STABILITY OF THIN CYLINDRICAL SHELLS SUBJECTED TO A CLASS OF AXISYMMETRIC MOVING LOADS

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ABSTRACT

The stability of an infinite-length cylindrical shell subjected to a broad class of axially symmetric moving loads with constant velocity is studied by utilizing a large deflection Donnell theory. Special cases of the general loading function include the moving ring, step and decayed step loads.

Stability is defined on the basis of the boundedness or divergence of an infinitesimal nonsymmetric disturbed motion about an initial nonlinear steady-state symmetric response. Following the determination of the symmetric response, under this concept of stability, the analysis is reduced to a study of a system of linear partial differential equations or so-called variational equations; these are analyzed by use of a double Laplace transform technique and the original stability problem is replaced by a simpler one of determining the location of the poles of a certain function. A scheme for accomplishing this task is outlined. Extension of the method to include more exact equations of motion and to a class of static problems involving finite length shells is discussed.

A related problem concerning a moving concentrated load on a nonlinear elastic cylindrical membrane (nonlinearity in both geometric and constitutive relations) and a string on a nonlinear foundation is discussed in an appendix to the text. Interesting analogies in both analysis and physical behavior of the string and shell systems are found.

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LIST OF SYMBOLS

Main Text

A_j, A_j^*	Matrix coefficients of the difference equation 3.41
B_j, B_j^*	Matrices defined by equation 3.48a
C_j, C_j^*	Coefficients of the series 3.17
$D = Eh^3/12(1 - \nu^2)$	Bending Stiffness
E	Young's Modulus
F	Stress function
$F_i(p)$, $F_i^*(p)$	Coefficients of the series 3.50 and 3.51 respectively
I	Unit matrix
L_j	Matrices defined by equation 3.40a, b
$M = V/\sqrt{E/\rho}$	Nondimensional load velocity
N_{xx}, N_{yy}, N_{xy}	Stress resultants
N_{xx}^o	Initial axial stress resultant
$N_{xx}^* = N_{xx}^o/Eh$	Nondimensional axial stress resultant
P(X, Y, T)	Lateral pressure
P_j, P_j^*, P_c	Load constants (see equation 2.6)
Q	Matrix defined by equation 3.68
S	Laplace transform parameter (see equations 3.33)
T	Time

U, V, W	Shell displacements
V	Load velocity
W_s	Symmetric radial displacement
X, Y, Z	Shell coordinates
$Z_n(\xi, \tau) = \begin{Bmatrix} \rho_n(\xi, \tau) \\ \eta_n(\xi, \tau) \end{Bmatrix}$	Perturbation vector
$\overline{Z}_n(\xi, p)$	Laplace transform of $Z_n(\xi, \tau)$ with respect to τ.
$\overline{\overline{Z}}_n(\xi, p)$	Laplace transform of $\overline{Z}_n(\xi, p)$ with respect to s
a	Shell midplane radius
c	phase velocity
$f(p)$	See equation 3.84
$f = F/a^2 Eh$	Nondimensional stress function
f_s	Symmetric case of above
$g(\xi)$	Green's function of the symmetric motion
$g_i(p)$	Constants (see equations 3.61, 3.62)
ℓ, ℓ^*	See equation 3.17
n	Number of circumferential half waves (see equation 3.22)
p	Laplace transform parameter
$q = Pa/Eh$	Nondimensional load function

$q_c = P_c/Eh$ $q_n = P_n \frac{a}{Eh}$ $q_n^* = P_n^* \frac{a}{Eh}$	Nondimensional load constants (see equation 2.10)
$r_i,\ r_i^*$	See equations 3.60
$t = \frac{T}{a}\sqrt{E/\rho}$	Nondimensional time
$w = W/a$	Nondimensional radial displacement
$w_s = W_s/a$	Symmetric case of above
$x = X/a$	Nondimensional axial coordinate
$\alpha_j,\ \alpha_j^*$	See equation 3.17
$\beta^4 = (\frac{h}{a})^2 \frac{1}{12(1-\nu^2)}$ $\bar{\beta} = \sqrt{2}\,\beta$	Nondimensional stiffness parameters
$\gamma_n,\ \gamma_n^*$	See equation 2.10
$\zeta,\ \eta$ $\zeta_n,\ \eta_n$	Perturbation quantities
$\bar{\zeta},\ \bar{\eta}$ $\bar{\zeta}_n,\ \bar{\eta}_n$	Laplace transform (with respect to τ) of the perturbation quantities
$\bar{\bar{\zeta}},\ \bar{\bar{\eta}}$ $\bar{\bar{\zeta}}_n,\ \bar{\bar{\eta}}_n$	Second Laplace transform (with respect to ξ) of the perturbations

$\theta = Y/a$	Circumferential coordinate
$\mathcal{M}^{(m-1)}_{(i)}$	See page 45
ν	Poisson's ratio
$\xi = x - Mt$ $\xi_1 = x - VT$	Moving coordinates
ρ	Shell mass density
$\rho_q(p)$, $\rho^*_q(p)$	Roots of $\Delta L_o(s) = 0$ (see equation 3.44)
$\tau = t$	Time in the moving coordinate system (see equation 3.10)
$\phi(s) = L_o^{-1}\psi(s)$	See equation 3.42
$\psi(s)$	Two-dimensional vector (see equation 3.40c)
ΔL_o	Determinant of L_o
ΔQ	Determinant of the matrix Q
Φ	See equation 3.48b

Appendix 5

A	Virtual work
A(p)	See equation 62
B(p) C(p)	See equation 63
D(p)	See equation 65b

E	Strain energy of the string
$F(a_1, b_1; c_1; \eta)$	Hypergeometric function
$G(Y)$	Nonlinear foundation (see equation 1)
G	Determinant of G_{ij}
G_{ij}	Metric tensor of deformed coordinates
K	Kinetic energy
L	Lagrangian function
$\vec{P}$	Surface traction
P	Modified load parameter (see equation 23)
P_1	Load constant (see equation 22)
$P_{1_{CR}}$	Load magnitude at which system first becomes unstable
P_2	Load constant
T	Time
U	Strain energy of the membrane
$U, V, W,$	Membrane displacements
$\vec{V}$	Displacement vector
X	Axial coordinate
Y	Transverse string displacement
Y_o	String displacement directly under the load
Z	Radial coordinate
a, b	Foundation constants (see equation 52)

c	String "sound speed"
$g(\Upsilon)$	Modified foundation (see equation 34)
k_1, k_2	Foundation constants (see page 123)
p	Laplace Transform parameter
g	Determinant of g_{ij}
g_{ij}	Metric tensor of deformed coordinates
$q(\xi, p)$	Periodic coefficient in Hill's equation 30
$t = c\Upsilon$	Nondimensional time
u_1, u_2	Normal solutions of Hill's equation 30
$v = V/c$	Nondimensional velocity
x	String coordinate
$\overline{y}$	Laplace transform of perturbation with respect to Υ
y	Perturbation quantity
$\alpha = v/\sqrt{1-v^2}$	Velocity parameter
$\gamma = \dfrac{1+\sqrt{1+\lambda^2 Y_o^2}}{Y_o}$	Deflection parameter
γ_{ij}	Covariant component of the strain tensor
$\epsilon = \lvert \lambda^2/\gamma^2 \rvert$	Load parameter
θ^i	Reference coordinates
$\lambda^2 = b/2a$	Foundation ratio
$\xi = \dfrac{x-vt}{\sqrt{v^2-1}}, \quad v>1$	See equation 23

$\xi = \dfrac{x-vt}{\sqrt{1 - v^2/G'(o)}}$ — See equation 34

ρ_o — Mass density of the undeformed body

τ^{ij} — Contravariant components of the stress tensor

τ — Time in moving coordinates

$\phi = (k_2/k_1)^{1/2}$ — Foundation parameter

$\delta L, \delta K, \delta U, \delta A$ — Variations of L, K, U and A respectively

CHAPTER I

INTRODUCTION

Since the first basic contribution by Fairbairns in 1858, the stability of thin elastic cylindrical shells has been the subject of investigation by many outstanding authors. The majority of this work, however, has been devoted to considerations of static loading. Recently, because of their widespread use in aerospace vehicles, a major emphasis has been placed on the dynamic stability of cylindrical shells. This emphasis is reflected in the increasing number of papers that have appeared on the subject since 1950. The earliest and larger part of these contributions have originated from Russian investigators such as Oniashvili (1), Agamirov (2), Vol'mir (3,4), Markov (5), Bolotin (6,7,8) and others. While Federhofer (12) appears to be the major German source, recent American contributions have emerged from Coppa and Nash (15), Yao (16,17,18), Roth and Klosner (14), Goodier and McIvor (19), Koval (20,21), Lindberg (22) and a few others. These works are chiefly concerned with impulsive, step, ramp and periodic load-time behavior under hydrostatic, external pressure and axial type loads. A representative cross section of this work can be found in references 1 through 22.

An examination of the literature on this subject indicates the state-of-the-art is indeed in its infancy, for there are many important problems yet to be considered. In particular, although the linear response of cylindrical shells to axially symmetric moving loads has been examined by several investigators (23-27), the associated stability problem has appeared in the literature through a sole source: Prisekin (26). Prisekin considers the problem of a ring load moving with constant velocity along an infinite cylindrical shell. The treatment, however, does not constitute a solution to the problem, but rather an engineering estimate of the ratio of dynamic to static critical loads based on the results of a linear symmetric response analysis.

The primary objective of this dissertation is to establish a method of solution to a class of stability problems involving thin elastic cylindrical shells subjected to axially symmetric moving loads with constant velocity. As this is a first treatment of the subject, the mathematical formulation will be simplified by assuming the shell length is infinite and the initial symmetric response of the shell has reached a steady-state value. Further, in the sequel we shall consider only the case where the load velocity is less than the minimum velocity for which axially symmetric sinusoidal waves can be propagated in the unloaded shell. For steel shells this velocity lies between 400-2000 f. p. s. for $h/a = 1/1000 - 1/40$ respectively. Below this velocity (if no axial compression is present) the steady-state motions are attenuated on both sides of the load and the results of the analysis will have significance for shells whose length is long compared to a characteristic attenuation length. Above this velocity the steady-state motions are not attenuated (unless damping is considered) and their physical significance (without damping) is questionable.

Formulation of the problem is based on the classical stability concept of Poincaré; stability is defined on the basis of the boundedness or divergence of an infinitesimal disturbed motion about an initial nonlinear state of motion. Under this concept of stability and once the initial symmetric response is obtained, the analysis is reduced to an investigation of a set of perturbation or so-called variational equations ("equation aux variations" of Poincaré). Under the assumption of small disturbances these equations are linearized. Nevertheless, considerable complexity is involved due to the existence of variable coefficients. To facilitate a solution the Laplace transform is used rather extensively and stability, with the introduction of a few theorems, is visualized entirely from the transform plane. The original stability problem is eventually replaced by a simpler problem of determining information regarding the zeros of a function in a Laplace transform plane. A method suitable for attaining this objective is discussed. All portions of the analysis, following the selection of the equations of motion, are exact.

In an appendix to the text a problem concerning the stability and response of a string on a nonlinear foundation subjected to a concentrated moving load is studied. Both the results and a major part of the analysis are closely related to the shell discussion. Investigation of the considerably simpler string formulation is instructive in that it allows one to obtain a basic understanding of problems of this type without excessive mathematical complication. The nonlinear one-dimensional wave equation considered is shown to also represent the radial displacements of a nonlinear (nonlinearity in both geometric and constitutive equations) elastic cylindrical membrane in the light of certain approximations. Under these approximations, the above string problem is equivalent to the stability and response of a nonlinear cylindrical membrane subjected to a ring load moving with constant velocity.

CHAPTER II

PRELIMINARY REMARKS

1. Formulation of the Problem

The Shell

In the ensuing analysis we shall consider an infinitely long, elastic, isotropic and homogeneous cylindrical shell with a uniform thickness h and whose middle surface has the radius a. The shell will be assumed to be thin so that $h/a \ll 1$.

All motions of the middle surface of the shell will be referred to a Lagrangian or fixed coordinate system as illustrated in Fig. 1. The longitudinal, circumferential and radial displacements of the middle surface will be denoted as U, V, and W respectively.

The Equations of Motion

A Donnell type nonlinear theory (30) will be employed as a mathematical model of the shell. Accordingly, the equilibrium equations of the cylinder are:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$$

$$D\nabla^4 W = p + N_x \frac{\partial^2 W}{\partial x^2} + 2N_{xy}\frac{\partial^2 W}{\partial x \partial y} + N_y \frac{\partial^2 W}{\partial x^2} + \frac{N_y}{a} - \rho h \frac{\partial^2 W}{\partial T^2}, \tag{2.1}$$

the stress (resultant)-strain relations are:

$$N_x = \frac{Eh}{1-\nu^2}\left[\epsilon_x + \nu\epsilon_y\right]$$

$$N_y = \frac{Eh}{1-\nu^2}\left[\epsilon_y + \nu\epsilon_x\right] \qquad (2.2)$$

$$N_{xy} = \frac{Eh}{2(1+\nu)}\gamma_{xy}$$

and the nonlinear strain-displacement relations are given by:

$$\epsilon_x = \frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2$$

$$\epsilon_y = \frac{\partial V}{\partial y} + \frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2 - \frac{W}{a} \qquad (2.3)$$

$$\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + \frac{\partial W}{\partial x}\frac{\partial W}{\partial y}$$

The first two equilibrium equations 2.1 can be satisfied by a stress function F defined by:

$$N_x = \frac{\partial^2 F}{\partial y^2} \qquad N_y = \frac{\partial^2 F}{\partial x^2} \qquad N_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y} \qquad (2.4)$$

which when introduced into the last of equations 2.1 yields:

$$D\nabla^4 W = p + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 W}{\partial x^2} - 2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 W}{\partial x\,\partial y}$$

$$+ \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + \frac{1}{a}\frac{\partial^2 F}{\partial x^2} - \rho h\frac{\partial^2 W}{\partial T^2} \qquad (2.5a)$$

By substituting equations 2. 3 into equations 2. 2 and eliminating the variables U and V, one obtains an additional equation governing W and F:

$$\nabla^4 F = Eh\left[\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2 - \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - \frac{1}{a}\frac{\partial^2 W}{\partial x^2}\right] \qquad (2.5b)$$

Equation 2. 5a represents the radial equilibrium of the shell; equation 2. 5b is the condition of compatability. These two equations will constitute the basic equations of motion of the shell. The domain of the problem is infinite, i. e.:

$$-\infty < x < \infty, \qquad 0 \leq y \leq 2\pi a, \qquad T > 0$$

It should be noted that it has been implicitly assumed in the derivation of these equations that strains and rotations are small compared to 1. Since the ratios of deformed areas to undeformed areas and deformed volumes to undeformed volumes differ from unity by magnitudes of the same order as the elongations and shears, one therefore (in the first approximation) need not differentiate between stresses on deformed and undeformed areas. (The reader is referred to Novozhilov (54) for amplification of these statements.) A further assumption (usually associated with Donnell (55)) is that $V << \partial W/\partial\theta$. Donnell has indicated that this latter approximation is valid if, upon deforming, the displacements of the middle surface are such that the square of the number of circumferential waves, n, is large compared to 1. For thin shells, $n > 5$ is considered as "large". (For the special case $n = 0$, i. e., axially symmetric motions, Donnell's approximation is not involved since V and $\partial W/\partial\theta$ are both identically zero.)

It is evident, from the equilibrium equations 2.1, that only the effects of radial inertia were included, i. e, we have neglected (1) tangential inertia, (2) circumferential inertia, (3) rotary inertia and (4) transverse shear deformation. (Since (3) and (4) were neglected, equations 2.5 have a diffusive character and one can expect energy transfer to take place at infinite velocity.) The neglect of these quantities will necessitate a restriction on the magnitude of the load velocity. The effects of these approximations on the symmetric motion of the shell will be discussed in the sequel.

The Loading Condition

The loading condition shall consist of a constant axial stress, N_x^o, and an axially symmetric lateral pressure distribution $P(X, T)$ moving with a constant velocity, V, so that

$$P(X,T) = P(X-VT)$$

The velocity of the load will be restricted to the following limits:

$$V < \left(\frac{E}{\rho}\,\frac{h}{a}\right)^{1/2}\left[3(1-\nu^2)\right]^{-1/4} \quad \text{IF} \quad N_x^{\circ} \geq 0$$

$$V < \left[\frac{h}{a}\,\frac{1}{\sqrt{3(1-\nu^2)}} + \frac{N_x^{\circ}}{Eh}\right]^{1/2}\sqrt{\frac{E}{\rho}} \quad \text{IF} \quad N_x^{\circ} < 0$$

These restrictions are associated with the approximations mentioned above and the assumed form of the initial motion of the shell; they will be discussed later.

Now, consider a coordinate system moving with the load as defined by the transformation:

$$\xi_1 = X - VT$$

Within the semi-infinite intervals

$$-\infty < \xi_1 < 0, \quad \text{AND} \quad 0 < \xi_1 < \infty$$

the load distribution will be assumed in the form:

$$P(\xi_1) = P_0 + \sum_{n=1}^{N} P_n e^{-\Omega_n \xi_1} \qquad 0 < \xi_1 < \infty$$

$$P(\xi_1) = P_0^* + \sum_{k=1}^{K} P_k^* e^{\Omega_k^* \xi_1} \qquad -\infty < \xi_1 < 0$$

Here N and K are finite, P_o and P_o^* are real constants, P_n, P_k^*, Ω_n, Ω_n^*, are in general complex valued and $\text{Re}\,\Omega_n > 0$, $\text{Re}\,\Omega_n^* > 0$. At the point $\xi_1 = 0$ we will allow a concentrated load $P_c\,\delta(\xi_1)$, where $\delta(\xi_1)$ denotes the Dirac delta function and P_c is a real constant. By use of the Heaviside step function, $H(\xi_1)$, the assumed lateral loading can be more compactly written as:

$$P(\xi_1) = P_c\,\delta(\xi_1) + H(\xi_1)\left[P_0 + \sum_{n=1}^{N} P_n e^{-\Omega_n \xi_1}\right] + H(-\xi_1)\left[P_0^* + \sum_{k=1}^{K} P_k^* e^{\Omega_k^* \xi_1}\right] \qquad (2.6)$$

Several special cases of equation 2.6 are illustrated in Figs. 2a, b, c, and d. They include the moving ring load, step load, decayed steps and pulse respectively. Note that the effect of an internal pressurization of the cylinder can be included by an appropriate choice of P_o and P_o^*.

Although we shall not consider it as part of our discussion, by a similar analysis one can generalize 2.6 considerably. Such a generalization consists of the sum of a finite number of delta functions and a function $\mathcal{P}(\xi_1)$. The function $\mathcal{P}(\xi_1)$ is assumed piecewise continuous on any finite interval, continuous on any semi-infinite interval and representable in any continuous ith interval by

$$P^{(i)}(\xi_1) = P_o^{(i)} + \sum_{n=1}^{N_i} P_n^{(i)} e^{-\Omega_n^{(i)} \xi_1}$$

where $P_o^{(i)}$ are real valued and $P_n^{(i)}$, and $\Omega_n^{(i)}$ are in general complex valued. Such a generalization, however, would not lend anything to the present analysis since (1) the method of handling the more general loading should be clear from the ensuing analysis and (2) the algebra becomes considerably more cumbersone.

The Axially Symmetric Response of the Shell

We shall seek bounded solutions to equations 2.5, under the loading condition 2.6, of the form

$$\begin{aligned} W(X, Y, T) &= W_s(X - VT) \\ F(X, Y, T) &= F_s(X - VT, Y) \end{aligned} \tag{2.7}$$

This class of solutions represent axially symmetric motions (denoted by the subscript "s") that are time invariant in a coordinate system moving with velocity, V. We shall denote these solutions as "steady-state". With the infinite shell in mind, such motions can be visualized as the limiting case $(T \to \infty)$ of a transient problem in which the load is applied and brought up to speed from rest in some manner. Our primary objective will be to determine the stability of these steady-state motions.

The System in Nondimensional Form

For future discussion it will be advantageous to recast the equations of motion 2.5 and the loading condition 2.6 in a non-dimensional form. For this purpose we define the quantities

$$w = \frac{W}{a}, \quad f = \frac{F}{a^2 Eh}, \quad x = \frac{X}{a}, \quad \theta = \frac{Y}{a}$$

$$t = \frac{T}{a}\sqrt{\frac{E}{\rho}}, \quad q = \frac{Pa}{Eh}, \quad M = \frac{V}{\sqrt{E/\rho}} \tag{2.8}$$

Under 2.8, and writing partial derivatives as $\frac{\partial^2(\ )}{\partial\theta^2} = (\ )_{\theta\theta}$, etc., equation 2.5 becomes

$$\beta^4 \nabla^4 w = q(x - Mt) + f_{\theta\theta} w_{xx} - 2 f_{x\theta} w_{x\theta} + f_{xx}(1 + w_{\theta\theta}) - w_{tt} \tag{2.9a}$$

$$\nabla^4 f = (w_{x\theta})^2 - w_{xx}(1 + w_{\theta\theta}) \tag{2.9b}$$

where $\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial\theta^2} + \frac{\partial^4}{\partial\theta^4}$, AND $\beta^4 = \frac{h^2}{12 a^2 (1 - \nu^2)}$

The loading function takes the following nondimensional form:

$$q(\xi) = q_c\, \delta(\xi) + H(\xi)\Big[q_0 + \sum_{m=1}^{N} q_m e^{-\gamma_m \xi}\Big] + H(-\xi)\Big[q_0^* + \sum_{k=1}^{K} q_k^* e^{+\gamma_k^* \xi}\Big] \tag{2.10}$$

and $N_x^* = \dfrac{N_x^\circ}{Eh}$

where

$$q_c = \frac{P_c}{Eh}, \quad q_n = \frac{P_n a}{Eh}, \quad q_n^* = \frac{P_n^* a}{Eh}, \quad \gamma_n = a\Omega_n, \quad \gamma_n^* = a\Omega_n^*$$

and

$$\xi = x - Mt$$

2. Definition of Stability. Physical Aspects of the Definition

Let us perturb the steady-state axially symmetric motions w_s and f_s by, respectively, the quantities $\varphi(x, \theta, t)$ and $\eta(x, \theta, t)$. The following restrictions will be placed on φ and η:

(1) φ and η will be considered as infinitesimal quantities so that their higher powers can be neglected in the analysis.

(2) φ and $\eta = 0$ for $|x| > x_0$ (The initial disturbance shall be confined to a finite interval of the x axis.)

(3) The perturbations shall satisfy the following quiescent condition:

$$\lim_{|x| \to \infty} \left\{ \begin{matrix} \varphi(x, \theta, t) \\ \eta(x, \theta, t) \end{matrix} \right\}_{\theta, t \text{ FIXED}} = 0 \qquad (2.11a)$$

If w_p and f_p denote the perturbed solutions, then we have:

$$\begin{aligned} w_p(x, \theta, t) &= w_s(x - Mt) + \varphi(x, \theta, t) \\ f_p(x, \theta, t) &= f_s(x - Mt, \theta) + \eta(x, \theta, t) \end{aligned} \qquad (2.11b)$$

Now, stability shall be defined on the following basis. If we insert w_p and f_p into the nonlinear equations of motion 2.9 and neglect powers of φ and η above the first, we obtain linear variational equations for φ and η. If all φ and η satisfying these equations and conditions (1), (2), and (3) above are bounded, w_s and f_s will be said to be stable, otherwise unstable. Note that since the variational equations will be linear, and if φ and η are bounded, then this bound will be directly proportional to the initial values $\varphi(x, \theta, t_o)$ and $\eta(x, \theta, t_o)$. Therefore an equivalent but probably more precise definition of stability is: w_s and f_s are stable if, given $\epsilon > 0$ and t_o, there exists a $\delta = \delta(\epsilon, t_o)$ such that

$$\begin{aligned} &(|\varphi(x,\theta,t_o)|,\ |\eta(x,\theta,t_o)|) < \delta(\epsilon,t_o) \\ \text{implies} \qquad & \qquad\qquad\qquad\qquad\qquad\qquad (2.12) \\ &(|\varphi(x,\theta,t)|,\ |\eta(x,\theta,t)|) < \epsilon \end{aligned}$$

In other words, stability is defined on the basis of whether an infinitesimal disturbed motion about an initial response state remains bounded or diverges.

Let us consider the physical consequences of the above definition of stability. If a thin elastic shell is statically loaded, i. e., if $V = 0$, the transition from stability to instability represents an upper bound on the buckling load. It is an upper bound in the sense that finite disturbance might conceivably lead to a buckling phenomena at a lower value of the load. (One might also argue, although it is surely remote, that conditions (2) and (3) are restrictions which if relaxed might lead to new unstable solutions. In such a case our definition of stability would still lead to an upper bound of the buckling load.) In the dynamic problem, a divergence may lead to either a buckling phenomena or simply to finite amplitude oscillations, depending on the nature of the loading. One cannot be distinguished from the other, however, directly from the linearized analysis. We cannot, of course, in either the static or dynamic case, obtain information regarding the postbuckled or finite amplitude oscillation state of the shell. The method, nevertheless, provides substantial information regarding the initiation of motions other than the initial symmetric motion and is particularly valuable in preliminary studies of shell stability problems. In the final analysis, its appropriateness depends on the question asked.

3. Basic Outline of the Solution

A preliminary sketch of the solution to the proposed stability problem is as follows:

1. The initial symmetric quantities $w_s(x - Mt)$ and $f_s(x - Mt, \theta)$ are determined from equations 2.9 subject to the loading defined by equation 2.10.

2. This symmetric solution is perturbed by the non-symmetrical quantities $\mathcal{Y}(x, \theta, t)$ and $\eta(x, \theta, t)$. Equations 2.11b, representing such perturbations, are substituted into the equations of motion 2.6 and the resulting equations governing $\mathcal{Y}$ and η are linearized on the assumption that $\mathcal{Y}$ and η are infinitesimal.

3. These linearized or variational equations are treated as follows:

a. φ and η are represented in a Fourier series in the θ variable. The result is an infinite system (uncoupled in sets of 2) of partial differential equations in x and t, θ being replaced by a parameter, n.

b. A Galilean transformation of the form $\xi = x - Mt$, $\tau = t$ is effected, rendering the variable coefficients in the variational equations a function of one variable, ξ, only. Upon the nth set of equations the Laplace transform is applied with respect to τ, replacing it with the parameter p and reducing the system to total differential equations with variable coefficients in ξ.

c. A second Laplace transform (unilateral) is applied with respect to the ξ variable in the nth set of subsidiary equations replacing ξ by the parameter s and yielding a system of functional difference equations in terms of s. This system of difference equations is solved under suitable restrictions and the second transform (s) is inverted, thus supplying a solution to the subsidiary equations.

d. The properties of the solution in the p-plane are ascertained. With the aid of a few theorems concerning stability and the Laplace transform, which we introduce, the stability analysis is reduced to locating the zeros of a certain function of p in the p-plane. A scheme for accomplishing this task is discussed.

CHAPTER III

GENERAL ANALYSIS

1. Steady-State Symmetric Response

The governing equations for the axially symmetric motion of the shell are obtained by setting:

$$w_\theta = 0, \quad f_{x\theta} = 0$$

and requiring that f_{xx} and $f_{\theta\theta}$ be independent of the variable θ. Under these restrictions, equations 2.9 take the form:

$$\beta^4 w_{s_{xxxx}} = q(x - Mt) + f_{s_{\theta\theta}} w_{s_{xx}} + f_{s_{xx}} - w_{s_{tt}} \qquad (3.1a)$$

$$\left(f_{s_{xx}} + f_{s_{\theta\theta}}\right)_{xx} = - w_{s_{xx}} \qquad (3.1b)$$

It is advantageous at this point to recall that equation 3.1b can be written as:

$$\left(\frac{N_x}{Eh} + \frac{N_y}{Eh}\right)_{xx} = - w_{s_{xx}}$$

From the first of equations 2.1 one observes that

$$\frac{\partial N_x}{\partial x} = 0$$

for the symmetric state of motion and therefore

$$N_x = N_x^\circ(t)$$

The quantity N_X^O therefore represents an initial compression or tension in the cylinder and is independent of the variable x. We shall assume N_X^O = constant and define

$$N_x^* = \frac{N_x^\circ}{Eh} \qquad (3.2)$$

Upon substituting equations 2.3 into equations 2.2 (imposing axial symmetry) and solving for the variable N_y, one obtains:

$$\frac{N_y}{Eh} = - w_s + \nu N_x^* \qquad (3.3)$$

Therefore equation 3.1b can be replaced by

$$f_{s_{xx}} = - w_s + \nu N_x^*, \qquad f_{s_{\theta\theta}} = N_x^* \qquad (3.4)$$

Substitution of equations 3.4 into 3.1a yields the following linear partial differential equation for the radial displacement w_s:

$$\beta^4 w_{s_{xxxx}} - N_x^* w_{s_{xx}} + w_s + w_{s_{tt}} = g(x - Mt) + \nu N_x^* \qquad (3.5)$$

The fact that equation 3.5 is linear is a consequence of the approximations used in deriving equations 2.6. If the effects of longitudinal inertia had been included this would not be the case. Let us attempt to estimate the validity of the approximations regarding the neglect of longitudinal inertia, shear deformation and rotational inertia. Consider the homogeneous portion of equation 3.5:

$$\beta^4 w_{s_{xxxx}} - N_x^* w_{s_{xx}} + w_s + w_{s_{tt}} = 0 \qquad (3.6)$$

One recognizes immediately that equation 3.6 represents a Bernoulli-Euler beam-column on an elastic foundation. This equation does not, of course, possess a correct hyperbolic character but rather, is diffusive in nature. Some light can be shed on the validity of equation 3.5 and indirectly on equation 2.5 by considering the phase velocity spectrum of equation 3.6. This is obtained by assuming steady-state wave train solutions of the form

$$w_s(x,t) = A e^{iK(x - ct)} \tag{3.7}$$

Here K represents the wave number and c the phase velocity. Substitution of the above relation into equation 3.6 yields

$$\beta^4 K^4 + K^2 (N_x^* - c^2) + 1 = 0 \tag{3.8}$$

A plot of K versus c constitutes the phase velocity spectrum. A comparison of the velocity spectrum of equation 3.6, under the condition $N_X^* = 0$, with that of a more exact theory (linear) corresponding to Timoshenko's theory of beam vibrations can be made by referring to the work of Tang (23). This comparison is illustrated in Figs. 3a and 3b for the case $\frac{h}{a} = 0.06$. When $c < \sqrt{2}\,\beta$, the velocity spectrum of both theories possess complex wave numbers and the spectrums are found to be practically identical. When $c > \sqrt{2}\,\beta$, the velocity spectrum from the Bernoulli-Euler theory exists with real wave numbers. The two spectrums agree, however, for only rather small wave numbers. In short, for $c < \sqrt{2}\,\beta$, one observes that, as far as steady-state solutions are concerned, the effects of rotary inertia and shear deformation are apparently negligible.

