A FORMULATION OF THE PROBLEM OF DISTRIBUTED VORTICITY
IN THE SHOCK WAVE BOUNDARY LAYER INTERACTION PROCESS

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ABSTRACT

The problem of shock wave boundary layer interaction is reviewed and attention focused upon the role of vorticity in the process. In order to simplify the physical considerations the two phenomena exhibited by vorticity in the interaction process—reflection and refraction of the disturbance, and transport of the vorticity from its original distribution—are divorced from one another. The reflection and refraction process is then considered apart from the other, and it is found that a boundary value problem can be formulated for it and formally solved for small perturbations from the undisturbed flow.

The perturbation component, which is associated with the pressure variation over the bounding surface, is set up and carried through to a point involving evaluation of a contour integral. This integral is so complex that its analytical evaluation would require many months of effort, and at this point it is thought that a re-examination of the original problem would be in order.

Although numerical results would be desirable, the effort expended would have to be weighed against their relative contribution to an understanding of the overall problem.
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INTRODUCTION

The problem of an external shock wave, incident upon a boundary layer of both subsonic and supersonic character, has attracted much attention in recent years. Several treatments of the subject are noteworthy in that they demonstrate essential features of the physical problem through simple mathematical models and, by means of these models, the researcher gains an insight into the contributions of specific physical parameters to the actual flow. One such parameter, vorticity, has been included as a discontinuity phenomenon which, however, may mask its true role in the problem. The role of vorticity in the overall interaction process consists of two parts: one, the reflection and refraction of the traversing wave, and two, the transport of the original vorticity induced by the traversing wave. The present investigation is undertaken to shed some light upon the former process.

An early contribution to an understanding of the problem was made by Howarth, (Ref.1) who considered two semi-infinite fluids of different, but constant, subsonic and supersonic Mach numbers, separated by a discontinuity surface. He introduced a shock wave into the supersonic flow field and then studied its reflection and pressure effects at the discontinuity surface. Through this he was able to demonstrate quantitatively that there is up-stream propagation of pressure through the subsonic half plane and therefore change in the flow conditions ahead of shock.

A later extension of the problem in the direction of the real flow was given by Tsien and Finston (Ref.2) where in addition to the assumptions of Howarth, they introduce a flat bounding surface into
the subsonic field. Two cases are discussed with these conditions, that of an incoming compression wave, and flow in a corner; both investigations confirming the upstream propagation of pressure through the subsonic layer. The order of magnitude of the results so obtained, however, is at variance with experiment. (see Ref. 3).

Another approach, based upon the Pohlhausen method and simple supersonic flow theory, was adopted by Lees (Ref. 4); while Marble, (Ref. 5) restricts himself to first order reflections in a purely supersonic flow. Reference 3, referred to above, summarized the work of these authors and in addition presents the experimental findings to date.

The problem treated in this paper is limited to the reflections and refractions of a weak wave traversing a shear layer. The medium is so chosen that the propagation velocity becomes imaginary for a part of the flow - i.e. the equation changes type, and there is no change during the interaction process from the original vorticity distribution.

Wave motion in such a medium is represented by the usual equation

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

where in this case \( \psi \) is identified with the stream function and the propagation velocity \( c \) with \( \sqrt{1-M^2} \). The boundary value problem may then be related to that of Tsien and Finston with the following additions:

a) The no-slip condition is applied at the bounding wall.

b) The supersonic or positive propagation velocity shear layer
is bounded by a uniform semi-infinite flow.

c) The shear is distributed linearly in each of the two parallel layers.

It is hoped that this formulation of one aspect of the vorticity question will shed some light upon the role of a distributed shear in the overall problem of shock wave boundary layer interaction. An answer to this question could lead to a better understanding of the interaction process as a whole, the difference between laminar and turbulent interaction, instability, and perhaps advances in the knowledge of the contribution of viscosity to the complete problem.
INTERACTION OF A WEAK WAVE AND A SHEAR LAYER IN THE ABSENCE
OF VORTICITY TRANSPORT

Mach Number Distribution

\[ M = \frac{\gamma}{\gamma} > 1 \]
\[ \gamma = 5 \]
\[ \gamma = 0 \]
\[ \gamma = -1 \]

\[ M^2 - 1 = \frac{(\gamma^2 - 1) \gamma}{\gamma} \]

Fig. 1

Differential equation:

\[ (1 - M^2) \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \]

1.

where \( \psi \) is the perturbation stream function and \( M \) is the local Mach number.

In region 3 the solutions of equation 1 are simple:

\[ \psi^{(3)} = f(x + \sqrt{M^2 - 1} \gamma) + g(x - \sqrt{M^2 - 1} \gamma) \]

2.

Following Tsien and Finston, the incident wave will be expressed
as a continuous disturbance which becomes a sharp wave only under a limiting process.

This wave will be taken as:

\[ \psi'_{(3)} = -\frac{\psi_0}{\beta} e^{-\beta \left(x + \sqrt{\frac{x^2}{\rho_0^2} + y^2}\right)}, \quad x + \sqrt{\frac{x^2}{\rho_0^2} + y^2} \geq 0 \]

\[ = 0, \quad x + \sqrt{\frac{x^2}{\rho_0^2} + y^2} < 0 \]

The computation of velocity components from the perturbation stream function requires some consideration. Let \( \overline{\psi} \) be the stream function corresponding to the main flow in the \( x \) direction so that:

\[ \frac{\partial \overline{\psi}}{\partial y} = \frac{\overline{\rho}}{\rho_0} \overline{u} \]

where \( \overline{u} \) is the main flow depending upon \( y \) and \( \overline{\rho} \) is the density of the main flow also depending upon \( y \). Then if \( \psi \) is the perturbation stream function, the resultant derivative in the \( y \) direction is:

\[ \frac{\partial}{\partial y} (\overline{\psi} + \psi) = \frac{\rho}{\rho_0} \left( \overline{u} + u \right) \]

where \( \rho \) is the density of the perturbed stream and \( u \) is the perturbation velocity. Writing this in the form:

\[ \frac{\partial}{\partial y} (\overline{\psi} + \psi) = \frac{\rho - \rho}{\rho_0} (\overline{u} + u) + \frac{\rho}{\rho_0} (\overline{u} + u) \]
it is clear that within the linearized approximation

\[ \frac{\partial}{\partial y} (\bar{y} + y) \approx \frac{\rho - \bar{\rho}}{\rho_0} (\bar{u} + u) + \frac{\bar{\rho}}{\rho_0} \bar{u} + \frac{\bar{\rho}}{\rho_0} u \]

Thus from equation 4 it follows that:

\[ \frac{\partial y}{\partial y} = \frac{\rho - \bar{\rho}}{\rho_0} \bar{u} + \frac{\bar{\rho}}{\rho_0} u \]

The perturbation is isentropic along the streamlines so that:

\[ dP = -\rho (\bar{u} + u) u \]

and

\[ \frac{dP}{\rho - \bar{\rho}} \approx a^2 \]

where \( a \) is the velocity of sound corresponding to the original flow. Hence:

\[ \rho - \bar{\rho} \approx -\frac{\bar{\rho} \bar{u}}{a^2} \]

or

\[ \frac{\partial y}{\partial y} = -\frac{\bar{\rho}}{\rho_0} \frac{\bar{u}^2}{a^2} u + \frac{\bar{\rho}}{\rho_0} u \]

\[ = \frac{\bar{\rho}}{\rho_0} (1 - \bar{M}^2) u \]

Then, finally the velocity perturbation in the direction of the
main stream may be computed from the perturbation stream function as:

$$\omega = - \frac{P_0}{P} \frac{\mu}{M^2 - 1} \frac{\partial \nu}{\partial y}$$

12.

