A FORMULATION OF THE PROBLEM OF DISTRIBUTED VORTICITY IN THE SHOCK WAVE BOUNDARY LAYER INTERACTION PROCESS

Thesis by

William C. Barker

In Partial Fulfillment of the Requirements

For the Degree of

Aeronautical Engineer

California Institute of Technology

Pasadena, California

1950

ACKNOWLEDGEMENT

The author wishes to express his gratitude to Doctor Frank E. Marble for the continued interest and guidance he has lent to this undertaking and for his invaluable aid and suggestions on the many questions which arose during the development of this thesis.

ABSTRACT

The problem of shock wave boundary layer interaction is reviewed and attention focused upon the role of vorticity in the process. In order to simplify the physical considerations the two phenomena exhibited by vorticity in the interaction process-reflection and refraction of the disturbance, and transport of the vorticity from its original distribution - are divorced from one another. The reflection and refraction process is then considered apart from the other, and it is found that a boundary value problem can be formulated for it and formally solved for small perturbations from the undisturbed flow.

The perturbation component, which is associated with the pressure variation over the bounding surface, is set up and carried through to a point involving evaluation of a contour integral. This integral is so complex that its analytical evaluation would require many months of effort, and at this point it is thoughtthat a re-examination of the original problem would be in order.

Although numerical results would be desirable, the effort expended would have to be weighed against their relative contribution to an understanding of the overall problem.

TABLE OF CONTENTS

INTRODUCTION

The problem of an external shock wave, incident upon a boundary layer of both subsonic and supersonic character, has attracted much attention in recent years. Several treatments of the subject are noteworthy in that they demonstrate essential features of the physical problem through simple mathematical models and, by means of these models, the researcher gains an insight into the contributions of specific physical parameters to the actual flow. One such parameter, vorticity, has been included as a discontinuity phenomenon which, however, may mask its true role in the problem. The role of vorticity in the overall interaction process consists of two parts: one, the reflection and refraction of the traversing wave, and two, the transport of the original vorticity induced by the traversing wave. The present investigation is undertaken to shed some light upon the former process.

An early contribution to an understanding of the problem was made by Howarth, (Ref.1) who considered two semi-infinite fluids of different, but constant, subsonic and supersonic Mach numbers, separated by a discontinuity surface. He introduced a shock wave into the supersonic flow field and then studied its reflection and pressure effects at the discontinuity surface. Through this he was able to demonstrate quantitatively that there is up-stream propagation of pressure through the subsonic half plane and therefore change in the flow conditions ahead of shock.

A later extension of the problem in the direction of the real flow was given by Tsien and Finston (Ref.2) where in addition to the assumptions of Howarth, they introduce a flat bounding surface into the subsonic field. Two cases are discussed with these conditions, that of an incoming compression wave, and flow in a corner; both investigations confirming the upstream propagation of pressure through the subsonic layer. The order of magnitude of the results so obtained, however, is at variance with experiment. (see Ref. 3).

Another approach, based upon the Pohlhausen method and simple supersonic flow theory, was adopted by Lees (Ref. 4); while Marble, (Ref. 5) restricts himself to first order reflections in a purely supersonic flow. Reference 3, referred to above, summarized the work of these authors and in addition presents the experimental findings to date.

The problem treated in this paper is limited to the reflections and refractions of a weak wave traversing a shear layer. The medium is so chosen that the propagation velocity becomes imaginary for a part of the flow - i.e. the equation changes type, and there is no change during the interaction process from the original vorticity distribution.

Wave motion in such a medium is represented by the usual equation

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

where in this case ψ is identified with the stream function and the propagation velocity c with $\sqrt{M^2-1}$. The boundary value problem may then be related to that of Tsien and Finston with the following additions:

a) The no-slip condition is applied at the bounding wall.

b) The supersonic or positive propagation velocity shear layer is bounded by a uniform semi-infinite flow.

c) The shear is distributed linearly in each of the two parallel layers.

It is hoped that this formulation of one aspect of the vorticity question will shed some light upon the role of a distributed shear in the overall problem of shock wave boundary layer interaction. An answer to this question could lead to a better understanding of the interaction process as a whole, the difference between laminar and turbulent interaction, instability, and perhaps advances in the knowledge of the contribution of viscosity to the complete problem.

INTERACTION OF A WEAK WAVE AND A SHEAR LAYER IN THE ABSENCE OF VORTICITY TRANSPORT

Mach Number Distribution

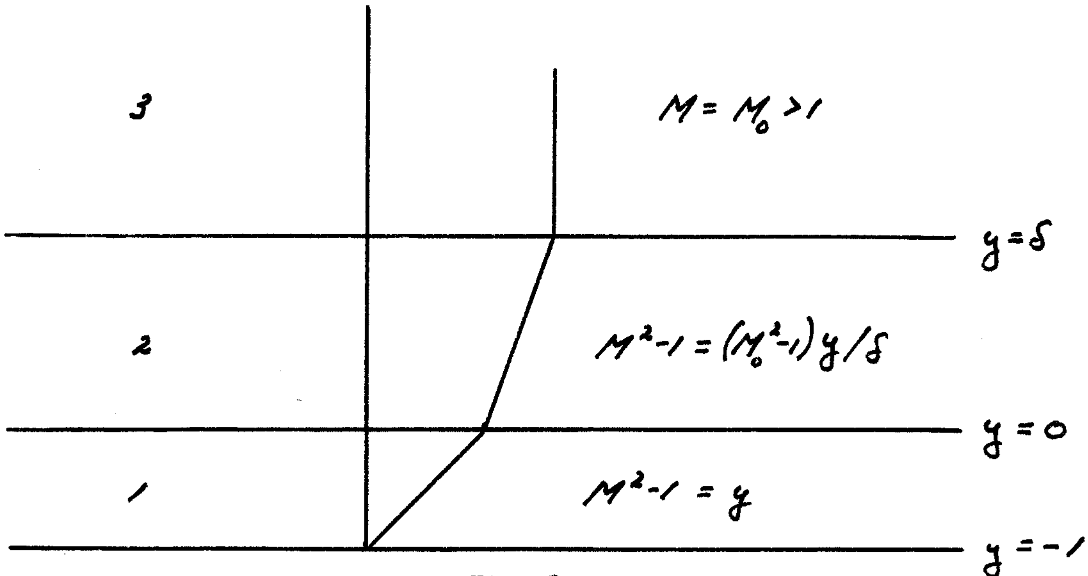

Fig. 1

Differential equation: -

$$(1 - M^2) \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad 1.$$

where ψ is the perturbation stream function and M is the local Mach number.

In region 3 the solutions of equation 1 are simple:

$$\psi^{(3)} = f\left(x + \sqrt{M_0^2 - 1}\, y\right) + g\left(x - \sqrt{M_0^2 - 1}\, y\right) \qquad 2.$$

Following Tsien and Finston, the incident wave will be expressed as a continuous disturbance which becomes a sharp wave only under a limiting process.

This wave will be taken as:

$$\psi_1^{(3)} = -\frac{u_0 \varepsilon}{\beta} e^{-\beta(x+\sqrt{M_0^2-1}\,y)} \qquad , x+\sqrt{M_0^2-1}\,y \geq 0$$

$$= 0 \qquad , x+\sqrt{M_0^2-1}\,y < 0$$

3.

