# Topics In Descriptive Set Theory Related To Number Theory and Analysis

Thesis by

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# Contents

# Abstract

Based on the point of view of descriptive set theory, we have investigated several definable sets from number theory and analysis.

In Chapter 1 we solve two problems due to Kechris about sets arising in number theory, provide an example of a somewhat natural $\mathcal{D}_2(\mathbf{\Pi}^0_3)$ set, and exhibit an exact relationship between the Borel class of a nonempty subset $X$ of the unit interval and the class of subsets of $\mathbb{N}$ whose densities lie in $X$.

In Chapter 2 we study the $A$, $S$, $T$ and $U$-sets from Mahler's classification of complex numbers. We are able to prove that $U$ and $T$ are $\mathbf{\Sigma}^0_3$-complete and $\mathbf{\Pi}^0_3$-complete respectively. In particular, $U$ provides a rare example of a natural $\mathbf{\Sigma}^0_3$-complete set.

In Chapter 3 we solve a question due to Kechris about $UCF$, the set of all continuous functions, on the unit circle, with Fourier series uniformly convergent. We further show that any $\mathbf{\Sigma}^0_3$ set, which contains $UCF$, must contain a continuous function with Fourier series divergent.

In Chapter 4 we use techniques from number theory and the theory of Borel equivalence relations to provide a class of complete $\mathbf{\Pi}^1_1$ sets.

Finally, in Chapter 5, we solve a problem due to Ajtai and Kechris. For each differentiable function $f$ on the unit circle, the Kechris-Woodin rank measures the failure of continuity of the derivative function $f'$, while the Zalcwasser rank measures how close the Fourier series of $f$ is to being a uniformly convergent series. We show that the Kechris-Woodin rank is finer than the Zalcwasser rank.

# Chapter 0

# Introduction

The purpose of this chapter is to provide an introduction to the results proved in the rest of the thesis.

One of the most interesting properties of a Borel set is its exact level in the Borel hierarchy. We can attempt to compute an upper bound and lower bound for the set. The upper bound is usually easier to find. It involves producing a calculation that witnesses a given level. On the other hand, since Borel complexities are preserved under continuous preimages, the notion of a continuous reduction yields a powerful technique for producing lower bounds.

A. Kechris asked whether the set of real numbers that are normal in base two is $\mathbf{\Pi}^0_3$-complete. In Chapter 1 we further study the relationship between the Borel class of $X \subset [0,1]$, and that of $D_X \subset 2^{\mathbb{N}}$, the collection of subsets of $\mathbb{N}$ whose densities lie in $X$. Given the exact location of $X$ in the Borel or difference hierarchy, we exhibit the exact location of $D_X$. For $\alpha \geq 3$, $X$ is properly $\mathcal{D}_\xi(\mathbf{\Pi}^0_\alpha)$ iff $D_X$ is properly $\mathcal{D}_\xi(\mathbf{\Pi}^0_{1+\alpha})$. We also show that for every non–empty set $X \subset [0,1]$, $D_X$ is $\mathbf{\Pi}^0_3$–hard. For each non–empty $\mathbf{\Pi}^0_2$ set $X \subset [0,1]$, in particular for $X = \{x\}$, $D_X$ is $\mathbf{\Pi}^0_3$–complete. For each $n \geq 2$, the collection of real numbers $x$ that are normal or simply normal to base $n$ is $\mathbf{\Pi}^0_3$–complete. And $D_{\mathbb{Q}}$, the subsets of $\mathbb{N}$ with rational densities, is $\mathcal{D}_2(\mathbf{\Pi}^0_3)$–complete. In particular $D_{\mathbb{Q}}$ provides one of few natural examples of a Borel set above the third level of the Borel hierarchy. Note that this chapter is a joint work with T. Linton. See [KL].

Mahler [Mah] divided complex numbers into classes $A$, $S$, $T$ and $U$ according to their properties of approximation by algebraic numbers. $A$ consists of algebraic numbers, while $S$, $T$, and $U$ provide a canonical partition of the transcendental numbers according to speed at which they are approached by a sequence of algebraic numbers. We calculate the possible locations of these sets in the Borel hierarchy. $A$ turns out to be $\mathbf{\Sigma_2^0}$–complete, while $U$ provides a rare example of a natural $\mathbf{\Sigma_3^0}$–complete set. We produce an upperbound of $\mathbf{\Sigma_4^0}$ for $S$ and show that $T$ is $\mathbf{\Pi_4^0}$ but not $\mathbf{\Sigma_3^0}$. Our main result is based on a deep theorem of Schmidt [Sc2] which guarantees the existence of $T$ numbers. These results are given in Chapter 2. See [Ki1].

We denote by $UCF$ the set of all continuous functions, on the unit circle, with uniformly convergent Fourier series. In Chapter 3 we answer a question from [Ke1]. $UCF$ is shown to be $\mathbf{\Pi_3^0}$-complete. This suggests an interesting question related to $UCF$. Namely, is it true that any $\mathbf{\Sigma_3^0}$ set, which includes $UCF$, has a continuous function with Fourier series divergent? We show that it is true in the course of the proof that $UCF$ is $\mathbf{\Pi_3^0}$-complete.

Let $X$ be a Polish space. A subset $A$ of $X$ is called a $\mathbf{\Pi_1^1}$ set if there exists a Borel function $f$ from the Cantor space to $X$ such that $X - A$ is the image of $f$. Thus a $\mathbf{\Pi_1^1}$ set is coanalytic. We say that a subset $A$ of $X$ is $\mathbf{\Pi_1^1}$-hard if for any $\mathbf{\Pi_1^1}$ subset $B$ of the Cantor space there exists a Borel function $f$ from the Cantor space to $X$ such that $B = f^{-1}(A)$. If, in addition, $A$ is $\mathbf{\Pi_1^1}$, then $A$ is $\mathbf{\Pi_1^1}$-complete. In Chapter 4 we provide new examples of complete $\mathbf{\Pi_1^1}$ sets from number theory and Borel equivalence relations. A set of real numbers $M$ is called a normal set if there exists a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ of reals such that for all $y \in \mathbb{R}$, $y \in M$ if and only if $\langle yx_n \rangle_{n\in\mathbb{N}}$ is uniformly distributed mod 1. A sequence of real numbers $\langle x_n \rangle_{n\in\mathbb{N}}$ is called a universal sequence if for all nonzero reals $y$, $\langle yx_n \rangle_{n\in\mathbb{N}}$ is uniformly distributed mod 1. We see that $US$ is $\mathbf{\Pi_1^1}$. A. Kechris suggested that we calculate the exact complexity of $US$, the set of universal sequences of reals. We show that $US$ is $\mathbf{\Pi_1^1}$-complete. Our result of $U$ is based on a theorem of Rauzy [Ra]. Let $E$ be a countable Borel equivalence relation on the Cantor space. We denote by $\mathcal{A}(E)$ ($\mathcal{F}(E)$) the set of all closed sets $K$ such that $E \cap (K \times K)$ is aperiodic (finite), i.e., for all $x \in K$, the equivalence class of $x$ is infinite (finite) in $K$. In many cases, we also show that $\mathcal{A}(E)$ and $\mathcal{F}(F)$ are $\mathbf{\Pi_1^1}$-complete.

In Chapter 5 we consider $\mathbf{\Pi_1^1}$ norms. A norm on a set $P$ is any function $\varphi$

taking $P$ into the ordinals. We only consider a regular norm $\varphi$, i.e., $\varphi$ maps $P$ onto some ordinal $\lambda$. Given a Polish space $X$ and a $\mathbf{\Pi}_1^1$ subset $P$ of $X$, we say that a norm $\varphi: P \to$ Ordinals is a $\mathbf{\Pi}_1^1$-norm if there are $\mathbf{\Pi}_1^1$ subsets $R$ and $Q$ of $X \times X$ such that

$$y \in P \Rightarrow [x \in P \;\&\; \varphi(x) \leq \varphi(y) \iff (x,y) \notin R \iff (x,y) \in Q].$$

From the previous relation, we see that in a uniform manner for $y \in P$, the set $\{x \in P : \varphi(x) \leq \varphi(y)\}$ is $\mathbf{\Pi}_1^1$ ( $(x,y) \in Q$) and the complement of a $\mathbf{\Pi}_1^1$ set ( $(x,y) \notin R$), hence a Borel set. In [Mo] it is shown that every $\mathbf{\Pi}_1^1$-norm is equivalent to one which takes values in $\omega_1$, the first uncountable ordinal. One of the basic facts is that every $\mathbf{\Pi}_1^1$ subset $P$ admits a $\mathbf{\Pi}_1^1$-norm $\varphi: P \to \omega_1$ (See [Mo].) Hence it is very natural to look for a canonical norm on $\mathbf{\Pi}_1^1$ sets that arise in analysis and topology.

Zalcwasser [Za] and Gillespie-Hurwitz [GH] introduced a rank that measures the uniform convergence of sequences of continuous functions on the unit interval. We call it the Zalcwasser rank and apply the Zalcwasser rank to the Fourier series of a continuous function on the unit circle. The Zalcwasser rank is a $\mathbf{\Pi}_1^1$ norm on the set of all continuous functions with convergent Fourier series. Kechris and Woodin [KeW] defined a rank that measures the uniform continuity of the derivative of a differentiable function. We shall refer to this rank as the Kechris-Woodin rank. In fact, they have shown that on the set of all differentiable functions, the Kechris-Woodin rank is a $\mathbf{\Pi}_1^1$-norm. Ajtai and Kechris [AK] conjectured that the Kechris-Woodin rank is finer than the Zalcwasser rank, meaning that for any function $f$, the Zalcwasser rank is less than or equal to the Kechris–Woodin rank. In the last chapter, we provide an affirmative answer to this conjecture.

# Chapter 1

# Normal numbers and subsets of $\mathbb{N}$ with given densities

## 1.1 Introduction

The collection of "naturally arising" or non "ad hoc" sets that are properly located in the Borel hierarchy (meaning for example $\mathbf{\Pi^0_2}$ non $\mathbf{\Sigma^0_2}$), is relatively small. In fact, only a small number of specific examples of any sort are known to be properly located above the third level of the Borel hierarchy. Recently, Kechris asked whether the set of real numbers that are normal in base two is $\mathbf{\Pi^0_3}$–complete. Ditzen then conjectured that if this were true for each base $n \geq 2$ then the set of real numbers that are normal to at least one base $n \geq 2$, should be $\mathbf{\Sigma^0_4}$–complete. Certainly this example is non ad–hoc. We found this set extremely difficult to manage, and hence we are inclined to agree with Ditzen's conjecture. There is some evidence supporting this conjecture; namely, results as in [Sc1] which suggest that normality base two and normality base three have a weak form of independence. Unfortunately, such proofs are non–constructive and the conjecture appears to be more number theoretic than set theoretic. It seemed reasonable to replace the set in the conjecture with the easier to manage collection of subsets of $\mathbb{N}$ with density $1/n$, for some (varying) $n \in \mathbb{N}$. However, in this case, the limit one computes is the same for all $n$, and the set is too simple. We then looked at the subsets of $\mathbb{N}$ with rational densities, $D_{\mathbb{Q}}$, and were able to show it was properly the difference of two $\mathbf{\Pi^0_3}$ sets, i.e., $\mathcal{D}_2(\mathbf{\Pi^0_3})$–complete. As $D_{\mathbb{Q}}$ is at least somewhat natural, this is rather surprising, since it lies above the third level of the Borel hierarchy. In continuing the study of the relationship between the Borel class of $X \subseteq [0, 1]$ and that of $D_X \subseteq 2^{\mathbb{N}}$, the collection of subsets of $\mathbb{N}$ whose densities lie in $X$, we were able to show that if $X$ is properly $\mathbf{\Pi^0_n}$ ($\mathbf{\Sigma^0_n}$), then $D_X$ is properly $\mathbf{\Pi^0_{n+1}}$ ($\mathbf{\Sigma^0_{n+1}}$) for $n \geq 3$. Furthermore, the relationship extended to the difference hierarchy of $\mathbf{\Delta^0_{n+1}}$ sets. If $X$ is properly $\mathcal{D}_\xi(\mathbf{\Pi^0_n})$, then $D_X$ is properly $\mathcal{D}_\xi(\mathbf{\Pi^0_{n+1}})$, so long as

$n \geq 2$. However, on the dual side, at the finite levels of the difference hierarchy for $n = 2$, an interesting phenomenon arises. For $m < \omega$, if $X$ is properly $\widetilde{\mathcal{D}}_m(\mathbf{\Pi}^0_2)$, then $D_X$ is properly $\mathcal{D}_{m+1}(\mathbf{\Pi}^0_3)$. So the analogy of $\mathbb{Q}$ to $D_{\mathbb{Q}}$ extends to all finite levels of the difference hierarchy, and no $D_X$ can be properly $\widetilde{\mathcal{D}}_m(\mathbf{\Pi}^0_3)$. If $\xi \geq \omega$ and $X$ is properly $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_2)$, then $D_X$ is properly $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_3)$. For $\alpha \geq 3$, if $\Gamma = \mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $\mathcal{D}_\xi(\mathbf{\Pi}^0_\alpha)$, or $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_\alpha)$, and $\Gamma^*$ is the class where the $\alpha$ in $\Gamma$ is replaced by $1+\alpha$, then $X$ is properly $\Gamma$ iff $D_X$ is properly $\Gamma^*$. In particular we are able to show that for every non–empty set $X \subseteq [0,1]$, $D_X$ is $\mathbf{\Pi}^0_3$–hard; for each nonempty $\mathbf{\Pi}^0_2$ set $X \subseteq [0,1]$, $D_X$ is $\mathbf{\Pi}^0_3$–complete; for each $n \geq 2$, the collection of real numbers $x$ that are normal or simply normal to base $n$ is $\mathbf{\Pi}^0_3$–complete; and as mentioned above, $D_{\mathbb{Q}}$, or $D_X$ for any $\mathbf{\Sigma}^0_2$–complete set $X$, is $\mathcal{D}_2(\mathbf{\Pi}^0_3)$–complete.

## 1.2 Notation and background information

For sets $A$ and $B$, $|A|$ is the cardinality of $A$, $\overline{A}$ denotes the topological closure of $A$, and we denote the set of all functions from $B$ into $A$ by $A^B$. If $X \subseteq A$, we denote the preimage under $f$ of $X$ by $f^{\leftarrow}(X)$. We sometimes identify $n \in \mathbb{N} = \{0,1,2,\dots\}$ with the set $\{0,1,\dots,n-1\}$. Thus $2^{\mathbb{N}}$ is the collection of functions $f\colon \mathbb{N} \to \{0,1\}$. We let $A^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} A^n$, denote all finite sequences from $A$, and $A^{\leq\mathbb{N}} = A^{<\mathbb{N}} \cup A^{\mathbb{N}}$. $\vec{0}$ and $\vec{1}$ denote the constant zero and constant one functions in $2^{\mathbb{N}}$. If $f \in A^{\leq\mathbb{N}}$ and $n \in \mathbb{N}$, $f{\restriction}_n = \langle f(0),\dots,f(n-1)\rangle$, and for $s,\ t \in A^{<\mathbb{N}}$, $|s|$ denotes the length of $s$ (the unique $n$ for which $s \in A^n$), $s \subseteq t$ ($t$ extends $s$) means $t{\restriction}_{|s|} = s$, and $s^\frown t$ is the sequence $s$ followed by the sequence $t$. We use $\mathbb{R}$ and $\mathbb{Q}$ to denote the reals and rationals, and $\mathbb{P}$ denotes the irrationals between zero and one.

We describe the Borel hierarchy using the standard modern terminology of Addison, and define the difference hierarchy, on the ambiguous classes of $\mathbf{\Delta}^0_{\alpha+1}$ sets, based on decreasing sequences of $\mathbf{\Pi}^0_\alpha$ sets. For Polish topological spaces $X$, let $\mathbf{\Sigma}^0_1(X)$ denote the collection of open subsets of $X$, and $\mathbf{\Pi}^0_1(X)$ denote the closed subsets of $X$. Inductively define for countable ordinals $\alpha \geq 2$,

$$\mathbf{\Sigma}^0_\alpha(X) = \{ A \subseteq X \mid A = \bigcup_{n\in\mathbb{N}} A_n, \text{ where each } A_n \in \mathbf{\Pi}^0_{\beta_n}(X) \text{ and } \beta_n < \alpha \}.$$

$$\mathbf{\Pi}^0_\alpha(X) = \{ A \subseteq X \mid A = \bigcap_{n\in\mathbb{N}} A_n, \text{ where each } A_n \in \mathbf{\Sigma}^0_{\beta_n}(X) \text{ and } \beta_n < \alpha \}.$$

$$\mathbf{\Delta}^0_\alpha(X) = \{ A \subseteq X \mid A \in \mathbf{\Pi}^0_\alpha(X) \cap \mathbf{\Sigma}^0_\alpha(X) \}.$$

If $X$ is known by context or irrelevant, we frequently drop it for notational convenience. Thus, $\mathbf{\Sigma}^0_1 = \text{Open}$, $\mathbf{\Pi}^0_1 = \text{Closed}$, $\mathbf{\Sigma}^0_2 = F_\sigma$, $\mathbf{\Pi}^0_2 = G_\delta$, and so on. The difference hierarchy, which is a finer two sided hierarchy on the $\mathbf{\Delta}^0_\alpha$ sets, extends the Borel hierarchy by including it as the first level ($\xi = 1$) for each countable ordinal $\alpha$. For $\xi$ a countable ordinal and any sequence of subsets of $X$, $\langle A_\beta \rangle_{\beta<\xi}$, where $A_\beta \supseteq A_{\beta'}$ if $\beta < \beta'$ (so $\langle A_\beta \rangle$ is decreasing) and for limit $\lambda < \xi$, $A_\lambda = \bigcap_{\beta<\lambda} A_\beta$ (so the sequence is continuous), define a set $A = \mathcal{D}_\xi(\langle A_\beta \rangle_{\beta<\xi})$, by

$$x \in A \Leftrightarrow \exists \beta < \xi \ \big(x \in A_\beta\big), \text{ and the largest such } \beta \text{ is even.}$$

A countable ordinal $\beta$ is even, if when we write $\beta = \lambda + n$, with $\lambda = 0$ or a limit ordinal, $n$ is even. Let $\mathcal{D}_\xi(\mathbf{\Pi}^0_\alpha)$ be the collection of sets of the form $\mathcal{D}_\xi(\langle A_\beta \rangle_{\beta<\xi})$, where $\langle A_\beta \rangle_{\beta<\xi}$ is a decreasing, continuous sequence of $\mathbf{\Pi}^0_\alpha$ sets (for $\xi < \omega$, the decreasing requirement is redundant). So $\mathcal{D}_1(\mathbf{\Pi}^0_\alpha) = \mathbf{\Pi}^0_\alpha$, $\mathcal{D}_2(\mathbf{\Pi}^0_\alpha) = \{ A - B \mid A, B \in \mathbf{\Pi}^0_\alpha, \text{ and } A \supseteq B \}$ (so in $\mathbb{R}$, $[0,2)$ is a typical $\mathcal{D}_2(\mathbf{\Pi}^0_1)$ set), and $\mathcal{D}_3(\mathbf{\Pi}^0_\alpha)$ is the collection of sets of the form

$$(A - B) \cup C \text{ where } A, B, C \in \mathbf{\Pi}^0_\alpha \text{ and } A \supseteq B \supseteq C.$$

For any class of sets $\Gamma$, let the dual class, $\widetilde{\Gamma}$, be the collection of complements of sets in $\Gamma$ (so $\widetilde{\mathcal{D}}_1(\mathbf{\Pi}^0_\alpha) = \mathbf{\Sigma}^0_\alpha$), and say $A$ is properly $\Gamma$, if $A \in \Gamma - \widetilde{\Gamma}$. We need the following elementary facts about the difference hierarchy classes.

The $\mathcal{D}_\xi(\mathbf{\Pi}^0_\alpha)$ sets are closed under:
(*i*) intersections with $\mathbf{\Pi}^0_\alpha$ sets;
(*ii*) intersections with $\mathbf{\Sigma}^0_\alpha$ sets, if $\xi$ is even;
(*iii*) unions with $\mathbf{\Pi}^0_\alpha$ sets, if $\xi$ is odd;
(*iv*) unions with $\mathbf{\Sigma}^0_\alpha$ sets, if $\xi \geq \omega$.

Each of these implies a dual property for the $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_\alpha)$ sets. For example, (*i*) says that the $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_\alpha)$ sets are closed under unions with $\mathbf{\Sigma}^0_\alpha$ sets. By combining the above properties with the fact that if $A$ is $\mathbf{\Pi}^0_\alpha$, and $B$ is $\mathbf{\Pi}^0_\beta$ ($\beta < \alpha$), then both $A - B$ and $A \cup B$ are $\mathbf{\Pi}^0_\alpha$, we also have (for $\alpha > \beta$, or $\alpha = \beta$ and $\xi \geq \omega$):

(*v*) the $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_\alpha)$ sets are closed under intersections with $\mathbf{\Pi}^0_\beta$ sets.

We will need this for $\beta = 3$ later.

In order to determine the exact location of a set in the above hierarchy, one must produce an upper bound, or prove membership in the class $\Gamma$, and then a lower bound, showing the set is not in $\widetilde{\Gamma}$. In general the lower bounds are more difficult, but since these classes are closed under continuous preimages, the notion of a continuous or Wadge reduction yields a powerful technique for producing lower bounds. The idea is take a set $C$ that is known to be a non $\widetilde{\Gamma}$ set, and find a continuous function $f$ such that $f^{\leftarrow}(A) = C$. Then $A$ cannot be in $\widetilde{\Gamma}$ either. The Cantor space $2^{\mathbb{N}}$ (with the usual product topology and $2 = \{0, 1\}$ discrete) is known to contain sets that are proper, in all the classes above. A subset, $A$, of a Polish topological space, $X$, is called $\Gamma$–hard (for $\Gamma = \mathcal{D}_\xi(\mathbf{\Pi}^0_\alpha)$ or $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_\alpha)$ ), if for every $C \in \Gamma(2^{\mathbb{N}})$ there is a continuous function, $f: 2^{\mathbb{N}} \to X$, such that $x \in C \Leftrightarrow f(x) \in A$, that is $f^{\leftarrow}(A) = C$. Thus if $A$ is $\Gamma$–hard, then $A \notin \widetilde{\Gamma}$. If in addition to being $\Gamma$–hard, $A$ is also in $\Gamma$, we say $A$ is $\Gamma$–complete. Wadge [Wa] (using Borel determinacy [Mar]), showed that in zero–dimensional Polish spaces, there is no difference between a set being $\Gamma$–complete or properly $\Gamma$. Let $X$ and $Y$ be Polish spaces, $C \subseteq X$, $A$ and $B$ disjoint subsets of $Y$; let $C \leq_{\mathbf{W}} (A; B)$ assert that there is a continuous function $f: X \to Y$ where

$$x \in C \Rightarrow f(x) \in A, \text{ and } x \notin C \Rightarrow f(x) \in B.$$

If $B = \neg A = Y - A$, we write $C \leq_{\mathbf{W}} A$ for $C \leq_{\mathbf{W}} (A; \neg A)$, and say $C$ is Wadge reducible to $A$. Wadge's result mentioned above was that for all Borel subsets $A$ and $B$ of zero–dimensional Polish spaces, either $A \leq_{\mathbf{W}} B$ or $\neg B \leq_{\mathbf{W}} A$. Louveau and Saint–Raymond [LS] later showed a similar result (which Wadge obtained using analytic determinacy) for $C \leq_{\mathbf{W}} (A; B)$, using closed games. It implies that for each class $\Gamma = \mathcal{D}_\xi(\mathbf{\Pi}^0_\alpha)$ or $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_\alpha)$ ($\alpha \geq 2$), there is a $\Gamma$–complete set $H_\Gamma \subseteq 2^{\mathbb{N}}$, such that for all disjoint analytic $A$ and $B$ (in any Polish space), either $H_\Gamma \leq_{\mathbf{W}} (A; B)$ (by a one–to–one continuous function), or there is a $\widetilde{\Gamma}$ set $S$ such that $A \subseteq S$ and $B \cap S = \emptyset$. For our classes (since we only work in Polish spaces), being $\Gamma$–hard, and being a non $\widetilde{\Gamma}$ set are the same thing. Hence a set will be properly $\Gamma$ iff it is $\Gamma$–complete. Notice also that if $C$ is $\Gamma$–hard, and $C \leq_{\mathbf{W}} B$, then $B$ is $\Gamma$–hard. And if $C \leq_{\mathbf{W}} (A; B)$, and $D$ is any set containing $A$ and disjoint from $B$, then $C \leq_{\mathbf{W}} D$.

## 1.3 Subsets of $\mathbb{N}$ with given densities

We describe here the basic facts and properties about the densities of subsets of the natural numbers that we need. This topic is covered in detail in [KN], for example.

**Definition 1.1** *For $A \subseteq \mathbb{N}$, let $\delta(A) = \lim_{n\to\infty} \frac{|A \cap [0,n)|}{n}$, if the limit exists, and say $\delta(A)$ does not exist, otherwise. We call $\delta(A)$ the density of $A$.*

Thus, whenever it exists, $\delta(A) \in \mathbb{R} \cap [0,1]$, and is roughly the frequency of occurrences of $A$ in $\mathbb{N}$. For nonempty $X \subseteq [0,1] \cap \mathbb{R}$, let

$$D_X = \{ A \subseteq \mathbb{N} \mid \delta(A) \in X \}$$

(if $X = \{ r \}$ we write $D_r$ for $D_{\{r\}}$). Let $DE = D_{[0,1]}$ denote the collection of subsets of $\mathbb{N}$ whose densities exist. If we identify $A \subseteq \mathbb{N}$ with its characteristic function

$$\chi_A : \mathbb{N} \to 2, \text{ given by } \chi_A(n) = \begin{cases} 1, & \text{if } n \in A; \\ 0, & \text{if } n \notin A, \end{cases}$$

then $D_X$ becomes a subset of the Cantor space $2^{\mathbb{N}}$ (with the usual product topology). For $s \in 2^{<\mathbb{N}}$, let $\| s \| = |\{ i \in Dom(s) : s(i) = 1 \}|$ and let $|s|$ denote the length of $s$. We can then define the density of $s$, as

$$\delta(s) = \frac{\| s \|}{|s|} \in \mathbb{Q} \cap [0,1] .$$

For $\alpha \in 2^{\mathbb{N}}$, the density of $\alpha$ exists iff the sequence $\{ \delta(\alpha\restriction_n) \}_{n\in\mathbb{N}}$ converges, in which case the limit of the sequence is the density of $\alpha$. This shows that $DE$ and $D_0$ are $\mathbf{\Pi}^0_3$, since

$$\alpha \in DE \Leftrightarrow \forall n \; \exists N \; \forall k \; \left( \left| \delta(\alpha\restriction_N) - \delta(\alpha\restriction_{N+k}) \right| < 1/n \right)$$

(that is the sequence of partial densities of $\alpha$ is Cauchy),

$$\Leftrightarrow \alpha \in \bigcap_{n\in\mathbb{N}} \; \bigcup_{N\in\mathbb{N}} \; \bigcap_{k\in\mathbb{N}} C(n,N,k),$$

where $C(n,N,k)$ is the collection of $\alpha \in 2^{\mathbb{N}}$ such that $|\delta(\alpha\restriction_N) - \delta(\alpha\restriction_{N+k})| < 1/n$, which is clopen (both closed and open). In the future we will not bother rewriting number quantifiers as countable intersections or countable unions, nor will we verify that the sets similar to $C(n, N, k)$ above are clopen, if it is clear that they are. $D_0$ is also $\mathbf{\Pi}^0_3$ since

$$\alpha \in D_0 \Leftrightarrow \forall n \; \exists N \; \forall k \geq N\big(\delta(\alpha\restriction_k) < 1/n\big).$$

The sequence $\{ \delta(\alpha\restriction_n) \}_{n\in\mathbb{N}}$ is very close to being a Cauchy sequence, meaning that if $n$ is large, then $\delta(\alpha\restriction_n)$ and $\delta(\alpha\restriction_{n+1})$ are very close. In fact,

$$\big|\delta(\alpha\restriction_n) - \delta(\alpha\restriction_{n+1})\big| < \frac{1}{n+1}\,. \tag{1.1}$$

This shows that if $I = \liminf_{n\to\infty} \{ \delta(\alpha\restriction_n) \}$, and $S = \limsup_{n\to\infty} \{ \delta(\alpha\restriction_n) \}$, then for every real number $r \in [I, S]$, $r$ is a limit point or cluster value of the sequence $\{ \delta(\alpha\restriction_n) \}_{n\in\mathbb{N}}$.