The perturbation normal to the direction of main flow may also be calculated:

$$\frac{\partial}{\partial x} (\bar{y} + \mu) = \frac{\partial \nu}{\partial x} = - \frac{P}{P_0} \nu - \frac{\bar{P}}{P_0} \nu$$

consequently:

$$\nu = - \frac{P_0}{P} \frac{\partial \nu}{\partial x}$$

13.

With the aid of formulas 12 and 13 the velocity perturbations corresponding to the stream function of equation 3 may be calculated:

$$\omega = \left( \frac{\dot{u}_o \varepsilon}{\sqrt{M_o^2 - 1}} \right)^{-} \beta(x + \sqrt{M_o^2 - 1} y)$$

$$= 0 \quad , x + \sqrt{M_o^2 - 1} y > 0$$

$$= \left( \frac{\dot{u}_o \varepsilon}{\sqrt{M_o^2 - 1}} \right)^{-} \beta(x + \sqrt{M_o^2 - 1} y)$$

$$= 0 \quad , x + \sqrt{M_o^2 - 1} y < 0$$

14.

and

$$\nu = - \left( \frac{\dot{u}_o \varepsilon}{\sqrt{M_o^2 - 1}} \right)^{-} \beta(x + \sqrt{M_o^2 - 1} y)$$

$$= 0 \quad , x + \sqrt{M_o^2 - 1} y > 0$$

$$= \left( \frac{\dot{u}_o \varepsilon}{\sqrt{M_o^2 - 1}} \right)^{-} \beta(x + \sqrt{M_o^2 - 1} y)$$

$$= 0 \quad , x + \sqrt{M_o^2 - 1} y < 0$$

15.

When $\beta \to 0$ these velocities change discontinuously across the
characteristic \( x + \sqrt{\mu_0 - 1} y = 0 \). This will be taken as the disturbance entering the shear flow from outside and consequently the only further disturbance in region 3 will be that caused by the deformations of the shear flow itself. These disturbances are then propagated in an outward direction and consequently may be represented as:

\[
\psi_2^{(3)} = \frac{1}{\sqrt{\mu_0 - 1}} (x + \sqrt{\mu_0 - 1} y)
\]

It is convenient to express the stream functions \( \psi_1^{(3)} \) and \( \psi_2^{(3)} \) (equations 3 and 16) as Fourier integrals. As shown by Tsien and Finston,

\[
\psi_i^{(3)} = -\frac{U_0 \varepsilon}{\nu} \int_0^\infty \left\{ \frac{1}{\beta^2 + \lambda^2} \cos \lambda (x + \sqrt{\mu_0 - 1} y) + \frac{1}{\beta(\beta^2 + \lambda^2)} \sin \lambda (x + \sqrt{\mu_0 - 1} y) \right\} d\lambda
\]

However since:

\[
\cos \lambda (x + \sqrt{\mu_0 - 1} y) = \cos \lambda x \cos \lambda \sqrt{\mu_0 - 1} y - \sin \lambda x \sin \lambda \sqrt{\mu_0 - 1} y
\]

and

\[
\sin \lambda (x + \sqrt{\mu_0 - 1} y) = \sin \lambda x \cos \lambda \sqrt{\mu_0 - 1} y + \cos \lambda x \sin \lambda \sqrt{\mu_0 - 1} y
\]

the integral may be written:

\[
\psi_i^{(3)} = -\frac{U_0 \varepsilon}{\nu} \int_0^\infty \left\{ \frac{1}{\beta^2 + \lambda^2} \cos \lambda \sqrt{\mu_0 - 1} y + \frac{1}{\beta(\beta^2 + \lambda^2)} \sin \lambda \sqrt{\mu_0 - 1} y \right\} \cos \lambda x d\lambda
\]

* This is neglected in the present analysis.
\[ + \frac{u_0 \Delta}{\pi} \int_0^\infty \frac{1}{\beta^2 + \lambda^2} \sin \lambda \sqrt{M_0^2 - 1} y - \frac{\lambda}{\beta(\beta^2 + \lambda^2)} \cos \lambda \sqrt{M_0^2 - 1} y \] \sin \lambda x d\lambda \]

18.

The stream function \( \psi_2^{(3)} \) may also be written as a Fourier integral:

\[ \psi_2^{(3)} = \psi_0 \int_0^\infty \left( A_3(\lambda) \sin \lambda (x - \sqrt{M_0^2 - 1} y) + B_3(\lambda) \cos \lambda (x - \sqrt{M_0^2 - 1} y) \right) d\lambda \]

19.

The complete stream function in region 3 is then \( \psi^{(3)} + \psi_2^{(3)} \).

Representation of the stream function in region 2 requires solution of equation 1 for the particular Mach number distribution

\[ M_2^2 - 1 = \left( \frac{M_0^2 - 1}{\delta} \right) y \]

20.

the equation is then

\[ \frac{M_0^2 - 1}{\delta} y \frac{\partial^2 \psi^{(2)}}{\partial x^2} - \frac{\partial^2 \psi^{(2)}}{\partial y^2} = 0 \]

21.

If the function \( \psi^{(2)} \) is assumed to be separable, \( \psi^{(2)} = \rho(x) g(y) \) it follows that:

\[ \rho''(x) + \lambda^2 \rho(x) = 0 \]

22.

\[ g''(y) + \lambda^2 \frac{M_0^2 - 1}{\delta} y g(y) = 0 \]

23.
The solutions of equation 22 are simply the trigonometric functions

\[ \Phi(x) = \sin \lambda x, \cos \lambda x \]

while the solutions of 23 are Bessel functions.

These are:

\[ G(y) = \begin{cases} \sqrt{\frac{4}{\pi}} J_{\nu_3} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0 - \gamma}{\delta}} y^{\frac{3}{2}} \right) & \\
\sqrt{\frac{4}{\pi}} J_{\nu_3} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0 - \gamma}{\delta}} y^{\frac{3}{2}} \right) & \end{cases} \]

The stream function \( \psi^{(2)} \) may then be represented in the Fourier integral form:

\[ \psi^{(2)}(x, y) = \mathcal{K} \int_0^\infty \left\{ A^{(2)}(\lambda) J_{\nu_3} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0 - \gamma}{\delta}} y^{\frac{3}{2}} \right) + B^{(2)}(\lambda) J_{\nu_3} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0 - \gamma}{\delta}} y^{\frac{3}{2}} \right) \right\} \sin \lambda x d\lambda + \int_0^\infty \left\{ C^{(2)}(\lambda) J_{\nu_3} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0 - \gamma}{\delta}} y^{\frac{3}{2}} \right) + D^{(2)}(\lambda) J_{\nu_3} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0 - \gamma}{\delta}} y^{\frac{3}{2}} \right) \right\} \cos \lambda x d\lambda \]