The computation of velocity components from the perturbation stream function requires some consideration. Let $\bar{\psi}$ be the stream function corresponding to the main flow in the x direction so that:

$$\frac{\partial \bar{\psi}}{\partial y} = \frac{\bar{\rho}}{\rho_0} \bar{u}$$

4.

where $\bar{u}$ is the main flow depending upon y and $\bar{\rho}$ is the density of the main flow also depending upon y. Then if ψ is the perturbation stream function, the resultant derivative in the y direction is:

$$\frac{\partial}{\partial y}(\bar{\psi} + \psi) = \frac{\rho}{\rho_0}(\bar{u} + u)$$

5.

where ρ is the density of the perturbed stream and u is the perturbation velocity. Writing this in the form:

$$\frac{\partial}{\partial y}(\bar{\psi} + \psi) = \frac{\rho - \bar{\rho}}{\rho_0}(\bar{u} + u) + \frac{\bar{\rho}}{\rho_0}(\bar{u} + u)$$

it is clear that within the linearized approximation

$$\frac{\partial}{\partial y}(\bar{\psi}+\psi) \approx \frac{\rho-\bar{\rho}}{\rho_0}(\bar{u}) + \frac{\bar{\rho}}{\rho_0}\bar{u} + \frac{\bar{\rho}}{\rho_0}u \qquad 6.$$

Thus from equation ψ it follows that:

$$\frac{\partial\psi}{\partial y} = \frac{\rho-\bar{\rho}}{\rho_0}\bar{u} + \frac{\bar{\rho}}{\rho_0}u \qquad 7.$$

The perturbation is isentropic along the streamlines so that:

$$dP = -\rho(\bar{u}+u)u \qquad 8.$$

and

$$\frac{dP}{\rho-\bar{\rho}} \approx a^2 \qquad 9.$$

where a is the velocity of sound corresponding to the original flow. Hence:

$$\rho-\bar{\rho} \approx -\frac{\bar{\rho}\,\bar{u}}{a^2}u \qquad 10.$$

or

$$\frac{\partial\psi}{\partial y} = -\frac{\bar{\rho}}{\rho_0}\frac{\bar{u}^2}{a^2}u + \frac{\bar{\rho}}{\rho_0}u$$

$$= \frac{\bar{\rho}}{\rho_0}(1-\bar{M}^2)u \qquad 11.$$

Then, finally the velocity perturbation in the direction of the main stream may be computed from the perturbation stream function as:

$$u = -\frac{\rho_0}{\bar{\rho}}\frac{1}{\bar{M}^2-1}\frac{\partial \psi}{\partial y}$$

12.

The perturbation normal to the direction of main flow may also be calculated:

$$\frac{\partial}{\partial x}(\bar{\psi}+\psi) \equiv \frac{\partial \psi}{\partial x} = -\frac{\rho}{\rho_0} v \approx -\frac{\bar{\rho}}{\rho_0} v$$

consequently:

$$v = -\frac{\rho_0}{\bar{\rho}}\frac{\partial \psi}{\partial x}$$

13.

With the aid of formulas 12 and 13 the velocity perturbations corresponding to the stream function of equation 3 may be calculated-

$$u_1^{(3)} = \left(\frac{U_0 \varepsilon}{\sqrt{M_0^2-1}}\right) e^{-\beta(x+\sqrt{M_0^2-1}\,y)} \quad , x+\sqrt{M_0^2-1}\,y \geq 0$$

$$= 0 \quad , x+\sqrt{M_0^2-1}\,y < 0$$

14.

and

$$v_1^{(3)} = -U_0 \varepsilon\, e^{-\beta(x+\sqrt{M_0^2-1}\,y)} \quad , x+\sqrt{M_0^2-1}\,y \geq 0$$

$$= 0 \quad , x+\sqrt{M_0^2-1}\,y < 0$$

15.

When $\beta \to 0$ these velocities change discontinuously across the characteristic $x + \sqrt{M_0^2 - 1}\, y = 0$. This will be taken as the disturbance entering the shear flow from outside and consequently the only further disturbance in region 3 will be that caused by the deformations of the shear flow itself.* These disturbances are then propagated in an outward direction and consequently may be represented as:

$$\varphi_2^{(3)} = g\left(x - \sqrt{M_0^2 - 1}\, y\right) \qquad 16.$$

It is convenient to express the stream functions $\varphi_1^{(3)}$ & $\varphi_2^{(3)}$ (equations 3 and 16) as Fourier integrals. As shown by Tsien and Finston,

$$\varphi_1^{(3)} = -\frac{U_0 \varepsilon}{\pi} \int_0^{\infty} \left\{ \frac{1}{\beta^2 + \lambda^2} \cos\lambda\left(x + \sqrt{M_0^2 - 1}\, y\right) + \frac{\lambda}{\beta(\beta^2 + \lambda^2)} \sin\lambda\left(x + \sqrt{M_0^2 - 1}\, y\right) \right\} d\lambda \qquad 17.$$

However since:

$$\cos\lambda\left(x + \sqrt{M_0^2 - 1}\, y\right) = \cos\lambda x \cos\lambda\sqrt{M_0^2 - 1}\, y - \sin\lambda x \sin\lambda\sqrt{M_0^2 - 1}\, y$$

and

$$\sin\lambda\left(x + \sqrt{M_0^2 - 1}\, y\right) = \sin\lambda x \cos\lambda\sqrt{M_0^2 - 1}\, y + \cos\lambda x \sin\lambda\sqrt{M_0^2 - 1}\, y$$

the integral may be written:

$$\varphi_1^{(3)} = -\frac{U_0 \varepsilon}{\pi} \int_0^{\infty} \left(\frac{1}{\beta^2 + \lambda^2} \cos\lambda\sqrt{M_0^2 - 1}\, y + \frac{\lambda}{\beta(\beta^2 + \lambda^2)} \sin\lambda\sqrt{M_0^2 - 1}\, y \right) \cos\lambda x \, d\lambda$$

* This is neglected in the present analysis.

$$+\frac{u_0\varepsilon}{\pi}\int_0^\infty\left(\frac{1}{\beta^2+\lambda^2}\sin\lambda\sqrt{M_0^2-1}\,y-\frac{\lambda}{\beta(\beta^2+\lambda^2)}\cos\lambda\sqrt{M_0^2-1}\,y\right)\sin\lambda x\,d\lambda$$

18.

The stream function $\psi_2^{(3)}$ may also be written as a Fourier integral:

$$\psi_2^{(3)}=u_0\int_0^\infty\left(A^{(3)}(\lambda)\sin\lambda(x-\sqrt{M_0^2-1}\,y)+B^{(3)}(\lambda)\cos\lambda(x-\sqrt{M_0^2-1}\,y)\right)d\lambda$$

19.

The complete stream function in region 3 is then $\psi_1^{(3)}+\psi_2^{(3)}$.

Representation of the stream function in region 2 requires solution of equation 1 for the particular Mach number distribution

$$M^2-1=\left(\frac{M_0^2-1}{\delta}\right)y \qquad 20.$$

the equation is then

$$\frac{M_0^2-1}{\delta}\,y\,\frac{\partial^2\psi^{(2)}}{\partial x^2}-\frac{\partial^2\psi^{(2)}}{\partial y^2}=0$$

21.

