{\mu}$, define $T : L^1(G) \to L^1(G) : f \mapsto f*\mu$. Then $\widehat{Tf} = \widehat{f*\mu} = \hat{\mu}\hat{f} = \lambda\hat{f}$ for each $f \in L^1(G)$, so $T = T_\lambda$ and $||T_\lambda|| = ||T|| \leqslant ||\mu|| \leqslant M$.

(2) $\Rightarrow$ (3). If $T_\lambda \in X$, then $\lambda \in \hat{X}_1$ and by Theorem 3.6 $F_{\lambda,f} \in L^1(G)$ for each $f \in C$. Also $||F_{\lambda,f}||_1 = ||T_\lambda f||_1 \leqslant ||T_\lambda||\ ||f||_1 \leqslant M||f||_1$.

(3) $\Rightarrow$ (1): Using the inversion formula

$$\lambda(\gamma)\hat{f}(\gamma) = \int_G F_{\lambda,f}(x)\,(-x,\gamma)dx$$

for each $f \in C$.

Let $c_1, \ldots, c_m$ be complex numbers and $\gamma_1, \ldots, \gamma_m \in \Gamma$. Then for each $f \in C$,

$$\left|\sum_{k=1}^{m} c_k \lambda(\gamma_k)\hat{f}(\gamma_k)\right| = \left|\sum_{k=1}^{m} c_k \int_G F_{\lambda,f}(x)\,(-x,\gamma_k)dx\right|$$

$$\leqslant \int_G \left|\sum_{k=1}^{m} c_k(-x,\gamma_k)\right|\ |F_{\lambda,f}(x)|dx$$

$$\leqslant \left|\left|\sum_{k=1}^{m} c_k\,(-x,\gamma_k)\right|\right|_\infty ||F_{\lambda,f}||_1$$

$$= \left|\left|\sum_{k=1}^{m} c_k(x,\gamma_k)\right|\right|_\infty ||F_{\lambda,f}||_1$$

$$\leqslant M\left|\left|\sum_{k=1}^{m} c_k(x,\gamma_k)\right|\right|_\infty ||f||_1 \quad .$$

Let $\epsilon > 0$. By Lemma 1.3 we can choose $f \in C$ such that $\hat{f}(\gamma_k) = 1$, $k = 1, \ldots, m$, and $\|f\|_1 < 1 + \epsilon$. So

$$\left|\sum_{k=1}^{m} c_k \lambda(\gamma_k)\right| \leq M \left\|\sum_{k=1}^{m} c_k (x, \gamma_k)\right\|_\infty \|f\|_1$$

$$< M \left\|\sum_{k=1}^{m} c_k (x, \gamma_k)\right\|_\infty (1 + \epsilon) \quad .$$

Since ϵ is arbitrary,

$$\left|\sum_{k=1}^{m} c_k \lambda(\gamma_k)\right| \leq M \left\|\sum_{k=1}^{m} c_k (x, \gamma_k)\right\|_\infty ,$$

and by Bochner's theorem $\lambda = \hat{\mu}$, where $\mu \in M(G)$ with $\|\mu\| \leq M$.

<u>Corollary:</u> $\hat{X}_1$ is the set of all functions which are locally the transform of a finite regular Borel measure on G.

IV. Generalized L^2 Multipliers

This chapter contains a complete characterization of X_2, the generalized multipliers on $L^2(G)$. The main result is that $\hat{X}_2$ is the set of all locally L^∞ functions on Γ. A function λ is locally L^∞ on Γ if $\lambda\chi_K \in L^\infty(\Gamma)$ for each compact subset K of Γ.

The cornerstone of this chapter is the Plancherel Theorem, which states that the Fourier transform is an L^2 isometry when restricted to $L^1 \cap L^2$, and since $L^1 \cap L^2$ is dense in L^2, it can be extended uniquely to an isometry from $L^2(G)$ to $L^2(\Gamma)$. The extension is called the Plancherel transform, and under this transform $L^2(G)$ and $L^2(\Gamma)$ are isometrically isomorphic.

Lemma 4.1: Let $f, g \in L^2(G)$, and $h \in L^1(G)$. Then $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$, and $\widehat{f*h} = \hat{f}\hat{h}$.