The stream function in region 1 may now be written down with little trouble, for it satisfies the differential equation:

\[ y \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \]
The solutions of this equation are then formally:

\[
\Psi (x,y) \sim \begin{cases} 
\sin \lambda x \\
\cos \lambda x 
\end{cases} \begin{cases} 
\left( \frac{2}{3} \lambda \gamma^{3/2} \right) \\
\left( \frac{2}{3} \lambda \gamma^{3/2} \right) 
\end{cases}
\]

where, however, the Bessel functions are to be employed only for negative values of \( \gamma \). The results will appear in a more usable form if the substitution \( z = i \gamma \) is made. Then

\[
\sqrt{\gamma} J_{\nu/3} \left( \frac{4}{3} \lambda \gamma^{3/2} \right) = i^{-1/3} \sqrt{\frac{z}{\gamma}} J_{\nu/3} \left( \frac{2}{3} \lambda i \gamma^{3/2} \right)
\]

\[
= i(-i)^{1/3} \sqrt{\frac{z}{\gamma}} J_{\nu/3} \left( \frac{2}{3} \lambda i \gamma^{3/2} \right)
\]

\[
= i(-i)^{1/3} i^{1/3} \sqrt{\frac{z}{\gamma}} I_{\nu/3} \left( \frac{2}{3} \lambda \gamma^{3/2} \right)
\]

\[
= -\sqrt{\frac{z}{\gamma}} I_{\nu/3} \left( \frac{4}{3} \lambda \gamma^{3/2} \right)
\]

Similarly:

\[
\sqrt{\gamma} J_{\nu/3} \left( \frac{2}{3} \lambda \gamma^{3/2} \right) = i^{-1/3} \sqrt{\frac{z}{\gamma}} J_{\nu/3} \left( \frac{2}{3} \lambda i \gamma^{3/2} \right)
\]

\[
= i(-i)^{-1/3} \sqrt{\frac{z}{\gamma}} J_{\nu/3} \left( \frac{2}{3} \lambda i \gamma^{3/2} \right)
\]

\[
= i(-i)^{-1/3} \sqrt{\frac{z}{\gamma}} I_{\nu/3} \left( \frac{2}{3} \lambda \gamma^{3/2} \right)
\]

\[
= \sqrt{\frac{z}{\gamma}} I_{\nu/3} \left( \frac{4}{3} \lambda \gamma^{3/2} \right)
\]
Hence, the solution is of the form:

\[
\psi''(x,y) \sim \begin{bmatrix} \sin \lambda x \\ \cos \lambda x \end{bmatrix} \begin{bmatrix} \sqrt{\lambda} I_{\frac{2}{3}}(\frac{2}{3} \lambda z^{\frac{3}{2}}) \\ \sqrt{\lambda} I_{-\frac{2}{3}}(\frac{2}{3} \lambda z^{\frac{3}{2}}) \end{bmatrix}
\]

and all of the functions are real for \(0 \leq z \leq 1\).

The actual solutions can now be written as a Fourier integral in the form:

\[
\psi''(x,z) = \mathcal{U} \int_{0}^{\alpha} \left[ A''(\lambda) I_{\frac{2}{3}}(\frac{2}{3} \lambda z^{\frac{3}{2}}) + B''(\lambda) I_{-\frac{2}{3}}(\frac{2}{3} \lambda z^{\frac{3}{2}}) \right] \sin \lambda x d\lambda
\]

\[
+ \mathcal{U} \int_{0}^{\alpha} \left[ C''(\lambda) I_{\frac{2}{3}}(\frac{2}{3} \lambda z^{\frac{3}{2}}) + D''(\lambda) I_{-\frac{2}{3}}(\frac{2}{3} \lambda z^{\frac{3}{2}}) \right] \cos \lambda x d\lambda
\]

The boundary condition to be satisfied on the wall at \(y = -1 = -z\), is simply that the vertical perturbation velocity vanish which is equivalent to the stream function being a constant (say zero) along the line \(z = 1\). Then there results:

\[
\psi''(x,z) = \mathcal{U} \int_{0}^{\alpha} \left[ I_{\frac{2}{3}}(\frac{2}{3} \lambda) I_{\frac{2}{3}}(\frac{2}{3} \lambda z^{\frac{3}{2}}) - I_{-\frac{2}{3}}(\frac{2}{3} \lambda) I_{-\frac{2}{3}}(\frac{2}{3} \lambda z^{\frac{3}{2}}) \right]
\times \left( A''(\lambda) \sin \lambda x + B''(\lambda) \cos \lambda x \right) d\lambda
\]

which satisfies this condition identically.
There are eight unknown functions to be determined by matching the solutions at the interfaces. Physically it is necessary that the flow angle and the pressure perturbations be matched along the lines $y = \delta$ and $y = 0$. However, some care must be exercised in applying the matching procedure at $y = 0$ for this is the sonic line of the unperturbed flow and the solutions may be unsatisfactory at this point. Therefore, a strip of width $2\delta$ will be omitted from the flow so that the sonic portion is initially omitted. The matching conditions are then:

$$\nu^{(3)}(x, \delta) = \nu^{(2)}(x, \delta)$$

$$\nu^{(3)}(x, \delta) = \nu^{(2)}(x, \delta)$$

$$\nu^{(2)}(x, \delta) = \nu^{(1)}(x, -\delta) \frac{U(\delta)}{U(-\delta)}$$

$$\rho(\delta) U(\delta) \nu^{(2)}(x, \delta) = \rho(-\delta) U(-\delta) U''(x, -\delta)$$

31.

In terms of the stream function these are:

$$\psi^{(3)}(x, \delta) = \psi^{(2)}(x, \delta)$$

$$\frac{\partial \psi^{(3)}}{\partial y}(x, \delta) = \frac{\partial \psi^{(2)}}{\partial y}(x, \delta)$$

$$\frac{1}{\rho(\delta) U(\delta)} \psi^{(2)}(x, \delta) = \frac{1}{\rho(-\delta) U(-\delta)} \psi''(x, -\delta)$$

$$\frac{U(\delta)}{M^2(\delta) - 1} \frac{\partial \psi^{(2)}}{\partial y} = \frac{U(-\delta)}{M^2(-\delta) - 1} \frac{\partial \psi''}{\partial y}$$

32.
These four relations will give eight relations among the unknown functions and hence sufficient information to determine them.

The first of equations 32 gives

\[
\left( -\frac{\epsilon}{\pi} \frac{1}{\beta^4 + \lambda^4} + B^{(3)}(\lambda) \right) \cos \lambda \sqrt{\mu_0^{-1} \delta} - \left( -\frac{\epsilon}{\pi} \frac{\lambda}{\beta^2 + \lambda^2} + A^{(3)}(\lambda) \right) \sin \lambda \sqrt{\mu_0^{-1} \delta}
\]

\[
= C^{(2)}(\lambda) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right) + D^{(2)}(\lambda) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right)
\]

and

\[
\left( -\frac{\epsilon}{\pi} \frac{1}{\beta^4 + \lambda^4} + B^{(3)}(\lambda) \right) \sin \lambda \sqrt{\mu_0^{-1} \delta} + \left( -\frac{\epsilon}{\pi} \frac{\lambda}{\beta^2 + \lambda^2} + A^{(3)}(\lambda) \right) \cos \lambda \sqrt{\mu_0^{-1} \delta}
\]

\[
= A^{(2)}(\lambda) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right) + B^{(2)}(\lambda) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right)
\]

