DISCRETE AND CONTINUOUS ESTIMATION IN CORRELATED NOISE WITH FINITE OBSERVATION TIME

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Lionel J. Skidmore

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ABSTRACT

In this thesis analytic formulas are derived for the elements of the inverse covariance matrix of sampled rational noise. It is shown that the number of terms composing these formulas is dependent only on the order of the noise and not on the dimension of the covariance matrix. Some special cases are worked out in detail.

The estimation of the parameter θ in the process $y(t) = \theta S(t) + n(t)$, where t is in the interval $[0,L]$, $n(t)$ is rational noise, and $S(t)$ is deterministic, is considered in detail for first and second order noise. A minimum variance continuous filter, $f(t)$, which gives an estimate of θ through

$$\hat{\theta} = \int_0^L f(t)y(t)dt$$

and its associated variance are computed. Also computed is a discrete minimum variance estimate of the form,

$$\hat{\theta} = \sum_{\mu} f_d(\mu T)y(\mu T)$$

where the $f_d(\cdot)$ are the "weights" for the sampled data and T is the sampling period. It is shown that the discrete weighting function and its variance approaches the continuous weighting function and its variance when the density of observations approaches infinity. It is seen that in general the discrete weighting function does not create the equivalent of a delta function and its derivatives by a simple differencing operation through the use of Kronecker deltas.

Asymptotic properties of the variance of the discrete estimate are considered. The asymptotic term is defined as the first order term in the power series expansion of the variance. It is seen that for a smooth S(t) and first order noise, the asymptotic term is zero. In the special case of S(t) equal to a constant and second order noise it is shown that the asymptotic term is zero if the noise has zeros in its spectral density and nonzero if the noise is all pole.

The connection between autoregressive noise and rational noise is considered in detail for second order noise. It is seen that rational noise will have autoregressive properties only for a special pole-zero configuration and a particular sampling rate. The advantages of sampling at this rate are discussed and a special case is considered.

It is shown that the results obtained for one parameter, one signal, and one noise can be easily extended to a vector of parameters, a matrix of signals, and a vector of noises. The only restriction is that the components of the noise vector be uncorrelated.

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DISCRETE AND CONTINUOUS ESTIMATION IN CORRELATED NOISE WITH FINITE OBSERVATION TIME

CHAPTER I

MINIMUM VARIANCE ESTIMATION

1.1 Introduction

The concept of minimum variance estimation is rather old and originated with investigators who should be classified as mathematical statisticians. They were primarily interested in obtaining "best" estimates of population parameters when a set of sample values was available, and minimum variance was one of the criterions decided upon. Another important criterion was that of maximum likelihood estimation which is credited to R. A. Fisher. It is well known that when the sample values are obtained from a gaussian population, the linear, unbiased minimum variance estimate is identical to the maximum likelihood estimate. Therefore, any statements concerning the variances of minimum variance estimates apply to the variances obtained by maximum likelihood estimation when the population is gaussian. The estimates just discussed are usually referred to as discrete estimates since they are obtained from a finite number of samples from a population.

In many problems which have been of interest over the last 25 years the observed information has been in the form of a continuous record. This record was considered to be composed of a desired signal plus an additive random process and the problem was to estimate the signal or some function of the signal at prescribed times. Wiener (1) and Kolmogoroff (2) are credited with pioneering work in the solution of the above problem. Many extensions and generalizations followed their basic work. Two fairly comprehensive lists of references are given in Reference 3 and Reference 4.

In Reference 5 it is shown that the solution of the continuous estimation problem, when the noise is generated by passing stationary white noise through a time-invariant linear system with the transfer function $\frac{N(S)}{D(S)}$ (S is the Laplace variable and $N(S)$ and $D(S)$ are polynomials in S)*, involves solving an integral equation for the optimum filter or weighting function. In Reference 5, Zadeh and Ragazzini find a solution of the integral equation which involves delta functions and the higher order derivatives of the delta functions. This solution raises the following problems: Suppose that a record which contains an additive combination of a deterministic signal with a multiplicative unknown parameter and a noise of the type discussed above is sampled at equally spaced intervals in time. Further, suppose that a discrete, linear, unbiased minimum variance estimate is made of the parameter and the variance of the estimate is computed, then:

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* Such a noise is called rational noise in this thesis. The order of the denominator polynomial is called the order of the noise.

1. Can analytic results be obtained for the form of the discrete estimate and its associated variance?

2. Does the limiting form of the discrete estimate and its variance approach the continuous estimate and its variance as the time between samples approaches zero and the number of samples approaches infinity?

3. If the answer to the above question is affirmative, then by what mechanism does the discrete estimate create the equivalent of delta functions and their higher order derivatives?

4. Can analytic results concerning the asymptotic properties of the variance of the discrete estimate be obtained?

The first question can be answered affirmatively for a large class of signals by use of an analytic inverse covariance matrix which is the subject of the next chapter. The answer to the second question was shown to be affirmative by an indirect method in a paper by Swerling (10) but he was unable to give any clues to the answers to the third and fourth questions. In this thesis, the second question is answered by a direct method for first and second order noise - namely, analytic formulas for the discrete estimate and its variance are determined as a function of the time between samples and the number of samples, and the limiting properties of these formulas are calculated. It will be seen that analytic formulas for the higher order noises are easily derived from results presented in this thesis but that an investigation of their limiting properties would become algebraically tedious.

In answering the third question posed above for the cases of first and second order noise, the mechanism for the creation of the discrete equivalent of delta functions and their higher order derivatives is exposed for the higher order noises. The fourth question is also answered for first and second order noise in some special cases. However, here, too, the method used is general, but algebraically tedious to apply.

It should be pointed out that quasi-discrete estimates involving orthogonal functions such as described in Chapter 14 of Reference 6 and the limiting properties of these estimates are not discussed in this thesis.

A brief summary of the contents of this thesis is given below. Some of the more pertinent work of other authors along the lines of discrete and continuous minimum variance estimation is discussed in the remaining two sections of this chapter. In Chapter II analytic formulas for the elements of the inverse covariance matrix of sampled rational noise are derived. The equivalence of the continuous and the limit of the discrete minimum variance estimators is shown in Chapter III. Some asymptotic properties of the variance of the discrete estimator are given in Chapter IV. A discussion of the connection between autoregressive noise and rational noise is given in Chapter V. A general formula for the variance of the discrete estimate of a constant in Butterworth noise is derived in Chapter VI. The work of the preceeding chapters is extended to the multiple dimensional case in Chapter VII. Chapter VIII gives a summary of the major results, draws some conclusions, and gives suggestions for further study. An expansion of second order all pole noise is derived in Appendix A and Appendix B discusses the connection between estimation problems and detection problems.

1.2 Discrete Minimum Variance Estimation

The basic discrete minimum variance estimation problem considered in this thesis is as follows:

Given the observed function of time

$$y(t) = \theta S(t) + n(t) \qquad 0 \le t \le L \qquad (1.2.1)$$

where θ is the parameter to be estimated, $S(t)$ is a deterministic signal, $n(t)$ is rational noise, and L is the length of the observation time, find the minimum variance estimate of θ using only the values of $y(t)$ at times T seconds apart where the ratio $\frac{L}{T}$ is an integer. It is shown in Reference 7 that the minimum variance estimate of θ is

$$\hat{\theta} = (\overline{S}' R^{-1} \overline{S})^{-1} (R^{-1}\overline{S})' \overline{y} \qquad (1.2.2)$$

and the variance of the estimate is

$$\sigma_{\theta}^2 = (\overline{S}' R^{-1} \overline{S})^{-1} \qquad (1.2.3)$$

where $\overline{y}$ and $\overline{S}$ are column vectors whose elements are the sampled values of the observed process and the value of the signal at the sample times, R is the covariance matrix of the sampled noise, that is $R = (R_{ij}) = (E\{n(t_i)\, n(t_j)\})$, and "prime" denotes transpose.

Equations 1.2.2 and 1.2.3 expose one of the major problems involved in minimum variance estimation - that the inversion of the covariance matrix of the sampled noise is necessary for minimum variance estimation. With the present state of the art of digital computation it is a practical impossibility to invert a noise covariance matrix of a dimension greater than 100 x 100. Adding to the major difficulties occurring simply because of the high dimension is the fact that a noise covariance matrix becomes highly singular as the dimension becomes large. Thus it is desirable and essentially a necessity to find analytic formulas for the inverse covariance matrix in order to accurately compute the minimum variance estimate, its associated variance, and their limiting and asymptotic properties.

One attempt at computing analytic formulas for the inverse covariance matrix was made by Janos in Reference 11. His basic attack involved a very novel idea, which is discussed in the next chapter, but he did not carry out the analysis correctly. In his paper he made about 30 errors, some of them very serious, which led him to completely incorrect results. Among other things, it can be shown that his resulting "inverse" matrix is not symmetric, which is a necessary condition for an inverse covariance matrix. In the next chapter the correct formulas for the elements of the inverse covariance matrix are derived.

1.3 Continuous Minimum Variance Estimation

Suppose that the random process given in 1.2.1 is observed and it is desired to estimate θ in the following manner:

$$\hat{\theta} = \int_0^L f(t)\ y(t)\ dt \qquad (1.2.4)$$

If $f(t)$ is to be chosen such that $\hat{\theta}$ defined by 1.2.4 is an unbiased, minimum variance estimate, then it can easily be shown by use of standard calculus of variation techniques that $f(t)$ must satisfy the integral equation

$$\int_0^L \phi(t - \tau)\ f(\tau)\ d\tau = \sigma_L^2\ S(t) \qquad (1.2.5)$$

and the constraint

$$\int_0^L f(t)\ S(t)\ dt = 1 \qquad (1.2.6)$$

where σ_L^2 is the variance of the estimate and $\phi(t)$ is the autocorrelation function of the noise.

A general method of solving the integral equation 1.2.5 was given by Zadeh and Raggazzini in Reference 5. One of the difficulties encountered in their method is that it requires the solution of a system of linear equations of an order equal to twice the order of the noise. This difficulty was eliminated in the case when the numerator polynomial was identical to a constant by Martel and Mathews (8). Both the methods discussed above are used to determine the continuous solutions which are compared with the limiting forms of the discrete solutions.

CHAPTER II

DERIVATION OF ANALYTIC FORMULAS FOR THE INVERSE COVARIANCE MATRIX OF SAMPLED RATIONAL NOISE

2.1 General Case

In this thesis rational noise is defined as the steady-state noise which is obtained when white noise is the input to a time-invariant system whose transfer function is the ratio of polynomials in S (the Laplace variable). It will also be assumed that the system just mentioned is stable in the sense that bounded inputs yield bounded outputs. This assumption eliminates the possibility of poles on the imaginary axis or in the right-half plane of the S-plane.

The random process or noise which is generated as just described has an autocorrelation function of the form

$$\phi(t) = \sum_{k=1}^{D} \sigma_k^2 \, e^{-\beta_k |t|} \tag{2.1.1}$$

where D is the order of the system, the σ_k^2 are constants such that their sum is the variance of the process, and the β_k are the poles of the system. (It is assumed in the following analysis that the β_k are distinct, however it will be clear that this assumption is not really a restriction since all the derivations can be modified to include multiple poles.) If a record of the process of L seconds duration is sampled at equally spaced intervals the covariance between the samples at time $t = mT$ and $t = \mu T$ is

$$\phi[(m - \mu)T] = \sum_{k=1}^{D} \sigma_k^2 e^{-\beta_k T|m - \mu|} \tag{2.1.2}$$

where T is the time between samples. The covariance matrix of the process is defined as the matrix whose m,μ^{th} element is $\phi[(m-\mu)T]$. It is the purpose of this section to compute a formula for the elements of the inverse covariance matrix.

In order to determine the elements of the inverse covariance matrix it is first necessary to consider the factorability property of the two-sided Z-transform of the sampled autocorrelation function. The sampled autocorrelation function is given by

$$\phi(mT) = \sum_{k=1}^{D} \sigma_k^2 e^{-\beta_k T|m|} \qquad -\infty \leq m \leq \infty \tag{2.1.3}$$

and its two-sided Z transform is

$$\phi^*(Z) = \sum_{-\infty}^{\infty} \phi(mT) Z^{-m} = \sum_{k=1}^{D} \sigma_k^2 \frac{1-e^{-2\beta_k T}}{(1-e^{-\beta_k T} Z)(1-e^{-\beta_k T} Z^{-1})} \tag{2.1.4}$$

From 2.1.4 it can be seen that if Z_o is a zero of $\phi^*(Z)$ then Z_o^{-1} is also a zero. Therefore $\phi^*(Z)$ can be expressed in factored form

$$\phi^*(Z) = \frac{\phi_N(Z)\phi_N(Z^{-1})}{\phi_D(Z)\phi_D(Z^{-1})} \tag{2.1.5}$$

where (the α's in the following expressions are functions of T)

$$\phi_D(Z) = k \prod_{j=1}^{D} \left(1-e^{-\beta_j T} Z\right) = D^{th} \text{ order polynomial} \quad (2.1.6)$$

$$\phi_N(Z) = \prod_{j=1}^{N} \left(1-e^{-\alpha_j T} Z\right) = N^{th} \text{ order polynomial} \quad (N \leq D - 1)$$

(2.1.7)

Equations 2.1.5, 2.1.6 and 2.1.7 express the factorability of 2.1.4.

Let $W_\mu(mT)$ denote the elements of the inverse covariance matrix, then by definition of an inverse matrix

$$\sum_{m'=0}^{M} W_\mu(m'T)\phi[(m-m')T] = \delta_{\mu m} \quad (2.1.8)$$

where $0 \leq \mu \leq M$, $0 \leq m \leq M$, and $M + 1$ is the number of equally spaced points in the sampled record. Now let a set of discrete functions be defined in the following manner

$$\begin{aligned} \overline{W}_\mu(mT) &= W_\mu(mT) && 0 \leq \mu, m \leq M \\ &= 0 && \text{otherwise} \end{aligned}$$

Therefore for fixed μ, the sequence of values of $\overline{W}_\mu(mT)$ are the elements in the μ^{th} row of the inverse covariance matrix when $0 \leq m \leq M$. By use of the above definitions, 2.1.8 can be written in the form

$$v_\mu(mT) = \sum_{m'=0}^{M} \overline{W}_\mu(m'T)\phi[(m-m')T] - \delta_{\mu m} \qquad (2.1.9)$$

where

$$v_\mu(mT) = 0 \qquad \text{for} \qquad 0 \le \mu, m \le M \qquad (2.1.10)$$

Outside of the interval $0 \le m \le M$ the first expression of 2.1.9 decays in a manner determined by the poles of $\phi^*(Z)$ since the convolved expression may be interpreted as the response of a nonrealizable digital filter (nonzero impulse response for both positive and negative time) to an input $\overline{W}_\mu(mT)$ which is nonzero only over $0 \le \mu, m \le M$. To explicitly exhibit the behavior discussed above, the Z transform of 2.1.9 may be written as

$$V_\mu^*(Z) = Z^{-(M+1)} \frac{P_\mu(Z^{-1})}{\phi_D(Z^{-1})} + Z \frac{P_\mu'(Z)}{\phi_D(Z)} \qquad (2.1.11)$$

where $P_\mu(Z^{-1})$ and $P_\mu'(Z)$ are polynomials of their respective arguments, each of degree not greater than D-1. (At this point the coefficients of $P_\mu(Z^{-1})$ and $P_\mu'(Z)$ are unknown.) The Z transform of $v_\mu(mT)$ can also be determined from 2.1.9. Upon noting that the first term on the right-hand side (RHS) of 2.1.9 is a convolution, the transform becomes

$$V_\mu^*(Z) = \sum_{-\infty}^{\infty} v_\mu(mT)Z^{-m} = \overline{W}^*(Z)\phi^*(Z) - Z^{-\mu} \qquad (2.1.12)$$

Solving 2.1.11 and 2.1.12 for $\overline{W}^*(Z)$ gives

$$\overline{W}_\mu^*(Z) = Z^{-\mu}\,\frac{\phi_D(Z)\phi_D\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} + Z^{-(M+1)}\,\frac{\phi_D(Z)P_\mu\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} + Z\,\frac{\phi_D\left(Z^{-1}\right)P_\mu^{'}(Z)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} \tag{2.1.13}$$

Inverting the transform 2.1.13 gives

$$\overline{W}_\mu(mT) = \left[\frac{1}{\phi^x(Z)}\right][(m-\mu)T] + \left[\frac{\phi_D(Z)P_\mu\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)}\right][(m-M-1)T] + \left[\frac{\phi_D\left(Z^{-1}\right)P_\mu^{'}(Z)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)}\right][(m+1)T] \tag{2.1.14}$$

where the large square brackets, in conjunction with the smaller square brackets, signify the time sequence whose transform is the enclosed expression.* It is now necessary to explicitly evaluate the components of 2.1.14 and determine the coefficients of $P_\mu\left(Z^{-1}\right)$ and $P_\mu^{'}(Z)$ in order to determine the elements of $\overline{W}_\mu(mT)$ and hence the elements of $W_\mu(mT)$.

The method of attack which will be used is to first expand 2.1.14 as a time sequence in terms of the unknown coefficients of $P_\mu\left(Z^{-1}\right)$ and $P_\mu^{'}(Z)$ and then use the fact that $\overline{W}_\mu(mT)$ is nonzero

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* For example, let $f(mT) = 1 + 2\delta_{m,1} + 5\delta_{m,2}$, then $F^*(Z) = 1 + 2Z^{-1} + 5Z^{-2}$ and $\left[F^*(Z)\right][(mT)] = 1 + 2\delta_{m,1} + 5\delta_{m,2} = f(mT)$

only in the interval $0 \le \mu, m \le M$ to obtain a set of simultaneous equations of order N which determine the coefficients. Actually it will turn out to be convenient and sufficient to determine a set of coefficients which are non-singularly related to the coefficients of $P_\mu\left(Z^{-1}\right)$ and $P'_\mu(Z)$.

Evaluation of $\left[\frac{1}{\phi^*(Z)}\right][(m-\mu)T]$

To facilitate the determination of the time sequences it is convenient to reduce the Z-transforms that have numerator polynomials of order equal to or greater than the order of their denominator polynomials by long division. This method of reduction will be used several times in the sequel.