## 1.4 Two methods for producing subsets with nice densities

We now give two methods for producing $\alpha \in 2^{\mathbb{N}}$ so that the density of $\alpha$ is easy to compute. The first involves copying the values of a sequence $\{ x_n \}_{n\in\mathbb{N}}$ with $x_n \in (0,1)$. The idea is to define $\alpha$ as a union, $\alpha = \bigcup\limits_{n\in\mathbb{N}} \alpha_n$, where for all $n \in \mathbb{N}$, $\alpha_{n+1}$ is a finite proper extension of $\alpha_n$, $\delta(\alpha_n) \approx x_n$, and $\delta(\alpha\restriction_{k+1})$ is between $\delta(\alpha_n)$ and $\delta(\alpha_{n+1})$, whenever $k$ is between $|\,\alpha_n\,|$ and $|\,\alpha_{n+1}\,|$. Thus the density of $\alpha$ will exist iff the sequence $\{ x_n \}_{n\in\mathbb{N}}$ converges, and the limit of this sequence will be the density of $\alpha$. Given any sequence $\{ x_n \}_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ we define $\alpha$, the result of running the canonical construction with input $\{ x_n \}_{n\in\mathbb{N}}$, inductively as follows:

Let $\alpha_0 = \langle 0, 1\rangle$. Given $\alpha_n$, if $\delta(\alpha_n) < x_{n+1}$, fix the least $k \in \mathbb{N}$ such that

$$\delta(\alpha_n{}^\frown 1^k) = \frac{\|\,\alpha_n\,\| + k}{|\,\alpha_n\,| + k} \geq x_{n+1}$$

($k$ exists since $\{ \delta(\alpha_n{}^\frown 1^k) \}_{k\in\mathbb{N}}$ starts at $\delta(\alpha_n)$ and increases to 1). Set $\alpha_{n+1} = \alpha_n{}^\frown 1^k$. If $\delta(\alpha_n) \geq x_{n+1}$, fix the least $k \in \mathbb{N} - \{ 0 \}$ such that

$$\delta(\alpha_n{}^\frown 0^k) = \frac{\|\,\alpha_n\,\|}{|\,\alpha_n\,| + k} \leq x_{n+1},$$

and set $\alpha_{n+1} = \alpha_n{}^\frown 0^k$. Let $\alpha = \bigcup\limits_{n\in\mathbb{N}} \alpha_n \in 2^{\mathbb{N}}$. Clearly, $|\,\alpha_{n+1}\,| \geq |\,\alpha_n\,| + 1 > n + 1$, for all $n \in \mathbb{N}$. Using the minimality of $k$ and (1.1), we see that

$$|\,x_{n+1} - \delta(\alpha_{n+1})\,| < \frac{1}{|\,\alpha_n\,|} < 1/n.$$

Since $\alpha_{n+1}$ is $\alpha_n$ followed $k$ zeros or $k$ ones, $\delta(\alpha\lceil_{m+1})$ is between $\delta(\alpha_n)$ and $\delta(\alpha_{n+1})$, whenever $m$ is between $|\alpha_n|$ and $|\alpha_{n+1}|$. So the density of $\alpha$ exists iff the sequence of partial densities of $\alpha$ is Cauchy iff the sequence of the densities of the $\alpha_n$'s is Cauchy iff $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. More precisely, for any convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$,

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} \delta(\alpha_{n_k}).$$

This gives then a canonical way to produce $\alpha$ with $\delta(\alpha) = r$ for any $r \in [0,1]$. Notice also that if $x_n$ happened to be zero or one, we could replace $x_n$ with $1/n$ or $1-1/n$ respectively, and hence we can run this construction for sequences in $[0,1]^{\mathbb{N}}$.

The second construction involves partitioning $\mathbb{N}$ into a finite or countably infinite collection of sets with positive densities, and placing a copy of some $\alpha_n \in 2^{\mathbb{N}}$ on the $n$th set in the partition. Then even when the partition is infinite, one can basically add the densities. In general this is not true, since the union of the singletons has density one, where as each singleton has density zero. But when the pieces being combined are contained in disjoint sets with positive densities, everything works out fine. Let $I \subseteq \mathbb{N}$ and $\{A_n\}_{n\in I}$ be a family of pairwise disjoint subsets of $\mathbb{N}$ such that $\bigcup_{n\in I} A_n = \mathbb{N}$ (i.e., a partition of $\mathbb{N}$); for each $n \in I$, the density of $A_n$ exists and is positive; and $\lim_{N\to\infty} \sum_{n=0}^{N} \delta(A_n) = 1$. For each $n \in I$, let $\alpha_n \in 2^{\mathbb{N}}$ be such that $\delta(\alpha_n)$ exists. Define $C \subseteq \mathbb{N}$, the set obtained by playing a copy of $\alpha_n$ on $A_n$, as follows:

First, since $\delta(A_n) > 0$, $A_n$ is infinite. Let $\{a_k^n\}_{k\in\mathbb{N}}$ be a one–to–one increasing enumeration of $A_n$. Then for each $m \in \mathbb{N}$ there is a unique $n$ and $k$ such that $m = a_k^n$. We put $m \in C$ iff $m = a_k^n$ and $\alpha_n(k) = 1$. It is straightforward to check that

$$\delta(C) = \sum_{n\in I} \delta(A_n) \cdot \delta(\alpha_n).$$

Thus, whenever we say "let $\alpha$ be the result of playing $\alpha_n$ on $A_n$," we mean that $\alpha$ is the characteristic function of the set $C$ defined above. Of course we can still make this definition even if $\delta(\alpha_n)$ does not exist. If at least two of the $\alpha_n$'s have divergent densities, the density of $C$ may or may not exist. However, if exactly one of $\alpha_n$'s has a divergent density, then the density of $C$ does not exist.

## 1.5 Some $\mathbf{\Pi}^0_3$–complete sets

In this section we establish a strong reduction of a $\mathbf{\Pi}^0_3$–complete set, to the set $D_0$. Thus we have an affirmative answer to a question of Kechris, who asked if $D_0$ was $\mathbf{\Pi}^0_3$–complete. Once this is done, we are able to show hardness for numerous other sets, including the collections of normal and simply normal numbers. It is known that the set

$$\mathcal{C}_3 = \{\, \beta \in \mathbb{N}^\mathbb{N} \mid \forall n,\ \beta^{\leftarrow}(n) \text{ is finite} \,\} = \{\, \beta \in \mathbb{N}^\mathbb{N} \mid \liminf_{n\to\infty} \beta(n) = \infty \,\}$$

is $\mathbf{\Pi}^0_3$–complete (see for example (24) in [Mi] for a proof).

**Theorem 1.2** $\mathcal{C}_3 \leq_\mathbf{W} (D_0; \neg DE)$. *In particular, both $D_0$ and $DE$ are $\mathbf{\Pi}^0_3$–complete.*

□ The second part of the theorem follows from the first, because $D_0 \subseteq DE$ and $DE \cap \neg DE = \emptyset$. The idea is to take $\beta \in \mathbb{N}^\mathbb{N}$ and define from it a sequence $\{\, x_n \,\}_{n\in\mathbb{N}}$, so that $x_n$ depends only on a finite initial segment of $\beta$. We then produce the canonical $\alpha$ with input $\{\, x_n \,\}_{n\in\mathbb{N}}$. The function $\beta \mapsto \alpha \ : \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$, will then be continuous since the first $N$ values of $\alpha$ depend only on the first $N$ values of $\{\, x_n \,\}_{n\in\mathbb{N}}$, which depend only on a finite initial segment of $\beta$. The sequence $x_n = \dfrac{1}{\beta(n)}$ almost works, but we must first fix $\beta$ so that $\beta(n) \geq 2$, and $\beta$ is not eventually constant (so that $\lim\limits_{n\to\infty} \dfrac{1}{\beta(n)}$ exists iff it is zero). For $\beta \in \mathbb{N}^\mathbb{N}$, define $\beta' \in \mathbb{N}^\mathbb{N}$ by

$$\beta'(n) = \begin{cases} \beta(n/2) + 2, & \text{if } n \text{ is even;} \\ n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Then $\beta \mapsto \beta'$ is continuous and $\beta \in \mathcal{C}_3 \Leftrightarrow \beta' \in \mathcal{C}_3$. Given $\beta \in \mathbb{N}^\mathbb{N}$, let $\alpha \in 2^\mathbb{N}$ be the result of running the canonical construction on input $\{\, 1/\beta'(n) \,\}_{n\in\mathbb{N}}$. Then the sequence $\{\, \delta(\alpha\restriction_n) \,\}_{n\in\mathbb{N}}$ always contains a subsequence which converges to zero, since $\beta'(2n+1) = 2n+2$. Hence the density of $\alpha$ exists iff it is zero. Thus,

$$\beta \in \mathcal{C}_3 \Leftrightarrow \beta' \in \mathcal{C}_3 \Leftrightarrow \lim_{n\to\infty} \frac{1}{\beta'(n)} = 0 \Leftrightarrow \delta(\alpha) = 0 \Leftrightarrow \alpha \in D_0 \Leftrightarrow \alpha \in DE.$$

This shows $\mathcal{C}_3 \leq_\mathbf{W} (D_0; \neg DE)$ and completes the proof. □

**Corollary 1.3** *For any nonempty $X \subseteq [0,1]$, $D_X$ is $\mathbf{\Pi}^0_3$–hard. In particular for each $r \in [0,1]$, $D_r$ is $\mathbf{\Pi}^0_3$–complete.*

□ It is clear that $D_r$ is $\mathbf{\Pi}^0_3$ for each $r \in [0,1]$, so it suffices to prove the first statement. Let $f$ denote the continuous function from Theorem 1.2. If $0 \in X$, $f$ shows $\mathcal{C}_3 \leq_{\mathbf{W}} D_X$. If $1 \in X$ let $g(\beta) = \phi(f(\beta))$, where $\phi$ is the bit switching homeomorphism of $2^{\mathbb{N}}$,

$$\phi(\alpha)(n) = \begin{cases} 0, & \text{if } \alpha(n) = 1; \\ 1, & \text{if } \alpha(n) = 0. \end{cases}$$

Then $g$ shows $\mathcal{C}_3 \leq_{\mathbf{W}} D_X$. Finally if $X \subseteq (0,1)$, let $x \in X$ be arbitrary. Let $A_0 \subseteq \mathbb{N}$ have density $x$. Fix $n \in \mathbb{N}$ such that $x + 1/n < 1$. Let $A_1$ be disjoint from $A_0$ and have density $1/n$. Given $\beta \in \mathbb{N}^{\mathbb{N}}$, let $\alpha$ be the characteristic function of $C = A_0 \cup C_1$, where $C_1$ is the result of playing $f(\beta)$ on $A_1$. Then $\beta \mapsto \alpha$ is continuous, $\delta(C)$ exists iff $\delta(C_1)$ exists, and

$$\beta \in \mathcal{C}_3 \Leftrightarrow \delta(C_1) \text{ exists } \Leftrightarrow \delta(C_1) = 0 \Leftrightarrow \delta(\alpha) = x + \frac{1}{n} \cdot 0 \Leftrightarrow \alpha \in D_x.$$

Hence $\mathcal{C}_3 \leq_{\mathbf{W}} (D_x; \neg DE)$, so $\mathcal{C}_3 \leq_{\mathbf{W}} D_X$, because $x \in X$ and $D_X \cap \neg DE = \emptyset$.

□

## 1.6 Normal numbers

For $x \in [0,1]$ and $n \geq 2$, the base $n$ expansion of $x$ is the sequence $\{ d_i \}_{i \in \mathbb{N}} \in n^{\mathbb{N}}$ such that $x = \sum_{i=1}^{\infty} \frac{d_i}{n^i}$, and $d_i \neq n-1$ for infinitely many $i$. For $x \in [0,1]$ and $n \geq 2$, say $x$ is simply normal base $n$, and write $x \in SN_n$, if for each $k = 0, 1, \ldots, n-1$,

$$\delta(\{ i \in \mathbb{N} \mid d_i = k \}) = 1/n.$$

Say $x \in [0,1]$ is normal to base $n$, and write $x \in N_n$, if for each $m \in \mathbb{N}$ and each $s \in n^{m+1}$,

$$\delta(\{ i \in \mathbb{N} \mid d_i = s(0), d_{i+1} = s(1), \ldots, d_{i+m} = s(m) \}) = 1/n^{m+1}.$$

Thus, $x$ is normal to base $n$, if in the base $n$ expansion of $x$, all the digits $k < n$ appear with equal frequency, all the pairs $\langle k, j \rangle$ appear with equal frequency, etc.

It is known that the set of numbers in $[0,1]$, that are normal to all bases $n \geq 2$ simultaneously, has Lebesgue measure one (see for example 8.11 in [Ni]). It is straightforward to see that $SN_n$ and $N_n$ are $\mathbf{\Pi}^0_3$, since $D_r$ is $\mathbf{\Pi}^0_3$. One of the main questions that motivated this study was to try to show that $N_{\mathbb{N}} = \bigcup_{n \in \mathbb{N}} N_n$ was $\mathbf{\Sigma}^0_4$–complete. We were unable to answer this, but we did manage to show that each $N_n$ and $SN_n$ are $\mathbf{\Pi}^0_3$–complete. As $N_n \subseteq SN_n$ the following result shows both of these simultaneously.

**Theorem 1.4** *For each $n \in \mathbb{N} - \{0, 1\}$, $D_0 \leq_{\mathbf{W}} (N_n; \neg SN_n)$. In particular, both $SN_n$ and $N_n$ are $\mathbf{\Pi}^0_3$–complete.*

□ Let $x = \sum_{i=1}^{\infty} \frac{d_i}{n^i}$ be any fixed number that is normal to base $n$. Let $\{ i_k \}_{k \in \mathbb{N}}$ be an increasing enumeration of the set $I_0 = \{ i \in \mathbb{N} \mid d_i = 0 \}$. Then $I_0$ has density $1/n$ since $x \in N_n$. Given $\alpha \in 2^{\mathbb{N}}$ let $x' \in [0,1]$ be given by the base $n$ expansion,

$$d'_i = \begin{cases} 1, & \text{if } i = i_k \text{ and } \alpha(k) = 1; \\ d_i, & \text{otherwise.} \end{cases}$$

That is $x' = x + \sum_{k \in \alpha^{-}(1)} \frac{1}{n^{i_k}}$. The function $\alpha \mapsto x'$ is continuous. If $\alpha \in D_0$, then $x'$ is the result of changing a subset of density zero of the 0's in the base $n$ expansion of $x$ to ones, leaving the rest of the base $n$ expansion of $x$ unchanged. Hence, $x'$ is still normal base $n$. And if $\alpha \notin D_0$, then $x' \notin SN_n$, since 0 and 1 no longer occur with density $1/n$ in the base $n$ expansion of $x'$. □

## 1.7 The Borel classes of $D_X$

We now turn to the problem of classifying the Borel class of $D_X$ in terms of the class of $X$. The fact that such an exact relationship exists is surprising. Basically, $D_X$ has one more quantifier and lies on the same side of the hierarchy as $X$. We start with the upper bounds.

**Proposition 1.5** *For nonempty $X \subseteq [0,1]$, if $X$ is $\mathbf{\Pi}^0_2$, then $D_X \subseteq 2^{\mathbb{N}}$ is $\mathbf{\Pi}^0_3$–complete.*

□ Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable basis of open sets for $\mathbb{R} \cap [0,1]$ (with the usual topology). Let $X = \bigcap_{k\in\mathbb{N}} G_k$ be any nonempty $\mathbf{\Pi^0_2}$ subset of $[0,1]$, where each $G_k$ is open. A moments reflection shows that

$$\alpha \in D_X \Leftrightarrow \alpha \in DE \text{ and } \forall k\ \exists n\ \exists m\ \forall p \geq m\ \big(\delta(\alpha\restriction_p) \in U_n \subseteq \overline{U}_n \subseteq G_k\big).$$

Since membership in $N(k,n,p) = \{\, \alpha \in 2^{\mathbb{N}} \mid \delta(\alpha\restriction_p) \in U_n \subseteq \overline{U}_n \subseteq G_k \,\}$ is completely determined by $\alpha\restriction_p$ (and whether or not $\overline{U}_n \subseteq G_k$, which is independent of $\alpha\restriction_p$), $N(k,n,p)$ is clopen for each $(k,n,p) \in \mathbb{N}^3$. Thus $D_X$ is $\mathbf{\Pi^0_3}$, and by Corollary 1.3, $D_X$ is $\mathbf{\Pi^0_3}$–complete. Furthermore, if we denote by $P(X)$ the set of $\alpha \in 2^{\mathbb{N}}$ such that

$$\forall k\ \exists n\ \exists m\ \forall p \geq m\ \big(\delta(\alpha\restriction_p) \in U_n \subseteq \overline{U}_n \subseteq G_k\big),$$

then $P(X)$ is $\mathbf{\Pi^0_3}$ and $\alpha \in D_X \Leftrightarrow \alpha \in DE \cap P(X)$. □

**Corollary 1.6** *Let $X \subseteq [0,1]$ be nonempty.*

- $(i)$ *If $X$ is $\mathbf{\Sigma^0_2}$, then $D_X$ is $\mathcal{D}_2(\mathbf{\Pi^0_3})$.*
- $(ii)$ *If $X$ is $\mathbf{\Pi^0_\alpha}$ ($\mathbf{\Sigma^0_\alpha}$) for $\alpha \geq 3$, then $D_X$ is $\mathbf{\Pi^0_{1+\alpha}}$ ($\mathbf{\Sigma^0_{1+\alpha}}$).*
- $(iii)$ *If $X$ is $\mathcal{D}_\xi(\mathbf{\Pi^0_\alpha})$, for $\alpha$ and $\xi \geq 2$, then $D_X$ is $\mathcal{D}_\xi(\mathbf{\Pi^0_{1+\alpha}})$.*
- $(iv)$ *If $X$ is $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi^0_\alpha})$ for $\alpha \geq 3$, or $\alpha = 2$ and $\xi \geq \omega$, then $D_X$ is $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi^0_{1+\alpha}})$.*
- $(v)$ *If $X$ is $\widetilde{\mathcal{D}}_m(\mathbf{\Pi^0_2})$, for $m < \omega$, then $D_X$ is $\mathcal{D}_{m+1}(\mathbf{\Pi^0_3})$.*

□ If $X$ is $\mathbf{\Sigma^0_2}$, $\neg X$ is $\mathbf{\Pi^0_2}$ and $D_X = DE - D_{\neg X} \in \mathcal{D}_2(\mathbf{\Pi^0_3})$ by Proposition 1.5, so $(i)$ holds. More precisely, for each $X \in \mathbf{\Sigma^0_2}([0,1])$, there is a $\mathbf{\Sigma^0_3}$ set $P'(X)$ (namely $\neg P(\neg X)$ from Proposition 1.5) such that $\alpha \in D_X \Leftrightarrow \alpha \in DE \cap P'(X)$. Clearly, for $W = X - X'$, $Y = \bigcup_{n\in\mathbb{N}} X_n$ and $Z = \bigcap_{n\in\mathbb{N}} X_n$,

$$D_W = D_X - D_{X'},\ D_Y = \bigcup_{n\in\mathbb{N}} D_{X_n}, \text{ and } D_Z = \bigcap_{n\in\mathbb{N}} D_{X_n}. \tag{1.2}$$

An easy induction then shows, for $n \geq 2$, that for each $\mathbf{\Pi^0_n}$ ($\mathbf{\Sigma^0_n}$) set $X \subseteq [0,1]$, there is a $\mathbf{\Pi^0_{n+1}}$ ($\mathbf{\Sigma^0_{n+1}}$) set $P(X) \subseteq 2^{\mathbb{N}}$, such that

$$\alpha \in D_X \Leftrightarrow \alpha \in DE \cap P(X). \tag{1.3}$$

This then gives $(ii)$ for $\alpha < \omega$, since the classes $\mathbf{\Pi^0_k}$ and $\mathbf{\Sigma^0_k}$, for $k \geq 4$, are closed under intersections with $\mathbf{\Pi^0_3}$ sets. The function $f\colon 2^{\mathbb{N}} \to \mathbb{R}$, given by

$$f(\alpha) = \begin{cases} \delta(\alpha), & \text{if } \delta(\alpha) \text{ exists;} \\ 2, & \text{otherwise.} \end{cases}$$

is a Baire class 4 function by Proposition 1.5. As $f^{\leftarrow}(X) = D_X$, for all $X \subseteq [0,1]$, if $X$ is $\mathbf{\Pi}^0_\alpha$ or $\mathbf{\Sigma}^0_\alpha$, then $D_X$ is $\mathbf{\Pi}^0_{4+\alpha}$ or $\mathbf{\Sigma}^0_{4+\alpha}$. Thus if $\alpha \geq \omega$, $4+\alpha = 1+\alpha = \alpha$, so (*ii*) holds (also the levels of the projective hierarchy do not increase from $X$ to $D_X$). Using (1.2) and (1.3), one shows that for each $\mathcal{D}_\xi(\mathbf{\Pi}^0_\alpha)$ set $X \subseteq [0,1]$ (with $\alpha$ and $\xi \geq 2$), there is a $\mathcal{D}_\xi(\mathbf{\Pi}^0_{1+\alpha})$ set $P(X) \subseteq 2^{\mathbb{N}}$, such that

$$D_X = DE \cap P(X).$$

Thus (*iii*) follows, since $\mathcal{D}_\xi(\mathbf{\Pi}^0_{1+\alpha})$ sets are closed under intersections with $\mathbf{\Pi}^0_3$ sets (as long as $\alpha \geq 2$). If $X$ is $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_\alpha)$, for $\alpha$ and $\xi \geq 2$, then $D_X = DE \cap \neg D_{\neg X}$ which is the intersection of a $\mathbf{\Pi}^0_3$ set with a $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_{1+\alpha})$ set. For $\alpha \geq 3$, or $\alpha = 2$ and $\xi \geq \omega$, the class $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_{1+\alpha})$ is closed under intersections with $\mathbf{\Pi}^0_3$ sets, and hence (*iv*) follows. Finally, if $X$ is $\widetilde{\mathcal{D}}_m(\mathbf{\Pi}^0_2)$, then $D_X = DE \cap \neg D_{\neg X}$, which by (*iii*) and the definition is a $\mathcal{D}_{m+1}(\mathbf{\Pi}^0_3)$ set, so (*v*) holds. □

The upper bound of $\mathcal{D}_2(\mathbf{\Pi}^0_3)$ for $X = \mathbb{Q}$ the rationals turns out to be a lower bound also. This is rather surprising, since very few sets are known to be properly located above the third level of the Borel hierarchy. We show now that $D_{\mathbb{Q}}$ is $\mathbf{\Sigma}^0_3$-hard. It turns out that no $D_X$ is $\mathbf{\Sigma}^0_3$-complete (except of course for $X = \emptyset$ in which case one might say $D_X = \neg DE$, which is $\mathbf{\Sigma}^0_3$-complete by Theorem 1.2), and our proof of this fact will be the second half of the proof that $D_{\mathbb{Q}}$ is $\mathcal{D}_2(\mathbf{\Pi}^0_3)$-complete. Our reduction here uses a non-standard $\mathbf{\Sigma}^0_3$-complete set, namely,

$$S_3 = \left\{ \alpha \in 2^{\mathbb{N}\times\mathbb{N}} \mid \exists R \; \forall r \geq R \; \exists c, \; \alpha(r,c) = 1 \right\}.$$

If one views $\alpha$ as an $\mathbb{N}\times\mathbb{N}$ matrix of zeros and ones whose entry in row $r$ and column $c$ is $\alpha(r,c)$, then $S_3$ is the set of matrices where all but finitely many rows contain a one, or equivalently with finitely many "all zero" rows. To prove that $\neg\mathcal{C}_3 \leq_{\mathbf{W}} S_3$, $\beta \in \mathbb{N}^{\mathbb{N}} \mapsto \alpha \in 2^{\mathbb{N}\times\mathbb{N}}$, one attempts to define $\alpha{\restriction}_{n\times n}$ so that it contains $\beta(n)$ partial "all zero" rows. With a little organization, this makes the number of rows in $\alpha$ without any ones equal to the lim inf of $\beta$, so $\beta \notin \mathcal{C}_3 \Leftrightarrow \liminf_{n\to\infty} \beta(n)$ is finite $\Leftrightarrow \alpha \in S_3$. We define inductively, $\alpha{\restriction}_{n\times n}$, from $\beta{\restriction}_n$ (so that $\beta \mapsto \alpha$ is continuous, and at stage $n$ we must define $\alpha$'s entries in column $n$ for the rows $0, 1, 2, \ldots, n-1$, as well as row $n$, columns $0, 1, \ldots, n$), as follows:

**Stage 0:** If $\beta(0) = 0$, set $\alpha(0,0) = 1$ and if $\beta(0) > 0$, set $\alpha(0,0) = 0$. Thus $Z_0$, the number of "all zero" rows in $\alpha\restriction_{1\times 1}$ is either $0 = \beta(0)$ or $1 \leq \beta(0)$.

**Stage n:** We are given $\alpha\restriction_{n\times n}$ and $\beta(n)$, and must define the first $n+1$ entries, in both column $n$ and row $n$, of $\alpha$. Let $Z_n$ be the number of partial rows in $\alpha\restriction_{n\times n}$ that are all zeros. If $\beta(n) \leq Z_n$, extend the "first" $\beta(n)$ many "all zero" rows of $\alpha\restriction_{n\times n}$ by adding a zero in column $n$; all the remaining rows (with index less than $n$) receive a one in column $n$; and define the first $n+1$ entries in row $n$ to be ones. Here, "first" is defined from the indices of the rows, so the first 5 rows refers to the 5 rows with lowest indices. If $\beta(n) > Z_n$, extend every "all zero" row by adding a zero in column $n$; every row that already has a one gets a one in column $n$; and make row $n$ begin with $n+1$ zeros. Hence $Z_{n+1}$, the number of "all zero" rows in $\alpha\restriction_{(n+1)\times(n+1)}$, is either $\beta(n)$ or $1 + Z_n \leq \beta(n)$. More precisely, for $r < n$, let $Z_n(r)$ denote the number of rows in $\alpha\restriction_{n\times n}$, with index $r' < r$, that are all zeros. Then (for $r < n$) we set

$$\alpha(r,n) = \begin{cases} 0, & \text{if } \forall c < n[\alpha(r,c) = 0] \text{ and } Z_n(r) < \beta(n); \\ 1, & \text{otherwise.} \end{cases}$$

And for $c \leq n$ set

$$\alpha(n,c) = \begin{cases} 0, & \text{if } \beta(n) > Z_n; \\ 1, & \text{otherwise.} \end{cases}$$

One sees that if for all $n \geq N$, $\beta(n) \geq k$, then $Z_m \geq k$ for all $m > N + k$, and the first $k$ many "all zero" rows in $\alpha\restriction_{(N+k+1)\times(N+k+1)}$ always receive a zero. Thus,

$$\liminf_{n\to\infty} \beta(n) \leq \text{ the number of "all zero" rows in } \alpha.$$

The reverse inequality is trivial if $\liminf \beta = \infty$, so assume $\liminf \beta = k < \infty$. Then $\beta$ takes the value $k$ infinitely often. Let $n_1 < n_2 < \ldots < n_{k+1}$ be any collection of $k+1$ natural numbers. We show that for some $i = 1$ to $k+1$, row $n_i$ of $\alpha$ contains a one. Since $\beta(n) = k$ infinitely often, fix $n > n_{k+1}$ such that $\beta(n) = k$. Then $Z_{n+1} \leq k$, so at least one of the rows with indices $n_i$ gets a one at stage $n$. Thus $\liminf_{n\to\infty} \beta(n) =$ the number of "all zero" rows in $\alpha$, which directly translates to

$$\beta \notin \mathcal{C}_3 \Leftrightarrow \alpha \in S_3,$$

and $\neg\mathcal{C}_3 \leq_{\mathbf{W}} S_3$. So $S_3$ is $\mathbf{\Sigma^0_3}$–hard and it is straightforward to see that $S_3 \in \mathbf{\Sigma^0_3}$.