The third condition gives, assuming \( \rho(-s) = \rho(s) \); \( \mu(-s) = \mu(s) \)

\[
A''(\lambda) \left\{ I_\lambda \left( \frac{2}{3} \lambda \right) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right) - I_\lambda \left( \frac{2}{3} \lambda \right) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right) \right\}
\]

\[
= A^{(2)}(\lambda) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right)^{3/2} + B^{(2)}(\lambda) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right)^{3/2}
\]

\[
B''(\lambda) \left\{ I_\lambda \left( \frac{2}{3} \lambda \right) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right) - I_\lambda \left( \frac{2}{3} \lambda \right) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right) \right\}
\]

\[
= C^{(2)}(\lambda) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right)^{3/2} + D^{(2)}(\lambda) \sqrt{\delta} J_\lambda \left( \frac{2}{3} \lambda \sqrt{\mu_0^{-1} \delta} \right)^{3/2}
\]
In order to complete the substitution into the matching conditions (32), it is necessary to calculate the following three partial derivatives:

\[
\frac{\partial A^{(0)}}{\partial y} = \frac{2}{\beta^2 + \lambda^2} \int_0^{\infty} \left( \frac{e^{\beta y}}{\beta^2 + \lambda^2} + B(\beta) \right) \cos \sqrt{\beta^2 + \lambda^2} y \left( \frac{e^{\beta y} \cdot \lambda}{\beta (\beta^2 + \lambda^2)} + A(\lambda) \right) \sin \sqrt{\beta^2 + \lambda^2} y \, \cos \lambda x \, d\lambda
\]

\[
+ \frac{2}{\beta^2 + \lambda^2} \int_0^{\infty} \left( \frac{e^{\beta y}}{\beta^2 + \lambda^2} + B(\beta) \right) \sin \sqrt{\beta^2 + \lambda^2} y \left( \frac{e^{\beta y} \cdot \lambda}{\beta (\beta^2 + \lambda^2)} + A(\lambda) \right) \cos \sqrt{\beta^2 + \lambda^2} y \, \sin \lambda x \, d\lambda
\]

\[
= \frac{2}{\beta^2 + \lambda^2} \int_0^{\infty} \left( \frac{e^{\beta y}}{\beta^2 + \lambda^2} - B(\beta) \right) \lambda \sqrt{\beta^2 + \lambda^2} \sin \sqrt{\beta^2 + \lambda^2} y \left( \frac{e^{\beta y} \cdot \lambda}{\beta (\beta^2 + \lambda^2)} + A(\lambda) \right) \lambda \sqrt{\beta^2 + \lambda^2} \cos \sqrt{\beta^2 + \lambda^2} y \, \cos \lambda x \, d\lambda
\]

\[
+ \frac{2}{\beta^2 + \lambda^2} \int_0^{\infty} \left( \frac{e^{\beta y}}{\beta^2 + \lambda^2} + B(\beta) \right) \lambda \sqrt{\beta^2 + \lambda^2} \cos \sqrt{\beta^2 + \lambda^2} y \left( \frac{e^{\beta y} \cdot \lambda}{\beta (\beta^2 + \lambda^2)} - A(\lambda) \right) \lambda \sqrt{\beta^2 + \lambda^2} \sin \sqrt{\beta^2 + \lambda^2} y \, \sin \lambda x \, d\lambda
\]

To differentiate \( A^{(2)} \) we must compute the quantity

\[
\frac{\partial^2}{\partial y^2} \left( \sqrt{\frac{\beta}{\gamma}} J_{\nu_3} (\gamma y^{3/2}) \right) = \frac{1}{2} \gamma^{-\nu_3/2} J_{\nu_3} (\gamma y^{3/2}) - \frac{\nu_3}{\gamma y^{\nu_3/2}} J_{\nu_3} (\gamma y^{3/2}) \cdot \frac{3}{2} \gamma y^{\nu_3/2}
\]

\[
+ J_{2\nu_3} (\gamma y^{3/2}) \cdot \frac{3}{2} \gamma y^{\nu_3/2} \gamma y^{\nu_3/2}
\]

\[
= \frac{3}{2} \gamma y^{\nu_3} J_{-\nu_3} (\gamma y^{3/2})
\]

\[
38.
\]
\[
\frac{2}{y} \left( \sqrt{y} \frac{I_{3/2}}{y^{3/2}} (r y^{3/2}) \right) = \frac{1}{2} y^{-2} \frac{I_{3/2}}{y^{3/2}} (r y^{3/2}) - \frac{1}{2} \frac{I_{3/2}}{y^{3/2}} (r y^{3/2}) \cdot \frac{2}{y} y^{3/2} y^{3/2} \\
- \frac{I_{3/2}}{y^{3/2}} (r y^{3/2}) \cdot \frac{3}{2} y^{3/2} y^{3/2} \\
= - \frac{3}{2} y^{3/2} \frac{I_{3/2}}{y^{3/2}} (r y^{3/2})
\]

The derivative of \( \zeta \) requires similar differentiation of the \( I_2 \) functions and these must be computed separately since they satisfy different recursion relations.

\[
\frac{d}{dy} \left( \sqrt{z} \frac{I_{3/2}}{z^{3/2}} (z^{3/2}) \right) = \frac{d}{dz} \left( \sqrt{z} \frac{I_{3/2}}{z^{3/2}} (z^{3/2}) \right) \\
= \frac{1}{2} z^{-2} \frac{I_{3/2}}{z^{3/2}} (z^{3/2}) - \frac{1}{2} \frac{I_{3/2}}{z^{3/2}} (z^{3/2}) \cdot \frac{2}{z} z^{3/2} z^{3/2} + \frac{I_{3/2}}{z^{3/2}} (z^{3/2}) \cdot \frac{3}{2} z^{3/2} z^{3/2} \\
= \frac{3}{2} z^{3/2} \frac{I_{3/2}}{z^{3/2}} (z^{3/2})
\]

\[\therefore \frac{d}{dy} \left( \sqrt{z} \frac{I_{3/2}}{z^{3/2}} (z^{3/2}) \right) = - \frac{3}{2} z^{3/2} \frac{I_{3/2}}{z^{3/2}} (z^{3/2}) \]

\[
\frac{d}{dy} \left( \sqrt{x} \frac{I_{3/2}}{x^{3/2}} (x^{3/2}) \right) = \frac{d}{dx} \left( \sqrt{x} \frac{I_{3/2}}{x^{3/2}} (x^{3/2}) \right) \\
= \frac{1}{2} x^{-2} \frac{I_{3/2}}{x^{3/2}} (x^{3/2}) - \frac{1}{2} \frac{I_{3/2}}{x^{3/2}} (x^{3/2}) \cdot \frac{2}{x} x^{3/2} x^{3/2} + \frac{I_{3/2}}{x^{3/2}} (x^{3/2}) \cdot \frac{3}{2} x^{3/2} x^{3/2} \\
= \frac{3}{2} x^{3/2} \frac{I_{3/2}}{x^{3/2}} (x^{3/2})
\]

\[\therefore \frac{d}{dy} \left( \sqrt{x} \frac{I_{3/2}}{x^{3/2}} (x^{3/2}) \right) = - \frac{3}{2} x^{3/2} \frac{I_{3/2}}{x^{3/2}} (x^{3/2}) \]
The derivative of $\kappa^{(a)}$ is now:

$$
\frac{\partial \kappa^{(a)}}{\partial \gamma} = C_0 \int_0^\infty \left\{ A^{(a)}(\lambda) \sqrt{\frac{\lambda^2 - 1}{\delta}} \right\} \frac{\partial}{\partial \lambda} \left[ \frac{1}{3} \lambda \sqrt{\frac{\lambda^2 - 1}{\delta}} \right] \sin \lambda x \, d\lambda 

+ C_0 \int_0^\infty \left\{ B^{(a)}(\lambda) \sqrt{\frac{\lambda^2 - 1}{\delta}} \right\} \frac{\partial}{\partial \lambda} \left[ \frac{1}{3} \lambda \sqrt{\frac{\lambda^2 - 1}{\delta}} \right] \cos \lambda x \, d\lambda
$$