The ratio of polynomials $\frac{\phi_D(Z)}{\phi_N(Z)}$ can be expressed by use of long division as

$$\frac{\phi_D(Z)}{\phi_N(Z)} = Q(Z) + \frac{R(Z)}{\phi_N(Z)} \tag{2.1.15}$$

where $Q(Z)$ is the quotient (of degree $D-N$) and $R(Z)$ is the remainder (of degree $N-1$). Using 2.1.15 $\frac{1}{\phi^*(Z)}$ can be expressed

$$\phi^{*^{-1}}(Z) = \frac{\phi_D(Z)\phi_D\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} = \left[Q(Z) + \frac{R(Z)}{\phi_N(Z)}\right]\left[Q\left(Z^{-1}\right) + \frac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)}\right]$$

$$= Q(Z)Q\left(Z^{-1}\right) + \frac{R(Z)}{\phi_N(Z)} Q\left(Z^{-1}\right) + \frac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)} Q(Z) +$$

$$+ \frac{R(Z)R\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} \tag{2.1.16}$$

Consider the term $Q(Z)Q\left(Z^{-1}\right)$. The polynomial $Q(Z)$ is of degree $D-N$, therefore define

$$Q(Z) = \sum_{k=o}^{D-N} Q_k Z^k \tag{2.1.17}$$

and hence

$$Q(Z)Q\left(Z^{-1}\right) = \sum_{k=o}^{D-N} \sum_{\ell=o}^{D-N} Q_k Q_\ell Z^{k-\ell}$$

$$= \sum_{k=o}^{D-N} Q_k^2 + \left(Z+Z^{-1}\right) \sum_{k=1}^{D-N} Q_k Q_{k-1} + \left(Z^2+Z^{-2}\right) \sum_{k=2}^{D-N} Q_k Q_{k-2} +$$

$$+ \cdots + \left(Z^{D-N} + Z^{-(D-N)}\right) Q_{D-N} Q_o$$

$$= \sum_{k=o}^{D-N} Q_k^2 + \sum_{\ell=1}^{D-N} \left(Z^\ell+Z^{-\ell}\right) \sum_{k=\ell}^{D-N} Q_k Q_{k-\ell}$$

$$= q_o + \sum_{\ell=1}^{D-N} q_\ell \left(Z^\ell+Z^{-\ell}\right) \tag{2.1.18}$$

where

$$q_o = \sum_{k=o}^{D-N} Q_k^2 \tag{2.1.19}$$

$$q_\ell = \sum_{k=\ell}^{D-N} Q_k Q_{k-\ell} \tag{2.1.20}$$

From 2.1.18

$$Q(Z)Q\left(Z^{-1}\right) = \sum_{\ell=-\infty}^{\infty} \left\{\left[Q(Z)Q\left(Z^{-1}\right)\right] [(\ell T)]\right\} Z^{-\ell} = q_o + \sum_{\ell=1}^{D-N} q_\ell \left(Z^{\ell}+Z^{-\ell}\right)$$

hence

$$\left[Q(Z)Q\left(Z^{-1}\right)\right] [(mT)] = q_o \delta_{m,o} + \sum_{\ell=1}^{D-N} q_\ell \left(\delta_{m,\ell} + \delta_{m,-\ell}\right) \tag{2.1.21}$$

Upon replacing m by $m-\mu$, 2.1.21 becomes

$$\left[Q(Z)Q\left(Z^{-1}\right)\right] [(m-\mu)T] = q_o \delta_{m-\mu,o} + \sum_{\ell=1}^{D-N} q_\ell \left(\delta_{m-\mu,\ell} + \delta_{m-\mu,-\ell}\right) \tag{2.1.22}$$

Next consider the terms $\frac{R(Z)}{\phi_N(Z)} Q\left(Z^{-1}\right)$ and $\frac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)} Q(Z)$. The ratio of polynomials $\frac{R(Z)}{\phi_N(Z)}$ can be expressed in a partial fraction expansion

$$\frac{R(Z)}{\phi_N(Z)} = \sum_{k=1}^{N} \frac{\rho_k}{1-e^{-\alpha_k T} Z} \qquad (2.1.23)$$

where the ρ_k are a new set of constants defined through 2.1.23.

Therefore

$$\frac{R(Z)}{Q_N(Z)} Q\left(Z^{-1}\right) = \sum_{k=1}^{N} \frac{\rho_k}{1-e^{-\alpha_k T} Z} \sum_{\ell=o}^{D-N} Q_\ell Z^{-\ell}$$

$$= \sum_{k=1}^{N} \sum_{\ell=o}^{D-N} \frac{Q_\ell \rho_k Z^{-\ell}}{1-e^{-\alpha_k T} Z} \qquad (2.1.24)$$

where 2.1.17 was used to obtain the expression for $Q(Z)$.

Inverting 2.1.24 and replacing m by $m-\mu$ gives*

$$\left[\frac{R(Z)}{Q_N(Z)} Q\left(Z^{-1}\right)\right][(m-\mu)T] = \sum_{k=1}^{N} \sum_{\ell=o}^{D-N} Q_\ell \rho_k e_{-}^{\alpha_k T(m-\mu-\ell)} \qquad (2.1.25)$$

Reasoning similar to that used in obtaining 2.1.25 gives*

$$\left[\frac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)} Q(Z)\right][(m-\mu)T] = \sum_{k=1}^{N} \sum_{\ell=o}^{D-N} Q_\ell \rho_k e_{+}^{-\alpha_k T(m-\mu+\ell)} \qquad (2.1.26)$$

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* The notation means:

$$e_{-}^{\alpha_k T(\gamma)} = \begin{cases} e^{+\alpha_k T\gamma} & \gamma \le 0 \\ 0 & \gamma > 0 \end{cases} \quad \text{and} \quad e_{+}^{-\alpha_k T(\gamma)} = \begin{cases} e^{-\alpha_k T\gamma} & \gamma \ge 0 \\ 0 & \gamma < 0 \end{cases}$$

Finally consider

$$\frac{R(Z)}{\phi_N(Z)}\ \frac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)} = \sum_{k=1}^{N}\ \sum_{\ell=1}^{N} \frac{\rho_k\rho_\ell}{\left(1-e^{-\alpha_k T}Z\right)\left(1-e^{-\alpha_k T}Z^{-1}\right)} \qquad (2.1.27)$$

$$= \sum_{k=1}^{N}\sum_{\ell=1}^{N} \frac{\rho_k\rho_\ell}{1-e^{-(\alpha_\ell+\alpha_k)T}}\left[\frac{Ze^{-\alpha_k T}}{1-e^{-\alpha_k T}Z} + \frac{1}{1-e^{-\alpha_k T}Z^{-1}}\right]$$

$$= \sum_{k=1}^{N}\sum_{\ell=1}^{N} \frac{\rho_k\rho_\ell}{1-e^{-(\alpha_\ell+\alpha_k)T}}\left[\frac{Ze^{-\alpha_k T}}{1-e^{-\alpha_k T}Z}\right] +$$

$$+ \sum_{k=1}^{N}\sum_{\ell=1}^{N} \frac{\rho_k\rho_\ell}{1-e^{-(\alpha_\ell+\alpha_k)T}}\left[\frac{1}{1-e^{-\alpha_\ell T}Z^{-1}}\right] \qquad (2.1.28)$$

Upon interchanging the dummy indices of the first sum in the above expression and combining the double sums

$$\frac{R(Z)R\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} = \sum_{k=1}^{N}\sum_{\ell=1}^{N} \frac{\rho_k\rho_\ell}{1-e^{-(\alpha_\ell+\alpha_k)T}}\left[\frac{1-e^{-2\alpha_\ell T}}{\left(1-e^{-\alpha_\ell T}Z\right)\left(1-e^{-\alpha_\ell T}Z^{-1}\right)}\right] \qquad (2.1.29)$$

Inverting 2.1.29 and replacing m by $m-\mu$ gives

$$\left[\frac{R(Z)R\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)}\right][(m-\mu)T] = \sum_{k=1}^{N}\sum_{\ell=1}^{N} \frac{\rho_k\rho_\ell}{1-e^{-(\alpha_\ell+\alpha_k)T}}\, e^{-\alpha_\ell T|m-\mu|} \qquad (2.1.30)$$

Therefore, from 2.16, 2.1.22, 2.1.25, 2.1.26 and 2.1.30

$$\left[\frac{1}{\phi^{*}(Z)}\right][(m-\mu)T] = \left[Q(Z)Q\left(Z^{-1}\right)\right][(m-\mu)T] + \left[\frac{R(Z)}{\phi_N(Z)}\,Q\left(Z^{-1}\right)\right][(m-\mu)T] +$$

$$+\left[\frac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)}\,Q(Z)\right][(m-\mu)T] + \left[\frac{R(Z)R\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)}\right][(m-\mu)T] \qquad (2.1.31)$$

$$= q_o\delta_{|m-\mu|,o} + \sum_{\ell=1}^{D-N} q_\ell\left(\delta_{m-\mu,\ell} + \delta_{m-\mu,-\ell}\right)$$

$$+ \sum_{k=1}^{N}\sum_{\ell=o}^{D-N} Q_\ell \rho_k e_{-}^{\alpha_k T(m-\mu-\ell)} + \sum_{k=1}^{N}\sum_{\ell=o}^{D-N} Q_\ell \rho_k e_{+}^{-\alpha_k T(m-\mu+\ell)}$$

$$+ \sum_{\ell=1}^{N} C_\ell e^{-\alpha_\ell T|m-\mu|} \qquad (2.1.32)$$

where

$$q_\ell = \sum_{k=\ell}^{D-N} Q_k Q_{k-\ell} \qquad \ell=0,\ \cdots,\ D-N \qquad (2.1.33)$$

$$C_\ell = \sum_{k=1}^{N} \frac{\rho_k \rho_\ell}{1-e^{-(\alpha_\ell+\alpha_k)T}} \qquad (2.1.34)$$

Evaluation of $\left[\frac{P_\mu(Z^{-1})\phi_D(Z)}{\phi_N(Z^{-1})\phi_N(Z)}\right]$ [(m-M-1)T]

The ratio of polynomials $\frac{P_\mu(Z^{-1})}{\phi_N(Z^{-1})}$ can be expressed by use of long division as

$$\frac{P_\mu(Z^{-1})}{\phi_N(Z^{-1})} = Q_\mu(Z^{-1}) + \frac{R_\mu(Z^{-1})}{\phi_N(Z^{-1})} \tag{2.1.35}$$

where in terms of the variable Z^{-1} , $Q_\mu(Z^{-1})$ is of degree D-N-1 and $R_\mu(Z^{-1})$ is of degree N-1. (The coefficients of Q_μ and R_μ are the new unknowns and they define the coefficients of P_μ through 2.1.35). The use of 2.1.15 and 2.1.35 gives

$$\frac{P_\mu(Z^{-1})\phi_D(Z)}{\phi_N(Z^{-1})\phi_N(Z)} = \left[Q_\mu(Z^{-1}) + \frac{R_\mu(Z^{-1})}{\phi_N(Z^{-1})}\right]\left[Q(Z) + \frac{R(Z)}{\phi_N(Z)}\right]$$

$$= Q_\mu(Z^{-1})\,Q(Z) + Q_\mu(Z^{-1})\frac{R(Z)}{\phi_N(Z)} + \frac{R_\mu(Z^{-1})}{\phi_N(Z^{-1})}\,Q(Z) +$$

$$+ \frac{R_\mu(Z^{-1})}{\phi_N(Z^{-1})}\;\frac{R(Z)}{\phi_N(Z)} \tag{2.1.36}$$

Consider the term $Q_\mu(Z^{-1})\,Q(Z)$. Define the coefficients $Q_\mu(Z^{-1})$ by

$$Q_\mu(Z^{-1}) = \sum_{k=0}^{D-N-1} Q_{\mu k}Z^{-k} \tag{2.1.37}$$

then

$$Q_\mu\left(Z^{-1}\right) Q(Z) = \sum_{k=0}^{D-N-1} Q_{\mu k} Z^{-k} \sum_{\ell=0}^{D-N} Q_\ell Z^\ell = \sum_{k=0}^{D-N-1} \sum_{\ell=0}^{D-N} Q_{\mu k} Q_\ell Z^{(\ell-k)}$$

$$= Z^0 \sum_{k=0}^{D-N-1} Q_{\mu k} Q_k + Z^1 \sum_{\ell=1}^{D-N} Q_\ell Q_{\mu,\ell-1} + \cdots + Z^{D-N} Q_{D-N} Q_{\mu,0}$$

$$+ Z^{-1} \sum_{k=1}^{D-N-1} Q_{\mu k} Q_{k-1} + \cdots + Z^{-(D-N-1)} Q_{\mu,D-N-1} Q_0$$

$$= \sum_{\ell=0}^{D-N-1} Z^{-\ell} \left\{ \sum_{k=\ell}^{D-N-1} Q_{\mu k} Q_{k-\ell} \right\} + \sum_{k=1}^{D-N} Z^k \left\{ \sum_{\ell=k}^{D-N} Q_\ell Q_{\mu,\ell-k} \right\}$$

$$= \sum_{\ell=0}^{D-N-1} E_{\mu\ell} Z^{-\ell} + \sum_{\ell=1}^{D-N} G_{\mu\ell} Z^\ell \qquad (2.1.38)$$

where

$$E_{\mu\ell} = \sum_{k=\ell}^{D-N-1} Q_{\mu k} Q_{k-\ell} \qquad (2.1.39)$$

and

$$G_{\mu\ell} = \sum_{k=\ell}^{D-N} Q_k Q_{\mu,k-\ell} \qquad (2.1.40)$$

Inverting 2.1.38 and replacing m by m-M-1 gives

$$\left[Q_\mu\left(Z^{-1}\right) Q(Z)\right][(m-M-1)T] = \sum_{\ell=0}^{D-N-1} E_{\mu\ell}\delta_{m-M-1,\ell} + \sum_{k=1}^{D-N} G_{\mu k}\delta_{m-M-1,-k} \tag{2.1.41}$$

Next consider the terms $Q_\mu\left(Z^{-1}\right)\frac{R(Z)}{\phi_N(Z)}$ and $\frac{R_\mu\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)} Q(Z)$. From 2.1.23 and 2.1.37

$$Q_\mu\left(Z^{-1}\right)\frac{R(Z)}{\phi_N(Z)} = \sum_{k=0}^{D-N-1} Q_{\mu k}Z^{-k}\sum_{\ell=1}^{N}\frac{\rho_\ell}{1-e^{-\alpha_\ell T}Z} = \sum_{k=0}^{D-N-1}\sum_{\ell=1}^{N}\frac{\rho_\ell Q_{\mu k}Z^{-k}}{1-e^{-\alpha_\ell T}Z} \tag{2.1.42}$$

Upon inverting 2.1.42

$$\left[Q_\mu\left(Z^{-1}\right)\frac{R(Z)}{\phi_N(Z)}\right][(m-M-1)T] = \sum_{k=0}^{D-N-1}\sum_{\ell=1}^{N}\rho_\ell Q_{\mu k}e_-^{\alpha_\ell T(m-M-1-k)} \tag{2.1.43}$$

Reasoning similar to that used in deriving 2.1.43 gives

$$\left[\frac{R_\mu\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)} Q(Z)\right][(m-M-1)T] = \sum_{k=0}^{D-N}\sum_{\ell=1}^{N}\rho_{\mu\ell}Q_k e_+^{-\alpha_\ell T(m-M-1+k)} \tag{2.1.44}$$

where the $\rho_{\mu\ell}$ are defined by the expansion

$$\frac{R_\mu\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)} = \sum_{\ell=1}^{N}\frac{\rho_{\mu\ell}}{1-e^{-\alpha_\ell T}Z^{-1}} \tag{2.1.45}$$

From 2.1.23 and 2.1.25

$$\frac{R_\mu\left(Z^{-1}\right)R(Z)}{\phi_N\left(Z^{-1}\right)\phi_N(Z)} = \sum_{\ell=1}^{N}\sum_{k=1}^{N}\frac{\rho_{\mu\ell}\rho_k}{\left(1-e^{-\alpha_\ell T}Z^{-1}\right)\left(1-e^{-\alpha_k T}Z\right)}$$

$$= \sum_{\ell=1}^{N}\sum_{k=1}^{N}\frac{\rho_{\mu\ell}\rho_k}{1-e^{-(\alpha_\ell+\alpha_k)T}}\left[\frac{1}{1-e^{-\alpha_\ell T}Z^{-1}}+\frac{Ze^{-\alpha_k T}}{1-e^{-\alpha_k T}Z}\right]$$

$$= \sum_{\ell=1}^{N}\sum_{k=1}^{N}\frac{\rho_{\mu\ell}\rho_k}{1-e^{-(\alpha_\ell+\alpha_k)T}}\left[\left(\frac{1}{1-e^{-\alpha_\ell T}Z^{-1}}-1\right)+\frac{1}{1-e^{-\alpha_k T}Z}\right] \tag{2.1.46}$$

$$= \sum_{\ell=1}^{N}H_{\mu\ell}\left[\frac{1}{1-e^{-\alpha_\ell T}Z^{-1}}-1\right]+\sum_{\ell=1}^{N}J_{\mu\ell}\frac{1}{1-e^{-\alpha_\ell T}Z} \tag{2.1.47}$$

where

$$H_{\mu\ell} = \rho_{\mu\ell}\sum_{k=1}^{N}\frac{\rho_k}{1-e^{-(\alpha_\ell+\alpha_k)T}} \tag{2.1.48}$$

and

$$J_{\mu\ell} = \rho_\ell\sum_{k=1}^{N}\frac{\rho_{\mu k}}{1-e^{-(\alpha_\ell+\alpha_k)T}} \tag{2.1.49}$$

Inversion of 2.1.47 with the required shift in argument gives

$$\left[\frac{R_\mu\left(Z^{-1}\right)R(Z)}{\phi_N\left(Z^{-1}\right)\phi_N(Z)}\right][(m-M-1)T] - \sum_{\ell=1}^{N} H_{\mu\ell}\left[e_+^{-\alpha_\ell T(m-M-1)} - \delta_{(m-M-1),o}\right] +$$

$$+ \sum_{k=1}^{N} J_{\mu k}\, e_-^{\alpha_k T(m-M-1)} \tag{2.1.50}$$

Therefore, from 2.1.36, 2.1.41, 2.1.43, 2.1.44 and 2.1.50

$$\left[\frac{R_\mu\left(Z^{-1}\right)\phi_D(Z)}{\phi_N\left(Z^{-1}\right)\phi_N(Z)}\right][(m-M-1)T] = \left[Q_\mu\left(Z^{-1}\right)Q(Z)\right][(m-M-1)T] +$$

$$+\left[Q_\mu\left(Z^{-1}\right)\frac{R(Z)}{\phi_N(Z)}\right][(m-M-1)T] + \left[\frac{R_\mu\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)}\, Q(Z)\right][(m-M-1)T] +$$

$$+\left[\frac{R_\mu\left(Z^{-1}\right)R(Z)}{\phi_N\left(Z^{-1}\right)\phi_N(Z)}\right][(m-M-1)T] \tag{2.1.51}$$

$$= \sum_{\ell=o}^{D-N-1} E_{\mu\ell}\,\delta_{m-M-1,\ell} + \sum_{k=1}^{D-N} G_{\mu k}\,\delta_{m-M-1,k} + \sum_{\ell=1}^{N}\sum_{k=o}^{D-N-1} \rho_\ell Q_{\mu k}\, e_-^{\alpha_\ell T(m-M-1-k)}$$

$$+ \sum_{\ell=1}^{N}\sum_{k=o}^{D-N} \rho_{\mu\ell} Q_k\, e_+^{-\alpha_\ell T(m-M-1+k)} + \sum_{\ell=1}^{N} J_{\mu\ell}\, e^{\alpha_\ell T(m-M-1)}$$

$$+ \sum_{\ell=1}^{N} H_{\mu\ell}\left[e_+^{-\alpha_\ell T(m-M-1)} - \delta_{(m-M-1),o}\right] \tag{2.1.52}$$