Proof: The first equality holds because the inner product in a Hilbert space is completely determined by the norm. The second equality holds for all $f \in L^1(G) \cap L^2(G)$, so it holds on $L^2(G)$.

Definition 4.1: Let λ be a complex valued function defined on Γ. λ is locally L^∞ on Γ if $\lambda\chi_K \in L^\infty(\Gamma)$ for each $K \subseteq \Gamma$, K compact.

Lemma 4.2: Let λ be locally L^∞ on Γ. Then λ is measurable.

Proof: Since $\lambda\chi_K \in L^\infty(\Gamma)$ for each compact K, $\lambda|_K$ is measurable for each compact K. Then λ is measurable by Hewitt and Ross I, Theorem (11.42).

Theorem 4.1: $\widehat{\mathrm{Bdd}(L^2(G)) \cap X_2} = L^\infty(\Gamma)$.

Proof: If $\lambda \in L^\infty(\Gamma)$ then $\lambda \in \hat{X}_2$ and $J^2_\lambda = L^2(G)$. Also for $f \in L^2(G)$ we have $||T_\lambda f||_2 = ||\lambda\hat{f}||_2 \leq ||\lambda||_\infty ||\hat{f}||_2 = ||\lambda||_\infty ||f||_2$, so $||T_\lambda|| \leq ||\lambda||_\infty$.

Let $T_\lambda \in \mathrm{Bdd}(L^2) \cap X_2$. Suppose $\lambda \notin L^\infty(\Gamma)$. Then there is $K \subseteq \Gamma$, K compact with $m(K) > 0$ such that $|\lambda| \geq ||T_\lambda||$ on K. Let

$$f = \frac{1}{m(K)^{\frac{1}{2}}} \check{\chi}_K \quad .$$

Then $f \in L^2(G)$, $||f||_2 = 1$, but

$$||T_\lambda f||_2^2 = ||\lambda\hat{f}||_2^2 = \frac{1}{m(K)} \int_\Gamma |\lambda\chi_K|^2 d\gamma > ||T_\lambda||^2 \quad ,$$

a contradiction.

To show $||\lambda||_\infty \leq ||T_\lambda||$, let $\epsilon > 0$. Then there is a compact $K_1 \subseteq \Gamma$ with $m(K_1) > 0$ and $|\lambda| > ||\lambda||_\infty - \epsilon$ on K_1. Then

$$(||\lambda||_\infty - \epsilon)^2 m(K_1) < \int_{K_1} |\lambda(\gamma)|^2 d\gamma = \int_\Gamma |\lambda(\gamma)\chi_{K_1}(\gamma)|^2 d\gamma$$

$$= \int_G |T_\lambda \check{\chi}_{K_1}(x)|^2 dx = ||T_\lambda \check{\chi}_{K_1}||_2^2$$

$\leq ||T_\lambda||^2 ||\check{\chi}_{K_1}||_2^2 = ||T_\lambda||^2 ||\chi_{K_1}||_2^2 = ||T_\lambda||^2 m(K_1)$. So $||\lambda||_\infty - \epsilon < ||T_\lambda||$, and since ϵ is arbitrary, $||\lambda||_\infty \leq ||T_\lambda||$. Combining with the previous result gives $||\lambda||_\infty = ||T_\lambda||$, so $X_2 \cap \mathrm{Bdd}(L^2)$ and $L^\infty(\Gamma)$ are isometrically isomorphic as Banach spaces.

Corollary: Let $T_\lambda \in X_2$, T_λ bounded. Then the spectrum of T_λ is the closure of the essential range of λ.

Theorem 4.2: $\hat{X}_2$ is the set of all locally L^∞ functions on Γ.

Proof: This follows from Theorems 4.1 and 2.5.

Theorem 4.3: X_2 is a commutative, self-adjoint algebra containing X_1, and the containment is proper if G is noncompact.

Proof: We have already shown that X_2 is a commutative algebra containing X_1. If G is noncompact then Γ is nondiscrete and there are noncontinuous functions in $L^\infty(\Gamma)$. Since $L^\infty(\Gamma) \subseteq \hat{X}_2$ and every $\lambda \in \hat{X}_1$ is continuous, the containment is proper.