Further enlightenment can be obtained from the analyses of Nachbar (27) and Jones and Bhuta (24) where the linear, axially symmetric response to moving loads on infinite-length cylindrical shells was studied. Extrapolating from their linear steady-state calculations, the effects of longitudinal inertia on the steady-state solutions are negligible for velocities (load or phase) that are considerably less than the plate speed which is given by

$$V_p^2 = \frac{E}{\rho(1-\nu^2)}$$

Now, the ratio of cutoff velocity ($N_x^* = 0$), i.e. the minimal velocity for which sinusoidal wave trains can be propagated in the shell, to plate speed is

$$\left(\frac{V_{cu}}{V_p}\right)^2 = \frac{Eh/\rho a\sqrt{3(1-\nu^2)}}{E/\rho(1-\nu^2)} = \frac{h}{a}\left(\frac{1-\nu^2}{3}\right)^{1/2}$$

Thus, since we are assuming the shell is thin so that $\frac{h}{a} << 1$, the effects of longitudinal inertia are apparently not important if $V < V_{cu}$ or in nondimensional form, if $c < \sqrt{2}\,\beta$.

Our theory can therefore be expected to satisfactorily model the steady-state motions if the shell (within the context of the usual strain displacement approximations given by equations 2.3) for $N_x^* = 0$ if the nondimensional load velocity, M, is restricted to $M < \sqrt{2}\,\beta$. Further, below the cutoff velocity, the symmetric response of the shell will be attenuated in space due to the existence of the complex arm of the phase velocity spectrum. For load velocities below the cutoff velocity, therefore, the steady-state response can have significant meaning if the shell is long compared to a characteristic attenuation length. However, for load velocities above the cutoff velocity, the displacements will not be attenuated and the effects of the load will be propagated throughout the entire shell (see Section 4 of Appendix 5). Thus the effects of boundaries, unless damping is considered, become important independent of the length of the shell and a transient problem should, in general, be examined.

If $N_x^* \neq 0$, the cutoff velocity for the Bernoulli-Euler theory is given by

$$c_{cu} = [2\beta^2 + N_x^*]^{1/2}$$

If axial compression is present we must require that

$$2\beta^2 + N_x^* > 0$$

so that a velocity region where the response will be attenuated exists. Note that $-N_x^* = 2\beta^2 = \frac{h}{a}\frac{1}{\sqrt{3(1-\nu^2)}} = -N_x/Eh$ represents the classical buckling load for a long cylinder in axial compression. Thus we are requiring that the axial load be less than the static buckling load. We shall therefore restrict the load velocity to

$$M < \sqrt{2}\beta \quad \text{OR} \quad V < \sqrt{\frac{E}{\rho}}\sqrt{\frac{h}{a}}\frac{1}{[3(1-\nu^2)]^{1/4}} \quad \text{IF } N_x^* \geq 0 \tag{3.9a}$$

and if axial compression is present we will require that

$$2\beta^2 + N_x^* > 0$$

and

$$M < [2\beta^2 + N_x^*]^{1/2} \quad \text{OR} \quad V < \left[\frac{h}{a\sqrt{3(1-\nu^2)}} + \frac{N_x^\circ}{Eh}\right]^{1/2}\left(\frac{E}{\rho}\right)^{1/2} \quad \text{IF } N_x^* < 0 \tag{3.9b}$$

If $N_x^* = 0$ and the shell material is steel, the cutoff velocity ranges from 400 f.p.s. – 2000 f.p.s. for $\frac{h}{a} = \frac{1}{1000} - \frac{1}{40}$ respectively.

In the ensuing analysis, it will be convenient to consider the equations of motion as referenced to a moving coordinate system defined by the Galilean transformation:

$$\begin{aligned} \xi &= x - Mt \\ \theta &= \theta \\ \tau &= t \end{aligned} \qquad (3.10)$$

Such a transformation simplifies the problem considerably since it renders the variable coefficients of the variational equations a function of one variable, ξ, only. Under the transformation 3.10, the equations of motion 2.9 become

$$\begin{aligned} \beta^4 \nabla^4 w &= q(\xi) + f_{\theta\theta} w_{\xi\xi} - 2 f_{\xi\theta} w_{\xi\theta} \\ &+ f_{\xi\xi}(1 + w_{\theta\theta}) - w_{\tau\tau} + 2M w_{\xi\tau} - M^2 w_{\xi\xi} \end{aligned} \qquad (3.11a)$$

$$\nabla^4 f = (w_{\xi\theta})^2 - w_{\xi\xi}(1 + w_{\theta\theta}) \qquad (3.11b)$$

where

$$\nabla^4 = \frac{\partial^4}{\partial \xi^4} + \frac{2\partial^4}{\partial \xi^2 \partial \theta^2} + \frac{\partial^4}{\partial \theta^4}$$

The symmetric equation 3.5 takes the following form in the moving coordinates:

$$\begin{aligned} &\beta^4 w_{s\,\xi\xi\xi\xi} - N_x^* w_{s\,\xi\xi} + w_s + w_{s\,\tau\tau} \\ &- 2M w_{s\,\xi\tau} + M^2 w_{s\,\xi\xi} = q(\xi) + \nu N_x^* \end{aligned} \qquad (3.12)$$

The steady-state symmetric response of the shell is obtained by solution of equation 3.12 under the condition that $w_{s\tau} = 0$. This leads to the following total differential equation for the radial displacement w_s:

$$\beta^4 \frac{d^4 w_s}{d\xi^4} + (M^2 - N_x^*) \frac{d^2 w_s}{d\xi^2} + w_s = q(\xi) + \nu N_x^* \tag{3.13}$$

As far as boundary conditions are concerned, we shall require only that solutions to 3.13 be bounded as $\xi \rightarrow \pm \infty$. Because of equations 3.9 and 2.10, this is equivalent to stating that (1) $w_s(\xi) \rightarrow 0$ as $\xi \rightarrow \pm \infty$ if $q(\xi)$ contains no constant part or (2) $w(\xi) \rightarrow$ constant parts of $q(\xi)$ as $\xi \rightarrow \pm \infty$. Under these conditions, the symmetric solution is unique. Since we are requiring 3.9 a, and b be satisfied the roots of the characteristic equation will have non-zero real parts and hence the bounded homogeneous solutions to equation 3.13 will be attenuated with distance from the load.

The solution to equation 3.13, subject to the loading function $q(\xi)$ as given by 2.10, and the boundness conditions at $\xi = \pm \infty$ can be written as

$$w_s(\xi) = \int_{-\infty}^{\infty} g(\xi, \lambda) [q(\lambda) + \nu N_x^*] d\lambda \tag{3.14}$$

where $g(\xi, \lambda)$ represents the Green's function of equation 3.13 and has the form:

$$g(\xi,\lambda) = \sum_{i=1}^{2} g_i \, e^{-\alpha_i(\xi-\lambda)}, \qquad (\xi-\lambda) > 0$$

$$g(\xi,\lambda) = \sum_{i=1}^{2} g_i \, e^{+\alpha_i(\xi-\lambda)}, \qquad (\xi-\lambda) < 0 \qquad (3.15)$$

where

$$g_{1,2} = \frac{1}{4\sqrt{\bar{\beta}^4 - \bar{M}^4}} \left[\sqrt{\bar{\beta}^2 + \bar{M}^2} \pm i\sqrt{\bar{\beta}^2 - \bar{M}^2}\right]$$

$$\alpha_{1,2} = \frac{1}{\bar{\beta}^2} \left[\sqrt{\bar{\beta}^2 + \bar{M}^2} \pm i\sqrt{\bar{\beta}^2 - \bar{M}^2}\right] \qquad (3.16)$$

and $\bar{\beta} = \sqrt{2}\,\beta$, $\bar{M}^2 = M^2 - N_x^*$. Note that the Green's function, which is representative of the symmetric motion resulting from a unit radial line load traveling with constant velocity, is symmetric in ξ. This occurs only for $M < \sqrt{2}\beta$ (see Appendix 5).

Upon evaluation of the integral 3.14, one obtains the steady-state solution, $w_s(\xi)$, as:

$$\left.\begin{aligned} w(\xi) &= c_0 + \sum_{j=1}^{\ell} c_j e^{-\alpha_j \xi}, \quad \ell = N+2, \quad \xi > 0 \\ w(\xi) &= c_0^* + \sum_{j=1}^{\ell^*} c_j^* e^{\alpha_j^* \xi}, \quad \ell^* = K+2, \quad \xi < 0 \end{aligned}\right\} \qquad (3.17)$$

where c_j, c_j^*, α_j, and α_j^* are in general complex valued (c_0 and c_0^* are real valued), $\alpha_1 = -\alpha_1^*$ and $\alpha_2 = -\alpha_2^*$ are given in equations 3.17 and in general

$$Re(\alpha_j) > 0, \quad Re(\alpha_j^*) > 0$$

An interesting observation can be made from the Green's function 3.15: a bending resonance* exists as the load velocity, M, approaches $\left[\bar{\beta}^2 + N^*\right]^{1/2}$. This value of M corresponds to the cutoff velocity discussed previously. Since we have not considered damping in our analysis, the amplitude of the motion at the resonance speed is of course unbounded. Some information regarding the effect of viscous damping on the magnitude of the amplitude at the resonance speed can be found in (27).

2. Variational Equations and Their Solutions

2.1 The Variational Equations

In the following paragraphs we shall consider the stability of the steady-state symmetric motion w_s and f_s given by equation 3.17.

Let us perturb the solutions w_s and f_s by the nonsymmetrical quantities φ and η as given by equations 2.11, but referred now to the ξ, τ coordinates:

$$\begin{aligned} w_p(\xi,\theta,\tau) &= w_s(\xi) + \varphi(\xi,\theta,\tau) \\ f_p(\xi,\theta,\tau) &= f_s(\xi,\theta) + \eta(\xi,\theta,\tau) \end{aligned} \tag{3.18}$$

where w_p and f_p represent the perturbed solutions. Substitution of 3.18 into the equations of motion 3.11, cancellation of quantities identically satisfied by the symmetric solution and subsequent neglect of powers of the perturbation quantities higher than the first, yields the following variational equations in the ξ, τ coordinates:

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* See (24) for a detailed discussion of this resonance condition for $N_x^* = 0$.

$$\beta^4 \nabla^4 \varphi = (N_x^* - M^2)\,\varphi_{\xi\xi} + \frac{d^2 w_s(\xi)}{d\xi^2}\,\eta_{\theta\theta}$$

$$- w_s(\xi)\,\varphi_{\theta\theta} + \eta_{\xi\xi} - \varphi_{\tau\tau} + 2M\,\varphi_{\xi\tau} \qquad (3.19a)$$

$$\nabla^4 \eta = -\varphi_{\theta\theta}\,\frac{d^2 w_s(\xi)}{d\xi^2} - \varphi_{\xi\xi} \qquad (3.19b)$$

where $-\infty < \xi < \infty$, $0 \le \theta \le 2\pi$, $\tau > 0$.

In view of equation 2.11a we shall require as a quiescent condition

$$\lim_{|\xi| \to \infty} \left\{ \begin{matrix} \varphi(\xi, \theta, \tau) \\ \eta(\xi, \theta, \tau) \end{matrix} \right\}_{\tau,\,\theta \text{ FIXED}} = 0 \qquad (3.21)$$

The subject of what initial conditions shall accompany equations 3.19 shall be put aside momentarily.

2.2 <u>Series Solution.</u> <u>First Laplace Transform</u>

Let us represent the functions φ and η by the following Fourier series*:

$$\varphi(\xi, \theta, \tau) = \sum_{n=0}^{\infty} \varphi_n(\xi, \tau)\cos n\theta$$

$$\eta(\xi, \theta, \tau) = \sum_{n=0}^{\infty} \eta_n(\xi, \tau)\cos n\theta \qquad (3.22)$$

Upon substituting the series 3.22 into the variational equations 3.19, one obtains the following set of coupled partial differential equations for each integer, n:

---

* It is sufficient to use either a cosine or sine series.

$$\beta^4 \varphi_{n_{\xi\xi\xi\xi}} + [M^2 - N_x^* - 2n^2\beta^4]\varphi_{n_{\xi\xi}} + [\beta^4 n^4 - n^2 \omega_s(\xi)]\varphi_n$$

$$+ \varphi_{n_{\tau\tau}} - 2M\varphi_{n_{\xi\tau}} = \eta_{n_{\xi\xi}} - n^2 \frac{d^2\omega_s(\xi)}{d\xi^2}\eta_n \qquad (3.23a)$$

$$\eta_{n_{\xi\xi\xi\xi}} - 2n^2\eta_{n_{\xi\xi}} + n^4\eta_n = n^2 \frac{d^2\omega_s(\xi)}{d\xi^2}\varphi_n$$

$$- \varphi_{n_{\xi\xi}} \qquad (3.23b)$$

Next, Laplace transform equations 3.23 with respect to τ. The philosophy regarding the transform will be to assume all properties necessary for its use. (In principle one can verify the solutions obtained for φ_n and η_n, written as a Bromwich inversion integral, by direct substitution into the original equations.) Denoting the transforms of φ_n and η_n by respectively:

$$\bar{\varphi}_n(\xi, p) = \int_0^\infty e^{-p\tau}\varphi_n(\xi, \tau)\, d\tau$$

$$\tau > 0 \quad Re\, p > c \qquad (3.24)$$

$$\bar{\eta}_n(\xi, p) = \int_0^\infty e^{-p\tau}\eta_n(\xi, \tau)\, d\tau$$

and transforming equations 3.23, we obtain the subsidiary equations

$$\beta^4 \frac{d^4\bar{\varphi}_n}{d\xi^4} + [M^2 - 2n^2\beta^4 - N_x^*] \frac{d^2\bar{\varphi}_n}{d\xi^2} - 2Mp \frac{d\bar{\varphi}_n}{d\xi}$$

$$+ [\beta^4 n^4 - n^2 w_s(\xi) + p^2]\bar{\varphi}_n - \frac{d^2\bar{\eta}_n}{d\xi^2} + n^2 \frac{d^2 w_s(\xi)}{d\xi^2}\bar{\eta}_n \tag{3.25a}$$

$$= [p\varphi_n(\xi,0) - 2M\frac{\partial \varphi_n(\xi,0)}{\partial \xi} + \frac{\partial \varphi_n(\xi,0)}{\partial \tau}]$$

$$\frac{d^4\bar{\eta}_n}{d\xi^4} - 2n^2 \frac{d^2\bar{\eta}_n}{d\xi^2} + n^4\bar{\eta}_n = n^2 \frac{d^2 w_s(\xi)}{d\xi^2}\bar{\varphi}_n - \frac{d^2\bar{\varphi}_n}{d\xi^2} \tag{3.25b}$$

From the quiescent conditions 3.21 we have in addition:

$$\lim_{|\xi| \to \infty} [\bar{\varphi}_n(\xi,n,p),\ \bar{\eta}_n(\xi,n,p)] = 0 \qquad Re(p) > c \tag{3.26}$$

The terms

$$\mathcal{A}(\xi,p) = [p\varphi_n(\xi,0) - 2M\varphi_{n_\xi}(\xi,0) + \varphi_{n_\tau}(\xi,0)] \tag{3.27}$$

which occur in the bracket of equation 3.25a represent the initial conditions of the problem, or, the form of the initial disturbance. We shall select a delta function in velocity, located at $\xi = 0$, as the initial disturbance, i.e.:

$$\varphi_{n_\tau}(\xi,0) = \delta(\xi)$$
$$\varphi_n(\xi,0) = 0 \tag{3.28}$$

The solution of equations 3.25 subject to the quiescent condition 3.26 and equation 3.28 is the solution of the boundary value problems in the domains $-\infty < \xi < 0$ and $0 < \xi < \infty$ consisting of the solution to equations 3.25 with the right hand side of 3.25 equal to zero, the quiescent condition 3.26, the continuity conditions:

$$\frac{d^\ell \bar{\zeta}_n(0^+)}{d\xi^\ell} = \frac{d^\ell \bar{\zeta}_n(0^-)}{d\xi^\ell}, \qquad \ell = 0, 1, 2$$
$$\frac{d^k \bar{\eta}_n(0^+)}{d\xi^k} = \frac{d^k \bar{\eta}_n(0^-)}{d\xi^k}, \qquad k = 0, 1, 2, 3 \tag{3.29}$$

and a jump condition:

$$\beta^4 \left[\frac{d^3 \bar{\zeta}_n(0^+)}{d\xi^3} - \frac{d^3 \bar{\zeta}_n(0^-)}{d\xi^3} \right] = 1 \tag{3.30}$$

Under conditions 3.26, 3.28, 3.29 and 3.30, the solution to equations 3.25a and b is unique.

2.3 A System of Difference Equations for the Second Laplace Transform

We must now construct the solution to the set of total differential equations 3.25. Because these equations possess variable coefficients, some complexity is involved. We begin by noting the form of the variable coefficients. Since the variable portion of $w_s(\xi)$ consists of a finite sum of exponentials (see equation 3.17) one observes that the variable coefficients of equations 3.25 also consist of a sum of exponentials. In view of this fact, it is possible to Laplace transform 3.25 with respect to ξ. We shall consider, for the present, only the interval $\xi \in (0, \infty)$, and shall apply a unilateral Laplace transform assuming the dependent variables are zero for $\xi < 0$. Inversion will yield a solution valid for $\xi > 0$ from which the solution for $\xi < 0$ can easily be deduced.

Upon transforming equations 3.25a and b, denoting the trans-forms of the dependent variables as

$$\bar{\bar{\varphi}}_n(s) = \int_0^\infty e^{-s\xi}\,\bar{\varphi}_n(\xi,t)\,d\xi \qquad \xi > 0 \qquad (3.31)$$

$$\bar{\bar{\eta}}_n(s) = \int_0^\infty e^{-s\xi}\,\bar{\eta}_n(\xi,t)\,d\xi \qquad \mathrm{Re}\, s > b \qquad (3.32)$$

and noting the shift property of the transform:

$$\int_0^\infty e^{-s\xi}\left[e^{-\alpha_j\xi}\,\bar{\varphi}_n(\xi,t)\right]d\xi = \bar{\bar{\varphi}}_n(s+\alpha_j) \qquad (3.33)$$

$$\int_0^\infty e^{-s\xi}\left[e^{-\alpha_j\xi}\,\bar{\eta}_n(\xi,t)\right]d\xi = \bar{\bar{\eta}}_n(s+\alpha_j) \qquad (3.34)$$

one obtains:

$$\bar{\bar{\varphi}}_n(s)\left[\beta^4 s^4 + s^2(M^2 - N_x^* - 2n^2\beta^4) - 2Mts + \beta^4 n^4 - n^2 c_0 + t^2\right] = s^2\bar{\bar{\eta}}_n(s) + \psi_1(s) - n^2\sum_{j=1}^{l} c_j\alpha_j^2\,\bar{\bar{\eta}}_n(s+\alpha_j) + n^2\sum_{j=1}^{l} c_j\,\bar{\bar{\varphi}}_n(s+\alpha_j) \qquad (3.35a)$$

and

$$(s^2-n^2)^2\,\bar{\bar{\eta}}_n(s) = -s^2\,\bar{\bar{\varphi}}_n(s) + n^2\sum_{j=1}^{l} c_j\alpha_j^2\,\bar{\bar{\varphi}}_n(s+\alpha_j) + \psi_2(s) \qquad (3.35b)$$

The functions $\psi_1(s)$ and $\psi_2(s)$ contain the initial conditions at $\xi = 0^+$ and are given by:

$$\Psi_1 = \beta^4 \sum_{i=0}^{3} s^{(3-i)} \frac{d^i \bar{\zeta}_n(0^+)}{d\xi^i} - 2Mp\,\bar{\zeta}_n(0^+) +$$

$$\sum_{i=0}^{1} s^{(1-i)} \left[(M^2 - N_x^+ - 2m^2\beta^4) \frac{d^i \bar{\zeta}_n(0^+)}{d\xi^i} - \frac{d^i \bar{\eta}_n(0^+)}{d\xi^i} \right] \qquad (3.36)$$

$$\Psi_2 = \sum_{i=0}^{3} s^{(3-i)} \frac{d^i \bar{\eta}_m(0^+)}{d\xi_i} + \sum_{i=0}^{1} s^{(1-i)} \left[\frac{d^i \bar{\zeta}_n(0^+)}{d\xi^i} - 2m^2 \frac{d^i \bar{\eta}_m(0^+)}{d\xi^i} \right]$$

Equations 3. 35 are linear functional difference equations with variable coefficients. There is a considerable advantage in dealing with the difference equations in place of the original differential equations; for, while the process of inversion is rather elementary, the difference equations are much easier to solve then equations 3. 25. For a discussion of the relationship between the Laplace transform and difference equations, and the solution of difference equations, reference is made to some of the works on the subject such as Van Der Pol and Bremmer (32), Milne-Thompson (33), and a recent paper by Valeev (34).

Since the variable coefficients of equation 3. 25, $w_s(\xi)$ and $\frac{d^2 w_s(\xi)}{d\xi^2}$ possess the property

$$\lim_{\xi \to \infty} w_s(\xi) = c_0$$

$$\lim_{\xi \to \infty} \frac{d^2 w_s(\xi)}{d\xi^2} = 0$$

it is not surprising that the solutions to equations 3. 25 are of exponential order*, i. e.,

$$|\bar{\varphi}_n(\xi,p)|,\ |\bar{\eta}_n(\xi,p)| \leq a e^{b\xi}$$

where a and b are constants. However, this implies

$$\left|\bar{\bar{\varphi}}_n(s)\right| = \left|\int_0^\infty e^{-s\xi}\,\bar{\varphi}_n(\xi,p)\,d\xi\right| \leq \int_0^\infty e^{-Re\,s\,\xi}|\bar{\varphi}_n|\,d\xi$$

$$\leq \int_0^\infty e^{-Re\,s\,\xi}(a e^{b\xi})\,d\xi = \frac{a}{Re\,s-b}, \quad \text{for } Re\,s > b$$

and a similar statement applies to $\bar{\bar{\eta}}_n(s)$. This being the case the following quiescent condition on the second transform must be satisfied:

$$|\bar{\bar{\varphi}}_n(s)|,\ |\bar{\bar{\eta}}_n(s)| \leq \frac{a}{Re\,s-b}, \quad Re\,s > b \qquad (3.37)$$

This condition is sufficient to render the second transform unique, or more specifically, the solution of the difference equation unique.

To facilitate a solution of the difference equations 3.35 it will be advantageous to rewrite these equations in a more compact matrix form. By denoting the two-dimensional vector $\bar{\bar{Z}}_n(s)$ as

$$\bar{\bar{Z}}_n(s) = \begin{Bmatrix} \bar{\bar{\varphi}}_n(s) \\ \bar{\bar{\eta}}_n(s) \end{Bmatrix} \qquad (3.38)$$

equations 3.35 can be written as:

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* See equations 3.61, 3.62 and 3.63.

$$L_0 \bar{\bar{Z}}_n(s) = \sum_{j=1}^{\ell} L_j \bar{\bar{Z}}_n(s+\alpha_j) + \Psi(s) \tag{3.39}$$

where $L_0, L_1, L_2, \cdots L_\ell$ are 2 x 2 matrices which are given by:

$$L_0 = \begin{bmatrix} \beta^4 s^4 + s^2[M^2 - N_x^* - 2n^2\beta^4] - 2Mps + [\beta^4 n^4 - n^2 c_0 + p^2] & -s^2 \\ s^2 & (s^2 - n^2)^2 \end{bmatrix} \tag{3.40a}$$

$$L_j = n^2 c_j \begin{bmatrix} 1 & -\alpha_j^2 \\ \alpha_j^2 & 0 \end{bmatrix} \tag{3.40b}$$

and $\Psi(s)$ is the two-dimensional vector:

$$\Psi(s) = \begin{Bmatrix} \Psi_1(s) \\ \Psi_2(s) \end{Bmatrix} \tag{3.40c}$$

Premultiplying 3.39 by the inverse of L_0, L_0^{-1}, we obtain the following system of difference equations:

$$\bar{\bar{Z}}_n(s) = \sum_{j=1}^{\ell} A_j(s) \bar{\bar{Z}}_n(s+\alpha_j) + \phi(s) \tag{3.41}$$

where

$$\phi(s) = L_0^{-1} \Psi(s) \tag{3.42}$$

$$A_j(s) = L_o^{-1} L_j = \frac{m^2 c_j}{\Delta(L_o)} \begin{bmatrix} (s^2-m^2)^2 + \alpha_j^2 s & -\alpha_j^2 (s^2-m^2)^2 \\ -s^2 + \alpha_j^2 [\beta^4 s^4 - 2M\not{p} s & \\ + s^2 (M^2 - N_x^* - 2m^2\beta^4) & \\ + (\beta^4 m^4 - m^2 c_o + \not{p}^2)] & s^2 \alpha_j^2 \end{bmatrix} \quad (3.43)$$

and $\Delta(L_o)$, the determinant of L_o , is given by:

$$\Delta(L_o) = \beta^4 s^8 + [M^2 - N_x^* - 4m^2\beta^4] s^6 - 2M\not{p} s^5$$
$$+ [1 + 6m^4\beta^4 - 2m^2(M^2 - N_x^*) - m^2 c_o + \not{p}^2] s^4$$
$$+ 4M\not{p} m^2 s^3 + [(M^2 - N_x^*)m^4 - 4m^6\beta^4 + 2m^4 c_o - 2m^2\not{p}^2] s^2 \quad (3.44)$$
$$- 2M\not{p} m^4 s + m^4(\beta^4 m^4 - m^2 c_o + \not{p}^2)$$

Now, let $\bar{\bar{Z}}_{m1}(s)$ and $\bar{\bar{Z}}_{m2}(s)$ be particular solutions of 3.41. Then $\bar{\bar{Z}}_{m1} - \bar{\bar{Z}}_{m2} = \bar{\bar{Z}}_{m3}$ satisfies the homogeneous difference equation

$$\bar{\bar{Z}}_{m3}(s) = \sum_{j=1}^{l} A_j(s)\, \bar{\bar{Z}}_{m3}(s + \alpha_j) \quad (3.45)$$

Thus, the difference between any two particular solutions is a solution of the homogeneous equation 3.45. Therefore, the most general solution of 3.41 is a particular solution of equation 3.41 plus the most general solution to the homogeneous equation, 3.45. However, all nontrivial solutions to the homogeneous difference equations 3.45 are unbounded in the limit as $\mathrm{Re}\, S \rightarrow \infty$. A proof of this statement can be found in Appendix 1. Thus only the trivial solution to the homogeneous equation can be accepted on the basis of the quiescent conditions 3.37. There is, therefore, a unique particular solution to be found.

The desired particular solution to 3.41 can be constructed by the method of ascending continued fractions (33). This method, by the way, is equivalent to the method of successive approximations (34) if $\phi(s)$ is selected as the first approximation. In either case, by repeated use of equation 3.41 and introduction of the indices $j_1, j_2, \cdots,$ one obtains the series:

$$\bar{\bar{Z}}_n(s) = \phi(s) + \sum_{j_1=1}^{l} A_{j_1}(s)\, \phi(s+\alpha_{j_1})$$

$$\sum_{j_1=1}^{l}\sum_{j_2=1}^{l} A_{j_1}(s)\, A_{j_2}(s+\alpha_{j_1})\, \phi(s+\alpha_{j_1}+\alpha_{j_2}) \qquad (3.46a)$$

$$+ \sum_{j_1=1}^{l}\sum_{j_2=1}^{l}\sum_{j_3=1}^{l} A_{j_1}(s)\, A_{j_2}(s+\alpha_{j_1})\, A_{j_3}(s+\alpha_{j_1}+\alpha_{j_2})\, \phi(s+\alpha_{j_1}+\alpha_{j_2}+\alpha_{j_3})$$

$$+ \cdots\cdots$$

The above series can also be written in a closed form as:

$$\bar{\bar{Z}}_n(s) = \phi(s) + \sum_{N=1}^{\infty} \prod_{k=1}^{N} \sum_{j_k=1}^{l} A_{j_k}\Big(s+\sum_{g=0}^{k-1}\alpha_{j_g}\Big)\, \phi\Big(s+\sum_{k=1}^{N}\alpha_{j_k}\Big) \qquad (3.46b)$$

where $\alpha_{j_0} \equiv 0$.

The component series for the vector $\bar{\bar{Z}}_n(s)$ are absolutely and uniformily (with respect to s) convergent and represent analytic functions of s when $s \in R_1$, where the region R_1 of the complex s-plane is defined by (see equations 3.47 for a definition of ρ_i)

$$|s - (\rho_i - m_1\alpha_1 - m_2\alpha_2 - \cdots - m_l\alpha_l)| \geq \epsilon > 0$$

$$m_j = 0, 1, 2, \cdots \qquad j = 1, 2, \cdots l$$

The singularities of $\bar{\bar{Z}}_n(s)$ are isolated poles, located at

$$s = \rho_i - m_1\alpha_1 - m_2\alpha_2 - \cdots m_l\alpha_l, \quad \begin{matrix} m_j = 0, 1, 2, \cdots \\ j = 1, 2, \cdots l \end{matrix}$$

Since these poles lie a finite distance to the right of the imaginary s-axis, $\bar{\bar{Z}}_n(s)$ is regular for Re $s > c_1$ = const. Additional details and a proof of these statements can be found in Appendix 2.

2.4 First Inversion

We shall now invert the series 3.46 with respect to the s-transform. Two methods of inversion will be utilized and two forms of the series for $\bar{Z}_n(\xi, p)$ will be obtained from the two methods. While the two forms are equivalent, one form allows easy observation of the behavior of $\bar{Z}_n(\xi, p)$ for large values of ξ while the other indicates the series truncates for $\xi = 0$. Recall that we must satisfy the boundary and continuity conditions defined by equations 3.26 and 3.29, 3.30 respectively. Thus the point $\xi = 0$ and the limit $\xi \to \infty$ are of particular importance.