**Proposition 1.7** *$S_3 \leq_{\mathbf{W}} (D_{\mathbb{Q}}; D_{\mathbb{P}})$, and hence $D_{\mathbb{Q}}$ is $\mathbf{\Sigma}^0_3$–hard.*

□ Let $\{ A_n \}_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ with $\delta(A_n) = 1/2^{n+1}$ (for example one can take $\{ 2^n(2p+1)-1 \}_{p\in\mathbb{N}}$ for $A_n$). Let $\{ a^n_k \}_{k\in\mathbb{N}}$ be an increasing enumeration of $A_n$. Let $B_n \subseteq A_n$ be the set $\left\{ a^n_{k\cdot(n!)} \mid k \in \mathbb{N} \right\}$, so that $\delta(B_n) = \dfrac{1}{n! 2^{n+1}}$. Let $B^* = \mathbb{N} - \bigcup_{n\in\mathbb{N}} B_n$, so that $B^* \cup \{ B_n \}_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}$ suitable for our second canonical construction. Given $\alpha \in 2^{\mathbb{N}\times\mathbb{N}}$, let

$$\alpha^*_r(c) = \begin{cases} 1, & \text{if for all } c' \leq c,\ \alpha(r,c') = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\alpha \mapsto \{ \alpha^*_r \}_{r\in\mathbb{N}}$ is continuous (from $2^{\mathbb{N}\times\mathbb{N}}$ to $\left(2^{\mathbb{N}}\right)^{\mathbb{N}}$) and $\alpha^*_r$ is eventually zero (hence $\delta(\alpha^*_r) = 0$) iff row $r$ of $\alpha$ has a one, and $\alpha^*_r$ is identically one (hence $\delta(\alpha^*_r) = 1$) iff row $r$ of $\alpha$ is identically zero. Let $f(\alpha) \in 2^{\mathbb{N}}$ be the result of playing $\alpha^*_r$ on $B_r$, for $r \in \mathbb{N}$, and $\vec{0}$ on $B^*$. Then, $\delta(f(\alpha)) = \sum\limits_{r=0}^{\infty} \dfrac{\delta(\alpha^*_r)}{r! \cdot 2^{r+1}}$ (which always exists). Also, $\sum\limits_{r=0}^{\infty} \dfrac{\delta(\alpha^*_r)}{r! \cdot 2^{r+1}}$ is rational iff $\delta(\alpha^*_r)$ is non–zero finitely often (see for example 1.7 in [Ni] or a proof that $e$ is irrational). Thus, $\alpha \in S_3$ iff all but finitely many rows of $\alpha$ contain a 1 iff for all but finitely many $r$, $\delta(\alpha^*_r) = 0$ iff $\delta(f(\alpha)) \in \mathbb{Q}$. Thus $S_3 \leq_{\mathbf{W}} (D_{\mathbb{Q}}; D_{\mathbb{P}})$, and $D_{\mathbb{Q}}$ is $\mathbf{\Sigma}^0_3$–hard, provided $f$ is continuous. Since $f(\alpha)\restriction_n$ is completely determined by $\alpha\restriction_{M\times n}$, where $M = \max\{ m \in \mathbb{N} \mid A_m \cap [0,n) \neq \emptyset \}$, $f$ is continuous and we are done. □

**Lemma 1.8** *For any set $C$, if $C \leq_{\mathbf{W}} (D_X; D_{\neg X})$, then $\mathcal{C}_3 \times C \leq_{\mathbf{W}} D_X$.*

□ . Let $f$ be continuous and witness $C \leq_{\mathbf{W}} (D_X; D_{\neg X})$. Assume $C \subseteq Y$ (some topological space), then $\delta(f(y))$ exists for all $y \in Y$. As in Theorem 1.2, we replace $\beta \in \mathbb{N}^{\mathbb{N}}$ with $\beta'$, where $\beta'(2n) = \beta(n) + 2$, and $\beta'(2n+1) = 2n+2$. So that $\beta \mapsto \beta'$ is continuous and does not alter membership in $\mathcal{C}_3$. We show $\mathcal{C}_3 \times C \leq_{\mathbf{W}} D_X$ by defining $\phi(\beta, y)$ to be the result of playing $f(y)$ (whose density always exists) on $A_0$, the evens, and $\alpha'$ on $A_1$, the odds, where $\alpha'$ comes from the canonical construction with input $\{ x_n \}_{n\in\mathbb{N}}$, where $x_n = (1 - 1/\beta'(n)) \cdot (\delta(f(y)\restriction_n) + 1/\beta'(n))$. This defines a continuous function, since $f$ is continuous and $\alpha'\restriction_n$ depends only on $\beta\restriction_n$ and the neighborhood of $y$ that determines $f(y)\restriction_n$ (which exists since $f$ is continuous). If $\beta \in \mathcal{C}_3$, then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left( [1 - 1/\beta'(n)] \cdot [\delta(f(y)\restriction_n) + 1/\beta'(n)] \right) = \delta(f(y)).$$

Hence, when $\beta \in \mathcal{C}_3$,

$$\delta(\phi(\beta, y)) = (1/2)\delta(f(y)) + (1/2)\delta(f(y)) = \delta(f(y)) \in X \Leftrightarrow y \in C.$$

When $\beta \notin \mathcal{C}_3$ the sequence $\{ x_n \}_{n \in \mathbb{N}}$ diverges. So $\delta(\alpha')$ does not exist and the density of $\phi(\beta, y)$ does not exist. Thus $\phi(\beta, y) \in D_X \Leftrightarrow (\beta, y) \in \mathcal{C}_3 \times C$. □

We now construct a sequence of complete sets for the differences of $\mathbf{\Pi}^0_3$ sets. Let $m \geq 1$ be a finite integer. In the space, $\left(\mathbb{N}^{\mathbb{N}}\right)^m$, consider the sets $A_0, A_1, \ldots, A_{m-1}$, where

$$\begin{aligned}
A_0 &= \mathcal{C}_3 \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \ldots \times \mathbb{N}^{\mathbb{N}} \\
A_1 &= \mathcal{C}_3 \times \mathcal{C}_3 \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \ldots \times \mathbb{N}^{\mathbb{N}} \\
&\vdots \\
A_i &= (\mathcal{C}_3)^{i+1} \times \left(\mathbb{N}^{\mathbb{N}}\right)^{m-i-1} \\
&\vdots \\
A_{m-1} &= \mathcal{C}_3 \times \mathcal{C}_3 \times \ldots \times \mathcal{C}_3 .
\end{aligned}$$

Then each $A_i \in \mathbf{\Pi}^0_3$ and $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_{m-1}$. Let $D^3_m = \mathcal{D}_m(\langle A_i \rangle_{i<m}) \in \mathcal{D}_m(\mathbf{\Pi}^0_3)$ and $\widetilde{D}^3_m = \neg D^3_m \in \widetilde{\mathcal{D}}_m(\mathbf{\Pi}^0_3)$. Then we have that

$$\begin{aligned}
D^3_m &= \left\{ \langle \beta_i \rangle_{i<m} \in \left(\mathbb{N}^{\mathbb{N}}\right)^m \mid \beta_0 \in \mathcal{C}_3 \text{ and } \max\{ i < m \colon \beta_0, \ldots, \beta_i \in \mathcal{C}_3 \} \text{ is even} \right\} \\
\widetilde{D}^3_m &= \left\{ \langle \beta_i \rangle_{i<m} \in \left(\mathbb{N}^{\mathbb{N}}\right)^m \mid \beta_0 \notin \mathcal{C}_3 \text{ or } \max\{ i < m \colon \beta_0, \ldots, \beta_i \in \mathcal{C}_3 \} \text{ is odd} \right\}.
\end{aligned}$$

We show now that for any $\mathcal{D}_m(\mathbf{\Pi}^0_3)$ set $B \subseteq 2^{\mathbb{N}}$, $B \leq_{\mathbf{W}} D^3_m$, so $D^3_m$ is $\mathcal{D}_m(\mathbf{\Pi}^0_3)$–complete and $\widetilde{D}^3_m$ is $\widetilde{\mathcal{D}}_m(\mathbf{\Pi}^0_3)$–complete. Given such a $B$, fix $B_0 \supseteq B_1 \supseteq \ldots \supseteq B_{m-1}$, $\mathbf{\Pi}^0_3$ subsets of $2^{\mathbb{N}}$ with $B = \mathcal{D}_m(\langle B_i \rangle_{i<m})$. Since $B_i \in \mathbf{\Pi}^0_3$ and $\mathcal{C}_3$ is $\mathbf{\Pi}^0_3$–complete, there is a continuous function, $f_i \colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, such that $\alpha \in B_i \Leftrightarrow f_i(\alpha) \in \mathcal{C}_3$. Define $f \colon 2^{\mathbb{N}} \to \left(\mathbb{N}^{\mathbb{N}}\right)^m$ by

$$f(\alpha) = (f_0(\alpha), f_1(\alpha), \ldots, f_{m-1}(\alpha)).$$

Since the $B_i$'s are decreasing, it is straightforward to check that $\alpha \in B_i \Leftrightarrow f_i(\alpha) \in \mathcal{C}_3 \Leftrightarrow f(\alpha) \in A_i$, which shows $B \leq_{\mathbf{W}} D^3_m$. Notice that $\mathcal{C}_3 \times \widetilde{D}^3_m = D^3_{m+1}$.

**Theorem 1.9** *For $1 \leq m < \omega$, if $D_X$ is $\widetilde{\mathcal{D}}_m(\mathbf{\Pi}^0_3)$-hard, then $D_X$ is $\mathcal{D}_{m+1}(\mathbf{\Pi}^0_3)$-hard. Thus no $D_X$ is $\widetilde{\mathcal{D}}_m(\mathbf{\Pi}^0_3)$-complete, and $D_{\mathbb{Q}}$ is $\mathcal{D}_2(\mathbf{\Pi}^0_3)$-complete.*

□ By Lemma 1.8, it suffices to show $\widetilde{D}^3_m \leq_{\mathbf{W}} (D_X; D_{\neg X})$. If this were not the case, then by the result of Louveau and Saint–Raymond [LS] mentioned earlier, there would be a $\mathcal{D}_m(\mathbf{\Pi}^0_3)$ set $S$ such that $D_X \subseteq S$ and $S \cap D_{\neg X} = \emptyset$. But then $D_X = S \cap DE$, a $\mathcal{D}_m(\mathbf{\Pi}^0_3)$ set, which is contrary to the assumption that $D_X$ is $\widetilde{\mathcal{D}}_m(\mathbf{\Pi}^0_3)$-hard. □

We shall now basically show that $X \leq_{\mathbf{W}} D_X$. Literally this cannot be true because $X$ lives in a connected space and $D_X$ lives in a zero–dimensional space. However, for large enough $\xi$ and $\alpha$, intersecting a $\mathcal{D}_\xi(\mathbf{\Pi}^0_\alpha)$ set $X$ with $\mathbb{P}$, the irrationals in $[0,1]$, does not change the Borel class. Since $\mathbb{P}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, if $X \subseteq \mathbb{P}$ one can show $X \leq_{\mathbf{W}} D_X$. The following material is well known (see [Ni] pp. 51–67 for example). Given $\beta \in \mathbb{N}^{\mathbb{N}}$, let $\beta^*(n) = \beta(n) + 1$. For each $n \in \mathbb{N}$, let

$$r_n(\beta) = r_n = \cfrac{1}{\beta^*(0) + \cfrac{1}{\beta^*(1) + \cfrac{1}{\ddots + \cfrac{1}{\beta^*(n)}}}} \tag{1.4}$$

so $r_n \in \mathbb{Q} \cap [0,1]$. Then, $\phi(\beta) = \lim_{n\to\infty} r_n$ exists, is irrational, and $\phi$ is a homeomorphism onto $\mathbb{P}$. In a way, the next two results, as well as Lemma 1.8, show that our canonical construction can absorb continuous functions. We shall see later that it can actually absorb some Baire class one functions too.

**Lemma 1.10** *For nonempty $X \subseteq \mathbb{N}^{\mathbb{N}}$, $X \leq_{\mathbf{W}} (D_{\phi(X)}; D_{\phi(\neg X)})$.*

□ Given $\beta \in \mathbb{N}^{\mathbb{N}}$, let $f(\beta) \in 2^{\mathbb{N}}$ be the result of running the canonical construction on input $\{x_n\}_{n\in\mathbb{N}}$, where $x_n$ is the $r_n(\beta)$ in (1.4) (which only depends on $\beta\restriction_{n+1}$). Then $\delta(\alpha) = \lim_{n\in\mathbb{N}} r_n = \phi(\beta)$. Hence, since $f$ is continuous, for any $X \subseteq \mathbb{N}^{\mathbb{N}}$, $f$ shows

$$X \leq_{\mathbf{W}} (D_{\phi(X)}; D_{\phi(\neg X)})$$

and we are done. □

**Theorem 1.11** *Let $\Gamma$ be one of the classes $\mathbf{\Pi}^0_\alpha$ or $\mathbf{\Sigma}^0_\alpha$ for $\alpha \geq 3$; $\mathcal{D}_\xi(\mathbf{\Pi}^0_\alpha)$ for $\alpha \geq 2$; $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_\alpha)$ for $\alpha \geq 3$; or $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_2)$ for $\xi \geq \omega$. If $X \subseteq [0,1]$ is $\Gamma$–hard, then so is $D_X$. In particular, if $\alpha \geq \omega$ and $X$ is $\Gamma$–complete, then so is $D_X$.*

□ The last part follows from the first and Corollary 1.6, since in this case $1 + \alpha = \alpha$. For each such $\Gamma$, let $\widetilde{\Gamma}$ denote the dual class $\{\, \neg X \mid X \in \Gamma \,\}$. Then $\Gamma$ is closed under intersections with $\mathbf{\Pi}^0_2$ sets and $\widetilde{\Gamma}$ is closed under unions with $\mathbf{\Sigma}^0_2$ sets. Since $X \subseteq [0,1]$ is $\Gamma$-hard, $X$ is not in $\widetilde{\Gamma}$. If $X \cap \mathbb{P} \in \widetilde{\Gamma}$, then

$$X = (X \cap \mathbb{P}) \cup (X \cap \mathbb{Q})$$

is also in $\widetilde{\Gamma}$, since $X \cap \mathbb{Q}$ is countable and thus $\mathbf{\Sigma}^0_2$. So $X \cap \mathbb{P} \notin \widetilde{\Gamma}$, and hence $X \cap \mathbb{P}$ is $\Gamma$–hard. As $\phi^{-1}$ is a homeomorphism, $\phi^{-1}(X \cap \mathbb{P})$ is $\Gamma$–hard. By Lemma 1.10,

$$\phi^{-1}(X \cap \mathbb{P}) \leq_{\mathbf{W}} (D_{X \cap \mathbb{P}}; D_{\mathbb{P} \cap \neg X}).$$

Thus $\phi^{-1}(X \cap \mathbb{P}) \leq_{\mathbf{W}} D_X$, and $D_X$ is $\Gamma$–hard. □

Notice that if $\Gamma$ is one of the projective hierarchy classes, then $X$ is $\Gamma$–complete iff $D_X$ is. We have already seen $D_X$ is $\mathbf{\Pi}^0_3$–complete, for any nonempty $\mathbf{\Pi}^0_2$ subset $X$ of $[0,1]$. We shall now show that for $\alpha \geq 3$, if $X \subseteq [0,1]$ is $\mathbf{\Pi}^0_\alpha$ ($\mathbf{\Sigma}^0_\alpha$)–complete, then $D_X$ is $\mathbf{\Pi}^0_{1+\alpha}$ ($\mathbf{\Sigma}^0_{1+\alpha}$)–complete. We need the complete sets, $\{\, H_n \subseteq 2^{\mathbb{N}} \mid n \in \mathbb{N} \,\}$, from [LS], and some basic properties of their function $\rho$. For $n$ and $m \in \mathbb{N}$, let

$$\langle n, m \rangle = (1/2)(n+m)(n+m+1) + m.$$

Thus $\langle\ ,\ \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is one–to–one and onto. Define $\rho \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, by

$$\rho(\alpha)(n) = 1 \Leftrightarrow \forall m \ \big(\alpha(\langle n, m \rangle) = 0\big).$$

Again, thinking of $\alpha$ as an $\mathbb{N} \times \mathbb{N}$ matrix of zeros and ones, with the entry in row $n$ and column $m$ being $\alpha(\langle n, m \rangle)$, then

$$\rho(\alpha)(n) = \begin{cases} 0, & \text{if row } n \text{ of } \alpha \text{ contains a 1;} \\ 1, & \text{if row } n \text{ of } \alpha \text{ is identically 0.} \end{cases}$$

Thus, if $\alpha_n$ is the binary sequence where $\alpha_n(m) = \alpha(\langle n, m \rangle)$, then $\rho(\alpha)(n) = \chi_{\{\,\vec{0}\,\}}(\alpha_n)$. One can extend $\rho$ to $2^{\leq \mathbb{N}}$, by defining for $s \in 2^k$, $\rho(s) = s^* \in 2^{<\mathbb{N}}$, where

$$Dom(s^*) = \{\, n \in \mathbb{N} \mid \langle n, 0 \rangle < k \,\}$$

(so that $Dom(s^*)$ is an initial segment of $\mathbb{N}$), and for $n \in Dom(s^*)$,

$$s^*(n) = 1 \Leftrightarrow \text{ for all } \langle n, m\rangle \in Dom(s),\ s(\langle n, m\rangle) = 0.$$

The properties of $\rho$ that we need are the following; all appear in [LS].

($i$) $\forall\alpha \in 2^{\mathbb{N}},\ \forall n \in \mathbb{N},\ \exists k\ \forall m \geq k\ \left(\rho(\alpha\restriction_m)\restriction_n = \rho(\alpha)\restriction_n\right)$.

($ii$) Let $H_1 = \{\vec{0}\} \subseteq 2^{\mathbb{N}}$ and $H_{n+1} = \rho^{\leftarrow}(H_n)$. Then $H_n$ is $\mathbf{\Pi^0_n}$–complete.

Thus ($i$) says that for each $i \in \mathbb{N}$, the approximations $\alpha^*_n(i) = \rho(\alpha\restriction_n)(i)$ are eventually equal to $\rho(\alpha)(i)$.

**Lemma 1.12** *For $H \subseteq 2^{\mathbb{N}}$ and $X \subseteq [0,1]$, if $H \leq_{\mathbf{W}} X$, then $\rho^{\leftarrow}(H) \leq_{\mathbf{W}} (D_X; D_{\neg X})$. In particular, for $n \geq 2$, and $X \subseteq [0,1]$, if $H_n \leq_{\mathbf{W}} X$, then $H_{n+1} \leq_{\mathbf{W}} D_X$, and if $\neg H_n \leq_{\mathbf{W}} X$, then $\neg H_{n+1} \leq_{\mathbf{W}} D_X$.*

□ Let $g: 2^{\mathbb{N}} \to [0,1]$ be a continuous function witnessing $H \leq_{\mathbf{W}} X$. Given $\alpha \in 2^{\mathbb{N}}$, apply $\rho$ to $\alpha\restriction_n$, yielding say $\alpha^*_n \in 2^{<\mathbb{N}}$ (this is a finitary process even though $\rho$ is Baire class one). Let $\alpha_n = \alpha^*_n{}^\frown\vec{0} \in 2^{\mathbb{N}}$, and set $x_n = g(\alpha_n) \in [0,1]$. We let $f(\alpha)$ be the canonical construction on input $\{x_n\}_{n\in\mathbb{N}}$. As usual, $x_n$ depends only on $\alpha\restriction_n$, so $f$ is continuous. Since $g$ is continuous, $x_n = g(\alpha_n)$, and $\{\alpha_n\}_{n\in\mathbb{N}}$ converges pointwise to $\rho(\alpha)$, we get that $\lim_{n\to\infty} x_n = g(\rho(\alpha))$ (and $\delta(f(\alpha))$ always exists). Hence,

$$\alpha \in \rho^{\leftarrow}(H) \Leftrightarrow \rho(\alpha) \in H \Leftrightarrow g(\rho(\alpha)) = \lim_{n\to\infty} x_n = \delta(f(\alpha)) \in X \Leftrightarrow f(\alpha) \in D_X.$$

So $\rho^{\leftarrow}(H) \leq_{\mathbf{W}} (D_X; D_{\neg X})$. □

**Theorem 1.13** ($i$) *If $X \subseteq [0,1]$ is $\mathbf{\Pi^0_\alpha}$–complete ($\mathbf{\Sigma^0_\alpha}$–complete), for $\alpha \geq 3$, then $D_X$ is $\mathbf{\Pi^0_{1+\alpha}}$–complete ($\mathbf{\Sigma^0_{1+\alpha}}$–complete).*

($ii$) *If $X \subseteq [0,1]$ is $\mathcal{D}_\xi(\mathbf{\Pi^0_\alpha})$–complete, for $\alpha \geq 2$, then $D_X$ is $\mathcal{D}_\xi(\mathbf{\Pi^0_{1+\alpha}})$–complete.*

($iii$) *If $X \subseteq [0,1]$ is $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi^0_\alpha})$–complete, for $\alpha \geq 3$, or for $\alpha = 2$ and $\xi \geq \omega$, then $D_X$ is $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi^0_{1+\alpha}})$–complete.*

($iv$) *If $X \subseteq [0,1]$ is $\widetilde{\mathcal{D}}_m(\mathbf{\Pi^0_2})$–complete, for $m < \omega$, then $D_X$ is $\mathcal{D}_{m+1}(\mathbf{\Pi^0_3})$–complete.*

*Likewise all these hold with hard replacing complete and all implications reverse for $\alpha \geq 3$.*

□ The upper bounds for $D_X$ are from Proposition 1.5 and Corollary 1.6. They show the reverse implications hold for $\alpha \geq 3$. If $\alpha \geq \omega$, Theorem 1.11 gives the above statements. Hence we need only work with $\alpha < \omega$, which we will denote by $n$. Now, $(i)$ is just the second part of Lemma 1.12. For the remaining cases consider the sets,

$$D_\xi^n = \left\{ \langle \alpha_\beta \rangle_{\beta<\xi} \in \left(2^{\mathbb{N}}\right)^\xi \mid \alpha_0 \in H_n \text{ and the least } \beta \text{ such that } \alpha_\beta \notin H_n \text{ is odd } \right\}$$
$$\widetilde{D}_\xi^n = \neg D_\xi^n = \left\{ \langle \alpha_\beta \rangle_{\beta<\xi} \in \left(2^{\mathbb{N}}\right)^\xi \mid \text{ the least } \beta \text{ such that } \alpha_\beta \notin H_n \text{ is even } \right\},$$

where $(H_n)^\xi$ is included in $D_\xi^n$ if $\xi$ is odd, and included in $\widetilde{D}_\xi^n$ when $\xi$ is even. Then $D_\xi^n$ is $\mathcal{D}_\xi(\mathbf{\Pi^0_n})$–complete, and $\widetilde{D}_\xi^n$ is $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi^0_n})$–complete. Furthermore, by applying $\rho$ coordinatewise to $D_\xi^{n+1}$, we obtain $D_\xi^n$. That is, $\vec{\alpha} = \langle \alpha_\beta \rangle_{\beta<\xi} \in D_\xi^{n+1} \Leftrightarrow \langle \rho(\alpha_\beta) \rangle_{\beta<\xi} \in D_\xi^n$. Thus, we simply mimic the proof of Lemma 1.12. Let $\Gamma$ be any of the classes $\mathcal{D}_\xi(\mathbf{\Pi^0_n})$ or $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi^0_n})$, mentioned in the hypothesis, where $X$ is $\Gamma$–complete or $\Gamma$–hard. Let $\Gamma^*$ be the class where the $n$ in $\Gamma$ is replaced by $n+1$. Let

$$D_\Gamma = \begin{cases} D_\xi^n, & \text{if } \Gamma = \mathcal{D}_\xi(\mathbf{\Pi^0_n}); \\ \widetilde{D}_\xi^n, & \text{if } \Gamma = \widetilde{\mathcal{D}}_\xi(\mathbf{\Pi^0_n}). \end{cases}$$

Then the assumptions give a continuous function $g$ witnessing $D_\Gamma \leq_{\mathbf{W}} X$. Since $\xi$ is countable, $\left(2^{\mathbb{N}}\right)^\xi$ is homeomorphic to $2^{\mathbb{N}}$ by some function $\phi\colon (2^{\mathbb{N}})^\xi \to 2^{\mathbb{N}}$. In fact, if we take $\langle\ ,\ \rangle\colon \xi \times \mathbb{N} \to \mathbb{N}$ to be any bijection such that for each $\beta < \xi$, the sequence $\langle \beta, n \rangle_{n\in\mathbb{N}}$ is increasing, then we can take $\phi(\vec{\alpha})(\langle \beta, n \rangle) = \alpha_\beta(n)$. If we then let $\vec{\alpha}\restriction_n = \{ \alpha_\beta(k) \mid \langle \beta, k \rangle < n \}$, this will be a finite set containing an initial segment of each $\alpha_\beta$. We can then apply $\rho$ to each initial segment, obtaining say $\langle \alpha^*_{\beta,n} \rangle_{\beta<\xi}$. Let $\vec{\alpha}^*_n$ be the extension of $\langle \alpha^*_{\beta,n} \rangle_{\beta<\xi}$ by setting all undefined values to zero. Then $\{ \vec{\alpha}^*_n \}_{n\in\mathbb{N}}$ converges pointwise to $\langle \rho(\alpha_\beta) \rangle_{\beta<\xi}$. Given $\vec{\alpha} \in \left(2^{\mathbb{N}}\right)^\xi$, let $x_n = g(\vec{\alpha}^*_n) \in [0,1]$ (where $\vec{\alpha}^*_n$ is as above). Then $x_n$ depends only on a finite piece of $\vec{\alpha}$. If we set $f(\vec{\alpha})$ to be the result of running the canonical construction on $\{ x_n \}_{n\in\mathbb{N}}$, $f$ is continuous and as in Lemma 1.12, $f$ witnesses $D_{\Gamma^*} \leq_{\mathbf{W}} (D_X; D_{\neg X})$. Hence $D_X$ is $\Gamma^*$–hard. Thus we are done, except for the last case where $\Gamma^* = \widetilde{\mathcal{D}}_m(\mathbf{\Pi^0_3})$, which follows immediately by Theorem 1.9. □

# Chapter 2

# The Borel classes of Mahler's $A, S, T$ and $U$–numbers

## 2.1 Introduction

Mahler [Mah] divided complex numbers into classes $A, S, T$ and $U$ according to their properties of approximation by algebraic numbers. Some studies were done on the structural properties of these sets. For example Kasch and Volkmann [KaV] verified that the $T$ numbers have Hausdorff dimension zero. Also in harmonic analysis, W. Morgan, C. E. M. Pearce and A. D. Pollington [MorPP] have shown that the set of $T$ and $U$ numbers support a measure whose Fourier transform vanishes at infinity. In the present paper we study the $A, S, T$, and $U$–sets from the point of view of Descriptive Set Theory. Among the few sets whose exact Borel class is known, a large percentage turn out to be $\mathbf{\Pi}^0_3$–complete. For example, the collection of reals that are normal or simply normal to base $n$ [KL], $C^\infty(\mathbb{T})$, the class of infinitely differentiable functions (viewed as a $2\pi$-periodic function on $\mathbb{R}$), and $UC_X$, the class of convergent sequences in a separable Banach space $X$, are $\mathbf{\Pi}^0_3$–complete [Kel]. Apparently, there are few known natural $\mathbf{\Sigma}^0_3$–complete sets. Of course, the complement of a $\mathbf{\Pi}^0_3$–complete set is $\mathbf{\Sigma}^0_3$–complete. But, the complement of a natural set need not be natural! Tom Linton [Li] has shown that the family of $H$–sets, a class of thin sets from harmonic analysis, is $\mathbf{\Sigma}^0_3$–complete, and this is the only $\mathbf{\Sigma}^0_3$–complete natural set we know of (whose complement is not also natural). A. Kechris proposed to find out what the Borel classes of the $A, S, T$ and $U$–sets are. It turns out that $A$ is rather simple, being $\mathbf{\Sigma}^0_2$–complete. On the other hand, $T$ is $\mathbf{\Pi}^0_3$–hard, while $U$ is $\mathbf{\Sigma}^0_3$–complete. Our main results are based on a theorem of W. M. Schmidt (see [Ba], p. 85-94). The exact Borel classes of the $S$ and $T$–sets are unknown to us.