If the function $\psi^{(2)}$ is assumed to be separable, $\psi^{(2)}=p(x)g(y)$ it follows that:

$$p''(x)+\lambda^2p(x)=0 \qquad 22.$$

$$g''(y)+\lambda^2\frac{M_0^2-1}{\delta}\,y\,g(y)=0 \qquad 23.$$

The solutions of equation 22 are simply the trigonometric functions

$$p(x) = \sin \lambda x, \ \cos \lambda x \tag{24}$$

while the solutions of 23 are Bessel functions. These are:

$$g(y) = \begin{cases} \sqrt{y}\, J_{1/3}\left(\frac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\right), \\ \sqrt{y}\, J_{-1/3}\left(\frac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\right) \end{cases} \tag{25}$$

The stream function $\psi^{(2)}$ may then be represented in the Fourier integral form:

$$\psi^{(2)}(x,y) = U_0\sqrt{y}\int_0^{\infty}\left\{A^{(2)}(\lambda)\, J_{1/3}\left(\lambda\frac{2}{3}\sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\right) + B^{(2)}(\lambda)\, J_{-1/3}\left(\lambda\frac{2}{3}\sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\right)\right\}\sin\lambda x\, d\lambda$$

$$+ U_0\sqrt{y}\int_0^{\infty}\left\{C^{(2)}(\lambda)\, J_{1/3}\left(\lambda\frac{2}{3}\sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\right) + D^{(2)}(\lambda)\, J_{-1/3}\left(\lambda\frac{2}{3}\sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\right)\right\}\cos\lambda x\, d\lambda \tag{26}$$

The stream function in region 1 may now be written down with little trouble, for it satifies the differential equation:

$$y\frac{\partial^2\psi^{(1)}}{\partial x^2} + \frac{\partial^2\psi^{(1)}}{\partial y^2} = 0 \tag{27}$$

The solutions of this equation are then formally:

$$\psi^{(1)}(x,y) \sim \begin{Bmatrix} \sin \lambda x \\ \cos \lambda x \end{Bmatrix} \begin{Bmatrix} \sqrt{y}\, J_{1/3}\left(\frac{2}{3}\lambda y^{3/2}\right) \\ \sqrt{y}\, J_{-1/3}\left(\frac{2}{3}\lambda y^{3/2}\right) \end{Bmatrix}$$

28.

where, however, the Bessel functions are to be employed only for negative values of y. The results will appear in a more usable form if the substitution $z = |y|$ is made. Then

$$\sqrt{y}\; J_{1/3}\left(\frac{2}{3}\lambda y^{3/2}\right) = i\sqrt{z}\, J_{1/3}\left(\frac{2}{3}\lambda (-i) z^{3/2}\right)$$

$$= i(-1)^{1/3}\sqrt{z}\, J_{1/3}\left(\frac{2}{3}\lambda i z^{3/2}\right)$$

$$= i(-1)^{1/3} i^{1/3} \sqrt{z}\, I_{1/3}\left(\frac{2}{3}\lambda z^{3/2}\right)$$

$$= -\sqrt{z}\, I_{1/3}\left(\frac{2}{3}\lambda z^{3/2}\right)$$

Similarly:

$$\sqrt{y}\; J_{-1/3}\left(\frac{2}{3}\lambda y^{3/2}\right) = i\sqrt{z}\, J_{-1/3}\left(\frac{2}{3}\lambda (-i) z^{3/2}\right)$$

$$= i(-1)^{-1/3}\sqrt{z}\, J_{-1/3}\left(\frac{2}{3}\lambda i z^{3/2}\right)$$

$$= i(-1)^{-1/3} i^{-1/3} \sqrt{z}\, I_{-1/3}\left(\frac{2}{3}\lambda z^{3/2}\right)$$

$$= \sqrt{z}\, I_{-1/3}\left(\frac{2}{3}\lambda z^{3/2}\right)$$

Hence, the solution is of the form:

$$\psi^{(1)}(x,y) \sim \begin{Bmatrix} \sin\lambda x \\ \cos\lambda x \end{Bmatrix} \begin{Bmatrix} \sqrt{z}\, I_{1/3}\left(\tfrac{2}{3}\lambda z^{3/2}\right) \\ \sqrt{z}\, I_{-1/3}\left(\tfrac{2}{3}\lambda z^{3/2}\right) \end{Bmatrix} \qquad 29.$$

and all of the functions are real for $0 \leq z \leq 1$.

The actual solutions can now be written as a Fourier integral in the form:

$$\psi^{(1)}(x,z) = U_0\sqrt{z}\int_0^\infty \left\{A^{(1)}(\lambda) I_{1/3}\left(\tfrac{2}{3}\lambda z^{3/2}\right) + B^{(1)}(\lambda) I_{-1/3}\left(\tfrac{2}{3}\lambda z^{3/2}\right)\right\} \sin\lambda x \, d\lambda$$

$$+ U_0\sqrt{z}\int_0^\infty \left\{C^{(1)}(\lambda) I_{1/3}\left(\tfrac{2}{3}\lambda z^{3/2}\right) + D^{(1)}(\lambda) I_{-1/3}\left(\tfrac{2}{3}\lambda z^{3/2}\right)\right\} \cos\lambda x \, d\lambda$$

The boundary condition to be satisfied on the wall at $y = -1 = -z$, is simply that the vertical perturbation velocity vanish which is equivalent to the stream function being a constant (say zero) along the line $z = 1$. Then there results:

$$\psi^{(1)}(x,z) = U_0\sqrt{z}\int_0^\infty \left\{I_{-1/3}\left(\tfrac{2}{3}\lambda\right) I_{1/3}\left(\tfrac{2}{3}\lambda z^{3/2}\right) - I_{1/3}\left(\tfrac{2}{3}\lambda\right) I_{-1/3}\left(\tfrac{2}{3}\lambda z^{3/2}\right)\right\} \times \left(A^{(1)}(\lambda)\sin\lambda x + B^{(1)}(\lambda)\cos\lambda x\right) d\lambda \qquad 30.$$

which satisfies this condition identically.

There are eight unknown functions to be determined by matching the solutions at the interfaces. Physically it is necessary that the flow angle and the pressure perturbations be matched along the lines $y = \delta$ and $y = 0$. However, some care must be exercised in applying the matching procedure at $y = 0$. for this is the sonic line of the unperturbed flow and the solutions may be unsatisfactory at this point. Therefore, a strip of width 2ξ will be omitted from the flow so that the sonic portion is initially omitted. The matching conditions are then:

$$v^{(3)}(x,\delta) = v^{(2)}(x,\delta)$$

$$u^{(3)}(x,\delta) = u^{(2)}(x,\delta)$$

$$v^{(2)}(x,\xi) = v^{(1)}(x,-\xi)\ \frac{U(\xi)}{U(-\xi)}$$

$$\rho(\xi)U(\xi)\,u^{(2)}(x,\xi) = \rho(-\xi)\,U(-\xi)\,u^{(1)}(x,-\xi) \qquad 31.$$

In terms of the stream function these are:

$$\psi^{(3)}(x,\delta) = \psi^{(2)}(x,\delta)$$

$$\frac{\partial \psi^{(3)}}{\partial y}(x,\delta) = \frac{\partial \psi^{(2)}}{\partial y}(x,\delta)$$

$$\frac{1}{\rho(\xi)U(\xi)}\,\psi^{(2)}(x,\xi) = \frac{1}{\rho(-\xi)U(-\xi)}\,\psi^{(1)}(x,-\xi)$$

$$\frac{U(\xi)}{M^2(\xi)-1}\,\frac{\partial \psi^{(2)}}{\partial y} = \frac{U(-\xi)}{M^2(-\xi)-1}\,\frac{\partial \psi^{(1)}}{\partial y} \qquad 32.$$

These four relations will give eight relations among the unknown functions and hence sufficient information to determine them.