Denote the roots of $\Delta L_o(s) = 0$ (see equation 3.44) as $\rho_2, \ldots, \rho_8$, so that

$$\Delta L_o(s) = \prod_{g=1}^{8} (s - \rho_g) \equiv \prod (s - \rho_g) \qquad (3.47)$$

and set

$$A_j(s) = \frac{B_j(s)}{\Delta L_o(s)} \qquad (3.48a)$$

$$\phi(s) = \frac{\Phi(s)}{\Delta L_o(s)} \qquad (3.48b)$$

where the meaning of B_j and $\Phi(s)$ should be clear from equations 3.42 and 3.43. To obtain the first form of the series, we carry out a term by term inversion of equation 3.46 (to be justified later) by residues, assuming the roots, ρ_g, as defined by equation 3.47, are non-repeated and assuming all factors in the denominator of each term of the series are non-repeated. This yields:

$$\bar{Z}_m(\xi, p) = \sum_{i=1}^{8} e^{\rho_i \xi} \left\{ \frac{\Phi(\rho_i)}{\prod\limits_{i \neq g} (\rho_i - \rho_g)} + \sum_{j_1=1}^{\ell} \frac{B_{j_1}(\rho_i)\, \Phi(\rho_i + \alpha_{j_1})}{\prod\limits_{i \neq g} (\rho_i - \rho_g)\, \prod (\rho_i - \rho_g + \alpha_{j_1})} + \right.$$

$$\left. \sum_{j_1, j_2 = 1}^{\ell} \frac{B_{j_1}(\rho_i)\, B_{j_2}(\rho_i + \alpha_{j_1})\, \Phi(\rho_i + \alpha_{j_1} + \alpha_{j_2})}{\prod\limits_{i \neq g} (\rho_i - \rho_g)\, \prod (\rho_i - \rho_g + \alpha_{j_1})} + \cdots \right\}$$

$$+ \sum_{i=1}^{8} \sum_{j_1=1}^{\ell} e^{(\rho_i - \alpha_{j_1})\xi} \left\{ \frac{B_{j_1}(\rho_i - \alpha_{j_1})\, \Phi(\rho_i)}{\prod (\rho_i - \rho_g - \alpha_{j_1})\, \prod (\rho_i - \rho_g)} + \right.$$

$$\sum_{j_2=1}^{\ell} \frac{B_{j_1}(\rho_i - \alpha_{j_1})\, B_{j_2}(\rho_i)\, \Phi(\rho_i + \alpha_{j_2})}{\prod (\rho_i - \rho_g - \alpha_{j_1})\, \prod\limits_{i \neq g} (\rho_i - \rho_g)\, \prod (\rho_i - \rho_g + \alpha_{j_2})} +$$

$$\left. \sum_{j_2, j_3 = 1}^{\ell} \frac{B_{j_1}(\rho_i - \alpha_{j_1})\, B_{j_2}(\rho_i)\, B_{j_3}(\rho_i + \alpha_{j_2})\, \Phi(\rho_i + \alpha_{j_2} + \alpha_{j_3})}{\prod (\rho_i - \rho_g - \alpha_{j_1})\, \prod\limits_{i \neq g} (\rho_i - \rho_g)\, \prod (\rho_i - \rho_g + \alpha_{j_2})\, \prod (\rho_i - \rho_g + \alpha_{j_2} + \alpha_{j_3})} + \cdots \right\}$$

$$+ \sum_{i=1}^{8} \sum_{j_1, j_2 = 1}^{\ell} e^{(\rho_i - \alpha_{j_1} - \alpha_{j_2})\xi} \left\{ \frac{B_{j_1}(\rho_i - \alpha_{j_1} - \alpha_{j_2})\, B_{j_2}(\rho_i - \alpha_{j_2})\, \Phi(\rho_i)}{\prod (\rho_i - \rho_g - \alpha_{j_1} - \alpha_{j_2})\, \prod (\rho_i - \rho_g - \alpha_{j_2})\, \prod\limits_{i \neq g} (\rho_i - \rho_g)} + \right.$$

$$\left. \sum_{j_3=1}^{\ell} \frac{B_{j_1}(\rho_i - \alpha_{j_1} - \alpha_{j_2})\, B_{j_2}(\rho_i - \alpha_{j_2})\, B_{j_3}(\rho_i)\, \Phi(\rho_i + \alpha_{j_3})}{\prod (\rho_i - \rho_g - \alpha_{j_1} - \alpha_{j_2})\, \prod (\rho_i - \rho_g - \alpha_{j_2})\, \prod\limits_{i \neq g} (\rho_i - \rho_g)} + \cdots \right\}$$

$$+ \cdots$$

where the notation $\Phi(\rho_i)$ indicates ρ_i is the argument of Φ, etc.

The above series can be re-arranged in the following form:

$$\bar{Z}_n(\xi,p) = \sum_{i=1}^{8} \left\{ e^{\rho_i \xi} \left[I + \sum_{j_1=1}^{\ell} \frac{B_{j_1}(\rho_i-\alpha_{j_1})\, e^{-\alpha_{j_1}\xi}}{\Pi(\rho_i-\rho_q-\alpha_{j_1})} + \right.\right.$$

$$\sum_{j_1,j_2=1}^{\ell} \frac{B_{j_1}(\rho_i-\alpha_{j_1}-\alpha_{j_2})\, B_{j_2}(\rho_i-\alpha_{j_2})\, e^{-(\alpha_{j_1}+\alpha_{j_2})\xi}}{\Pi(\rho_i-\rho_q-\alpha_{j_1}-\alpha_{j_2})\,\Pi(\rho_i-\rho_q-\alpha_{j_2})} +$$

$$\left.\left. \sum_{j_1,j_2,j_3=1}^{\ell} \frac{B_{j_1}(\rho_i-\alpha_{j_1}-\alpha_{j_2}-\alpha_{j_3})\, B_{j_2}(\rho_i-\alpha_{j_2}-\alpha_{j_3})\, B_{j_3}(\rho_i-\alpha_{j_3})\, e^{-(\alpha_{j_1}+\alpha_{j_2}+\alpha_{j_3})\xi}}{\Pi(\rho_i-\rho_q-\alpha_{j_1}-\alpha_{j_2}-\alpha_{j_3})\,\Pi(\rho_i-\rho_q-\alpha_{j_2}-\alpha_{j_3})\,\Pi(\rho_i-\rho_q-\alpha_{j_3})} + \cdots \right] \right\}$$

$$\left\{ \frac{\Phi(\rho_i)}{\prod_{i\neq q}(\rho_i-\rho_q)} + \sum_{j_1=1}^{\ell} \frac{B_{j_1}(\rho_i)\, \Phi(\rho_i+\alpha_{j_1})}{\prod_{i\neq q}(\rho_i-\rho_q)\,\Pi(\rho_i-\rho_q+\alpha_{j_1})} + \right. \tag{3.49}$$

$$\left. + \sum_{j_1,j_2=1}^{\ell} \frac{B_{j_1}(\rho_i)\, B_{j_2}(\rho_i+\alpha_{j_1})\, \Phi(\rho_i+\alpha_{j_1}+\alpha_{j_2})}{\prod_{i\neq q}(\rho_i-\rho_q)\,\Pi(\rho_i-\rho_q+\alpha_{j_1})\,\Pi(\rho_i-\rho_q+\alpha_{j_1}+\alpha_{j_2})} + \cdots \right\}$$

The last bracket, $\{\ \}$, in equation 3.49 is independent of ξ and represents a two-dimensional vector. Denoting this vector by $\mathcal{F}_i(p)$ and recalling the substitution 3.48a, the series 3.49 can be written as

$$\bar{Z}_n(\xi,p) = \sum_{i=1}^{8} e^{\rho_i \xi}\Big[I + \sum_{j_1=1}^{\ell} A_{j_1}(\rho_i-\alpha_{j_1})\, e^{-\alpha_{j_1}\xi}$$

$$+ \sum_{j_1=1}^{\ell}\sum_{j_2=1}^{\ell} A_{j_1}(\rho_i-\alpha_{j_1}-\alpha_{j_2})\, A_{j_2}(\rho_i-\alpha_{j_2})\, e^{-(\alpha_{j_1}+\alpha_{j_2})\xi} \tag{3.50a}$$

$$+ \sum_{j_1=1}^{\ell}\sum_{j_2=1}^{\ell}\sum_{j_3=1}^{\ell}\{A_{j_1}(\rho_i-\alpha_{j_1}-\alpha_{j_2}-\alpha_{j_3})\, A_{j_2}(\rho_i-\alpha_{j_2}-\alpha_{j_3})$$

$$A_{j_3}(\rho_i-\alpha_{j_3})\, e^{-(\alpha_{j_1}+\alpha_{j_2}+\alpha_{j_3})\xi}\} + \cdots\Big]\mathcal{F}_i(p)$$

$$\xi > 0$$

or in a closed form as

$$\bar{Z}_n(\xi,p) = \sum_{i=1}^{8} e^{\rho_i\xi}\Big[I + \tag{3.50b}$$

$$\sum_{N=1}^{\infty}\prod_{k=1}^{N}\sum_{j_k=1}^{\ell} A_{j_k}\Big(\rho_i - \sum_{t=1}^{k}\alpha_{j_t}\Big)\, e^{-\alpha_{j_k}\xi}\Big]\mathcal{F}_i(p), \quad \xi \geq 0$$

The solution to equations 3.25 for $\xi < 0$ can be deduced by inspection from 3.50. One need only replace ℓ, α_{j_k}, A_{j_k} and ρ_i by the quantities ℓ^*, $-\alpha^*_{j_k}$, $A^*_{j_k}$ and ρ^*_i which are obtained as follows:

1. ℓ^* and $\alpha^*_{j_k}$ are given by equation 3.17
2. ρ^*_i are the roots of $\Delta L_o(s)$ (see 3.44) with c_o replaced by c_o^*; if c_o and c_o^* are zero or equal, $\rho^*_i = \rho_i$.

3. $A^*_{j_k}$ are the matrices 3.43 with c_o replaced by c_o^*, c_j by c_j^* (see 3.17 again), and with ρ_i^* used as the roots of $\Delta(L_o) = 0$.

4. $\mathcal{F}_i$ is replaced by $\mathcal{F}_i^*$ which is obtained from $\mathcal{F}_i$ by substituting ρ_i^* for ρ_i, $-\alpha^*_{j_k}$ for α_{j_k}, and all initial values at 0^+ by 0^- (see 3.36)

Thus,

$$\bar{Z}_n(\xi, p) = \sum_{i=1}^{8} e^{\rho_i^* \xi} \Big[I + \sum_{N=1}^{\infty} \prod_{k=1}^{N} \sum_{j_k=1}^{\ell^*} A^*_{j_k} \Big(\rho_i^* + \sum_{t=1}^{k} \alpha^*_{j_t} \Big) e^{\alpha^*_{j_k} \xi} \Big] \mathcal{F}_i^*(p), \quad \xi < 0 \tag{3.51}$$

A discussion of the properties of the series 3.50 and 3.51 and a justification of the term by term inversion of 3.46 will be delayed until the vectors $\mathcal{F}_i(p)$ and $\mathcal{F}_i^*(p)$ have been determined. Until then, all properties necessary for differentiation of the series and interchanging limits will be assumed.

The unknown quantities in the components of the vectors $\mathcal{F}_i(p)$ and $\mathcal{F}_i^*(p)$ are the values of the dependent variables, $\bar{\varphi}_n$ AND $\bar{\eta}_n$, and their derivatives (up to the 3rd) at $\xi = 0^+$ and $\xi = 0^-$ (see 3.36). While one can evaluate $\mathcal{F}_i$ and $\mathcal{F}_i^*$ by first determining the initial values 3.36, this results in a very cumbersome operation. A better approach is to assume the vectors are arbitrary, then evaluate them directly by use of the differential equations 3.25, the quiescent condition 3.26 and the continuity, and jump requirements 3.29 and 3.30 respectively.

Let us begin with the quiescent condition. To satisfy 3.26 it is necessary to determine the asymptotic (large $|p|$) behavior of the roots $\rho_i(p)$ and $\rho_i^*(p)$. This can be accomplished by noting that $\Delta L_o(s,p) = 0$ (see 3.44) is satisfied by the asymptotic series

$$s = \sqrt{p}\left[s_o + \frac{s_1}{\sqrt{p}} + \frac{s_2}{(\sqrt{p})^2} + \frac{s_3}{(\sqrt{p})^3} + \cdots\right] \qquad (3.52)$$

Equations governing the coefficients s_n are obtained by substituting 3.52 into $\Delta L_o(s,p) = 0$ and equating terms of the same p-order. Solution of those equations for the leading terms of 3.52 yield the following asymptotic results for the roots:

$$\begin{aligned} &\rho_1, \rho_1^* \sim \frac{\sqrt{p}(1+i)}{\sqrt{2}\beta}; \quad \rho_2, \rho_2^* \sim \frac{\sqrt{p}(1-i)}{\sqrt{2}\beta}; \quad \rho_3, \rho_3^* \sim m \\ &\rho_4, \rho_4^* \sim m; \quad \rho_5, \rho_5^* \sim -\frac{\sqrt{p}(1+i)}{\sqrt{2}\beta} \\ &\rho_6, \rho_6^* \sim -\frac{\sqrt{p}(1-i)}{\sqrt{2}\beta}; \quad \rho_7, \rho_7^* \sim -m; \quad \rho_8, \rho_8^* \sim -m \end{aligned} \qquad (3.53)$$

From equation 3.53 it is evident that the conditions 3.26 can be fulfilled only if

$$\begin{aligned} &\mathcal{F}_i(p) \equiv 0, \quad i = 1, 2, 3, 4 \\ &\mathcal{F}_i^*(p) \equiv 0, \quad i = 5, 6, 7, 8 \end{aligned} \qquad (3.54)$$

Since the characteristic polynomial governing the roots $\rho_i(p)$ and $\rho_i^*(p)$ is of eighth order, the explicit functional dependence of the roots in terms of the parameter p is not at our disposal. For calculation purposes, however, these roots must be properly identified.

This can be accomplished as follows: For all points in the p-plane such that $\frac{\partial \Delta L_o(s,p)}{\partial s} \neq 0$, the roots of $\Delta L_o(s,p)$ are non-repeated. It can be seen, in Appendix 4, that the condition $\frac{\partial \Delta L_o(s,p)}{\partial s} = 0$ is satisfied only at branch points of the roots in the p-plane. By introducing branch cuts the roots can be made analytic function of p. On any contour in the p-plane not passing through a branch point or cut, the roots are thus analytic functions of p and are non-repeated. A root, therefore, can be identified at any point on the contour by tracing it back to its asymptotic value while requiring that the root be a smooth function of the path traced. There is, of course, a certain degree of freedom in identifying the roots with the first term asymptotic values $\pm$ n. Here, although two roots have similar asymptotic values, one need only make an initial choice (which is arbitrary) of the branch and then be consistent.

The elements of each of the eight vectors that remain are not independent, but are related through the differential equations 3.25. These equations can be written in matrix form as:

$$\begin{bmatrix} \beta^4 & 0 \\ 0 & 1 \end{bmatrix} \bar{Z}_n'''' + \begin{bmatrix} M^2 - N_x^* - 2n^2\beta^4 & -1 \\ 1 & -2n^2 \end{bmatrix} \bar{Z}_n'' + \begin{bmatrix} -2Mp & 0 \\ 0 & 0 \end{bmatrix} \bar{Z}_n' + \begin{bmatrix} \beta^4 n^4 - n^2\omega_s + p^2 & -n^2\omega_s'' \\ n^2\omega_s'' & n^4 \end{bmatrix} \bar{Z}_n = 0 \qquad (3.55)$$

where the prime indicates differentiation with respect to ξ. If the elements of each vector are to be related, this relation must be invariant with ξ. Thus we may, for convenience select $\xi \rightarrow +\infty$

and $\xi \to -\infty$ in equation 3.55 to determine the relationship between the elements of each of the vectors $\mathcal{F}_i(p)$ and $\mathcal{F}_i^*(p)$ respectively. For large positive ξ we have from equation 3.50b, since $\mathrm{Re}\,\alpha_{jk} > 0$,

$$\lim_{\xi \to +\infty} \frac{d^m \bar{Z}_n(\xi, p)}{d\xi^m} = \sum_{i=5}^{8} \rho_i^m e^{\rho_i \xi} \mathcal{F}_i(p) \qquad (3.56a)$$

and from equation 3.51, since $\mathrm{Re}\,\alpha^*_{jk} > 0$,

$$\lim_{\xi \to -\infty} \frac{d^m \bar{Z}_n(\xi, p)}{d\xi^m} = \sum_{i=1}^{4} \rho_i^{*m} e^{\rho_i^* \xi} \mathcal{F}_i^*(p) \qquad (3.56b)$$

Now, in the differential equations 3.55, the function $w_s(\xi)$ has the property (see equation 3.17):

$$\begin{aligned} &\lim_{\xi \to \infty} w_s(\xi) = c_o && \lim_{\xi \to \infty} w_s''(\xi) = 0 \\ &\lim_{\xi \to -\infty} w_s(\xi) = c_o^* && \lim_{\xi \to -\infty} w_s''(\xi) = 0 \end{aligned} \qquad (3.57)$$

By substituting 3.56a into 3.55, with consideration of 3.57, one obtains

$$0 = \begin{bmatrix} \beta^4 \rho_i^4 + [M^2 - N_x^* - 2n^2\beta^4]\rho_i^2 - 2Mp\rho_i + (\beta^4 n^4 - n^2 c_o + p^2) & -\rho_i^2 \\ \rho_i^2 & (\rho_i^2 - n^2)^2 \end{bmatrix} \begin{Bmatrix} \mathcal{F}_{i_1} \\ \mathcal{F}_{i_2} \end{Bmatrix} \qquad (3.58)$$

However, from 3.47 and 3.44 one notes that the determinant of the coefficients is zero and thus $\mathcal{F}_{i_1}$ and $\mathcal{F}_{i_2}$ are related according to

$$\mathcal{F}_{i_2} = - \frac{\rho_i^{\,2}}{(\rho_i^{\,2} - m^2)^2} \mathcal{F}_{i_1} \tag{3.59a}$$

In a similar fashion one obtains a relationship between $\mathcal{F}_{i_1}^{*}$ and $\mathcal{F}_{i_2}^{*}$ of the form:

$$\mathcal{F}_{i_2}^{*} = - \frac{\rho_i^{*2}}{(\rho_i^{*2} - m^2)^2} \mathcal{F}_{i_1}^{*} \tag{3.59b}$$

Let

$$- \frac{\rho_i^{\,2}}{(\rho_i^{\,2} - m^2)^2} = r_i , \quad - \frac{\rho_i^{*2}}{(\rho_i^{*2} - m^2)^2} = r_i^{*} \tag{3.60}$$

Then the vector $\bar{Z}_m(\xi, \phi)$ can be written:

$$\bar{Z}_m(\xi, \phi) = \sum_{i=5}^{8} e^{\rho_i \xi} \Big[I + \sum_{N=1}^{\infty} \prod_{k=1}^{N} \sum_{j_k=1}^{\ell} A_{j_k} \Big(\rho_i - \sum_{t=1}^{k} \alpha_{j_t}\Big) e^{-\alpha_{j_k} \xi} \Big] \begin{Bmatrix} 1 \\ r_i \end{Bmatrix} g_i(\phi) \tag{3.61}$$

FOR $\xi > 0$

$$\bar{Z}_m(\xi, \phi) = \sum_{i=1}^{4} e^{\rho_i^{*} \xi} \Big[I + \sum_{N=1}^{\infty} \prod_{k=1}^{N} \sum_{j_k=1}^{\ell^{*}} A_{j_k}^{*} \Big(\rho_i^{*} + \sum_{t=1}^{k} \alpha_{j_t}^{*}\Big) e^{\alpha_{j_k}^{*} \xi} \Big] \begin{Bmatrix} 1 \\ r_i^{*} \end{Bmatrix} g_i(\phi) \tag{3.62}$$

FOR $\xi < 0$

where the substitution $\mathcal{F}_{i_1} = g_i$, i = 5, . . . , 8 and $\mathcal{F}_{i_2} = g_i$, i = 1,.. 4 has been made.

The remaining eight constants, $q_i(p)$, are determined by application of the continuity and jump conditions 3.29 and 3.30 respectively. To accomplish this, however, it is of great advantage to recast the series 3.61 and 3.62 in a different form. For this purpose we again invert the series 3.46 term by term, but this time the following quantities of the Nth term (see equation 3.46b):

$$\frac{1}{\Pi(s-\rho_g)}, \quad \frac{1}{\Pi[s-(\rho_g-\alpha_{j_1})]}, \quad \cdots, \quad \frac{1}{\Pi[s-(\rho_g-\alpha_{j_1}-\cdots-\alpha_{j_{N-1}})]},$$

$$\frac{B_{j_1}(s)\, B_{j_2}(s+\alpha_{j_1}) \cdots B_{j_N}(s+\alpha_{j_1}+\cdots \alpha_{j_{N-1}})\, \Phi(s+\alpha_{j_1}+\alpha_{j_2}+\cdots+\alpha_{j_N})}{\Pi[s-(\rho_g-\alpha_{j_1}-\cdots-\alpha_{j_N})]}$$

will be inverted individually and the inversion of the entire term will be obtained by use of the convolution integral. This yields the following series for $\overline{Z}_n(\xi, p)$:

$$\overline{Z}_n(\xi,p) = \sum_{i=5}^{8} \Bigg\{ e^{\rho_i \xi} I + \sum_{j_1=1}^{\ell} \sum_{i_1=1}^{8} \frac{B_{j_1}(\rho_i-\alpha_{j_1})}{\underset{i_1\neq g}{\Pi}(\rho_{i_1}-\rho_g)} \int_0^{\xi} e^{(\rho_i-\alpha_{j_1})(\xi-\lambda_1)+\rho_{i_1}\lambda_1}\, d\lambda_1$$

$$+\sum_{j_1,j_2=1}^{\ell} \sum_{i_1,i_2=1}^{8} \frac{B_{j_1}(\rho_i-\alpha_{j_1}-\alpha_{j_2})\, B_{j_2}(\rho_i-\alpha_{j_2})}{\underset{i_1\neq g}{\Pi}(\rho_{i_1}-\rho_g)\, \Pi(\rho_{i_2}-\rho_g)} \int_0^{\xi}\int_0^{\lambda_1} e^{(\rho_i-\alpha_{j_1}-\alpha_{j_2})(\xi-\lambda_1)+(\rho_{i_1}-\alpha_{j_1})(\lambda_1-\lambda_2)+\rho_{i_2}\lambda_2}\, d\lambda_2\, d\lambda_1 \tag{3.63}$$

$$+\sum_{j_1,j_2,j_3=1}^{\ell} \sum_{i_1,i_2,i_3=1}^{8} \frac{B_{j_1}(\rho_i-\alpha_{j_1}-\alpha_{j_2}-\alpha_{j_3})\, B_{j_2}(\rho_i-\alpha_{j_2}-\alpha_{j_3})\, B_{j_3}(\rho_i-\alpha_{j_3})}{\underset{i_1\neq g}{\Pi}(\rho_{i_1}-\rho_g)\, \underset{i_2\neq g}{\Pi}(\rho_{i_2}-\rho_g)\, \underset{i_3\neq g}{\Pi}(\rho_{i_3}-\rho_g)} \times$$

$$\times \int_0^{\xi}\int_0^{\lambda_1}\int_0^{\lambda_2} e^{(\rho_i-\alpha_{j_1}-\alpha_{j_2}-\alpha_{j_3})(\xi-\lambda_1)+(\rho_{i_1}-\alpha_{j_1}-\alpha_{j_2})(\lambda_1-\lambda_2)+(\rho_{i_2}-\alpha_{j_1})(\lambda_2-\lambda_3)+\rho_{i_3}\lambda_3}\, d\lambda_3\, d\lambda_2\, d\lambda_1 + \cdots \Bigg\} \begin{Bmatrix} 1 \\ \eta_i \end{Bmatrix} q_i, \quad \xi > 0$$

In the above, we have included the quiescent condition and the relation between the elements of the vector $\mathcal{F}_i$, given by equation 3.59a.

The solution for $\xi < 0$ is obtained from equation 3.63 by summing i from 1 to 4, replacing α_{jk} by $-\alpha^*_{jk}$ and all remaining quantities by the corresponding "starred" values.

The advantage of the form of the series, 3.63, is now evident. The series truncates at $\xi = 0$! If one differentiates the series 3.63 with respect to ξ, one finds that the series representing the derivatives also truncates after a certain number of terms. With the continuity and jump conditions across the origin in mind, the function $\bar{Z}_m(\xi,p)$ and its first three derivatives on either side of the origin can be written, by use of equation 3.63, in the form

$$\frac{d^{m-1}\bar{Z}_m(0^+,p)}{d\xi^{m-1}} = \sum_{i=5}^{8} \mathcal{M}^{(m-1)}(i)\, g_i \tag{3.64a}$$

$$\frac{d^{m-1}\bar{Z}_m(0^-,p)}{d\xi^{m-1}} = \sum_{i=1}^{4} \mathcal{M}^{(m-1)}(i)\, g_i \tag{3.64b}$$

where, for i = 5, ..., 8 we have:

$$\mu^{(1)}(i) = \begin{Bmatrix} 1 \\ r_i \end{Bmatrix}$$

$$\mu^{(2)}(i) = \left\{ \rho_i I + \sum_{j_1=1}^{l} a_1 B_{j_1}(\rho_i - \alpha_{j_1}) \right\} \begin{Bmatrix} 1 \\ r_i \end{Bmatrix}$$

$$\mu^{(3)}(i) = \left\{ \rho_i^2 I + \sum_{j_1=1}^{l} [(\rho_i - \alpha_{j_1}) a_1 + a_2] B_{j_1}(\rho_i - \alpha_{j_1}) + \sum_{j_1, j_2=1}^{l} [a_1^2 B_{j_1}(\rho_i - \alpha_{j_1} - \alpha_{j_2}) B_{j_2}(\rho_i - \alpha_{j_2}) \right\} \begin{Bmatrix} 1 \\ r_i \end{Bmatrix}$$

$$\mu^{(4)}(i) = \left\{ \rho_i^3 I + \sum_{j_1=1}^{l} [(\rho_i - \alpha_{j_1})^2 a_1 + (\rho_i - \alpha_{j_1}) a_2 + a_3] B_{j_1}(\rho_i - \alpha_{j_1}) + \sum_{j_1, j_2=1}^{l} [(\rho_i - 2\alpha_{j_1} - \alpha_{j_2}) a_1^2 + 2 a_1 a_2] B_{j_1}(\rho_i - \alpha_{j_1} - \alpha_{j_2}) B_{j_2}(\rho_i - \alpha_{j_2}) + \sum_{j_1, j_2, j_3=1}^{l} [a_1^3 B_{j_1}(\rho_i - \alpha_{j_1} - \alpha_{j_2} - \alpha_{j_3}) B_{j_2}(\rho_i - \alpha_{j_2} - \alpha_{j_3}) B_{j_3}(\rho_i - \alpha_{j_3})] \right\} \begin{Bmatrix} 1 \\ r_i \end{Bmatrix} \quad (3.65a)$$

where

$$a_1 = \sum_{k=1}^{8} \frac{1}{\prod_{k \neq g} (\rho_k - \rho_g)} \quad ; \quad a_2 = \sum_{k=1}^{8} \frac{\rho_k}{\prod_{k \neq g} (\rho_k - \rho_g)}$$

$$a_3 = \sum_{k=1}^{8} \frac{\rho_k^2}{\prod_{k \neq g} (\rho_k - \rho_g)}$$

and for i = 1, ... , 4 we obtain

$$\mathcal{M}^{(1)}(i) = \begin{Bmatrix} 1 \\ r_i^* \end{Bmatrix}$$

$$\mathcal{M}^{(2)}(i) = \left\{ \rho_i^* I + \sum_{j_1=1}^{\ell^*} a_1^* B_{j_1}^* (\rho_i^* + \alpha_{j_1}^*) \right\} \begin{Bmatrix} 1 \\ r_i^* \end{Bmatrix}$$

$$\mathcal{M}^{(3)}(i) = \left\{ \rho_i^{*2} I + \sum_{j_1=1}^{\ell^*} [(\rho_i^* + \alpha_{j_1}^*) a_1^* + a_2^*] B_{j_1}^* (\rho_i^* + \alpha_{j_1}^*) \right. \tag{3.65b}$$

$$\left. + \sum_{j_1, j_2=1}^{\ell^*} a_1^{*2} B_{j_1}^* (\rho_i^* + \alpha_{j_1}^* + \alpha_{j_2}^*) B_{j_2}^* (\rho_i^* + \alpha_{j_2}^*) \right\} \begin{Bmatrix} 1 \\ r_i^* \end{Bmatrix}$$

$$\mathcal{M}^{(4)}(i) = \left\{ \rho_i^{*3} I + \sum_{j_1=1}^{\ell^*} [(\rho_i^* + \alpha_{j_1}^*)^2 a_1^* + (\rho_i^* + \alpha_{j_1}^*) a_2^* + a_3^*] B_{j_1}^* (\rho_i^* + \alpha_{j_1}^*) \right.$$

$$+ \sum_{j_1, j_2=1}^{\ell^*} [(\rho_i^* + 2\alpha_{j_1}^* + \alpha_{j_2}^*) a_1^{*2} + 2 a_1^* a_2^*] B_{j_1}^* (\rho_i^* + \alpha_{j_1}^* + \alpha_{j_2}^*) B_{j_2}^* (\rho_i^* + \alpha_{j_2}^*)$$

$$\left. + \sum_{j_1, j_2, j_3=1}^{\ell^*} a_1^{*3} B_{j_1}^* (\rho_i^* + \alpha_{j_1}^* + \alpha_{j_2}^* + \alpha_{j_3}^*) B_{j_2}^* (\rho_i^* + \alpha_{j_2}^* + \alpha_{j_3}^*) B_{j_3}^* (\rho_i^* + \alpha_{j_3}^*) \right\} \begin{Bmatrix} 1 \\ r_i^* \end{Bmatrix}$$

where a_1^*, a_2^*, and a_3^* are obtained from a_1, a_2, and a_3 by replacing the ρ_k and ρ_q by ρ_k^* and ρ_q^*.

The remaining eight unknown constants, $q_i(p)$, can now be determined from the continuity and jump conditions 3.29 and 3.30 respectively. These can be written in vector form as

$$\frac{d^{m-1}\bar{Z}_n(0^-,p)}{d\zeta^{m-1}} - \frac{d^{m-1}\bar{Z}_n(0^+,p)}{d\zeta^{m-1}} = 0 \qquad \text{for } m = 1,2,3 \tag{3.66}$$

$$\frac{d^3\bar{Z}_n(0^-,p)}{d\zeta^3} - \frac{d^3\bar{Z}_n(0^+,p)}{d\zeta^3} = \begin{Bmatrix} -\frac{1}{\beta^4} \\ 0 \end{Bmatrix}$$

By application of 3.66 one obtains eight equations for the eight unknown functions $q_i(p)$ of the form

$$Qq = e \tag{3.67}$$

where Q is an 8 x 8 matrix, the elements of which are given by

$$Q \equiv \left[\begin{array}{c|c} Q_{m,i} = M_1^{(m-1)}(i) & Q_{m,i} = -M_1^{(m-1)}(i) \\ m = 1,\dots,4 & m = 1,\dots,4 \\ i = 1,\dots,4 & i = 5,\dots,8 \\ \hline Q_{(m+4),i} = M_2^{(m-1)}(i) & Q_{(m+4),i} = -M_2^{(m-1)}(i) \\ m = 1,\dots,4 & m = 1,\dots,4 \\ i = 1,\dots,4 & i = 5,\dots,8 \end{array}\right] \tag{3.68}$$

where as usual the first subscript, m, indicates the row and the second subscript, i, the column. In the above, $\mu_1^{(m-1)}(i)$ and $\mu_2^{(m-1)}(i)$ denote the elements of the two-dimensional vector $\mu^{(m-1)}(i)$:

$$\mu^{(m-1)}(i) = \begin{Bmatrix} \mu_1^{(m-1)}(i) \\ \mu_2^{(m-1)}(i) \end{Bmatrix} \tag{3.69}$$

The quantities g and e of equation 3.67 represent the following eight-dimensional vectors:

$$g = \begin{Bmatrix} g_1 \\ g_2 \\ \vdots \\ g_8 \end{Bmatrix} \qquad e = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\beta^4} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.70}$$

Premultiplying equation 3.67 by Q^{-1}, we obtain the vector g as

$$g = Q^{-1} e \qquad \text{DET } Q \neq 0 \tag{3.71}$$

The solution for $\bar{Z}_m(\xi, p)$ is now complete. We next discuss a few of the basic properties of the series 3.61 and 3.63 (Because of their similar form, the remarks apply also to the series representing $\bar{Z}_m(\xi, p)$ for $\xi < 0$.)

2.4 <u>Properties of $\bar{Z}_m(\xi, p)$ in the p-plane</u>

Let us define the region R of the complex p-plane by:

(1) $|p - p_b| \geq \epsilon_1 > 0$, where p_b are branch points of the roots, $\rho_q(p)$ and $\rho_q^*(p)$ in the p-plane.