Evaluation of $\left[\frac{P'_\mu(Z)\phi_D(Z^{-1})}{\phi_N(Z)\phi_N(Z^{-1})}\right][(m+1)T]$

The ratio of polynomials $\frac{P'_\mu(Z)}{\phi_N(Z)}$ can be expressed (the technique just used is now going to be repeated)

$$\frac{P'_\mu(Z)}{\phi_N(Z)} = Q'_\mu(Z) + \frac{R'_\mu(Z)}{\phi_N(Z)} \tag{2.1.53}$$

where $Q'_\mu(Z)$ is of degree D-N-1 and $R'_\mu(Z)$ is of degree N-1. Using 2.1.15 and 2.1.53 gives

$$\frac{P'_\mu(Z)\phi_D(Z^{-1})}{\phi_N(Z)\phi_N(Z^{-1})} = \left[Q'_\mu(Z) + \frac{R'_\mu(Z)}{\phi_N(Z)}\right]\left[Q(Z^{-1}) + \frac{R(Z^{-1})}{\phi_N(Z^{-1})}\right]$$

$$= Q'_\mu(Z)Q(Z^{-1}) + \frac{R'_\mu(Z)}{\phi_N(Z)}Q(Z^{-1}) + Q'_\mu(Z)\frac{R(Z^{-1})}{\phi_N(Z^{-1})} +$$

$$+ \frac{R'_\mu(Z)R(Z^{-1})}{\phi_N(Z)\phi_N(Z^{-1})} \tag{2.1.54}$$

Consider the term $Q'_\mu(Z)Q(Z^{-1})$. Let

$$Q'_\mu(Z) = \sum_{k=o}^{D-N-1} Q'_{\mu k}Z^k \tag{2.1.55}$$

then

$$Q_\mu'(Z)Q(Z) = \sum_{k=o}^{D-N-1} \sum_{\ell=o}^{D-N} Q_{\mu k} Q_\ell Z^{(k-\ell)}$$

$$= \sum_{\ell=o}^{D-N-1} Z^\ell \left\{ \sum_{k=\ell}^{D-N-1} Q_{\mu k}' Q_{k-\ell} \right\} + \sum_{\ell=1}^{D-N} Z^{-\ell} \left\{ \sum_{k=\ell}^{D-N} Q_k Q_{\mu, k-\ell}' \right\}$$

$$= \sum_{\ell=o}^{D-N-1} E_{\mu\ell}' Z^\ell + \sum_{\ell=1}^{D-N} G_{\mu\ell}' Z^{-\ell} \qquad (2.1.56)$$

where

$$E_{\mu\ell}' = \sum_{k=\ell}^{D-N-1} Q_{\mu k}' Q_{k-\ell} \qquad (2.1.57)$$

$$G_{\mu\ell}' = \sum_{k=\ell}^{D-N} Q_k Q_{\mu, k-\ell}' \qquad (2.1.58)$$

Inverting 2.1.56 and replacing m by $m + 1$ gives

$$\left[Q_\mu'(Z)Q\left(Z^{-1}\right)\right] [(m+1)T] = \sum_{\ell=o}^{D-N-1} E_{\mu\ell}' \delta_{m+1,-\ell} + \sum_{\ell=1}^{D-N} G_{\mu\ell}' \delta_{m+1,\ell} \qquad (2.1.59)$$

Next consider the terms $Q_\mu'(Z) \dfrac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)}$ and $\dfrac{R_\mu'(Z)}{\phi_N(Z)} Q\left(Z^{-1}\right)$.

From 2.1.23 and 2.1.55

$$Q'_\mu(Z)\,\frac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)} = \sum_{k=o}^{D-N-1}\ \sum_{\ell=1}^{N} \frac{\rho_\ell Q'_{\mu k} Z^k}{1-e^{-\alpha_\ell T} Z^{-1}} \tag{2.1.60}$$

or in the time domain

$$\left[Q'_\mu(Z)\,\frac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)}\right][(m+1)T] = \sum_{k=o}^{D-N-1}\ \sum_{\ell=1}^{N} \rho_\ell Q'_{\mu k} e_+^{-\alpha_\ell T(m+1+k)} \tag{2.1.61}$$

Reasoning similar to that used in deriving 2.1.61 gives

$$\left[\frac{R'_\mu(Z)}{\phi_N(Z)}\, Q\left(Z^{-1}\right)\right][(m+1)T] = \sum_{k=o}^{D-N}\ \sum_{\ell=1}^{N} \rho'_{\mu\ell} Q_k e_-^{\alpha_\ell T(m+1-k)} \tag{2.1.62}$$

where the $\rho'_{\mu\ell}$ are defined by the expansion

$$\frac{R'_\mu(Z)}{\phi_N(Z)} = \sum_{\ell=1}^{N} \frac{\rho'_{\mu\ell}}{1-e^{-\alpha_\ell T} Z} \tag{2.1.63}$$

From 2.1.23 and 2.1.63

$$\frac{R'_\mu(Z) R\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} = \sum_{\ell=1}^{N}\ \sum_{k=1}^{N} \frac{\rho'_{\mu\ell}\rho_k}{\left(1-e^{-\alpha_\ell T} Z\right)\left(1-e^{-\alpha_k T} Z^{-1}\right)}$$

$$= \sum_{\ell=1}^{N}\ \sum_{k=1}^{N} \frac{\rho'_{\mu\ell}\rho_k}{1-e^{-(\alpha_\ell+\alpha_k)T}} \left[\frac{Ze^{-\alpha_\ell T}}{1-e^{-\alpha_\ell T} Z} + \frac{1}{1-e^{-\alpha_k T} Z^{-1}}\right]$$

$$= \sum_{\ell=1}^{N}\ \sum_{k=1}^{N} \frac{\rho'_{\mu\ell}\rho_k}{1-e^{-(\alpha_\ell+\alpha_k)T}} \left[\frac{1}{1-e^{-\alpha_\ell T} Z} + \left(\frac{1}{1-e^{-\alpha_k T} Z^{-1}} - 1\right)\right]$$

$$= \sum_{\ell=1}^{N} \left\{ \sum_{k=1}^{N} \frac{\rho'_{\mu\ell}\rho_k}{1-e^{-(\alpha_\ell+\alpha_k)T}} \right\} \left[\frac{1}{1-e^{-\alpha_\ell T} Z} \right] +$$

$$+ \sum_{\ell=1}^{N} \left\{ \sum_{k=1}^{N} \frac{\rho'_{\mu k}\rho_\ell}{1-e^{-(\alpha_\ell+\alpha_k)T}} \right\} \left[\frac{1}{1-e^{-\alpha_\ell T} Z^{-1}} - 1 \right]$$

$$= \sum_{\ell=1}^{N} H'_{\mu\ell} \frac{1}{1-e^{-\alpha_\ell T} Z} + \sum_{\ell=1}^{N} J'_{\mu\ell} \left(\frac{1}{1-e^{-\alpha_\ell T} Z^{-1}} - 1 \right) \qquad (2.1.64)$$

where

$$H'_{\mu\ell} = \rho'_{\mu\ell} \sum_{k=1}^{N} \frac{\rho_k}{1-e^{-(\alpha_\ell+\alpha_k)T}} \qquad (2.1.65)$$

$$J'_{\mu\ell} = \rho_\ell \sum_{k=1}^{N} \frac{\rho'_{\mu k}}{1-e^{-(\alpha_\ell+\alpha_k)T}} \qquad (2.1.66)$$

Inversion of 2.1.64 with the required shift in argument gives

$$\left[\frac{R'_\mu(Z) R\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} \right] [(m+1)T] = \sum_{\ell=1}^{N} H'_{\mu\ell}\, e_-^{\alpha_\ell T(m+1)} +$$

$$+ \sum_{\ell=1}^{N} J'_{\mu\ell} \left(e_+^{-\alpha_\ell T(m+1)} - \delta_{m+1,o} \right) \qquad (2.1.67)$$

Therefore, from 2.1.54, 2.1.59, 2.1.61, 2.1.62 and 2.1.67

$$\left[\frac{R_\mu^{'}(Z)R\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)}\right][(m+1)T] = \left[Q_\mu^{'}(Z)Q\left(Z^{-1}\right)\right][(m+1)T] +$$

$$+\left[\frac{R_\mu^{'}(Z)}{\phi_N(Z)}\,Q\left(Z^{-1}\right)\right][(m+1)T] + \left[Q_\mu^{'}(Z)\,\frac{R\left(Z^{-1}\right)}{\phi_N\left(Z^{-1}\right)}\right][(m+1)T] +$$

$$+\left[\frac{R_\mu^{'}(Z)R\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)}\right][(m+1)T] \tag{2.1.68}$$

$$= \sum_{\ell=o}^{D-N-1} E_{\mu\ell}^{'}\delta_{m+1,-\ell} + \sum_{\ell=1}^{D-N} G_{\mu\ell}^{'}\delta_{m+1,\ell} + \sum_{\ell=1}^{N}\sum_{k=o}^{D-N-1} \rho_\ell Q_{\mu k}^{'} e_+^{-\alpha_\ell T(m+1+k)}$$

$$+ \sum_{\ell=1}^{N}\sum_{k=o}^{D-N} \rho_{\mu\ell}^{'} Q_k e_-^{\alpha_\ell T(m+1-k)} + \sum_{\ell=1}^{N} H_{\mu\ell}^{'} e_-^{\alpha_\ell T(m+1)} \tag{2.1.69}$$

$$+ \sum_{\ell=1}^{N} J_{\mu\ell}^{'}\left[e_+^{-\alpha_\ell T(m+1)} - \delta_{m+1,o}\right]$$

Expression for $\overline{W}_\mu(mT)$

From 2.1.14, 2.1.32, 2.1.52 and 2.1.69

$$\overline{W}_\mu(mT) = \sum_{\ell=o}^{D-N} q_\ell \delta_{|m-\mu|,\ell} + \sum_{\ell=1}^{N} \sum_{k=o}^{D-N} Q_k \rho_\ell \left[e_+^{-\alpha_\ell T(m-\mu+k)} + e_-^{\alpha_\ell T(m-\mu-k)} \right] +$$

$$+ \sum_{\ell=1}^{N} C_\ell e^{-\alpha_\ell T|m-\mu|} + \sum_{\ell=o}^{D-N-1} E_{\mu\ell} \delta_{m-M-1,\ell} + \sum_{k=1}^{D-N} G_{\mu k} \delta_{m-M-1,-k} +$$

$$+ \sum_{\ell=1}^{N} \sum_{k=o}^{D-N-1} \rho_\ell Q_{\mu k} e_-^{\alpha_\ell T(m-M-1-k)} + \sum_{\ell=1}^{N} \sum_{k=o}^{D-N} \rho_{\mu\ell} Q_k e_+^{-\alpha_\ell T(m-M-1+k)} +$$

$$+ \sum_{\ell=1}^{N} J_{\mu\ell} e_-^{\alpha_\ell T(m-M-1)} + \sum_{\ell=1}^{N} H_{\mu\ell} \left[e_+^{-\alpha_\ell T(m-M-1)} - \delta_{(m-M-1)o} \right]$$

$$+ \sum_{\ell=o}^{D-N-1} E'_{\mu\ell} \delta_{m+1,-\ell} + \sum_{\ell=1}^{D-N} G'_{\mu\ell} \delta_{m+1,\ell} + \sum_{\ell=1}^{N} \sum_{k=o}^{D-N-1} \rho_\ell Q'_{\mu k} e_+^{-\alpha_\ell T(m+1+k)}$$

$$+ \sum_{\ell=1}^{N} \sum_{k=o}^{D-N} \rho'_{\mu\ell} Q_k e_-^{\alpha_\ell T(m+1-k)} + \sum_{\ell=1}^{N} H'_{\mu\ell} e_-^{\alpha_\ell T(m+1)} +$$

$$+ \sum_{\ell=1}^{N} J'_{\mu\ell} \left[e_+^{-\alpha_\ell T(m+1)} - \delta_{m+1,o} \right] \qquad (2.1.70)$$

where

$$q_\ell = \sum_{k=\ell}^{D-N} Q_k Q_{k-\ell} \quad ; \qquad C_\ell = \sum_{k=1}^{N} \frac{\rho_k \rho_\ell}{1-e^{-(\alpha_\ell+\alpha_k)T}}$$

$$E_{\mu\ell} = \sum_{k=\ell}^{D-N-1} Q_{\mu k} Q_{k-\ell} \quad ; \quad G_{\mu\ell} = \sum_{k=\ell}^{D-N} Q_k Q_{\mu,k-\ell} \quad ; \quad E'_{\mu\ell} = \sum_{k=\ell}^{D-N-1} Q'_{\mu k} Q_{k-\ell} \ ;$$

$$G'_{\mu\ell} = \sum_{k=\ell}^{D-N} Q_k Q'_{\mu,k-\ell} \qquad (2.1.71)$$

$$H_{\mu\ell} = \rho_{\mu\ell} \sum_{k=1}^{N} \frac{\rho_k}{1-e^{-(\alpha_k+\alpha_\ell)T}} \quad ; \quad J_{\mu\ell} = \rho_\ell \sum_{k=1}^{N} \frac{\rho_{\mu k}}{1-e^{-(\alpha_\ell+\alpha_k)T}} \ ;$$

$$H'_{\mu\ell} = \rho'_{\mu\ell} \sum_{k=1}^{N} \frac{\rho_k}{1-e^{-(\alpha_k+\alpha_\ell)T}} \quad ; \quad J'_{\mu\ell} = \rho_\ell \sum_{k=1}^{N} \frac{\rho'_{\mu k}}{1-e^{-(\alpha_\ell+\alpha_k)T}}$$

Equation 2.1.70 and the definitions 2.1.71 express the final result in terms of the quantities $Q_{\mu k}$, $Q'_{\mu k}$, $\rho_{\mu k}$, and $\rho'_{\mu k}$ which are at present unknown. The fact that $\overline{W}_\mu(mT)$ is nonzero only for $0 \le m \le M$ will now be used to determine these unknown quantities. It should be noted that to insure that $\overline{W}_\mu(mT)$ is nonzero only for $0 \le m \le M$ with arbitrary α_ℓ , it is necessary and sufficient to set the coefficients of the Kronecker delta terms and the exponential terms which occur outside of the interval separately equal to zero.

Upon examination of the Kronecker delta terms for $m \geq M + 1$ (with the exception of the term in $\delta_{m-M-1,0}$ which will be discussed later) it is seen that

$$\sum_{\ell=0}^{D-N} q_\ell \delta_{|m-\mu|,\ell} + \sum_{\ell=0}^{D-N-1} E_{\mu\ell} \delta_{m-M-1,\ell} = 0 \qquad \begin{matrix} m \geq M+1 \\ 0 \leq \mu \leq M \end{matrix} \tag{2.1.72}$$

must hold. But from 2.1.22 and 2.1.41, for the values of m and μ being considered, 2.1.72 is equivalent to

$$\left[Q(Z)Q\left(Z^{-1}\right)\right][(m-\mu)T] + \left[Q_\mu\left(Z^{-1}\right)Q(Z)\right][(m-M-1)T] = 0 \tag{2.1.73}$$

or after a shift in argument

$$\left[Q(Z)Q\left(Z^{-1}\right)\right][(m+M+1-\mu)T] + \left[Q_\mu\left(Z^{-1}\right)Q(Z)\right][(mT)] = 0 \tag{2.1.74}$$

Because of the form of 2.1.74 and the polynomial nature of $Q(Z)$, $Q\left(Z^{-1}\right)$, and $Q_\mu\left(Z^{-1}\right)$ the solution of 2.1.74 is

$$Q_{\mu n} = \left[Q_\mu\left(Z^{-1}\right)\right][(nT)] = -\left[Q\left(Z^{-1}\right)\right][(n+M+1-\mu)T] \qquad 0 \leq n \leq D-N-1$$

$$= - Q_{n+M+1-\mu} \qquad 0 \leq n + M + 1 - \mu \leq D-N \tag{2.1.75}$$

As before, examination of the Kronecker delta terms for $m \leq -1$ (with the exception of the term in $\delta_{m+1,0}$ which will be discussed later) shows that

$$\sum_{\ell=0}^{D-N} q_\ell \delta_{|m-\mu|,\ell} + \sum_{\ell=0}^{D-N-1} E'_{\mu\ell}\delta_{m+1,-\ell} = 0 \qquad \begin{matrix} m \le -1 \\ 0 \le \mu \le M \end{matrix} \tag{2.1.76}$$

But from 2.1.22 and 2.1.59, for the values of m and μ being considered, 2.1.76 is equivalent to

$$\left[Q(Z)Q\left(Z^{-1}\right)\right][(m-\mu)T] = -\left[Q'_\mu(Z)Q\left(Z^{-1}\right)\right][(m+1)T], \tag{2.1.77}$$

which by reasoning similar to that just used, has the solution

$$\left[Q'_\mu(Z)\right][(nT)] = -\left[Q(Z)\right][(n-\mu-1)T] \qquad 0 \ge n \ge -(D-N-1)$$

or

$$Q'_{\mu n} = -Q_{\mu+1+n} \quad . \qquad \begin{matrix} 0 \le \mu + 1 + n \le D-N \\ 0 \le n \le (D-N-1) \end{matrix} \tag{2.1.78}$$

To summarize, Equations 2.1.75 and 2.1.78 give the $Q_{\mu k}$ and $Q'_{\mu k}$ explicitly in terms of the Q_k . It is now necessary to determine the $\rho_{\mu k}$ and $\rho'_{\mu k}$.