The first of equations 32 gives

$$\left(-\frac{\epsilon}{\pi}\frac{1}{\beta^2+\lambda^2}+B^{(3)}(\lambda)\right)\cos\lambda\sqrt{M_0^2-1}\,\delta-\left(\frac{\epsilon}{\pi}\frac{\lambda}{\beta(\beta^2+\lambda^2)}+A^{(3)}(\lambda)\right)\sin\lambda\sqrt{M_0^2-1}\,\delta$$

$$=C^{2}(\lambda)\sqrt{\delta}\,J_{1/3}\left(\tfrac{2}{3}\lambda\sqrt{M_0^2-1}\,\delta\right)+D^{(2)}(\lambda)\sqrt{\delta}\,J_{-1/3}\left(\tfrac{2}{3}\lambda\sqrt{M_0^2-1}\,\delta\right)$$

33.

and

$$\left(\frac{\epsilon}{\pi}\frac{1}{\beta^2+\lambda^2}+B^{(3)}(\lambda)\right)\sin\lambda\sqrt{M_0^2-1}\,\delta+\left(-\frac{\epsilon}{\pi}\frac{\lambda}{\beta(\beta^2+\lambda^2)}+A^{(3)}(\lambda)\right)\cos\lambda\sqrt{M_0^2-1}\,\delta$$

$$=A^{(2)}(\lambda)\sqrt{\delta}\,J_{1/3}\left(\tfrac{2}{3}\lambda\sqrt{M_0^2-1}\,\delta\right)+B^{(2)}(\lambda)\sqrt{\delta}\,J_{-1/3}\left(\tfrac{2}{3}\lambda\sqrt{M_0^2-1}\,\delta\right)$$

34.

The third condition gives, assuming $p(-\zeta)\approx p(\zeta);\ u(-\zeta)\approx u(\zeta)$

$$A^{(1)}(\lambda)\left\{I_{-1/3}\left(\tfrac{2}{3}\lambda\right)\sqrt{\zeta}\,I_{1/3}\left(\tfrac{2}{3}\lambda\zeta\right)-I_{1/3}\left(\tfrac{2}{3}\lambda\right)\sqrt{\zeta}\,I_{-1/3}\left(\tfrac{2}{3}\lambda\zeta\right)\right\}$$

$$=A^{(2)}(\lambda)\sqrt{\zeta}\,J_{1/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\zeta^{3/2}\right)+B^{(2)}(\lambda)\sqrt{\zeta}\,J_{-1/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\zeta^{3/2}\right)$$

35.

$$B^{(1)}(\lambda)\left\{I_{-1/3}\left(\tfrac{2}{3}\lambda\right)\sqrt{\zeta}\,I_{1/3}\left(\tfrac{2}{3}\lambda\zeta\right)-I_{1/3}\left(\tfrac{2}{3}\lambda\right)\sqrt{\zeta}\,I_{-1/3}\left(\tfrac{2}{3}\lambda\zeta\right)\right\}$$

$$=C^{(2)}(\lambda)\sqrt{\zeta}\,J_{1/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\zeta^{3/2}\right)+D^{(2)}(\lambda)\sqrt{\zeta}\,J_{-1/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\zeta^{3/2}\right)$$

36.

In order to complete the substitution into the matching conditions 32, it is necessary to calculate the following three partial derivatives:

$$\frac{\partial \psi^{(3)}}{\partial y} = \frac{\partial}{\partial y}\left\{ u_0 \int_0^\infty \left[\left(\frac{-\varepsilon/\pi}{\beta^2+\lambda^2} + \overset{(3)}{B}(\lambda)\right)\cos\lambda\sqrt{M_0^2-1}\,y - \left(\frac{\varepsilon/\pi\cdot\lambda}{\beta(\beta^2+\lambda^2)} + \overset{(3)}{A}(\lambda)\right)\sin\lambda\sqrt{M_0^2-1}\,y\right]\cos\lambda x\,d\lambda\right.$$

$$\left. + u_0\int_0^\infty\left[\left(\frac{\varepsilon/\pi}{\beta^2+\lambda^2} + \overset{(3)}{B}(\lambda)\right)\sin\lambda\sqrt{M_0^2-1}\,y + \left(\frac{-\varepsilon/\pi\cdot\lambda}{\beta(\beta^2+\lambda^2)} + \overset{(3)}{A}(\lambda)\right)\cos\lambda\sqrt{M_0^2-1}\,y\right]\sin\lambda x\,d\lambda\right\}$$

$$= u_0\int_0^\infty\left\{\left(\frac{\varepsilon/\pi}{\beta^2+\lambda^2} - \overset{(3)}{B}(\lambda)\right)\lambda\sqrt{M_0^2-1}\sin\lambda\sqrt{M_0^2-1}\,y - \left(\frac{\varepsilon/\pi\cdot\lambda}{\beta(\beta^2+\lambda^2)} + \overset{(3)}{A}(\lambda)\right)\lambda\sqrt{M_0^2-1}\cos\lambda\sqrt{M_0^2-1}\,y\right\}\cos\lambda x\,d\lambda$$

$$+ u_0\int_0^\infty\left\{\left(\frac{\varepsilon/\pi}{\beta^2+\lambda^2} + \overset{(3)}{B}(\lambda)\right)\lambda\sqrt{M_0^2-1}\cos\lambda\sqrt{M_0^2-1}\,y + \left(\frac{\varepsilon/\pi\cdot\lambda}{\beta(\beta^2+\lambda^2)} - \overset{(3)}{A}(\lambda)\right)\lambda\sqrt{M_0^2-1}\sin\lambda\sqrt{M_0^2-1}\,y\right\}\sin\lambda x\,d\lambda$$

37.

To differentiate $\psi^{(2)}$ we must compute the quantity

$$\frac{\partial}{\partial y}\left(\sqrt{y}\,J_{1/3}(\gamma y^{3/2})\right) = \frac{1}{2}y^{-1/2}J_{1/3}(\gamma y^{3/2}) - \frac{1/3}{\gamma y^{3/2}}J_{1/3}(\gamma y^{3/2})\cdot\frac{3}{2}\gamma y^{1/2}y^{1/2}$$

$$+ J_{-2/3}(\gamma y^{3/2})\cdot\frac{3}{2}\gamma y^{1/2}y^{1/2}$$

$$= \frac{3}{2}\gamma y\,J_{-2/3}(\gamma y^{3/2})$$

38.

and

$$\frac{\partial}{\partial y}\left(\sqrt{y}\, J_{-1/3}(\gamma y^{3/2})\right) = \frac{1}{2} y^{-1/2} J_{-1/3}(\gamma y^{3/2}) - \frac{1/3}{\gamma y^{3/2}} J_{-1/3}(\gamma y^{3/2}) \cdot \frac{3}{2}\gamma y^{1/2} y^{1/2}$$

$$- J_{2/3}(\gamma y^{3/2}) \cdot \frac{3}{2}\gamma y^{1/2} y^{1/2}$$

$$= -\frac{3}{2}\gamma y\, J_{2/3}(\gamma y^{3/2})$$

39.