(2) $|p - p_q| \geq \epsilon_2 > 0$, where p_q are zeros of the determinant of Q.

Then if $p \in R$, the series 3.61 or 3.63 are absolutely and uniformly convergent with respect to both ξ and p when $\xi \in (0, \infty)$ or $\xi \in (-\infty, 0)$*.

The uniform convergence of the series with respect to ξ is sufficient to justify the term by term inversion of the series 3.46 (see (32), page 147). Also since the series obtained from 3.61 or 3.63 by an nth term by term ξ-derivative possesses the same property of uniform convergence with repect to ξ, our term by term differentiation of these series was justified.

If appropriate branch cuts in the p-plane are made to render the roots $\rho_i(p)$ and $\rho_i^*(p)$ analytic functions of p and if $p \in R$, then each term of the series is an analytic function of p. The uniform convergence with repect to p thus indicates $\bar{Z}_n(\xi, p)$ is an analytic function of p in the subregion of R defined by the branch cuts. The points for which ΔQ (the determinant of Q) $= 0$, $p = p_Q$, represent poles of $\bar{Z}_n(\xi, p)$ in the p-plane.

Additional details and proofs of the above statements can be found in Appendix 2.

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* The form of the series 3.61 becomes indeterminate at points in the p-plane (p = 0 is one) where $\rho_i - \rho_j = k_1 \alpha_1 + k_2 \alpha_2 + \ldots + k_l \alpha_l$, $i, j = 1, \ldots, 8$, $k_r = 0, 1, 2, \ldots$ At such points in the p-plane, repeated factors occur in the denominator of the terms of the series 3.64, which is contrary to our assumption regarding the first form of the inversion. The form of the series 3.63, however, requires only that the roots ρ_q be non repeated, which is guaranteed by (1) above. This form of the series indicates each term is regular at such points where repeated factors occur.

3. The Determination of Stability

3.1 General

Consider the definition of stability as given on page 12 and the x, t plane. If the length of our shell were finite, it would be sufficient to investigate stability by determining the boundedness of $Z_n(x,t)$ along lines of constant x. Because we are considering an infinite x interval, however, this is not sufficient since motions may exist that are bounded in any finite ξ-interval but are unbounded in an infinite ξ-interval. (An example of this is a pulse, traveling with constant velocity, with amplitude growing continuously with distance.) One method of covering the entire upper half of the t-x plane is to employ rays from the origin, i. e., examine the boundedness of $Z_n(x, t)$ along an arbitrary ray in the x, t plane as $t \rightarrow \infty$. One of these rays (displaced from the origin if $\xi \neq 0$) is defined by ξ = constant. In view of the nonlinear string analysis contained in Appendix 5, it is reasonable to assume that if the perturbation quantities, $Z_n(\xi, \tau)$ are found to be bounded for ξ = constant as $\tau = t \rightarrow \infty$, the same is true of any ray in the x, t plane. This implies it is sufficient to ascertain stability by determining the boundedness of $Z_n(\xi, \tau)$ along lines of constant ξ in the τ-ξ plane. We shall consider this as part of our definition of stability, i. e., that the stability of the system be defined on the basis of the boundedness or divergence of $Z_n(\xi, \tau)$ along lines of constant ξ in the ξ-τ plane.

3.2 Boundedness Along the Rays ξ = constant

The quantity $\overline{Z}_n(\xi, p)$ need not be inverted to obtain stability information. Indeed, the boundedness of $Z_n(\xi, \tau)$ along a line of constant ξ is governed entirely by the location and type of singularities of $\overline{Z}_n(\xi, p)$ in the p-plane. In general, one can state:

(1) If $\overline{Z}_n(\xi, p)$ is regular on the imaginary axis and within the right half of the p-plane, then $\lim_{\tau \to \infty} Z_n(\xi, \tau) = 0$ and the system is stable.

(2) If $\overline{Z}_n(\xi, p)$ possesses singularities in the right half plane (Re $p > 0$), $Z_n(\xi, \tau)$ is unbounded and the system is unstable.

(3) For the present transform, we have if $\overline{Z}_n(\xi, p)$ is regular in the right half plane (Re $p > 0$) and all poles on the imaginary axis are:

a) of first order, the system is stable

b) of higher than first, unstable.

These statements are based on theorems and a discussion presented in Appendix 3.

3.3 Location of the Singularities of $\overline{Z}_n(\xi, p)$

In view of the above statements, it is clear that one need only consider the imaginary axis and the right half of the p-plane. The singularities of $\overline{Z}_n(\xi, p)$ in this region consist of branch points and poles. Consider first the branch points. While location of the branch points of the roots $\rho_i(p)$ and $\rho_i^*(p)$ does not pose a great problem, location of the branch points of $\overline{Z}_n(\xi, p)$ is very difficult and represents an impassé from a purely mathematical point of view. To proceed it will be necessary to, in part, resort to the physics of the problem. It will be assumed that the equations of motion we have selected represent a reasonable approximation to the physical problem and the philosophy taken will be one of what can be done with the mathematical model under this assumption. With this in mind we continue.

The vector function $\bar{Z}_n(\xi, p)$ will possess no more branch points than those of the roots $\rho_i(p)$ and $\rho_i^*(p)$. However, all root branch points may not be branch points of $\bar{Z}_n(\xi, p)$, due to cancellation, squaring, etc. As far as stability is concerned, we are especially interested in the possibility of branch points existing in the right half of the p-plane. To begin, set $M = N_x^* = C_o = C_o^* = 0$, indicating the load is statically applied and decays as $|\xi| \rightarrow \infty$. Then equation 3.44, which governs the roots $\rho_i(p)$ and $\rho_i^*(p)$ is independent of quantities c_j, c_j^* and α_j, α_j^* (the latter for $j > 2$). This implies the location of the branch points depends only on β and n and is completely independent of the magnitude and distribution of the load. If we are to assume that our equations of motion represent a reasonable mathematical model of the shell, then clearly those branch points of the roots appearing in the right half plane cannot be branch points of $\bar{Z}_n(\xi, p)$; if this were true, by theorem 1 of Appendix 3, we would deduce that the system was unstable independent of the loading condition, which is absurd physically. If N_x^* , C_o and C_o^* are non zero, then equation 3.44 depends only on the constant portion of the initial deflection, $w_s(\xi)$. Since we shall consider values of N_x^* , $C_o = C_o^*$ less than those required to buckle the shell alone, stability must depend on the magnitude and distribution of $w_s(\xi)$. Again we must conclude that, for such cases, $\bar{\bar{Z}}_n(\xi, p)$ can possess no branch points in the right half of the p-plane. If M is non-zero, then the branch points are independent of the load magnitude and distribution but do depend on M. Recall, however that we are considering only those velocities below the cutoff velocity. This being the case one must expect that stability will depend on all load parameters, not just velocity alone; once again this leads to the conclusion that $\bar{\bar{Z}}_n(\xi, p)$ possesses no branch points in the right-half plane. A similar argument indicates branch points appearing on the imaginary axis do not contribute to instability. Stability therefore depends entirely on the location of the poles of $\bar{\bar{Z}}_n(\xi, p)$ in Re $p \geqslant 0$.

On a purely mathematical basis, the above statements should be placed in the form of the following hypothesis: Those singularities determining the stability of the system must depend on the load parameters describing the load magnitude and distribution. (The poles of $\bar{\bar{Z}}_n(\xi, p)$ satisfy this requirement). In the event that for some obscure reason the above statement should not be true, the analysis will still provide an upper bound to the stability of the system.

Poles of $\bar{\bar{Z}}_n(\xi, p)$ in the p-plane occur only when the determinant of the matrix Q, ΔQ, vanishes. The stability analysis therefore reduces to the determination of the conditions for which zeros of $\Delta Q(p)$ exist when Re $p \geqslant 0$. Recall that the infinite series for $\bar{\bar{Z}}_n(\xi, p)$ and its derivatives truncated at $\xi = 0$ and therefore the elements of the matrix Q do not involve infinite series, but on the contrary, consist of a finite number of terms. Thus there is no question of the rate of convergence of the series for $\bar{Z}_m(\xi, p)$ involved in the stability analysis; the matrix Q is exact and is written in a closed form. Because of the complexity of Δ Q, however, and since one cannot in general obtain the eight roots $\rho_i(p)$ or $\rho_i^*(p)$ explicitly as a function of p, the investigation of ΔQ must in the final analysis be primarily of a numerical nature (this is in contrast to the analysis of Appendix 5 where exact results were directly obtainable). We discuss the location of the zeros of ΔQ next.

First we indicate that the zeros of ΔQ must occur as complex conjugates in the p-plane. To see this, consider the roots $\rho_i(p)$ and $\rho_i^*(p)$. The latter possess branch points in the p-plane. Now, $\rho_i(p)$ and $\rho_i^*(p)$ are the roots of (with c_o replaced by c_o^* in the case of $\rho_i^*(p)$)

$$\Delta L_o(s, p) = 0 \tag{3.72a}$$

The necessary and sufficient condition* that these roots possess branch points in the p-plane is that

$$\frac{dp}{ds} = 0 \tag{3.72b}$$

simultaneously be satisfied with equation 3.72a. The relations 3.72a and 3.72b can be combined to yield the following polynomials in s (see equation 3.44)

$$s\left\{ (s^2)^4 - 4m^2(s^2)^3 + 6m^4(s^2)^2 - 4m^2\left(m^4 + \frac{1}{4\beta^4}\right)s^2 + m^8 \right\} = 0 \qquad \text{FOR } M = 0 \tag{3.73}$$

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* See Appendix 4 for a discussion of the location of branch points.

$$\sum_{i=0}^{9} c_i \, (s^2)^i = 0 \qquad \text{FOR } M \neq 0 \tag{3.74}$$

where

$$c_0 = \frac{M^2 m^{16}}{4} \qquad c_1 = m^{14}(m^2\beta^4 - 2M^2)$$

$$c_2 = m^8\left(-2m^2 - 8m^6\beta^4 + 7m^4M^2 + \frac{M^2}{4\beta^4}\right)$$

$$c_3 = m^4\left(8m^4 + 28m^8\beta^4 + \frac{1}{\beta^4} - 14m^6M^2 - \frac{M^2m^2}{\beta^4}\right)$$

$$c_4 = m^4\left(-56\beta^4m^6 - 12m^2 + \frac{35}{2}M^2m^4 + \frac{3}{2}M^2\right)$$

$$c_5 = m^2\left(72m^6\beta^4 + 8m^2 - 14M^2m^4 - \frac{M^2}{\beta^4}\right)$$

$$c_6 = 7m^4M^2 - 2m^2 - 56m^6\beta^4 + \frac{M^2}{\beta^4}$$

$$c_7 = 2m^2(14\beta^4m^2 - M^2) \qquad c_8 = \frac{M^2}{4} - 8m^2\beta^4 \qquad c_9 = \beta^4$$

and the branch points can be calculated from

$$p = Ms \pm \left[\frac{1}{s^2 - m^2}\right]\Big\{-\beta^4s^8 + 4m^2\beta^4s^6 - (1 + 6m^4\beta^4)s^4 + 4m^6\beta^4s^2 - m^8\beta^4\Big\}^{1/2} \tag{3.75}$$

utilizing the roots of 3.73 or 3.74 (some of the roots of 3.74 are extraneous). By use of the last three equations, one can show that the branch points of each root occur as complex conjugates and that the collection of all root branch points form a pattern which is symmetric about both the real and imaginary axes. A typical configuration of the total collection is shown in Fig. 4a.

If we provide branch cuts in the p-plane as illustrated in Fig. 4b, then the roots will be analytic functions of p in the region defined by the cuts. This of course implies $\Delta Q(p)$ and $\bar{Z}_n(\zeta, p)$ are analytic in the same region. Now if p is real valued the coefficients of $\Delta L_o(s) = 0$ (see equations 3.44 again) are real valued and the roots $\rho_i(p)$, $\rho_i^*(p)$ therefore occur in complex conjugate pairs. Little loss in generality occurs if we assume that the constants P_n, P_n^*, Ω_n, Ω_n^* (see equation 2.2) occur in complex conjugate pairs or are real valued, since the loading itself must be real valued. This being the case, the constants c_j, c_j^*, α_j, α_j^* will occur in complex conjugate pairs or be real valued. These properties (complex conjugate roots and above complex conjugate constants) are sufficient to guarantee that $\bar{Z}_n(\zeta, p)$ and $\Delta Q(p)$ are real valued when p is real valued. Therefore, since $\Delta Q(p)$ and $\bar{Z}_n(\zeta, p)$ are analytic in a region which is symmetric with respect to the real p-axis (and contains that axis) and both are real valued when p is real valued, the reflection principle indicates

$$\Delta Q(\bar{p}) = \overline{\Delta Q}(p) \tag{3.76a}$$

$$\bar{Z}_n(\zeta, \bar{p}) = \overline{\bar{Z}_n(\zeta, p)} \tag{3.76b}$$

where the darker bar denotes "complex conjugate". The property 3.76a implies the roots of $\Delta Q(p) = 0$ occur as complex conjugates.

We next show that roots of $\Delta Q(p) = 0$ can occur in the right-half plane only on the real p-axis if $M = 0$.

Let

$$\bar{Z}_n(\xi, p) = \frac{1}{[\Delta Q(p)]^2} \begin{Bmatrix} a(\xi, p) \\ b(\xi, p) \end{Bmatrix}, \quad \xi > 0 \qquad (3.77)$$

Then in the interval $\xi > 0$ the vector components a and b satisfy the equations (see equation 3.55):

$$\beta^4 \frac{d^4 a}{d\xi^4} - 2n^2\beta^4 \frac{d^2 a}{d\xi^2} + (\beta^4 n^4 - n^2 w_5(\xi) + p^2) a = \frac{d^2 b}{d\xi^2} - n^2 \frac{d^2 w_5(\xi)}{d\xi^2} b \qquad (3.78a)$$

$$\frac{d^4 b}{d\xi^4} - 2n^2 \frac{d^2 b}{d\xi^2} + n^4 b = n^2 \frac{d^2 w_5(\xi)}{d\xi^2} a - \frac{d^2 a}{d\xi^2} \qquad (3.78b)$$

with the boundary conditions:

$$\frac{d^m}{d\xi^m} a(\xi, p) = \frac{d^m b(\xi, p)}{d\xi^m} = 0 \quad \text{for } \xi \to \infty, \quad \operatorname{Re} p > 0^{*}, \quad m = 0, 1 \qquad (3.79a)$$

$$\frac{d^m}{d\xi^m} \begin{Bmatrix} a(\xi,p) \\ b(\xi,p) \end{Bmatrix}_{\xi = 0^+} = (\Delta Q)^2 \frac{d^m}{d\xi^m} \bar{Z}_m(0^+, p) = \Delta Q \frac{d^m}{d\xi^m} \bar{R}_m(0^+, p), \qquad (3.79b)$$

$$m = 0, 1$$

---

* Let n, β, C_0, C_0^*, and N_X^* be selected such that the system is stable for appropriate values of c_j, c_j^*, and d_j, d_j^* $(j > 2)$ (d_1 and d_2 are determined by β). Hold all constants fixed except the c_j, c_j^*, $j > 0$. Let the stability of the system be controlled by these latter constants. Now, the roots $\rho_i(p)$ do not depend on c_j, c_j^*, $j > 0$. Therefore, if Re $\rho_i(p) < 0$ (i = 5, . . . , 8) for $Z_n(\xi, \tau)$ bounded, the same is true if the system is unstable. Assume $|Z_m(\xi, \tau)| \le M$. Then $\left|\int_0^\infty e^{-p\tau} Z_m \, d\tau\right| \le \int_0^\infty e^{-\operatorname{Re} p \tau} |Z_m| \, d\tau \le M / \operatorname{Re} p$. Thus if Re $p \ge \epsilon > 0$, the definition integral (by the Weierstrass M-test) is uniformly convergent with respect to ξ. Assuming $Z_m(\xi, \tau)$ a continuous function of ξ and τ we therefore have $\lim_{\xi \to \infty} \bar{Z}_m = \int_0^\infty e^{-p\tau} \lim_{\xi \to \infty} Z_m(\xi, \tau) \, d\tau = 0$ by equation 3.21. This implies Re $\rho_i(p) < 0$ for i = 5, . . . , 8 if Re $p > 0$ and Re $p > 0$ in 3.79a follows.

where $\overline{R}_n(0^+, p)$ is regular when p is in the region of the complex p-plane defined by the branch cuts (see Fig. 4b) and its meaning should be clear from the functions $\dfrac{d^m \overline{Z}_n(0^+, p)}{d\xi^m}$ as given in equation 3.64a.

Let a_1 and b_1 correspond to a point p_1 and similarly let a_2 and b_2 correspond to another point, p_2. Assume p_1 and p_2 are roots of $\Delta Q(p) = 0$. Next multiply equation 3.78a for a_1 by a_2 and that for a_2 by a_1, subtract the latter from the former and integrate the result from $\xi = 0^+$ to $\xi = \infty$. This yields:

$$\int_0^\infty \Big\{ \beta^4 \left(a_2 \frac{d^4 a_1}{d\xi^4} - a_1 \frac{d^4 a_2}{d\xi^4} \right) + (p_1^2 - p_2^2)\, a_1 a_2 - \left(a_2 \frac{d^2 b_1}{d\xi^2} - a_1 \frac{d^2 b_2}{d\xi^2} \right) + n^2 \frac{d^2 w_s}{d\xi^2} (a_2 b_1 - a_1 b_2) \Big\} d\xi = 0$$

Following a few integrations by parts and application of 3.79a and b, noting that $\Delta Q(p_1) = \Delta Q(p_2) = 0$, one obtains

$$(p_1^2 - p_2^2) \int_0^\infty a_1 a_2 \, d\xi = \int_0^\infty \Big[\frac{da_1}{d\xi} \frac{db_2}{d\xi} - \frac{da_2}{d\xi} \frac{db_1}{d\xi} - n^2 \frac{d^2 w_s}{d\xi^2} (a_2 b_1 - a_1 b_2) \Big] d\xi \tag{3.80}$$

In a similar fashion, multiply equation 3.78b for b_1 by b_2 and that for b_2 by b_1, subtract one from the other and integrate the remaining equation from $\xi = 0^+$ to $\xi = \infty$. The result, following integration by parts and application of 3.79a and b, is

$$\int_{0^+}^{\xi}\left[\frac{da_1}{d\xi}\frac{db_2}{d\xi} - \frac{da_2}{d\xi}\frac{db_1}{d\xi} - n^2\,\frac{d^2 w_s(\xi)}{d\xi^2}\,(a_2 b_1 - a_1 b_2)\right]d\xi = 0 \qquad (3.81)$$

Therefore equation 3.80 becomes

$$(p_1^2 - p_2^2)\int_{0^+}^{\infty} a_1 a_2\, d\xi = 0 \qquad (3.82)$$

However p_1 and p_2 were assumed to be roots of $\Delta Q(p) = 0$ and therefore $p_1 = \bar{p}_2$. Since $a(\bar{p}) = \bar{a}(p)$, $a_1 a_2 = |a_1|^2 = |a_2|^2$ and

$$(p_1^2 - p_2^2)\int_{0^+}^{\infty} |a_1|^2\, d\xi = 0 \qquad (3.83)$$

Equation 3.83 represents a contradiction unless $p_1^2 = p_2^2$, which can occur for complex conjugate roots in Re $p > 0$ only if $p_1 = p_2$ are real valued. Therefore the zeros of $\Delta Q(p)$ in the right-half plane must lie on the real p-axis if $M = 0$.

While in the case of the string analysis of Appendix 5, the same result applied for $M \neq 0$, the author was unable to prove this to be true for the shell. On the basis of the string analysis, however, one might suspect that the same should be true of the shell.

The fact that zeros of $\Delta Q(p)$ must lie on the real p-axis for Re $p > 0$ brings forth the following question: does the transition between stability and instability, for $M = 0$, take place at $p = 0$? The answer is yes. This can be seen as follows. For $M = 0$, the variational equations represent a conservative system. Therefore the energy method of analyzing stability and the present method are equivalent (53). If one calculates the potential energy of the shell (under the same approximations associated with the derivation of equations 2.1) assuming ψ and η are virtual displacements from the loaded state, then one finds that equations 3.19 with $\frac{\partial}{\partial \tau} = 0$ are obtained by setting the second variation of the potential energy equal to zero (a necessary condition for transition from stability to instability). Now, the function 3.63 and its counterpart for $\xi < 0$ with $p = 0$ and g_i considered as arbitrary constants represents a solution to these equations (see 3.22) satisfying $\psi(\pm\infty) = \eta(\pm\infty) = 0$. The solution is completed by requiring continuity of ψ and η and their derivatives with respect to ξ, up to and including the third, at $\xi = 0$. This condition is represented by equations 3.64 with the right side of 3.66 set equal to zero. Upon applying these conditions, one obtains

$$Q g = 0$$

whence we must require that $\Delta Q = 0$ which implies the transition takes place at $p = 0$ in the p-plane. Any increase in the load magnitude will force the zero of ΔQ into the right half plane on the real p-axis, insuring an unstable system.

If $M \neq 0$, the situation is more complex since one must search for zeros of ΔQ within the right-half plane and on the imaginary axis. Recall first equation 3.55. This equation possesses the same form for $M \neq 0$ as it does for $M = 0$ when $p = 0$. The parameter M, if $p = 0$, occurs everywhere in the combination $M^2 - N_x^*$ (see equation 3.16); it therefore has the same effect as an axial compression of the cylinder. Thus the determinant, $\Delta Q_{M \neq 0}^{(p=0)}$

has the same properties as $\Delta Q_{M=0}(p = 0)$, the only difference between the two determinants being an effective change in N_x^*, i. e., N_x^* is replaced by $N_x^* - M^2$ when $M \neq 0$.

Let the load and shell parameters be fixed except for one parameter, say λ, which characterizes the "magnitude" of the load. One concludes that a zero of $\Delta Q_{M \neq 0}$ will appear at $p = 0$ for an appropriate value of λ. In all probability $p = 0$ again represents the transition from stability to instability. To verify this one must demonstrate that this pole moves into the right-half plane when λ is increased, that no zeros of ΔQ exist in $\mathrm{Re}\, p > 0$ when $\lambda < \lambda_{CRM}$ where λ_{CRM} corresponds to $\Delta Q_{M \neq 0}(p = 0) = 0$, and that all poles of $\frac{1}{\Delta Q}$ on the imaginary axis (if any exist) are of first order when $\lambda < \lambda_{CRM}$.

Zeros in $\mathrm{Re}\ p \geqslant 0$ can be detected with the aid of a theorem which is sometimes called the principle of the argument. This theorem (37) states:

> If $f(p)$ is analytic in a region s, bounded by a contour c, and does not vanish on c, then the number of zeros minus the number of poles of $f(p)$ within c is $1/2\pi$ times the increase in arg $f(p)$ as p goes once around c in the positive direction. (The positive direction is defined such that the enclosed region s appears to the left of an observer moving along c.)

Let us consider the function $f(p)$ defined by

$$f(p) = \frac{(p+1)^3}{\Delta Q} \tag{3.84}$$

and a contour c, as shown in Fig. 5a, which covers the right-half plane as $R \rightarrow \infty$. The dotted lines in this figure indicate no calculation need be made on that portion of the contour due to the reflection property 3.76a, i.e., f(p) can be obtained on the dotted part of the contour by a reflection in the real axis of the f(p) plane. The zeros of ΔQ are of course the poles of f(p) and f(p) possesses the same number of poles in Re $p \geqslant 0$ as does $\frac{1}{\Delta Q}$. The function f(p), however, is non zero (approaches a constant) on the circular portion of the contour as $R \to \infty$. This is known since ΔQ can be shown to behave asymptotically as Kp^3, where K = constant. Since ΔQ is finite for $p < \infty$, f(p) possesses no zeros in Re $p \geqslant 0$ (assuming branch points are circumvented). The function f(p) can be made analytic in the region of interest by use of the branch cuts discussed previously. Assuming the branch cuts have been made, the contour shown in Fig. 5a can be deformed so that f(p) is an analytic function on the new contour and within the enclosed region. (On this new contour the roots $\rho_i(p)$ and $\rho_i^*(p)$ will not be repeated since the branch points have been circumvented and therefore $\frac{\partial \Delta L_o(s,p)}{\partial s} \neq 0$) To determine if poles exist in Re $p \geqslant 0$ one therefore can map the function f(p) as p goes from point A to point B of Fig. 5b and apply the theorem. This is illustrated in Fig. 6. If any poles on the imaginary axis are encountered, their order can be detected by observation of the growth of f(p) in the neighborhood of the poles. In the neighborhood of a pole p_1, of order n we can write

$$f(p) = \frac{1}{(p-p_1)^n} g(p)$$

where $g(p)$ is regular in the neighborhood of p_1. Let $p - p_1 = \epsilon e^{i\theta}$

then

$$f(p) = \frac{1}{\epsilon^n} e^{-in\theta} \left[g(p_1) + O(\epsilon) \right]$$

Therefore, when p goes around a semicircular arc (from $\theta = \pi/2$ to $\theta = -\pi/2$) of radius ϵ and as $\epsilon \to 0$, the change in phase of f(p) is $n\pi$.

The basic steps involved in the determination of the stability of the system are summarized in the following paragraphs.

3.4 Summary of the Procedure

We have reduced the original rather complex stability problem to a simpler problem of locating the poles of a certain function, f(p), in $\mathrm{Re}\, p \geqslant 0$. The occurrence of poles in $\mathrm{Re}\; p > 0$ indicate instability, as do poles of second order or greater on the imaginary axis of the p-plane. If no poles arise in $\mathrm{Re}\; p > 0$ and those on the imaginary axis are of first order, the system is stable. Because of the complexity involved, the poles of f(p) must be located numerically by the use of a digital computer. It is appropriate at this point to summarize the basic steps involved.

(1) Select the parameters n, $\mathcal{M}$, and β, as well as the load distribution 2.6. By virtue of equation 3.14 one can obtain the parameters c_j, c_j^*, α_j, α_j^*, ℓ, and ℓ^* as illustrated in equation 3.17. Let there exist a parameter λ, which characterizes the "magnitude" of the loading such that λ is a function only of the c_j, c_j^*; and the lateral loading is either zero or a constant value (internal pressurization) when $\lambda = 0$.

Case i, M = 0

(2) If M = 0, set p = 0 and calculate the roots $\rho_i(p)$ and $\rho_i^*(p)$ of $\Delta L_0(s, p) = 0$, where $\Delta L_0(s, p)$ is given by equation 3.44. (ρ_i^*

is obtained by replacing C_o by C_o^* in 3.44). Select the four roots ρ_i with negative real parts and denote them as ρ_i(i = 5 to 8), in any order. Select the four roots ρ_i^* with positive real parts and denote them as ρ_i^* (i = 1 to 4), in any order.

(3) Calculate the function f(p = 0), given by equation 3.84 (see also the relation 3.68) and plot the value of f(p = 0) against the parameter λ , beginning with a sufficiently small value so that the system is initially stable. (Note that the ρ_i and ρ_i^* do not depend on λ and thus they need not be recalculated for each λ). Increase λ until a pole of f(p = 0) is first obtained. This represents the transition from stability to instability for the particular value of n considered.

(4) Repeat the process with various values of n until a minimum λ has been found for a wide range of n values. This value represents the critical value or buckling load.

Case ii, $M \neq 0$

(5) Repeat the above procedure for each value of M. Denote the critical value of λ by λ_{cRM} .

(6) Determine the branch point locations of the roots $\rho_i(p)$ and $\rho_i^*(p)$ by numerically solving the polynomials 3.73 or 3.74 and evaluating 3.75.

(7) Select a contour in the p-plane as illustrated in Fig. 5b, circumventing the branch points on the imaginary axis and in the right-half plane. A value of R should be selected by trial such that

it renders f(p) approximately a constant on the circular portion of the contour.

(8) Select appropriate increments of p on the contour and for each value of p calculate the roots of $\Delta L_o(s, p) = 0$. Order these roots at each station by requiring that they be smooth functions of the path and approach the asymptotic values 3.53 along the path of radius R.

(9) Verify that $\lambda > \lambda_{cRM}$ is unstable by mapping the function f(p) as p goes from point A to point B of Fig. 5b for a value of $\lambda > \lambda_{cRM}$, utilizing the principle of the argument to show that a pole exists in Re $p > 0$. Verify that $\lambda < \lambda_{cRM}$ is stable by again mapping f(p) for a value of $\lambda < \lambda_{cRM}$ and showing that no zeros exist in Re $p > 0$ (and all those on the imaginary axis, if any, are of first order).

CHAPTER IV

REFINEMENTS AND EXTENSIONS OF THE ANALYSIS

1. Remarks on More Exact Equations of Motion

If equations of motion of a more exact nature are desired, it will probably be necessary to obtain an approximate solution of the initial steady-state equations. If one obtains this approximation in the form of a finite series of exponentials, as in equation 3.17, the variable coefficients of the variational equations will again be of exponential form. Therefore, assuming the asymptotic behavior of the roots to the characteristic equation can be ascertained, the same stability analysis can be applied with minor alterations. At worst, if the equations are written in terms of the three displacements as dependent variables, the matrices A_j will be 3 x 3 instead of the present 2 x 2 matrices.

2. If Linear Damping is Included

Linear viscous damping can be included in the present analysis with virtually no additional complications, other than introducing another parameter. If damping is considered in conjunction with a more exact set of equations of motion which include the effects of rotational inertia and transverse shear deformation, etc., then the restriction we have applied to the magnitude of the velocity can be eliminated. Since, if damping is included, the steady-state motions will be attenuated in space, again the same stability analysis can be applied if the approximate symmetric solutions are sought in the form of a series of exponentials.

3. Remarks on Extending the Method to Static Problems Involving Finite Length Shells

In many static buckling problems of cylindrical shells, the pre-loaded state is such as to cause initial symmetric displacements with an axial variation of exponential form. For example, assuming equations 2. 5 describe the shell adequately, if a cylindrical shell is clamped at its ends, then loaded axially by a constant force which is applied symmetrically, the axial variation of the initial displacement field (caused by a Poisson's expansion) is the sum of exponentials. The effects of such an initial displacement field on the buckling load are surprisingly large as Stein and Fisher have shown. Again, in such cases, the present analysis, with some variations, can be utilized to solve the variational equations and ascertain stability. If the system is conservative, the problem is especially simple since one can neglect the effect of time in the variational equations and reduce the analysis to an eigenvalue problem, i. e, the usual bifurcation analysis. The point is, the difficulty encountered when variable coefficients arise in the perturbation equations can be overcome if the coefficients are of the form we have discussed. It should be noted, however, that for the case of a finite-length shell the elements of the characteristic determinant (our matrix Q) will involve infinite series.

CHAPTER V

SUMMARY AND CONCLUDING REMARKS

In the main text of this dissertation, the problem of determining the stability of a thin cylindrical shell subjected to a class of moving loads was discussed. As a mathematical model of the shell, a non-linear Donnell theory was employed.

The class of moving loads considered were those representable by a finite series of exponential functions and a delta function, all possessing arguments of $X-VT$ where X and T are respectively the axial coordinate and time, and V represents the load velocity. A constant axial compression or tension of the shell was also included. The shell was idealized as infinite in length and the initial symmetric response was assumed to have reached a steady-state value. The load velocity was restricted to less than the minimal velocity for which sinusoidal wave trains can be propagated in the unloaded shell. For such velocities, the steady-state solutions were found to be attenuated with distance from the load indicating the analysis has meaning for shells whose length is long compared to a characteristic attenuation length.