Let $\lambda \in \hat{X}_2$. Then $\bar{\lambda} \in \hat{X}_2$, and $J_\lambda^2 = J_{\bar{\lambda}}^2$. Also if f, $g \in J_\lambda^2$, then $\langle T_\lambda f, g\rangle = \langle \widehat{T_\lambda f}, \hat{g}\rangle = \langle \lambda \hat{f}, \hat{g}\rangle = \langle \hat{f}, \bar{\lambda}\hat{g}\rangle = \langle \hat{f}, \widehat{T_{\bar{\lambda}} g}\rangle = \langle f, T_{\bar{\lambda}} g\rangle$. Therefore $T_\lambda{}^* = T_{\bar{\lambda}}$.

Corollary 1: X_2 is a normal algebra, i.e., $T_\lambda T_\lambda{}^* = T_\lambda{}^* T_\lambda$ for each $T_\lambda \in X_2$.

Corollary 2: $T_\lambda \in X_2$ is unitary iff $\lambda\bar{\lambda} = 1$.

Corollary 3: $T_\lambda \in X_2$ is Hermitian iff λ is real. T_λ is skew Hermitian iff λ is pure imaginary.

Corollary 4: $T_\lambda \in X_2$ is positive definite iff $\lambda \geqslant 0$ and $\lambda = 0$ only on a set of measure 0.

Corollary 5: $T_\lambda \in X_2$ is invertible iff λ is locally bounded away from 0.

Proof: If $1/\lambda$ is locally L^{∞} on Γ then $1/\lambda \in \hat{X}_2$ and $T_\lambda T_{1/\gamma} = I$. $1/\lambda$ is locally L^{∞} on Γ iff λ is locally bounded away from 0 a.e.

Remark: Note that $T_\lambda \in X_1$ is invertible iff λ is never zero, while $T_\lambda \in X_2$ is invertible iff λ is locally bounded away from zero (except on sets of measure 0). Since $X_1 \subseteq X_2$, this seems to conflict, but recall that every function in X_1 is continuous, so λ never zero implies λ locally bounded away from 0.

V. The Compact Abelian Case

Throughout this section we will take G to be a compact Abelian group. Haar measure m on G will be normalized so that $m(G) = 1$. Then the dual measure $\mathfrak{m}$ on Γ is counting measure, i.e., $\mathfrak{m}(\{0\}) = 1$. Γ is discrete, so $C = C_p = C_q = C_2 = \{\text{trigonometric polynomials}\}$ for $1 \leq p \leq q \leq 2$. Also by Lemma 5.1 below we have the Fourier transform available for all the spaces $L^p(G)$, $1 \leq p \leq \infty$, and we can study all generalized (p, q) multipliers for $1 \leq p, q \leq \infty$.

Lemma 5.1: Let $1 \leq p \leq q \leq \infty$. Then $L^q(G) \subseteq L^p(G)$ and if $f \in L^q(G)$, $||f||_p \leq ||f||_q$.

Proof: This is a fact contained in any good real variables book.

Lemma 5.2: Let $f \in C$. Then

$$f(x) = \sum_{\gamma \in \Gamma} \hat{f}(\gamma)(x, \gamma) \quad .$$

Proof: Γ is discrete, so compact subsets are finite subsets. Therefore the sum above has only a finite number of nonzero terms and is absolutely convergent. If γ_1 is a character on G, then $||\gamma_1||_1 = 1$ since $m(G) = 1$, so the sum above is in $L^1(G)$. Finally

$$\hat{\gamma}_1(\gamma) = \int_G (x,\gamma_1)(-x,\gamma)dx = \int_G (x,\gamma_1-\gamma)dx$$

$$= \begin{cases} 1 & \text{if } \gamma = \gamma_1 \\ 0 & \text{otherwise} \end{cases} .$$

So both f and

$$\sum_{\gamma\in\Gamma} \hat{f}(\gamma)(\cdot,\gamma)$$

have the same Fourier transforms, and equality holds.

Remark: Note that we have the Fourier transform available for $L^p(G)$ for $1 \leq p \leq \infty$. Our definitions of $\hat{X}_{p,q}$, $J_\lambda^{p,q}$, and $X_{p,q}$ now make sense for $1 \leq p, q \leq \infty$, and from now on we assume this extension.