The derivative of $\psi^{(1)}$ requires similar differentiation of the I_ν functions and these must be computed separately since they satisfy different recursion relations.

$$-\frac{\partial}{\partial y}\left(\sqrt{z}\, I_{1/3}(\gamma z^{3/2})\right) = \frac{\partial}{\partial z}\left(\sqrt{z}\, I_{1/3}(\gamma z^{3/2})\right)$$

$$= \frac{1}{2} z^{-1/2} I_{1/3}(\gamma z^{3/2}) - \frac{1/3}{\gamma z^{3/2}} I_{1/3} \cdot \frac{3}{2}\gamma z^{1/2} z^{1/2} + I_{-2/3} \cdot \frac{3}{2}\gamma z^{1/2} z^{1/2}$$

$$= \frac{3}{2}\gamma z\, I_{-2/3}(\gamma z^{3/2})$$

$$\therefore \quad \frac{\partial}{\partial y}\left(\sqrt{z}\, I_{1/3}(\gamma z^{3/2})\right) = -\frac{3}{2}\gamma z\, I_{-2/3}(\gamma z^{3/2}) \qquad 40.$$

$$-\frac{\partial}{\partial y}\left(\sqrt{z}\, I_{-1/3}(\gamma z^{3/2})\right) = \frac{\partial}{\partial z}\left(\sqrt{z}\, I_{-1/3}(\gamma z^{3/2})\right)$$

$$= \frac{1}{2} z^{-1/2} I_{-1/3}(\gamma z^{3/2}) - \frac{1/2}{\gamma z^{3/2}} I_{-1/3}(\gamma z^{3/2}) \cdot \frac{3}{2}\gamma z^{1/2} z^{1/2}$$

$$+ I_{2/3}(\gamma z^{3/2}) \cdot \frac{3}{2}\gamma z^{1/2} z^{1/2}$$

$$\therefore \frac{\partial}{\partial y}\left(\sqrt{z}\, I_{-1/3}(\gamma z^{3/2})\right) = -\frac{3}{2}\gamma z\, I_{2/3}(\gamma z^{3/2}) \qquad 41.$$

The derivative of $\psi^{(2)}$ is now

$$\frac{\partial \psi^{(2)}}{\partial y} = U_0 \int_0^\infty \Big\{ A^{(2)}(\lambda)\, \lambda \sqrt{\frac{M_0^2-1}{\delta}}\, y\, J_{-2/3}\Big(\tfrac{2}{3}\lambda \sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\Big) - B^{(2)}(\lambda)\, \lambda \sqrt{\frac{M_0^2-1}{\delta}}\, y\, J_{2/3}\Big(\tfrac{2}{3}\lambda \sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\Big)\Big\} \sin \lambda x \, d\lambda$$

$$+ U_0 \int_0^\infty \Big\{ C^{(2)}(\lambda)\, \lambda \sqrt{\frac{M_0^2-1}{\delta}}\, y\, J_{-2/3}\Big(\tfrac{2}{3}\lambda \sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\Big) - D^{(2)}(\lambda)\, \lambda \sqrt{\frac{M_0^2-1}{\delta}}\, y\, J_{2/3}\Big(\tfrac{2}{3}\lambda \sqrt{\frac{M_0^2-1}{\delta}}\, y^{3/2}\Big)\Big\} \cos \lambda x \, d\lambda \qquad 42.$$

Finally the derivative of $\psi^{(1)}$ is just:

$$\frac{\partial \psi^{(1)}}{\partial y} = U_0 \int_0^\infty \Big\{ -I_{-1/3}(\tfrac{2}{3}\lambda)\, \lambda z\, I_{-2/3}(\tfrac{2}{3}\lambda z^{3/2}) + I_{1/3}(\tfrac{2}{3}\lambda)\, \lambda z\, I_{2/3}(\tfrac{2}{3}\lambda z^{3/2}) \Big\}$$

$$\cdot \Big(A^{(1)}(\lambda) \sin \lambda x + B^{(1)}(\lambda) \cos \lambda x \Big)\, d\lambda \qquad 43.$$

Using these relations, the remaining matching relations may be completed. The second of equations 32 becomes:

$$\Big(\frac{\varepsilon/\pi}{\beta^2+\lambda^2} - B^{(3)}(\lambda)\Big)\lambda\sqrt{M_0^2-1}\,\sin\lambda\sqrt{M_0^2-1}\,\delta - \Big(\frac{\varepsilon/\pi\cdot\lambda}{\beta(\beta^2+\lambda^2)} + A^{(3)}(\lambda)\Big)\lambda\sqrt{M_0^2-1}\,\cos\lambda\sqrt{M_0^2-1}\,\delta$$

$$= C^{(3)}(\lambda)\,\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta\, J_{-2/3}\Big(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta^{3/2}\Big) - D^{(2)}(\lambda)\,\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta\, J_{2/3}\Big(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta^{3/2}\Big) \qquad 44.$$

$$\Big(\frac{\varepsilon/\pi}{\beta^2+\lambda^2} + B^{(3)}(\lambda)\Big)\lambda\sqrt{M_0^2-1}\,\cos\lambda\sqrt{M_0^2-1}\,\delta + \Big(\frac{\varepsilon/\pi\cdot\lambda}{\beta(\beta^2+\lambda^2)} - A^{(3)}(\lambda)\Big)\lambda\sqrt{M_0^2-1}\,\sin\lambda\sqrt{M_0^2-1}\,\delta$$

$$= A^{(2)}(\lambda)\,\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta\, J_{-2/3}\Big(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta^{3/2}\Big) - B^{(2)}(\lambda)\,\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta\, J_{2/3}\Big(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta^{3/2}\Big) \qquad 45.$$

The final relation, the last of equations 32, is assuming $U(\xi)=U(-\xi)$

$$\overset{(2)}{A}(\lambda)\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi\, J_{-2/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi^{3/2}\right)-\overset{(2)}{B}(\lambda)\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi\, J_{2/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi^{3/2}\right)$$

$$=\frac{M^2(\xi)-1}{M^2(-\xi)-1}\overset{(1)}{A}(\lambda)\Big\{-I_{-1/3}(\tfrac{2}{3}\lambda)\lambda\,\xi\, I_{-2/3}(\tfrac{2}{3}\lambda\,\xi^{3/2})$$
$$+I_{1/3}(\tfrac{2}{3}\lambda)\lambda\,\xi\, I_{2/3}(\tfrac{2}{3}\lambda\,\xi^{3/2})\Big\}$$

46.

$$\overset{(2)}{C}(\lambda)\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi\, J_{-2/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi^{3/2}\right)-\overset{(2)}{D}(\lambda)\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi\, J_{2/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi^{3/2}\right)$$

$$=\frac{M^2(\xi)-1}{M^2(-\xi)-1}\overset{(1)}{B}(\lambda)\Big\{-I_{-1/3}(\tfrac{2}{3}\lambda)\lambda\,\xi\, I_{-2/3}(\tfrac{2}{3}\lambda\,\xi^{3/2})$$
$$+I_{1/3}(\tfrac{2}{3}\lambda)\lambda\,\xi\, I_{2/3}(\tfrac{2}{3}\lambda\,\xi^{3/2})\Big\}$$

47.

Now equations 33,34,35,36 and equations 44,45,46,47 are the eight linear relations among the eight functions and may be solved.