As far as the equations of motion were concerned, only the effects of radial inertia were included. However, it was indicated that in the load velocity range considered, longitudinal inertia, transverse shear deformation and rotational inertia played a negligible role in the steady-state response.

Stability of the system was defined on the basis of boundedness or divergence of an infinitesimal perturbation about an initial nonlinear state of motion. Following the selection of the equations of motion, all portions of the analysis were exact (all effects of prebuckled displacements, usually neglected in shell stability discussions, were included). By virtue of a double Laplace transform technique and the introduction of some theorems regarding stability and the Laplace transform, the original stability problem was reduced to one of determining the location of the zeros of a determinant of a certain matrix in the Laplace transform domain. The elements of the matrix were presented in a closed and exact form, and hence may be calculated with little difficulty with the aid of a digital computer. A method of determining the necessary information regarding the zeros of the said function and hence the stability information of the system was outlined*.

In the case of static problems, the method outlined should serve as a powerful tool for solving a wide variety of axially symmetric buckling problems. One notes that the effects of prebuckled displacements and bending stresses, which are neglected in many studies, are easily accounted for in the present analysis. In applying the method to shells of finite length, it is probably conservative to limit the shell length to not less than a value defined by a 90 per cent attenuation of the initial displacements as measured from its peak

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* A numerical investigation of some special cases of the general loading function is presently underway and the results of a parametric study of these cases will be presented in a forthcoming paper.

value (on either side of the peak value) and not less than twice the shell radius.

As far as the dynamic problem is concerned, and aside from the intrinsic value of the analysis, the method should provide a point from which one can extrapolate to obtain useful information regarding the effects of moving loads on finite length shells which are (1) long compared to a characteristic attenuation length of the symmetric displacements, (2) long compared to the product of the local velocity and a characteristic response time or characteristic period of oscillation of the shell and (3) not less than twice the shell radius. Unfortunately, for dynamic loads under the present method of analysis (linearized perturbation equations), it is not possible to differentiate between instabilities that lead to a buckling phenomena and those that lead only to finite amplitude oscillations. A logical extension of the present analysis would therefore be to include certain nonlinear terms in the perturbation equations.

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APPENDIX 1

THE HOMOGENEOUS DIFFERENCE EQUATION

In this section we show that, to satisfy the quiescent conditions 3.37 of Chapter 3, only the trivial solution to 3.41 can be accepted. For this purpose we define the norm, $|\overline{\overline{Z}}_n(s)|$, of the vector $\overline{\overline{Z}}_n(s)$ by:

$$|\overline{\overline{Z}}_m(s)| \equiv |\overline{\overline{\zeta}}_m(s)| + |\overline{\overline{\eta}}(s)| \tag{1}$$

Clearly to satisfy 3.37 we must require that

$$|\overline{\overline{Z}}_m(s)| \leq \frac{2a}{\mathrm{Re}\, s - b} \quad , \mathrm{Re}\, s > b \tag{2}$$

Now consider the following theorem and its proof.

Theorem: Given the difference equations

$$\overline{\overline{Z}}_m(s) = \sum_{j=1}^{\ell} A_j(s)\, \overline{\overline{Z}}_m(s+\alpha_j) \tag{3}$$

where $A_j(s)$ are given by 3.43 and $\mathrm{Re}\,\alpha_j > 0$ (see 3.17). We have, for all non trivial solutions

$$\lim_{\mathrm{Re}\, s \to \infty} |\overline{\overline{Z}}_m(s)| = \infty$$

Proof: Let us define the norms, $|A_j(s)|$, of the 2 x 2 matrices A_j by

$$|A_j| = \sum_{\substack{q=1 \\ \max r}}^{2} |A^{(j)}_{qr}| \tag{4}$$

where $A^{(j)}_{qr}$ represents the element in the qth row and rth column of the jth matrix. Then, from 3 we have

$$|\bar{\bar{Z}}_m(s)| \leq \sum_{j=1}^{\ell} |A_j| \, |\bar{\bar{Z}}_m(s+\alpha_j)| \tag{5}$$

Now, assume $Z_n(s)$ is bounded as $\mathrm{Re}\, s \to \infty$, i.e.

$$|\bar{\bar{Z}}_m(s)| \leq M \quad , \quad \mathrm{Re}\, s > C \tag{6}$$

where M and C are positive constants. Since $\mathrm{Re}\, \alpha_j > 0$, equation 6 also indicates

$$|\bar{\bar{Z}}_m(s+\alpha_j)| \leq M \quad , \quad \mathrm{Re}\, s > C \tag{7}$$

Combining 5 and 7 we obtain

$$|\bar{\bar{Z}}_m(s)| \leq M \sum_{j=1}^{\ell} |A_j| \tag{8}$$

The matrices $A_j(s)$ have the following property:

$$|A_j(s)| \leq \frac{K_j}{|s|^m} \, , \; m > 1 \quad \text{if} \quad \mathrm{Re}\, s > d_j \tag{9}$$

where K_j and d_j are constants. Choose $C > d_{j\,max}$. Then 8 and 9 yield

$$\left| \overline{\overline{Z}}_m(s) \right| \le \frac{M}{|s|^m} \sum_{j=1}^{\ell} K_j \tag{10}$$

This implies $\exists$ $s = s_o \ni$ Re $s_o > c > d_{j\ max}$ yields

$$\left| \overline{\overline{Z}}_m(s_o) \right| = \epsilon(s_o)$$

where $\epsilon(s_o)$ can be made arbitrarily small. Equation 10 also implies

$$\left| \overline{\overline{Z}}_m(s_o + \alpha_j) \right| \le \epsilon(s_o)$$

However from 5 we obtain

$$\epsilon(s_o) \le \sum_{j=1}^{\ell} \frac{K_j}{|s|^m} \epsilon(s_o)$$

or

$$1 \le \sum_{j=1}^{\ell} \frac{K_j}{|s_o|^m} \tag{11}$$

But since s_o can be selected arbitrarily large, this represents a contradiction unless $Z_n(s)$ is identically zero. Thus, all nontrivial solutions to 3.43 are unbounded as Re $s \to \infty$. Therefore only the trivial solution to 3.41 can be accepted on the basis of equation 2.

APPENDIX 2

PROPERTIES OF THE SERIES FOR $\overline{\overline{Z}}_n(s)$ AND $\overline{Z}_n(\xi, p)$

1. Properties of the $\overline{\overline{Z}}_n(s)$ Series (equation 3.45)

The series 3.45 formally satisfies equation 3.41. In this section we investigate a few of the properties of the series for $Z_n(s)$.

Denoting

$$A_{j_k}(s) = \frac{B_{j_k}(s)}{\prod_{q=1}^{8}(s-\rho_q)} \quad \text{AND} \quad \phi(s) = \frac{\Phi(s)}{\prod_{q=1}^{8}(s-\rho_q)} \tag{1}$$

where ρ_q are the 8 roots of $\Delta L_o(s) = 0$ and the meaning of $B_{j_k}(s)$ and $\Phi(s)$ is clear from equations 3.43, the series 3.46 can be written as

$$\overline{\overline{Z}}_n(s) = \frac{\Phi(s)}{\prod_{q=1}^{8}(s-\rho_q)} + \sum_{j_1=1}^{\ell} \frac{B_{j_1}(s)\,\Phi(s+\alpha_{j_1})}{\prod_{q=1}^{8}(s-\rho_q)\prod_{q=1}^{8}(s+\alpha_{j_1}-\rho_q)} +$$

$$\sum_{j_1,j_2=1}^{\ell} \frac{B_{j_1}(s)\,B_{j_2}(s+\alpha_{j_1})\quad \Phi(s+\alpha_{j_1}+\alpha_{j_2})}{\prod_{q=1}^{8}(s-\rho_q)\prod_{q=1}^{8}(s+\alpha_{j_1}-\rho_q)\prod_{q=1}^{8}(s+\alpha_{j_1}+\alpha_{j_2}-\rho_q)} + \cdots \tag{2}$$

Since the known elements of the matrices $B_j(s)$ and the vector $\Phi(s)$ are entire functions of s, the singularities of $\overline{\overline{Z}}_{n_N}(s)$ in the s-plane (the Nth terms of the series 2) consist of isolated poles at the locations

$$s = (\rho_q - m_1\alpha_1 - m_2\alpha_2 - \cdots - m_\ell\alpha_\ell), \qquad m_i = 0, 1, 2, \ldots \quad , i = 1, 2, \ldots, \ell \tag{3}$$

Defining the region R of the complex s-plane by

$$\left| s - (\rho_g - m_1\alpha_1 - m_2\alpha_2 - \cdots - m_\ell \alpha_\ell) \right| \geq \epsilon > 0 \tag{4}$$

we see that $\bar{\bar{Z}}_{m_N}(s)$ is regular if $s \in R$. Also, since the poles lie a finite distance to the right of the imaginary axis, $\bar{\bar{Z}}_{m_N}(s)$ is regular if Re $s > C_1$ = const.

With the vector and matrix norms as defined in Appendix 1, we have from equation 3.46 or 2 above:

$$|\bar{\bar{Z}}_m(s)| \leq |\phi(s)| + \sum_{j_1=1}^{\ell} |A_{j_1}(s)|\,|\phi(s+\alpha_{j_1})| + \sum_{j_1, j_2=1}^{\ell} |A_{j_1}(s)|\,|A_{j_2}(s+\alpha_{j_1})|\,|\phi(s+\alpha_{j_1}+\alpha_{j_2})| + \cdots \tag{3}$$

If $s \in R$, $|\phi(s)| \leq M$ = const. Further, if Re $s > b_1$ = const.

$$\sum_{j_1=1}^{\ell} |A_{j_1}(s)| \leq \delta < 1 \tag{5}$$

Since Re $\alpha_j > 0$ there exists an $N > r$ such that

$$\mathrm{Re}\,(s + \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_{N-1}}) > b_1 \tag{6}$$

for all combinations, $j_k = 1, 2, \ldots, \ell$. For all terms less than the Nth we have

$$|A_j(s+\alpha_{j_1}+\alpha_{j_2}+\cdots)| < M^* = \text{CONSTANT}$$

if $s \in R$. Therefore, from 3 we obtain

$$|\bar{\bar{Z}}_n(s)| \leq \mathcal{S} + MM^{*r}\delta + MM^{*r}\delta^2 + MM^{*r}\delta^3 + \cdots \qquad (7)$$

where $\mathcal{S}$ denotes the sum of the first r terms. The series (less $\mathcal{S}$) 7 is a geometrical series whose common ratio is $\delta < 1$ and is therefore convergent. Thus, by the Weierstrass M-test the series $\bar{\bar{Z}}_n(s)$ is uniformly (with respect to s) and absolutely convergent when $s \in R$. Since each term of the series is regular when $s \in R$, the uniform convergence indicates $\bar{\bar{Z}}_n(s)$ is unique and represents a regular function in R.

Returning to the form of the series 1, it is clear that the poles of each term of the series are of finite order. Further, any pole can be removed from all terms by defining a new function which is the product of $\bar{\bar{Z}}_n(s)$ and $(s - \rho q + m_1 d_1 + \cdots + m_\ell d_\ell)^{r_1}$ where r_1 is a finite integer and m_i indicates the pole in question. Clearly the new series will be absolutely and uniformly convergent in R and also in the circle $|s - (\rho q - m_1 d_1 - \cdots - m_\ell d_\ell)| < \epsilon$ where m_i corresponds to the above pole. One thus deduces the series 3.46 represents an analytic function of s with isolated poles at the locations given by the relation 3.

2. Properties of the $\bar{Z}_n(\xi, p)$ Series

It will suffice to investigate only the series 3.61 since 3.62 is similar in form.

Substituting

$$A_{j_k}(s) = B_{j_k}(s) \Big/ \prod_{g=1}^{8} (s - \rho_g) \tag{8}$$

one obtains 3.61 as

$$\bar{Z}_m(\xi, p) = \sum_{i=5}^{8} e^{\rho_i \xi} \Big[I + \sum_{j_1=1}^{\ell} \frac{B_{j_1}(\rho_i - \alpha_{j_1}) e^{-\alpha_{j_1} \xi}}{\prod_{g=1}^{8} (\rho_i - \rho_g - \alpha_{j_1})} + \tag{9}$$

$$+ \sum_{j_1, j_2 = 1}^{\ell} \frac{B_{j_1}(\rho_i - \alpha_{j_1} - \alpha_{j_2}) B_{j_2}(\rho_i - \alpha_{j_2}) \; e^{-(\alpha_{j_1} + \alpha_{j_2})\xi}}{\prod_{g=1}^{8} (\rho_i - \rho_g - \alpha_{j_1} - \alpha_{j_2}) \prod_{g=1}^{8} (\rho_i - \rho_g - \alpha_{j_2})} + \cdots \Big] \begin{Bmatrix} 1 \\ r_i \end{Bmatrix} g_i$$

Define the region R_{b_1} of the complex p-plane by

$$|p - p_b| \geq \epsilon_1 > 0 \tag{10}$$

where p_b denotes a branch point of any root ρ_g in the p-plane. If $p \in R_{b_1}$, $\frac{\partial \Delta L_o}{\partial s}(s) \neq 0$ and the roots ρ_g of $L_o(s) = 0$ are non-repeated. Therefore, if $p \in R_{b_1}$, the assumption of non-repeated roots utilized in the inversion of $Z_n(s)$ was justified.

If in addition to 10, we provide appropriate branch cuts in the p-plane, the roots $\rho_g(p)$ can be made analytic functions of p. For definiteness assume the branch cuts to be parallel to the real p axis and running leftward from each branch point to negative infinity. In addition to the branch cuts let us require that

$$p \ni |\rho_i - \rho_g - m_1 d_1 - m_2 d_2 - \cdots - m_\ell d_\ell| \geq \epsilon_2 > 0, \quad i = 5 \text{ to } 8$$

$$p \ni |\rho_i^* - \rho_g^* - m_1 d_1^* - m_2 d_2^* - \cdots - m_\ell^* d_\ell^*| \geq \epsilon_3 > 0, \quad i = 1 \text{ to } 4 \qquad (11)$$

$$m_i = 0, 1, 2, \ldots$$

and $p \ni \Delta Q \geq \epsilon_4 > 0$. Define this new region, which is contained within R_{b_1} and R_{b_2}. If $p \in R_{b_2}$, the series 9 can be shown, by the same procedure as in Section 1, to be absolutely and uniformly convergent with respect to both ξ and p. Further $\overline{Z}_n(\xi, p)$ is an analytic function of p and a continuous function of ξ. Also, all series derived from equation by term by term differentiation with respect to ξ possess the same property of uniform convergence with ξ. Thus the term by term differentiation of the series in the analysis was justified, assuming $p \in R_{b_2}$.

With reference the equivalent form of the series given by equation 3.63, one observes that each term of the series for $\overline{Z}_n(\xi, p)$ is regular within the regions defined by 11. Since the series converges uniformly on the boundary of the region, by the Weierstrass limit theorem one concludes the series converges uniformly within the region and thus $\overline{Z}_n(\xi, p)$ is regular within the region.

The points for which $\Delta Q = 0$ (see 3.68) represent poles of $Z_n(\xi, p)$ in the p-plane.

APPENDIX 3

STABILITY AS VISUALIZED FROM THE LAPLACE TRANSFORM PLANE

Necessary and sufficient conditions on the image function for stable or unstable originals are well known for transforms that are the ratio of polynomials. These conditions can easily be obtained by a direct inversion of the transform. For transforms that are transcendental, theorems exist that are applicable if the transform satisfies certain conditions. An important inversion theorem for large time by Erdelyi, for example, is discussed by Fung in reference 37. This theorem yields the asymptotic form of the original and thus from it stability information can be deduced. The transform, however, must be expressible in a certain asymptotic form, which is inconvenient here.

Since we are not interested in the actual form of the original, it is possible to construct a few theorems, which can easily be proved, regarding the boundedness of the original. We discuss these below. Note that the first theorem requires no use of an inversion theorem and is completely general.

Theorem 1: Let $\bar{f}(p)$ denote the Laplace transform of a function $f(t)$. Consider the region s of the p-plane defined by $\mathrm{Re}\, p > \epsilon > 0$ where ϵ is an arbitrarily small constant. Then if $\exists$ singularities of $\bar{f}(p)$ in s, $f(t)$ is unbounded as a function of t.

Proof: By definition

$$\bar{f}(p) = \int_0^\infty e^{-p\tau} f(\tau)\, d\tau$$

assume f(t) is a bounded function of t. Call the u. b $|f(t)| = M$. This assumption and the hypothesis that $p \in s$ indicates the definition integral converges uniformly in s since:

$$\left| \int_0^\infty e^{-p\tau} f(\tau)\, d\tau \right| \leq \int_0^\infty |e^{-p\tau}||f(\tau)|\, d\tau \leq M \int_0^\infty e^{-\varepsilon\tau}\, d\tau = \frac{M}{\varepsilon}$$

and therefore, by the Weierstrass M-test, the integral converges uniformly in s.

Next we show $\bar{f}(p)$ is regular if $p \in s$. For this purpose we perform a contour integration with respect to p over an arbitrary simple closed curve C in s:

$$\int_C \bar{f}(p)\, dp = \int_C \left\{ \int_0^\infty e^{-p\tau} f(\tau)\, d\tau \right\} dp$$

By virtue of the uniform convergence of the integral and the properties of the integrand when $p \in s$, the order of integration can be inverted and we obtain

$$\int_C \bar{f}(p)\, dp = \int_0^\infty f(\tau) \left[\int_C e^{-p\tau}\, dp \right] d\tau = 0$$

since by Cauchy's integral theorem, $\int_C e^{-p\tau}\, dp = 0$.

Because the path c is arbitrary in s, Morera's theorem establishes that f(p) is regular in s.

Therefore, boundedness of f(t) implies $\bar{f}(p)$ is regular in s. This conclusion is logically equivalent to: if $\bar{f}(p)$ is not regular in s, f(t) is not bounded. (The proof of this last statement is simple: assume $\bar{f}(p)$ not regular in s but f(t) bounded; this however contradicts the first conclusion, thus f(t) cannot be bounded.)

<u>Theorem 2:</u> Let f(t) be defined by:

$$f(t) = \lim_{b \to \infty} \frac{1}{2\pi i} \int_{c-ib}^{c+ib} e^{pt} \bar{f}(p)\, dp, \quad t > 0 \qquad (i)$$

Assume $\bar{f}(p)$ possesses the properties:

(1) $\bar{f}(p)$ regular in Re $p \geq 0$ (all singularities lie to the left of the imaginary axis)

(2) $f(p) \sim O\left(\frac{1}{p^k}\right)$, $k > 0$ when $|p| \to \infty$ in $-\frac{\pi}{2} \leq \arg p \leq \frac{\pi}{2}$

(3) One or both of the following:

(a) $\bar{f}(p) \sim O\left(\frac{1}{p^k}\right)$, $k > 1$

(b) $\bar{f}^{\prime}(p) \sim O\left(\frac{1}{p^m}\right)$, $m > 1$

when $|p| \to \infty$ in $-\frac{\pi}{2} \leq \text{ARG}\, p \leq \frac{\pi}{2}$

Then $\lim_{t \to \infty} f(t) = 0$.

<u>Proof:</u> Consider the contour shown below. By hypothesis (1) and Cauchy's integral formula we have:

$$I_{BR_1} + I_{C_1} + (-I_{BR_2}) + I_{C_2} = 0$$

where I_{BR_1} is the integral (i) along the Bromwich contour BR_1, I_{C_1}, and I_{C_2} the same integral along the circular arc paths C_1 and C_2 if radius R, and I_{BR_2} (i) with C = 0. By hypothesis (2), Jordon's lemma can be applied to yield:

$$\lim_{R \to \infty} \left\{ \begin{matrix} I_{c_1} \\ I_{c_2} \end{matrix} \right\} = 0$$

Thus (i) is equivalent to:

$$f(t) = \lim_{b \to \infty} \frac{1}{2\pi} \int_{-b}^{b} e^{i\sigma t} \bar{f}(\sigma)\, d\sigma, \qquad Im(f) = \sigma \tag{ii}$$

Now, assume first that (3-a) holds. The real and imaginary component integrals of (ii) are therefore absolutely convergent and the Riemann Lebesque lemma can be directly applied yielding

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\sigma t} \bar{f}(\sigma)\, d\sigma = 0$$

Next assume (3-b) holds but that (3-a) does not. Integrating (ii) by parts and setting $\alpha = \dfrac{e^{i\sigma t}}{it}$ we have

$$2\pi\,(ii) = \lim_{b \to \infty} \left\{ \bar{f}(b)\,\alpha(b) - \bar{f}(-b)\,\alpha(-b) - \int_{-b}^{b} \alpha \frac{d\bar{f}}{d\sigma}\, d\sigma \right\}$$

By hypothesis (2) and, setting $t > \epsilon > 0$ so that α is bounded,

$$\lim_{b \to \infty} \left\{ \begin{matrix} \bar{f}(b)\,\alpha(b) \\ \bar{f}(-b)\,\alpha(-b) \end{matrix} \right\} \to 0$$

By (3-b) $\int_{-\infty}^{\infty} |\alpha||\bar{f}'(\sigma)|\, d\sigma$ exists and therefore $\int_{-\infty}^{\infty} \alpha \bar{f}'(\sigma)\, d\sigma$ is absolutely convergent. We can therefore apply the Riemann Lebesque lemma again to obtain

$$\lim_{t \to \infty} \int_{-\infty}^{\infty} \alpha \frac{d\bar{f}}{d\sigma}\, d\sigma = 0$$

We therefore conclude under the imposed conditions on f(p):

$$\lim_{t \to \infty} f(t) = 0$$

and the theorem is proved.

If one now assumes f(t) is recoverable from its transform by the integral (i)*, then it has been shown that $f(\infty) = 0$ if its transform satisfies the conditions of the theorem.

We have not yet discussed the case of singularities appearing on the imaginary axis. Assume f(p) is regular in $\operatorname{Re} p > 0$, but possesses isolated singularities on the imaginary axis. It is of little loss in generality to assume f(t) is recoverable from its transform by (i) of theorem 2 above and that f(p) satisfies properties (2) and (3). Then, one can select the Bromwich 2 contour up the imaginary axis, circumventing any singularities by a half circle of radius ϵ in the usual procedure. By Riemann's lemma the only contribution to f(t) for $t \to \infty$ arises from the integrations around the singularities. It is easy to show that contribution from poles of order 1 are bounded and from poles of higher order than 1 are unbounded.

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* Sufficient conditions for f(t) to be recoverable from its transform by (i) are

1. f(t) be of bounded variation in any finite interval
2. there exists a positive number, a, such that the integral $f(p) = \int_0^\infty e^{-p\tau} f(t)\, d\tau$ converges absolutely for $\operatorname{Re} p > a$, with $c > a$ in (i)
3. f(t) is equal to its mean value at a point of discontinuity.

APPENDIX 4

LOCATION OF THE ROOT BRANCH POINTS

The roots $\rho_i(p)$ and $\rho_i^*(p)$ are the roots of $\Delta L_o(s,p) = 0$. This expression can be thought of as a second order polynomial in p and one can solve for p in terms of s as

$$p = \frac{ms \pm \left[-\beta^4 s^8 + 4m^2\beta^4 s^6 - (1+6m^4\beta^4)s^4 + 4m^6\beta^4 s^2 - m^8\beta^4\right]^{1/2}}{s^2 - m^2} \quad (1)$$

Each branch of the inverse function s(p) in the p-plane represents a root of $\Delta L_o(s,p) = 0$. The branch points of the roots in the p-plane are those of s(p).

With appropriate branch cuts, one can render p an analytic function of s (for each of the above signs). Having made such cuts, p(s) will be regular except at the branch points (in the s-plane) associated with the above radical and at the poles $s = \pm n$. Now, let s_o be a regular point of one of the branches of p(s) at which it takes the value p_o. Assume first that $p'(s_o) \neq 0$. Then p(s) has a unique inverse s(p), regular in a neighborhood of p_o (36). Thus if $p'(s_o) \neq 0$, p_o is not a branch point of s(p) in the p-plane. Next, assume $p'(s_o) = 0$. Since s_o is a regular point, p(s) can be represented in a neighborhood of s_o by a Taylor series

$$p(s) = p_o + a_2(s-s_o)^2 + a_3(s-s_o)^3 + \cdots$$

When s is sufficiently near s_o, one obtains the approximate equality

$$p(s) \doteq p_o + a_2(s-s_o)^2$$

or

$$s - s_o = (a_2)^{-1/2}(p-p_o)^{1/2}$$

This last expression indicates that a branch point of the inverse function s(p) exists at $p(s_o)$. We conclude, therefore, that for a regular point, s_o, a necessary and sufficient condition that p_o be a branch point of s(p) in the p-plane is that $p'(s_o) = 0$.

Now, consider the poles at $s = \pm n$. Let s_o denote a pole. Since $s = \pm n$ are not branch points of the radical, p(s) can be expanded in a Laurent series about s_o. Noting the poles are of first order we have

$$p(s) = b_{-1}(s-s_o)^{-1} + b_o + b_1(s-s_o)^{+1} + \cdots$$

When s is near s_o, this can be reduced to the approximate equality

$$p(s) = b_{-1}(s-s_o)^{-1}$$

or

$$(s-s_o) = b_{-1}^{-1}\, p$$

This last relation indicates a branch point does not occur in the case of the simple poles $s = \pm n$.

Next, let s_o be a zero of g(s), where

$$g(s) = -\beta^4 s^8 + 4n^2\beta^4 s^6 - (1+6n^4\beta^4)s^4 + 4n^6\beta^4 s^2 - n^8\beta^4$$

It is easy to show that $g'(s) = 0$ and $g''(s) = 0$ cannot simultaneously be satisfied for any real values of β or n. Thus the roots of $g(s) = 0$ are at most double roots, if they are repeated at all. If a double root occurs, the root is a regular point of p(s). Therefore there is no loss in generality in the present discussion in assuming $g(s) = 0$ does not possess repeated roots. Under this assumption s_o is a branch point of p(s) in the s-plane. Consider a neighborhood of s_o defined by $s-s_o = \epsilon e^{i\phi}$ where ϵ = constant. Equation (1) can be rewritten as

$$p = m(\epsilon e^{i\phi} + s_o) + \epsilon^{1/2} e^{i\phi/2} [(\epsilon e^{i\phi} + s_o - s_1) \cdots (\epsilon e^{i\phi} + s_o - s_\gamma)]$$

where s_o, $s_1, \ldots, s_\gamma$ are the zeros of g(s). As ϵ is allowed to become vanishingly small we obtain the approximate equality

$$p = m s_o + \epsilon^{1/2} e^{i\phi/2} K \; , \quad K = \text{CONSTANT}$$

Noting that $s - s_o = \epsilon e^{i\phi}$, we obtain by squaring both sides

$$s - s_o = K^{-2} (p - m s_o)^2$$

which indicates the inverse point $p = ms_o$ is not a branch point of s(p).

In view of the preceding discussion, $p'(s) = 0$ is a necessary and a sufficient condition for the existence of branch points of s(p) in the p-plane.

APPENDIX 5

A NONLINEAR CYLINDRICAL MEMBRANE AND/OR A NONLINEAR STRING SUBJECTED TO A CONCENTRATED MOVING LOAD

1. Introduction

The text of this dissertation has been concerned with moving loads on cylindrical shells. A closely related problem is that of a moving load on a cylindrical membrane or a nonlinear string. We shall present at this time a brief version of an analysis on the response and stability of a nonlinear cylindrical membrane and/or a string on a nonlinear foundation, subjected to a concentrated radial line load or concentrated load, respectively, moving with constant velocity.

We will again consider the domain of the problem to be infinite and seek steady-state solutions. There is, of course, no question of buckling involved here but rather the analysis of the stability of the steady-state motions, in general, serves to indicate whether or not such motions can be expected as the limiting case of a transient problem.

The mathematical model we shall consider will be shown to represent both a nonlinear string and a nonlinear cylindrical membrane depending upon certain approximations. A brief summary of past work dealing with nonlinear strings can be found in a review of the subject by J. O. Eisley (38). Further papers can be found in references 39 to 42. Previous analyses having the most in common with the present discussion are those of B. Fleischman (39) and P. Ungar (40). Fleischman discussed traveling waves with constant velocity in an infinite string on a nonlinear foundation. Ungar proved such wave motions can be unstable under certain conditions.

2. Formulation of the Problem

Consider the nonlinear one-dimensional wave equation

$$Y_{XX} - \frac{1}{c^2} Y_{TT} + F(X,T) - G(Y) = 0 \qquad (1)$$

Equation 1 is the governing equation for the small transverse displacements of a string or flexible cable under constant initial tension and with uniform mass per unit length, carrying a distributed transverse load, $F(X,T)$, and supported by a nonlinear foundation whose restoring force is $G(Y)$. Both $F(X,T)$ and $G(Y)$ are considered as normalized on the initial tension in the string. The quantity c^2 is the "sound speed" and is the ratio of the initial tension to the mass per unit length. It will be shown that equation 1 also represents the symmetric radial motions of an elastic cylindrical membrane where nonlinearity of both the geometric and constitutive type is allowed.

As the objective* of this discussion we shall investigate the stability of a class of solutions of 1 for which

(i) $Y(X,T) = Y(X-VT)$, where V = constant

(ii) $F(X,T) = P_1\ \delta(X-VT)$, where $\delta(X-VT)$ is the Dirac delta function and P_1 is constant.

(iii) $G(Y)$ will, in general, be assumed a smooth function of Y with $G(0) = 0$ and $G'(0) > 0$.

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* See page 128 for a summary of the results.

Our choice of F(X, T) represents a concentrated load moving in the positive x direction with a constant velocity. The solutions we seek represent time invariant motions in a coordinate system moving with the load; such motion will, as before, be denoted as steady-state.

Stability will be based, as in the shell discussion, on the boundedness of perturbations from the initial state of motion. The Laplace transform will be used to determine the stability for $V < c$; however, the abundant theory of Hill's equation indicates a separation of variables approach to be more appropriate if $V > c$.

3. Derivation of the Membrane Equation

We show next that under certain approximations, equation 1 can be assumed as a mathematical model for the symmetric motion of an initially taut (axially) elastic cylindrical membrane.