Examination of the exponential terms of 2.1.70 outside of the interval $0 \le m \le M$ shows that for there to be no exponential components outside of the interval it is necessary and sufficient that

$$\sum_{\ell=1}^{N}\sum_{k=o}^{D-N} Q_k \rho_\ell e^{-\alpha_\ell T(m-\mu+k)} + \sum_{\ell=1}^{N} C_\ell e^{-\alpha_\ell T(m-\mu)} +$$

$$+ \sum_{\ell=1}^{N}\sum_{k=o}^{D-N} \rho_{\mu\ell} Q_k e^{-\alpha_\ell T(m-M-1+k)}$$

$$+ \sum_{\ell=1}^{N} \rho_{\mu\ell} \sum_{k=1}^{N} \frac{\rho_k}{1-e^{-(\alpha_k+\alpha_\ell)T}} e^{-\alpha_\ell T(m-M-1)} +$$

$$+ \sum_{\ell=1}^{N}\sum_{k=o}^{D-N-1} \rho_\ell Q'_{\mu k} e^{-\alpha_\ell T(m+1+k)}$$

$$+ \sum_{\ell=1}^{N} \rho_\ell \sum_{k=1}^{N} \frac{\rho'_{\mu k} e^{-\alpha_\ell T(m+1)}}{1-e^{-(\alpha_\ell+\alpha_k)T}} = 0 \qquad m \geq M+1 \qquad (2.1.79)$$

and

$$\sum_{\ell=1}^{N}\sum_{k=o}^{D-N} Q_k \rho_\ell e^{\alpha_\ell T(m-\mu-k)} + \sum_{\ell=1}^{N} C_\ell e^{-\alpha_\ell T(\mu-m)} +$$

$$+ \sum_{\ell=1}^{N}\sum_{k=o}^{D-N-1} \rho_\ell Q_{\mu k} e^{\alpha_\ell T(m-M-1-k)}$$

$$+ \sum_{\ell=1}^{N} \rho_\ell \sum_{k=1}^{N} \frac{\rho_{\mu k}}{1-e^{-(\alpha_\ell+\alpha_k)T}} e^{\alpha_\ell T(m-M-1)} + \sum_{\ell=1}^{N}\sum_{k=o}^{D-N} \rho'_{\mu\ell} Q_k e^{\alpha_\ell T(m+1-k)} +$$

$$+\sum_{\ell=1}^{N}\rho_{\mu\ell}'\sum_{k=1}^{N}\frac{\rho_k e^{\alpha_\ell(m+1)}}{1-e^{-(\alpha_k+\alpha_\ell)T}}=0 \qquad m\le -1 \tag{2.1.80}$$

The definitions for the J's and H's given in 2.1.71 were used in obtaining 2.1.79 and 2.1.80. Also, it was noted that

$$\sum_{\ell=1}^{N} J_{\mu\ell}=\sum_{\ell=1}^{N} H_{\mu\ell} \quad \text{and} \quad \sum_{\ell=1}^{N} J_{\mu\ell}'=\sum_{\ell=1}^{N} H_{\mu\ell}'$$ in eliminating the

terms is $\delta_{m+1,o}$ and $\delta_{m-M-1,o}$.

Collecting the coefficients of $e^{-\alpha_\ell T}$ in 2.1.79 and the coefficient of $e^{\alpha Tm}$ in 2.1.80 and setting them separately equal to zero gives

$$e^{\mu\alpha_\ell T}\sum_{k-o}^{D-N} Q_k\rho_\ell e^{-k\alpha_\ell T}+C_\ell e^{\mu\alpha_\ell T}+\sum_{k=o}^{D-N-1}\rho_\ell Q_{\mu k}' e^{-\alpha_\ell T(1+k)}$$

$$+\rho_{\mu\ell}e^{\alpha_\ell MT}\left[\sum_{k=o}^{D-N} Q_k e^{-\alpha_\ell T(k-1)}+\sum_{k=1}^{N}\frac{\rho_k}{1-e^{-(\alpha_k+\alpha_\ell)T}}e^{\alpha_\ell T}\right]$$

$$+\rho_\ell e^{-\alpha_\ell T}\sum_{k=1}^{N}\frac{\rho_{\mu k}'}{1-e^{-(\alpha_\ell+\alpha_k)T}}=0 \tag{2.1.81}$$

and

$$e^{-\mu\alpha_\ell T}\sum_{k=o}^{D-N} Q_k\rho_\ell e^{-k\alpha_\ell T} + C_\ell e^{-\mu\alpha_\ell T} + e^{-\alpha_\ell MT}\sum_{k=o}^{D-N-1} \rho_\ell Q_{\mu k} e^{-\alpha_\ell T(1+k)}$$

$$+ \rho'_{\mu\ell}\left[\sum_{k=o}^{D-N} Q_k e^{-\alpha_\ell T(k-1)} + \sum_{k=1}^{N} \frac{\rho_k}{1-e^{-(\alpha_k+\alpha_\ell)T}} e^{\alpha_\ell T}\right]$$

$$+ \rho_\ell e^{-(M+1)\alpha_\ell T}\sum_{k=o}^{N} \frac{\rho_{\mu k}}{1-e^{-(\alpha_k+\alpha_\ell)T}} = 0 \qquad (2.1.82)$$

After multiplication of 2.1.81 by $e^{-\alpha_\ell MT}$ and using 2.1.75 and 2.1.78, Equations 2.1.81 and 2.1.82 become

$$e^{-(M-\mu)\alpha_\ell T}\sum_{k=o}^{D-N} Q_k\rho_\ell e^{-k\alpha_\ell T} + C_\ell e^{-(M-\mu)\alpha_\ell T} - \rho_\ell e^{-\alpha_\ell T(M+1)}\sum_{k=o}^{D-N-1} Q_{\mu+1+k} e^{-\alpha_\ell Tk}$$

$$+ \rho_{\mu\ell} e^{\alpha_\ell T}\left[\sum_{k=o}^{D-N} Q_k e^{-\alpha_\ell Tk} + \sum_{k=1}^{N} \frac{\rho_k}{1-e^{-(\alpha_\ell+\alpha_k)T}}\right] +$$

$$+ \rho_\ell e^{-(M+1)\alpha_\ell T}\sum_{k=1}^{N} \frac{\rho'_{\mu k}}{1-e^{-(\alpha_\ell+\alpha_k)T}} = 0 \qquad (2.1.83)$$

and

$$e^{-\mu\alpha_\ell T}\sum_{k=o}^{D-N} Q_k\rho_\ell e^{-k\alpha_\ell T} + C_\ell e^{-\mu\alpha_\ell T} - \rho_\ell e^{-\alpha_\ell T(M+1)}\sum_{k=o}^{D-N-1} Q_{k+M+1-\mu} e^{-\alpha_\ell Tk}$$

$$+ \rho'_{\mu\ell} e^{\alpha_\ell T}\left[\sum_{k=o}^{D-N} Q_k e^{-\alpha_\ell Tk} + \sum_{k=1}^{N}\frac{\rho_k}{1-e^{-(\alpha_\ell+\alpha_k)T}}\right]+$$

$$+ \rho_\ell e^{-(M+1)\alpha_\ell T}\sum_{k=o}^{N}\frac{\rho_{\mu k}}{1-e^{-(\alpha_\ell+\alpha_k)T}} = 0 \qquad (2.1.84)$$

(It should be noted that the largest subscript value for the Q's is D-N; hence Q's with a larger subscript are considered to be zero.)

The system of equations of order 2N, 2.1.83 and 2.1.84, in the 2N unknowns $\rho_{\mu\ell}$ and $\rho'_{\mu\ell}$ must be solved to complete the solution for the inverse matrix. From the form of these equations it can be seen that $\rho'_{\mu\ell}$ can be obtained from $\rho_{\mu\ell}$ by replacing μ with $M-\mu$. To exhibit some of the other properties of 2.1.83 and 2.1.84 they will now be placed in matrix-vector notation and solved in terms of a N^{th} order inverse matrix.

In terms of the following definitions

$$\vec{\rho}_\mu = \begin{bmatrix} \rho_{\mu 1} \\ \rho_{\mu 2} \\ \cdot \\ \cdot \\ \cdot \\ \rho_{\mu N} \end{bmatrix}, \qquad \vec{\rho}'_\mu = \begin{bmatrix} \rho'_{\mu 1} \\ \rho'_{\mu 2} \\ \cdot \\ \cdot \\ \cdot \\ \rho'_{\mu N} \end{bmatrix}$$

$$A_{\ell k} = e^{\alpha_\ell T}\left[\sum_{k=o}^{D-N} Q_k e^{-\alpha_\ell Tk} + \sum_{k=1}^{N} \frac{\rho_k}{1-e^{-(\alpha_k+\alpha_\ell)T}}\right]\delta_{k\ell} \quad , \quad A = (A_{\ell k})$$

$$B_{\ell k} = \frac{\rho_\ell e^{-(M+1)\alpha_\ell T}}{1-e^{-(\alpha_\ell+\alpha_k)T}} \quad , \quad B = (B_{\ell k}) \tag{2.1.85}$$

$$\Gamma_{\mu\ell} = -e^{-(M-\mu)\alpha_\ell T}\sum_{k=o}^{D-N} Q_k \rho_\ell e^{-k\alpha_\ell T} - C_\ell e^{-(M-\mu)\alpha_\ell T} +$$

$$+ \rho_\ell e^{-\alpha_\ell T(M+1)} \sum_{k=o}^{D-N-1} Q_{\mu+1+k} e^{-\alpha_\ell Tk}$$

$$\Gamma'_{\mu\ell} = -e^{-\mu\alpha_\ell T}\sum_{k=o}^{D-N} Q_k \rho_\ell e^{-k\alpha_\ell T} - C_\ell e^{-\mu\alpha_\ell T} +$$

$$+ \rho_\ell e^{-\alpha_\ell T(M+1)} \sum_{k=o}^{D-N-1} Q_{k+M+1-\mu} e^{-\alpha_\ell Tk}$$

$$\vec{\Gamma}_\mu = \begin{bmatrix} \Gamma_{\mu 1} \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_{\mu N} \end{bmatrix} \quad , \quad \Gamma'_\mu = \begin{bmatrix} \Gamma'_{\mu 1} \\ \cdot \\ \cdot \\ \cdot \\ \Gamma'_{\mu N} \end{bmatrix}$$

2.1.83 and 2.1.84 become

$$A\vec{\rho}_{\mu} + B\vec{\rho}\,'_{\mu} = \vec{\Gamma}_{\mu} \tag{2.1.86}$$

and

$$B\vec{\rho}_{\mu} + A\vec{\rho}\,'_{\mu} = \vec{\Gamma}\,'_{\mu} \tag{2.1.87}$$

which have the solutions

$$\vec{\rho}_{\mu} = \left[A - BA^{-1} B\right]^{-1} \left[\vec{\Gamma}_{\mu} - BA^{-1} \vec{\Gamma}\,'_{\mu}\right] \tag{2.1.88}$$

$$\vec{\rho}\,'_{\mu} = \left[A - BA^{-1} B\right]^{-1} \left[\vec{\Gamma}\,'_{\mu} - BA^{-1} \vec{\Gamma}_{\mu}\right] \tag{2.1.89}$$

In 2.1.88 and 2.1.89 it is important to note that $A - BA^{-1} B$ does not depend on μ and is a N^{th} order matrix and that A is a diagonal matrix. This means that to invert the $M+1$ order covariance matrix only a N^{th} order matrix must be inverted where $N \leqq D - 1$ and D is the order of the noise.

The final answer for the inverse covariance matrix is

$$W_{\mu}(mT) = \sum_{\ell=o}^{D-N} q_{\ell}\delta_{|m-\mu|,\ell} + \sum_{\ell-1}^{N} \sum_{k=o}^{D-N} Q_k \rho_{\ell}\left[e_{+}^{-\alpha_{\ell}T(m-\mu+\ell)} + e_{-}^{\alpha_{\ell}T(m-\mu-\ell)}\right] +$$

$$+ \sum_{\ell=1}^{N} C_{\ell} e^{-\alpha_{\ell}T|m-\mu|} +$$

$$+ \sum_{k=1}^{D-N} G_{\mu k}\delta_{m-M-1,-k} + \sum_{\ell=1}^{N}\sum_{k=o}^{D-N-1} \rho_\ell Q_{\mu k} e_-^{\alpha_\ell T(m-M-1-k)} +$$

$$+ \sum_{\ell=1}^{N}\sum_{k=o}^{D-N} \rho_{\mu\ell} Q_k e_+^{-\alpha_\ell T(m-M-1+k)}$$

$$+ \sum_{\ell=1}^{N} J_{\mu\ell} e_-^{\alpha_\ell T(m-M-1)} + \sum_{k=1}^{D-N} G'_{\mu k}\delta_{m+1,k} + \sum_{\ell=1}^{N}\sum_{k=o}^{D-N-1} \rho_\ell Q'_{\mu k} e_+^{-\alpha_\ell T(m+1+k)}$$

$$+ \sum_{\ell=1}^{N}\sum_{k=o}^{D-N} \rho'_{\mu\ell} Q_k e_-^{\alpha_\ell T(m+1-k)} + \sum_{\ell=1}^{N} J'_{\mu\ell} e_+^{-\alpha_\ell T(m+1)} \qquad \begin{array}{l} 0 \le m \le M \\ 0 \le \mu \le M \end{array}$$

(2.1.90)

where the components of 2.1.90 are determined through 2.1.71, 2.1.75, 2.1.78, 2.1.85, 2.1.88 and 2.1.89, and only the terms which are non-zero in the interval $0 \le m \le M$ have been retained.

2.2 First Order Noise

In this thesis first order noise is defined as noise with the autocorrelation function

$$\phi(t) = \sigma_1^2 e^{-\beta_1 |t|} \qquad (2.2.1)$$

and hence the m,μ^{th} element of its covariance matrix is

$$\phi[(m-\mu)T] = \sigma_1^{2} e^{-\beta_1 T|m-\mu|} \tag{2.2.2}$$

The sampled autocorrelation function is

$$\phi(mT) = \sigma_1^{2} e^{-\beta_1 T|m|} \tag{2.2.3}$$

and the two-sided transform of 2.2.3 is

$$\phi^*(Z) = \sigma_1^{2} \frac{1-e^{-2\beta T}}{\left(1-e^{-\beta T}Z\right)\left(1-e^{-\beta T}Z^{-1}\right)}$$

$$= \frac{1}{\sqrt{\frac{\sigma_1^{-2}}{1-e^{-2\beta T}}}\left(1-e^{-\beta T}Z\right)} \quad \frac{1}{\sqrt{\frac{\sigma_1^{-2}}{1-e^{-2\beta T}}}\left(1-e^{-\beta T}Z^{-1}\right)}$$

$$= \frac{\phi_N(Z)\phi_N\left(Z^{-1}\right)}{\phi_D(Z)\phi_D\left(Z^{-1}\right)} \tag{2.2.4}$$

where

$$\phi_N(Z) = 1 \quad , \quad N = 0$$

$$\phi_D(Z) = K\left(1-e^{-\beta T}Z\right) \quad , \quad D = 1 \tag{2.2.5}$$

$$K = \sqrt{\frac{\sigma_1^{-2}}{1-e^{-2\beta T}}}$$

Now that the preliminaries have been disposed of, the elements of the inverse covariance matrix will be calculated. From 2.1.15 and 2.2.5

$$\frac{\phi_D(Z)}{\phi_N(Z)} = K\left(1-e^{-\beta T}Z\right) = Q(Z) + \frac{R(Z)}{\phi(Z)} \quad , \qquad (2.2.6)$$

therefore

$$Q_o = K$$

$$Q_1 = -Ke^{-\beta T} \qquad (2.2.7)$$

$$R(Z) \equiv 0 \qquad \text{(which implies } \rho_\ell \equiv 0\text{)}$$

Since $\rho_\ell \equiv 0$, inspection of 2.1.81 and 2.1.82 shows that $\rho_{\mu\ell} = \rho'_{\mu\ell} = 0$, therefore only the $Q_{\mu k}$ and $Q'_{\mu k}$ need be determined. Using 2.1.75 and 2.1.78 gives

$$Q_{\mu o} = -Q_{M+1-\mu} = -Q_1 \delta_{\mu M} \qquad (2.2.8)$$

$$Q'_{\mu o} = -Q_{\mu+1} = -Q_1 \delta_{\mu o} \qquad (2.2.9)$$

From the final form of the inverse matrix, 2.1.90,

$$W_\mu(mT) = q_o \delta_{|m-\mu|,o} + q_1 \delta_{|m-\mu|,1} + G_{\mu 1} \delta_{m-M-1,-1} + G'_{\mu 1} \delta_{m+1,1}$$

$$= q_o \delta_{|m-\mu|,o} + q_1 \delta_{|m-\mu|,1} + G_{\mu 1} \delta_{m,M} + G'_{\mu 1} \delta_{m,o} \qquad (2.2.10)$$

where from 2.1.71, 2.2.8, 2.2.9

$$q_o = Q_o^{\,2} + Q_1^{\,2} = \sigma_1^{-2} \frac{1+e^{-2\beta_1 T}}{1-e^{-2\beta_1 T}}$$

$$q_1 = Q_o Q_1 = - \sigma_1^{-2} \frac{e^{-\beta_1 T}}{1-e^{-\beta_1 T}}$$

$$G_{\mu 1} = Q_1 Q_{\mu o} = - Q_1^{\,2} \delta_{\mu M} = - \frac{\sigma_1^{-2} e^{-2\beta_1 T}}{1-e^{-2\beta_1 T}} \delta_{\mu M}$$

$$G'_{\mu 1} = Q_1 Q'_{\mu o} = - Q_1^{\,2} \delta_{\mu o} = - \frac{\sigma_1^{-2} e^{-2\beta T}}{1-e^{-2\beta T}} \delta_{\mu o} \qquad ,$$

therefore

$$W_\mu(mT) = \frac{\sigma_1^{-2}}{1-e^{-2\beta_1 T}} \left[\left(1+e^{-2\beta_1 T}\right) \delta_{|m-\mu|,o} - e^{-\beta_1 T} \delta_{|m-\mu|,1} - e^{-2\beta_1 T} \delta_{\mu M} \delta_{m,M} - \right.$$

$$\left. - e^{-2\beta T} \delta_{mo} \delta_{\mu o} \right] \qquad (2.2.11)$$

Equation 2.2.11 is the final result and explicitly shows that the inverse covariance matrix has nonzero elements only on the main diagonal and the two adjacent diagonals.