## 2.2 Definitions and background

For spaces $X$ and $Y$, $X^Y$ denotes the set of all functions $f$ from $Y$ to $X$, with the usual product topology, $X$ and $Y$ being endowed with their usual topologies ($2 = \{0, 1\}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$ being discrete). For sets $U$ and $V$, if $S$ is a function from $X^{n+1} \times Y^{n+1}$ to $U^{n+1} \times V^{n+1}$ and $n \in \mathbb{N}$, then $S|_n$ is the function from $X^{n+1} \times Y^{n+1}$ to $U^n \times V^n$ such that if $S((x_1, \cdots, x_{n+1}), (y_1, \cdots, y_{n+1})) = ((u_1, \cdots, u_{n+1}), (v_1, \cdots, v_{n+1}))$, $S|_n((x_1, \cdots, x_{n+1}), (y_1, \cdots, y_{n+1})) = ((u_1, \cdots, u_n), (v_1, \cdots, v_n))$. $\mathbb{P} = \{x \in \mathbb{R} : x > 1\}$ and $\mathbb{A}$ denotes the class of all non-zero real algebraic numbers in $\mathbb{C}$. We briefly describe the Borel hierarchy. Thus the multiplicative sets of level $n$ are denoted by $\mathbf{\Pi^0_n}$, while the additive class of level $n$ is denoted by $\mathbf{\Sigma^0_n}$. In particular, $\mathbf{\Sigma^0_1}$ = Open, $\mathbf{\Pi^0_1}$ = Closed, $\mathbf{\Sigma^0_2} = F_\sigma$, $\mathbf{\Pi^0_2} = G_\delta$. In addition, the countable union of $\mathbf{\Pi^0_n}$ sets is $\mathbf{\Sigma^0_{n+1}}$; the countable intersection of $\mathbf{\Pi^0_n}$ sets is a $\mathbf{\Sigma^0_{n+1}}$ set; the complement of a $\mathbf{\Pi^0_n}$ set is $\mathbf{\Sigma^0_n}$; the $\mathbf{\Sigma^0_n}$ sets are closed under finite intersection and countable union; while the $\mathbf{\Pi^0_n}$ sets are closed under finite union and countable intersection. If the context demands it, we use $\mathbf{\Pi^0_n}(X)$ to denote the $\mathbf{\Pi^0_n}$ subsets of a space $X$.

Now we define the $A, S, T$ and $U$ sets, from Mahler's classification. For convenience we use Koksma's notation which is equivalent to that of Mahler. Given algebraic $\alpha \in \mathbb{C}$, let $p(x) \in \mathbb{Z}[x]$ be its minimal polynomial. Fix $d, h \in \mathbb{N}$. Let $X_{d,h}$ be the finite collection of polynomials with degree $\leq d$ whose largest coefficient has absolute value $\leq h$. Let the height of a polynomial, $ht(p)$, be the maximum of the absolute values of the coefficients. Let $A_{d,h}$ be the finite collection of algebraic numbers $\alpha$ such that for some $p \in X_{d,h}$, $p(\alpha)$ is zero (recall that $0 \notin \mathbb{N}$). Thus, $A_{d,h}$ is the finite collection of algebraic (complex) numbers whose minimal polynomial has degree $\leq d$ and $ht \leq h$. Let $\xi$ be any complex number and let $\alpha$ belong to $A_{d,h}$ such that $|\xi - \alpha|$ takes the smallest positive value; and define $\omega^*_d(\xi, h)$ by

$$|\xi - \alpha| = \frac{1}{h^{d\omega^*_d(\xi,h)+1}}.$$

Set

$$\omega_d^*(\xi) = \limsup_{h\to\infty} \omega_d^*(\xi, h) \text{ and } \omega^*(\xi) = \limsup_{d\to\infty} \omega_d^*(\xi).$$

So the values of $\omega_d^*(\xi)$ and $\omega^*(\xi)$ measure how fast $\xi$ is approximated by algebraic numbers. We define, according to the values of $\omega_d^*(\xi)$ and $\omega^*(\xi)$, the $A, S, T$ and $U$–sets as follows:

$$\begin{aligned} A &= \{\xi \in \mathbb{C} : \omega^*(\xi) = 0\}, \\ S &= \{\xi \in \mathbb{C} : 0 < \omega^*(\xi) < \infty\}, \\ T &= \{\xi \in \mathbb{C} : \omega^*(\xi) = \infty \text{ and } \forall d \in \mathbb{N}\ (\omega_d^*(\xi) < \infty)\}, \\ U &= \{\xi \in \mathbb{C} : \omega^*(\xi) = \infty \text{ and } \exists d \in \mathbb{N}\ (\omega_d^*(\xi) = \infty)\}. \end{aligned}$$

Thus, the $A$ numbers are slowly approximated by algebraic numbers. The $S$ numbers are approximated a bit more quickly than $A$ numbers. On the other hand, the $T$ numbers and $U$ numbers are very rapidly approximated, i.e., the value of $\omega^*(\xi)$ is infinite. In particular, the approximation of the $U$ numbers is so quick that for some $d \in \mathbb{N}$, $\omega_d^*(\xi)$ diverges. For these reasons, we claim that the set of complex numbers is naturally partitioned by the $A, S, T$ and $U$ numbers.

## 2.3 Results

**Lemma 2.1** $\xi \in A \iff \xi$ *is an algebraic number.*

(See [Ba], p. 85-94.)

**Proposition 2.2**

*(i) The* $A$ *numbers are* $\mathbf{\Sigma_2^0}$*–complete, and the* $U$ *numbers are* $\mathbf{\Sigma_3^0}$.

*(ii) The* $S$ *numbers are* $\mathbf{\Sigma_4^0}$*, while the collection of* $T$ *numbers are* $\mathbf{\Pi_4^0}$.

**Proof of Proposition 2.2**(i) For each $d \in \mathbb{N}$, let $U_d$ be the collection of $\xi \in \mathbb{C}$

such that $\omega_d^*(\xi) = \infty$. Then $U_d$ is $\mathbf{\Pi_2^0}$, since

$$\begin{aligned}
\xi \in U_d &\iff \omega_d^*(\xi) = \infty \\
&\iff \forall a \in \mathbb{N}\ \forall b \in \mathbb{N}\ \exists c \in \mathbb{N}\ (\omega_d^*(\xi, b+c) > a) \\
&\iff \forall a \in \mathbb{N}\ \forall b \in \mathbb{N}\ \exists c \in \mathbb{N}\ \exists \alpha \in A_{d,b+c} \left(0 < |\xi - \alpha| < \frac{1}{(b+c)^{ad+1}}\right) \\
&\iff \xi \in \bigcap_{a\in\mathbb{N}} \bigcap_{b\in\mathbb{N}} \bigcup_{c\in\mathbb{N}} \bigcup_{\alpha\in A_{d,b+c}} V(a,b,c,\alpha),
\end{aligned}$$

where $V(a,b,c,\alpha)$ is the collection of $\xi \in \mathbb{C}$ such that $0 < |\xi - \alpha| < \dfrac{1}{(b+c)^{ad+1}}$, which is open. Since it is easy to see that for each $d$, $\omega_d^*(\xi) = \infty$ implies $\omega_{d+1}^*(\xi) = \infty$, we have $U = \bigcup_{d=1}^{\infty} U_d$ and $U$ is $\mathbf{\Sigma_3^0}$. It is well-known that if $D$ is a countable dense set in a perfect Polish space, then $D$ is $\mathbf{\Sigma_2^0}$–complete. Thus, by Lemma 2, $A$ is $\mathbf{\Sigma_2^0}$–complete.

(ii) By definition, $T$ is the collection of $\xi \in \mathbb{C}$ such that $\omega^*(\xi) = \infty$ and $\forall a \in \mathbb{N}\ (\omega_a^*(\xi) < \infty)$. Thus, $T = M \bigcap N$, where $M = \{\xi \in \mathbb{C} : \omega^*(\xi) = \infty\}$ and $N = \{\xi \in \mathbb{C} : \forall a \in \mathbb{N}\ (\omega_a^*(\xi) < \infty)\}$. Now $M$ is $\mathbf{\Pi_4^0}$, since

$$\begin{aligned}
\xi \in M &\iff \forall a \in \mathbb{N}\ \forall b \in \mathbb{N}\ \exists c \in \mathbb{N}\ (\omega_{b+c}^*(\xi) > a) \\
&\iff \forall a \in \mathbb{N}\ \forall b \in \mathbb{N}\ \exists c \in \mathbb{N}\ \exists d \in \mathbb{N}\ \forall e \in \mathbb{N}\ \exists f \in \mathbb{N} \\
&\qquad \left(\omega_{b+c}^*(\xi, e+f) > a + \frac{1}{d+1}\right) \\
&\iff \xi \in \bigcap_{a\in\mathbb{N}} \bigcap_{b\in\mathbb{N}} \bigcup_{c\in\mathbb{N}} \bigcup_{d\in\mathbb{N}} \bigcap_{e\in\mathbb{N}} \bigcup_{f\in\mathbb{N}} W(a,b,c,d,e,f),
\end{aligned}$$

where $W(a,b,c,d,e,f)$ is the collection of $\xi \in \mathbb{C}$ such that $\omega_{b+c}^*(\xi, e+f) > a + \dfrac{1}{d+1}$, which is open by the argument above. So $N$ is $\mathbf{\Pi_3^0}$, since by (i), $U$ is $\mathbf{\Sigma_3^0}$ and

$$\begin{aligned}
\xi \in N &\iff \forall a \in \mathbb{N}\ (\omega_a^*(\xi) < \infty) \\
&\iff \xi \in \mathbb{C} - U.
\end{aligned}$$

Hence $T$ is $\mathbf{\Pi_4^0}$, being the intersection of two $\mathbf{\Pi_4^0}$ sets. Since $\xi \in S \iff \xi \notin T, \xi \notin U$ and $\xi \notin A$, $S$ is $\mathbf{\Sigma_4^0}$. □

In $2^{\mathbb{N}}$, Q is the collection of sequences which end in zeros.

**Lemma 2.3** *There exists a continuous function $\nu$ from $2^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ such that*

*(i) for each $d \in \mathbb{N}, \alpha \in 2^{\mathbb{N}}$, $\nu(\alpha)(d) \leq \nu(\alpha)(d+1)$;*

*(ii) $\alpha \in Q \iff \lim_{d\to\infty} \nu(\alpha)(d) < \infty$.*

**Proof of Lemma 2.3** Let $\alpha \in 2^{\mathbb{N}}$. We produce $\beta = \nu(\alpha)$ recursively. First $\beta(2.1) = \alpha(2.1)$. Suppose that we have defined $\beta(i)$ for all $i \leq k$. Put $\beta(k+1) = \beta(k)$ if $\alpha(k+1) = 0$ and $\beta(k+1) = \beta(k) + 1$ otherwise. It is easy to see that the function $\nu$ satisfies (i). As long as $\alpha$ ends in zeros, so does $\nu(\alpha)$ in constants. Otherwise, $\nu(\alpha)(d)$ goes to the infinity as $d \to \infty$, because infinitely many $d$'s, $\nu(\alpha)(d+1) = \nu(\alpha)(d) + 1$. So (ii) is valid. For given $d \in \mathbb{N}, \alpha_1, \alpha_2 \in 2^{\mathbb{N}}$, such that $\alpha_1(i) = \alpha_2(i)$ for all $i \leq d$, $\nu(\alpha_1)(i) = \nu(\alpha_2)(i)$ for all $i \leq d$. So $\nu$ is continuous. This completes Lemma 2.3. □

From Lemma 2.3, $\alpha \notin Q \iff \lim_{d\to\infty} \nu(\alpha)(d) = \infty$. To prove our main theorem, we need a standard example of the $\mathbf{\Pi}^0_3$–complete set.

**Lemma 2.4** *The set $P_3 = \{\alpha = (\alpha_d) \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall d \in \mathbb{N}\ (\alpha_d \in Q)\}$ is $\mathbf{\Pi}^0_3$–complete.*

(See [Ke1].)

The following theorem is the main result of the paper.

**Theorem 2.5** *There is a continuous function $f$ from $(2^{\mathbb{N}})^{\mathbb{N}}$ to $\mathbb{C}$ such that*

$$\alpha \in P_3 \iff f(\alpha) \in T \text{ and } \alpha \notin P_3 \iff f(\alpha) \in U.$$

*In particular, $T$ is $\mathbf{\Pi}^0_3$–hard and $U$ is $\mathbf{\Sigma}^0_3$–complete.*

Roughly speaking, the original statment of a theorem of Schmidt is the following: Let $\alpha_1, \alpha_2, \cdots$ be any non zero algebraic numbers and let $\nu_1, \nu_2, \cdots$ be any real numbers exceeding 1. Then we may find $\xi \in \mathbb{C}$ such that according to $\alpha_1, \alpha_2, \cdots$ and $\nu_1, \nu_2, \cdots$, $\xi$ is a $U$ number or $T$ number.

By using $\nu$, which is constructed in Lemma 2.3, we shall effectively control $\nu_i$'s so that we are able to prove Theorem 2.5. In order to make it work, we need to state the reformulated version of a theorem of Schmidt which will play a crucial role in the proof of Theorem 2.5.

**Theorem S**[Schmidt] *There exists a sequence* $< S_n >$ *such that for each* $n \in \mathbb{N}$,

*(i)* $S_n$ *is a function from* $\mathbb{A}^n \times \mathbb{P}^n$ *to* $\mathbb{A}^n \times (0,1)^n$ *and* $S_{n+1}|_n = S_n$,

*(ii) Suppose that* $S_n((\theta_1, \cdots, \theta_n), (\nu_1, \cdots, \nu_n)) = ((\gamma_1, \cdots, \gamma_n), (\lambda_1, \cdots, \lambda_n))$. *Then for each* $j < n$, $\gamma_j/\theta_j$ *is rational,* $H_{j+1} > 2H_j$ *and* $\frac{1}{4}H_j^{-1} < \gamma_{j+1} - \gamma_j < \frac{1}{2}H_j^{-1}$, *where* $H_j = h_j^{\nu_j}$ *and* $h_j = ht(\gamma_j)$, *and furthermore, we have* $|\gamma_j - \beta| > B^{-1}$ *for all algebraic numbers* $\beta$ *with degree* $d \leq j$ *distinct from* $\gamma_1, \cdots, \gamma_j$, *where* $B = \lambda_d^{-1} b^{(3d)^4}$ *and* $b$ *denotes the height of* $\beta$.

(See [Ba], p. 85-94.)

Using Theorem S we define the function $S^*$ from $\mathbb{A}^{\mathbb{N}} \times \mathbb{P}^{\mathbb{N}}$ to $\mathbb{A}^{\mathbb{N}} \times (0,1)^{\mathbb{N}}$ as follows: $S^*((\theta_1, \theta_2, \cdots), (\nu_1, \nu_2, \cdots)) = ((\gamma_1, \gamma_2, \cdots), (\lambda_1, \lambda_2, \cdots))$, where for each $n$, $S_n((\theta_1, \cdots, \theta_n), (\nu_1, \cdots, \nu_n)) = ((\gamma_1, \cdots, \gamma_n), (\lambda_1, \cdots, \lambda_n))$. $S^*$ is well-defined by Theorem S (i).

**Proof of Theorem 2.5** Let $\alpha \in (2^{\mathbb{N}})^{\mathbb{N}}$. Fix a bijection $<,>$ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For each $d, k \in \mathbb{N}$, define

$$\nu_{<d,k>} = (\nu(\alpha_d)(k) + 1)(3d)^5 \text{ and } \theta_{<d,k>} = \theta_{d,k},$$

where the function $\nu$ is constructed in Lemma 2.3. Put $\mathbb{A} = \{\theta_{d,k}\}$ and $\deg(\theta_{d,k}) = d$. Say $S^*((\theta_1, \theta_2, \cdots), (\nu_1, \nu_2, \cdots)) = ((\gamma_1, \gamma_2, \cdots), (\lambda_1, \lambda_2, \cdots))$. Then by Theorem S (ii), $\gamma_1, \gamma_2, \cdots$ tends to a limit $\xi$ which is a real number and satisfies

$$(2.1) \qquad |\xi - \beta| \geq B^{-1} \text{ for all algebraic numbers } \beta \text{ distinct from } \gamma_1, \gamma_2, \cdots,$$

and also

$$(2.2) \qquad \frac{1}{4}H_j^{-1} \leq \xi - \gamma_j \leq H_j^{-1} \text{ for all } j.$$

Define

$$f(\alpha) = \lim_{j \to \infty} \gamma_j = \xi.$$

**Claim.** $f$ *is continuous from* $(2^{\mathbb{N}})^{\mathbb{N}}$ *to* $\mathbb{C}$.

**Proof of the Claim.** Suppose $(\alpha_d^{(m)}) \to (\alpha_d)$ as $m \to \infty$, where for each $m$, $(\alpha_d^{(m)}) \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $(\alpha_d) \in (2^{\mathbb{N}})^{\mathbb{N}}$. Say for each $m$,

$$f((\alpha_d^{(m)})) = \xi_m = \lim_{k\to\infty} \gamma_k^{(m)} \text{ and } f((\alpha_d)) = \xi = \lim_{k\to\infty} \gamma_k,$$

where for each $k \in \mathbb{N}$, $\gamma_k^{(m)}$ and $\gamma_k$ are defined by $S^*$, according to $(\alpha_d^{(m)})$ and $(\alpha_d)$. Let $\epsilon > 0$. Choose $a_0$ such that $\frac{1}{2^{a_0-2}} < \epsilon$. Since $(\alpha_d^{(m)})$ goes to $(\alpha_{(d)})$ as $m \to \infty$, by the definition of $\gamma_k^{(m)}$ and $\gamma_k$, we may find $N_0 \in \mathbb{N}$ such that $|\gamma_{a_0}^{(m)} - \gamma_{a_0}| = 0$ for all $m \geq N_0$. Then for all $m \geq N_0$, we have the following inequality:

$$|\xi_m - \xi| \leq |\xi_m - \gamma_{a_0}^{(m)}| + |\gamma_{a_0}^{(m)} - \gamma_{a_0}| + |\gamma_{a_0} - \xi| < \frac{1}{2^{a_0-2}} < \epsilon,$$

since from (2.2) and Theorem S (ii), $|\xi_m - \gamma_a^{(m)}| \leq (H_a^{(m)})^{-1} < \frac{1}{2^{a-1}}(H_1^{(m)})^{-1} \leq \frac{1}{2^{a-1}}$ and $|\xi - \gamma_a| \leq H_a^{-1} < \frac{1}{2^{a-1}} H_1^{-1} \leq \frac{1}{2^{a-1}}$ for all $a \geq 1$. So $f$ is a continuous function. □

Now we show the main part of the theorem. Depending on the properties of $\nu$, Theorem S guarantees that we produce a $T$ number or $U$ number. So we divide the following two cases so that one can have more intuitive ideas.

Case 1. $\alpha = (\alpha_d) \notin P_3$, i.e., $\exists d \in \mathbb{N}\ (\alpha_d \notin Q)$.

Fix such $d$, i.e., $\alpha_d \notin \mathbb{Q}$. Then by Lemma 2.3, we have $\lim_{k\to\infty} (\nu(\alpha_d)(k)+1) = \infty$. It is clear that for all $k$, $h = h_{<d,k>}$,

$$h^{-d\omega_d^*(\xi,h)-1} \leq |\xi - \gamma_{<d,k>}| \leq h^{-\nu_{<d,k>}} \text{ from (2.2) and the definition of } \omega_d^*(\xi, h),$$

where $f(\alpha) = \xi$. So $d\omega_d^*(\xi, h_{<d,k>}) \geq \nu_{<d,k>} - 1$, i.e.,

$$\text{(2.3)} \qquad \omega_d^*(\xi, h_{<d,k>}) \geq \frac{\nu_{<d,k>} - 1}{d} \geq (\nu(\alpha_d)(k) + 1)3^5 d^4 - \frac{1}{d} \text{ for all } k.$$

It is easy to see that $\limsup_{k\to\infty} h_{<d,k>} = \infty$, since the right side of (2.3) goes to infinity as $k \to \infty$. This shows that we may choose $\{k_m\}$ such that $k_m \to \infty$ and $h_{<d,k_m>} \to \infty$ as $m \to \infty$. From (2.3), we get the following inequality:

$$\begin{aligned}\omega_d^*(\xi) &= \limsup_{h\to\infty} \omega_d^*(\xi, h) \geq \limsup_{m\to\infty} \omega_d^*(\xi, h_{<d,k_m>}) \\ &\geq \lim_{m\to\infty} (\nu(\alpha_d)(k_m) + 1)3^5 d^4 - \frac{1}{d} = \infty.\end{aligned}$$

Therefore, $\omega_d^*(\xi) = \infty$ and $f(\alpha) = \xi \in U$. So we derive $\alpha \notin P_3 \Longrightarrow f(\alpha) = \xi \in U$.

Case 2. $\alpha = (\alpha_d) \in P_3$ i.e. $\forall d \in \mathbb{N}\ (\alpha_d \in Q)$.

Fix $d \in \mathbb{N}$. Then for all $h, k, m$, we have

$$
\begin{aligned}
\xi - \gamma_{<m,k>} &\geq \frac{1}{4} h_{<m,k>}^{-(\nu(\alpha_m)(k)+1)(3m)^5} \\
|\xi - \beta| &\geq \lambda_{\deg(\beta)} (\mathrm{ht}(\beta))^{-(3\deg(\beta))^4}
\end{aligned}
\tag{2.4}
$$

for all algebraic numbers $\beta$ distinct from $\gamma_1, \gamma_2, \cdots$ from (2.1) and (2.2), where $\xi$ is the image of $f$ of $\alpha$. In fact, all nonzero algebraic numbers appear in these two inequalities. Let $h$ be a given natural number. Then from (2.4) and the definition of $\omega_d^*(\xi, h)$, we have the following inequality:

$$
h^{-d\omega_d^*(\xi,h)} \geq \min\{\frac{1}{4} h^{-M_0(3d)^5}, \lambda(d) h^{-(3d)^4}\},
\tag{2.5}
$$

where $M_0 = \sup\{\nu(\alpha_s)(k) + 1 : s \leq d \text{ and } k < \infty\}$ and $\lambda(d) = \min\{\lambda_s : s \leq d\}$. Even if for $s \leq d$, there is no $k$ such that $h_{<s,k>} = h$, this inequality can be applied. The value $\lambda(d)$ is positive and $1 \leq M_0 < \infty$, since $\{\lambda_s : s \leq d\}$ is the finite set of positive values and by assumption and Lemma 2.3, $\forall d \in \mathbb{N}\ (\lim_{k\to\infty} \nu(\alpha_d)(k) < \infty)$. So from (2.5), we get

$$
\omega_d^*(\xi, h) \leq \max\{\frac{\log 4}{\log h} + 3^5 M_0 d^4, \frac{\log \frac{1}{\lambda(d)}}{d \log h} + 3^5 d^4\} < \infty
$$

and

$$
\omega_d^*(\xi) = \limsup_{h\to\infty} \omega_d^*(\xi, h) \leq \max\{3^5 M_0 d^4, 3^5 d^4\} = 3^5 M_0 d^4 < \infty.
$$

Hence we can see that the inequality

$$
\omega_d^*(\xi) = \limsup_{h\to\infty} \omega_d^*(\xi, h) < \infty
\tag{2.6}
$$

holds for all $d$. But for all $d, k$, we obtain

$$
\omega_d^*(\xi, h_{<d,k>}) \geq \frac{\nu_{<d,k>} - 1}{d} \geq (\nu(\alpha_d)(k) + 1) 3^5 d^4 - \frac{1}{d}.
$$

As in case 1, $\omega_d^*(\xi) \geq 3^5 d^4 M_1 - \frac{1}{d}$, where $M_1 = \lim_{k\to\infty} \nu(\alpha_s)(k) + 1 \geq 1$. Therefore,

$$
\omega_d^*(\xi) \geq (3d)^4 \text{ and } \omega^*(\xi) = \limsup_{d\to\infty} \omega_d^*(\xi) = \infty.
\tag{2.7}
$$

From (2.6) and (2.7), for all $d \in \mathbb{N}$, $\omega_d^*(\xi) < \infty$ and $\omega^*(\xi) = \infty$, i.e., $f(\alpha) = \xi \in T$. So we derive $\alpha \in P_3 \Longrightarrow f(\alpha) = \xi \in T$.

By case 1 and case 2, we obtain $\alpha \in P_3 \Longrightarrow f(\alpha) \in T$ and $\alpha \notin P_3 \Longrightarrow f(\alpha) \in U$. By definition of $T, U$, it is easy to see that they are disjiont. So the continuous function $f$ satisfies $P_3 = f^{-1}(T)$ and $\mathbb{C} - P_3 = f^{-1}(U)$. This fact implies that $T, U$ are $\mathbf{\Pi_3^0}$–hard, $\mathbf{\Sigma_3^0}$–complete, respectively, since by Lemma 2.4, $P_3$ is $\mathbf{\Pi_3^0}$–complete. We complete the proof of Theorem 2.5. $\square$

**Remark.** We conjecture that $S, T$ are $\mathbf{\Sigma_4^0}$–complete, $\mathbf{\Pi_4^0}$–complete, respectively.

# Chapter 3

# On the set of all continuous functions with uniformly convergent Fourier series

## 3.1 Introduction

There are many criteria for uniform convergence of a Fourier series on the unit circle. One can find those tests in [Zy]. In the present paper, we study $UCF$ from the point of view of Descriptive Set Theory. In [Ke1], it was conjectured that $UCF$ is properly $\mathbf{\Pi}^0_3$ ($\mathbf{\Pi}^0_3$ non $\mathbf{\Sigma}^0_3$). Several natural properly $\mathbf{\Pi}^0_3$ sets have been found. For example, the collection of reals that are normal or simply normal to base $n$ [KL]; $C^\infty(\mathbb{T})$, the class of infinitely differentiable functions (viewed as $2\pi$-periodic functions on $\mathbb{R}$), and $UC_X$, the class of convergent sequences in a separable Banach space $X$, are properly $\mathbf{\Pi}^0_3$ [Ke1]. It turns out that $UCF$ is properly $\mathbf{\Pi}^0_3$. We give two different proofs for it. [AK] Ajtai and Kechris have shown that $EC$, the set of all continuous functions with everywhere Fourier series convergent, is properly $CA$, i.e., coanalytic non Borel. We show that there is no $\mathbf{\Sigma}^0_3$ set $A$ such that $UCF \subseteq A \subseteq EC$. Hence any $\mathbf{\Sigma}^0_3$ set, which includes $UCF$, must contain a continuous function with Fourier series divergent.