Theorem 5.1: $\hat{X}_{p,q} = \Gamma^{\mathbb{C}} = \{$all complex valued functions defined on $\Gamma\}$. Denote $\hat{X}_{p,q}$ by $\hat{X}$ for every p,q, $1 \leq p,q \leq \infty$.

Proof: Let λ be a complex valued function defined on Γ, and $f \in C$. Let

$$g(x) = \sum_{\gamma\in\Gamma} \lambda(\gamma)\hat{f}(\gamma)(x,\gamma) .$$

Then $g \in C$ and $\hat{g} = \lambda\hat{f}$, so $J_\lambda^{p,q} \supseteq C$ and $\lambda \in \hat{X}_{p,q}$.

Theorem 5.2: Let $1 \leq p,q \leq \infty$, and $T_\lambda \in X_{p,q}$. Then T_λ is linear, (p,q) closed, and commutes with convolution by elements of M(G).

Proof: If $f, g \in J_\lambda^{p,q}$, α, β are complex, and $\mu \in M(G)$, then $\lambda(\alpha\hat{f} + \beta\hat{g}) = \alpha\lambda\hat{f} + \beta\lambda\hat{g}$ and $\lambda\hat{f}\hat{\mu} = \lambda\widehat{(f*\mu)}$, so $T_\lambda(\alpha f + \beta g) = \alpha T_\lambda f + \beta T_\lambda g$ and $(T_\lambda f) * \mu = T_\lambda(f * \mu)$.

Suppose $\{f_n\}_{n=1}^{\infty} \subseteq J_\lambda^{p,q}$, $f \in L^p(G)$, $g \in L^q(G)$, and $f_n \xrightarrow{p} f$, $T_\lambda f_n \xrightarrow{q} g$. Let $h \in C$. Then

$$\begin{aligned}
&||T_\lambda(f*h) - g*h||_q \\
&\qquad \leq ||T_\lambda(f*h) - T_\lambda(f_n*h)||_q + ||T_\lambda f_n * h - g*h||_q \\
&\qquad \leq ||T_\lambda h||_q\, ||f_n - f||_1 + ||h||_1\, ||T_\lambda f_n - g||_q \\
&\qquad \leq ||T_\lambda h||_q\, ||f_n - f||_p + ||h||_1\, ||T_\lambda f_n - g||_q \to 0
\end{aligned}$$

as $n \to \infty$. So $T_\lambda(f*h) = g*h$ or $\lambda\hat{f}\hat{h} = \hat{g}\hat{h}$ for each $h \in C$. Therefore $\lambda\hat{f} = \hat{g}$, so $f \in J_\lambda^{p,q}$, $T_\lambda f = g$, and T_λ is (p,q) closed.

Theorem 5.3: Let $1 \leq p, q < \infty$, and T a (p,q) closed, linear operator with domain $D(T)$ a translation invariant subspace of $L^p(G)$, $D(T) \supseteq C$, range $R(T) \subseteq L^q(G)$, such that T commutes with all translations. Then $T \in X_{p,q}$.

Proof: Let $\gamma \in \Gamma$. Define $\lambda(\gamma) = T(\cdot, \gamma)(\gamma)$. Then $T_\lambda \in X_{p,q}$, and on C $T = T_\lambda$. Let $f \in D(T)$, and $\{f_n\} \subseteq C$ such that $f_n \xrightarrow{p} f$. Then for $h \in C$, $f_n * h \xrightarrow{p} f*h$, and $Tf_n * h \xrightarrow{q} Tf*h$, since $||Tf_n*h - Tf*h||_q \leq ||Th||_q\, ||f_n - f||_p$. So $T_\lambda f_n * h \xrightarrow{q} Tf*g$, and since T_λ is (p,q) closed, $f*h \in J_\lambda^{p,q}$ and $T_\lambda(f*h) = Tf*h$. Taking Fourier transforms we get $\lambda\hat{f}\hat{h} = \widehat{Tf}\hat{h}$ for each $h \in C$, or $\lambda\hat{f} = \widehat{Tf}$. So $f \in J_\lambda^{p,q}$ and $T_\lambda f = Tf$. Therefore $D(T) \subseteq J_\lambda^{p,q}$ and $T = T_\lambda$ on $D(T)$.