Finally, the derivative of $\kappa^{(b)}$ is just:

$$
\frac{\partial \kappa^{(b)}}{\partial \gamma} = C_0 \int_0^\infty \left\{ - I^{(a)}(\frac{2}{3} \lambda) \frac{\lambda}{3} \sqrt{\frac{\lambda^2 - 1}{\delta}} \right\} \sin \lambda x \left[ A^{(a)}(\lambda) \sin \lambda x + B^{(a)}(\lambda) \cos \lambda x \right] \, d\lambda
$$

Using these relations, the remaining matching relations may be completed. The second of equations 32 becomes:

$$
\left( \frac{e^{\Pi}}{\delta^{2} + \lambda^2} - \frac{\partial^2}{\partial \lambda^2} \right) \sqrt{\frac{\lambda^2 - 1}{\delta}} \sin \lambda x \delta \frac{\partial^2}{\partial \lambda^2} \sin \lambda x \delta
$$

$$
= C^{(a)}(\lambda) \sqrt{\frac{\lambda^2 - 1}{\delta}} \frac{\lambda}{3} \sqrt{\frac{\lambda^2 - 1}{\delta}} \frac{\partial^2}{\partial \lambda^2} \left[ \frac{2}{3} \lambda \sqrt{\frac{\lambda^2 - 1}{\delta}} \right] - B^{(a)}(\lambda) \sqrt{\frac{\lambda^2 - 1}{\delta}} \frac{\partial^2}{\partial \lambda^2} \left[ \frac{2}{3} \lambda \sqrt{\frac{\lambda^2 - 1}{\delta}} \right]
$$

$$
\left( \frac{e^{\Pi}}{\delta^{2} + \lambda^2} + \frac{\partial^2}{\partial \lambda^2} \right) \lambda \sqrt{\frac{\lambda^2 - 1}{\delta}} \sin \lambda x \delta + \left( \frac{e^{\Pi} \cdot \lambda}{\delta^{2} + \lambda^2} - A^{(a)}(\lambda) \right) \lambda \sqrt{\frac{\lambda^2 - 1}{\delta}} \sin \lambda x \delta
$$

$$
= A^{(a)}(\lambda) \sqrt{\frac{\lambda^2 - 1}{\delta}} \frac{\lambda}{3} \sqrt{\frac{\lambda^2 - 1}{\delta}} \frac{\partial^2}{\partial \lambda^2} \left[ \frac{2}{3} \lambda \sqrt{\frac{\lambda^2 - 1}{\delta}} \right] - B^{(a)}(\lambda) \sqrt{\frac{\lambda^2 - 1}{\delta}} \frac{\partial^2}{\partial \lambda^2} \left[ \frac{2}{3} \lambda \sqrt{\frac{\lambda^2 - 1}{\delta}} \right]
$$

44.
The final relation, the last of equations 32, is assuming $u(\psi) = u(\psi^*)$

\[
\frac{\mu^2}{\hat{A}(\lambda)} \sqrt{\frac{\mu^2}{\hat{A}(\lambda^*)}} \frac{\lambda}{2} \int \left( \frac{2}{3} \lambda \sqrt{\frac{\mu^2}{\hat{A}(\lambda^*)}}\right)^{\frac{3}{2}} - \frac{\mu^2}{\hat{B}(\lambda)} \sqrt{\frac{\mu^2}{\hat{B}(\lambda^*)}} \frac{\lambda}{2} \int \left( \frac{2}{3} \lambda \sqrt{\frac{\mu^2}{\hat{B}(\lambda^*)}}\right)^{\frac{3}{2}}
\]

\[
= \frac{\mu^2}{\hat{A}(\lambda)} \sqrt{\frac{\mu^2}{\hat{A}(\lambda^*)}} \left\{ - \frac{\lambda}{2} \int \frac{1}{\lambda} \int \left( \frac{2}{3} \lambda \sqrt{\frac{\mu^2}{\hat{A}(\lambda^*)}}\right)^{\frac{3}{2}} \right\}
\]

46.

\[
\frac{\mu^2}{\hat{C}(\lambda)} \sqrt{\frac{\mu^2}{\hat{C}(\lambda^*)}} \frac{\lambda}{2} \int \left( \frac{2}{3} \lambda \sqrt{\frac{\mu^2}{\hat{C}(\lambda^*)}}\right)^{\frac{3}{2}} - \frac{\mu^2}{\hat{D}(\lambda)} \sqrt{\frac{\mu^2}{\hat{D}(\lambda^*)}} \frac{\lambda}{2} \int \left( \frac{2}{3} \lambda \sqrt{\frac{\mu^2}{\hat{D}(\lambda^*)}}\right)^{\frac{3}{2}}
\]

\[
= \frac{\mu^2}{\hat{C}(\lambda)} \sqrt{\frac{\mu^2}{\hat{C}(\lambda^*)}} \left\{ - \frac{\lambda}{2} \int \frac{1}{\lambda} \int \left( \frac{2}{3} \lambda \sqrt{\frac{\mu^2}{\hat{C}(\lambda^*)}}\right)^{\frac{3}{2}} \right\}
\]

47.

Now equations 33, 34, 35, 36 and equations 44, 45, 46, 47 are the eight linear relations among the eight functions and may be solved.
Define

\[ A_1^{(0)} = A_1 \quad C_1^{(0)} = A_5 \]
\[ B_1^{(0)} = A_2 \quad D_2^{(0)} = A_6 \]
\[ A_3^{(3)} = A_3 \quad A_7^{(3)} = A_7 \]
\[ B_4^{(3)} = A_4 \quad B_8^{(3)} = A_8 \]

Then rewrite the matching equations

\[ A_1 f_{11} - A_3 f_{13} - A_4 f_{14} = 0 \quad 35a \]
\[ A_1 f_{21} - A_3 f_{23} + A_4 f_{24} = 0 \quad 46a \]
\[ A_2 f_{32} - A_5 f_{35} - A_6 f_{36} = 0 \quad 36a \]
\[ A_2 f_{42} - A_5 f_{45} + A_6 f_{46} = 0 \quad 47a \]
\[ A_3 f_{53} + A_4 f_{54} - A_7 f_{57} - A_8 f_{58} = F_5 \quad 34a \]
\[ A_3 f_{63} - A_4 f_{64} + A_7 f_{67} - A_8 f_{68} = F_6 \quad 45a \]
\[ A_5 f_{75} + A_6 f_{76} + A_7 f_{77} - A_8 f_{78} = F_7 \quad 33a \]
\[ A_5 f_{85} - A_6 f_{86} + A_7 f_{87} + A_8 f_{88} = F_8 \quad 44a \]