## 3.2 Definitions and background

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of positive integers and $\mathbb{N}^\mathbb{N}$ the Polish space with the usual product topology taking $\mathbb{N}$ discrete. Let $X$ be a Polish space. A subset $A$ of $X$ is $CA$ if there is a Borel funtion from $\mathbb{N}^\mathbb{N}$ to $X$ such that the image of

$\mathbb{N}^{\mathbb{N}}$ of $f$ is $X - A$, i.e., $f(\mathbb{N}^{\mathbb{N}}) = X - A$. A $CA(\mathbf{\Pi}^0_3)$ subset $A$ of $X$ is called properly $CA(\mathbf{\Pi}^0_3)$ if for any $CA(\mathbf{\Pi}^0_3)$ subset $B$ of $\mathbb{N}^{\mathbb{N}}$, there is a Borel (continuous) function $f$ from $\mathbb{N}^{\mathbb{N}}$ to $X$ such that the preimage of $A$ of $f$ is $B$, i.e., $B = f^{-1}(A)$. From the definition it is easy to see that no properly $CA(\mathbf{\Pi}^0_3)$ set is Borel($\mathbf{\Sigma}^0_3$). In paticular, if $\mathbf{\Pi}^0_3$ subset $A$ of a Polish space is properly $\mathbf{\Pi}^0_3$ and the continuous preimage of a subset $B$ of a Polish space, then so is $B$.

Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{T}$ denote the unit circle and $I$, the unit interval. Let $E$ be $\mathbb{T}$ or $I$. We denote by $C(E)$ the Polish space of continuous functions on $E$ with the uniform metric

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in E\}.$$

$C(\mathbb{T})$ can also be considered as the space of all continuous $2\pi$-periodic functions on $\mathbb{R}$, viewing $\mathbb{T}$ as $\mathbb{R}/2\pi\mathbb{Z}$. Let $UC$ denote the set of all sequences of continuous functions on $I$ that are uniformly convergent, i.e.,

$$UC = \{(f_n) \in C(I)^{\mathbb{N}} : (f_n) \text{ converges uniformly}\}.$$

To each $f \in C(\mathbb{T})$, we associate its Fourier series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx},$$

where $\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(t)e^{-int}dt$. Let

$$S_n(f,t) = \sum_{k=-n}^{n} \hat{f}(n)e^{ikt}$$

be that $n$th partial sum of the Fourier series of $f$. We say the Fourier series of $f$ converges at a point $t \in \mathbb{T}$ if the sequence $(S_n(f,t))$ converges. Similarly, we define the uniform convergence of the Fourier series of $f$. Let $EC$ denote the set of all continuous functions with Fourier series convergent. According to a standard theorem [Kat], the Fourier series of $f$ at $t$ converges to $f(t)$ if it converges. Hence we have

$$\begin{aligned} EC &= \{f \in C(\mathbb{T}) : \forall\, t \in [0,2\pi] \left((S_n(f,t)) \text{ converges }\right)\} \\ &= \{f \in C(\mathbb{T}) : \forall\, t \in [0,2\pi] \left(f(t) = \lim_{n\to\infty} S_n(f,t)\right)\}. \end{aligned}$$

We define by $NF$ the complement of $EC$. Let $UCF$ denote the set all continuous functions with Fourier series uniformly convergent, i.e.,

$$UCF = \{f \in C(\mathbb{T}) : \text{ the Fourier series of } f \text{ converges uniformly}\}.$$

## 3.3 Results

**Theorem [AK]** *$EC$ is properly $CA$.*

(See [AK].)

**Proposition 3.1** *$UCF$ and $UC$ are* $\mathbf{\Pi}^0_3$.

**Proof of Proposition 3.1.** Let $\mathbb{Q}$ be the set of all rational numbers. We consider $\mathbb{T}$ as $[0, 2\pi]$ with identifying $0 = 2\pi$. By the definition of $UCF$,

$$\begin{aligned} f \in UCF &\iff S_N(f) \text{ converges uniformly} \\ &\iff \forall\, a \in \mathbb{N}\ \exists\, b \in \mathbb{N}\ \forall\, c, d \in \mathbb{N}\ \forall\, e \in \mathbb{Q} \\ &\qquad \left(|S_{b+c}(f, e) - S_{b+d}(f, e)| \leq \frac{1}{a}\right) \\ &\iff f \in \bigcap_{a \in \mathbb{N}} \bigcup_{b \in \mathbb{N}} \bigcap_{c,d \in \mathbb{N}} \bigcap_{e \in \mathbb{Q} \cap [0, 2\pi]} V(a, b, c, d, e), \end{aligned}$$

where $V(a, b, c, d, e)$ is the collection of $f \in C(\mathbb{T})$ such that $|S_{b+c}(f, e) - S_{b+d}(f, e)| \leq 1/a$, which is closed, since the function $f \longmapsto \hat{f}(n)$ is continuous. Hence $UFC$ is $\mathbf{\Pi}^0_3$. Similarly, so is $UC$, and we are done. □

**Lemma 3.2** *The set $C_3 = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \lim_{n \to \infty} \alpha(n) = \infty\}$ is properly* $\mathbf{\Pi}^0_3$.

(See [Ke1].)

This set will be used to prove our main theorem.

**Proposition 3.3** *$UC$ is properly* $\mathbf{\Pi}^0_3$.

**Proof.** We define the function $F$ from $\mathbb{N}^{\mathbb{N}}$ to $C(I)^{\mathbb{N}}$ as follows: for each $\beta \in \mathbb{N}^{\mathbb{N}}$,

$$F(\beta) = \left(\frac{1}{\beta(n)}\right).$$

Then it is easy to see that

$$\beta \in C_3 \iff F(\beta) \text{ converges} \iff F(\beta) \text{ converges uniformly,}$$

since $F(\beta)$ is a sequence of constant functions. Clearly, $F$ is continuous. Hence $UC$ is the continuous preimage of $C_3$. By Proposition 3.1 and Lemma 3.2, $UC$ is properly $\mathbf{\Pi}^0_3$. □

**Theorem 3.4** *There is a continuous function $H$ from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ such that*

$$\beta \in C_3 \text{ implies } H(\beta) \in UCF, \text{ and}$$
$$\beta \notin C_3 \text{ if and only if} H(\beta) \in NF.$$

*In particular, $UCF$ is properly $\mathbf{\Pi}^0_3$.*

By this theorem, we have the following corollary.

**Corollary 3.5** *There is no $\mathbf{\Sigma}^0_3$ set $A$ such that*

$$UCF \subseteq A \subseteq EC,$$

*i.e., any $\mathbf{\Sigma}^0_3$ set, which includes $UCF$, must contain a continuous function with Fourier series divergent.*

**Proof.** Suppose a $\mathbf{\Sigma}^0_3$ set $A$ satisfies $UCF \subseteq A \subseteq EC$. Then by Theorem 3.4 we obtain $H^{-1}(A) = C_3$. Since $A$ is $\mathbf{\Sigma}^0_3$, so is $C_3$. By Lemma 3.2, it contradicts our assumption. □

From a basic fact of Descriptive Set Theory [Kel], any Borel set is coanalytic. So by Theorem [AK], since $EC$ is properly $CA$, it is a very natural guess that the complement of $C_3$ can be reducible to $EC - UCF$. In fact, we have the following theorem.

**Theorem 3.6** *There is a continuous function $\widetilde{H}$ from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ such that*

$$\beta \in C_3 \text{ implies } \widetilde{H}(\beta) \in UCF, \text{ and}$$
$$\beta \notin C_3 \text{ implies } \widetilde{H}(\beta) \in EC - UCF.$$

*In particular, $UCF$ is properly $\mathbf{\Pi}^0_3$.*

In order to prove Theorem 3.4 and Theorem 3.6, we need the following criterion due to Dini-Lipschitz [Zy]. Let $f$ be defined in a closed interval $J$, and let

$$\omega(\delta) = \omega(\delta; f) = \sup\{|f(x) - f(y)| : x, y \in J \text{ and } |x - y| \leq \delta\}.$$

The function $\omega(\delta)$ is called the *modulus of continuity* of $f$.

**The Dini-Lipschitz test.** *If $f$ is continuous and its modulus of continuity $\omega(\delta)$ satisfies the condition $\omega(\delta)\log\delta \to 0$, then the Fourier series of $f$ converges uniformly.*

We introduce the Féjer polynomials, for given $0 < n < N \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\begin{aligned} Q(x, N, n) &= 2\sin Nx \sum_{k=1}^{n} \frac{\sin kx}{k} \\ R(x, N, n) &= 2\cos Nx \sum_{k=1}^{n} \frac{\sin kx}{k}. \end{aligned}$$

These two polynomials were used to prove that there exists a continuous function whose Fourier series diverges at a point.

**Lemma 3.7** *There are positive numbers $C_1, C_2 > 0$ such that*

$$|Q| < C_1 \text{ and } |R| < C_2,$$

*i.e., these polynomials are uniformly bounded in $x, N, n$.*

From Lemma 3.7, we immediately have the following.

**Proposition 3.8** *Let $(N_k)$ and $(n_k)$ be any two sequences of positive integers, with $n_k < N_k$ and let $\alpha_k$ be such that $\alpha_1 + \alpha_2 + \alpha_3 \cdots < \infty$. Then the series*

$$\sum \alpha_k Q(x, N_k, n_k) \text{ and } \sum \alpha_k R(x, N_k, n_k)$$

*converge to continuous functions.*

**Proof of Theorem 3.4** Let $\alpha_k = 2^{-k}$, $n_k = 1/2N_k = 2^{2^k}$ $(k = 1, 2, 3, \cdots)$. We define $H$ from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ as follows: for all $\beta \in \mathbb{N}^{\mathbb{N}}$,

$$H(\beta) = \sum \alpha_k \frac{1}{\beta(k)} Q(x, N_k, n_k).$$

**Claim 1** *H is continuous and well-defined.*

**Proof of Claim 1** By Proposition 3.8, $H$ is well-defined. By Lemma 3.7, it is easy to see that $H$ is continuous. □

We divide the rest of proof in two parts.

**Case 1** $\lim_{n\to\infty}\beta(n)\neq\infty$.

We want to show that $H(\beta)\in NF$. For each $k\in\mathbb{N}$,

$$
\begin{aligned}
&\left|S_{N_k+n_k}(H(\beta),0)-S_{N_k}(H(\beta),0)\right| = \left|\sum_{|l|\le N_k+n_k}\widehat{H(\beta)}(l)-\sum_{|l|\le N_k}\widehat{H(\beta)}(l)\right| \\
(3.1)\qquad &= \alpha_k\frac{1}{\beta(k)}\left(1+\frac{1}{2}+\cdots+\frac{1}{n_k}\right) > \alpha_k\frac{1}{\beta(k)}\log n_k = 2^{-k}\frac{1}{\beta(k)}\log 2^{2^k} \\
&= \frac{1}{\beta(k)}\log 2
\end{aligned}
$$

holds. Since $\lim_{n\to\infty}\beta(n)\neq\infty$, there exists a $p\in\mathbb{N}$ such that for infinitely many $k$'s, $\beta(k)=p$. Hence the Fourier series of $H(\beta)$ does not converge, since in (3.1), we have $1/p\log 2$ for infinitely many $k$'s. Thus $H(\beta)\in NF$.

**Case 2** $\lim_{n\to\infty}\beta(n)=\infty$.

We show that $H(\beta)\in UCF$. We will demonstrate that $\omega(\delta;H(\beta))\log\delta\to 0$ as $\delta\to 0$. Then by the Dini-Lipschitz test, this shows that the Fourier series of $H(\beta)$ converges uniformly. We take any $0<\delta\le 1/2$ and define $\nu=\nu(\delta)$ as the largest integer $k$ satifying $2^{2^k}\le 1/\delta$. By Lemma 5, we have the following inequality:

$$
\begin{aligned}
&\left|\sum_{k=\nu+1}^{\infty}\alpha_k\frac{1}{\beta(k)}Q(x+\delta,N_k,n_k)-\sum_{k=\nu+1}^{\infty}\alpha_k\frac{1}{\beta(k)}Q(x,N_k,n_k)\right| \\
(3.2)\qquad &\le 2C\sum_{k=\nu+1}^{\infty}\alpha_k\frac{1}{\beta(k)}\le 2C\sup\{\frac{1}{\beta(k)}:k>\nu\}\sum_{k=\nu+1}^{\infty}\alpha_k \\
&= 4C\sup\{\frac{1}{\beta(k)}:k>\nu\}2^{-\nu-1}=4C\sup\{\frac{1}{\beta(k)}:k>\nu\}\frac{\log 2}{|\log\delta|}.
\end{aligned}
$$

Now we calculate the rest of $H(\beta)$. We clearly have

$$
Q'(x,N,n)=NR(x,N,n)+2\sin Nx\sum_{k=1}^{n}\cos kx,\ |Q'|\le NC+2n=nC',
$$

for $N = 2n$ and $C' = 2C + 2$. By the mean value theorem, we have the following inequality:

$$\begin{aligned}
&\left|\sum_{k\le\nu}\alpha_k\frac{1}{\beta(k)}Q(x+\delta,N_k,n_k)-\sum_{k\le\nu}\alpha_k\frac{1}{\beta(k)}Q(x,N_k,n_k)\right|\\
(3.3)\qquad &\le C'\delta\left(2^{-1}2^{2^1}\frac{1}{\beta(3.1)}+\cdots+2^{-\nu}2^{2^\nu}\frac{1}{\beta(\nu)}\right)\\
&\le C'2^{2^{-\nu}}\sum_{k\le\nu}2^{-k}2^{2^k}\frac{1}{\beta(k)}\le C'\frac{1}{|\log\delta|}2^{2^{-\nu}}\sum_{k\le\nu}2^{2^k-k}\frac{1}{\beta(k)}.
\end{aligned}$$

By (3.2) and (3.3), we have the following:

$$(3.4)\qquad \left|\omega(\delta;H(\beta))\log\delta\right|\le\max\{4C\sup\{\frac{1}{\beta(k)}:k>\nu\},C'2^{2^{-\nu}}\sum_{k\le\nu}2^{2^k-k}\frac{1}{\beta(k)}\}.$$

Now if $\delta\to0$, then $\nu\to\infty$. So it suffices to show that the right part of (3.4) goes to 0 as $\nu\to\infty$. Since $\beta(\nu)\to\infty$ as $\nu\to\infty$, $\sup\{1/\beta(k):k>\nu\}$ goes to 0. We need to show that the rest goes to zero as $\nu$ diverges to infinity. It requires the following easy fact.

**Claim 2** $\sum_{k\le\nu}2^{2^k-k}\le2^{2^\nu-\nu+4}$.

**Proof of Claim 2** Use induction on $\nu$. For $\nu=1$, $2^{2-1}=2\le2^{2-1+4}=2^5$. Suppose it is true for $\nu$. By the induction assumption, $\sum_{k\le\nu}2^{2^k-k}+2^{2^{\nu+1}-(\nu+1)}\le 2^{2^\nu-\nu+4}+2^{2^{\nu+1}-(\nu+1)}$. It is enough to show that $2^{2^\nu-\nu+4}+2^{2^{\nu+1}-(\nu+1)}\le2^{2^{\nu+1}+4}$. Letting $\theta=2^{2^\nu}$, one can verify this inequality. □

Fix $\epsilon$. Take $N_0$ such that $1/\beta(k)<\epsilon$ for all $k\ge N_0$. For this $N_0$, we choose $N>N_0$ so that $2^{-2^\nu+\nu}\sum_{k\le N_0}2^{2^k-k}<\epsilon$ for all $\nu\ge N$. Then for all $\nu\ge N$, by claim 2, the following inequality is valid:

$$\begin{aligned}
&2C'2^{-2^\nu}2^\nu\sum_{k\le\nu}2^{2^k-k}\\
&<2C'\left(2^{-2^\nu+\nu}\sum_{k\le N_0}2^{2^k-k}\frac{1}{\beta(k)}+2^{-2^\nu+\nu}\sum_{N_0<k\le\nu}2^{2^k-k}\frac{1}{\beta(k)}\right)\\
&<2C'\left(\epsilon+2^{-2^\nu+\nu}\epsilon\sum_{N_0<k\le\nu}2^{2^k-k}\right)<2\epsilon C'\left(1+\frac{2^{2^\nu-\nu+4}}{2^{2^\nu-\nu}}\right)\\
&=34\epsilon.
\end{aligned}$$

Hence the right side of (3.4) converges to zero as $\nu$ goes to the infinity, i.e., as $\delta \to 0$. So we derive $H(\beta) \in UCF$.

By cases, we obtain

$$\beta \notin C_3 \Rightarrow \; H(\beta) \in NF, \text{ and}$$
$$\beta \in C_3 \Rightarrow \; H(\beta) \in UCF$$

respectively. We have shown the first part. In particular, $C_3$ is the preimage of $UCF$. Hence by Lemma 3.2, the second assertion follows. We have thus completed the proof of Theorem 3.4. □

**Proof of Theorem 3.6** Instead of $Q$, we use $R$. As in the proof of Theorem 3.4, we define $\widetilde{H}$ from $\mathbb{N}^{\mathbb{N}}$ to $C(\mathbb{T})$ as follows: for each $\beta \in \mathbb{N}^{\mathbb{N}}$,

$$\widetilde{H}(\beta) = \sum \alpha_k \frac{1}{\beta(k)} R(x, N_k, n_k).$$

The same proof as before will demonstrate that this function is continuous, well-defined and if $\lim_{n\to\infty} \beta(n) = \infty$, then the Fourier series of $\widetilde{H}(\beta)$ converges uniformly. So it suffices to show that if $\lim_{n\to\infty} \beta(n) \neq \infty$, then $\widetilde{H}(\beta) \in EC - UCF$. Suppose $\lim_{n\to\infty} \beta(n) \neq \infty$. The representation of $\widetilde{H}(\beta)$ as Fourier series is $\sum a_v \sin vx$. We see that $\sum a_v \sin vx$ converges uniformly for $\delta \leq |x| \leq \pi$ for any $\delta > 0$, since the partial sums of $R(x, N_k, n_k)$ are uniformly bounded in $k$ and $x$, $\delta \leq |x| \leq \pi$. The series $\sum a_v \sin vx$ contains sines only, and hence it converges for $x = 0$, and so everywhere. Now we will show that $\sum a_v \sin vx$ does not converge uniformly. It is easy to see that

$$\sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx = \sum_{v=2n_k+1}^{3n_k} a_v \sin vx = 2^{-k} \frac{1}{\beta(k)} \sum_{v=1}^{n_k} \frac{\sin(2n_k + v)x}{v}.$$

So if we let $x = \dfrac{\pi}{4n_k}$, then we have

$$\left| \sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx \right| = 2^{-k} \frac{1}{\beta(k)} \sum_{v=1}^{n_k} \frac{\sin(2n_k + v)x}{v} \geq 2^{-k} \frac{1}{\beta(k)} \sin \frac{\pi}{4} \sum_{v=1}^{n_k} \frac{1}{v},$$

since for all $1 \leq v \leq n_k$, $\frac{3}{4}\pi \geq \frac{\pi}{4n_k}(2n_k + v) \geq \frac{\pi}{2}$. So finally,

$$
\begin{aligned}
\left|\sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx\right| &\geq 2^{-k}\frac{1}{\beta(k)} \sin\frac{\pi}{4}\sum_{v=1}^{n_k}\frac{1}{v} \\
&\geq 2^{-k}\frac{1}{\beta(k)} \log n_k \sin\frac{\pi}{4} \\
&= \frac{\log 2}{\sqrt{2}}\frac{1}{\beta(k)}.
\end{aligned} \tag{3.5}
$$

Hence $\sum a_v \sin vx$ does not converge uniformly, since in (3.5), the same value appears for infinitely many $k$'s. Hence as in the proof of Theorem 3.4, we finish the proof of Theorem 3.6. □

# Chapter 4

# Complete coanalytic sets

## 4.1 Introduction

The following are some examples of $\mathbf{\Pi}^1_1$-complete sets in Analysis and Topology. The set of all differentiable functions on the unit interval [Maz]; the set of all nowhere dense differentiable functions on the unit interval [Mau]; the set of all sequences $\langle f_n \rangle_{n\in\mathbb{N}}$ of continuous functions on the unit interval such that $\langle f_n \rangle_{n\in\mathbb{N}}$ converges pointwise; the set of all continuous functions $f$ on the unit interval such that the Fourier series of $f$ converges everwhere [AK]; the set of all compact countable subsets of an uncountable Polish space [Hu]; the set of all compact subsets $K$ of $\mathbb{R}^2$ such that $K$ is simply connected [Be]; the set of all compact subsets $K$ of the unit interval such that $K$ is a set of uniqueness [Kau] and Solovay (unpublished), are all $\mathbf{\Pi}^1_1$-complete. In this paper, we give some natural complete $\mathbf{\Pi}^1_1$ sets occurring in Number Theory and in the study of countable Borel equivalence relations.

A set of real numbers $M$ is called a normal set if there exists a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ of real numbers such that for all $y \in \mathbb{R}$, $y \in M$ if and only if $\langle yx_n \rangle_{n\in\mathbb{N}}$ is uniformly distributed mod 1. Rauzy [Raz] found a sufficient and necessary condition for a set to be normal. By a theorem of Hahn [Kel], we can get a more exact necessary and sufficient condition. Namely, for a given subset $M$ of real numbers, $M$ is normal if and only if for all nonzero integer $q$, $qB \subset B$, $0 \notin B$ and $B$ is $F_{\sigma\delta}$. A sequence of real numbers is universal if and only if $\langle x_n \rangle_{n\in\mathbb{N}}$, i.e., for all nonzero real numbers $y$, $\langle yx_n \rangle_{n\in\mathbb{N}}$ is uniformly distributed mod 1. Using results of [Rau] we show that the set $US$, of universal sequence of real numbers, is $\mathbf{\Pi}^1_1$-complete. A Borel equivalence relation on a Polish space $X$ is countable if all its equivalence classes are countable.

For a given countable Borel equivalence relation $E$ on $2^{\mathbb{N}}$, we denote by $\mathcal{A}(E)$ ($\mathcal{F}(E)$) the set of all closed sets $K$ such that $E \cap (K \times K)$ is aperiodic (finite), i.e., for all $x \in K$, the equivalence class of $x$ is infinite (finite) in $K$. In many cases, we show that $\mathcal{A}(E)$ and $\mathcal{F}(E)$ are $\mathbf{\Pi}^1_1$-complete. We also prove that $US$ is $\mathbf{\Pi}^1_1$-complete.

## 4.2 Notations and background

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of natural numbers, integers, rational numbers and real numbers. For such a space $X$ and a $Y$, $X^Y$ denotes the set of all functions $f$ from $Y$ to $X$, with the usual product topology, $X$ being endowed with its usual topologies ($2 = \{0, 1\}$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$ being discrete). For given set $X$, $X^{\times}$ is the subset of $X$ without zero, i.e., $X^{\times} = X - \{0\}$. Let $\mathbb{N}^{-1} = \{0\} \cup \{\frac{1}{n+1}\}_{n \in \mathbb{N}}$. We consider the space $\mathbb{N}^{-1}$ as the subspace of $\mathbb{R}$. We denote by $\mathbb{N}^{<\mathbb{N}}$ the set of all finite sequences of natural numbers. We consider the space $2^{\mathbb{N}^{<\mathbb{N}}}$ with the usual product topology with $\mathbb{N}^{<\mathbb{N}}$ being discrete. For $s \in \mathbb{N}^{<\mathbb{N}}$, $lh(s)$ is the length of $s$. Let $s, t \in \mathbb{N}^{<\mathbb{N}}$. We say $t \subset s$ if there is $k \leq lh(s)$ such that $t = s \upharpoonright k$. We denote by $s\hat{\ }t$ the concatenation of $s$ and $t$, i.e., $lh(s\hat{\ }t) = lh(s) + lh(t)$ and $s\hat{\ }t(i) = s(i)$ if $i < lh(s)$, and $s\hat{\ }t(j + lh(s)) = t(j)$ if $j < lh(t)$. We put for each $n \in \mathbb{N}$ and $s \in \mathbb{N}^{<\mathbb{N}}$, $s\hat{\ }\emptyset = s$ and $s\hat{\ }\langle n \rangle = s\hat{\ }n$. For each $n \in \mathbb{N}$ and $s_0, s_1, \cdots, s_{n+1} \in \mathbb{N}^{<\mathbb{N}}$, we inductively define $s_0\hat{\ }s_1\hat{\ }\cdots\hat{\ }s_{n+1} = (s_0\hat{\ }s_1\hat{\ }\cdots\hat{\ }s_n)\hat{\ }s_{n+1}$. For infinitely many nonempty finite sequences $s_0, s_1, s_2 \cdots \in \mathbb{N}^{<\mathbb{N}}$, we may similarly consider $s_0\hat{\ }s_1\hat{\ }s_2\hat{\ }\cdots$ as an element in $\mathbb{N}^{\mathbb{N}}$. Let $T \subset \mathbb{N}^{<\mathbb{N}}$. $T$ is called a tree if $T$ is nonempty and for all $s \in \mathbb{N}^{<\mathbb{N}}$, $s \in T \Rightarrow \forall t \subset s\ (t \in T)$. We denote by $Tr$ the set of all trees on $\mathbb{N}$. We may think that any tree $T$ on $\mathbb{N}$ is an element of $2^{\mathbb{N}^{<\mathbb{N}}}$, i.e., $Tr \subset 2^{\mathbb{N}^{<\mathbb{N}}}$. Then we see that $Tr$ is a closed subset of $2^{\mathbb{N}^{<\mathbb{N}}}$, so a Polish space. $T$ is called a wellfounded tree if for all $\alpha \in \mathbb{N}^{\mathbb{N}}$ there is $n \in \mathbb{N}$ such that $\alpha \upharpoonright n \notin T$. We denote by $WF$ the set of all wellfounded trees on $\mathbb{N}$. Let $X$ and $Y$ be sets. Let $x \in X$, $B \subset Y$ and $C \subset X \times Y$. Put

$$C_x = \{y \in Y : (x, y) \in C\} \text{ and } \forall^B C = \{x \in X : \forall y \in B\ \big((x, y) \in C\big)\}.$$

Let $X$ be a Polish space. Let $A$ be a subset of $X$. $A$ is called a $\mathbf{\Pi}^1_1$ subset if there is a Borel set $C$ of $X \times 2^{\mathbb{N}}$ such that $A = \forall^{2^{\mathbb{N}}} C$. Equivalently, there exists a Borel function from $2^{\mathbb{N}}$ to $X$ such that $f(2^{\mathbb{N}}) = X - A$. Thus $\mathbf{\Pi}^1_1$ is coanalytic. Note that for any Polish space $Y$, any Borel subset $A$ of $Y$ and any Borel subset $B$ of $X \times Y$, $\forall^A B$ is $\mathbf{\Pi}^1_1$. Hence any Borel set is $\mathbf{\Pi}^1_1$. $A$ is called $\mathbf{\Pi}^1_1$-hard if for any Polish space $Y$ and any $\mathbf{\Pi}^1_1$ subset $B$ of $Y$ there exists a Borel function $f$ from $Y$ to $X$ such that $f^{-1}(A) = B$. If in addition to being $\mathbf{\Pi}^1_1$-hard, $A$ is also $\mathbf{\Pi}^1_1$, then we say $A$ is $\mathbf{\Pi}^1_1$-complete. Note that any $\mathbf{\Pi}^1_1$-hard set is non Borel.

For a given set $C \subseteq X$, in order to calculate the exact complexity of $C$, one must first calculate an upper bound for $C$, by showing for example that $C$ is $\mathbf{\Pi}^1_1$. And then prove a lower bound for $C$ , for example by showing that $C$ is $\mathbf{\Pi}^1_1$-hard. Usually, finding the upperbound is fairly easy. However, it can be difficult to prove the hardness of $C$. Since the $\mathbf{\Pi}^1_1$ classes are closed under preimages of Borel functions, if $B$ is $\mathbf{\Pi}^1_1$-hard ($\mathbf{\Pi}^1_1$-complete) and $B = f^{-1}(C)$, where $f$ is a Borel function, then $C$ is $\mathbf{\Pi}^1_1$-hard ($\mathbf{\Pi}^1_1$-complete, if also $C \in \mathbf{\Pi}^1_1$). This remark is the basis of a common method for showing that a given set $B$ is $\mathbf{\Pi}^1_1$-hard: Choose an already known $\mathbf{\Pi}^1_1$-hard set $B$ and show that there is a Borel function $f$ such that $B = f^{-1}(C)$.

By a standard theorem in [Kel] or [Mos], $WF$ is $\mathbf{\Pi}^1_1$-complete. We will use $WF$ to prove the last theorem.