Now suppose $f \in J_\lambda^{p,q}$. Using Lemmas 1.1 and 1.6 we can find a sequence $\{h_n\} \subseteq C$ such that $f*h_n \xrightarrow{p} f$ and $T_\lambda f*h_n \xrightarrow{q} T_\lambda f$. But $T(f*h_n) = T_\lambda(f*h_n) = T_\lambda f*h_n$, so $f \in D(T)$ and $Tf = T_\lambda f$. Therefore $J_\lambda^{p,q} = D(T)$ and $T = T_\lambda$.

<u>Theorem 5.4</u>: Let $1 \leq p,q < \infty$, and $T_\lambda \in X_{p,q}$. Then T_λ is (p, q) bounded iff $||F_{\lambda,f}||_q \leq M||f||_p$ for each $f \in C$ and some $M \geq 0$.

Proof: Note that

$$F_{\lambda,f}(x) = \sum_{\gamma\in\Gamma} \lambda(\gamma)\hat{f}(\gamma)(x,\gamma) = T_\lambda f(x) \quad .$$

If $T_\lambda \in \mathrm{Bdd}(p,q)$, then $||F_{\lambda,f}||_q \leq ||T_\lambda||_{p,q}\,||f||_q$ for each $f \in C$. If T_λ is (p, q) bounded on the dense subspace C, then T_λ is (p, q) bounded since it is (p, q) closed.

<u>Theorem 5.5</u>: Let $\lambda \in \hat{X}$.

(1) If $\lambda \notin L^\infty(\Gamma)$ then $T_\lambda \notin \mathrm{Bdd}(p,q)$ for any p, q, $1 \leq p,q \leq \infty$.

(2) If $\lambda \in \widehat{M(G)}$ then $T_\lambda \in \mathrm{Bdd}(p,q)$ for $1 \leq q \leq p \leq \infty$.

(3) If $\lambda \in L^r(\Gamma)$ for some r, $1 \leq r \leq 2$, then $T_\lambda \in \mathrm{Bdd}(p,q)$ for $r \leq p \leq \infty$ and $1 \leq q \leq \infty$.

(4) (Larsen) If $\lambda \in L^r(\Gamma)$ for some r, $2 < r < \infty$ then $T_\lambda \in \mathrm{Bdd}(p,q)$ for $1 \leq q \leq \frac{2r}{r-2} \leq p \leq \infty$.

(5) If $\lambda \in L^\infty(\Gamma)$ then $T_\lambda \in \mathrm{Bdd}(p,q)$ for $1 \leq q \leq 2 \leq p \leq \infty$.

Proof: (1) If $\lambda \notin L^\infty(X)$, then for each n there is a $\gamma_n \in \Gamma$ such that $|\lambda(\gamma_n)| > n$. Let $f_n = (\cdot,\gamma_n)$ (i.e., $f_n(x) = (x,\gamma_n)$). Then $f_n \in C$, and

$||T_\lambda f_n||_q = ||\lambda(\gamma_n)(\cdot,\gamma_n)||_q = |\lambda(\gamma_n)| > n$, but $||f_n||_p = 1$ for every p, $1 \leq p \leq \infty$. So $T_\lambda \notin \mathrm{Bdd}(p,q)$.

(2) If $\lambda \in \widehat{M(G)}$, then $\lambda = \hat{\mu}$ for some $\mu \in M(G)$ and $T_\lambda f = f*\mu$ for each $f \in L^p(G)$. Then from Lemma 5.1 we get for $q \leq p$ that $||T_\lambda f||_q = ||f*\mu||_q \leq ||f||_q\, ||\mu|| \leq ||f||_p\, ||\mu||$, so $T_\lambda \in \mathrm{Bdd}(p,q)$ and $||T_\lambda||_{p,q} \leq ||\mu||$.