In general, the equations of motion of an elastic body can be determined by Hamilton's variational principle:

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{2}$$

where the variations in displacements vanish at t_1 and t_2 and L is the Lagrangian function, the variation of which is

$$\delta L = \delta K - \delta U + \delta A \tag{3}$$

where

$$\delta K = \iiint_{\tau_0} \left(\frac{\partial \vec{V}}{\partial t}\right) \cdot \delta\left(\frac{\partial \vec{V}}{\partial t}\right) \rho_0 \sqrt{g} \, d\theta^1 d\theta^2 d\theta^3 \tag{4}$$

is the variation of the kinetic energy of the body,

$$\delta U = \iiint_{\tau_o} \tau^{ij} \frac{\sqrt{G}}{\sqrt{g}} \delta \gamma_{ij} \sqrt{g}\, d\theta^1 d\theta^2 d\theta^3 \tag{5}$$

is the variation of the elastic strain energy, and

$$\delta A = \iint_{S_o} (\vec{P} \cdot \delta \vec{V})\, dS_o \tag{6}$$

is the virtual work. We shall consider, as reference coordinates, the undeformed coordinates of the body and therefore all integrations will be referenced to undeformed coordinates. The notation is that of (43) where

θ^i = reference coordinates

g_{ij} = metric tensor of undeformed coordinates

G_{ij} = metric tensor of deformed coordinates

γ_{ij} = $\frac{1}{2}(G_{ij}-g_{ij})$ = covariant components of the strain tensor

$\vec{V}$ = displacement vector

τ^{ij} = contravariant components of the stress tensor

$\vec{P}$ = surface traction, referred to undeformed area

ρ_o = mass density of the undeformed body

g = determinant of g_{ij}

G = determinant of G_{ij}

Let us write 5 in terms of physical stresses σ_{ij} and physical strains, ε_{ij}, assuming the θ^i are taken as <u>orthogonal</u> undeformed coordinates:

$$\delta U = \iiint_{\tau_o} \sigma_{ij} \left[\frac{g_{ii}\, g_{jj}\, G}{G_{ii}\, G_{jj}\, g} \right]^{1/2} \delta \epsilon_{ij} \sqrt{g}\, d\theta^1 d\theta^2 d\theta^3 \tag{7}$$

(i, j, NOT SUMMED)

where $\sigma_{ij} = \sqrt{G_{ii}G_{jj}}\ \tau^{ij}$ (since the θ^i are orthogonal) (i, j not summed)

$\epsilon_{ij} = \gamma_{ij}/\sqrt{g_{ii}g_{jj}}$

If we define

$$\sigma_{ij}^{*} = \sigma_{ij}\left[\frac{g_{ii}\,g_{jj}\,G}{G_{ii}\,G_{jj}\,g}\right]^{1/2} \quad (i, j \text{ NOT SUMMED}) \tag{8}$$

then 7 becomes

$$\delta U = \iiint_{\tau_0} \sigma_{ij}^{*}\,\delta\epsilon_{ij}\,\sqrt{g}\,d\theta^1 d\theta^2 d\theta^3 \tag{9}$$

The meaning of σ_{ij}^{*} becomes clear if one considers that the element of area on the Θ^i-surface of the undeformed body is

$$(dS_i)_0 = \sqrt{g\,g^{ii}}\,d\theta^j d\theta^k \quad (i \text{ NOT SUMMED}, i \neq j \neq k)$$

and in the deformed body becomes

$$dS_i = \sqrt{G\,G^{ii}}\,d\theta^j d\theta^k \quad (i \text{ NOT SUMMED}, i \neq j \neq k)$$

Thus

$$\left[\frac{G\,g_{ii}}{g\,G_{ii}}\right]^{1/2} = \frac{dS_i}{(dS_i)_0} \quad (i \text{ NOT SUMMED})$$

is the ratio of the deformed to undeformed areas. Further, since the extension e_i of a line element ds_i along a curvilinear coordinate curve θ^i is defined by

$$e_i = \frac{ds_i - (ds_i)_0}{(ds_i)_0} \quad (i \text{ NOT SUMMED})$$

and

$$ds_i = \sqrt{G_{ii}}\,d\theta^i, \quad (ds_i)_0 = \sqrt{g_{ii}}\,d\theta^i \quad (i \text{ NOT SUMMED})$$

we have

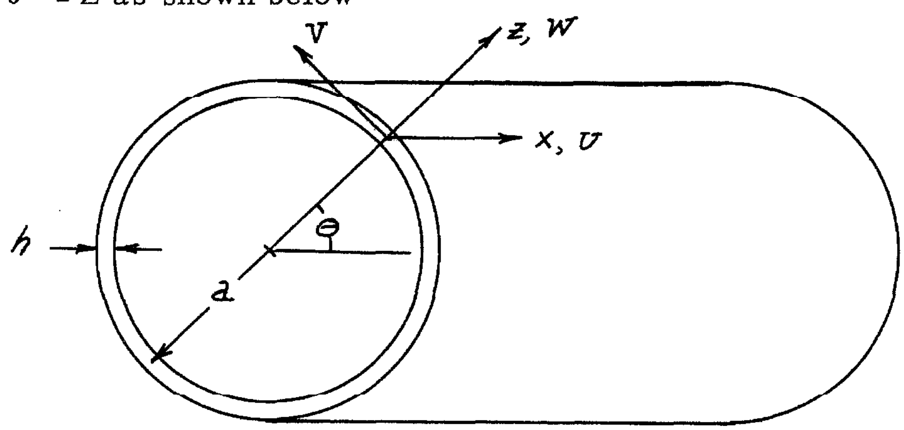
$$1 + e_i = \left(\frac{G_{ii}}{g_{ii}}\right)^{1/2} \quad (i \text{ NOT SUMMED})$$

Therefore σ_{ij}^* can be written

$$\sigma_{ij}^* = \frac{\sigma_{ij}\, ds_i}{(1+e_i)(ds_i)_0} \qquad (i \text{ NOT SUMMED}) \tag{10}$$

and $\sigma_{ij}^* (1 + e_i)$ is recognized as being obtained from the force vector acting on a face of the deformed element by dividing by the face area before deformation.

Consider now as coordinates for the problem: $\theta^1 = X$, $\theta^2 = \theta$, $\theta^3 = Z$ as shown below



The Lagrangian components of strain, ε_{ij}, can be written as

$$\varepsilon_{ij} = \frac{1}{2}(g_{ii}\, g_{jj})^{-\frac{1}{2}} \left(V_i|_j + V_j|_i + V^r|_i \, V_r|_j \right) \tag{11}$$

where $V_1 = U$, $V_2 = V$, $V_3 = W$.

Let us now assume $h/a << 1$ and approximate the displacements by neglecting any variation through the thickness:

$$V_i(x, \theta, z) \doteq V_i(x, \theta) \tag{12}$$

Further, considering only symmetric motions we obtain

$$V_i = V_i(x), \quad V_2 \equiv 0$$

Under these assumptions, the strain components 11 can be written:

$$\varepsilon_{xx} = \frac{\partial U}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2\right]$$

$$\varepsilon_{\theta\theta} = \frac{W}{a+z} + \frac{1}{2}\frac{W^2}{(a+z)^2} \qquad (13)$$

$$\varepsilon_{x\theta} = 0,\ \varepsilon_{zz} = 0,\ \varepsilon_{z\theta} = 0,\ \varepsilon_{zx} = 0$$

Consider P_z as the only component of the surface traction. Then, after defining the quantities

$$q = P_z\left(1+\frac{z}{a}\right)\Bigg|_{z=-\frac{h}{2}}^{z=\frac{h}{2}}$$

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}^{*}\, dz \qquad N_{\theta\theta} = \int_{-h/2}^{h/2} \sigma_{\theta\theta}^{*}\left(1+\frac{z}{a}\right) dz \qquad (14)$$

and carrying out the variation 2 in the usual manner, one obtains the following approximate nonlinear equations of motion:

$$\frac{\partial}{\partial x}\left[\left(1+\frac{\partial U}{\partial x}\right) N_{xx}\right] = \rho_o h \frac{\partial^2 U}{\partial T^2} \qquad (15a)$$

$$-\frac{N_{\theta\theta}}{a} + \frac{\partial}{\partial x}\left[N_{xx}\frac{\partial W}{\partial x}\right] + q = \rho_o h \frac{\partial^2 W}{\partial T^2} \qquad (15b)$$

If we assume the tangential inertia term, $\rho_o h \frac{\partial^2 u}{\partial T^2}$, is negligible, then equation 15a can be integrated once yielding

$$\left[1+\frac{\partial U}{\partial x}\right] N_{xx} = C_1(t) \qquad (16)$$

If we assume further that $\partial u/\partial x \ll 1$, and the membrane tension is held constant, then

$$N_{xx} = c_2 \tag{17}$$

For an elastic, homogeneous, and isotropic membrane, the constitutive relations can be written formally as

$$N_{xx} = F_1 (\varepsilon_{xx}, \varepsilon_{\theta\theta}) \tag{18a}$$

$$N_{\theta\theta} = F_2 (\varepsilon_{xx}, \varepsilon_{\theta\theta}) \tag{18b}$$

Utilizing the approximation 17, assuming one can solve explicitly for ε_{xx} in terms of $\varepsilon_{\theta\theta}$ from 18a, and substitution of the result into 18b yields

$$\frac{N_{\theta\theta}}{a} = F_3 (\varepsilon_{\theta\theta})$$

which, by 13, can be written as

$$\frac{N_{\theta\theta}}{a} = G_1 (W) \tag{19}$$

Therefore, from equation 15b, we finally obtain

$$N_{xx} \frac{\partial^2 W}{\partial X^2} - \rho_0 h \frac{\partial^2 W}{\partial T^2} + g(X,T) - G(W) = 0 \tag{20}$$

under the approximations;

(1) displacements = FNC (X,T) ONLY

(2) $\partial U/\partial x << 1$

(3) tangential inertia can be neglected.

Now, if we define

$$F = \frac{g}{N_{xx}}, \qquad G = \frac{G_1}{N_{xx}}, \qquad c^2 = \frac{N_{xx}}{\rho_0 h}$$

then equation 20 takes the same form as equation 1:

$$W_{XX} - \frac{1}{C^2} W_{TT} + F(X,T) - G(W) = 0 \tag{21}$$

4. Relationship to the Shell Problem

Let us consider again the case of a cylindrical shell subjected to a ring load traveling with constant velocity along its generatrix. We stated previously that if the velocity of such a load were to exceed the minimal velocity for which axially symmetric sinusoidal wave trains are propagated in the shell, the steady-state motions would not be attenuated with distance behind the load. Extrapolating from the linear steady-state analysis of Tang (23) and the "Timoshenko Beam on an Elastic Foundation" work of Crandall (44), where the effects of rotational inertia and the first thickness-shear mode were included, one can expect the following results in four regions of velocity: when $0 < V < V_o$, the head and tail waves (ahead of and behind the concentrated load) will be damped sinusoids, symmetric with respect to the load (the case we studied); when $V_o < V < V_1$, the head wave will be a sinusoid of short wave length and the tail wave a long wave length sinusoid; when $V_1 < V \lessgtr V_2$, the head wave will decay exponentially and the tail wave will be a decaying exponential superimposed on a long wavelength sinusoid; and when $V > V_2$, there will exist no head wave and the tail wave will be the sum of a long wave length and short wave length sinusoid. In the above, the velocity V_o is the bending resonance velocity given by $V_o = \sqrt{E/\rho}\sqrt{h/a} \; / \; \sqrt[4]{3(1-\nu^2)}$; $V_1 = \sqrt{kG/\rho}$ is the modified shear wave velocity, where k is a shear correction factor; and $V_2 = \sqrt{E/\rho(1-\nu^2)}$ is the plate dilatational wave velocity.

The similar problem of the ring load over the cylindrical membrane (acting outward, or inward if internal pressure exists) or the concentrated load on the string, possesses the following properties: assuming sgn G(Y) = sgn Y, for $V < c$ the head and tail waves are symmetric with respect to the load, possess a maximum value directly under the load and decay monotonically on either side of the load. If $V > c$, there exists no head wave and the tail wave is a periodic function (in X-VT) which, as $P_1 \rightarrow 0$, approaches a sinusoid.

The membrane or string possess only two regions of velocity where the character of the solutions are different, whereas there are four in the case of the shell. Nevertheless, there is a similarity in behavior in the two regions $0 < V < V_o$ and $V > V_2$. The interesting fact is that the steady-state motions of the membrane or string for $\underline{V > c}$ (except for the linear case, G(Y) = aY) are unstable with respect to small superimposed disturbances. Mathematically this arises from the periodic coefficients (in X-VT) of the variational equation which are the result of the steady-state motions being periodic in X-VT for $V > c$. Since (if one assumes the linear solution is accurate enough) the shell variational equations will possess variable coefficients which are the sum of sinusoids, for $V > V_2$, one might suspect that such motions are unstable also.

If $V < c$, we will find that the motions of the string/membrane system are in general stable if sgn $G(Y)$ = sgn Y. For those cases where $G(Y) < 0$, $Y >$ positive constant or $G(Y) > 0$, $Y <$ negative constant, the system can be unstable.

5. The Analysis for $V > c$

We begin by transforming the independent variables X and T by

$$x = X \quad , \quad \tau = cT$$

whence equation 1 becomes

$$Y_{xx} - Y_{\tau\tau} - G(Y) + P_1 \delta(x - v\tau) \tag{22}$$

where $v = V/c$.

By a further transformation:

$$\xi = \frac{x - v\tau}{\sqrt{v^2 - 1}} \quad , \quad \tau' = \tau$$

we obtain 22 as

$$Y_{\xi\xi} - 2\bar{\alpha}\, Y_{\xi\tau'} + Y_{\tau'\tau'} + G(Y) = P\delta(\xi) \tag{23}$$

where $\bar{\alpha} = v/\sqrt{v^2 - 1}$ and $P = P_1/\sqrt{v^2 - 1}$.

Steady-State Motions

Steady-state motions of the form $Y = Y(x-vt)$ are obtained from equation 23 by setting $\partial/\partial\tau' = 0$. This yields

$$\frac{d^2 Y_s}{d\xi^2} + G(Y_s) = P\delta(\xi) \tag{24}$$

where the subscript "s" denotes steady-state.

and since $v = V/c > 1$, we shall require that $Y = 0$ for $\xi \geqslant 0$, i. e., no propagation of disturbances ahead of the load will be allowed. Then, by consideration of the jump condition across the origin, $\xi = 0$, due to the delta function, and by requiring that $Y_s(\xi)$ be continuous, one obtains

$$\frac{d^2 Y_s}{d\xi^2} + G(Y_s) = 0 \tag{25a}$$

with the initial conditions

$$Y_s(0) = 0, \quad \left(\frac{dY_s}{d\xi}\right)_{0^+} = -P \tag{25b}$$

and for $\xi \geqslant 0$

$$Y_s(\xi) \equiv 0 \tag{26}$$

If $\operatorname{sgn} G(Y) = \operatorname{sgn} Y$* the solutions of equation 25 are periodic in ξ. One notes that equations 25 are analogous to the free vibration of a unit mass on a nonlinear spring with a given initial velocity. As an example of what might be expected, suppose $G(Y)$ is given by

$$G(Y) = Y + \alpha_1 Y^3$$

Then if $\alpha_1 > 0$, so that $G(Y)$ is a "hardening" foundation.

$$Y_s(\xi) = \frac{-P}{\left[1+2\alpha_1 P^2\right]^{1/4}} \operatorname{sd}\left[(1+2\alpha_1 P^2)^{1/4}\xi \,\middle|\, \frac{\sqrt{1+2\alpha_1 P^2} - 1}{2\sqrt{1+2\alpha_1 P^2}}\right], \quad \xi < 0$$

---

* This is the only case we shall discuss if $v > 1$.

where $sd(u|m) = sn(u|m)/dn(u|m)$. The quantities $sn(u|m)$ and $dn(u|m)$ are the Jacobian elliptic functions with argument u and parameter m and the notation is that of (51).

If $\alpha_1 < 0$, indicating a "softening" foundation, we obtain

$$Y_s(\xi) = A\, sn\left(-\frac{P\xi}{A} \,\Big|\, \frac{\alpha_1 A^2}{1-\alpha_1 A^2}\right), \quad |Y_s| < \frac{1}{\sqrt{|\alpha_1|}}$$

where $A = -\sqrt{2}\, P/[1+\sqrt{1-2\alpha_1 P^2}]^{\frac{1}{2}}$. If $|Y_s| > \frac{1}{\sqrt{|\alpha_1|}}$, bounded steady-state solutions do not exist.

Stability

Let y denote a perturbation of the steady-state motion, so that

$$Y_p = Y_s(\xi) + y(\xi, \tau)$$

where Y_p is the perturbed solution. Since G(Y) was assumed a smooth function of Y, it can be represented in a Taylor series about $Y_s(\xi)$:

$$G(Y_p) = G(Y_s) + G'(Y_s)\, y + O(y^2)$$

Substitution of these last two relations into equation 23, cancellation of terms identically satisfied by the steady-state response, and retention of only those terms of O(y), yields the variational equation:

$$y_{\xi\xi} - 2\bar{\alpha}\, y_{\xi\tau} - y_{\tau\tau} + G'[Y_s(\xi)]\, y = 0 \qquad (27)$$

Consider separated solutions of the variational equations of the form

$$y(\xi, \tau) = e^{\sigma\tau} f(\xi)$$

where $f(\xi)$ satisfies the equation

$$\frac{d^2 f}{d\xi^2} - 2\bar{\alpha}\sigma\frac{df}{d\xi} + \{\sigma^2 + G'[Y_s(\xi)]\} f = 0$$

Substitute $f(\xi) = e^{\bar{\alpha}\sigma\xi} u(\xi)$, then $u(\xi)$ satisfies the equation:

$$\underline{\xi < 0}: \quad \frac{d^2 u}{d\xi^2} + \{ G'[Y_s(\xi)] + \Psi \} u = 0; \quad \Psi = \frac{-\sigma^2}{\nu^2 - 1} \tag{28a}$$

$$\underline{\xi > 0}: \quad \frac{d^2 u}{d\xi^2} + \{ G'(0) + \Psi \} u = 0 \tag{28b}$$

For $\xi > 0$, equation 28b possesses the solutions

$$u = C_1 \sin [G'(0) + \Psi]^{1/2} \xi + C_2 \cos [G'(0) + \Psi]^{1/2} \xi$$

Continuity of $f(\xi)$ and $\frac{df(\xi)}{d\xi}$ at $\xi = 0$ can, of course, always be guaranteed since there will exist two further constants in the solution for $\xi < 0$. For purposes of stability it will suffice to consider only $\xi < 0$.

Now, assume $G(Y_s)$ is a nonlinear function of Y_s. Then, for $\xi < 0$, since $Y_s(\xi)$ is a periodic function of ξ with period $\Omega(P)$, $G'(Y_s(\xi))$ will be a periodic function with period $\Omega^*(P)$ and can be written as

$$G'[Y_s(\xi)] = \Lambda(P) + \epsilon(P) q(\xi, P) \tag{29}$$

where $\Lambda(P)$ is the average value of $G'(Y_s)$ over one period, i.e., $q(\xi, P)$ is assumed to have zero average value over one period. Substitution of equation 29 into 28a yields

$$\frac{d^2 u}{d\xi^2} + [\delta(P, \Psi) + \epsilon(P) q(\xi, P)] u = 0, \quad \xi < 0 \tag{30}$$

where $\delta(P,\psi) = \Lambda(P) + \psi$.

Now, consider P as a fixed quantity in equation 30. Then, Floquet's theory indicates equation 30 possesses normal solutions of the form

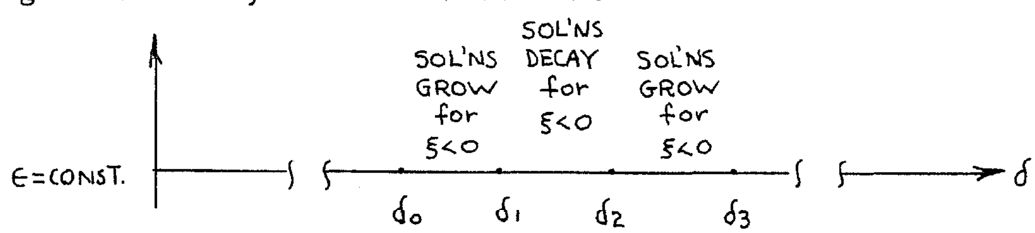
$$u_1(\xi) = e^{\nu_1 \xi}\phi_1(\xi) \; ; \; u_2(\xi) = e^{\nu_2 \xi}\phi_2(\xi) \; ; \; \nu_1 \neq \nu_2 \qquad (31)$$

where $\phi_1(\xi)$ and $\phi_2(\xi)$ are periodic functions of period $\bar{\Omega}$ and ν is in general complex valued. If $\nu_1 = \nu_2 = \nu$, the solutions have the form

$$u_1(\xi) = e^{\nu \xi}\phi_1(\xi) \; ; \; u_2(\xi) = e^{\nu \xi}\left[\frac{a\xi}{\sigma\bar{\Omega}}\phi_1(\xi) + \psi_1(\xi)\right] \qquad (32)$$

where $\phi_1(\xi)$ and $\psi_1(\xi)$ have period $\bar{\Omega}$. Clearly if bounded solutions are to exist as $\xi \to -\infty$, $\mathrm{Re}\,\nu_i > 0$ (and $a = 0$ in 32).

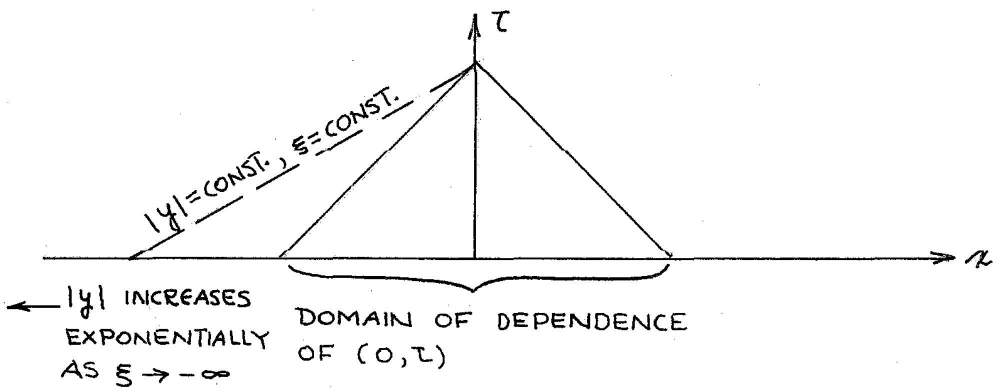
As Stoker (45) has indicated, O. Haupt (46) has shown that, for each fixed ϵ, there exists an infinite set, δ_i, of isolated values of δ in a δ,ϵ plane which separate solutions with $\mathrm{Re}\,\nu_i > 0$ from those with $\mathrm{Re}\,\nu_i < 0$; they are bounded on the negative side of the δ-axis but unbounded on the positive side. Upon traversing from left to right along the δ-axis, one encounters regions where the solutions 31 or 32 grow and decay as illustrated below.



Let us choose a positive ψ, therefore, such that $\delta(P,\psi)$ lies in a region where the solutions grow exponentially as $\xi \to -\infty$. In the x,t space the separated solution can be written

$$y(x,t) = e^{\sigma\frac{(vx-t)}{v^2-1}} f\left(\sigma; \frac{x-vt}{\sqrt{v^2-1}}\right) \quad (33)$$

Since ψ is positive, σ is pure imaginary. Although equation 27 has unbounded initial values for this case, as P. Ungar discussed in (39), an inspection of the geometry indicates the Riemann function of the equation must grow exponentially. To see this, observe the figure below. From equation 23 we see that $|y|$ grows exponentially on the



x-axis as $x \to -\infty$. However, $|y|$ is constant along lines of $\xi = \frac{x-vt}{\sqrt{v^2-1}}$ = constant. Considering $y(0,t)$, the line $|y|$ = constant intersects the x-axis to the left of the domain of dependence of $y(0,t)$. Thus $y(0,t)$ grows faster than the largest initial value in its domain of dependence and one must conclude that the Riemann function of the equation is growing exponentially in time. (The details of a precise proof can be found in Ungar's paper.) We therefore conclude that $Y_s(\xi)$ is unstable for $V > c$.

6. <u>The Analysis for $V < c$</u>

Under the transformation

$$x - v\tau = \xi\sqrt{1 - v^2/G'(0)}, \quad \tau = T/G'(0)$$

equation 22 becomes

$$Y_{\xi\xi} + 2\alpha Y_{\xi T} - Y_{TT} - g(Y) = -\frac{P}{2}\delta(\xi) \quad (34)$$

where $\alpha = \dfrac{v}{\sqrt{1-v^2}}$ $\quad P_2 = P_1\left[\dfrac{G'(0)}{1-v^2}\right]^{1/2}$

$$g(Y) = \frac{G(Y)}{G'(0)}$$

Steady-State Motions

Setting $\partial/\partial\tau = 0$ in 34 one obtains

$$\frac{d^2 Y_s}{d\xi^2} - g[Y_s(\xi)] = -P\delta(\xi) \tag{35}$$

We will consider only those solutions that satisfy

$$Y_s(\pm\infty) = 0 \tag{36}$$

Noting from the symmetry of the problem that

$$Y_s(\xi) = Y_s(-\xi) \tag{37}$$

the solution to 35 in the interval $0 < \xi < \infty$ is the solution of the boundary value problem

$$\frac{d^2 Y_s}{d\xi^2} - g[Y_s(\xi)] = 0 \qquad 0 < \xi < \infty \tag{38a}$$

$$\left.\frac{dY_s}{d\xi}\right|_{\xi=0^+} = -\frac{P}{2}, \quad Y(\infty) = 0 \tag{38b}$$

where the slope condition at $\xi = 0^+$ was obtained by integrating 35 across the origin, requiring that $Y_s(\xi)$ be continuous, and by use of 36, where the solution is obtained for $-\infty < \xi < 0$.

In general $Y_s(\xi)$ is symmetric with respect to ξ, has a maximum value at $\xi = 0$, and decreases monotonically to zero as $|\xi| \to \infty$. This can easily be seen as follows. Equation 38a can be integrated once to give

$$\frac{dY_s}{d\xi} = \pm \left[2 \int g(Y_s)\, dY_s + c_1 \right]^{1/2} \tag{39}$$

A further integration yields

$$\int \frac{dY_s}{\sqrt{2 \int g(Y_s)\, dY_s + c_1}} = \pm \xi + c_2 \tag{40}$$

From equation 38b, $dY_s/d\xi < 0$ for $\xi = 0^+$ and therefore only the negative sign in 39 and 40 can be accepted for $\xi > 0$. We in turn must require the positive sign for $\xi < 0$. Since $Y_s = 0$ implies $G(Y_s) = 0$ it is clear from 40 that c_1 must be set equal to zero to satisfy $Y_s(\pm\infty) = 0$. Then, requiring

$$\int g(Y_s)\, dY_s \geqslant 0$$

so that a steady-state solution satisfying 36 exists, we observe from 39 with $c_1 = 0$ that $dY_s/d\xi$ is necessarily of one sign and $dY_s/d\xi = 0$ when $Y_s = 0$. Since $Y_s(+\xi) = Y_s(-\xi)$, the only possibility of a maximum in $-\infty < \xi < \infty$ is at $\xi = 0$ and $Y_s(\xi)$ must decrease monotonically to zero as $|\xi| \rightarrow \infty$.

Stability

The variational equation for $V < c$ has the form

$$y_{\xi\xi} + 2\alpha y_{\xi\tau} - y_{\tau\tau} - g'[Y_s(\xi)]y = 0 \tag{41}$$

as referred to the y-ξ coordinates.

Let us now Laplace transform equation 41 with respect to τ. Denoting the Laplace transform of $y(\xi, \tau)$ as

$$\bar{y}(\xi, p) = \int_0^\infty e^{-p\tau}\, y(\xi, \tau)\, d\tau$$

and transforming 41 one obtains the subsidiary equation

$$\frac{d^2\bar{y}}{d\xi^2} + 2\alpha p \frac{d\bar{y}}{d\xi} - [p^2 + g'(Y_s)]\bar{y} = p\, y(\xi, 0) + y_\tau(\xi, 0) - 2\alpha y_\xi(\xi, 0) \tag{42}$$

Assuming the initial conditions are confined to a finite interval of the ξ or x axis, we have, since the variational equation is hyperbolic:

$$y(\infty, \tau) = 0$$
$$y(-\infty, \tau) = 0$$

whereby we obtain

$$\bar{y}(\infty, p) = 0, \quad \bar{y}(-\infty, p) = 0 \tag{43}$$

The initial disturbance will consist of a localized initial velocity at $\xi = 0$ (at the point of load application), i.e., we shall set

$$y(\xi, 0) = 0, \quad y_\tau(\xi, 0) = \delta(\xi)$$

and consider:

$$\frac{d^2\bar{y}}{d\xi^2} + 2\alpha p \frac{d\bar{y}}{d\xi} - [p^2 + g'(Y_s)]\bar{y} = \delta(\xi) \tag{44}$$

WHERE $\bar{y}(\pm\infty, p) = 0$, AND $Y_s = Y_s(\xi)$.

With the substitution

$$\bar{y}(\xi, p) = e^{-\alpha p \xi}\, \bar{u}(\xi, p)$$

equation 44 becomes

$$\frac{d^2\bar{u}}{d\xi^2} - [p^2(1+\alpha^2) + g'(Y_s)]\bar{u} = e^{-\alpha p \xi}\delta(\xi)$$

$$\bar{u}(\infty, p) = 0 \tag{45}$$

The solution to 45 in the interval $0 < \xi < \infty$ is the solution to the boundary value problem

$$\frac{d^2\bar{u}}{d\xi^2} - [p^2(1+\alpha^2) + g'(Y_s)]\bar{u} = 0 \tag{46a}$$

$$\left.\frac{d\bar{u}}{d\xi}\right|_{\xi=0^+} = \frac{1}{2}, \quad \bar{u}(\infty, p) = 0 \qquad 0 < \xi < \infty \tag{46b}$$

In $-\infty < \xi < 0$, $\bar{u}$ is obtained from $\bar{u}(-\xi, p) = \bar{u}(\xi, p)$. For $\nu < 1$, $g'(Y_s(\xi)) \to 1$ as $\xi \to \infty$ and therefore there exist solutions of 46a with the asymptotic property (47)

$$\bar{u}_1 \sim A(p)\, e^{\sqrt{p^2(1+\alpha^2)+1}\,\xi}, \quad \bar{u}_2 \sim B(p)\, e^{-\sqrt{p^2(1+\alpha^2)+1}\,\xi}$$

Therefore the quiescent condition $\bar{u}(\infty, p) = 0$ can always be satisfied by setting $A = 0$. The constant $B(p)$ is determined by the slope condition.

Since p enters 46 only in the combination $p\sqrt{1+\alpha^2}$, the solution to 46 will be of the form

$$\bar{y}(\xi, p) = e^{-\alpha p \xi}\,\bar{u}(p\sqrt{1+\alpha^2}, \xi)$$

From the shift property of the transform, if

$$\mathcal{L}\{u(\xi, \tau)\} = \bar{u}(\xi, p)$$

then

$$\mathcal{L}^{-1}\{e^{-\alpha p \xi}\,\bar{u}(\xi, p)\} = u[\xi, \tau - \alpha\xi]\,H(\tau - \alpha\xi)$$

where $H(t-\alpha\xi)$ is the Heaviside step function. Further, by the similarity rule:

$$\mathcal{L}^{-1}\{\bar{u}(p\sqrt{1+\alpha^2}, \xi)\} = \frac{1}{\sqrt{1+\alpha^2}}\, u\left(\xi, \frac{\tau}{\sqrt{1+\alpha^2}}\right)$$

Therefore the solution of 46 will have the form

$$y(\xi,\tau) = \frac{1}{\sqrt{1+\alpha^2}}\, u\left[\xi, \frac{\tau-\alpha\xi}{\sqrt{1+\alpha^2}}\right] H(\tau-\alpha\xi) \tag{47}$$

Equation 47 indicates that the boundedness of $y(\xi,\tau)$ depends only on the boundedness of $u(\xi,\tau)$. Therefore one concludes that if the solutions of the variational equation, 34, (satisfying our initial conditions) are bounded with the term $2\alpha y_{\xi\tau}$ missing, the solutions with this term included are also bounded, i.e., for purposes of determining stability, this term of the variational equation can be neglected. Consider next

$$y_{\xi\xi} - y_{\tau\tau} - g'(Y_s)y = 0 \tag{48}$$

One can construct the following energy integral for this equation:

$$E(\tau) = \int_{-\infty}^{\infty} \left[(y_\xi)^2 + (y_\tau)^2 + g'(Y_s)\, y^2 \right] d\xi$$

whereby $E'(\tau) = 0$ is obtained by use of equation 48 and $y(\pm\infty, \tau) = 0$. Thus E is a constant and therefore if $g'(Y_s) > 0$, $-\infty < \xi < \infty$, then $y(\xi, \tau)$ is bounded and the system is stable.