Where the \( f_{ij} \) are defined as follows

\[ f_{11} = \frac{I_{13}}{J_{12}} \left( \frac{3\lambda}{\frac{3}{2}} \right) \sqrt{5} \left( \frac{3\lambda}{\frac{3}{2}} \right)^5 - \frac{I_{13}}{J_{12}} \left( \frac{3\lambda}{\frac{3}{2}} \right) \sqrt{5} \left( \frac{3\lambda}{\frac{3}{2}} \right)^5 \]
\[ f_{13} = \sqrt{5} \left( \frac{3\lambda}{\frac{3}{2}} \right) \sqrt{\frac{n}{\frac{n}{2}}} 5^{\frac{3n}{2}} \]
\[ f_{14} = \sqrt{5} \left( \frac{3\lambda}{\frac{3}{2}} \right) \sqrt{\frac{n}{\frac{n}{2}}} 5^{\frac{3n}{2}} \]

\[ f_{21} = \frac{\lambda (5)}{n^2 (-5)} \left[ -I_{13} \left( \frac{3\lambda}{\frac{3}{2}} \right) \lambda 5 I_{32} \left( \frac{3\lambda}{\frac{3}{2}} \right) + I_{13} \left( \frac{3\lambda}{\frac{3}{2}} \right) \lambda 5 I_{32} \left( \frac{3\lambda}{\frac{3}{2}} \right) \right] \]
\[ f_{23} = \lambda \sqrt{\frac{n}{\frac{n}{2}}} 5 \left( \frac{3\lambda}{\frac{3}{2}} \right) \sqrt{\frac{n}{\frac{n}{2}}} 5^{\frac{3n}{2}} \]
\[ f_{23} = \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta J_{\bar{y}} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta^{3/2} \right) \]

\[ f_2 = \xi_{\bar{y}} \left( \frac{2}{3} \lambda \right) \sqrt{\frac{M_0^{4.1}}{\delta}} \xi_{\bar{y}} \left( \frac{2}{3} \lambda \delta \right) - \xi_{\bar{y}} \left( \frac{2}{3} \lambda \right) \sqrt{\frac{M_0^{4.1}}{\delta}} \xi_{\bar{y}} \left( \frac{2}{3} \lambda \delta \right) \]

\[ f_{35} = \sqrt{\delta} J_{\bar{y}} \left( \frac{2}{3} \lambda \left( \frac{M_0^{4.1}}{\delta} \right)^{3/2} \right) \]

\[ f_{36} = \sqrt{\delta} J_{\bar{y}} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta^{3/2} \right) \]

\[ f_{\nu 2} = \frac{M_0^{4.1}}{\delta} \left[ -\frac{2}{3} \lambda \delta J_{\bar{y}} \left( \frac{2}{3} \lambda \delta \right) + \frac{2}{3} \lambda \delta J_{\bar{y}} \left( \frac{2}{3} \lambda \delta \right) \right] \]

\[ f_{\nu 5} = \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta J_{\bar{y}} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta^{3/2} \right) \]

\[ f_{\nu 6} = \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta J_{\bar{y}} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta^{3/2} \right) \]

\[ f_{53} = \sqrt{\delta} J_{\bar{y}} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta \right) \]

\[ f_{54} = \sqrt{\delta} J_{\bar{y}} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta \right) \]

\[ f_{57} = \cos \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta \]

\[ f_{58} = \sin \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta \]

\[ f_{63} = \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta J_{\bar{y}} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta^{3/2} \right) \]

\[ f_{64} = \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta J_{\bar{y}} \left( \frac{2}{3} \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta^{3/2} \right) \]

\[ f_{67} = \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \sin \lambda \sqrt{\frac{M_0^{4.1}}{\delta}} \delta \]
\[ f_{68} = \lambda \sqrt{\nu_{0}^{2} - 1} \cos \lambda \sqrt{\nu_{0}^{2} - 1} \delta \]
\[ f_{75} = \sqrt{\delta} J_{\frac{1}{2}} \left( \frac{2}{3} \lambda \sqrt{\nu_{0}^{2} - 1} \delta \right) \]
\[ f_{76} = \sqrt{\delta} J_{\frac{1}{2}} \left( \frac{2}{3} \lambda \sqrt{\nu_{0}^{2} - 1} \delta \right) \]
\[ f_{77} = \sin \lambda \sqrt{\nu_{0}^{2} - 1} \delta \]
\[ f_{78} = \cos \lambda \sqrt{\nu_{0}^{2} - 1} \delta \]
\[ f_{85} = \lambda \sqrt{\frac{\nu_{0}^{2} - 1}{\delta}} J_{\frac{1}{2}} \left( \frac{2}{3} \lambda \sqrt{\frac{\nu_{0}^{2} - 1}{\delta}} \delta^{3/2} \right) \]
\[ f_{86} = \lambda \sqrt{\frac{\nu_{0}^{2} - 1}{\delta}} J_{\frac{1}{2}} \left( \frac{2}{3} \lambda \sqrt{\frac{\nu_{0}^{2} - 1}{\delta}} \delta^{3/2} \right) \]
\[ f_{87} = \lambda \sqrt{\nu_{0}^{2} - 1} \cos \lambda \sqrt{\nu_{0}^{2} - 1} \delta \]
\[ f_{88} = \lambda \sqrt{\nu_{0}^{2} - 1} \sin \lambda \sqrt{\nu_{0}^{2} - 1} \delta \]
\[ F_{5} = \frac{\epsilon_{1} \mu}{\beta^{2} + \lambda^{2}} \sin \lambda \sqrt{\nu_{0}^{2} - 1} \delta - \frac{\epsilon_{1} \mu \cdot \lambda}{\beta (\beta^{2} + \lambda^{2})} \cos \lambda \sqrt{\nu_{0}^{2} - 1} \delta \]
\[ F_{6} = \frac{\epsilon}{\pi} \frac{1}{\beta^{2} + \lambda^{2}} \lambda \sqrt{\nu_{0}^{2} - 1} \cos \lambda \sqrt{\nu_{0}^{2} - 1} \delta + \frac{\epsilon}{\pi} \frac{\lambda}{\beta (\beta^{2} + \lambda^{2})} \lambda \sqrt{\nu_{0}^{2} - 1} \sin \lambda \sqrt{\nu_{0}^{2} - 1} \delta \]
\[ F_{7} = -\frac{\epsilon_{1} \mu}{\beta^{2} + \lambda^{2}} \cos \lambda \sqrt{\nu_{0}^{2} - 1} \delta - \frac{\epsilon_{1} \mu \cdot \lambda}{\beta (\beta^{2} + \lambda^{2})} \sin \lambda \sqrt{\nu_{0}^{2} - 1} \delta \]
\[ F_{8} = \frac{\epsilon}{\pi} \frac{1}{\beta^{2} + \lambda^{2}} \lambda \sqrt{\nu_{0}^{2} - 1} \sin \lambda \sqrt{\nu_{0}^{2} - 1} \delta + \frac{\epsilon}{\pi} \frac{\lambda}{\beta (\beta^{2} + \lambda^{2})} \lambda \sqrt{\nu_{0}^{2} - 1} \cos \lambda \sqrt{\nu_{0}^{2} - 1} \delta \]
From 35a and 46a

\[ R_1 \left( f_{14} t_{24} + f_{24} t_{14} \right) - R_3 \left( f_{13} t_{24} + f_{23} t_{14} \right) = 0 \]
\[ R_1 \left( f_{21} t_{14} - R_3 t_{24} f_{14} + R_4 t_{24} f_{14} \right) = 0 \]

\[ \therefore A_1 \left( f_{14} t_{24} + f_{24} t_{14} \right) - A_3 \left( f_{13} t_{24} + f_{23} t_{14} \right) = 0 \]

Also

\[ R_1 \left( f_{14} t_{24} - f_{13} t_{23} - A_4 t_{14} f_{24} \right) = 0 \]
\[ R_1 \left( f_{24} t_{14} - f_{23} t_{23} + A_4 t_{24} f_{14} \right) = 0 \]

\[ \therefore A_3 \left( f_{14} t_{24} - f_{13} t_{23} \right) - A_4 \left( f_{14} t_{24} + f_{13} t_{24} \right) = 0 \]