(3) Let $f \in C$. Then for each $x \in G$,

$$\begin{aligned}
|T_\lambda f(x)| &= |F_{\lambda,f}(x)| = \Big|\sum_{\gamma\in\Gamma} \lambda(\gamma)\hat{f}(\gamma)(x,\gamma)\Big| \\
&\leq \sum_{\gamma\in\Gamma} |\lambda(\gamma)\hat{f}(\gamma)| \leq \left(\sum_{\gamma\in\Gamma} |\lambda(\gamma)|^r\right)^{1/r} ||\hat{f}||_{r'} \\
&\leq ||\lambda||_r\, ||f||_r \quad ,
\end{aligned}$$

using Hölders inequality and the Hansdorff-Young theorem. Therefore $T_\lambda f \in L^\infty(G)$ and $||T_\lambda f||_\infty \leq ||\lambda||_r\, ||f||_r$, and if $1 \leq q < \infty$, and $r \leq p < \infty$, $||T_\lambda f||_q \leq ||T_\lambda f||_\infty \leq ||\lambda||_r\, ||f||_r \leq ||\lambda||_r\, ||f||_p$. Therefore by Theorem 5.4, $T_\lambda \in \mathrm{Bdd}(p,q)$ for $1 \leq q < \infty$ and $r \leq p \leq \infty$. Now for any $g \in L^\infty(G)$, $||g||_\infty = \lim ||g||_p$.
So in particular $||T_\lambda g||_q \leq ||\lambda||_r\, ||g||_r$ for every q, $1 \leq q < \infty$, and we have $||T_\lambda g||_\infty \leq ||\lambda||_r\, ||g||_r$ for each $g \in L^\infty(G)$. Therefore $T_\lambda \in \mathrm{Bdd}(p,\infty)$ for $r \leq p \leq \infty$.

Finally, if $h \in L^p(G)$, for $r \leq p \leq \infty$, and $1 \leq q \leq \infty$, $||T_\lambda h||_q \leq ||T_\lambda h||_\infty \leq ||\lambda||_r\, ||h||_r \leq ||\lambda||_r\, ||h||_p$. So $T_\lambda \in \mathrm{Bdd}(p,q)$ and $||T_\lambda||_{p,q} \leq ||\lambda||_r$.

(4) (Proof due to Larsen). Let $t = \frac{2r}{r-2}$. Then $t > 2$, and $L^t(G) \subseteq L^2(G)$. Let $f \in L^t(G)$. Then $\hat{f} \in L^2(\Gamma)$. Let $s = \frac{2r}{r+2}$ and $\alpha = r/2 + 1$. Then $1 < s < 2$, $1 < \alpha < \infty$, and

$$\sum_{\gamma \in \Gamma} |\lambda(\gamma)\hat{f}(\gamma)|^s \leqslant \left(\sum_{\gamma\in\Gamma} |\lambda(\gamma)|^{s\alpha}\right)^{1/\alpha} \left(\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)|^{s\alpha'}\right)^{1/\alpha'}$$

$$= \left(\sum_{\gamma\in\Gamma} |\lambda(\gamma)|^r\right)^{1/\alpha} \left(\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)|^2\right)^{1/\alpha'} < \infty .$$

Therefore $\lambda\hat{f} \in L^S(\Gamma)$, and by the Hansdorff-Young theorem there is $g \in L^{s'}(G)$ such that $\hat{g} = \lambda\hat{f}$. But $s' = \frac{2r}{r-2} = t$, and $\lambda\hat{f} \in L^t(G)$, or $T_\lambda f \in L^t(G)$. Since T_λ is (t,t) closed and $J_\lambda^t = L^t(G)$, $T_\lambda \in \mathrm{Bdd}(t,t)$ $= \mathrm{Bdd}(\frac{2r}{r-2}, \frac{2r}{r-2})$. Then by arguments similar to those in part (3) of this proof, $T_\lambda \in \mathrm{Bdd}(p,q)$ for $1 \leqslant q \leqslant \frac{2r}{r-2} \leqslant p \leqslant \infty$.

(5) By Theorem 4.3 $T_\lambda \in \mathrm{Bdd}(2,2)$. So if $1 \leqslant q \leqslant 2 \leqslant p \leqslant \infty$, $T_\lambda \in \mathrm{Bdd}(p,q)$ by arguments similar to those above.

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