Define

$$A^{(1)} = A_1 \qquad C^{(2)} = A_5$$
$$B^{(1)} = A_2 \qquad D^{(2)} = A_6$$
$$A^{(2)} = A_3 \qquad A^{(3)} = A_7$$
$$B^{(2)} = A_4 \qquad B^{(3)} = A_8$$

Then rewrite the matching equations

$$A_1 f_{11} - A_3 f_{13} - A_4 f_{14} = 0 \qquad 35a$$
$$A_1 f_{21} - A_3 f_{23} + A_4 f_{24} = 0 \qquad 46a$$
$$A_2 f_{32} - A_5 f_{35} - A_6 f_{36} = 0 \qquad 36a$$
$$A_2 f_{42} - A_5 f_{45} + A_6 f_{46} = 0 \qquad 47a$$
$$A_3 f_{53} + A_4 f_{54} - A_7 f_{57} - A_8 f_{58} = F_5 \qquad 34a$$
$$A_3 f_{63} - A_4 f_{64} + A_7 f_{67} - A_8 f_{68} = F_6 \qquad 45a$$
$$A_5 f_{75} + A_6 f_{76} + A_7 f_{77} - A_8 f_{78} = F_7 \qquad 33a$$
$$A_5 f_{85} - A_6 f_{86} + A_7 f_{87} + A_8 f_{88} = F_8 \qquad 44a$$

Where the f_{ij} are defined as follows

$$f_{11} = I_{-1/3}\left(\tfrac{2}{3}\lambda\right)\sqrt{\zeta}\, I_{1/3}\left(\tfrac{2}{3}\lambda\zeta\right) - I_{1/3}\left(\tfrac{2}{3}\lambda\right)\sqrt{\zeta}\, I_{-1/3}\left(\tfrac{2}{3}\lambda\zeta\right)$$

$$f_{13} = \sqrt{\zeta}\, J_{+1/3}\left(\tfrac{2}{3}\lambda\sqrt{\tfrac{M_0^2-1}{\delta}}\,\zeta^{3/2}\right)$$

$$f_{14} = \sqrt{\zeta}\, J_{-1/3}\left(\tfrac{2}{3}\lambda\sqrt{\tfrac{M_0^2-1}{\delta}}\,\zeta^{3/2}\right)$$

$$f_{21} = \frac{M^2(\zeta)-1}{M^2(-\zeta)-1}\left\{-I_{-1/3}\left(\tfrac{2}{3}\lambda\right)\lambda\zeta\, I_{-2/3}\left(\tfrac{2}{3}\lambda\zeta^{3/2}\right) + I_{1/3}\left(\tfrac{2}{3}\lambda\right)\lambda\zeta\, I_{2/3}\left(\tfrac{2}{3}\lambda\zeta^{3/2}\right)\right\}$$

$$f_{23} = \lambda\sqrt{\tfrac{M_0^2-1}{\delta}}\,\zeta\, J_{-2/3}\left(\tfrac{2}{3}\lambda\sqrt{\tfrac{M_0^2-1}{\delta}}\,\zeta^{3/2}\right)$$

$$f_{24} = \lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi\, J_{2/3}\left(\frac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi^{3/2}\right)$$

$$f_{32} = I_{-1/3}\left(\frac{2}{3}\lambda\right)\sqrt{\xi}\, I_{1/3}\left(\frac{2}{3}\lambda\xi\right) - I_{1/3}\left(\frac{2}{3}\lambda\right)\sqrt{\xi}\, I_{-1/3}\left(\frac{2}{3}\lambda\xi\right)$$

$$f_{35} = \sqrt{\xi}\, J_{1/3}\left(\frac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi^{3/2}\right)$$

$$f_{36} = \sqrt{\xi}\, J_{-1/3}\left(\frac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi^{3/2}\right)$$

$$f_{42} = \frac{M^2(\xi)-1}{M^2(-\xi)-1}\left\{-I_{-1/3}\left(\frac{2}{3}\lambda\right)\lambda\xi\, I_{-2/3}\left(\frac{2}{3}\lambda\,\xi^{3/2}\right) + I_{1/3}\left(\frac{2}{3}\lambda\right)\lambda\xi\, I_{2/3}\left(\frac{2}{3}\lambda\,\xi^{3/2}\right)\right\}$$

$$f_{45} = \lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi\, J_{-2/3}\left(\frac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi^{3/2}\right)$$

$$f_{46} = \lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi\, J_{2/3}\left(\frac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\xi^{3/2}\right)$$

$$f_{53} = \sqrt{\delta}\, J_{1/3}\left(\frac{2}{3}\lambda\sqrt{M_0^2-1}\,\delta\right)$$

$$f_{54} = \sqrt{\delta}\, J_{-1/3}\left(\frac{2}{3}\lambda\sqrt{M_0^2-1}\,\delta\right)$$

$$f_{57} = \cos\lambda\sqrt{M_0^2-1}\,\delta$$

$$f_{58} = \sin\lambda\sqrt{M_0^2-1}\,\delta$$

$$f_{63} = \lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta\, J_{-2/3}\left(\frac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta^{3/2}\right)$$

$$f_{64} = \lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta\, J_{2/3}\left(\frac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta^{3/2}\right)$$

$$f_{67} = \lambda\sqrt{M_0^2-1}\,\sin\lambda\sqrt{M_0^2-1}\,\delta$$

$$f_{68} = \lambda\sqrt{M_0^2-1}\,\cos\lambda\sqrt{M_0^2-1}\,\delta$$

$$f_{75} = \sqrt{\delta}\,J_{1/3}\left(\tfrac{2}{3}\lambda\sqrt{M_0^2-1}\,\delta\right)$$

$$f_{76} = \sqrt{\delta}\,J_{-1/3}\left(\tfrac{2}{3}\lambda\sqrt{M_0^2-1}\,\delta\right)$$

$$f_{77} = \sin\lambda\sqrt{M_0^2-1}\,\delta$$

$$f_{78} = \cos\lambda\sqrt{M_0^2-1}\,\delta$$

$$f_{85} = \lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta\,J_{-2/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta^{3/2}\right)$$

$$f_{86} = \lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta\,J_{2/3}\left(\tfrac{2}{3}\lambda\sqrt{\frac{M_0^2-1}{\delta}}\,\delta^{3/2}\right)$$

$$f_{87} = \lambda\sqrt{M_0^2-1}\,\cos\lambda\sqrt{M_0^2-1}\,\delta$$

$$f_{88} = \lambda\sqrt{M_0^2-1}\,\sin\lambda\sqrt{M_0^2-1}\,\delta$$

$$F_5 = \frac{\varepsilon/\pi}{\beta^2+\lambda^2}\sin\lambda\sqrt{M_0^2-1}\,\delta - \frac{\varepsilon/\pi\cdot\lambda}{\beta(\beta^2+\lambda^2)}\cos\lambda\sqrt{M_0^2-1}\,\delta$$

$$F_6 = \frac{\varepsilon}{\pi}\frac{1}{\beta^2+\lambda^2}\lambda\sqrt{M_0^2-1}\,\cos\lambda\sqrt{M_0^2-1}\,\delta + \frac{\varepsilon}{\pi}\frac{\lambda}{\beta(\beta^2+\lambda^2)}\lambda\sqrt{M_0^2-1}\,\sin\lambda\sqrt{M_0^2-1}\,\delta$$

$$F_7 = -\frac{\varepsilon/\pi}{\beta^2+\lambda^2}\cos\lambda\sqrt{M_0^2-1}\,\delta - \frac{\varepsilon/\pi\cdot\lambda}{\beta(\beta^2+\lambda^2)}\sin\lambda\sqrt{M_0^2-1}\,\delta$$

$$F_8 = \frac{\varepsilon}{\pi}\frac{1}{\beta^2+\lambda^2}\lambda\sqrt{M_0^2-1}\,\sin\lambda\sqrt{M_0^2-1}\,\delta + \frac{\varepsilon}{\pi}\frac{\lambda}{\beta(\beta^2+\lambda^2)}\lambda\sqrt{M_0^2-1}\,\cos\lambda\sqrt{M_0^2-1}\,\delta$$

From 35a and 46a

$$A_1 f_{11} f_{24} - A_3 f_{13} f_{24} - A_4 f_{14} f_{24} = 0$$

$$A_1 f_{21} f_{14} - A_3 f_{23} f_{14} + A_4 f_{24} f_{14} = 0$$

$$\therefore A_1 (f_{11} f_{24} + f_{21} f_{14}) - A_3 (f_{13} f_{24} + f_{23} f_{14}) = 0 \qquad 48$$