Similarly from equations 36a and 47a

\[ A_2 \left( f_{32} f_{16} - A_5 \left( f_{35} f_{16} - A_6 f_{36} f_{16} \right) = 0 \]
\[ A_2 \left( f_{32} f_{16} - A_5 \left( f_{45} f_{36} + A_6 f_{46} f_{36} \right) = 0 \]

\[ \therefore A_2 \left( f_{32} f_{16} + f_{32} f_{46} \right) - A_5 \left( f_{35} f_{16} + f_{45} f_{36} \right) = 0 \]

and

\[ A_2 \left( f_{32} f_{42} - A_5 \left( f_{35} f_{42} - A_6 f_{36} f_{42} \right) = 0 \]
\[ A_2 \left( f_{32} f_{42} - A_5 \left( f_{45} f_{36} + A_6 f_{46} f_{36} \right) = 0 \]

\[ \therefore A_2 \left( f_{32} f_{42} - A_5 \left( f_{35} f_{42} \right) - A_6 \left( f_{36} f_{42} \right) = 0 \]

Now \( A_8 \) can be eliminated between the equations 34a and 45a

\[ A_3 \left( f_{33} f_{64} + A_4 f_{34} f_{66} - A_7 f_{54} f_{66} - A_8 f_{56} f_{66} = f_5 f_{66} \right) \]
\[ A_3 \left( f_{63} f_{54} - A_4 f_{64} f_{56} + A_7 f_{54} f_{66} - A_8 f_{56} f_{66} = f_6 f_{56} \right) \]

\[ \therefore A_3 \left( f_{33} f_{64} - f_{63} f_{34} \right) + A_4 \left( f_{34} f_{66} + f_{64} f_{36} \right) - A_7 \left( f_{54} f_{66} + f_{64} f_{56} \right) = f_5 f_{66} - f_6 f_{56} \]
And likewise between equations 33a and 44a
\[ R_5 \frac{t_{56}}{t_{56}} + R_6 \frac{t_{65}}{t_{65}} + R_7 \frac{t_{75}}{t_{75}} - R_8 \frac{t_{85}}{t_{85}} = \bar{F}_5 \frac{t_{56}}{t_{56}} \]
\[ R_5 \frac{t_{56}}{t_{56}} - R_6 \frac{t_{65}}{t_{65}} + R_7 \frac{t_{75}}{t_{75}} + R_8 \frac{t_{85}}{t_{85}} = \bar{F}_6 \frac{t_{56}}{t_{56}} \]
\[ R_7 (\frac{t_{56} - t_{56}}{t_{56}} \bar{F}_6 + \frac{t_{56} - t_{56}}{t_{56}}) + R_6 (\frac{t_{65} - t_{65}}{t_{65}} \bar{F}_5 + \frac{t_{65} - t_{65}}{t_{65}}) = \bar{F}_5 \frac{t_{56}}{t_{56}} + \bar{F}_6 \frac{t_{56}}{t_{56}} \]

Now eliminate \( R_7 \) between equations 52 and 53
\[ R_3 (\frac{f_{53} - f_{53}}{f_{53}} f_{56} + \frac{f_{53} - f_{53}}{f_{53}} f_{56} ) + R_5 (\frac{f_{55} - f_{55}}{f_{55}} f_{56} + \frac{f_{55} - f_{55}}{f_{55}} f_{56} ) + R_6 (\frac{f_{66} - f_{66}}{f_{66}} f_{56} + \frac{f_{66} - f_{66}}{f_{66}} f_{56} ) + R_7 (\frac{f_{77} - f_{77}}{f_{77}} f_{56} + \frac{f_{77} - f_{77}}{f_{77}} f_{56} ) = \bar{F}_5 \frac{t_{56}}{t_{56}} + \bar{F}_6 \frac{t_{56}}{t_{56}} \]
\[ + (\bar{F}_7 \frac{t_{56}}{t_{56}} + \bar{F}_7 \frac{t_{56}}{t_{56}}) (\frac{f_{56} - f_{56}}{f_{56}} f_{56} + \frac{f_{56} - f_{56}}{f_{56}} f_{56} ) \]

Another equation of this set can be obtained by working with equations 34a, 45a, 53a, and 44a.

Between 33a and 34a
\[ R_3 \frac{t_{56}}{t_{56}} + R_4 \frac{t_{55}}{t_{55}} - R_7 \frac{t_{75}}{t_{75}} - R_8 \frac{t_{85}}{t_{85}} = F_5 \frac{t_{56}}{t_{56}} \]
\[ R_3 \frac{t_{56}}{t_{56}} - R_4 \frac{t_{55}}{t_{55}} + R_7 \frac{t_{75}}{t_{75}} + R_8 \frac{t_{85}}{t_{85}} = F_7 \frac{t_{56}}{t_{56}} \]
\[ \therefore R_3 \frac{t_{56}}{t_{56}} - R_5 \frac{t_{55}}{t_{55}} + R_4 \frac{t_{55}}{t_{55}} - R_8 \frac{t_{85}}{t_{85}} = F_5 \frac{t_{56}}{t_{56}} \]
\[ \quad \quad \quad + \frac{t_{75} f_{75}}{t_{75}} = F_5 \frac{t_{56}}{t_{56}} - F_7 \frac{t_{56}}{t_{56}} \]

Between 44a and 45a
\[ R_3 \frac{f_{66}}{f_{66}} + R_4 \frac{f_{65}}{f_{65}} + R_7 \frac{f_{75}}{f_{75}} - R_8 \frac{f_{85}}{f_{85}} = F_5 \frac{f_{66}}{f_{66}} \]
\[ R_3 \frac{f_{66}}{f_{66}} - R_4 \frac{f_{65}}{f_{65}} + R_7 \frac{f_{75}}{f_{75}} + R_8 \frac{f_{85}}{f_{85}} = F_7 \frac{f_{66}}{f_{66}} \]
\[ \therefore R_3 \frac{f_{66}}{f_{66}} - R_5 \frac{f_{65}}{f_{65}} + R_4 \frac{f_{65}}{f_{65}} - R_8 \frac{f_{85}}{f_{85}} = F_5 \frac{f_{66}}{f_{66}} \]
\[ \quad \quad \quad + R_8 (f_{75} f_{75} + f_{75} f_{75}) = F_5 \frac{f_{66}}{f_{66}} + F_7 \frac{f_{66}}{f_{66}} \]
Now eliminate $A_7$ between 55 and 56

$$
A_3 \left\{ f_{53} f_{75} \left( f_{7} f_{86} + f_{87} f_{65} \right) + f_{63} f_{66} \left( f_{57} f_{76} + f_{77} f_{56} \right) \right\}
+ A_4 \left\{ f_{54} f_{76} \left( f_{7} f_{86} + f_{87} f_{65} \right) - f_{64} f_{66} \left( f_{57} f_{76} + f_{77} f_{56} \right) \right\}
+ A_5 \left\{ f_{65} f_{66} \left( f_{57} f_{76} + f_{77} f_{56} \right) - f_{75} f_{76} \left( f_{67} f_{66} + f_{67} f_{55} \right) \right\}
- A_6 \left\{ f_{76} f_{56} \left( f_{67} f_{66} + f_{67} f_{55} \right) + f_{66} f_{66} \left( f_{57} f_{76} + f_{77} f_{56} \right) \right\}
= \left( f_{5} f_{75} - F_{7} f_{75} \right) \left( f_{7} f_{86} + f_{87} f_{65} \right) + \left( F_{7} f_{86} + F_{6} f_{66} \right) \left( f_{57} f_{76} + f_{77} f_{56} \right)
$$

Now use equations 49 and 51 to express equations 54 and 57 in terms of $A_3$ and $A_5$ alone.