## 4.3 The set of all universal sequences of real numbers

We introduce a coanalytic set from Number Theory. Let $\langle x_n \rangle_{n \in \mathbb{N}}$ be a given sequence of real numbers. For a positive integer $N$ and a subset $T$ of the unit interval, let the counting function $A\big(T; N; \langle x_n \rangle_{n \in \mathbb{N}}\big)$ be defined as the number of terms $x_n$, $1 \le n \le N$, for which the fractional part of $x_n$ is in $T$. The sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ of real numbers is said to be uniformly distributed modulo 1 if for every pair $a, b$ of real numbers with $0 \le a < b \le 1$, we have

$$\lim_{N \to \infty} \frac{A\big([a, b); N; \langle x_n \rangle_{n \in \mathbb{N}}\big)}{N} = b - a.$$

A set $M$ of real numbers is called a normal set if there exists a sequence $\langle \lambda_n \rangle_{n \in \mathbb{N}}$ in $\mathbb{R}$ such that $\langle x\lambda_n \rangle_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 if and only if $x \in M$.

We denote by $\mathcal{U}$ the set of all pairs $(\langle\lambda_n\rangle_{n\in\mathbb{N}}, x)$ of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}$ such that $\langle x\lambda_n\rangle_{n\in\mathbb{N}}$ is uniformly distributed modulo 1, i.e.,

$$\mathcal{U} = \{(\langle\lambda_n\rangle_{n\in\mathbb{N}}, x) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R} : \langle x\lambda_n\rangle_{n\in\mathbb{N}} \text{ is uniformly distributed mod } 1\}.$$

Thus for each $\langle\lambda_n\rangle_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $\mathcal{U}_{\langle\lambda_n\rangle_{n\in\mathbb{N}}}$ is a normal set. Also for each normal set $M$, there exists $\langle\lambda_n\rangle_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $M = \mathcal{U}_{\langle\lambda_n\rangle_{n\in\mathbb{N}}}$. We call $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ a universal sequence if for every nonzero real $x$, $\langle x\lambda_n\rangle_{n\in\mathbb{N}}$ is uniformly distributed mod 1. Hence for each $\langle\lambda_n\rangle_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ is a universal sequence iff $\mathcal{U}_{\langle\lambda_n\rangle_{n\in\mathbb{N}}} = \mathbb{R}^{\times}$. We denote by $US$ the set of all universal sequences, i.e.,

$$US = \{\langle\lambda_n\rangle_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \ \forall x \in \mathbb{R}^{\times}(\langle x\lambda_n\rangle_{n\in\mathbb{N}} \text{ is uniformly distributed mod } 1)\}.$$

Thus $US = \forall^{\mathbb{R}^{\times}}\mathcal{U}$.

Let $X$ be a Polish space. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of continuous functions from $X$ to $\mathbb{R}$. We denote by $C(\langle f_n\rangle_{n\in\mathbb{N}})$ the set of all real numbers $x$ such that $f_n(x)$ converges to zero as $n \to \infty$, i.e.,

$$C(\langle f_n\rangle_{n\in\mathbb{N}}) = \{x \in \mathbb{R} : f_n(x) \to 0 \text{ as } n \to \infty\}.$$

A continuous function $f : \mathbb{R} \to \mathbb{C}$ is said continuously (c.d.p.) definited positive if for all finite sets $A$ of real numbers and all functions $c$ from $A$ to $\mathbb{C}$, we have the inequality $\sum_{(x,y)\in A\times A} f(x-y)c(x)\bar{c}(y) \geq 0$. We recall $\mathcal{L}$ the set of c.d.p. functions, $\mathcal{L}_0$ the set of c.d.p. functions such that $f(0) = 1$, and $\mathcal{L}_0^+$ the set of $f \in \mathcal{L}_0$ such that $\forall x \in \mathbb{R}$ , $f(x) \geq 0$. Note that if $f \in \mathcal{L}_0$, we have $\overline{f(x)} = f(-x)$ and $f(x) \leq 1$; in particular if $f \in \mathcal{L}_0^+$, $f$ is even and $\forall x \in \mathbb{R}$ , $0 \leq f(x) \leq 1$.

Let $\nu$ be a function from $\mathbb{N}$ to $\mathbb{N}$. Let $\{w_{i,j} : i,j \in \mathbb{N}\}$ be a family. Then $\langle u_k\rangle_{k\in\mathbb{N}} = \langle w_{n,0}, \cdots, w_{n,\nu(n)}\rangle_{n\in\mathbb{N}}$ means that for each $j \leq \nu(0)$, $u_j = w_{0,j}$ and for each $i \geq 1$, $j \leq \nu(i)$, $u_{\nu(0)+\cdots+\nu(i-1)+j} = w_{i,j}$. We call by $\langle u_k\rangle_{k\in\mathbb{N}}$ the composition of $\langle w_{n,0}, \cdots, w_{n,\nu(n)}\rangle_{n\in\mathbb{N}}$.

**Theorem H**[Hahn] *Let $X$ be Polish.*

*A subset $A \subseteq X$ is $F_{\sigma\delta}$ iff there exists $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of continuous functions from $X$ to $\mathbb{R}$ such that $A = C(\langle f_n\rangle_{n\in\mathbb{N}})$.*

(See [Kel].)

We recall now a famous theorem of Weyl [We].

**Theorem W** *The sequence $\langle x_n \rangle$ is uniformly distributed mod 1 iff*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0$$

*for all integers $h \neq 0$.*

We use Theorem W to prove that $US$ is $\mathbf{\Pi}^1_1$. By Theorem W, we obtain the following:

$$\begin{aligned} (\langle \lambda_n \rangle_{n\in\mathbb{N}}, x) \in \mathcal{U} &\iff \langle x\lambda_n \rangle_{n\in\mathbb{N}} \text{ is uniformly distributed mod 1} \\ &\iff \forall a \in \mathbb{Z}^{\times} \left( \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i a x \lambda_n} = 0 \right) \\ &\iff \forall a \in \mathbb{Z}^{\times}\ \forall b \in \mathbb{N}\ \exists c \in \mathbb{N}\ \forall d \in \mathbb{N} \\ &\iff \left| \frac{1}{c+d} \sum_{n=1}^{c+d} e^{2\pi i a x \lambda_n} \right| \leq \frac{1}{b+1} \\ &\iff \bigcap_{a\in\mathbb{Z}^{\times}} \bigcap_{b\in\mathbb{N}} \bigcup_{c\in\mathbb{N}} \bigcap_{d\in\mathbb{N}} V_{a,b,c,d}, \end{aligned}$$

where $V_{a,b,c,d}$ is the set of all elements $(\langle \lambda_n \rangle, x)$ such that $\left| \frac{1}{c+d} \sum_{n=1}^{c+d} e^{2\pi i a x \lambda_n} \right| \leq \frac{1}{b+1}$. Clearly $V_{a,b,c,d}$ is a closed subset of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}$. So $\mathcal{U}$ is a $F_{\sigma\delta}$ set, i.e., a Borel set. Hence $US$ is $\mathbf{\Pi}^1_1$, for $US = \forall^{\mathbb{R}^{\times}} \mathcal{U}$ and $\mathcal{U}$ is a Borel set.

By a theorem of Hahn [Kel], we immediately obtain the following reformulation theorem of Rauzy [Raz].

**Theorem R**[Rauzy] *For given $B \subseteq \mathbb{R}$, $B$ is normal iff $\forall q \in \mathbb{Z}^{\times}$ $(qB \subseteq B)$, $0 \notin B$ and $B$ is $F_{\sigma\delta}$.*

**Lemma 4.1** *There exists $F_{\sigma\delta}$ set $B \subseteq \mathbb{R} \times \mathbb{R}$ such that $\forall^{\mathbb{R}^{\times}} B$ is $\mathbf{\Pi}^1_1$-complete, $(x, y) \in B \iff (-x, y) \in B$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $\forall x$ $(B_x$ is normal).*

**Proof of Lemma 4.1** We can choose a $F_{\sigma\delta}$ subset $C$ of $\mathbb{R} \times \mathbb{R}$ such that $\forall x \in \mathbb{R}$ $((x, 0) \notin C)$, $(x, y) \in B \iff (-x, y) \in B$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $\forall^{\mathbb{R}^{\times}} C$ is

$\mathbf{\Pi}^1_1$-complete. Let

$$B = \{(x,y) \in C : \forall q \in \mathbb{Z}^\times \ \big((x,qy) \in C\big)\}.$$

Then $B$ is also a $F_{\sigma\delta}$ set. Clearly for given $x \in \mathbb{R}$, $B_x$ is $F_{\sigma\delta}$. Let $(x,y) \in B$ and $q \in \mathbb{Z}^\times$. Then by the definition of $B$, $(x,iqy) \in C$ for all $i \in \mathbb{Z}^\times$. Hence by the definition of $B$, we obtain $(x,qy) \in B$. We thus have $qB_x \subset B_x$ for all $q \in \mathbb{Z}^\times$. Since for all $x \in \mathbb{R}$, $\forall q \in \mathbb{Z}^\times \ \big(qB_x \subseteq B_x\big)$, $B_x$ is $F_{\sigma\delta}$ and $0 \notin B_x$, by Theorem R, $B_x$ is normal. Clearly $\forall^{\mathbb{R}^\times} B \subseteq \forall^{\mathbb{R}^\times} C$. Suppose $x \in \forall^{\mathbb{R}^\times} C$. Then $\forall y \in \mathbb{R}^\times \ \big((x,y) \in C\big)$. Thus $\forall y \in \mathbb{R}^\times \ \forall q \in \mathbb{Z}^\times \ \big((x,qy) \in C\big)$, i.e., $\forall y \in \mathbb{R}^\times \ \big((x,y) \in B\big)$. So $x \in \forall^{\mathbb{R}^\times} B$. Hence we derive $\forall^{\mathbb{R}^\times} B = \forall^{\mathbb{R}^\times} C$. □

**Theorem 4.2** *For $B$ as is in Lemma 4.1, there is a Borel function $R$ from $\mathbb{R}$ to $\mathbb{R}^{\mathbb{N}}$ such that for all $x, y \in \mathbb{R}$,*

$$(x,y) \in B \iff \big(R(x), y\big) \in \mathcal{U}.$$

By Theorem 4.2, we have $x \in \forall^{\mathbb{R}^\times} B$ iff $B_x = \mathbb{R}^\times$ iff $\mathcal{U}_{R(x)} = \mathbb{R}^\times$ iff $R(x) \in \forall^{\mathbb{R}^\times} \mathcal{U}$ iff $R(x) \in US$. So we obtain $\forall^{\mathbb{R}^\times} B = R^{-1}(US)$. Hence the function $R$ witnesses that $US$ is $\mathbf{\Pi}^1_1$-hard, i.e., $\mathbf{\Pi}^1_1$-complete, since $US$ is $\mathbf{\Pi}^1_1$. Thus we conclude:

**Corollary 4.3** *$US$ is $\mathbf{\Pi}^1_1$-complete.*

**Proof of Theorem 4.2** We fix $B$ in Lemma 4.1. By Theorem H, we have a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of continuous functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that $(x,y) \in B \iff f_n(x,y) \to 0$ as $n \to \infty$. Let $x \in \mathbb{R}$. Then $B_x = C\big(\langle (f_n)_x \rangle_{n \in \mathbb{N}}\big)$. Replace $\langle f_n \rangle_{n \in \mathbb{N}}$ by $\langle g_n \rangle_{n \in \mathbb{N}}$ where for each $n \in \mathbb{N}$ and for all $x, y \in \mathbb{R}$, $g_n(x,y) = f_n(|x|,|y|)$. Note that for all $(x,y) \in \mathbb{R} \times \mathbb{R}$, $(x,y) \in B$ iff $(|x|,|y|) \in B$. Thus $C\big(\langle g_n \rangle_{n \in \mathbb{N}}\big) = C\big(\langle f_n \rangle_{n \in \mathbb{N}}\big)$. We can suppose that for each $n \in \mathbb{N}$, $g_n$ is even. We fix $x \in \mathbb{R}$. Clearly, $g_n(x,0)$ does not tend to zero. So choose the least $N_x \in \mathbb{N}$ such that for infinitely many $n$, $|g_n(x,0)| \geq 1/N_x$.

**Lemma 4.4** *Let $0 < 2r < \frac{1}{N_x}$. Then in a Borel way, there exists a sequence of elements of $\mathcal{L}^+$ such that if $B_{x,r} = C\big(\langle f^r_{k,x} \rangle_{k \in \mathbb{N}}\big)$, we have*

*(i) If for all enough large $k$, $|g_k(x,y)| \leq r$, then $y \in B_{x,r}$,*

(ii) *If for infinitely many $k$, $|g_k(x,y)| > 2r$, then $y \notin B_{x,r}$.*

**Proof of Lemma 4.4** Let $F_n^x = \{y \in \mathbb{R} : |g_n(x,y)| \leq r \text{ and } |y| \geq \frac{1}{n+1}\}$, and $K_n^x = \{y \in [-n,n] : |g_n(x,y)| \geq 2r\}$. Note that we can check in a Borel way whether or not $F_n^x$ and $K_n^x$ are nonempty, since a continuous function on $\mathbb{R}$ is totally determined on the domain $\mathbb{Q}$. Hence without loss generality, we may assume that $F_n^x$ and $K_n^x$ are nonempty. Then $F_n^x, K_n^x$ are disjoint symetric to the origin and $0 \notin F_n^x$. Take the least $M(x,n) \in \mathbb{N}$ such that $M(x,n) > M(x,i)$ for all $i < n$ and $0 < 2\frac{1}{M(x,n)} < \inf_{y \in F_n^x} |y|$ and $0 < 2\frac{1}{M(x,n)} < \inf_{(y,z)\in F_n^x \times K_n^x} |y-z|$. For each $n \in \mathbb{N}$, set $\mathbb{Q}_n = [-n,n] \cap \mathbb{Q}$. We enumerate $\mathbb{Q}_n = \{q_p^{(n)}\}_{p\in\mathbb{N}}$. We define a sequence $\langle M_{n,p}^x \rangle_{p\in\mathbb{N}}$ as follows:

$$M_{n,p}^x = \{z \in K_n^x : |q_p^{(n)} - z| = \inf_{y\in K_n^x} |q_p^{(n)} - y|\}.$$

Then it is easy to see that for each $p \in \mathbb{N}$, $M_{n,p}^x$ is non empty and has at most two elements. We define a sequence $\langle c_{n,p}^x \rangle$ as follows:

$$c_{n,p}^x = \sup_{z \in M_{n,p}^x} z.$$

**Claim 4.1** *We can choose $\langle c_{n,p}^x \rangle_{p\in\mathbb{N}}$ in a Borel way so that $\{c_{n,p}^x\}_{p\in\mathbb{N}}$ is a countable dense subset of $K_n^x$.*

**Proof of Claim 4.1** Since $g_n(x,\cdot)$ can be determined on the domain $\mathbb{Q}$, we can proceed in a Borel way to choose $c_{n,p}^x$ for each $p \in \mathbb{N}$. So the first assertion follows. We show that $\{c_{n,p}^x\}_{p\in\mathbb{N}}$ is dense in $K_n^x$. Let $k \in K_n^x$. Then there exists a sequence $\langle p_l \rangle_{l\in\mathbb{N}}$ such that $q_{p_l}^{(n)}$ converges to $k$. It is enough to show that $c_{n,p_l}^x$ converges to $k$. For all $l \in \mathbb{N}$, we obtain

$$\begin{aligned}|c_{n,p_l}^x - k| \leq& |c_{n,p_l}^x - q_{p_l}^{(n)}| + |q_{p_l}^{(n)} - k| = \inf_{y\in K_n^x} |y - q_{p_l}^{(n)}| + |q_{p_l}^{(n)} - k| \\ \leq& |k - q_{p_l}^{(n)}| + |k - q_{p_l}^{(n)}| \to 0,\end{aligned}$$

as $l \to \infty$. Hence $c_{n,p_l}^x \to k$. So we have shown that $\{c_{n,p}^x\}$ is a countable dense subset of $K_n^x$. □

We let $\epsilon = \frac{1}{M(x,n)}$. For each $n \in \mathbb{N}$, choose the least $N_n \in \mathbb{N}$ such that

$$C_n^x \subset \bigcup_{i \leq N_n} \left(c_{n,i}^x - \frac{\epsilon}{2}, c_{n,i}^x + \frac{\epsilon}{2}\right) \text{ but } C_n^x \not\subset \bigcup_{i < N_n} \left(c_{n,i}^x - \frac{\epsilon}{2}, c_{n,i}^x + \frac{\epsilon}{2}\right).$$

The least $N_n$ exists, since $K_n^x$ is compact and $C_n^x$ is dense in $K_n^x$. Here we see that this procedure can be done in a Borel way, since $C_n^x$ is countable. Since $C_n^x$ is dense in $K_n^x$, we then obtain the following:

$$K_n^x \subset \bigcup_{i\leq N_n}\left[c_{n,i}^x-\frac{\epsilon}{2},c_{n,i}^x+\frac{\epsilon}{2}\right]\subset \bigcup_{i\leq N_n}\left(c_{n,i}^x-\epsilon,c_{n,i}^x+\epsilon\right).$$

Say $\lambda_0(x,n),\lambda_1(x,n),\cdots,\lambda_{N_n}(x,n)$ are the centers of corresponding intervals $(c_{n,0}^x-\epsilon,c_{n,0}^x+\epsilon),\cdots,(c_{n,N_n}^x-\epsilon,c_{n,N_n}^x+\epsilon)$. Then $\lambda_i(x,n)\in K_n^x$ for all $i\leq N_n$. Now let

$$\phi_n^x(y)=\begin{cases}1-\frac{M(x,n)|y|}{2}, & \text{for } |y|\leq 2\frac{1}{M(x,n)};\\ 0, & \text{for } |y|>2\frac{1}{M(x,n)}.\end{cases}$$

Then for all $n\in\mathbb{N}$ and $x\in\mathbb{R}$, $\phi_n^x\in\mathcal{L}^+$. Let

$$G_{n,i}^x(y)=\phi_n^x(y)+\frac{1}{2}\left|\phi_n^x(y-\lambda_i(x,n))+\phi_n^x(y+\lambda_i(x,n))\right|.$$

Then $G_{n,i}^x$ is c.d.p. Let

$$f_{n,i,x}^r=\frac{G_{n,i}^x}{G_{n,i}^x(0)},$$

where $i\leq N_n$. Clearly $f_{n,i,x}^r\in\mathcal{L}^+$. We set $\langle f_{k,x}^r\rangle=\langle f_{1,1,x}^r,\cdots,f_{1,N_n,x}^r\rangle_{n\in\mathbb{N}}$. We verify (i) and (ii). Suppose $|g_k(x,y)|\leq r$ for all enough large $k$. Then $y\in F_k^x$ for all sufficiently large $k$. We fix such $k$. Then $\phi_{M(x,k)}(y)=0$. Since for all $i\leq N_k$, $|y-\lambda_i(x,k)|,|y+\lambda_i(x,k)|>2\frac{1}{M(x,k)}$, $\phi_{M(x,k)}\big(y-\lambda_i(x,k)\big)=\phi_{M(x,k)}\big(y+\lambda_i(x,k)\big)=0$. Hence $f_{k,i,x}^r(y)=0$. Therefore, for sufficiently large $k$ and for all $i\leq N_k$, $f_{k,i,x}^r(y)=0$, i.e., for enough large $k$, $h_k^r(y)=0$. So $y\in B_{x,r}$. We have finished (i). Now suppose $|g_k(x,y)|>2r$ for infinitely many $k$. If $y=0$, then clearly $y\notin B_{x,r}$. So suppose $y\neq 0$. Then for infinitely many $k$, $y\in K_k^x$ and $2\frac{1}{M(x,k)}<|y|$. We fix such $k$. Then for some $i\leq N_k$, $|y-\lambda_i(x,k)|<\frac{1}{M(x,k)}$ holds. Hence by the definition of $\phi_{M(x,k)}$, $\phi_{M(x,k)}\big(y-\lambda_i(x,k)\big)$ is bigger than $1/2$. Since $2\frac{1}{M(x,k)}<|y|$, we have $\phi_{M(x,k)}(y)=0$. It is easy to see that $G_{n,i}^x\geq\frac{1}{4}$. Since by the definition of $\phi_{M(x,k)}$, for all $y\in\mathbb{R}$, $\phi_{M(x,k)}(y)\leq 1$, we obtain $\frac{1}{2}|\phi_{M(x,k)}(z_1)+\phi_{M(x,k)}(z_2)|\leq 1$. Hence we get

$$\frac{1}{8}\leq f_{k,i,x}^r(y)\leq\frac{1}{4}.$$

So we conclude $y \notin B_{x,r}$. We have finished (ii). It is easy to check that our construction was carried out in the Borel way. $\square$

We will use Lemma 4.4 countably many times. It will be easy to see that we are constructing our function $R$ in a Borel way. We let $r(n) = 2\frac{1}{N_x(n+1)}$. Then by Lemma 4.4, if $B_{x,r(n)} = C(\langle f_{k,x}^{r(n)}\rangle_{k\in\mathbb{N}})$, we have:

(i) If for all enough large $k$, $|g_k(x,y)| \le r(n)$, then $y \in B_{x,r(n)}$,

(ii) If for infinitely many $k$, $|g_k(x,y)| > 2r(n)$, then $y \notin B_{x,r(n)}$.

Clearly $B_x = \bigcap_{n\in\mathbb{N}} B_{x,r(n)}$. We take

$$h_k^x = \frac{\left(\sum_{j=0}^k 2^{-j} f_{k,x}^{r(j)}\right)}{\left(\sum_{j=0}^k 2^{-j}\right)}$$

for each $k \in \mathbb{N}$. Then $\langle h_k^x\rangle$ is the sequence of c.d.p. functions with $h_k^x(0) = 1$. It is easy to see that $C(\langle h_n^x\rangle_{n\in\mathbb{N}}) = \bigcap_{n\in\mathbb{N}} B_{x,r(n)} = B_x$. We need the following lemma [Raz]:

**Lemma 4.5** *Let $f$ be a c.d.p function with $f(0) = 1$, let $K$ a compact set of $\mathbb{R}$ and $\epsilon > 0$. Then there exists an integer $N \ge 1$ and a finite sequence $u_0, u_1, \cdots, v_{N-1}$ such that*

$$(*) \qquad \sup_{x\in K}\left| f(x) - \frac{1}{N}\sum_{k=0}^{N-1} e(xu_k)\right| < \epsilon.$$

We let $K = [-n, n]$. For each $s, n \in \mathbb{N}$ and $s \ge 1$, we denote by $V_{s,n}$ the set of all $(w_k)_{k\le s-1}$ satisfying $(*)$, i.e.,

$$V_{s,n} = \{\langle w_k\rangle_{k\le s-1} : \sup_{y\in[-n,n]}\left| h_n^x(y) - \frac{1}{s}\sum_{k=0}^{s-1} c_k e(yw_k)\right| < \frac{1}{n+1}\}.$$

We know that for each $n \in \mathbb{N}$, $h_n^x$ is c.d.p. and $h_n^x(0) = 1$. Lemma 4.5 thus implies that there exists a least $\nu(n) \ge 1$ such that $V_{\nu(n),n}$ is a nonempty open set. Since we have the countable dense set $\bigcup_{n\in\mathbb{N}} \mathbb{Q}^n$ of $\bigcup_{n\in\mathbb{N}} \mathbb{R}^n$ and for each $s, n \in \mathbb{N}$, $V_{s,n}$ is open, we can calculate in the Borel way whether or not $V_{s,n}$ is empty. We define a well order $\prec_{\nu(n)}$ in $\mathbb{N}^{\nu(n)}$ as follows:

$$(k_0,\cdots,k_{\nu(n)-1}) \prec_{\nu(n)} (k'_0,\cdots,k'_{\nu(n)-1}) \in \mathbb{N}^{\nu(n)} \iff (k_0 < k'_0) \text{ or } (k_0 = k'_0 \text{ and } k_1 < k'_1) \text{ or } \big(\forall i < \nu(n)\ (k_i = k'_i) \text{ and } k_{\nu(n)-1} < k'_{\nu(n)-1}\big)$$

for all $(k_0,\cdots,k_{\nu(n)-1})$,$(k'_0,\cdots,k'_{\nu(n)-1}) \in \mathbb{N}^{\nu(n)}$. We take, in terms of $\prec_{\nu(n)}$, the least $(l_0^{(n)},\cdots,l_{\nu(n)-1}^{(n)}) \in \mathbb{N}^{\nu(n)}$ such that $(q_{l_0^{(n)}},\cdots,q_{l_{\nu(n)-1}^{(n)}}) \in V_{\nu(n),n}$, where $\mathbb{Q} = \{q_m\}_{m\in\mathbb{N}}$. We set $q_{l_i^{(n)}} = w_{n,i}$ for each $i \leq \nu(n)-1$. Set

$$S_n^x(y) = \frac{1}{\nu(n)} \sum_{k=0}^{\nu(n)-1} e(yw_{n,k}) \text{ for all } y \in \mathbb{R}.$$

Then we evidently have $C\big(\langle S_n^x\rangle_{n\in\mathbb{N}}\big) = C\big(\langle (h_n^x)_x\rangle_{n\in\mathbb{N}}\big) = C\big(\langle (f_n)_x\rangle_{n\in\mathbb{N}}\big)$. Make $\langle u_n^x\rangle_{n\in\mathbb{N}}$ the sequence composition corresponding to $\langle w_{n,0}\cdots,w_{n,\nu(n)-1}\rangle_{n\in\mathbb{N}}$. For each $n \geq 1$, set

$$t_n^x(y) = \frac{1}{n}\sum_{k=1}^{n} e(yu_k^x) \text{ for all } y \in \mathbb{R}.$$

We require the following lemma [Raz]:

**Lemma 4.6** *If $\langle u_n\rangle_{n\in\mathbb{N}}$ is composed from the sequence $\langle w_{n,0},\cdots,w_{n,\nu(n)-1}\rangle_{n\in\mathbb{N}}$ and if $f$ is a function from $\mathbb{R}$ to $\mathbb{C}$, the sequence $\big\langle \frac{1}{\nu(n)}\sum_{k=0}^{\nu(n)-1} f(w_{n,k})\big\rangle$ converges to zero iff so does for $\big\langle \frac{1}{n}\sum_{k=0}^{n-1} f(w_k)\big\rangle$.*

By this lemma, we have $C\big(\langle S_n^x\rangle_{n\in\mathbb{N}}\big) = C\big(\langle t_n^x\rangle\big)$. We finally set

$$R(x) = \langle u_n^x\rangle_{n\in\mathbb{N}}.$$

By Theorem W, for each $y \in \mathbb{R}$, the sequence $\langle yu_n^x\rangle$ is uniformly distributed mod 1 iff

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} e(hyu_n^x) = 0,$$

for all $q \in \mathbb{Z}^\times$, i.e., $y \in \mathcal{U}_{\langle u_n^x\rangle} = \mathcal{U}_{R(x)}$ iff for all $q \in \mathbb{Z}^\times$ $\big(t_n^x(qy) \to 0 \text{ as } n \to \infty\big)$. Hence for all $x \in \mathbb{R}$, $C\big(\langle (f_n)_x\rangle_{n\in\mathbb{N}}\big) = \mathcal{U}_{\langle u_n^x\rangle} = \mathcal{U}_{R(x)}$. It is easy (but somewhat complicated) to see that $R$ is a Borel function from $\mathbb{R}$ to $\mathbb{R}^{\mathbb{N}}$. So we complete Theorem 4.2. $\square$

## 4.4 The sets arising countable Borel equivalence relations

Let $X$ be a Polish space. Let $\mathcal{K}(X)$ be the Polish space of all closed sets of $X$ with the Hausdorff metric. Given Borel equivalence relation $E$ on $X$, $E$ is called countable if for all $x \in X$, $[x]_E$, the equivalence class of $x$, is countable. Let $E$ be a countable equivalence relation on $2^{\mathbb{N}}$. We denote by $\mathcal{A}(E)$ ($\mathcal{F}(E)$) the set of all closed sets $K$ such that $E \cap K \times K$ is aperiodic (resp. finite), i.e., for all $x \in K$, the equivalence class of $x$ is infinite (resp. finite) in $K$. Hence

$$\mathcal{A}(E) = \{K \in \mathcal{K}(2^{\mathbb{N}}) : E \restriction K \text{ is aperiodic}\}$$
$$\mathcal{F}(E) = \{K \in \mathcal{K}(2^{\mathbb{N}}) : E \restriction K \text{ is finite,}\}$$

where $E \restriction K = E \cap K \times K$.