If the foundation is of the type for which $g'(Y_s) < 0$ for $Y_s >$ positive constant then the system will be unstable for those values of P and M such that $g(Y_o) < 0$ where $Y_o = Y_s(\xi = 0)$. To see this consider the potential energy of the steady-state wave form itself. Choosing Y_o as the generalized coordinate describing the potential energy, V, we have

$$V(Y_o, P) = E(Y_o) - PY_o \tag{49}$$

where

$$E(Y_o) = \int_{-\infty}^{\infty} \left[\frac{1}{2} Y_s^2(Y_o, \xi) + \int_0^{Y_s} g(\eta)\, d\eta \right] d\xi$$

From 49 one obtains

$$\frac{\partial^2 V}{\partial Y_o^2} = \frac{d^2 E}{dY_o^2}$$

however since the system is in equilibrium, $\frac{\partial V}{\partial Y_o} = 0$, whence

$$P = \frac{d\,E(Y_o)}{d\,Y_o}$$

and therefore

$$\frac{\partial^2 V}{\partial Y_o^2} = \frac{dP}{dY_o}$$

By use of equation 39 we obtain

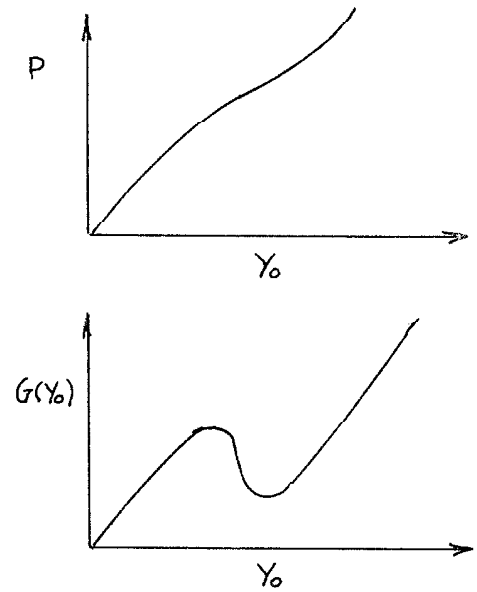
$$\frac{dP}{dY_o} = \frac{2\,g(Y_o)}{P} \tag{50}$$

and thus

$$\frac{\partial^2 V}{\partial Y_o^2} = \frac{2\,g(Y_o)}{P} \tag{51}$$

If $P > 0$, and $g(Y_o) < 0$, $\frac{\partial^2 V}{\partial Y_o^2} < 0$ and the system is unstable. If $g(Y_o) = 0$, then $Y_s(\xi)$ is unstable if the first non zero derivative, $d^n g(Y_o)/dY_o^n < 0$.

The relationship between P and Y_o given by equation 50 brings forth an interesting point. Suppose one considers a foundation that, by itself, exhibits a "snap-thru" characteristic. For $g(Y_o) > 0$ equation 50 indicates $dP/dY_o > 0$. Thus if the restriction



$G(Y) > 0$, $Y > 0$ is made, the string and foundation combination under a concentrated load cannot exhibit such a snap behavior. This can be shown to be a peculiarity of the concentrated load case.

Any further investigation stability for $V > c$ apparently requires consideration of the variational equation in detail. Unfortunately, since this equation possesses variable coefficients, it is difficult to make further generalities. We will therefore consider a few specific cases.

The Cubic Foundation

Let the foundation be described by

$$G(Y) = aY + bY^3 \qquad a > 0 \tag{52}$$

By virtue of equations 38, the steady-state solution is obtained as

$$Y_s(\xi) = \frac{2e^{-|\xi|}}{\gamma\left[1 - \frac{\lambda^2}{\gamma^2} e^{-2|\xi|}\right]} \qquad -\infty < \xi < \infty \tag{53}$$

where $\gamma = \dfrac{1 + \sqrt{1 + \lambda^2 Y_o^2}}{Y_o}$, $\lambda^2 = b/2a$

A quadratic equation for Y_o^2 is determined from the slope condition 38b as

$$\left(\frac{1}{Y_o^2}\right)^2 - \left(\frac{2}{P}\right)^2\left(\frac{1}{Y_o}\right)^2 - \lambda^2\left(\frac{2}{P}\right)^2 = 0$$

where $P = P_1\sqrt{a/(1-v^2)}$ from which we obtain the roots

$$Y_{o_i} = \frac{P}{\sqrt{2}}\left[1 + (-1)^{1+i}\sqrt{1+\lambda^2 P^2}\right]^{-\frac{1}{2}} \qquad i = 1, 2 \tag{54}$$

(Two roots are extraneous)

Having determined Y_o from 54, the parameter, λ, can be evaluated and the solution is complete. Now consider the following cases:

<u>Case i.</u> Suppose $b > 0$ in equation 52. Then $G(Y)$ is a "hardening" foundation and $\lambda^2 > 0$. Since Y_o must be real valued, we can accept only the root Y_{o_1} of equation 54. It can be shown that $(\frac{\lambda}{\gamma})^2 < 1$ for $P < \infty$ in equation 53 and thus the solution is well behaved.

<u>Case ii.</u> Suppose $b < 0$. Then $G(Y)$ is a "softening" foundation and $\lambda^2 < 0$. In this case both roots, Y_{o_i}, are acceptable. Their meaning is as follows: since Y_o and γ must be real valued quantities, we see that

$$1 + \lambda^2 Y_o^2 > 0, \qquad 1 + \lambda^2 P^2 > 0$$

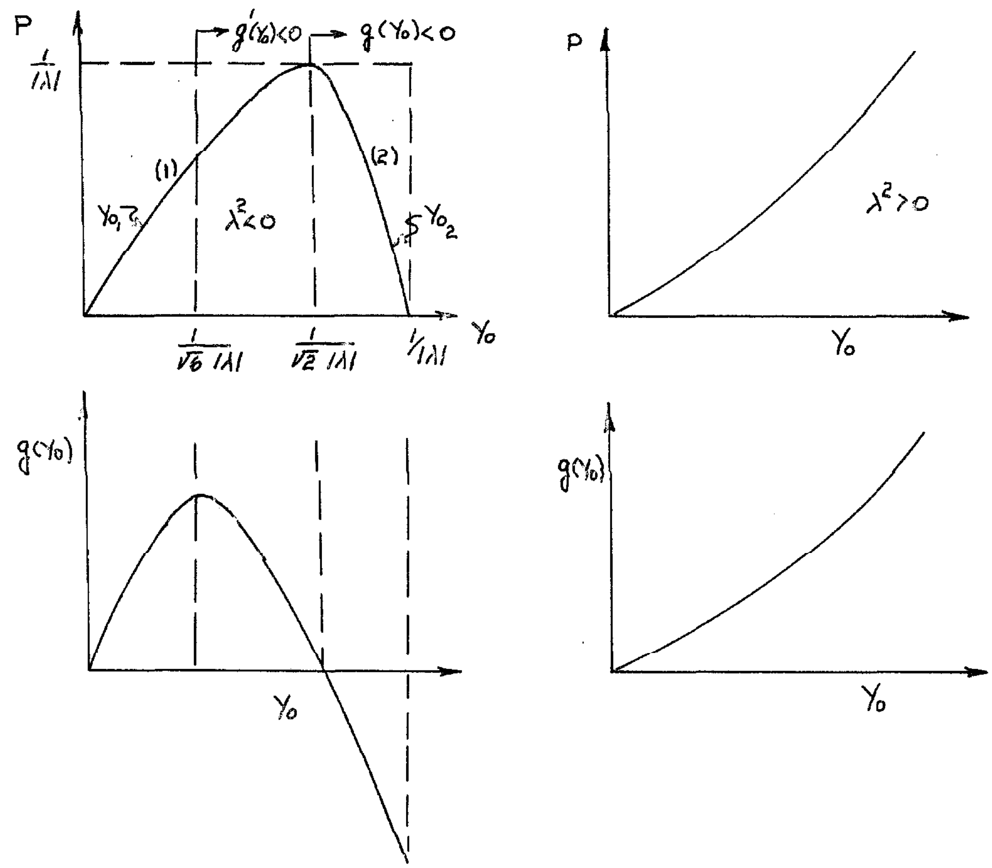
Therefore if $\lambda^2 < 0$, $P \leq 1/|\lambda|$, i.e., for loads above this magnitude solutions of the form 53 do not exist. Substituting $P_{max} = 1/|\lambda|$ into equation 54 one obtains $Y_{o_{1max}} = \frac{1}{\sqrt{2}\,|\lambda|}$. Considering Y_{o_2} we see $Y_{o_{2min}} = \frac{1}{\sqrt{2}\,|\lambda|} = Y_{o_{1max}}$. From 54 we note that for real γ we must require that $Y_{o_2} \leq 1/|\lambda|$. A sketch of P vs. Y_o for $\lambda^2 < 0$ is shown below along with the foundation behavior at $\xi = 0$.

<u>Stability</u>

If $\lambda^2 > 0$, $G'(Y) > 0$, $Y > 0$ and the system is stable.

If $\lambda^2 < 0$, the modified variational equation 48 is of the form

$$y_{\xi\xi} - y_{zz} - \left[1 - \frac{24\epsilon\, e^{-2|\xi|}}{(1+\epsilon\, e^{-2|\xi|})^2}\right] y = 0 \tag{56}$$



where $\epsilon = |\lambda^2/\gamma^2|$.

Laplace transforming equation 56 with respect to τ and seeking the Green's function of the subsidiary equations, as discussed previously, we obtain the boundary value problem

$$\frac{d^2\bar{y}}{d\xi^2} - \left[(p^2+1) - \frac{24\,\epsilon\, e^{-2\xi}}{(1+\epsilon\, e^{-2\xi})^2}\right]\bar{y} = 0 \qquad (57a)$$

$$0 < \xi < \infty$$

$$\left.\frac{d\bar{y}}{d\xi}\right|_{\xi=0^+} = \frac{1}{2}, \qquad \bar{y}(\infty, p) = 0 \qquad (57b)$$

where for $-\infty < \xi < 0$ $\bar{y}(\xi, p)$ is obtained from

$$\bar{y}(-\xi, p) = \bar{y}(\xi, p) \qquad (57c)$$

By the change of variables

$$\eta = -\epsilon e^{-2\xi}, \quad \bar{y} = e^{-\sqrt{p^2+1}\,\xi}(1-\eta)^{-2}V(\eta) \quad (58)$$

equation 57a becomes

$$\eta(1-\eta)V_{\eta\eta} + [c_1 - (a_1+b_1)\eta]V_\eta - a_1 b_1 V = 0 \quad (59)$$

where $a_1 = -2$, $b_1 = \sqrt{p^2+1} - 2$, $c_1 = \sqrt{p^2+1} + 1$.

Equation 59 is the standard form of the hypergeometric equation. The desired solution to 58 is

$$V(\eta) = F(-2, \sqrt{p^2+1} - 2 ; \sqrt{p^2+1} + 1 ; \eta) \quad (60)$$

where F(a,b;c;n) denotes the hypergeometric function. The particular hypergeometric function 60 can be represented in terms of elementary functions by the identity*

$$\frac{d^m}{d\eta^m}\left[\eta^{m+c-1}(1-\eta)^{b-c}\right] = \text{CONST.}\ \eta^{c-1}(1-\eta)^{b-c-m} F(-m,b;c;\eta)$$

from which we obtain

$$F(-2,b;c;\eta) = \text{CONST.}\left[1 + 2\epsilon\frac{(\rho-2)}{(\rho+1)}e^{-2|\xi|} + \epsilon^2\frac{(\rho-1)(\rho-2)}{(\rho+1)(\rho+2)}e^{-4|\xi|}\right]$$

Our solution can now be written

$$\bar{y}(\xi,p) = \frac{A(p)e^{-\rho|\xi|}}{(1+\epsilon e^{-2|\xi|})^2}\left[1 + 2\epsilon\frac{(\rho-2)}{(\rho+1)}e^{-2|\xi|} + \epsilon^2\frac{(\rho-1)(\rho-2)}{(\rho+1)(\rho+2)}e^{-4|\xi|}\right] \quad (61)$$

---

* See (48), page 113.

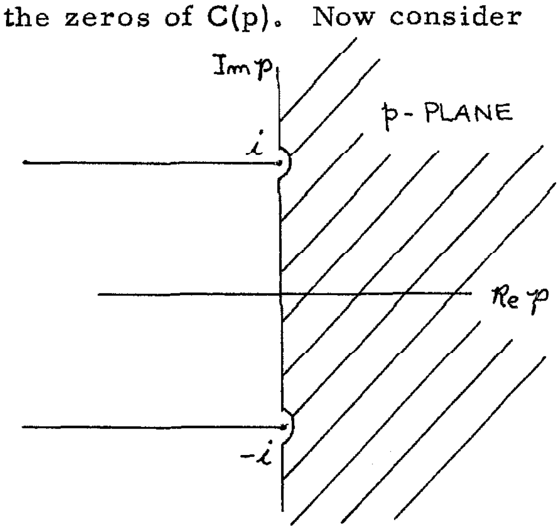
where $\rho = \sqrt{p^2+1}$ and A(p) is determined from the slope condition, 57b, as

$$2A(p) = \frac{(\rho+1)(\rho+2)(1+\epsilon)^2}{\left[(\rho+1)(\rho+2)+2\epsilon(\rho-2)(\rho+2)+\epsilon^2(\rho-1)(\rho-2)\right]\left[\frac{4\epsilon}{1+\epsilon}-\rho\right]-4\epsilon(\rho-2)\left[(\rho+2)-\epsilon(\rho-1)\right]} \qquad (62)$$

From the properties of the Laplace transform (see Appendix 3) $y(\xi,\tau)$ is unbounded along a line of constant ξ only if poles of A(p) of order greater than one appear on the imaginary axis or if poles of any order appear in the right half of the p-plane. Next we show, by a method similar to that of Tranter (49), that the poles of A(p) in Re $p > 0$ can lie only on the real axis of the p-plane and therefore only the real p axis and the imaginary axis need be investigated. Let

$$A(p) = B(p)/C(p) \qquad (63)$$

where B(p) and C(p) are respectively the numerator and denominator of 62. Then the poles of A(p) are the zeros of C(p). Now consider the region R comprising the right half plane (Re $p > 0$) but circumventing the branch points $p = \pm i$ as shown below, where we have provided branch cuts to render ρ an analytic function of p in R.



In R

(i) c(p) and r(p) are analytic

(ii) c(p) and r(p) are real when p is real

Since the region R is symmetric with respect to the real axis of the p-plane we have, by the reflection principle:

$$c(\bar{p}) = \bar{c}(p) \quad , \quad r(\bar{p}) = \bar{r}(p) \quad , \quad p \in R$$

and the roots of c(p) = 0 must therefore be complex conjugates.

Now, let:

$$\bar{r}(\xi, p) = c^2(p)\, \bar{y}(\xi, p) \tag{64}$$

Then $\bar{r}$ (ξ, p) satisfies

$$\frac{d^2\bar{r}(\xi,p)}{d\xi^2} - \left[p^2 - g'(Y_S) \right] \bar{r}(\xi,p) = 0 \tag{65a}$$

with boundary conditions of

$$\begin{aligned} \bar{r}\,(0,p) &= B(p)\,c(p)\,D(p) \\ \bar{r}\,(\infty,p) &= 0 \quad , \quad \mathcal{R}e\, p > 0 \end{aligned} \tag{65b}$$

where $\qquad D(p) = \left[1 + 2\epsilon \frac{(p-2)}{p+1} + \epsilon^2 \frac{(p-1)(p-2)}{(p+1)(p+2)} \right] (1+\epsilon)^{-2}$

Let $\bar{r}_1$ correspond to $p = p_1$

Let r_2 correspond to $p = p_2$, $p_1, p_2 \in R$

Multiplying equation 65a for $\bar{r}_1$ by $\bar{r}_2$ and that for $\bar{r}_2$ by $\bar{r}_1$ and integrating over the interval 0 to ∞ yields

$$\int_0^\infty \bar{r}_2 \frac{d}{d\xi}\left\{ \frac{d\bar{r}_1}{d\xi} \right\} d\xi - \int_0^\infty \left[p_1^2 - g' \right] \bar{r}_1 \bar{r}_2 \, d\xi = 0 \tag{66a}$$

$$\int_0^\infty \bar{\kappa}_1 \frac{d}{d\xi}\left[\frac{d\bar{\kappa}_2}{d\xi}\right] d\xi - \int_0^\infty \left[p_2^2 - g'(\xi)\right] \bar{\kappa}_1 \bar{\kappa}_2 \, d\xi = 0 \qquad (66b)$$

Subtracting 66a from 66b, and following an integration by parts and application of 65b one obtains

$$B(p_2)C(p_2)D(p_2) \left.\frac{d\bar{\kappa}_2}{d\xi}\right|_{\xi=0^+} - B(p_1)C(p_1)D(p_1) \left.\frac{d\bar{\kappa}_1}{d\xi}\right|_{\xi=0^+}$$

$$- (p_1^2 - p_2^2) \int_0^\infty \bar{\kappa}_1 \bar{\kappa}_2 \, d\xi = 0$$

Now let p_1 and p_2 be roots of $C(p_i) = 0$. They must therefore be complex conjugates and we obtain

$$(p_1^2 - p_2^2) \int_0^\infty |\kappa|^2 \, d\xi = 0 \qquad (67)$$

representing a contradiction unless $p_1^2 - p_2^2 = 0$, which can occur only on the real axis for $\mathrm{Re}\,p > 0$. Therefore the poles of $A(p)$ can lie only on the real p-axis in $\mathrm{Re}\,p > 0$.

An investigation of the zeros of $C(p)$ on the two axes indicates (i) poles occur in $\mathrm{Re}\,p > 0$ if $\epsilon > \epsilon_{cr}$, (ii) a second order pole occurs at $p = 0$ if $\epsilon = \epsilon_{cr}$, and (iii) no poles occur in $\mathrm{Re}\,p > 0$ and those on the imaginary axis are of first order if $\epsilon < \epsilon_{cr}$, where

$$\epsilon_{cr} = 3 - 2\sqrt{2}$$

assuming* that boundedness along a line of constant ξ in the ψ, ξ plane implies boundedness throughout the entire plane we conclude that the system is

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* See page 125

(1) stable if $\epsilon > \epsilon_{cr}$

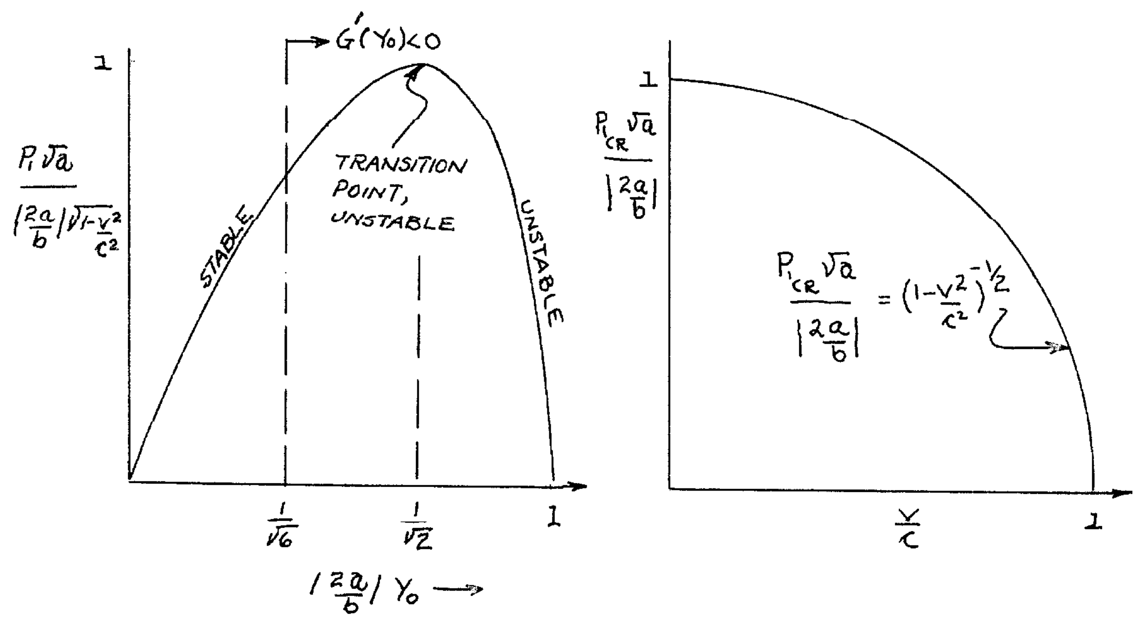
(2) unstable if $\epsilon \leq \epsilon_{cr}$

The point ϵ_{cr} represents the transition from stability to instability. If one considers the plot of P vs. Y_o, it represents the transition from positive to negative slope.

In terms of physical quantities, if $G(Y) = aY + bY^3$, $a > 0$, and $F(X,T) = P_1\delta(X - VT)$, then one concludes that

(1) The steady-state motions are stable for $V < c$ if $b > 0$.

(2) If $b < 0$, then stability is best illustrated by the figures below:



Behavior at the Point of Application of the Load

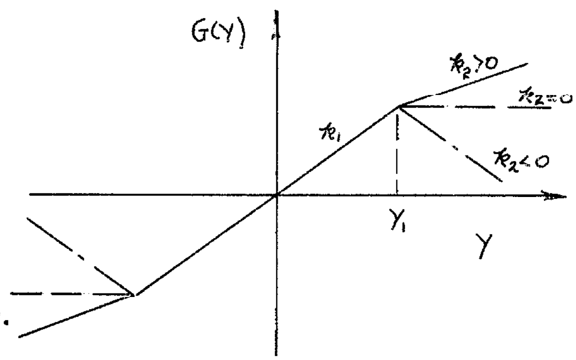
Critical Load Versus Load Velocity, V

Note that $G'(Y_s(0)) < 0$ on a portion of the stable curve (compare the above figures with those on page 117). This is a peculiarity of the concentrated load case. For any finitely distributed load, the system can be shown to be unstable, with the above foundation somewhat above the $G(Y_s(0)) = 0$ point. For any finitely distributed load, the transition point will occur in $1/\sqrt{6} \leq |\frac{2a}{b}|\ Y_s(0) \leq 1/\sqrt{2}$.

Note on a Bilinear Foundation

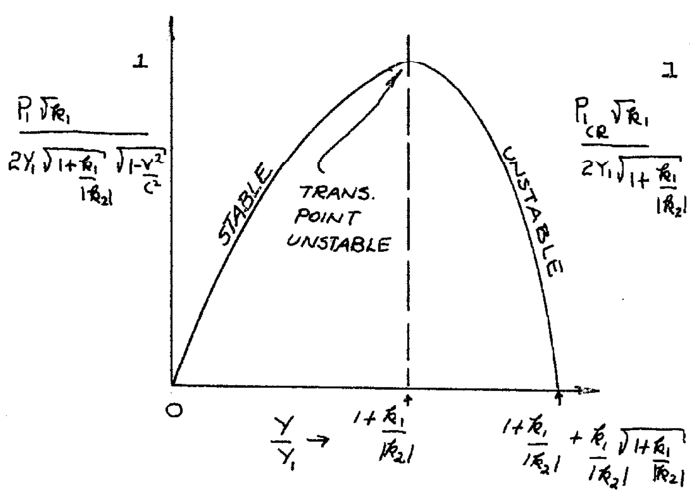
Suppose
$$G(Y) = k_1 Y, \quad |Y| < Y_1$$
$$= k_1 Y_1 + k_2 (Y - Y_1), \quad |Y| > Y_1$$

where $k_1, Y_1 > 0$. Then $G(Y)$ represents a bilinear foundation and is illustrated below:

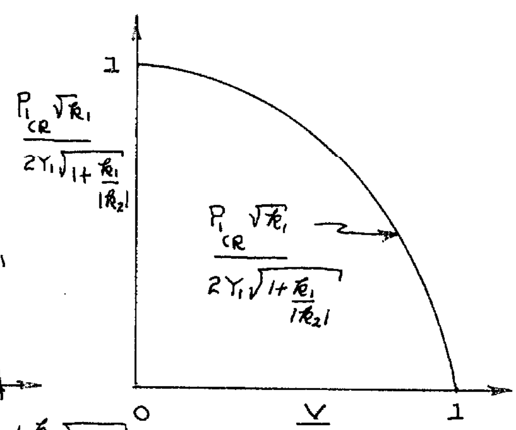


By a similar analysis as was conducted for the case of the cubic foundation, one can demonstrate that the steady-state motions are stable if $k_2 > 0$.

If $k_2 < 0$, the stability picture is as illustrated below:



Behavior at the Point of Application of the Load, (Bilinear Foundation)



Critical Load Versus Load Velocity, V (Bilinear Foundation)

The two systems, cubic and bilinear, are seen to behave in a similar manner. Both are stable if $G(Y_s(0)) > 0$. The unstable branches of the curves would, of course, not be obtainable as a limit ($\tau \to \infty$) of a transient problem.

If $k_2 > 0$, the steady-state motions for the bilinear case are given by

$$\begin{cases} Y(\xi) = e^{-|\xi| + \xi_0} & |\xi| > \xi_0 \\ \quad = -\frac{P}{2\phi}\sinh\phi|\xi| + \left\{\left[\frac{1}{\phi^2}\operatorname{sech}\phi\xi_0 + \frac{P}{2\phi}\tanh\phi\xi_0\right]\cosh\phi\xi\right\} + \left(1 - \frac{1}{\phi^2}\right) & \text{for } |\xi| < \xi_0 \end{cases}$$

where $\phi^2 = k_2/k_1$

$$\phi\xi_0 = \sinh^{-1}\left(\frac{\phi P}{2\sqrt{1-\phi^2}}\right) - \tanh^{-1}\phi, \qquad 0 < \phi^2 < 1$$

$$\phi\xi_0 = \cosh^{-1}\left(\frac{\phi P}{2\sqrt{1-\phi^2}}\right) - \tanh^{-1}\left(\frac{1}{\phi}\right), \qquad \phi^2 > 1$$

and if $k_2 = 0$

$$\begin{cases} Y(\xi) = e^{-|\xi| + \xi_0} & |\xi| > \xi_0 \\ \quad = \frac{\xi^2}{2} - \frac{P}{2}|\xi| + \left(1 - \frac{\xi_0^2}{2} + \frac{P\xi_0}{2}\right), & |\xi| < \xi_0 \end{cases}$$

where

$$\xi_0 = \frac{P}{2} - 1, \qquad \frac{P}{2} > 1$$

and finally, if $k_2 < 0$,

$$\begin{cases} Y(\xi) = e^{-|\xi| + \xi_0} & |\xi| > \xi_0 \\ \quad = -\frac{P}{2|\phi|}\sin|\phi\xi| \\ \qquad + \frac{\cos|\phi\xi|}{\cos|\phi\xi_0|}\left[\frac{P}{2|\phi|}\sin(\phi\xi_0) - \frac{1}{|\phi|^2}\right] \end{cases}$$

where $|\phi \xi_o| = \sin^{-1} \dfrac{|\phi| P}{2\sqrt{1+|\phi|^2}} - \tan^{-1} |\phi|$

Boundedness of $y(\xi, \tau)$ Along an Arbitrary Ray in the τ-ξ Plane

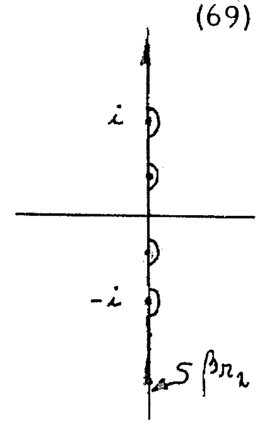
We assumed above that boundedness of $y(\xi, \tau)$ along a line of constant ξ in the τ, ξ plane implied boundedness of $y(\xi, \tau)$ over the entire τ-ξ plane, as $\tau \to \infty$. We show this to be the case next.

The t, x plane, or equivalently, the τ, ξ plane can be "covered" by arbitrary rays from the origin. Let us consider the τ, ξ plane. Assume that $\varepsilon < \varepsilon_{cr}$ (If $\varepsilon \geq \varepsilon_{cr}$ $y(\xi, \tau)$ is unbounded as $\tau \to \infty$ for ξ = constant and therefore the system is unstable.) Inverting the transform, $\bar{y}(\xi, \tau)$ by use of the Bromwich inversion integral we have

$$y(\xi, \tau) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{p\tau} \bar{y}(\xi, p)\, dp \tag{68}$$

Since $\varepsilon < \varepsilon_{cr}$, by theorem 1, Appendix 3, $\bar{y}(\xi, p)$ is regular in $\operatorname{Re} p > 0$. In addition, $\bar{y}(\xi, p)$ satisfies Jordan's lemma, and therefore 68 is equivalent to

$$y(\xi, \tau) = \frac{1}{2\pi i} \int_{\beta r_2} e^{p\tau} \bar{y}(\xi, p)\, dp \tag{69}$$



where βr_2 is a path up the imaginary axis, circumventing any singularities as shown below.

Consider the contribution to 69 from integrations on the paths between singularities. Each of these contributions is of the form

$$I_1 = \frac{1}{2\pi} \int_a^b e^{i\sigma\tau} y(\xi, \sigma)\, d\sigma \tag{70}$$

where a = $-\infty$, b = $\sigma_o - \delta$ for $\sigma = -\infty$ to the first singularity,

a = $\sigma_i + \delta$, b = $\sigma_{i+1} - \delta$ between singularities, and

a = $\sigma_\ell + \delta$, b = ∞ from the last singularity to $\sigma = \infty$.

Substituting $\overline{y}(\xi, p)$ from equation 61 into equation 70 and letting $|\xi| = b\tau$, $b > 0$, we obtain

$$I_2 = (1 + \varepsilon e^{-2b\tau})^2 I_1$$

$$= \frac{1}{2\pi} \int_a^b \left\{ A(\sigma) e^{i\sigma\tau - \sqrt{1-\sigma^2}\, b\tau} [1 + 2\varepsilon B^*(\sigma) e^{-2b\tau} + \varepsilon^2 c^*(\sigma) e^{-4b\tau}] \right\} d\sigma \tag{71}$$

where $A(\sigma)$, $B^*(\sigma)$ and $c^*(\sigma)$ are analytic on the path and their meaning should be clear from equation 61.