Equation 54 becomes

$$
A_3 \left\{ \left( f_{53} f_{66} f_{65} f_{56} \right) \left( f_{57} f_{76} + f_{77} f_{56} \right) + \frac{f_{11} f_{23} - f_{11} f_{31}}{f_{11} f_{22} + f_{11} f_{33}} \left( f_{54} f_{76} + f_{87} f_{65} \right) \right\}
\times \left( f_{54} f_{76} + f_{87} f_{65} \right)
+ A_5 \left\{ \left( f_{76} f_{56} + f_{87} f_{65} \right) \left( f_{57} f_{76} + f_{77} f_{56} \right) + \frac{f_{15} f_{25} - f_{15} f_{35}}{f_{15} f_{22} + f_{15} f_{33}} \left( f_{54} f_{76} + f_{87} f_{65} \right) \right\}
\times \left( f_{54} f_{76} + f_{87} f_{65} \right)
= \left( f_{5} f_{75} - F_{7} f_{75} \right) \left( f_{7} f_{86} + f_{87} f_{65} \right) + \left( F_{7} f_{86} + F_{5} f_{56} \right) \left( f_{57} f_{76} + f_{77} f_{56} \right)
$$

and Equation 57 becomes

$$
A_3 \left\{ f_{53} f_{76} \left( f_{7} f_{86} + f_{87} f_{65} \right) + f_{63} f_{66} \left( f_{57} f_{76} + f_{77} f_{56} \right) \right. \\
\left. + \frac{f_{11} f_{23} - f_{11} f_{31}}{f_{11} f_{22} + f_{11} f_{33}} \left( f_{54} f_{76} \left( f_{67} f_{66} + f_{67} f_{55} \right) - f_{75} f_{76} \left( f_{57} f_{76} + f_{77} f_{56} \right) \right) \right\}
+ A_5 \left\{ f_{65} f_{66} \left( f_{57} f_{76} + f_{77} f_{56} \right) - f_{75} f_{76} \left( f_{67} f_{66} + f_{67} f_{55} \right) \right. \\
\left. - \frac{f_{15} f_{25} - f_{15} f_{35}}{f_{15} f_{22} + f_{15} f_{33}} \left( f_{54} f_{76} \left( f_{67} f_{66} + f_{67} f_{55} \right) + f_{66} f_{66} \left( f_{57} f_{76} + f_{77} f_{56} \right) \right) \right\}
$$
\[
\begin{align*}
&= (f_5 f_6 - f_7 f_5)(f_6 f_8 + f_7 f_6) + (f_6 f_8 + f_5 f_7)(f_5 f_7 + f_7 f_5)
\end{align*}
\]

Now we can solve for \( R_3 \) and \( R_5 \) independently.

Make the substitution

\[
G_1 = (f_5 f_6 - f_6 f_5)(f_7 f_8 + f_8 f_7) + \frac{f_6 f_8 - f_8 f_6}{f_5 f_7 + f_7 f_5} \times (f_5 f_8 + f_8 f_5)(f_7 f_8 + f_8 f_7)
\]

\[
G_2 = (f_5 f_8 + f_8 f_5)(f_5 f_6 + f_6 f_5) + \frac{f_5 f_2 - f_2 f_5}{f_5 f_2 - f_2 f_5} \times (f_5 f_2 - f_2 f_5)(f_5 f_6 + f_6 f_5)
\]

\[
G_3 = (f_5 f_6 - f_6 f_5)(f_5 f_8 + f_8 f_5) + (f_5 f_8 + f_8 f_5)(f_5 f_6 + f_6 f_5)
\]

\[
G_4 = f_5 f_3 f_8 (f_6 f_8 + f_7 f_6) + f_6 f_3 f_8 (f_5 f_7 + f_7 f_5)
\]

\[
+ \frac{f_5 f_2 - f_2 f_5 f_8}{f_5 f_7 + f_7 f_5} \times (f_5 f_8 + f_8 f_5)(f_5 f_7 + f_7 f_5)
\]

\[
G_5 = f_5 f_8 f_6 (f_5 f_7 + f_8 f_6) - f_7 f_5 (f_5 f_7 + f_7 f_5)
\]

\[
- \frac{f_5 f_2 - f_2 f_5 f_8}{f_5 f_2 - f_2 f_5} \times (f_5 f_8 + f_8 f_5)(f_5 f_7 + f_7 f_5)
\]

\[
G_6 = (f_5 f_6 - f_7 f_5)(f_6 f_8 + f_7 f_6) + (f_6 f_8 + f_7 f_6)(f_5 f_7 + f_7 f_5)
\]

Then

\[
R_3 G_1 + R_5 G_2 = G_1
\]

\[
R_3 G_3 + R_5 G_4 = G_2
\]
\[ R_3 = \frac{G, \delta - G_2 \delta_2}{\delta, \delta - \delta_3 \delta_2} \]

\[ R_5 = \frac{G_1 \delta_3 - G_2 \delta_1}{\delta_2 \delta_3 - \delta_2 \delta_1} \]

Then finally the required coefficients are

\[ A_1 = A'' \equiv \frac{t_{12} + t_{21} t_{12}}{t_{11} + t_{21} t_{12}} \left( \frac{G, \delta - G_2 \delta_2}{\delta_1 \delta - \delta_2 \delta_3} \right) \]

\[ A_2 = B'' \equiv \frac{t_{22} t_{12} + t_{21} t_{12}}{t_{22} t_{12} + t_{21} t_{12}} \left( \frac{G_1 \delta_3 - G_2 \delta_1}{\delta_2 \delta_3 - \delta_2 \delta_1} \right) \]

With these we can write down the value of the stream function \( \psi''(x, z) \) explicitly. However, we actually wish the pressure distribution along the surface \( \gamma = z = \) . The stream velocity vanishes at this point and therefore the pressure variations can not be calculated by the linearized formula. However, the velocity variation parallel to the plate is

\[ \omega''(x, -1) = \frac{\rho_0}{\rho(-1)} \frac{\partial \psi''}{\partial y} \]

\[ = \frac{\rho_0}{\rho(-1)} \left[ \int_{-\infty}^{\infty} \left( -I_{\frac{2}{3}}(\frac{1}{3} \lambda) I_{\frac{3}{2}}(\frac{2}{3} \lambda) + I_{\frac{2}{3}}(\frac{2}{3} \lambda) \lambda I_{\frac{3}{2}}(\frac{2}{3} \lambda) \right) \right] \]

\[ \times \left( A''(\lambda) \sin \lambda x + B''(\lambda) \cos \lambda x \right) d\lambda \]
DISCUSSION

As indicated in the discussion preceding equation 65, the pressure variation at the plate cannot be calculated by the linearized formula \( \Delta P = -\rho U \) because of the vanishing of \( U \). One can employ the linearized formula at a finite distance off the plate, however, by using equation 43 for \( \partial \rho / \partial y \); this should yield a pressure perturbation in the subsonic region not too much different from the value at the plate.

In connection with the evaluation of equation 65, it is apparent that although an analytic expression has been obtained for the velocity variation sought, still its calculation will be a tedious task. If numerical results are desired, a contour integration seems the most logical method of attack, however, the question then arises as to whether a straight numerical integration would not be more feasible.

In conclusion, it follows that any positive results ensuing from this analysis will depend upon evaluation of the above integral, which, however, is outside the present scope of the investigation.
REFERENCES