We denote by $F$ the countable Borel equivalence relation on $2^{\mathbb{N}} \times \mathbb{N}^{-1}$ such that for all $(x,a),(y,b) \in 2^{\mathbb{N}} \times \mathbb{N}^{-1}$, $(x,a)F(y,b) \iff x = y$. We also introduce the basic equivalence relation $E_0$ on $2^{\mathbb{N}}$ as follows: for all $x,y \in 2^{\mathbb{N}}$, $xE_0y \iff$ there is a $m \in \mathbb{N}$ such that for all $n \geq m$, $x(n) = y(n)$.

Let $E$ be a countable Borel equivalence relation on $2^{\mathbb{N}}$. We calculate the upper bounds of complexities of $\mathcal{A}(E)$ and $\mathcal{F}(E)$. By definition, we have the following:

$$\begin{aligned}
K \in \mathcal{A}(E) &\iff E \restriction K \text{ is aperiodic} \\
&\iff \forall x \in 2^{\mathbb{N}} \left(x \notin K \text{ or } [x]_E \text{ is infinite in } K\right) \\
&\iff \forall x \in 2^{\mathbb{N}} \big(x \notin K \text{ or } \forall n \in \mathbb{N}\ \exists x_0, x_1, \cdots, x_n \in [x]_E \\
&\qquad (\forall i,j \leq n\ (i \neq j \Rightarrow x_i \neq x_j) \text{ and } \forall i \leq n\ (x_i \in K))\big) \\
&\iff \forall x \in 2^{\mathbb{N}} (x \notin K \text{ or } (x,K) \in V);
\end{aligned}$$

$$
\begin{aligned}
K \in \mathcal{F}(E) &\iff E \restriction K \text{ is finite} \\
&\iff \forall x \in 2^{\mathbb{N}} \ \big(x \notin K \text{ or } [x]_E \text{ is finite in } K\big) \\
&\iff \forall x \in 2^{\mathbb{N}} \ \Big(x \notin K \text{ or} \\
&\qquad \exists n \in \mathbb{N} \ \exists x_0, x_1, \cdots, x_n \in [x]_E \ \big(\forall i \leq n \ (x_i \in K)\big) \text{ and} \\
&\qquad \forall y \in [x]_E \ \big(\forall i \leq n \ (y \neq x_i) \Rightarrow y \notin K\big)\Big) \\
&\iff \forall x \in 2^{\mathbb{N}} \ \big(x \notin K \text{ or } (K, x) \in W\big),
\end{aligned}
$$

where $V$ is the set of all elements $(K, x)$ such that $\forall n \in \mathbb{N} \ \exists x_0, x_1, \cdots, x_n \in [x]_E \ \big(\forall i, j \leq n \ (i \neq j \Rightarrow x_i \neq x_j) \text{ and } \forall i \leq n \ (x_i \in K)\big)$ and $W$ the set of all elements $(K, x)$ such that $\exists n \in \mathbb{N} \ \exists x_0, x_1, \cdots, x_n \in [x]_E \ \Big(\forall i \leq n \ (x_i \in K) \text{ and } \forall y \in [x]_E \ \big(\forall i \leq n \ (y \neq x_i) \Rightarrow y \notin K\Big)$. Let $X$ and $Y$ be Polish spaces. Let $A$ be a Borel subset of $Y \times X$ such that for all $y \in Y$, $A_y$ is countable. Then by a standard theorem in [Ke1], $\{y \in Y : \exists x \in A_y \ \big((x, y) \in A)\big)\}$ is Borel. The relation '$x \in K$', i.e., $\{(K, x) \in \mathcal{K}(2^{\mathbb{N}}) \times 2^{\mathbb{N}} : x \in K\}$, is a closed subset of $\mathcal{K}(2^{\mathbb{N}}) \times 2^{\mathbb{N}}$. Hence by the above two facts, it is easy to see that $V$ and $W$ are Borel. Finally, $\mathcal{A}(E)$ and $\mathcal{F}(E)$ are $\mathbf{\Pi}^1_1$.

We need the following proposition to make the proofs of our theorems easy.

**Proposition 4.7** *Let $E$ be a countable Borel equivalence relation on $2^{\mathbb{N}}$. Assume $W = \{x \in 2^{\mathbb{N}} : [x]_E \cap [x]'_E \neq \emptyset\}$ is uncountable. Then there is one-to-one continuous function $f$ from $2^{\mathbb{N}} \times \mathbb{N}^{-1}$ to $2^{\mathbb{N}}$ such that*

$$(x, a)F(y, b) \iff f(x, a)Ef(y, b)$$

*for all $(x, a), (y, b) \in 2^{\mathbb{N}} \times \mathbb{N}^{-1}$.*

**Proof of Proposition 4.7** We may find a Cantor subset $C$ of $W$ such that

(i) For all $x \in C$, $x$ is a limit point in $[x]_E$,

(ii) For all $x, y \in C$, $xEy$ implies $x = y$.

By a standard theorem of Feldman and Moore [FM], there exists a countable group $G$ and a Borel action of $G$ such that $E = E_G$. We enumerate $G = \{g_m\}_{m \in \mathbb{N}}$. We define the function from $C \times \mathbb{N}^{-1}$ to $2^{\mathbb{N}}$ as follows: for all $x \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$,

$T(x,0)=x$, $T(x,\frac{1}{n+1})=g_m.x$ where $m$ is the least $m$ such that $0<d(x,g_m.x)<\frac{1}{n+1}$ and $g_m.x$ is different from $T(x,0),\cdots,T(x,\frac{1}{n+1})$. Then $T$ is an one-to-one Borel such that for all $(x,a),(y,b)\in 2^{\mathbb{N}}\times\mathbb{N}^{-1}$, $(x,a)F(y,b) \iff T(x,a)ET(y,b)$. So for each $a\in\mathbb{N}^{-1}$, there exists a dense $G_\delta$ subset $S_a$ of $2^{\mathbb{N}}$ such that $T\restriction S_a\times\{a\}$ is continuous. Let $S=\bigcap_{a\in\mathbb{N}^{-1}}S_a$. Then $S$ is a dense $G_\delta$ subset in $2^{\mathbb{N}}$. It is enough to show that $T$ is continuous on $S\times\mathbb{N}^{-1}$. Suppose $(x_n,a_n)\to(x,a)$ in $S\times\mathbb{N}^{-1}$. If $a_n$'s end in the same value, it is obious. So $a_n\to 0$ and for infinitely many $n$'s $a_n\neq 0$. Clearly we can assume $a_n\neq 0$ for all $n\in\mathbb{N}$. Then we have

$$(**)\quad \begin{aligned} d\big(T(x_n,a_n),T(x,a)\big) &\le d\big(T(x_n,a_n),T(x_n,0)\big)+d\big(T(x_n,0),T(x,0)\big)\\ &\quad+d\big(T(x,0),T(x,a_n)\big)+d\big(T(x,a_n),T(x,a)\big)\\ &\le a_n+d\big(T(x,0),T(x,a_n)\big)+a_n+a_n\\ &=3a_n+d\big(T(x_n,0),T(x,0)\big)\to 0, \end{aligned}$$

as $n\to\infty$. Hence $T$ is continuous in $S\times\mathbb{N}^{-1}$. We choose a Cantor subset $K$ of $S$ which is homeomorphic to $2^{\mathbb{N}}$. We take $f=T\restriction K\times\mathbb{N}^{-1}$. Then $f$ witnesses Propostion 7, since $K$ is homeomorphic to $2^{\mathbb{N}}$. So we are done. □

**Theorem 4.8** *Under the same assumptions as Proposition 4.7. $\mathcal{A}(E)$ and $\mathcal{F}(E)$ are $\mathbf{\Pi}^1_1$-complete.*

Recall now the following theorem:

**Theorem D**[HKL] *For any Borel equivalence relation $E$ on $2^{\mathbb{N}}$,*

*(a) either $E$ is smooth or,*

*(b) there is an one-to-one continuous function $h$ from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$ such that for all $x,y\in 2^{\mathbb{N}}$,*

$$xE_0y \iff f(x)Ef(y)$$

*holds.*

Here $E$ is called smooth if there is a countable Borel separates family. By Theorem D, for nonsmooth $E$, it is easy to see that $\mathcal{A}(E)$ and $\mathcal{F}(E)$ are $\mathbf{\Pi}^1_1$-hard using Theorem 4.8. In addition, if $E$ is a countable Borel equivalence relation, then $\mathcal{A}(E)$ and $\mathcal{F}(E)$ are $\mathbf{\Pi}^1_1$-complete.

**Proof of Theorem 4.8** We have seen that $\mathcal{A}(E)$ and $\mathcal{F}(E)$ are $\mathbf{\Pi}^1_1$. We show the hardness of $\mathcal{A}(E)$ and $\mathcal{F}(E)$. By Proposition 4.7, it is enough to show for $\mathcal{A}(2^{\mathbb{N}}\times\mathbb{N}^{-1},F)$ and $\mathcal{F}(2^{\mathbb{N}}\times\mathbb{N}^{-1},F)$. We write $\mathcal{A}(2^{\mathbb{N}}\times\mathbb{N}^{-1},F)=\mathcal{A}(F)$ and $\mathcal{F}(2^{\mathbb{N}}\times\mathbb{N}^{-1},F)=\mathcal{F}(F)$. We recall that $WF$ is $\mathbf{\Pi}^1_1$-complete. We will somehow construct Borel functions from $Tr$ to $\mathcal{K}(2^{\mathbb{N}}\times\mathbb{N}^{-1})$ such that the preimages of $\mathcal{A}(F)$ and $\mathcal{F}(F)$ of these functions are precisely $WF$. This will prove that $\mathcal{A}(F)$ and $\mathcal{F}(F)$ are $\mathbf{\Pi}^1_1$-hard. First we show that $\mathcal{A}(F)$ is $\mathbf{\Pi}^1_1$-hard. We will construct a Borel fuction from $Tr$ to $\mathcal{K}(2^{\mathbb{N}}\times\mathbb{N})$ which witnesses that $\mathcal{A}(F)$ is $\mathbf{\Pi}^1_1$-hard. Fix a bijection $\langle,\rangle$ from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$. Let $T\in Tr$. We define $\langle A_{s,k}\rangle_{s\in\mathbb{N}^{<\mathbb{N}},k\in\mathbb{N}}$ as follows: for all $s\in\mathbb{N}^{<\mathbb{N}}$ and $k\in\mathbb{N}$,

$$A_{s,k}=\begin{cases}\{1^{s(0)+\cdots+s(k)+k+1}{}^\frown 0^\frown 1^{s(0)+\cdots+s(n)+k+1}{}^\frown 0^\frown\cdots\}\cup\{1^\infty\}, & \text{if } k<lh(s);\\ 2^{\mathbb{N}}, & \text{o.w.}\end{cases}$$

Denote by $A_s$ the set of all elements $\alpha$ of $2^{\mathbb{N}}$ such that for all $k\in\mathbb{N}$, $(\alpha)_k\in A_{s,k}$, i.e.,

$$A_s=\{\alpha\in 2^{\mathbb{N}}:\forall k\in\mathbb{N}\ \big((\alpha)_k\in A_{s,k}\big)\},$$

where for all $k\in\mathbb{N}$, $(\alpha)_k(m)=\alpha\big(\langle k,m\rangle\big)$. Inductively, we define $\langle A^T_s\rangle_{s\in\mathbb{N}^{<\mathbb{N}}}$ as follows:

$$A^T_\emptyset=2^{\mathbb{N}};$$

for each $n\in\mathbb{N}$,

$$A^T_{\langle n\rangle}=\begin{cases}A_{\langle n\rangle}, & \text{if } \langle n\rangle\in T;\\ \emptyset, & \text{o.w.},\end{cases}$$

for each $s\in\mathbb{N}^{<\mathbb{N}}, m\in\mathbb{N}$,

$$A^T_{s^\frown m}=\begin{cases}A_{s^\frown m}, & \text{if } s^\frown m\in T;\\ 2^{\mathbb{N}}, & \text{if } \exists t\subset s\ \big(t\neq\emptyset, t\in T, \forall n\in\mathbb{N}\ (t^\frown n\notin T) \text{ and } s^\frown m\notin T\big);\\ \emptyset, & \text{if } \forall t\subset s\ \big(t\neq\emptyset \text{ and } t\in T\Rightarrow\exists n\in\mathbb{N}\ (t^\frown n\in T)\big),\\ & \exists t\subset s(t\neq\emptyset \text{ and } t\in T)\ \&\ s^\frown m\notin T\\ & \text{or } \forall t\subset s\ (t\neq\emptyset \text{ and } t\notin T).\end{cases}$$

We define $\langle B^T_s\rangle_{s\in\mathbb{N}^{<\mathbb{N}}}$ as follows: for all $s\in\mathbb{N}^{<\mathbb{N}}$,

$$B^T_s=\begin{cases}\{0\}\cup\{\frac{1}{k+1}\}_{k\geq lh(s)}, & \text{if } s\in T;\\ \mathbb{N}^{-1} & \text{o.w.}\end{cases}$$

For each $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $C^T_\alpha = \bigcap_{n\in\mathbb{N}} A^T_{\alpha\restriction n} \times B^T_{\alpha\restriction n}$. We define the function $H$ from $Tr$ to $\mathcal{K}(2^{\mathbb{N}} \times \mathbb{N}^{-1})$ as follows: for all $T \in Tr$,

$$H(T) = \bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} C^T_\alpha.$$

We should verify that the function $H$ is well-defined, Borel and $H^{-1}(\mathcal{A}(F)) = WF$.

(i) $H$ is Borel and well-defined.

**Proof of (i)** Let $T \in Tr$. Inductively, we define $\langle T^{(n)}\rangle_{n\in\mathbb{N}}$ as follows:

$T^{(0)} = \{\langle m_0\rangle \in \mathbb{N}^{<\mathbb{N}} : \langle m_0\rangle \in T\}$;

$T^{(n+1)} = \{\langle m_0, \cdots, m_{n+1}\rangle \in \mathbb{N}^{<\mathbb{N}} : \langle m_0, \cdots, m_{n+1}\rangle \in T\} \bigcup T^{(n)}$.

Note that for all $T \in Tr$ and $n \in \mathbb{N}$,

$$H(T) \subset H(T^{(n)}) \text{ and } H(T^{(n+1)}) \subset H(T^{(n)}).$$

**Claim 4.2** $H(T) = \bigcap_{n\in\mathbb{N}} H(T^{(n)})$.

**Proof of Claim 4.2** Clearly $H(T) \subset \bigcap_{n\in\mathbb{N}} H(T^{(n)})$. Suppose $(x,a) \in H(T^{(n)})$ for all $n \in \mathbb{N}$. Then there exists $\{\alpha_n\} \subset \mathbb{N}^{\mathbb{N}}$ such that $(x,a) \in C^{T^{(n)}}_{\alpha_n}$, i.e., $(x,a) \in A^{T^{(n)}}_{\alpha_n\restriction k} \times B^{T^{(n)}}_{\alpha_n\restriction k}$ for all $k, n \in \mathbb{N}$. We take an $\alpha$ from the closure set of $\{\alpha_n\}$. Then there is a strictly increasing sequence $\langle N_n\rangle$ of natural numbers such that $\alpha_{N_n}$ converges to $\alpha$. We fix $k$ in $\mathbb{N}$. It is enough to show that $x \in A^T_{\alpha\restriction k}$ and $a \in B^T_{\alpha\restriction k}$. We let $s = \alpha_k$. Suppose $s \in T$. Then for large enough $n \in \mathbb{N}$, $s \in T^{(N_n)}$, i.e., by the definition, $A^T_s = A^{T^{(N_n)}}_s$ and $B^T_s = B^{T^{(N_n)}}_s$. We are done for this case. Suppose $s \notin T$. Clearly $a \in B^T_s$. We choose a sufficiently large $n \in \mathbb{N}$ such that $N_n > lh(s) + 5$. Since $x \in A^{T^{(N_n)}}_s$ and $s \notin T^{(N_n)}$, by the definition, there is $t \subset s$ such that

$$t \neq \emptyset, t \in T^{(N_n)} \text{ and } \forall m \in \mathbb{N}\ (t\hat{\ }m \notin T^{(N_n)}).$$

Clearly the previous relation is true for $T$ instead of $T^{(N_n)}$. Hence $A^T_s = 2^{\mathbb{N}}$, i.e., $x \in A^T_s$. So we obtain $\bigcap_{n\in\mathbb{N}} H(T^{(n)}) \subset H(T)$. This completes Claim 2. □

Using Claim 2, we show (i). We refer to the fact that if $X$ is metrizable and $K_n \in \mathcal{K}(X)$, $\cdots \subset K_1 \subset K_0$, then $\lim_{n\to\infty} K_n = \bigcap_{n\in\mathbb{N}} K_n$. By Claim 2 and this fact, we obtain

$$H(T) = \lim_{n\to\infty} H(T^{(n)}),$$

since $\cdots \subset H(T^{(1)}) \subset H(T^{(0)})$. For each $n \in \mathbb{N}$, we define a function $H_n$ from $Tr$ to $\mathcal{K}(2^{\mathbb{N}} \times \mathbb{N}^{-1})$ as follows: for all $T \in Tr$, $H_n(T) = H(T^{(n)})$. We denote by $Tr^{(n)}$ the set of all trees $T$ such that for all $s \in T$, $lh(s) \leq n$. Note that for any $T \in Tr^{(n)}$, $H_n(T) = \bigcup_{s\in T_M} A_s^T \times B_s^T$, where $T_M = \{s \in T : \forall n \in \mathbb{N}\ (s\hat{\ }n \notin T)\}$. Then it is easy to see that for each $n \in \mathbb{N}$, $H_n$ is well-defined. We need the following claim.

**Claim 4.3** *Let $n \in \mathbb{N}$. Then $H_n$ is a Borel function.*

**Proof of Claim 4.3** Since the function $T \longmapsto T^{(n)}$ is continuous, it is enough to show that $H_n \restriction Tr^{(n)}$ is a Borel function. Set $R = H_n \restriction Tr^{(n)}$. Let $U$ be open in $2^{\mathbb{N}} \times \mathbb{N}^{-1}$. Then it suffices to show that $R^{-1}(\{k \in \mathcal{K}(X) : K \subset U\})$ and $R^{-1}(\{k \in \mathcal{K}(X) : K \cap U \neq \emptyset\})$ are Borel sets. We observe the following:

$$\begin{aligned} K \in R^{-1}(\{k \in \mathcal{K}(X) : K \subset U\}) &\iff R(T) = \bigcup_{s\in T_M} A_s^T \times B_s^T \subset U \\ &\iff \forall s \in T_M\ (A_s^T \times B_s^T \subset U); \\ K \in R^{-1}(\{k \in \mathcal{K}(X) : K \subset U\}) &\iff (\bigcup_{s\in T_M} A_s^T \times B_s^T) \cap U \neq \emptyset \\ &\iff \exists s \in T_M\ (A_s^T \times B_s^T \cap U \neq \emptyset). \end{aligned}$$

Hence we need only show that for given $s \in \mathbb{N}^{<\mathbb{N}}$ with $lh(s) \leq n$, $\{T \in Tr^{(n)} : A_s^T \times B_s^T \subset U\}$ and $\{T \in Tr^{(n)} : A_s^T \times B_s^T \cap U \neq \emptyset\}$ are Borel sets. Let $s \in \mathbb{N}^{<\mathbb{N}}$. We define two functions $P_{1,s}$ anf $P_{2,s}$ from $Tr^{(n)}$ to $\mathcal{K}(2^{\mathbb{N}})$ and $\mathcal{K}(\mathbb{N}^{-1})$ as follows: for each $T \in Tr^{(n)}$, $P_{1,s}(T) = A_s^T$ and $P_{2,s}(T) = B_s^T$. Clearly $P_{1,s}$ and $P_{2,s}$ are continuous. So is $P_{1,s} \times P_{2,s}$. Hence $\{T \in Tr^{(n)} : A_s^T \times B_s^T \subset U\}$ and $\{T \in Tr^{(n)} : A_s^T \times B_s^T \cap U \neq \emptyset\}$ are Borel sets. We have finished the proof of Claim 4.3. □

Since $H = \lim_{n\to\infty} H_n$, by Claim 2, $H$ is the limit of Borel functions $H_n$. So $H$ is a Borel function. This completes the proof of (i) □

(ii) $H^{-1}(\mathcal{A}(F)) = WF$.

**Proof of (ii)** It is enough to show that for all $T \in Tr$, $T \in WF \Rightarrow H(T) \in \mathcal{A}(F)$ and $T \notin WF \Rightarrow H(T) \notin \mathcal{A}(F)$. Suppose $T \in WF$. For all $\alpha \in 2^{\mathbb{N}}$, $A_\alpha^T \neq \emptyset$ if

$\exists t \in T\ (\forall m \in \mathbb{N}\ (t^\frown m \notin T)$ and $t \subset \alpha)$, or $\emptyset$ if o.w. Hence either $C^T_\alpha = A^T_{\alpha\restriction n} \times B^T_{\alpha\restriction n}$ for some $n \in \mathbb{N}$ or $C^T_\alpha = \emptyset$. So it is easy to see that $H(T) \in \mathcal{A}(F)$. Suppose $T \notin WF$. We take $\alpha \in [T]$. Note that for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$, either $C^T_\alpha = C^T_\beta$ or for all $x \in C^T_\alpha$ and $y \in C^T_\beta$, $x$ and $y$ are not $F$-equivalent. Clearly $C^T_\alpha \neq \emptyset$ and $C^T_\alpha \subset 2^\mathbb{N} \times \{0\}$, i.e., for all $z \in 2^\mathbb{N} \times \mathbb{N}^{-1}$, $C^T_\alpha \cap [z]_F$ contains at most one element. Hence $H(T) \notin \mathcal{A}(F)$. □

We have finished the first part of the theorem 4.8, i.e., $\mathcal{A}(F)$ is $\mathbf{\Pi}^1_1$-hard. We show that $\mathcal{F}(F)$ is $\mathbf{\Pi}^1_1$-hard. We slightly modify the proof of the first part. We construct $\widetilde{H}$ from $Tr$ to $\mathcal{K}(2^\mathbb{N} \times \mathbb{N}^{-1})$ which proves that $\mathcal{F}(F)$ is $\mathbf{\Pi}^1_1$-hard. Let $T \in Tr$. We define $\langle D^T_s \rangle_{s \in \mathbb{N}^{<\mathbb{N}}}$ as follows: for all $s \in \mathbb{N}^{<\mathbb{N}}$,

$$D^T_s = \begin{cases} \{0\} \cup \{\frac{1}{k+1}\}_{k \leq lh(s)}, & \text{if } s \in T; \\ \emptyset & \text{o.w.} \end{cases}$$

For each $\alpha \in \mathbb{N}^\mathbb{N}$, we define $C^T_{1,\alpha}$ and $C^T_{2,\alpha}$ as follows: $C^T_{1,\alpha} = \bigcap_{n\in\mathbb{N}} A^T_{\alpha\restriction n}$ and $C^T_{2,\alpha} = \bigcup_{n\in\mathbb{N}} D^T_{\alpha\restriction n}$. We define $\widetilde{H}$ from $Tr$ to $\mathcal{K}(2^\mathbb{N} \times \mathbb{N}^{-1}))$ as follows: for each $T \in Tr$,

$$\widetilde{H}(T) = \bigcup_{\alpha\in\mathbb{N}^\mathbb{N}} C^T_{1,\alpha} \times C^T_{2,\alpha}.$$

We define $\langle \widetilde{H}_m \rangle_{n\in\mathbb{N}}$ and $\langle \widetilde{H}_{m,n} \rangle_{n\in\mathbb{N}}$ as follows: for each $T \in Tr$ and $m, n \in \mathbb{N}$,

$$\widetilde{H}_m(T) = \bigcup_{\alpha\in\mathbb{N}^\mathbb{N}} C^T_{1,\alpha} \times C^{T^{(m)}}_{2,\alpha}$$

$$\widetilde{H}_{m,n}(T) = \bigcup_{\alpha\in\mathbb{N}^\mathbb{N}} C^{T^{(n)}}_{1,\alpha} \times C^{T^{(m)}}_{2,\alpha}.$$

Then as in the first part, for each $m, n \in \mathbb{N}$, one can show that $\widetilde{H}_{m,n}$ is Borel, $\widetilde{H}_m = \lim_{n\to\infty} \widetilde{H}_{m,n}$ and for each $T \in Tr$, $\widetilde{H}_m(T) \subset \widetilde{H}_{m+1}(T)$. It is easy to see that $\widetilde{H}(T) = \bigcup_{m\in\mathbb{N}} \widetilde{H}_m(T)$ and as in the first part, $\widetilde{H}$ is well-defined.

**Claim 4.4** *Let $X$ be a Polish space. Let $K_0 \subset K_1 \subset \cdots$, $K_n \in \mathcal{K}(X)$ for all $n \in \mathbb{N}$. Suppose $K_\infty = \bigcup_{n\in\mathbb{N}} K_n \in \mathcal{K}(X)$. Then $\lim_{n\to\infty} K_n = K_\infty$.*

**Proof of Claim 4.4** Let $U_0, U_1, \cdots, U_k$ be open sets in $X$. We denote by $B$ the basic open set related to $U_0, U_1, \cdots, U_k$, i.e.,

$$B = \{K \in \mathcal{K}(X) : K \subset U_0\ \&\ K \cap U_1 \neq \emptyset\ \& \cdots \&\ K \cap U_k \neq \emptyset\}.$$

Suppose $K_\infty \in B$. It suffices to show that for large enough $n \in \mathbb{N}$, $K_n \in B$. Since $K_\infty \in B$, we have

$$K_\infty \subset U_0 \ \&\ K_\infty \cap U_1 \neq \emptyset \ \& \cdots \&\ K_\infty \cap U_k \neq \emptyset.$$

As $K_\infty$ is the increasing union of $K_n$'s, for sufficiently large $n \in \mathbb{N}$, the previous relation is true for $K_n$, i.e., $K_n \in B$. Hence $K_n$ converges to $K_\infty = \bigcup_{n\in\mathbb{N}} K_n$. So we are done. □

We have seen that for each $T \in Tr$, $\widetilde{H}(T)$ is the increasing union of $\widetilde{H}_m$'s. By Claim 4.4, we obtain that $\widetilde{H}(T) = \lim_{m\to\infty} \widetilde{H}_m(T)$. Hence $\widetilde{H}$ is the limit of Borel functions $\widetilde{H}_m$. So $\widetilde{H}$ is a Borel function. Similarly to the first part, we can show that $\widetilde{H}^{-1}(\mathcal{F}(F)) = WF$. We have finished the second part. Hence we completed the proof of Theorem 4.8. □

# Chapter 5

# The Kechris-Woodin rank is finer than the Zalcwasser rank

## 5.1 Introduction

Zalcwasser [Za] introduced a rank that measures the uniform convergence of sequences of continuous functions on the unit interval. We apply the Zalcwasser rank to the Fourier series of a continuous function on the unit circle. Throughout this paper, we will only consider the Zalcwasser rank on the Fourier series of a continuous function. In [AK] it is shown that on $EC$ (the set of all continuous functions, on the unit circle, with convergent Fourier series), the Zalcwasser rank is a $\mathbf{\Pi}_1^1$ norm which is unbounded below $\omega_1$, i.e., functions in $EC$ are arbitrarily bad in terms of this rank. Kechris and Woodin [KeW] defined a rank that measures the uniform continuity of the derivative of a differentiable function. We shall refer to this rank as the Kechris-Woodin rank. In fact, they have shown that on the set of all differentiable functions, the Kechris-Woodin rank is a $\mathbf{\Pi}_1^1$-norm which is unbounded below $\omega_1$.