2.3 Second Order Noise

In this section the inverse covariance matrix of equally spaced samples of second order noise will be determined by the use of formulas derived in 2.1. By definition, the autocorrelation function of second order noise is

$$\phi(t) = \sigma_1^2 e^{-\beta_1|t|} + \sigma_2^2 e^{-\beta_2|t|} \tag{2.3.1}$$

where the β's and σ's can be complex conjugates or one of the σ^2's could be negative. The sampled autocorrelation function is

$$\phi(mT) = \sigma_1^2 e^{-\beta_1 T|m|} + \sigma_2^2 e^{-\beta_2|m|T}$$

and its two-sided Z-transform is

$$\phi^*(Z) = \sigma_1^2 \frac{1-e^{-2\beta_1 T}}{\left(1-e^{-\beta_1 T}Z\right)\left(1-e^{-\beta_1 T}Z^{-1}\right)} +$$

$$+ \sigma_2^2 \frac{1-e^{-2\beta_2 T}}{\left(1-e^{-\beta_2 T}Z\right)\left(1-e^{-\beta_2 T}Z^{-1}\right)}$$

$$= \frac{B_0 - A_0\left(Z+Z^{-1}\right)}{\left(1-e^{-\beta_1 T}Z\right)\left(1-e^{-\beta_2 T}Z\right)\left(1-e^{-\beta_1 T}Z^{-1}\right)\left(1-e^{-\beta_2 T}Z^{-1}\right)} \tag{2.3.2}$$

where

$$A_o = \sigma_1^2\left(1-e^{-2\beta_1 T}\right)e^{-\beta_2 T} + \sigma_2^2\left(1-e^{-2\beta_2 T}\right)e^{-\beta_1 T} \qquad (2.3.3)$$

$$B_o = \sigma_1^2\left(1-e^{-2\beta_1 T}\right)\left(1+e^{-2\beta_2 T}\right) + \sigma_2^2\left(1-e^{-2\beta_2 T}\right)\left(1+e^{-2\beta_1 T}\right) \qquad (2.3.4)$$

Upon factoring the numerator of 2.3.2 and using the definitions

$$e^{\pm\alpha_1 T} = \frac{B_o}{2A_o} \pm \sqrt{\frac{B_o^2}{4A_o^2} - 1} \qquad (2.3.5)$$

$$k = A_o e^{\alpha_1 T} = \frac{1}{2}\left(B_o + \sqrt{B_o - 4A_o^2}\right) \qquad (2.3.6)$$

2.3.2 can be expressed

$$\phi^*(Z) = \frac{1-e^{-\alpha_1 T}Z^{-1}}{\sqrt{\frac{1}{k}}\left(1-e^{-\beta_1 T}Z^{-1}\right)\left(1-e^{-\beta_2 T}Z^{-1}\right)} \; \frac{1-e^{-\alpha_1 T}Z}{\sqrt{\frac{1}{k}}\left(1-e^{-\beta_1 T}Z\right)\left(1-e^{-\beta_2 T}Z\right)}$$

$$= \frac{\phi_N\left(Z^{-1}\right)}{\phi_D\left(Z^{-1}\right)} \; \frac{\phi_N(Z)}{\phi_D(Z)} \qquad (2.3.7)$$

where

$$\phi_D(Z) = \frac{1}{\sqrt{k}}\left[1-\left(e^{-\beta_1 T}+e^{-\beta_2 T}\right)Z + e^{-(\beta_1+\beta_2)T}Z^2\right], \quad D=2 \qquad (2.3.8)$$

$$\phi_N(Z) = 1 - e^{-\alpha_1 T} Z \qquad N = 1 \tag{2.3.9}$$

By long division the ratio $\dfrac{\phi_D(Z)}{\phi_N(Z)}$ can be expressed

$$\frac{\phi_D(Z)}{\phi_N(Z)} = Q_1 Z + Q_o + \frac{\rho_1}{1-e^{-\alpha T}Z} \tag{2.3.10}$$

where

$$Q_1 = -k^{-\frac{1}{2}} e^{-(\beta_1+\beta_2-\alpha)T} \tag{2.3.11}$$

$$Q_o = -k^{-\frac{1}{2}}\left[e^{-(\beta_1+\beta_2-2\alpha)T} - e^{-(\beta_1-\alpha)T}\right] \tag{2.3.12}$$

$$\rho_1 = k^{-\frac{1}{2}}\left[1+e^{-(\beta_1+\beta_2-2\alpha)T} - e^{-(\beta_1-\alpha)T} - e^{-(\beta_2-\alpha)T}\right] \tag{2.3.13}$$

and the subscript on α will be omitted in the remainder of this section.

The Q_μ's and the Q_μ''s will now be determined. From (2.1.75)

$$Q_{\mu o} = -Q_{M+1-\mu} = -Q_1 \delta_{\mu M} \tag{2.3.14}$$

and from (2.1.78)

$$Q'_{\mu o} = -Q_{\mu+1} = -Q_1 \delta_{\mu o} \tag{2.3.15}$$

Since $N = 1$ only $\rho_{\mu 1}$ and $\rho'_{\mu 1}$ are present and they will now be calculated. It is convenient to use the quantities (see 2.3.9 and 2.3.10)

$$\frac{\phi_D\left(e^{-\alpha T}\right)}{\phi_N\left(e^{-\alpha T}\right)} = Q_o + Q_1 e^{-\alpha T} + \frac{\rho_1}{1-e^{-2\alpha T}} \tag{2.3.16}$$

$$\phi_N\left(e^{-\alpha T}\right) = 1-e^{-2\alpha T} \tag{2.3.17}$$

in the solution. From (2.1.85)

$$A = e^{\alpha T} \frac{\phi_D\left(e^{-\alpha T}\right)}{\phi_N\left(e^{-\alpha T}\right)}$$

$$B = \frac{\rho_1 e^{-(M+1)\alpha T}}{\phi_N\left(e^{-\alpha T}\right)}$$

and hence

$$BA^{-1} = \frac{1}{\phi_D} \rho_1 e^{-(M+2)\alpha T} \tag{2.3.18}$$

$$\left[A-BA^{-1}B\right]^{-1} = \frac{\phi_N \phi_D}{\phi_D^2 - \rho_1^2 e^{-2(M+2)\alpha T}} \tag{2.3.19}$$

where the arguments of $\phi_N\left(e^{-\alpha T}\right)$ and $\phi_D\left(e^{-\alpha T}\right)$ will be omitted in the remainder of this section.

Also from 2.1.85 (see 2.1.71 for the definition of C_1)

$$\Gamma_{\mu 1} = -\rho_1\left[\frac{\phi_D}{\phi_N} e^{-(M-\mu)\alpha T} - Q_1\delta_{\mu o}e^{-\alpha T(M+1)}\right] \qquad (2.3.20)$$

$$\Gamma_{\mu 1}' = -\rho_1\left[\frac{\phi_D}{\phi_N} e^{-\mu\alpha T} - Q_1\delta_{\mu M}e^{-\alpha T(M+1)}\right] \qquad (2.3.21)$$

Therefore (2.1.88) gives

$$\rho_{\mu 1} = \frac{-\rho_1}{\phi_D^{\,2}-\rho_1^{\,2}e^{-2\alpha T(M+2)}}\left\{\phi_D\left[\phi_D e^{-\alpha T(M+1-\mu)}-Q_1\delta_{\mu o}\phi_N e^{-\alpha T(M+2)}\right]\right.$$
$$\left. -\rho_1 e^{-\alpha T(M+2)}\left[\phi_D e^{-\alpha T(1+\mu)} - Q_1\delta_{\mu M}\phi_N e^{-\alpha T(M+2)}\right]\right\} \qquad (2.3.22)$$

As previously mentioned $\rho_{\mu 1}'$ can be obtained from $\rho_{\mu 1}$ by replacing μ by $M-\mu$. The result of performing this operation in 2.3.22 is

$$\rho_{\mu 1}' = \frac{-\rho_1}{\phi_D^{\,2}-\rho_1^{\,2}e^{-2\alpha T(M+2)}}\left\{\phi_D\left[\phi_D e^{-\alpha T(1+\mu)}-Q_1\delta_{\mu M}\phi_N e^{-\alpha T(M+2)}\right]\right.$$
$$\left. -\rho_1 e^{-\alpha T(M+2)}\left[\phi_D\, e^{-\alpha T}\, e^{-\alpha T(M-\mu)}-Q_1\delta_{\mu o}\phi_N e^{-\alpha T(M+2)}\right]\right\} \qquad (2.3.23)$$

The following quantities are necessary to specify (2.1.90), which is the final answer (see 2.1.71 for definitions and note that $0 \le m,\mu \le M$)

1. $$q_\ell = \sum_{k=\ell}^{D-N} Q_k Q_{k-\ell}$$

$$q_o = Q_o^2 + Q_1^2$$

$$q_1 = Q_o Q_1$$

and

$$\sum_{\ell=o}^{D-N} q_\ell \delta_{|m-\mu|,\ell} = \left(Q_o^2 + Q_1^2\right)\delta_{|m-\mu|,o} + Q_o Q_1 \delta_{|m-\mu|,1} \qquad (2.3.24)$$

2. $$\sum_{\ell=1}^{N} \sum_{k=o}^{D-N} Q_k \rho_\ell \left[e_+^{-\alpha_\ell T(m-\mu+\ell)} + e_-^{\alpha_\ell T(m-\mu-\ell)} \right] =$$

$$= Q_o \rho_1 \left[e_+^{-\alpha T(m-\mu)} + e_+^{-\alpha T(\mu-m)} \right] +$$

$$+ Q_1 \rho_1 \left[e_+^{-\alpha T[1+(m-\mu)]} + e_+^{-\alpha T[1-(m-\mu)]} \right]$$

$$= Q_o \rho_1 \left[\delta_{m\mu} + e^{-\alpha T|m-\mu|} \right] + Q_1 \rho_1 \left\{ e^{-\alpha T} \left[e^{-\alpha T|m-\mu|} + \delta_{|m-\mu|,o} \right] \right.$$

$$\left. + \delta_{|m-\mu|,1} \right\} \qquad (2.3.25)$$

3. $$C_\ell = \sum_{k=1}^{N} \frac{\rho_k \rho_\ell}{1-e^{-(\alpha_\ell+\alpha_k)T}} = \frac{\rho_1^{\,2}}{1-e^{-2\alpha T}} = C_1$$

$$\sum_{\ell=1}^{N} C_\ell e^{-\alpha_\ell T|m-\mu|} = \frac{\rho_1^{\,2}}{1-e^{-2\alpha T}}\, e^{-\alpha T|m-\mu|} \qquad (2.3.26)$$

4. $$G_{\mu k} = \sum_{\ell=k}^{D-N} Q_\ell Q_{\mu,\ell-k}$$

$$G_{\mu 1} = -Q_1^{\,2}\delta_{\mu M}$$

$$\sum_{k=1}^{D-N} G_{\mu k}\delta_{m-M-1,-k} = -Q_1^{\,2}\delta_{\mu M}\delta_{mM} \qquad (2.3.27)$$

5. $$\sum_{\ell=1}^{N}\sum_{k=0}^{D-N-1} \rho_\ell Q_{\mu k} e_-^{\alpha_\ell T(m-M-1-k)} = -Q_1 \rho_1 \delta_{\mu M} e^{-\alpha T(M+1-m)} \qquad (2.3.28)$$

6. $$\sum_{\ell=1}^{N}\sum_{k=0}^{D-N} \rho_{\mu\ell} Q_k e_+^{-\alpha_\ell T(m-M-1+k)} = \rho_{\mu 1} Q_1 \delta_{mM} \qquad (2.3.29)$$

7. $$J_{\mu\ell} = \rho_\ell \sum_{k=1}^{N} \frac{\rho_{\mu k}}{1-e^{-(\alpha_\ell+\alpha_k)T}} = \frac{\rho_1 \rho_{\mu 1}}{1-e^{-2\alpha T}} = J_{\mu 1}$$

$$\sum_{\ell=1}^{N} J_{\mu\ell} e_-^{\alpha_\ell T(m-M-1)} = \frac{\rho_1 \rho_{\mu 1}}{1-e^{-2\alpha T}}\, e^{-\alpha T(M+1-m)} \qquad (2.3.30)$$

8. $$G'_{\mu k} = \sum_{\ell=k}^{D-N} Q_\ell Q'_{\mu,\ell-k}$$

$$G_{\mu 1} = -Q_1{}^2 \delta_{\mu o}$$

$$\sum_{k=1}^{D-N} G'_{\mu k} \delta_{m+1,k} = -Q_1{}^2 \delta_{\mu o} \delta_{mo} \qquad (2.3.31)$$

9. $$\sum_{\ell=1}^{N} \sum_{k=o}^{D-N-1} \rho_\ell Q'_{\mu k}\, e_+^{-\alpha_\ell T(m+1+k)} = -Q_1 \rho_1 \delta_{\mu o} e^{-\alpha T(m+1)} \qquad (2.3.32)$$

10. $$\sum_{\ell=1}^{N} \sum_{k=o}^{D-N} \rho'_{\mu\ell} Q_k e_-^{\alpha_\ell T(m+1-k)} = \rho_{\mu 1} Q_1 \delta_{m,o} \qquad (2.3.33)$$

11. $$J'_{\mu\ell} = \rho_\ell \sum_{k=1}^{N} \frac{\rho'_{\mu k}}{1-e^{-(\alpha_\ell+\alpha_k)T}} = \frac{\rho_1 \rho'_{\mu 1}}{1-e^{-2\alpha T}} = J'_{\mu 1}$$

$$\sum_{\ell=1}^{N} J'_{\mu\ell} e_+^{-\alpha_\ell T(m+1)} = \frac{\rho_1 \rho'_{\mu 1}}{1-e^{-2\alpha T}}\, e^{-\alpha T(m+1)} \qquad (2.3.34)$$

Therefore from 2.1.90 and 2.3.24 through 2.3.34

$$W_\mu(mT) = \left(Q_o^{\ 2}+Q_1^{\ 2}\right)\delta_{|m-\mu|,o}+Q_1Q_o\delta_{|m-\mu|,1}+Q_o\rho_1\left[e^{-\alpha T|m-\mu|}+\delta_{|m-\mu|,o}\right]$$

$$+\ Q_1\rho_1\left\{e^{-\alpha T}\left[e^{-\alpha T|m-\mu|}+\delta_{|m-\mu|,o}\right]+\delta_{|m-\mu|,1}\right\}$$

$$+\ \frac{\rho_1^{\ 2}}{1-e^{-2\alpha T}}\,e^{-\alpha T|m-\mu|}\ -\ Q_1^{\ 2}\left(\delta_{\mu M}\delta_{mM}+\delta_{\mu o}\delta_{mo}\right)$$

$$-\ Q_1\rho_1\left[\delta_{\mu M}e^{-\alpha T(M+1-m)}\ +\ \delta_{\mu o}e^{-\alpha T(m+1)}\right]+\rho_{\mu 1}Q_1\delta_{mM}+\rho_{\mu 1}^{'}Q_1\delta_{mo}$$

$$+\ \frac{\rho_1}{1-e^{-2\alpha T}}\left[\rho_{\mu 1}e^{-\alpha T(M+1-m)}\ +\ \rho_{\mu 1}^{'}e^{-\alpha T(m+1)}\right] \qquad (2.3.35)$$

where $\rho_{\mu 1}$ and $\rho_{\mu 1}^{'}$ are defined in 2.3.22 and 2.3.23. Upon substituting 2.3.22 and 2.3.23 in 2.3.35 and performing some algebraic manipulation, $W_\mu(mT)$ can be expressed

$$W_\mu(mT) = -Q_1^2\left[\delta_{\mu M}\delta_{mM}+\delta_{\mu o}\delta_{mo}\right] - \frac{\rho_1 Q_1^2 \phi_N e^{-\alpha T(M+2)}}{\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}}\left\{\rho_1 e^{-\alpha T(M+2)}\left[\delta_{\mu M}\delta_{mM}+\delta_{mo}\delta_{\mu o}\right]\right.$$

$$\left. -\phi_D\left[\delta_{mM}\delta_{\mu o}+\delta_{\mu M}\delta_{mo}\right]\right\} + \rho_1 \frac{\phi_D}{\phi_N} e^{-\alpha T|m-\mu|}$$

$$-\frac{\rho_1 Q_1 \phi_D}{\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}}\left\{\phi_D\left[\delta_{\mu M}e^{-\alpha T(M+1-m)}+\delta_{mM}e^{-\alpha T(M+1-\mu)}+\right.\right.$$

$$\left. +\delta_{\mu o}e^{-\alpha T(1+m)}+\delta_{mo}e^{-\alpha T(1+\mu)}\right] - \rho_1 e^{-\alpha T(M+2)}\left[\delta_{\mu M}e^{-\alpha T(1+m)}+\right.$$

$$\left.\left. +\delta_{mM}e^{-\alpha T(1+\mu)}+\delta_{\mu o}e^{-\alpha T(M+1-m)}+\delta_{mo}e^{-\alpha T(M+1-\mu)}\right]\right\}$$

$$+\left[Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T}\right]\delta_{|m-\mu|,o} + \left[Q_1 Q_o + Q_1\rho_1\right]\delta_{|m-\mu|,1}$$

$$-\frac{\rho_1^2 e^{-2\alpha T}\frac{\phi_D}{\phi_N}}{\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}}\left\{\phi_D\left[e^{-\alpha T[(M-m)+(M-\mu)]} + e^{-\alpha T(m+\mu)}\right]-\right.$$

$$\left. -\rho_1 e^{-\alpha T(M+2)}\left[e^{-\alpha T(M+\mu-m)} + e^{-\alpha T(M+m-\mu)}\right]\right\} \quad (2.3.36)$$

where

$$Q_1 = -k^{-\frac{1}{2}} e^{-(\beta_1+\beta_2-\alpha)T} \qquad \text{(see 2.3.11)}$$

$$\rho_1 = k^{-\frac{1}{2}}\left[1-e^{-(\beta_1-\alpha)T}\right]\left[1-e^{-(\beta_2-\alpha)T}\right] \qquad \text{(see 2.3.13)}$$

$$Q_o = k^{-\frac{1}{2}} - \rho_1 \qquad \text{(see 2.3.12 and 2.3.13)}$$

$$\phi_D \equiv \phi_D\left(e^{-\alpha T}\right) = k^{-\frac{1}{2}}\left[1-e^{-(\beta_1+\alpha)T}\right]\left[1-e^{-(\beta_2+\alpha)T}\right] \qquad \text{(see 2.3.8)}$$

$$(2.3.37)$$

$$e^{-\alpha T} = \frac{1}{2A_o}\left(B_o - \sqrt{B_o^2 - 4A_o^2}\right)$$

$$2k = B_o + \sqrt{B_o^2 - 4A_o^2}$$

$$B_o = \sigma_1^2\left(1-e^{-2\beta_1 T}\right)\left(1+e^{-2\beta_2 T}\right) + \sigma_2^2\left(1-e^{-2\beta_2 T}\right)\left(1+e^{-2\beta_1 T}\right)$$

$$A_o = \sigma_1^2\left(1-e^{-2\beta_1 T}\right)e^{-\beta_2 T} + \sigma_2^2\left(1-e^{-2\beta_2 T}\right)e^{-\beta_1 T}$$

$$\frac{\phi_D}{\phi_N} = Q_o + Q_1 e^{-\alpha T} + \frac{\rho_1}{1-e^{-2\alpha T}}$$