If $\sigma < 1$, then a and b are finite. Since the integrand is a continuous function of ξ and an analytic function of p on the interval (a, b), we have

$$\lim_{\tau \to \infty} I_2 = \frac{1}{2\pi} \int_a^b \lim_{\tau \to \infty} \{ \quad \} d\sigma = 0, \quad (\sigma < 1) \tag{72}$$

If a and b (or a segment of the σ interval) are such that $|\sigma| > 1$, then we have:

$$I_2 = \frac{1}{2\pi} \int_a^b \left\{ A(\sigma) e^{i\tau[\sigma - \sqrt{\sigma^2 - 1}\, b]} [1 + 2\varepsilon B^* e^{-2b\tau} + \varepsilon^2 c^* e^{-4b\tau}] \right\} d\sigma \tag{73}$$

Thus we must consider the asymptotic behavior of the following integrals

$$I_{2_1} = \frac{1}{2\pi}\int_a^b A(\sigma)\, e^{i\tau[\sigma - \sqrt{\sigma^2-1}\, b]}\, d\sigma$$

$$I_{2_2} = \frac{\varepsilon e^{-2b\tau}}{\pi}\int_a^b A(\sigma)\, B^*(\sigma)\, e^{i\tau[\sigma - \sqrt{\sigma^2-1}\, b]}\, d\sigma$$

$$I_{2_3} = \frac{\varepsilon^2 e^{-4b\tau}}{2\pi}\int_a^b A(\sigma)\, c^*(\sigma)\, e^{i\tau[\sigma - \sqrt{\sigma^2-1}\, b]}\, d\sigma$$

The above integrals are of a type for which the method of stationary phase (50) can be used to obtain the asymptotic approximations for large τ. Because of their similar form, it will suffice to consider only I_{2_1}.

Set $\psi(\sigma) = \Theta = \sigma - \sqrt{\sigma^2 - 1}\, b$ in I_{2_1}. This yields

$$I_{2_1} = \frac{1}{2\pi}\int_{a^*}^{b^*} \frac{A(\sigma)\, e^{i\tau\theta}}{\psi'(\sigma)}\, d\theta \tag{74}$$

On those intervals where $A(\sigma)/\psi'(\sigma)$ is of bounded variation, Riemann's lemma indicates the integral is $O(\tau^{-1})^*$; thus we need be concerned only with the contribution from those points where $\psi'(\sigma)$ vanishes on the path. Now $\psi(0) = 0$ when $\sigma = \pm 1/\sqrt{1-b^2}$ and since real roots occur only when $b < 1$, we need not consider $b \geq 1$. Note that we are considering the modified variational equation 48 which has characteristic directions, $d\tau/d\xi = \pm 1$. Therefore since the slope of an arbitrary ray is $d\tau/d\xi = \pm 1/b$, we need only consider $b < 1$.

---

*Since $[A(\sigma)/\psi'(\sigma)] \sim O(1/\sigma^n)$, $n > 1$ as $|\sigma| \to \infty$, the two infinite intervals can be handled exactly as in the proof of theorem 2, Appendix 3.

Let a point where $\psi'(\sigma) = 0$ be denoted by σ_o (saddle points of $\psi(\sigma)$). Then we have: $\psi''(\sigma_o) = \sigma_o(1-b^2)/(\sigma_o^2 - 1) \neq 0$. Since $\psi(\sigma)$ is analytic in the neighborhood of $\psi(\sigma_o)$ we can write:

$$\psi(\sigma) - \psi(\sigma_o) = \tfrac{1}{2}\,\psi''(\sigma_o)\,[\sigma-\sigma_o]^2$$

Set $\psi(\sigma) - \psi(\sigma_o) = u^2$ and consider the integral from $\sigma_o - \delta$ to $\sigma_o + \delta$:

$$I_\delta = \frac{1}{2\pi}\int_{\sigma_o-\delta}^{\sigma_o+\delta} \frac{A(\sigma)}{\psi'(\sigma)} \cdot 2u\, e^{i\tau[\psi(\sigma_o)+u^2]}\, du$$

Jeffries (50), page 506, has shown that I_δ has the asymptotic character:

$$\lim_{\tau\to\infty} I_\delta \sim O(\tau^{-1/2})$$

and thus $\lim_{\tau\to\infty} I_2 = 0$.

It is easy to show that the contributions to the inversion integral from either the first order poles (since $\epsilon < \epsilon_{cr}$) on the imaginary axis or the branch points at $\pm i$ are bounded as $\tau \to \infty$. One therefore concludes that along any ray $|\xi| = b\tau$, the solution $y(b\tau, \tau)$ is bounded as $\tau \to \infty$ if $\epsilon < \epsilon_{cr}$. Thus, a stability investigation along lines of constant ξ is sufficient.

7. Summary

In conclusion we have found that :

V > c:

(1) If $V > c$ and sgn $G(Y)$ = sgn Y the steady-state motions are zero ahead of the load and a periodic function of X-VT behind the load.

(2) Such motions are always unstable if $G(Y)$ is a nonlinear

function of Y. For the nonlinear case, they therefore cannot be considered as the limit ($\tau \to \infty$) case of a transient problem.

V < c:

(3) If $V < c$, and $\int_0^{Y_0} G(Y)dY > 0$ where $Y_o = Y_s(\xi = 0)$, then the steady-state motions are symmetric with respect to the load, have a maximum value directly under the load, and decrease monotonically on either side of the load.

(4) A sufficient condition for stability is that $G'(Y) > 0$, $|Y| > 0$.

(5) A sufficient condition for instability is that $G(Y_o) < 0$. If $G(Y_o) = 0$, then $Y_s(\xi)$ is unstable if the first non-zero derivative

$$\frac{d^m G(Y_o)}{dY_o^m} < 0$$

(6) For the two examples cited, $G(Y_o) > 0$ was both necessary and sufficient for stability. Reference is made to the figures contained on pages 122-3. In both cases the critical load, $P_{1_{cr}}$, i.e., the load magnitude indicating a transition from stability to instability, was inversely proportional to $\sqrt{1-v^2/c^2}$. Again, the unstable branches of the curves on page 122 cannot be expected as a limit case of a transient problem as $\tau \to \infty$. One can say, therefore, that the steady-state solutions have physical significance for $V > 0$, only if $P_1 < P_{1_{cr}}$.

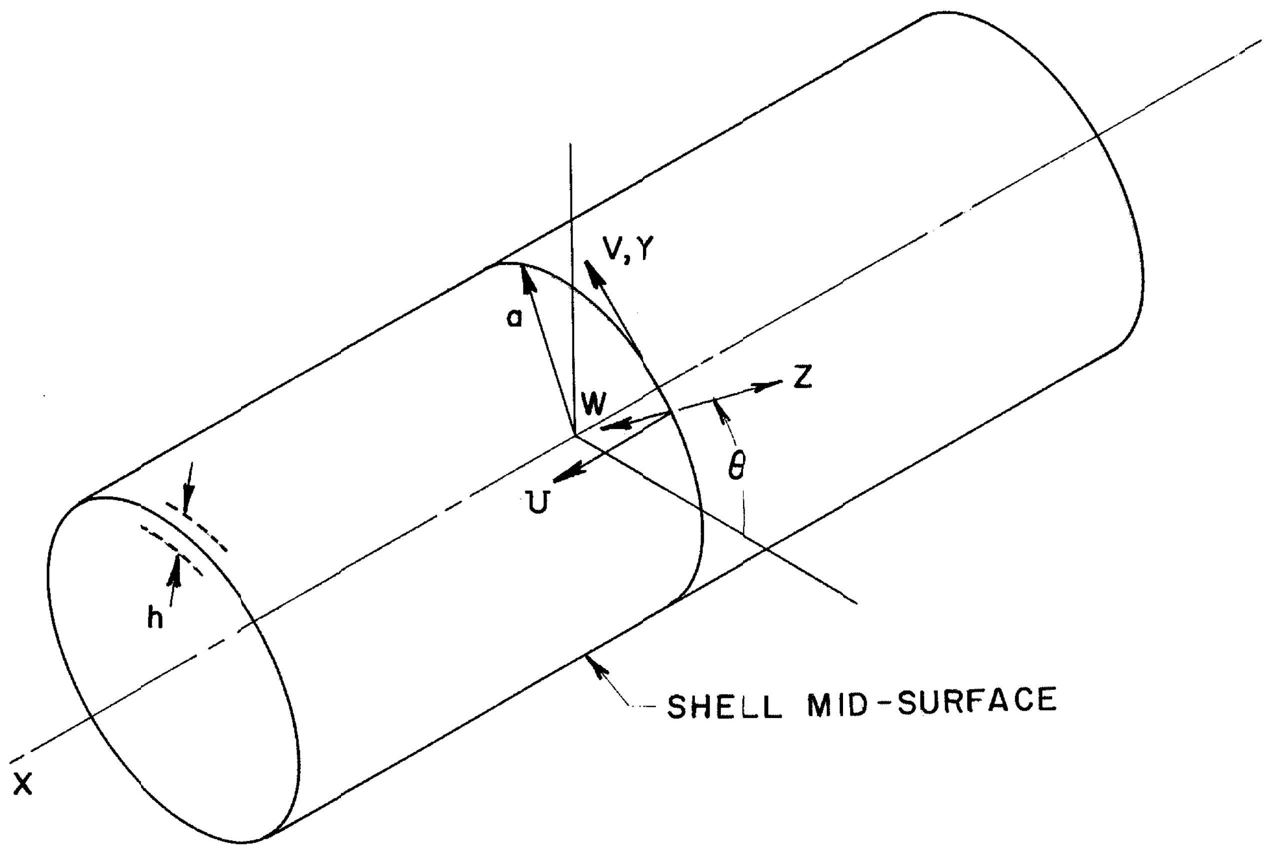


FIG. I COORDINATE SYSTEM

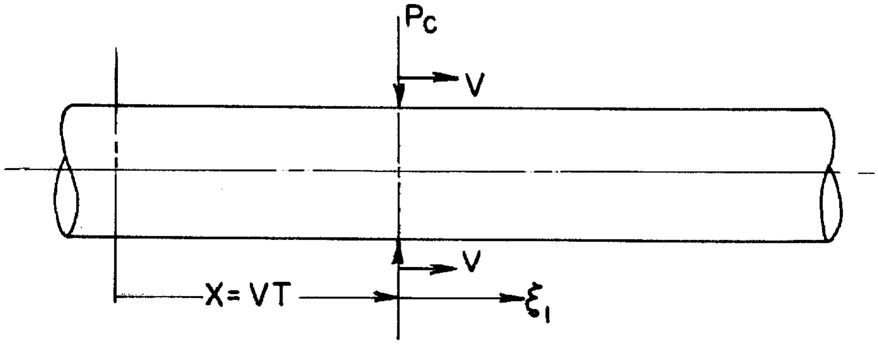


FIG. 2a MOVING CONCENTRATED LOAD : $P(\xi_1) = P_c\,\delta(\xi_1)$

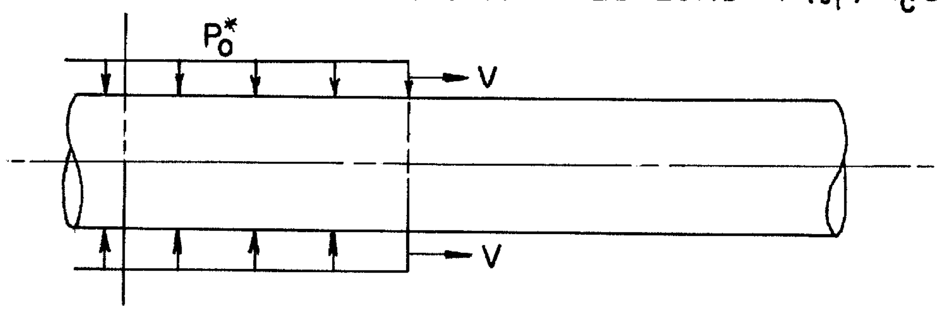


FIG. 2b MOVING STEP LOAD : $P(\xi_1) = P_0^*\,H(-\xi_1)$

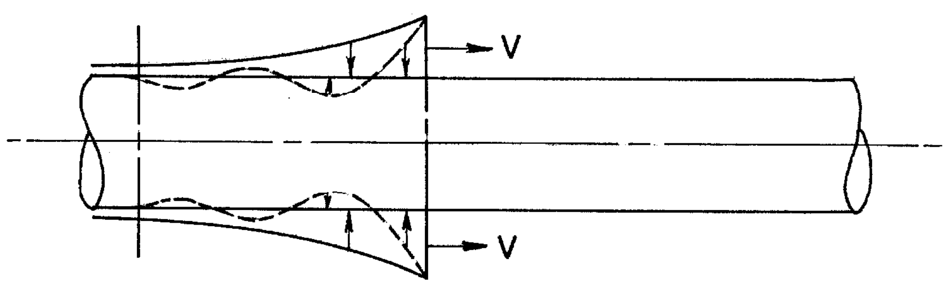


FIG. 2c MOVING DECAYED STEP LOADS: $P(\xi_1) = H(-\xi_1)\sum_{k=1}^{K} P_k^* e^{\Omega_k^* \xi_1}$

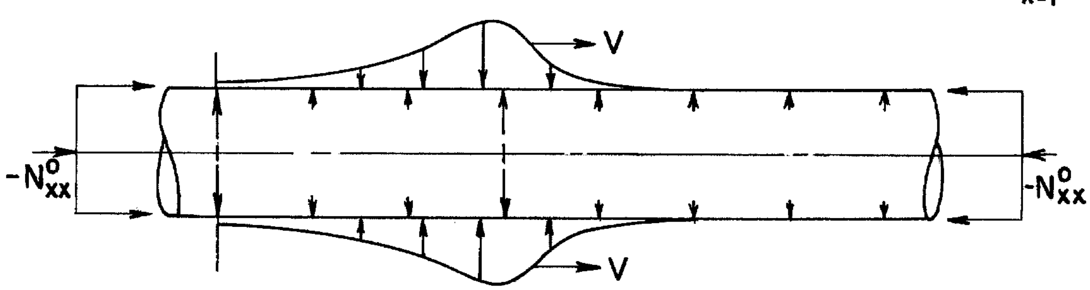


FIG. 2d GENERAL MOVING PULSE WITH INTERNAL PRESSURE AND AXIAL COMPRESSION:

$$P(\xi_1) = H(-\xi_1)\sum_{k=1}^{K} P_k^* e^{\Omega^*_k \xi_1} + H(\xi_1)\sum_{n=1}^{N} P_n\, e^{-\Omega_n \xi_1} - P_0$$

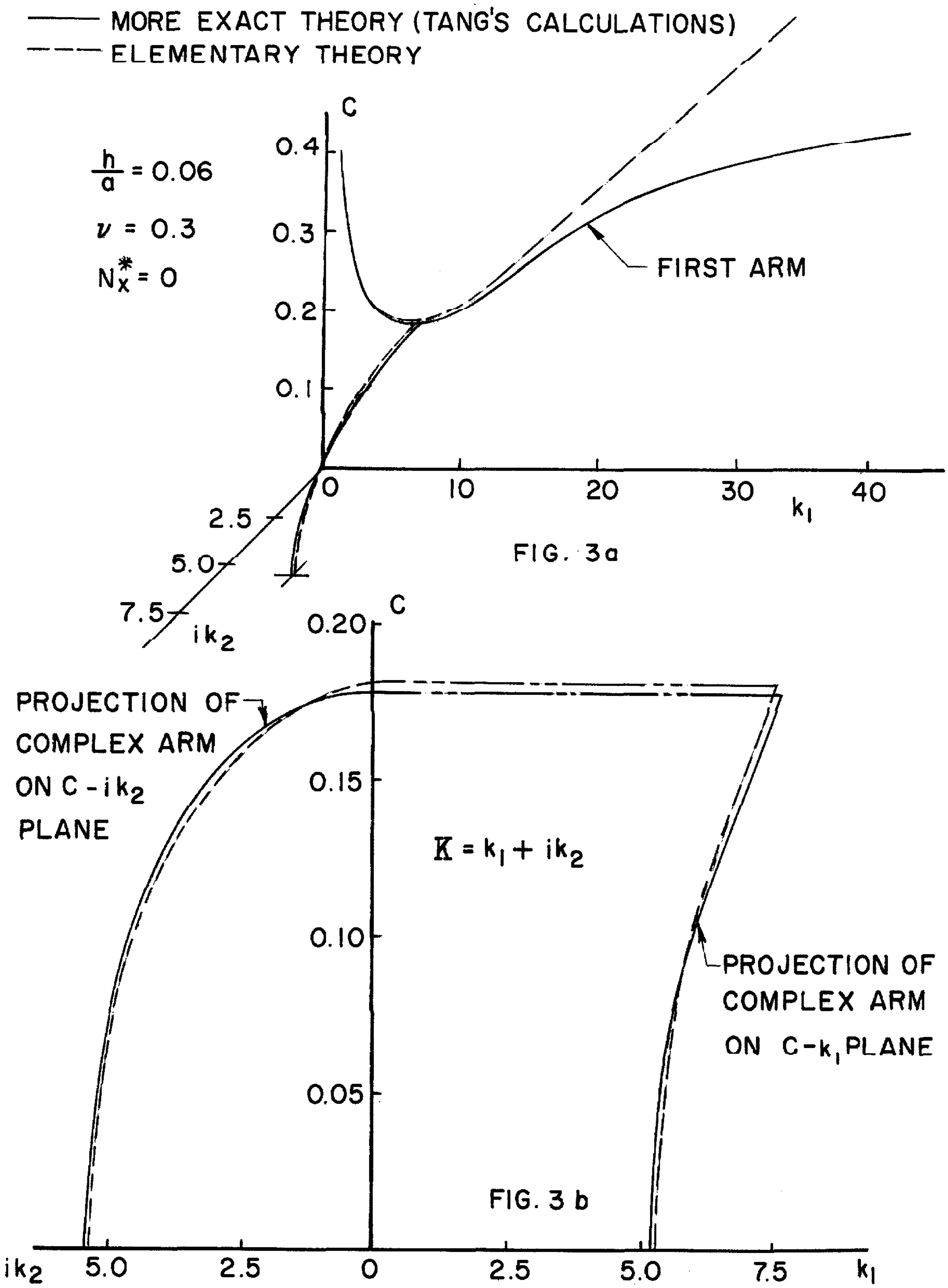


FIG. 3a, b PHASE VELOCITY SPECTRUM FOR $h/a = 0.06$
DATA OBTAINED FROM TANG (23)

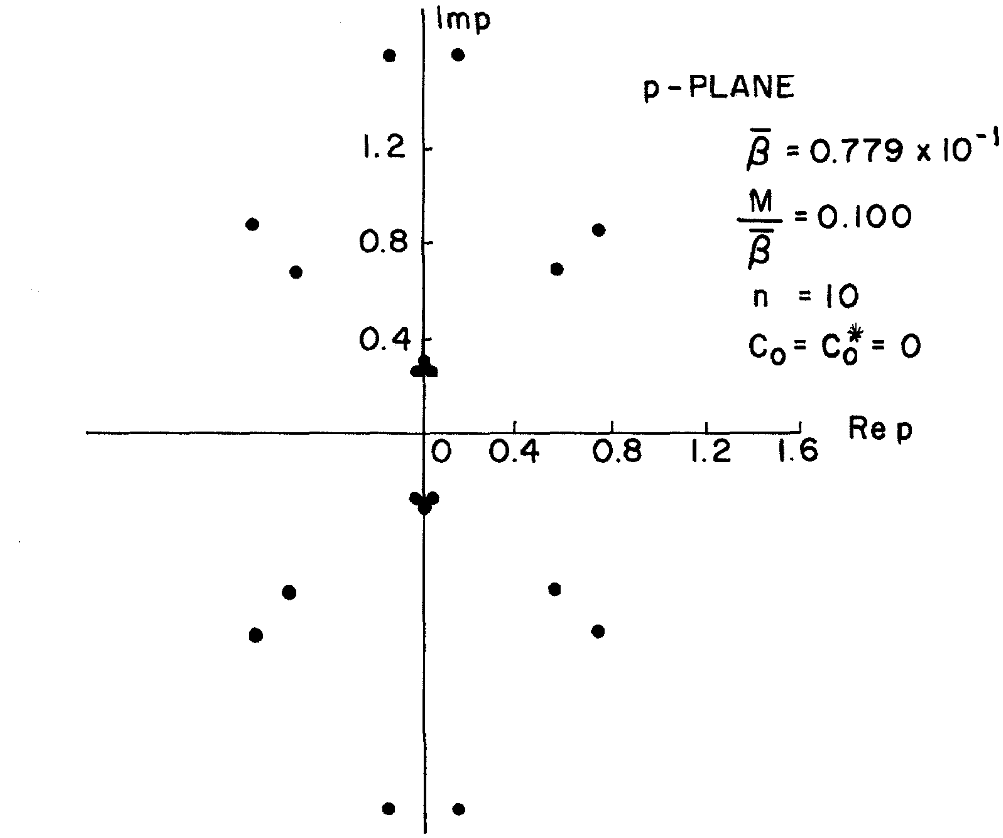


FIG. 4a TYPICAL SET OF ROOT BRANCH POINTS

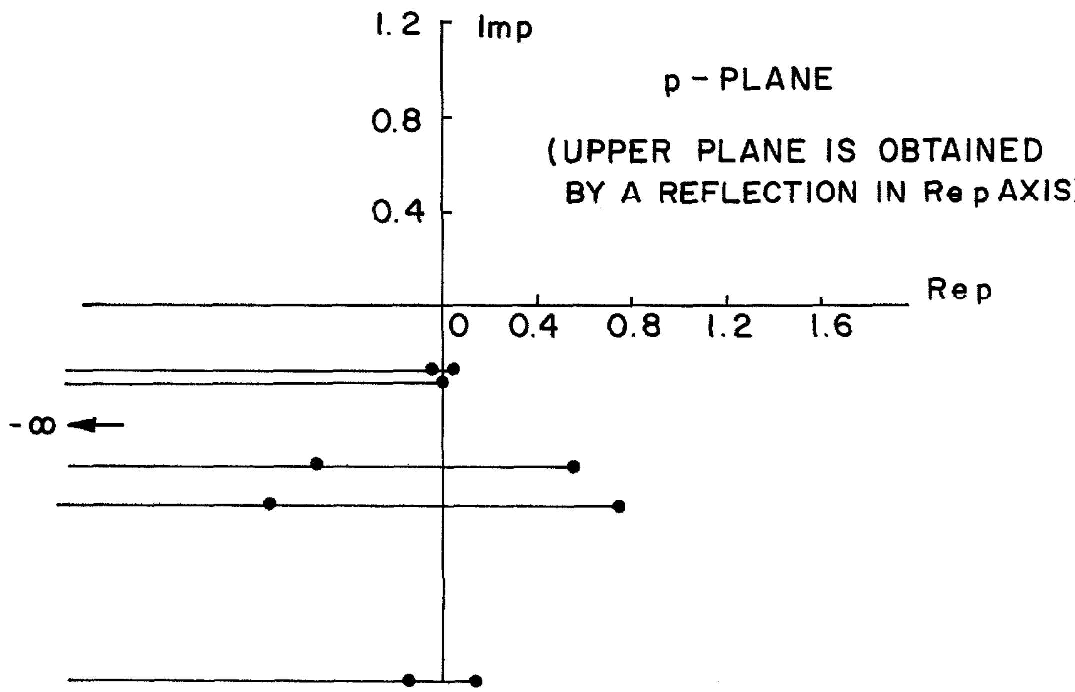


FIG. 4b TYPICAL SET OF BRANCH CUTS

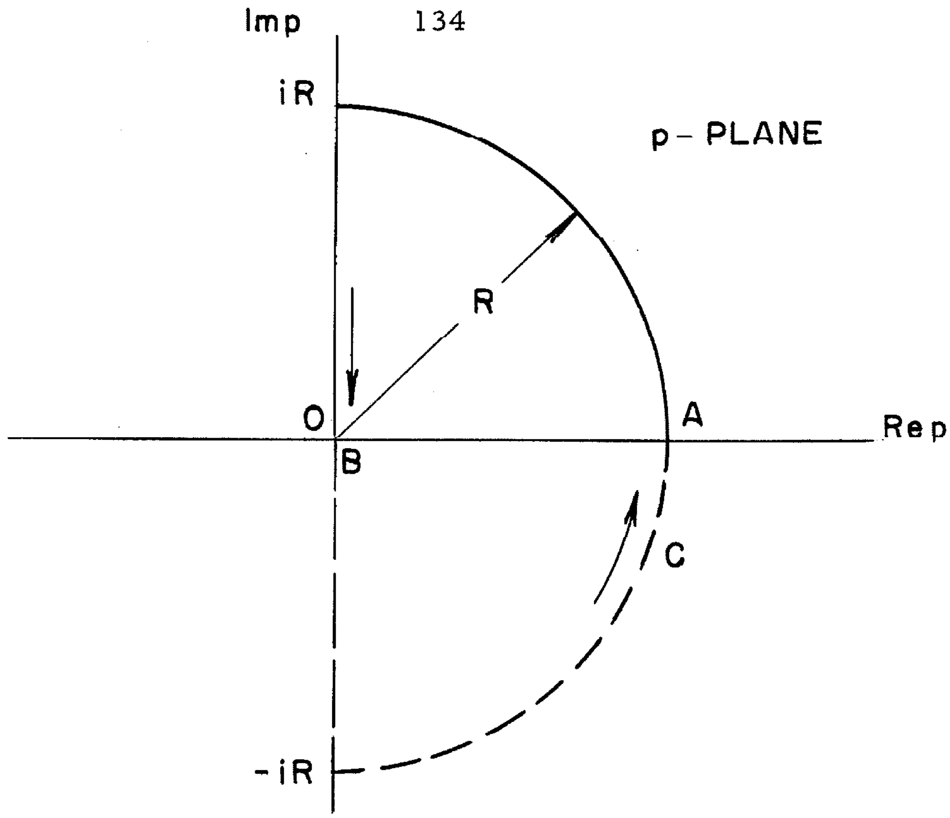


FIG. 5a ORIGINAL CONTOUR

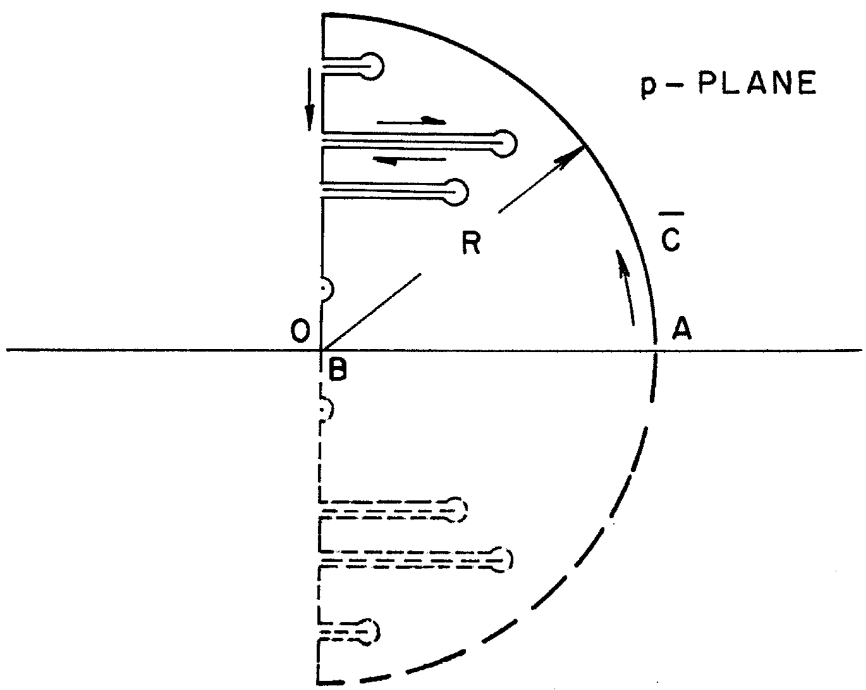


FIG. 5b DEFORMED CONTOUR - BRANCH POINTS CIRCUMVENTED

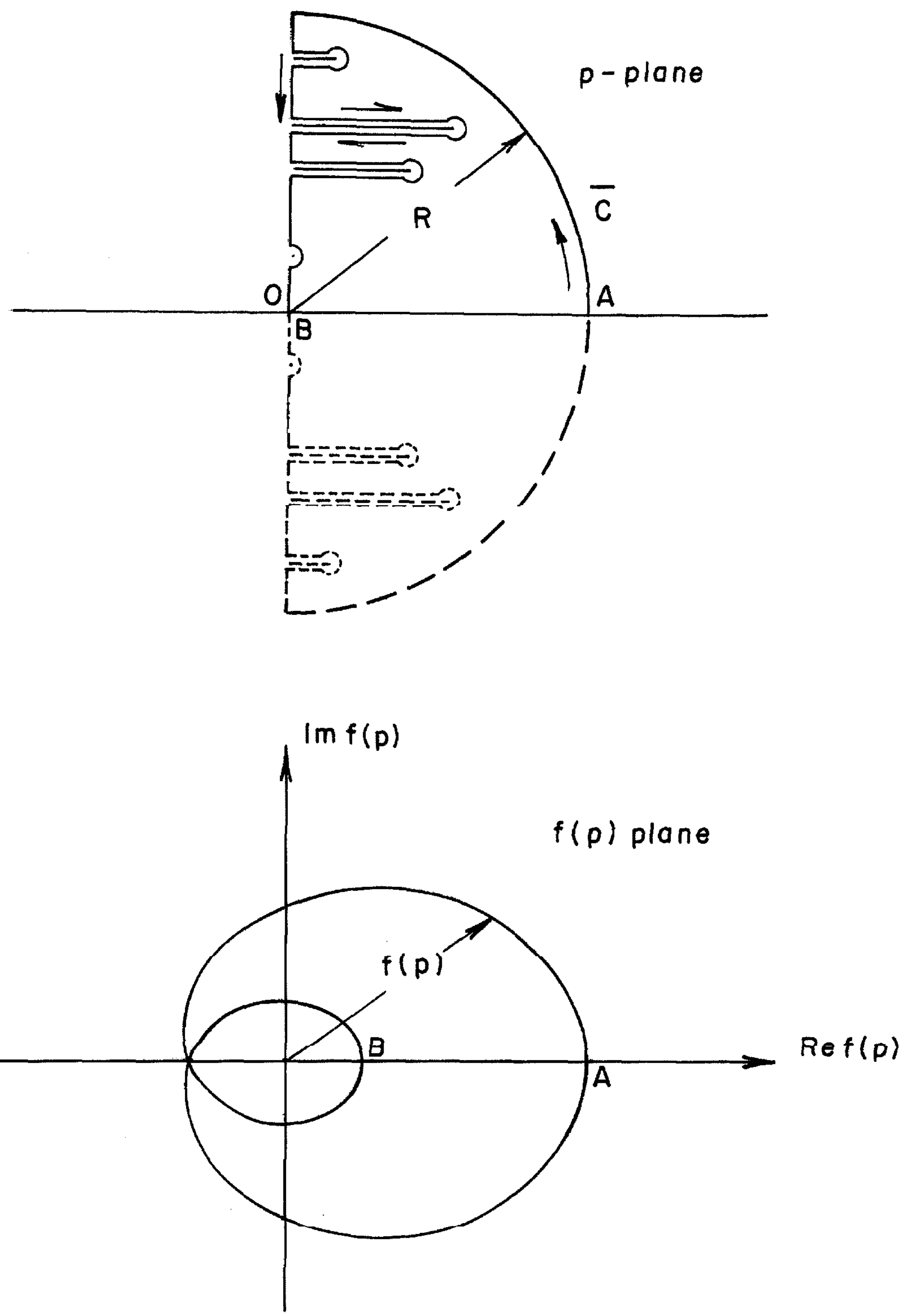


FIG. 6 LOCUS OF f(p) AS p TRACES THE CURVE C