Also

$$A_1 f_{11} f_{21} - A_3 f_{13} f_{21} - A_4 f_{14} f_{21} = 0$$

$$A_1 f_{21} f_{11} - A_3 f_{11} f_{23} + A_4 f_{11} f_{24} = 0$$

$$\therefore A_3 (f_{11} f_{23} - f_{13} f_{21}) - A_4 (f_{11} f_{24} + f_{14} f_{21}) = 0 \qquad 49$$

Similarly from equations 36a and 47a

$$A_2 f_{32} f_{46} - A_5 f_{35} f_{46} - A_6 f_{36} f_{46} = 0$$

$$A_2 f_{42} f_{36} - A_5 f_{45} f_{36} + A_6 f_{36} f_{46} = 0$$

$$\therefore A_2 (f_{32} f_{46} + f_{42} f_{36}) - A_5 (f_{35} f_{46} + f_{45} f_{36}) = 0 \qquad 50$$

and

$$A_2 f_{32} f_{42} - A_5 f_{35} f_{42} - A_6 f_{36} f_{42} = 0$$

$$A_2 f_{42} f_{32} - A_5 f_{45} f_{32} + A_6 f_{46} f_{32} = 0$$

$$\therefore A_5 (f_{45} f_{32} - f_{35} f_{42}) - A_6 (f_{36} f_{42} - f_{46} f_{32}) = 0 \qquad 51$$

Now A_8 can be eliminated between the equations 34a and 45a

$$A_3 f_{53} f_{68} + A_4 f_{54} f_{68} - A_7 f_{57} f_{68} - A_8 f_{58} f_{68} = F_5 f_{68}$$

$$A_3 f_{63} f_{58} - A_4 f_{64} f_{58} + A_7 f_{67} f_{58} - A_8 f_{58} f_{68} = F_6 f_{58}$$

$$\therefore A_3 (f_{53} f_{68} - f_{63} f_{58}) + A_4 (f_{54} f_{68} + f_{64} f_{58}) - A_7 (f_{57} f_{68} + f_{67} f_{58}) = F_5 f_{68} - F_6 f_{58}$$

52

And likewise between equations 33a and 44a

$$A_5 f_{75} f_{88} + A_6 f_{76} f_{88} + A_7 f_{77} f_{88} - A_8 f_{78} f_{88} = F_7 f_{88}$$

$$A_5 f_{85} f_{78} - A_6 f_{86} f_{78} + A_7 f_{87} f_{78} + A_8 f_{88} f_{78} = F_8 f_{78}$$

$$\therefore A_5 (f_{75} f_{88} + f_{85} f_{78}) + A_6 (f_{76} f_{88} - f_{86} f_{78}) + A_7 (f_{77} f_{88} + f_{87} f_{78}) = F_7 f_{88} + F_8 f_{78} \qquad 53$$

Now eliminate A_7 between equations 52 and 53

$$A_3 (f_{53} f_{68} - f_{63} f_{58})(f_{77} f_{88} + f_{87} f_{78}) + A_5 (f_{75} f_{88} + f_{85} f_{78})(f_{57} f_{68} + f_{67} f_{58}) + A_4 (f_{54} f_{68} + f_{64} f_{58})(f_{77} f_{88} + f_{87} f_{78}) + A_6 (f_{76} f_{88} - f_{86} f_{78})(f_{57} f_{68} + f_{67} f_{58}) = (F_5 f_{68} - F_6 f_{58})(f_{77} f_{88} + f_{87} f_{78}) + (F_7 f_{88} + F_8 f_{78})(f_{57} f_{68} + f_{67} f_{58}) \qquad 54$$

Another equation of this set can be obtained by working with equations 34a, 45a, 33a, and 44a.

Between 33a and 34a

$$A_3 f_{53} f_{78} + A_4 f_{54} f_{78} - A_7 f_{57} f_{78} - A_8 f_{58} f_{78} = F_5 f_{78}$$

$$A_5 f_{75} f_{58} + A_6 f_{76} f_{58} + A_7 f_{77} f_{58} - A_8 f_{58} f_{78} = F_7 f_{58}$$

$$\therefore A_3 f_{53} f_{78} - A_5 f_{75} f_{58} + A_4 f_{54} f_{78} - A_6 f_{76} f_{58} - A_7 (f_{57} f_{78} + f_{77} f_{58}) = F_5 f_{78} - F_7 f_{58} \qquad 55$$

Between 44a and 45a

$$A_3 f_{63} f_{88} - A_4 f_{64} f_{88} + A_7 f_{67} f_{88} - A_8 f_{68} f_{88} = F_6 f_{88}$$

$$A_5 f_{85} f_{68} - A_6 f_{86} f_{68} + A_7 f_{87} f_{68} + A_8 f_{88} f_{68} = F_8 f_{68}$$

$$\therefore A_3 f_{63} f_{88} - A_4 f_{64} f_{88} + A_5 f_{85} f_{68} - A_6 f_{86} f_{68} + A_7 (f_{67} f_{88} + f_{87} f_{68}) = F_6 f_{88} + F_8 f_{68} \qquad 56$$

Now eliminate A_7 between 55 and 56

$$A_3\left\{f_{53}f_{78}(f_{67}f_{88}+f_{87}f_{68})+f_{63}f_{88}(f_{57}f_{78}+f_{77}f_{58})\right\}$$

$$+A_4\left\{f_{54}f_{78}(f_{67}f_{88}+f_{87}f_{68})-f_{64}f_{88}(f_{57}f_{78}+f_{77}f_{58})\right\}$$

$$+A_5\left\{f_{85}f_{68}(f_{57}f_{78}+f_{77}f_{58})-f_{75}f_{58}(f_{67}f_{88}+f_{87}f_{68})\right\}$$

$$-A_6\left\{f_{76}f_{58}(f_{67}f_{88}+f_{87}f_{68})+f_{86}f_{68}(f_{57}f_{78}+f_{77}f_{58})\right\}$$

$$=(F_5f_{78}-F_7f_{58})(f_{67}f_{88}+f_{87}f_{68})+(F_6f_{88}+F_8f_{68})(f_{57}f_{78}+f_{77}f_{58}) \qquad 57$$

Now use equations 49 and 51 to express equations 54 and 57

in terms of A_3 and A_5 alone.

Equation 54 becomes

$$A_3\left\{(f_{53}f_{68}-f_{63}f_{58})(f_{57}f_{88}+f_{87}f_{78})+\frac{f_{11}f_{23}-f_{13}f_{21}}{f_{11}f_{24}+f_{14}f_{21}}\times(f_{54}f_{68}+f_{64}f_{58})(f_{77}f_{88}+f_{87}f_{78})\right\}$$

$$+A_5\left\{(f_{75}f_{88}+f_{85}f_{78})(f_{57}f_{68}+f_{67}f_{58})+\frac{f_{45}f_{32}-f_{35}f_{42}}{f_{36}f_{42}-f_{46}f_{32}}\times(f_{76}f_{88}+f_{86}f_{78})(f_{57}f_{68}+f_{67}f_{58})\right\}$$

$$=(F_5f_{68}-F_6f_{58})(f_{77}f_{88}+f_{87}f_{78})+(F_7f_{88}+F_8f_{78})(f_{57}f_{68}+f_{67}f_{58}) \qquad 58$$