Equation 2.3.36 gives the solution in terms of the definitions 2.3.37 and the autocorrelation function

$$\phi(t) = \sigma_1^2 e^{-\beta_1|t|} + \sigma_2^2 e^{-\beta_2|t|} \qquad (2.3.38)$$

Some specific examples of 2.3.38 will now be considered. Suppose a noise has the spectral density

$$S(\omega) = \frac{K^2}{\left(\omega^2+\beta_1^2\right)\left(\omega^2+\beta_2^2\right)} = \frac{K^2}{\beta_2^2-\beta_1^2}\left[\frac{1}{\omega^2+\beta_1^2} - \frac{1}{\omega^2+\beta_1^2}\right] \qquad (2.3.39)$$

then it has the autocorrelation function

$$\phi(t) = \int_{-\infty}^{\infty} e^{-j\omega t} S(\omega)\,\frac{d\omega}{2\pi}$$

$$= \frac{K^2}{2\left(\beta_2^2-\beta_1^2\right)}\left[\frac{1}{\beta_1} e^{-\beta_1|t|} - \frac{1}{\beta_2} e^{-\beta_2|t|}\right] \qquad (2.3.40)$$

Therefore from 2.3.38

$$\sigma_1^2 = \frac{K^2}{2\beta_1\left(\beta_2^2-\beta_1^2\right)} \quad , \quad \sigma_2^2 = -\frac{K^2}{2\beta_2\left(\beta_2^2-\beta_1^2\right)} \qquad (2.3.41)$$

Another noise of interest is one that has the spectral density

$$S(\omega) = K^2 \frac{\left(\omega^2+a^2\right)}{\left(\omega^2+\beta_1^2\right)\left(\omega^2+\beta_2^2\right)} \qquad (2.3.42)$$

and the autocorrelation function

$$\phi(t) = \frac{K^2}{2\left(\beta_2^2-\beta_1^2\right)}\left[\frac{1}{\beta_1}\left(a^2-\beta_1^2\right)e^{-\beta_1|t|} - \frac{1}{\beta_2}\left(a^2-\beta_2^2\right)e^{-\beta_2|t|}\right] \qquad (2.3.43)$$

and hence

$$\sigma_1^2 = K^2 \frac{a^2-\beta_1^2}{2\beta_1\left(\beta_2^2-\beta_1^2\right)} \quad , \qquad \sigma_2^2 = -K^2 \frac{a^2-\beta_2^2}{2\beta_2\left(\beta_2^2-\beta_1^2\right)} \qquad (2.3.44)$$

Finally to consider an example involving complex numbers let

$$\beta_1 = a + jb$$

$$\beta_2 = a - jb \qquad (2.3.46)$$

$$\sigma_1^2 = \sigma_2^2 = \frac{1}{2}$$

then

$$\phi(t) = \sigma_1^2 e^{-\beta_1|t|} + \sigma_2^2 e^{-\beta_2|t|}$$

$$= \frac{1}{2}\left[e^{-a|t|}e^{-jb|t|} + e^{-a|t|}e^{jb|t|}\right]$$

$$= e^{-a|t|}\text{Cos } bt \qquad (2.3.47)$$

which is a well known autocorrelation function. It can be shown that all the quantities defined in (2.3.37) are real when the substitutions 2.3.46 are made.

2.4 Autoregressive Noise

Let $\chi_o, \chi_1, \cdots \chi_m$ be observations of a random process at times $t = 0, T, 2T, \ldots MT$ and let the model for the generation of these samples be

$$a_o x_t + a_1 x_{t-T} + \cdots + a_D x_{t-DT} = Z_t \tag{2.4.1}$$

where the Z's are independent with zero mean and the roots of the equation

$$\sum_{j=o}^{D} a_j y^i = 0$$

lie inside the unit circle $|y| = 1$ in the complex plane, then the random process is called an autoregressive process of order D(9). It is shown in Reference 9 that the inverse covariance matrix of this noise has nonzero elements only on the main diagonal and the $2D$ adjacent diagonals. It will be shown in this section that the inverse covariance matrix of the noise considered in 2.1 can also have this property when the rate is chosen properly.

Referring back to 2.1.4, suppose that T is chosen such that $\phi^*(Z)$ has no zeros in the finite Z plane (that such a rate can be found will be demonstrated in Chapter V in the case of second order noise), then

$$\phi_N(Z) = \phi_N\left(Z^{-1}\right) = 1 \tag{2.4.2}$$

$$N = 0$$

This means that

$$\frac{\phi_D(Z)}{\phi_N(Z)} = \phi_D(Z) = Q(Z) \tag{2.4.3}$$

and hence

$$R(Z) = 0$$

which implies

$$\rho_k = 0 = \rho_{\mu k} = \rho_{\mu k}' = C_\ell = J_{\mu\ell} = J_{\mu\ell}' \tag{2.4.4}$$

Using 2.4.4 in 2.1.90 gives

$$W_\mu(mT) = \sum_{\ell=0}^{D} q_\ell \delta_{|m-\mu|,\ell} + \sum_{k=1}^{D} G_{\mu k}\delta_{m-M-1,-k} + \sum_{k=1}^{D} G_{\mu k}'\delta_{m+1,k}$$

$$= \sum_{\ell=0}^{D} q_\ell \delta_{|m-\mu|,\ell} + \sum_{k=1}^{D} G_{\mu k}\delta_{m,M+1-k} + \sum_{k=1}^{D} G_{\mu k}'\delta_{m,k-1} \tag{2.4.5}$$

Consider the term $\sum_{\ell=0}^{D} q_\ell \delta_{|m-\mu|,\ell}$. This term is nonzero only on the main diagonal and the 2D adjacent diagonals. It will now be shown that the last two terms of 2.4.5 are nonzero only on the intersection of the last D rows and columns and the first D rows and columns respectively. Once this is demonstrated, it will have been shown that the elements of the inverse covariance matrix are nonzero only on the main diagonal and the 2D adjacent diagonals.

Consider the term $\sum_{k=1}^{D} G_{\mu k}\delta_{m,M+1-k}$. This term is nonzero only on the last D rows (which are denoted by $m \geq M-(D-1)$ because of the Kronecker delta term. From 2.1.75

$$Q_{\mu n} = -Q_{n+M+1-\mu} = -\sum_{p=n+1}^{D} Q_p \delta_{\mu, M+1+n-p}$$

and hence

$$G_{\mu\ell} = \sum_{k=\ell}^{D} Q_k Q_{\mu,k-\ell} = -\sum_{k=\ell}^{D}\left[\sum_{p=k-\ell+1}^{D} Q_p \delta_{\mu,M+1+k-\ell-p}\right] Q_k \qquad (2.4.6)$$

Inspection of the limits on the sums and the Kronecker delta term shows that $\mu \geq M-(D-1)$ for the nonzero components of 2.4.6. Therefore, $\sum_{k=1}^{D} G_{\mu k}\delta_{m,M+1-k}$ is nonzero only for $m, \mu \geq M-(D-1)$ which is the intersection of the last D rows and columns.

Finally consider $\sum_{k=1}^{D} G'_{\mu k}\delta_{m,k-1}$. This term is nonzero only on the first D rows (which are denoted by $m \leq D-1$) because of the Kronecker delta term. From 2.1.78

$$Q'_{\mu n} = -Q_{\mu+1+n} = -\sum_{p=n+1}^{D} Q_p \delta_{\mu,p-1-n}$$

and hence

$$G'_{\mu\ell} = \sum_{k=\ell}^{D} Q_k Q'_{\mu,k-\ell} = -\sum_{k=\ell}^{D}\left[\sum_{p=k-\ell+1}^{D} Q_p \delta_{\mu,p-1-k+\ell}\right] Q_k \qquad (2.4.7)$$

As before, inspection of the limits on the sums and the Kronecker delta term shows that $\mu \leq D-1$ for the nonzero components of 2.4.6.

Therefore, $\sum_{k=\ell}^{D} G'_{\mu k}\delta_{m,k-1}$ is nonzero only for $m,\mu \leq D-1$ which is the intersection of the first D rows and columns which completes the demonstration that the $W_\mu(mT)$ is nonzero only on the main diagonal and the 2D adjacent ones.

CHAPTER III

EQUIVALENCE OF THE INTEGRAL EQUATION SOLUTION AND THE LIMIT OF THE DISCRETE SOLUTION

3.1 Minimum Variance Estimation in First Order Noise

The problem of minimum variance estimation when an infinity of data points are available is treated from two points of view in this section. The noise is assumed to be first order. In the first point of view it is assumed that a continuous record of the process

$$y(t) = \theta s(t) + n(t) \qquad 0 \le t \le L \tag{3.1.1}$$

is available and it is desired to find an unbiased minimum variance estimate of θ of the form

$$\hat{\theta} = \int_0^L f(t)\, y(t)\, dt \tag{3.1.2}$$

and the variance of the above estimate. The solution of this problem is well known and leads to an integral equation which is solved by the method of Reference 8. Thus in part 3.1.1 of this section the optimum weighting function $f(t)$ is determined and the associated variance is calculated.

In the second point of view it is assumed that the record is sampled at equally spaced intervals of time T seconds apart, and that a minimum variance estimate and its associated variance are calculated. The limiting form of the estimate and its variance are then determined as the duration of the record is held fixed, the number of samples approaches infinity and hence the time interval between samples approaches zero. It is shown that the limiting form of the discrete estimate and its variance is identical to the form of the "continuous" estimate $f(t)$ and its variance.

3.1.1 Integral Equation Approach to the Continuous Estimation Problem.

It has previously been discussed that finding an estimate of the form

$$\hat{\theta} = \int_0^L f(t)y(t)dt$$

leads to an integral equation of the form

$$\int_0^L \phi(t-\tau)f(\tau)d\tau = \sigma_L^2 s(t) \tag{3.1.3}$$

with the constraint

$$\int_0^L f(t)s(t)dt = 1 \tag{3.1.4}$$

The constraint 3.1.4 forces the estimate to be unbiased.

If the noise is first order it is easily shown that it has an autocorrelation function of the form

$$\phi(t) = \sigma^2 e^{-\beta|t|} \tag{3.1.5}$$

and a corresponding spectral density

$$G(\omega) = \frac{2\beta}{\omega^2+\beta^2}\,\sigma^2 \tag{3.1.6}$$

Inspection of the spectral density 3.1.6 shows that it has no finite zeros, therefore the integral equation 3.1.3 can be solved directly from the results of Reference 8.

Using the formulas* of Reference 8 the solution of 3.1.3 is found to be

$$f(t) = \frac{\sigma_L^2}{2}\left[\beta s(t) - \frac{1}{\beta}\ddot{s}(t)\right] + \frac{\sigma_L^2}{2}\left[s(o) - \frac{1}{\beta}\dot{s}(o)\right]\delta(t) + \frac{\sigma_L^2}{2}\left[s(L) + \frac{1}{\beta}\dot{s}(L)\right]\delta(L-t) \tag{3.1.7}$$

where

$$\frac{\sigma_L^2}{\sigma^2} = 2\left[\beta\int_o^L s^2(t)dt + \frac{1}{\beta}\int_o^L\left(\frac{ds}{dt}\right)^2 dt + s^2(o) + s^2(L)\right]^{-1} \tag{3.1.8}$$

Equations 3.1.7 and 3.1.8 express the desired result for the continuous cases.

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* A further discussion of these formulas is included in Chapter VII.

3.1.2 Limiting Form of the Discrete Minimum Variance Estimator

In this part of Section 3.1 it is assumed that the process

$$y(t) = \theta s(t)+n(t) \qquad 0 \le t \le L \tag{3.1.1}$$

is observed at equally spaced points in time which are T seconds apart. It is desired to make an unbiased minimum variance estimate of the unknown parameter θ. As discussed in Section 1.2 this estimate is

$$\hat{\theta} = (\overline{S}'R^{-1}\overline{S})^{-1}(R^{-1}\overline{S})'\overline{y} \tag{1.2.2}$$

and the variance of the estimate is

$$\sigma_\theta^2 = (\overline{S}'R^{-1}\overline{S})^{-1} \tag{1.2.3}$$

Since the noise is first order it has an autocorrelation function of the form

$$\phi(t) = \sigma^2 e^{-\beta|t|} \tag{3.1.5}$$

The inverse covariance matrix of the sampled values of first order noise was determined in Section 2.2. The elements of that matrix, which are denoted as $W_\mu(mT)$, are

$$W_\mu(mT) = \frac{\sigma^{-2}}{1-e^{-2\beta T}}\left[\left(1+e^{-2\beta T}\right)\delta_{|m-\mu|,o} - e^{-\beta T}\delta_{|m-\mu|,1} - e^{-2\beta T}\delta_{\mu M}\delta_{m,M} - \right.$$

$$\left. - e^{-2\beta T}\delta_{mo}\delta_{\mu o}\right] \qquad 0 \le \mu, m \le M \qquad (2.2.11)$$

The elements of the vector $\overline{S}$ are defined by

$$\overline{S} = \left(S(\mu T)\right) \equiv \left(S_\mu\right) \qquad 0 \le \mu \le M$$

Therefore from 1.2.3 and 2.2.11

$$\frac{\sigma^2}{\sigma_\theta^2} = \sum_{\mu=o}^{M}\sum_{m=o}^{M} W_\mu(mT)S_m S_\mu$$

$$= \frac{1}{1-e^{-2\beta T}}\left[\left(1+e^{-2\beta T}\right)\sum_{\mu=o}^{M}\sum_{m=o}^{M}\delta_{m,\mu}S_m S_\mu - \right.$$

$$-e^{-\beta T}\sum_{\mu=o}^{M}\sum_{m=o}^{M}\left(\delta_{m,\mu+1}+\delta_{m,\mu-1}\right)S_m S_\mu - e^{-2\beta T}\sum_{\mu=o}^{M}\sum_{m=o}^{M}\left(\delta_{\mu M}\delta_{m,M}\right.$$

$$\left.\left.+\delta_{mo}\delta_{\mu o}\right)S_m S_\mu\right]$$

$$= \frac{1}{1-e^{-2\beta T}}\left[\left(1+e^{-2\beta T}\right)\sum_{\mu=o}^{M}S_\mu^2 - e^{-2\beta T}\left(S_o^2+S_m^2\right) - 2e^{-\beta T}\sum_{\mu=o}^{M-1}S_\mu S_{\mu+1}\right]$$

$$= \frac{1}{1-e^{-2\beta T}}\left[S_o^2+S_M^2-2e^{-\beta T}\sum_{\mu=o}^{M-1}S_\mu S_{\mu+1}+\left(1+e^{-2\beta T}\right)\sum_{\mu=1}^{M-1}S_\mu^2\right] \qquad (3.1.9)$$

Equation 3.1.9 gives the variance of the estimate of the parameter θ in terms of the sampled values of the general signal $S(t)$.

To find the limiting form of (3.1.9), let

$$T \to 0 \quad , \quad M \to \infty \quad \text{with} \quad MT = L \quad .$$

Carrying out these operations gives

$$2\frac{\sigma^2}{\sigma_\theta^2} \cong \frac{2}{1-\left[1-\beta T+2\beta^2T^2-\frac{8}{6}\beta^3T^3\right]}\left\{S_o^2+S_M^2-2\left[1-\beta T+\frac{1}{2}\beta^2T^2\right]\sum_{\mu=o}^{M-1} S_\mu S_{\mu+1} + \right.$$

$$\left. + 2\left[1-\beta T+\beta^2T^2\right]\sum_{\mu=1}^{M-1} S_\mu^2\right\}$$

$$\cong \left\{S_o^2+S_M^2-2\left[1-\beta T+\frac{1}{2}\beta^2T^2\right]\sum_{\mu=o}^{M-1} S_\mu S_{\mu+1}+ 2\left[1-\beta T+\beta^2T^2\right]\sum_{\mu=1}^{M-1} S_\mu^2\right\}\left[\frac{1}{\beta T}+1+\frac{1}{3}\beta T\right]$$

$$\cong \frac{1}{\beta T}\left[S_o^2+S_M^2-2\sum_{\mu=o}^{M-1} S_\mu S_{\mu+1}+2\sum_{\mu=1}^{M-1} S_\mu^2\right]+\left(S_o^2+S_M^2\right)+\frac{1}{3}\beta T\left[S_o^2+S_M^2 + \right.$$

$$\left. + \sum_{\mu=o}^{M-1} S_\mu S_{\mu+1}+ 2\sum_{\mu=1}^{M-1} S_\mu^2\right]$$

which is correct to order T. Now as $T \to 0$

$$\frac{1}{\beta T}\left[S_o^2+S_M^2- 2\sum_{\mu=o}^{M-1} S_\mu S_{\mu+1}+ 2\sum_{\mu=1}^{M-1} S_\mu^2\right] = \frac{1}{\beta}T\left[\sum_{o}^{M-1}\left(\frac{S_{\mu+1}-S_\mu}{T}\right)^2\right] \to \frac{1}{\beta}\int_o^L \dot{S}^2 dt$$

$$S_o^2+S_M^2 = S^2(o) + S^2(L)$$

$$\frac{1}{3}\beta T\left[S_o^2+S_M^2+\sum_{\mu=o}^{M-2} S_\mu S_{\mu+1}+2\sum_{\mu=2}^{M-1} S_\mu^2\right]\to\frac{1}{3}\beta T\left[S_o^2+S_M^2+\sum_{\mu=1}^{M-1} S_\mu^2+2\sum_{\mu=1}^{M-1} S_\mu^2\right]$$

$$\to\frac{1}{3}\beta T\left[3\sum_{\mu=2}^{M-1} S_\mu^2\right]=\beta T\sum_{\mu=2}^{M-1} S_\mu^2\to\beta\int_o^L S^2 dt$$

Therefore

$$\frac{\sigma_\theta^2}{\sigma^2}\to 2\left[\beta\int_o^L S^2 dt+\frac{1}{\beta}\int_o^L \dot{S}^2 dt+S^2(o)+S^2(L)\right]^{-1} \tag{3.1.10}$$

which agrees with 3.1.8.