Ajtai and Kechris [AK] conjectured that the Kechris-Woodin rank is finer than the Zalcwasser rank, meaning that for any function $f$, the Zalcwasser rank is less than or equal to the Kechris–Woodin rank. There is a fair amount of evidence supporting this conjecture. For example, the Zalcwasser rank is 1, i.e., the smallest possible number, for all differentiable functions $f$, whose derivative $f'$ is bounded. On the other hand, on the set of all differentiable functions with bounded derivatives, the Kechris-Woodin rank is unbounded below $\omega_1$ (See [KeW]). Our main result is an affirmative answer to this conjecture of Ajtai and Kechris.

## 5.2 Definitions and background

Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{T}$ denote the unit circle and $C(\mathbb{T})$ the Polish space of continuous functions on $\mathbb{T}$ with the uniform metric

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in \mathbb{T}\}.$$

$C(\mathbb{T})$ can also be considered as the space of all continuous $2\pi$-periodic functions on $\mathbb{R}$, by viewing $\mathbb{T}$ as $\mathbb{R}/2\pi\mathbb{Z}$. We denote by $D(\mathbb{T})$ the set of differentiable functions on $\mathbb{T}$. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of positive integers and $\mathbb{N}^{\mathbb{N}}$ the Polish space with the usual product topology, where $\mathbb{N}$ is given the discrete topology.

We briefly recall the definition of a complete $\mathbf{\Pi}_1^1$ set. Let $X$ be a Polish space. Let $A$ be a subset of $X$. A $\mathbf{\Pi}_1^1$ set $A$ is called $\mathbf{\Pi}_1^1$-hard if for any Polish space $Y$ and $\mathbf{\Pi}_1^1$ subset $B$ of $Y$ there exists a Borel function $f$ from $Y$ to $X$ such that $f^{-1}(A) = B$. If also $A$ is $\mathbf{\Pi}_1^1$ it is called $\mathbf{\Pi}_1^1$-complete. Clearly any $\mathbf{\Pi}_1^1$-hard set is non Borel. A norm on a set $P$ is any function $\varphi$ taking $P$ into the ordinals. For each such $\varphi$, we associate the prewellordering $\leq_\varphi$ on $P$, $x \leq_\varphi y \iff \varphi(x) \leq \varphi(y)$. $\varphi$ is regular if $\varphi$ maps $P$ onto some ordinal $\lambda$. Two norms $\varphi$ and $\psi$ on $P$ are equivalent if the two associated prewellorderings are the same ($\leq_\varphi = \leq_\psi$), i.e., $\varphi(x) \leq \varphi(y) \iff \psi(x) \leq \psi(y)$. Every norm is equivalent to a unique regular norm. Given a Polish space $X$ and a $\mathbf{\Pi}_1^1$ subset $P$ of $X$, we say that a norm $\varphi\colon P \to$ Ordinals is a $\mathbf{\Pi}_1^1$-norm if there are $\mathbf{\Pi}_1^1$ subsets $R$ and $Q$ of $X \times X$ such that

$$(5.0) \qquad y \in P \Rightarrow [x \in P \;\&\; \varphi(x) \leq \varphi(y) \iff (x, y) \notin R \iff (x, y) \in Q].$$

It is well known that if a subset $A$ of a Polish space and its complement are both $\mathbf{\Pi}_1^1$, then $A$ is Borel (See [Mos]). In (5.0), we see that in a uniform manner for $y \in P$, the set $\{x \in P : \varphi(x) \leq \varphi(y)\}$ is $\mathbf{\Pi}_1^1$ ( $(x, y) \in Q$) and the complement of a $\mathbf{\Pi}_1^1$ set ( $(x, y) \notin R$), hence a Borel set. In [Mos] it is shown that every $\mathbf{\Pi}_1^1$-norm is equivalent to one which takes values in $\omega_1$, the first uncountable ordinal. One of the basic facts is that every $\mathbf{\Pi}_1^1$ subset $P$ admits a $\mathbf{\Pi}_1^1$-norm $\varphi\colon P \to \omega_1$. (See

[Mos].) Hence it is very natural to look for a canonical norm on $\mathbf{\Pi}^1_1$ sets that arise in analysis, topology, etc. We will introduce $\mathbf{\Pi}^1_1$-norms on the set of continuous functions with everywhere convergent Fourier series and the set of differentiable functions. From norm theory, we have the following fundamental theorem. (See Chapter 4, [Mos].)

**Boundedness Principle.** *Let $X$ be a Polish space. Let $P$ be a $\mathbf{\Pi}^1_1$ subset of $X$ and $\varphi: P \to \omega_1$ be a $\mathbf{\Pi}^1_1$-norm on $P$. Then $P$ is Borel if and only if $\varphi$ is bounded below $\omega_1$.*

With this basic principle, one can prove that a $\mathbf{\Pi}^1_1$ set $P$ is $\mathbf{\Pi}^1_1$ non Borel by showing that some $\mathbf{\Pi}^1_1$-norm on $P$ is unbounded below $\omega_1$.

## 5.3 The Kechris-Woodin rank

We define a $\mathbf{\Pi}^1_1$-norm on $D(\mathbb{T})$, which we refer to as the Kechris-Woodin rank [KW]. We consider $\mathbb{T}$ as $[0, 2\pi]$ identifying 0 with $2\pi$. When we say $U$ is an open neighborhood in $\mathbb{T}$, $U$ is considered as the usual open set in $\mathbb{R}$. Let $f$ be a function and $I$ an interval with endpoints $a$ and $b$. We define the following:

$$\Delta f(I) = \frac{f(b) - f(a)}{b - a.}$$

Fix $f \in C(\mathbb{T})$ and $\epsilon > 0$. For each closed subset $P$ of $\mathbb{T}$, we define the K–W derived set of $P$ by

$$\partial^{KW}_{f,\epsilon}(P) = \{x \in P : \forall \text{ open neighborhood } U \text{ of } x,\ \exists \text{ closed intervals } I, J \subseteq U \text{ such that } I \cap J \cap P \neq \emptyset \text{ and } |\Delta f(I) - \Delta f(J)| \geq \epsilon\}.$$

$\partial^{KW}_{f,\epsilon}(P)$ consists of all $\epsilon$ badly behaved points of $P$ in terms of the derivative of $f$. Clearly, $\partial^{KW}_{f,\epsilon}(P)$ is closed. We can then define the sequence $\langle \partial^{KW}_{f,\epsilon}(P, \alpha)\rangle_{\alpha<\omega_1}$ by transfinite induction. Let

$$\partial^{KW}_{f,\epsilon}(P, 0) = P.$$

$$\partial^{KW}_{f,\epsilon}(P, \alpha + 1) = \partial^{KW}_{f,\epsilon}\big(\partial^{KW}_{f,\epsilon}(P, \alpha)\big).$$

For $\lambda$ a limit ordinal, $\partial^{KW}_{f,\epsilon}(P, \lambda) = \bigcap_{\alpha<\lambda} \partial^{KW}_{f,\epsilon}(P, \alpha)$.

Note that $\bigcup_{\epsilon>0} \partial_{f,\epsilon}^{KW}(P,\alpha) = \bigcup_{n\in\mathbb{N}} \partial_{f,\frac{1}{n}}^{KW}(P,\alpha)$. By transfinite induction, we define the sequence $\langle \partial_f^{KW}(P,\alpha)\rangle_{\alpha<\omega_1}$ by setting

$$\partial_f^{KW}(P,\alpha) = \bigcup_{n\in\mathbb{N}} \partial_{f,\frac{1}{n}}^{KW}(P,\alpha).$$

**Fact 5.1** $f \in D(\mathbb{T}) \iff \exists\alpha < \omega_1, \partial_f^{KW}(\mathbb{T},\alpha) = \emptyset$.

Using this fact, we can define the Kechris–Woodin rank on $D(\mathbb{T})$. For each $f \in D(\mathbb{T})$, let $|f|_{KW}$ = the least ordinal $\alpha$ for which $\partial_f^{KW}(\mathbb{T},\alpha) = \emptyset$. We let $b_1D(\mathbb{T})$ be the set of all functions whose derivatives are bounded in absolute value by 1. The following two facts appear in [KeW].

**Fact 5.2** *For each $\alpha < \omega_1$, there is a function $f$ in $b_1D(\mathbb{T})$ with $|f|_{KW} = \alpha$.*

**Fact 5.3** $|\cdot|_{KW}: D(\mathbb{T}) \to \omega_1$ *is a $\mathbf{\Pi}_1^1$-norm.*

By these two facts and the Boundedness Principle, we have the following:

**Corollary**[KeW] *The sets $D(\mathbb{T})$ and $b_1D(\mathbb{T})$ are $\mathbf{\Pi}_1^1$ non Borel subsets of $C(\mathbb{T})$.*

## 5.4 The Zalcwasser rank

We associate to each $f \in C(\mathbb{T})$, its Fourier series $S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$, where $\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(t)e^{-int}dt$. Let

$$S_n(f,t) = \sum_{k=-n}^{n} \hat{f}(n)e^{ikt}$$

be the $n$th partial sum of the Fourier series of $f$. We say "the Fourier series of $f$ converges at a point $t \in \mathbb{T}$" if the sequence $\langle S_n(f,t)\rangle_{n\in\mathbb{N}}$ converges. We will give a rank on $EC$, the collection of all continuous functions with everywhere convergent Fourier series. According to a standard theorem [Kat], if the Fourier series of $f$ at $t$ converges, then it must converge to $f(t)$. Hence,

$$\begin{aligned} EC =& \{f \in C(\mathbb{T}) : \forall\, t \in [0,2\pi], \langle\, S_n(f,t)\,\rangle_{n\in\mathbb{N}} \text{ converges }\} \\ =& \{f \in C(\mathbb{T}) : \forall\, t \in [0,2\pi], f(t) = \lim_{n\to\infty} S_n(f,t)\,\}. \end{aligned}$$

Let $f \in C(\mathbb{T})$, $P \subseteq \mathbb{T}$ be a closed set, and let $x \in P$. We define the value of the oscillation function of $f$ on $P$ at $x$ as follows.

$$\omega(x, f, P) = \inf_{\delta>0} \inf_{p\in\mathbb{N}} \sup\{|S_m(f, y) - S_n(f, y)| : m, n \geq p \;\&\; y \in P \;\&\; |x - y| < \delta\}.$$

Thus the oscillation function of $f$ on $P$ measures how bad the uniform convergence of the Fourier series of $f$, near $x$, is on $P$. For each $f \in C(\mathbb{T})$ and each $\epsilon > 0$, define the Z derived set of $P$ by

$$\partial^Z_{f,\epsilon}(P) = \{x \in P : \omega(x, f, P) \geq \epsilon\}.$$

Fix $f \in C(\mathbb{T})$ and $\epsilon > 0$. We define $\langle \partial^Z_{f,\epsilon}(P, \alpha)\rangle_{\alpha<\omega_1}$ by transfinite induction as follows. Let

$\partial^Z_{f,\epsilon}(P, 0) = P.$

$\partial^Z_{f,\epsilon}(P, \alpha + 1) = \partial^Z_{f,\epsilon}(\partial^Z_{f,\epsilon}(P, \alpha)).$

For limit ordinals $\lambda$, $\partial^Z_{f,\epsilon}(P, \lambda) = \bigcap_{\alpha<\lambda} \partial^Z_{f,\epsilon}(P, \alpha).$

Note that $\bigcup_{\epsilon>0} \partial^Z_{f,\epsilon}(P, \alpha) = \bigcup_{n\in\mathbb{N}} \partial^Z_{f,\frac{1}{n}}(P, \alpha)$. Define the sequence $\langle \partial^Z_f(P, \alpha)\rangle_{\alpha<\omega_1}$ by

$$\partial^Z_f(P, \alpha) = \bigcup_{n\in\mathbb{N}} \partial^Z_{f,\frac{1}{n}}(P, \alpha).$$

**Fact 5.4** $f \in EC \iff \exists\alpha < \omega_1, \partial^Z_f(\mathbb{T}, \alpha) = \emptyset.$

Using this fact, we define the Zalcwasser rank as follows. For each $f \in EC$, let $|f|_Z =$ the least ordinal $\alpha$ for which $\partial^Z_f(\mathbb{T}, \alpha) = \emptyset$.

**Fact 5.5** $|\cdot|_Z\colon EC \to \omega_1$ *is a* $\mathbf{\Pi}^1_1$*-norm.*

**Fact 5.6** *For each* $\alpha < \omega_1$*, there is a differentiable function* $f$ *such that* $|f|_Z \geq \alpha$.

In particular, by these facts and the Boundedness Principle, $D(\mathbb{T})$ is $\mathbf{\Pi}^1_1$ non Borel. Also the following theorem is true.

**Theorem** [Ajtai-Kechris] *EC is* $\mathbf{\Pi}^1_1$*-complete.* (See [AK].)

For a reference to the previous three facts and theorem, see [AK].

## 5.5 An equivalent definition of the Zalcwasser rank

As we have seen, the definition of the Zalcwasser rank is very natural. But when we compare the Zalcwasser rank to other ranks or attempt to calculate the Z derived set of a given closed subset and continuous function, the definition of the Zalcwasser rank is extremely difficult to work with. We give an equivalent definition of the Zalcwasser rank which is more practical. We need the following formula for Fourier series (See [Zy]).

**Proposition 5.7** *Let $\delta$ be a fixed positive number less than $\pi$. Then*

$$S_n(f,x) - f(x) = \frac{2}{\pi}\int_0^{\delta} \phi_x(t)\frac{\sin nt}{t}\,dt + o(1), \tag{1}$$

*where* $\phi_x(t) = \dfrac{f(x+t)+f(x-t)-2f(x)}{2}$.

*In this formula, $o(1)$ tends to 0 for any $x$ and the convergence to zero is uniform in every interval where $f$ is bounded.*

Let $f \in C(\mathbb{T})$, $P \subseteq \mathbb{T}$ be a closed set, and let $x \in P$. We define $\Omega(x,f,P)$ the analogous definition of the oscillation function as follows:

$$\Omega(x,f,P) = \inf_{\delta>0}\inf_{p\in\mathbb{N}}\sup\{\left|\int_0^{\delta}\phi_y(t)\frac{\sin nt}{t}dt\right| : n \geq p \;\&\; y \in P \;\&\; |x-y| < \delta\}.$$

In order to calculate $\Omega(x,f,P)$, we only need to know the local behavior of $f$. But for $\omega(x,f,P)$, we have to calculate the $n$th partial sum of the Fourier series of $f$ (which usually is not easy) before we can calculate $\omega(x,f,P)$. From this point of view, $\Omega(x,f,P)$ is more practical than $\omega(x,f,P)$. For each $f \in C(\mathbb{T})$ and each $\epsilon > 0$, we define the K derived set of $P$ by

$$\partial^K_{f,\epsilon}(P) = \{x \in P : \Omega(x,f,P) \geq \epsilon\}.$$

As in the definition of the Zalcwasser rank, we define $\langle\partial^K_{f,\epsilon}(P,\alpha)\rangle_{\alpha<\omega_1}$ for each $\epsilon > 0$ and then $\langle\partial^K_f(P,\alpha)\rangle_{\alpha<\omega_1}$ by transfinite induction.

**Theorem 5.8** *Let $f \in C(\mathbb{T})$ and $P \subseteq \mathbb{T}$ be a closed set. For each $\alpha < \omega_1$,*

*if $f \in EC$, then $\partial^K_f(P,\alpha) = \partial^Z_f(P,\alpha)$ and if $f \notin EC$, then $\partial^K_f(P,\alpha) \neq \emptyset$.*

*In particular, instead of $\partial_f^Z(P,\alpha)$, we can use $\partial_f^K(P,\alpha)$ to define the Zalcwasser rank.*

**Proof of Theorem 5.8** We fix $f \in EC$. Let $P$ be a closed subset of $\mathbb{T}$. By transfinite induction on $\alpha$, it is enough to show that for each $\epsilon > 0$,

$$\partial_{f,\epsilon}^K(P,\alpha) = \partial_{f,\frac{2}{\pi}\epsilon}^Z(P,\alpha),$$

since $\partial_f^K(P) = \bigcup_{\epsilon>0} \partial_{f,\epsilon}^K(P)$ and $\partial_f^Z(P) = \bigcup_{\epsilon>0} \partial_{f,\epsilon}^Z(P)$. Hence, it suffices to show that for $x \in P$,

$$\Omega(x,f,P) \geq \epsilon \iff \omega(x,f,P) \geq \frac{2}{\pi}\epsilon.$$

Let $x \in P$. By the definition of $\omega(x,f,P)$, for each $\delta > 0, p \in \mathbb{N}$,

$$\omega(x,f,P) \leq |S_n(f,y) - S_m(f,y)| \tag{5.2}$$

for all $n,m \geq p$ & $y \in P$ & $|x-y| < \delta$. In (5.2) letting $m \to \infty$, by (5.1) we have

$$\omega(x,f,P) \leq |S_n(f,y) - f(y)| \leq \frac{2}{\pi}\left|\int_0^\delta \phi_y(t)\frac{\sin nt}{t}dt\right| + o(1). \tag{5.3}$$

Since $f$ is continuous, in (5.3) $o(1)$ tends to 0 uniformly on all of $\mathbb{T}$. Hence by (5.3), we have

$$\omega(x,f,P) \leq \frac{2}{\pi}\Omega(x,f,P). \tag{5.4}$$

So $\omega(x,f,P) \geq 2\pi^{-1}\epsilon$ implies $\Omega(x,f,P) \geq \epsilon$. For the other direction, suppose $\omega(x,f,P) < 2\pi^{-1}\epsilon$. Let $\epsilon_0 > 0$ be such that $\omega(x,f,P) < 2\pi^{-1}\epsilon_0 < 2\pi^{-1}\epsilon$. Let $\epsilon_1 > 0$. Then for some $\delta > 0$ and $p \in \mathbb{N}$,

$$|S_n(f,y) - S_m(f,y)| \leq \frac{2}{\pi}\epsilon_0 + \epsilon_1 \tag{5.5}$$

for all $n,m \geq p$ & $y \in P$ & $|x-y| < \delta$. Here we can take $\delta \leq \pi$, since $f$ is periodic. In (5.5) letting $m \to \infty$, we have

$$|S_n(f,y) - f(y)| \leq \frac{2}{\pi}\epsilon_0 + 2\epsilon_1 \tag{5.6}$$

for all $n \geq p$ & $y \in P$ & $|x-y| < \delta$. Since (5.1) holds uniformly in $\mathbb{T}$, by (5.1) and (5.6), we have the following

$$\left| \frac{2}{\pi} \int_0^\delta \phi_y(t) \frac{\sin nt}{t} dt \right| \leq \frac{2}{\pi}\epsilon_0 + 3\epsilon_1,$$

for sufficiently large $n$ and for all $y \in P$ & $|x-y| < \delta$. Hence we conclude $\Omega(x,f,P) \leq \epsilon_0 + 3/2\pi^{-1}\epsilon_1$. Since $\epsilon_1$ is arbitrary, $\Omega(x,f,P) \leq \epsilon_0 < \epsilon$. It is not hard to see that $\partial_f^K(P) \neq \emptyset$ if $f \notin EC$. It is easy to see then that the second part is a consequence of the first part. □

## 5.6 The Kechris-Woodin rank is finer than the Zalcwasser rank

By Fact 5.2, the set $b_1D(\mathbb{T})$ has arbitrary Kechris-Woodin ranks below $\omega_1$. But for any $f \in b_1D(\mathbb{T})$, it is easy to see that the Fourier series of $f$ converges uniformly, i.e., the Zalcwasser rank of $f$ is 1. Hence it is natural to guess that the Kechris-Woodin rank is finer than the Zalcwasser rank. We verify this now.

**Theorem 5.9** *For given* $f \in D(\mathbb{T})$,

$$|f|_Z \leq |f|_{KW},$$

*i.e., the Kechris-Woodin rank is finer than the Zalcwasser rank.*

In order to prove this, we need the following lemma.

**Lemma 5.10** *Let* $f \in D(\mathbb{T})$ *and* $P$ *be a closed set in* $\mathbb{T}$. *Then for given* $\epsilon_1, \epsilon_2 > 0$,

$$\partial^Z_{f,\epsilon_1}(P) \subseteq \partial^{KW}_{f,\epsilon_2}(P).$$

**Proof of Lemma 5.10** Suppose $x \in P - \partial^{KW}_{f,\epsilon_2}(P)$. Then by the definition, $\exists\ \delta > 0$ such that $\forall\ p < q, r < s$ in $(-\delta + x, \delta + x) \cap [0, 2\pi]$ with $[p,q] \cap [r,s] \cap P \neq \emptyset$

$$(5.7) \qquad |\Delta f([p,q]) - \Delta f([r,s])| < \epsilon_2.$$

We fix positive values $\delta_1, \delta_2$ such that $\delta_2 \leq \pi$ and $x-\delta < x-(\delta_1+\delta_2)$. In particular, by (5.7),

$$2\frac{|\phi_y(t)|}{t} = |\Delta f([y-t,y]) - \Delta f([y,y+t])| \leq \epsilon_2 \tag{5.8}$$

holds for all $0 < t < \delta_2$ and all $y \in P \cap [x-\delta_1, x+\delta_1]$. Hence by the formula (5.1) and (5.8), we have the following: for all $y \in P \cap [x-\delta_1, x+\delta_1]$,

$$\begin{aligned} |S_n(f,y) - f(y)| &= \left| \frac{2}{\pi} \int_0^{\delta_2} \phi_x(t) \frac{\sin nt}{t} dt \right| + o(1) \\ &\leq \frac{1}{\pi} \int_0^{\delta_2} |\Delta f([y-t,y]) - \Delta f([y,y+t])| |\sin nt| dt + o(1) \\ &\leq \frac{1}{\pi} \int_0^{\delta_2} \epsilon_2 dt + o(1) = \frac{1}{\pi}\delta_2\epsilon_2 + o(1). \end{aligned}$$

Since our function $f$ is differentiable, Proposition 5.7 says that $o(1)$ tends to 0 uniformly in every interval, i.e., the value $o(1)$ is dependent on $n$ only. Hence for sufficiently large $n$ and all $y \in [x-\delta_1, x+\delta_1] \cap P$, we have

$$|S_n(f,y) - f(y)| \leq \frac{1}{\pi}\delta_2\epsilon_2 + \delta_2,$$

i.e., $\Omega(x,f,P) \leq \pi^{-1}\delta_2\epsilon_2 + \delta_2$ by the definition. Since $\delta_2$ is arbitrary, we have $\Omega(x,f,P) = 0$. Hence by Theorem 5.8, $x \notin \partial^Z_{f,\epsilon_1}(P)$. So we are done. □

**Proof of Theorem 5.9** Fix $f \in D(\mathbb{T})$. Suppose that for all ordinals $\alpha < \omega_1$, $\partial^Z_f(P,\alpha) \subseteq \partial^{KW}_f(P,\alpha)$. Since $f$ is in $D(\mathbb{T})$, by Fact 5.1, there is an $\alpha < \omega_1$ such that $\partial^{KW}_f(\mathbb{T},\alpha) = \emptyset$. Thus by our assumption, $\partial^Z_f(\mathbb{T},\alpha)$ must be the empty set, i.e., $|f|_Z$ is less than or equal to $\alpha$. Hence $|f|_Z \leq |f|_{KW}$. It is enough to show that for any $\epsilon > 0$, $\partial^Z_{f,\epsilon}(P,\alpha) \subseteq \partial^{KW}_{f,\epsilon}(P,\alpha)$ by transfinite induction on $\alpha$. For $\alpha = 0$ or $\alpha$, a limit ordinal, this is obvious. Suppose it is true for $\alpha$. Then by Lemma 5.10, we have

$$\partial^Z_{f,\epsilon}(\mathbb{T},\alpha+1) = \partial^Z_{f,\epsilon}(\partial^Z_{f,\epsilon}(\mathbb{T},\alpha)) \subseteq \partial^{KW}_{f,\epsilon}(\partial^Z_{f,\epsilon}(\mathbb{T},\alpha)).$$

It is easy to see that for all closed subsets $A$ and $B$ of $\mathbb{T}$ with $A \subseteq B$, $\partial^{KW}_{f,\epsilon}(A,\alpha) \subseteq \partial^{KW}_{f,\epsilon}(B,\alpha)$. Hence by the inductive assumption,

$$\partial^{KW}_{f,\epsilon}(\partial^Z_{f,\epsilon}(\mathbb{T},\alpha)) \subseteq \partial^{KW}_{f,\epsilon}(\partial^{KW}_{f,\epsilon}(\mathbb{T},\alpha)) = \partial^{KW}_{f,\epsilon}(\mathbb{T},\alpha+1).$$

Thus $\partial_f^Z(P,\alpha+1) \subseteq \partial_f^{KW}(P,\alpha+1)$. Hence the theorem is established. □

## 5.7 The Denjoy rank and remark

For each $f \in D(\mathbb{T})$, there is a canonical rank, which is called the Denjoy rank, $|f|_{DJ}$, from $D(\mathbb{T})$ to $\omega_1$ which measures how long it takes to recover $f$ from $f'$ via the Denjoy process (See [Br]). We briefly introduce the Denjoy rank. Let $g$ be a measurable function on $\mathbb{T}$ and $P$ be a closed subset of $\mathbb{T}$. We define the set of all singular points of $g$ over $P$ by

$$S(g,P) = \{x \in P : g \text{ is not Lebesgue integrable on } I \cap P \text{ for any open interval } I \text{ with } x \in I\}.$$

Let $f \in C(\mathbb{T})$. Let $\langle (a_n, b_n) \rangle$ be the sequence of open intervals complementing $P$ in $\mathbb{T}$. We define the set of divergence points of $f$ over $P$ by

$$D(f,P) = \{x \in P : \sum_I |f(b_n) - f(a_n)| \text{ diverges for every open interval } I \text{ with } x \in I\}.$$

Here $\sum_I$ indicates that the sum is to be taken over all the intervals $(a_n, b_n)$ which are contained in $I$. For $f \in D(\mathbb{T})$ and each closed subset $P$ of $\mathbb{T}$, we define the DJ derived set of $P$ by

$$\partial_f^{DJ}(P) = S(f',P) \cup D(f,P).$$

As before, we define the transfinite sequence $\langle \partial_f^{DJ}(P,\alpha) \rangle_{\alpha<\omega_1}$. For each $f \in D(\mathbb{T})$, let $|f|_{DJ} =$ the least ordinal $\alpha$ for which $\partial_f^{DJ}(\mathbb{T},\alpha) = \emptyset$. For $f \in D(\mathbb{T})$, it is known that $|f|_{DJ} = 1$ if and only if $f'$ is integrable. Hence $|f|_{DJ} = 1$ implies that the Fourier series of $f$ converges uniformly, i.e., $|f|_Z = 1$. So we might guess that $|f|_Z \leq |f|_{DJ}$.

**Conjecture** For each $f \in D(\mathbb{T})$, $|f|_Z \leq |f|_{DJ}$.

T. Ramsamujh [Ra] has shown that $|f|_{DJ} \leq |f|_{KW}$. In fact he proved that for any $\epsilon > 0$ and $\alpha < \omega_1$, $\partial_f^{DJ}(P,\alpha) \subseteq \partial_{f,\epsilon}^{KW}(P,\alpha)$. We have shown that for any $\epsilon_1, \epsilon_2 > 0$ and $\alpha < \omega_1$, $\partial_{f,\epsilon_1}^Z(P,\alpha) \subseteq \partial_{f,\epsilon_2}^{KW}(P,\alpha)$. So it seems likely that

$\partial^Z_{f,\epsilon}(P,\alpha) \subseteq \partial^{DJ}_f(P,\alpha)$ for any $\epsilon > 0$ and $\alpha$, i.e., $|f|_Z \leq |f|_{DJ}$. Recently, Ki [Ki3] has shown that this conjecure is not true. Namely for any ordinal $\alpha$, any nonzero ordinal $\beta$ and any countable ordinal $\gamma$ with $\alpha, \beta < \gamma$, one can construct a differentiable function $f$ on the unit circle such that

$$|f|_Z = \alpha + 1, \ |f|_{DJ} = \beta + 1 \text{ and } |f|_{KW} = \gamma.$$

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