and Equation 57 becomes

$$A_3\left\{f_{53}f_{78}(f_{67}f_{88}+f_{87}f_{68})+f_{63}f_{88}(f_{57}f_{78}+f_{77}f_{58}) + \frac{f_{11}f_{23}-f_{13}f_{21}}{f_{11}f_{24}+f_{14}f_{21}}\Big(f_{54}f_{78}(f_{67}f_{88}+f_{87}f_{68})-f_{64}f_{88}(f_{57}f_{78}+f_{77}f_{58})\Big)\right\}$$

$$+A_5\left\{f_{85}f_{68}(f_{57}f_{78}+f_{77}f_{58})-f_{75}f_{58}(f_{67}f_{88}+f_{87}f_{68}) - \frac{f_{45}f_{32}-f_{35}f_{42}}{f_{36}f_{42}-f_{46}f_{32}}\Big(f_{76}f_{58}(f_{67}f_{88}+f_{87}f_{68})+f_{86}f_{68}(f_{57}f_{78}+f_{77}f_{58})\Big)\right\}$$

$$= (F_5 f_{78} - F_7 f_{58})(f_{67} f_{88} + f_{87} f_{68}) + (F_6 f_{88} + F_8 f_{68})(f_{57} f_{78} + f_{77} f_{58}) \qquad 59$$

Now we can solve for A_3 and A_5 independently.

Make the substitution

$$g_1 = (f_{53} f_{68} - f_{63} f_{58})(f_{57} f_{88} + f_{87} f_{58}) + \frac{f_{11} f_{23} - f_{13} f_{21}}{f_{11} f_{24} + f_{14} f_{21}} \times (f_{54} f_{68} + f_{64} f_{58})(f_{77} f_{88} + f_{87} f_{78})$$

$$g_2 = (f_{75} f_{88} + f_{85} f_{78})(f_{57} f_{68} + f_{67} f_{58}) + \frac{f_{45} f_{32} - f_{35} f_{42}}{f_{36} f_{42} - f_{46} f_{32}} \times (f_{76} f_{88} - f_{86} f_{78})(f_{57} f_{68} + f_{67} f_{58})$$

$$G_1 = (F_5 f_{68} - F_6 f_{58})(f_{77} f_{88} + f_{87} f_{78}) + (F_7 f_{88} + F_8 f_{78})(f_{57} f_{68} + f_{67} f_{58})$$

$$g_3 = f_{53} f_{78}(f_{67} f_{88} + f_{87} f_{68}) + f_{63} f_{88}(f_{57} f_{78} + f_{77} f_{58}) + \frac{f_{11} f_{23} - f_{13} f_{21}}{f_{11} f_{24} + f_{14} f_{21}} \Big(f_{54} f_{78}(f_{67} f_{88} + f_{87} f_{68}) - f_{64} f_{88}(f_{57} f_{78} + f_{77} f_{58}) \Big)$$

$$g_4 = f_{85} f_{68}(f_{57} f_{78} + f_{77} f_{58}) - f_{75} f_{58}(f_{67} f_{88} + f_{87} f_{68}) - \frac{f_{45} f_{32} - f_{35} f_{42}}{f_{36} f_{42} - f_{46} f_{32}} \Big(f_{76} f_{58}(f_{67} f_{88} + f_{87} f_{68}) + f_{86} f_{68}(f_{57} f_{78} + f_{77} f_{58}) \Big)$$

$$G_2 = (F_5 f_{78} - F_7 f_{58})(f_{67} f_{88} + f_{87} f_{68}) + (F_6 f_{88} + F_8 f_{68})(f_{57} f_{78} + f_{77} f_{58})$$

Then

$$A_3 g_1 + A_5 g_2 = G_1 \qquad 60$$

$$A_3 g_3 + A_5 g_4 = G_2$$

$$\therefore A_3 = \frac{G_1 g_4 - G_2 g_2}{g_1 g_4 - g_3 g_2} \qquad 61$$

$$A_5 = \frac{G_1 g_3 - G_2 g_1}{g_2 g_3 - g_4 g_1} \qquad 62$$

Then finally the required coefficients are

$$A_1 \equiv A^{(1)} = \frac{f_{13} f_{24} + f_{23} f_{14}}{f_{11} f_{24} + f_{21} f_{14}} \left(\frac{G_1 g_4 - G_2 g_2}{g_1 g_4 - g_3 g_2} \right) \qquad 63$$

$$A_2 \equiv B^{(1)} = \frac{f_{35} f_{46} + f_{45} f_{36}}{f_{32} f_{46} + f_{42} f_{36}} \left(\frac{G_1 g_3 - G_2 g_1}{g_2 g_3 - g_4 g_1} \right) \qquad 64$$

With these we can write down the value of the stream function $\psi^{(1)}(x,z)$ explicitly. However, we actually wish the pressure distribution along the surface $-y = z = 1$. The stream velocity vanishes at this point and therefore the pressure variations can not be calculated by the linearized formula. However, the velocity variation parallel to the plate is

$$u^{(1)}(x,-1) = \frac{\rho_0}{\rho(-1)} \frac{\partial \psi^{(1)}}{\partial y}$$

$$= \frac{\rho_0 u_0}{\rho(-1)} \int_0^\infty \left\{ -I_{-1/3}(\tfrac{2}{3}\lambda)\lambda I_{-2/3}(\tfrac{2}{3}\lambda) + I_{1/3}(\tfrac{2}{3}\lambda)\lambda I_{2/3}(\tfrac{2}{3}\lambda) \right\}$$

$$\times \left(A^{(1)}(\lambda) \sin\lambda x + B^{(1)}(\lambda) \cos\lambda x \right) d\lambda$$

65

DISCUSSION

As indicated in the discussion preceding equation 65, the pressure variation at the plate cannot be calculated by the linearized formula $\Delta p = -\rho U u$ because of the vanishing of U . One can employ the linearized formula at a finite distance off the plate, however, by using equation 43 for $\partial \varphi'''/\partial y$; this should yield a pressure perturbation in the subsonic region not too much different from the value at the plate.

In connection with the evaluation of equation 65, it is apparent that although an analytic expression has been obtained for the velocity variation sought, still its calculation will be a tedious task. If numerical results are desired, a contour integration seems the most logical method of attack, however,the question then arises as to whether a straight numerical integration would not be more feasible.

In conclusion, it follows that any positive results ensuing from this analysis will depend upon evaluation of the above integral, which, however, is outside the present scope of the investigation.

REFERENCES

1. Howarth, L. : "The Propagation of Steady Disturbances in a Supersonic Stream Bounded on One Side by a Parallel Subsonic Stream". Proceedings of the Cambridge Philosophical Society, 1947, Vol. 44.

2. Tsien, H.S. and Finston, M. : "Interaction Between Parallel Streams of Subsonic and Supersonic Velocities". Journal of Aeronautical Sciences, (Sept. 1949).

3. Liepman, H.W.; Roshko and Dhawan, S. : "On the Reflection of Shock Waves From Boundary Layers". Galcit Final Report for Contract NAw-5631, (Aug. 1949).

4. Lees, Lester, : "Interaction Between the Laminar Boundary Layer Over a Plane Surface and an Incident Oblique Shock Wave". Princeton University Aeronautical Engineering Laboratory Report # 143, (Jan. 1949).

5. Galcit Transonic Research Group: "Problems on Shock Reflection". Final Report for Contract W33-038 ac 1717 (11592); Section III, Marble, F. : "The Reflection of a Weak Shock From a Supersonic Shear Layer."

6. Magnus, W. and Oberhettinger, F. :"Formulas and Theorems for the Special Functions of Mathematical Physics", Chelsea Publishing Co., New York, (1949).