It remains to determine the limiting form of the estimate. From 1.2.2 and 1.2.3 the minimum variance weighting vector is

$$\overline{f}=\sigma_0^2 R^{-1}\overline{S} \tag{3.1.11}$$

The μ^{th} component of the vector $\overline{f}$ is

$$f_\mu=\sigma_\theta^2\sum_{m=o}^{M} W_\mu(mT)S_m \tag{3.1.12}$$

and using 2.2.11 $\overline{f}$ can be expressed

$$\bar{f} = \sigma_\theta^2 R^{-1}\bar{S} = \frac{\sigma_\theta^2}{1-e^{-2\beta T}} \begin{bmatrix} S_o - e^{-\beta T}S_1 \\ \vdots \\ -e^{-\beta T}S_{\mu-1} + \left(1+e^{-2\beta T}\right)S_\mu - e^{-\beta T}S_{\mu+1} \\ \vdots \\ S_M - e^{-\beta T}S_{M-1} \end{bmatrix} \qquad (3.1.13)$$

From 3.1.13 it is seen that the following limits are needed

$$\frac{S_o - e^{-\beta T}S_1}{1-e^{-2\beta T}} \to \frac{1}{2}\left\{S_o - S_1\left[1-\beta T+\frac{1}{2}\beta^2T^2\right]\right\}\left\{\frac{1}{\beta T}+1+\frac{1}{3}\beta T\right\}$$

$$\to \frac{1}{2}\left[\frac{1}{\beta T}\left(S_o - S_1\right) + \left(S_o - S_1 + S_1\right)\right] = \frac{1}{2}\left[S_o - \frac{1}{\beta}\left(\frac{S_1 - S_o}{T}\right)\right] \to \frac{1}{2}\left[S(o) - \frac{1}{\beta}\dot{S}(o)\right]$$

$$\frac{S_M - e^{-\beta T}S_{M-1}}{1-e^{-2\beta T}} \to \frac{1}{2}\left[S_M + \frac{1}{\beta}\left(\frac{S_M - S_{M-1}}{T}\right)\right] \to \frac{1}{2}\left[S(L) + \frac{1}{\beta}\dot{S}(L)\right]$$

$$\frac{-e^{-\beta T}S_{\mu-1} + \left(1+e^{-2\beta T}\right)S_\mu - e^{-\beta T}S_{\mu+1}}{1-e^{-2\beta T}} \to \frac{\beta T}{2}\left[\frac{1}{6}\, S_{\mu-1} + \frac{4}{6}\, S_\mu + \frac{1}{6}\, S_{\mu+1}\right] -$$

$$- \frac{T}{2\beta}\left[\frac{S_{\mu+1} - 2S_\mu + S_{\mu-1}}{T^2}\right]$$

$$\to \frac{T}{2}\left[\beta S(\mu T) - \frac{1}{\beta}\,\ddot{S}(\mu T)\right]$$

Therefore

$$\overline{f}'\overline{y} \rightarrow \frac{\sigma_L^2}{2}\left[S(o) - \frac{1}{\beta}\dot{S}(o)\right]y(o) + \frac{\sigma_L^2}{2}\int_0^L\left[\beta S(t) - \frac{1}{\beta}\ddot{S}(t)\right]y(t)dt +$$

$$+ \frac{\sigma_L^2}{2}\left[S(L) + \frac{1}{\beta}\dot{S}(L)\right]y(L)$$

which agrees with 3.1.7, the solution of the continuous problem, in conjunction with 3.1.2. This completes the demonstration of the equivalence between the integral equation solution and the limit of the discrete solution for first order noise.

3.2 Minimum Variance Estimation in Second Order Noise

The remarks made in 3.1 concerning first order noise apply to the second order noise case treated in this section. However, it will be necessary to apply the method of Reference 5 when the noise has two finite zeros in its spectral density. The method of Reference 8 is used in the case of all-pole second order noise. It will also be convenient to determine the limiting form of the discrete minimum variance estimate separately in the cases of noise with and without finite zeros in its spectrum.

3.2.1 Integral Equation Approach to the Continuous Estimation Problem.

In 3.1.1 it has been noted that to obtain the continuous filter $f(t)$ it is necessary to solve the integral equation

$$\int_0^L \phi(t-\tau)f(\tau)d\tau = \sigma_L^2 S(t) \tag{3.1.3}$$

with the constraint

$$\int_0^L f(t)S(t)dt = 1 \tag{3.1.4}$$

This problem is solved in 3.2.1.1 when the noise has the spectral density

$$G(\omega) = K\frac{\omega^2+a^2}{\left(\omega^2+\beta_1^2\right)\left(\omega^2+\beta_2^2\right)} = K\frac{\omega^2+a^2}{\omega^4+\left(\beta_1^2+\beta_2^2\right)\omega^2+\beta_1^2\beta_2^2} \tag{3.2.1}$$

and in 3.2.1.2 when the noise has the spectral density

$$G(\omega) = K_p\frac{1}{\left(\omega^2+\beta_1^2\right)\left(\omega^2+\beta_2^2\right)} = K_p\frac{1}{\omega^4+\left(\beta_1^2+\beta_2^2\right)\omega^2+\beta_1^2\beta_2^2} \tag{3.2.2}$$

3.2.1.1 Noise with Two Zeros in its Spectrum.

The integral equation 3.1.3 is now going to be solved for the case of a noise with the spectral density 3.2.1. Upon noting that the spectral density and the autocorrelation function are Fourier Transform pairs, 3.1.3 can be written

$$\int_0^L \phi(t-\tau)f(\tau)d\tau = \sigma_L^2 S(t) \qquad 0 \le t \le L \tag{3.1.3}$$

$$\int_0^L f(\tau) \int_{-\infty}^{\infty} K \frac{\omega^2+a^2}{\omega^4+\left(\beta_1^2+\beta_2^2\right)\omega^2+\beta_1^2\beta_2^2} e^{j\omega(t-\tau)} \frac{d\omega}{2\pi}\, d\tau = \sigma_L^2 S(t) \tag{3.2.3}$$

Formal differentiation of 3.1.7 gives

$$\left\{\frac{d^2}{dt^2} - a^2\right\} \int_0^L f(\tau) \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} \frac{d\omega}{2\pi}\, d\tau = - \frac{\sigma_L^2}{K}\left[\frac{d^4 S}{dt^4} - \left(\beta_1^2+\beta_2^2\right)\frac{d^2 S}{dt^2} + S\beta_1^2\beta_2^2\right] \tag{3.2.4}$$

and upon use of the formal identity

$$\delta(t-\tau) = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} \frac{d\omega}{2\pi} \tag{3.2.5}$$

3.2.4 becomes

$$\frac{d^2 f}{dt^2} - a^2 f = - \frac{\sigma_L^2}{K}\left[\frac{d^4 S}{dt^4} - \left(\beta_1^2+\beta_2^2\right)\frac{d^2 S}{dt^2} + S\beta_1^2\beta_2^2\right] \tag{3.2.6}$$

It is well known that the function of time which satisfies 3.2.6 is not the complete solution of the integral equation (see Reference 5) but is only part of the solution which is denoted herein by $f_c(t)$. In Reference 5 it is shown that the complete solution is of the form

$$f(t) = f_c(t) + B_1\delta(t) + B_2\delta(t-L) \tag{3.2.7}$$

The method of attack used in solving 3.1.3 is then as follows: First find the solution of 3.2.6 and hence determine 3.2.7, then substitute 3.2.7 back into the integral equation 3.1.3 to determine the unknown constants. The differential equation 3.2.6 will now be solved. (For notational convenience the substitutions $\beta_1 = \beta$ and $\beta_2 = \gamma$ are used in the remainder of this chapter.)

As discussed in the above paragraph the differential equation to be solved is

$$\frac{d^2 f_c}{dt^2} - a^2 f = -\frac{\sigma_L^2}{K}\left[\frac{d^4 S}{dt^4} - \left(\gamma^2+\beta^2\right)\frac{d^2 S}{dt^2} + \gamma^2\beta^2 S(t)\right] \quad (3.2.8)$$

The particular solution of 3.2.8 is

$$f_{cp} = -\frac{\sigma_L^2}{K}\left\{\frac{d^2 S}{dt^2} + [a^2-\left(\beta^2+\gamma^2\right)]S(t) + \left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)g(t)\right\} \quad (3.2.9)$$

where

$$g(t) = \frac{1}{2a}\left[e^{at}\int_L^t e^{-at}S(t)dt - e^{-at}\int_0^t e^{at}S(t)dt\right]$$

$$= -\frac{1}{2a}\int_0^L e^{-a|t-u|}S(u)du \quad (3.2.10)$$

The "transient component" of the solution is

$$f_{ct} = \frac{\sigma_L^2}{K}\left[A_1 e^{-at} + A_2 e^{at}\right] \quad (3.2.11)$$

therefore, from 3.2.7, 3.2.9 and 3.2.11

$$f(t) = \frac{\sigma_L^2}{K}\left\{-\frac{d^2S}{dt^2} + [\beta^2+\gamma^2-a^2]S(t) + \frac{1}{2a}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\int_0^L e^{-a|t-u|}S(u)du\right.$$

$$\left. + A_1e^{-at} + A_2e^{at} + B_1\delta(t) + B_2\delta(t-L)\right\} \qquad (3.2.12)$$

The result 3.2.12 must now be substituted into 3.1.3 to determine the unknown constants. It should be noted that noise with the spectral density 3.2.1 has an autocorrelation function of the form

$$\phi(t) = \sigma_1^2e^{-\beta|t|} + \sigma_2^2e^{-\gamma|t|} \qquad (3.2.13)$$

where

$$\sigma_1^2 = \frac{K}{2\beta}\,\frac{a^2-\beta^2}{\gamma^2-\beta^2} \qquad (3.2.14)$$

$$\sigma_2^2 = \frac{K}{2\gamma}\,\frac{\gamma^2-a^2}{\gamma^2-\beta^2} \qquad (3.2.15)$$

Upon substituting 3.2.12 in 3.1.3, using 3.2.13, and considerable algebra and reduction of integrals, the following result is obtained

$$\frac{1}{\sigma_L^2}\int_o^L \phi(t-u)f(u)du = S(t)+\frac{\sigma_1^2}{K}\left\{-[\beta S(o)-\dot{S}(o)]+(a-\beta)\left(a^2-\gamma^2\right)g(o)-\frac{A_1}{\beta-a}\frac{A_2}{\beta+a}\right.$$

$$\left.+B_1\right\}e^{-\beta t}+\frac{\sigma_2^2}{K}\left\{-[\gamma S(o)-\dot{S}(o)]+(a-\gamma)\left(a^2-\beta^2\right)g(o)\right. -$$

$$-\frac{A_1}{\gamma-a}-\frac{A_2}{\gamma+a}+\left.B_1\right\}e^{-\gamma t}-\frac{\sigma_1^2}{K}\left\{[\beta S(L)+\dot{S}(L)]\right. -$$

$$-(a-\beta)\left(a^2-\gamma^2\right)g(L)+\frac{A_1}{\beta+a}\quad e^{-aL}+\frac{A_2e^{aL}}{\beta-a}$$

$$-\left.B_2\right\}e^{-\beta(L-t)}-\frac{\sigma_2^2}{K}\left\{[\gamma S(L)+\dot{S}(L)]-(a-\gamma)\left(a^2-\beta^2\right)g(L)\right.$$

$$+\frac{A_1}{\gamma+a}e^{-aL}+\frac{A_2e^{aL}}{\gamma-a}-\left.B_2\right\}e^{-\gamma(L-t)}$$

$$= S(t) \tag{3.2.16}$$

where $g(t)$ is defined in 3.2.10. It is seen from 3.2.16 that for 3.2.12 to be a solution of the integral equation the system of equations

$$\frac{A_1}{\beta-a} + \frac{A_2}{\beta+a} - B_1 = (a-\beta)\left(a^2-\gamma^2\right)g(o)-[\beta S(o)-\dot{S}(o)]$$

$$\frac{A_1}{\gamma-a} + \frac{A_2}{\gamma+a} - B_1 = (a-\gamma)\left(a^2-\beta^2\right)g(o)-[\gamma S(o)-\dot{S}(o)] \qquad (3.2.17)$$

$$\frac{A_1}{\beta+a}\, e^{-aL} + \frac{A_2}{\gamma-a}\, e^{aL} - B_2 = (a-\beta)\left(a^2-\gamma^2\right)g(L)-[\beta S(L)+\dot{S}(L)]$$

$$\frac{A_1}{\gamma+a}\, e^{-aL} + \frac{A_2 e^{aL}}{\gamma-a} - B_2 = (a-\gamma)\left(a^2-\beta^2\right)g(L)-[\gamma S(L)+\dot{S}(L)]$$

must hold. The solution of the system of equations 3.2.17 is

$$A_1 = \frac{1}{\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\left\{\left[\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)g(o)-(a-\beta)^2(a-\gamma)^2e^{-aL}g(L)\right]\right.$$

$$\left. + \left[(a+\beta)(a+\gamma)S(o)-e^{-aL}(a-\beta)(a-\gamma)S(L)\right]\right\}$$

$$A_2 = \frac{1}{\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)e^{-aL}\left\{\left[\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)g(L)-(a-\beta)^2(a-\gamma)^2e^{-aL}g(o)\right]\right.$$

$$\left. + \left[(a+\beta)(a+\gamma)S(L)-e^{-aL}(a-\beta)(a-\gamma)S(o)\right]\right\} \qquad (3.2.18)$$

$$B_1 = \frac{1}{\Delta}\,[-2a(a+\beta)(a+\gamma)\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)g(o) +$$

$$+ 2a\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)(a-\beta)(a-\gamma)e^{-aL}g(L)$$

$$-2a(a-\beta)^2(a-\gamma)^2e^{-2aL}S(o)+2ae^{-aL}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)S(L)]$$

$$+ (\beta+\gamma-a)S(o)-\dot{S}(o)$$

$$B_2 = \frac{1}{\Delta}\,[-2a(a+\beta)(a+\gamma)\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)g(L) +$$

$$+ 2a\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)(a-\beta)(a-\gamma)e^{-aL}g(o)-2a(a-\beta)^2(a-\gamma)^2e^{-2aL}S(L)$$

$$+ 2ae^{-aL}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)S(o)] + (\beta+\gamma-a)S(L) + \dot{S}(L)$$

The quantities of 3.2.18 determine the solution up to the constant σ_L^2 which is the variance of the estimate. The variance σ_L^2 can be determined from the constraint 3.1.4 and the result, in terms of $S(t)$, is

$$\sigma_L^2 = K\left\{-\int_o^L \frac{d^2S}{dt^2}\,S(t)dt + \left(\beta^2+\gamma^2-a^2\right)\int_o^L S^2(t)dt + \int_o^L [A_1e^{-at}+A_2e^{at}]S(t)dt\right.$$

$$+ \frac{1}{2a}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\int_o^L\int_o^L e^{-a|t-u|}S(t)S(u)dudt + B_1S(o)$$

$$\left. + B_2S(L)\right\}^{-1} \qquad (3.2.19)$$

Equations 3.2.12, 3.2.18 and 3.2.19 complete the solution for the case of the spectral density 3.2.1.

3.2.1.2 All Pole Second Order Noise

It now remains to solve the integral equation 3.1.3 when the noise has the "all pole" spectral density 3.2.2. Explicit formulas* are available for this problem in Reference 8. Using these formulas gives

$$f(t) = \frac{\sigma_L^2}{K_p}\Bigg\{\frac{d^4S}{dt^4} - \left(\beta^2+\gamma^2\right)\frac{d^2S}{dt^2} + \beta^2\gamma^2 S(t) +$$

$$+\left[\dddot{S}(o)+\left(\beta\gamma-(\beta+\gamma)^2\right)\dot{S}(o)+\beta\gamma(\beta+\gamma)S(o)\right]\delta(t)$$

$$-\left[\dddot{S}(L)+\left(\beta\gamma-(\beta+\gamma)^2\right)\dot{S}(L)-\beta\gamma(\beta+\gamma)S(L)\right]\delta(t-L)$$

$$+\left[\ddot{S}(o)-(\beta+\gamma)\dot{S}(o)+\beta\gamma S(o)\right]\delta'(t)-\left[\ddot{S}(L)+(\beta+\gamma)\dot{S}(L)\right.$$

$$\left. + \beta\gamma S(L)\right]\delta'(t-L)\Bigg\} \qquad (3.2.20)$$

and the variance of the estimate is

---

* A discussion of these formulas is included in Chapter VII.

$$\sigma_L^2 = K_p \left\{ \int_o^L \left[\frac{d^4S}{dt^4} - \left(\beta^2+\gamma^2\right)\frac{d^2S}{dt^2} + \beta^2\gamma^2 S(t) \right] S(t)dt \right.$$

$$+ \left[\dddot{S}(o) + \left(\beta\gamma - (\beta+\gamma)^2\right)\dot{S}(o) + \beta\gamma(\beta+\gamma)S(o) \right] S(o)$$

$$- \left[\dddot{S}(L) + \left(\beta\gamma - (\beta+\gamma)^2\right)\dot{S}(L) - \beta\gamma(\beta+\gamma)S(L) \right] S(L)$$

$$\left. - \left[\ddot{S}(o) - (\beta+\gamma)\dot{S}(o) + \beta\gamma S(o) \right] \dot{S}(o) + \left[\ddot{S}(L) + (\beta+\gamma)\dot{S}(L) + \beta\gamma S(L) \right] \dot{S}(L) \right\}^{-1} \quad (3.2.21)$$

which completes the solution for this case.

3.2.2 Limiting Form of the Discrete Minimum Variance Estimator

In this part of the thesis, as in part 3.1.2, it is assumed that the process

$$y(t) = \theta S(t) + n(t) \qquad 0 \le t \le L \qquad (3.1.1)$$

is observed at equally spaced points in time which are T seconds apart. It is desired to make a linear, unbiased, minimum variance estimate of the parameter θ. As discussed in Section 1.2, this estimate is

$$\hat{\theta} = (\overline{S}'R^{-1}\overline{S})^{-1}(R^{-1}S)'\overline{y} \qquad (1.2.2)$$

and the variance of the estimate is

$$\sigma_\theta^2 = (\overline{S}'R^{-1}\overline{S})^{-1} \tag{1.2.3}$$

It should be noted that since the estimate is unbiased, the discrete filter

$$\overline{f} = (\overline{S}'R^{-1}\overline{S})^{-1}R^{-1}\overline{S} = \sigma_\theta^2 R^{-1}\overline{S} \tag{3.2.22}$$

satisfies the constraint

$$\overline{f}'\overline{S} = 1 \tag{3.2.23}$$

Since the noise considered in this part is second order it has an autocorrelation function of the form

$$\phi(t) = \sigma_1^2 e^{-\beta|t|} + \sigma_2^2 e^{-\gamma|t|} \tag{3.2.24}$$

The inverse covariance matrix of the sampled values of second order noise was determined in Section 2.3 and this result will be applied in this part. The same method of attack that was used in 3.1.2 will be used here also, that is, obtaining the limiting form of the filter as $T \to 0$, $M \to \infty$, with $MT = L$. It is necessary to treat the two noises with the spectral densities 3.2.1 and 3.2.2 separately since for the all pole noise, the condition

$$\beta\sigma_1^2 + \gamma\sigma_2^2 = 0 \tag{3.2.25}$$

holds. As will be seen shortly, it is important to know whether or not 3.2.25 holds when carrying out the required limiting operations.

3.2.2.1 Noise with Two Zeros in its Spectral Density

The limiting form of 3.2.22 is now going to be determined when the autocorrelation function is 3.2.24 and the condition 3.2.25 does not hold. Only the limiting form of $R^{-1}\overline{S}$ will be found since if two unbiased filters differ only by a multiplicative constant, then the constants must be identical. For convenient reference the continuous filter is summarized below. The numbers on the terms are for identification with parts of the limiting discrete solution.

$$f(t) = \frac{\sigma_L^2}{K}\left\{-\overset{24.)}{\frac{d^2S}{dt^2}} + \overset{24.)}{\left(\beta^2+\gamma^2-a^2\right)S(t)} - \overset{7.)}{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)g(t)} + A_1e^{-at} + A_2e^{at} + B_1\delta(t) + B_2\delta(t-L)\right\} \tag{3.2.26}$$

where

$$1 = \int_0^L f(t)S(t)dt \tag{3.2.27}$$

$$g(t) = -\frac{1}{2a}\int_0^L e^{-a|t-u|}S(u)du \tag{3.2.28}$$

$$A_1 = \frac{1}{\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\left\{\left[\overset{11.)}{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)g(o)}-\overset{11.)}{(a-\beta)^2(a-\gamma)^2e^{-aL}g(L)}\right] + \left[\underset{13.)}{(a+\beta)(a+\gamma)S(o)}-\underset{15.)}{e^{-aL}(a-\beta)(a-\gamma)S(L)}\right]\right\} \qquad (3.2.29)$$

$$A_2 = \frac{e^{-aL}}{\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\left\{\left[\overset{11.)}{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)g(L)}-\overset{11.)}{(a-\beta)^2(a-\gamma)^2e^{-aL}g(o)}\right] + \left[\underset{13.)}{(a+\beta)(a+\gamma)S(L)}-\underset{15.)}{e^{-aL}(a-\beta)(a-\gamma)S(o)}\right]\right\} \qquad (3.2.30)$$

$$\Delta = (\beta+a)^2(\gamma+a)^2-(\beta-a)^2(\gamma-a)^2e^{-2aL} \qquad (3.2.31)$$

$$B_1 = \frac{2a}{\Delta}\left\{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\left[\overset{15.)}{(a-\beta)(a-\gamma)e^{-aL}g(L)}-\overset{13.)}{(a+\beta)(a+\gamma)g(o)}\right] \overset{16.)}{-(a-\beta)^2(a-\gamma)^2e^{-2aL}S(o)}+\overset{17.)}{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)e^{-aL}S(L)}\right\} + \overset{21.)}{(\beta+\gamma-a)S(o)} - \overset{21.)}{\dot{S}(o)} \qquad (3.2.32)$$

$$B_2 = \frac{2a}{\Delta}\left\{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\left[\overset{15.)}{(a-\beta)(a-\gamma)e^{-aL}g(o)}-\overset{13.)}{(a+\beta)(a+\gamma)g(L)}\right] \overset{17.)}{-(a-\beta)^2(a-\gamma)^2e^{-2aL}S(L)}+\overset{16.)}{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)e^{-aL}S(L)}\right\} + \overset{21.)}{(\beta+\gamma-a)S(L)} + \overset{21.)}{\dot{S}(L)} \qquad (3.2.33)$$

The quantities β, γ, a, σ_1^2, σ_2^2, and K are not all independent. The relations defining them are

$$\phi(t) = \sigma_1^2 e^{-\beta|t|} + \sigma_2^2 e^{-\gamma|t|} \tag{3.2.24}$$

$$G(\omega) = K \frac{\omega^2+a^2}{\omega^4+\left(\beta^2+\gamma^2\right)\omega^2+\beta^2\gamma^2} \tag{3.2.1}$$

$$a^2 = \beta\gamma \frac{\gamma\sigma_1^2 + \beta\sigma_2^2}{\beta\sigma_1^2 + \gamma\sigma_2^2} \tag{3.2.34}$$

$$K = 2\left(\beta\sigma_1^2 + \gamma\sigma_2^2\right) \tag{3.2.35}$$

The limiting form of $R^{-1}\overline{S}$ is now going to be determined

1.) From 2.3.37

$$e^{-\alpha T} = \frac{1}{2A_o}\left(B_o - \sqrt{\left(B_o+2A_o\right)\left(B-2A_o\right)}\right)$$

where

$$B_o = \sigma_1^2\left(1-e^{-2\beta T}\right)\left(1+e^{-2\gamma T}\right) + \sigma_2^2\left(1-e^{-2\gamma T}\right)\left(1+e^{-2\beta T}\right)$$

$$A_o = \sigma_1^2\left(1-e^{-2\beta T}\right)e^{-\gamma T} + \sigma_2^2\left(1-e^{-2\gamma T}\right)e^{-\beta T}$$

$$MT = L$$

Expanding A_o and B_o gives (using 3.2.34 and 3.2.35 to simplify the results)

$$B_o \cong 4T\left\{\tfrac{1}{2}K-\tfrac{1}{2}K(\gamma+\beta)T + \left[\beta\sigma_1^2\left(\gamma^2+\beta\gamma+\tfrac{2}{3}\beta^2\right) + \gamma\sigma_2^2\left(\beta^2+\gamma\beta+\tfrac{2}{3}\gamma^2\right)\right]T^2\right\}$$

$$A_o \cong 2T\left\{\tfrac{1}{2}K-\tfrac{1}{2}K(\gamma+\beta)T + \left[\beta\sigma_1^2\left(\tfrac{1}{2}\gamma^2+\beta\gamma+\tfrac{2}{3}\beta^2\right)+ \gamma\sigma_2^2\left(\tfrac{1}{2}\beta^2+\gamma\beta+\tfrac{2}{3}\gamma^2\right)\right]T^2\right\} \quad ,$$

therefore

$$B_o - 2A_o \cong 2\beta\gamma T^3\left[\gamma\sigma_1^2+\beta\sigma_2^2\right] = Ka^2T^3$$

$$B_o + 2A_o \cong 4T\left[\frac{K}{2} + \frac{K}{2}\right] = 4KT$$

and hence

$$\sqrt{\left(B_o+2A_o\right)\left(B_o-2A\right)} = 2KaT^2$$

From the above

$$e^{-\alpha T} = \frac{1}{2A_o}\left[B_o - \sqrt{B_o^2 - 4A_o^2}\right] \to \frac{2KT - 2K(\gamma+\beta)T^2 - 2KaT^2}{2TK - 2K(\gamma+\beta)T^2}$$

$$\to \frac{2KT[1-(a+\gamma+\beta)T]}{2KT[1-(\gamma+\beta)T]} \to 1 - aT + O\left(T^2\right)^* \qquad (3.2.36)$$

and

$$e^{-\alpha TM} \to \left[1 - aT + O\left(T^2\right)\right]^M = \left[1 - \frac{aL}{M} + O\left(T^2\right)\right]^M$$

$$\to e^{-aL}\,(1 + O(T)) \qquad (3.2.37)$$

This completes the expansion of $e^{-\alpha T}$ and $e^{-\alpha TM}$

2.) From 2.3.37

$$k = \frac{1}{2}\left[B_o + \sqrt{B_o^2 - 4A_o^2}\right]$$

$$\to \frac{1}{2}\left[2KT - 2K(\gamma+\beta)T^2 + 2KaT^2\right]$$

$$\to KT\left[1 - (\beta+\gamma-a)T + O\left(T^2\right)\right] \qquad (3.2.38)$$

where the expansions of A_o and B_o in 1.) were used.

---

* The notation $O\left(T^2\right)$ denotes a power series whose lowest order term is of an order at least T^2.

3.) From 2.3.37

$$k^{\frac{1}{2}}\rho_1 = \left[1-e^{-(\beta-\alpha)T}\right]\left[1-e^{-(\gamma-\alpha)T}\right] \rightarrow (a-\beta)(a-\gamma)T^2 \qquad (3.2.39)$$

and

4.)

$$k^{\frac{1}{2}}\phi_D = \left[1-e^{-(\beta+\alpha)T}\right]\left[1-e^{-(\gamma+\alpha)T}\right] \rightarrow (a+\beta)(a+\gamma)T^2 \qquad (3.2.40)$$

5.)

$$k^{\frac{1}{2}}Q_o = -\left[e^{-(\beta+\gamma-2\alpha)T} - e^{-(\beta-\alpha)T} - e^{-(\gamma-\alpha)T}\right] = 1-k^{\frac{1}{2}}\rho_1$$

$$\rightarrow 1 - (a-\beta)(a-\gamma)T^2 \qquad (3.2.41)$$

6.)

$$k^{\frac{1}{2}}Q_1 = -e^{\alpha T}\, e^{-(\beta+\gamma)T} \rightarrow -\left[1-(\beta+\gamma-a)T\right] \qquad (3.2.42)$$

7.) Now consider the term

$$\rho_1 \frac{\phi_D}{\phi_N} e^{-\alpha T|m-\mu|} = \rho_1\left[Q_o+Q_1 e^{-\alpha T} + \frac{\rho_1}{1-e^{-2\alpha T}}\right]e^{-\alpha T|m-\mu|}$$

which appears in 2.3.36. Expanding gives

$$\rho_1\left[Q_o+Q_1e^{-\alpha T}+\frac{\rho_1}{1-e^{-2\alpha T}}\right]\to k^{-1}(a-\beta)(a-\gamma)T^2\left[1-1+(\rho+\gamma)T+\frac{(a-\beta)(a-\gamma)T^2}{2aT}\right]$$

$$\to\frac{T^2\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)}{2aK} \qquad (3.2.43)$$

where the expansions of k, Q_o, Q, $e^{-\alpha T}$, and ρ_1 were used.

Therefore the contribution of this term to $R^{-1}\overline{S}$ is

$$\sum_{m=o}^{M}\rho_1\left[Q_o+Q_1e^{-\alpha T}+\frac{\rho_1}{1-e^{-2\alpha T}}\right]e^{-\alpha T|m-\mu|}S_m\to$$

$$\to T\,\frac{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)}{2aK}\sum_{m=o}^{M}Te^{-\alpha T|\mu-m|}S_m \qquad (3.2.44)$$

$$\to\frac{T}{K}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\frac{1}{2a}\int_o^L e^{-a|t-u|}S(u)du=-\frac{T}{K}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)g(t) \qquad (3.2.45)$$

where $\mu T\to t$ and $mT\to u$.

Equation 3.2.45 gives the term marked 7.) in 3.2.26. It should be noted that the quantity T which occurs in 3.2.45 is similar to the dt in the integral

$$\hat{\theta}=\int_o^L f(t)y(t)dt$$

Some other quantities which are needed are

8.)

$$\phi_D^2 - \rho_1^2 e^{-2\alpha T(M+2)} \to k^{-1}\left[(a+\beta)^2(a+\gamma)^2 - (a-\beta)^2(a-\gamma)^2 e^{-2aL}\right]T^4$$

$$= \Delta k^{-1} T^4 \qquad (3.2.46)$$

9.)

$$-\frac{\rho_1^2\phi_D^2 e^{-2\alpha T}}{\left(1-e^{-2\alpha T}\right)\left[\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}\right]} \to -\frac{T^2(a-\beta)^2(a-\gamma)^2(a+\beta)^2(a+\gamma)^2}{2aK\Delta} \qquad (3.2.47)$$

10.)

$$\frac{\rho_1^3\phi_D e^{-2\alpha T}e^{-\alpha T(M+2)}}{\left(1-e^{-2\alpha T}\right)\left[\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}\right]} \to \frac{T^2}{2aK\Delta}(a-\beta)^3(a-\gamma)^3(a+\beta)(a+\gamma)e^{-\alpha L} \qquad (3.2.48)$$

11.) Now consider the term

$$-\frac{\rho_1^2\phi_D e^{-2\alpha T}}{\left(1-e^{-2\alpha T}\right)\left[\phi_D^2-\rho_1^2 e^{-\alpha T(M+2)}\right]}\left\{\phi_D\left[e^{-2\alpha MT}e^{\alpha T(m+\mu)}+e^{-\alpha T(m+\mu)}\right]\right.$$

$$\left. - \rho_1 e^{-\alpha T(M+2)}\left[e^{-\alpha T(M+\mu-m)}+e^{-\alpha T(M+m-\mu)}\right]\right\}$$

which occurs in 2.3.36.

Note that

$$\sum_{m=o}^{M} Te^{-2\alpha MT} e^{\alpha T(m+\mu)} S_m \rightarrow e^{-2aL} e^{at} \int_{o}^{L} e^{au} S(u)du = -2ae^{-aL} e^{at} g(L)$$

$$\sum_{m=o}^{M} Te^{-\alpha T(m+\mu)} S_m \rightarrow e^{-at} \int_{o}^{L} e^{-au} S(u)du = -2ag(o)e^{-at}$$

$$\sum_{m=o}^{M} Te^{-\alpha TM} e^{-\alpha T\mu} e^{\alpha Tm} S_m \rightarrow e^{-aL} e^{-at} \int_{o}^{L} e^{au} S(u)du = -2ag(L)e^{-at}$$

$$\sum_{m=o}^{M} Te^{-\alpha TM} e^{\alpha T\mu} e^{-\alpha Tm} S_m \rightarrow e^{-aL} e^{at} \int_{o}^{L} e^{-au} S(u)du = -2ae^{-aL} g(o)e^{at}$$

Using 9.), 10.) and the above gives the contribution of this term to $R^{-1}\overline{S}$ as

$$\frac{T}{K\Delta} (a-\beta)^2 (a-\gamma)^2 (a+\beta)^2 (a+\gamma)^2 \left[e^{-aL} g(L)e^{at} + g(o)e^{-at} \right]$$

$$- \frac{T}{K\Delta} (a-\beta)^3 (a-\gamma)^3 (a+\beta)(a+\gamma) \left[g(L)e^{-at} + g(o)e^{-aL} e^{at} \right] e^{-aL} \tag{3.2.49}$$

which gives the terms marked 11.) in 3.2.29 and 3.2.30.

12.)

$$-\frac{\rho_1 Q_1 \phi_D^2}{\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}} \rightarrow \frac{1}{K\Delta}\left\{\left(\rho^2-a^2\right)\left(\gamma^2-a^2\right)(\beta+a)(\gamma+a)\right\}T$$

13.) Consider the term of 2.3.36

$$-\frac{\rho_1 Q_1 \phi_D^2}{\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}}\left[\delta_{\mu M} e^{-\alpha T(M+1-m)} + \delta_{mM} e^{-\alpha T(M+1-\mu)}\right.$$

$$\left.+ \delta_{\mu o} e^{-\alpha T(1+m)} + \delta_{mo} e^{-\alpha T(1+\mu)}\right]$$

Note that

$$\sum_{m=o}^{M} Te^{-\alpha T(M+1-m)} S_m \rightarrow e^{-aL}\int_o^L e^{-au} S(u)du$$

$$\sum_{m=o}^{M} T\delta_{mM} S_m = TS(L) = \text{1st order term (approaches zero)}$$

$$\sum_{m=o}^{M} Te^{-\alpha T(1+m)} S_m \rightarrow \int_o^L e^{-au} S(u)du$$

$$\sum_{m=o}^{M} T\delta_{mo} S_m = TS(o) = \text{1st order term (approaches zero)}$$

Using 12.) and the above, the contribution of this term to $R^{-1}S$ is

$$\frac{1}{K\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)(a+\beta)(a+\gamma)\left\{\delta_{\mu M}e^{-aL}\int_0^L e^{-au}S(u)du + \delta_{\mu o}\int_0^L e^{-au}S(u)du\right.$$

$$\left. + Te^{-aL}e^{at}S(L) + Te^{-at}S(o)\right\}$$

$$= -\frac{2\alpha}{K\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)(a+\beta)(a+\gamma)\left\{g(o)\delta_{\mu o} + g(L)\delta_{\mu M}\right\}$$

$$+ \frac{T}{K\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)(a+\beta)(a+\gamma)\left[e^{-aL}e^{at}S(L) + e^{-at}S(o)\right] \qquad (3.2.50)$$

which gives the terms marked 13.) in 3.2.29 through 3.2.33. Note that in this case the discrete equivalent of a delta function is a Kronecker delta operating on a sampled signal.

14.)

$$\frac{\rho_1^2 Q_1 \phi_D e^{-\alpha T(M+2)}}{\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}} \rightarrow \frac{1}{K\Delta}\left\{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)(a-\beta)(a-\gamma)\right\}Te^{-aL}$$

15.) Consider the term

$$\frac{\rho_1^2 Q_1 \phi_D e^{-\alpha T(M+2)}}{\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}}\left[\delta_{\mu M}e^{-\alpha T(1+m)} + \delta_{mM}e^{-\alpha T(1+\mu)} + \delta_{\mu o}e^{-\alpha T(M+1-m)}\right.$$

$$\left. + \delta_{mo}e^{-\alpha T(M+1-\mu)}\right]$$

Upon using 14.) and the sums of 13.) the contribution of this term to $R^{-1}S$ becomes

$$- \frac{1}{K\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)(a-\beta)(a-\gamma)e^{-aL}\left\{\int_0^L e^{-aL}S(u)du + \delta_{\mu o}e^{-aL}\int_0^L e^{au}S(u)du + Te^{-at}S(L) + Te^{-aL}e^{at}S(o)\right\}$$

$$= \frac{1}{K\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)(a-\beta)(a-\gamma)e^{-aL}\Big[2ag(L)\delta_{\mu o} + 2ag(o)\delta_{\mu M} - - Te^{-at}S(L) - Te^{-aL}e^{at}S(o)\Big] \tag{3.2.51}$$

See terms marked 15.) in 3.2.29 through 3.2.33.

16.) Consider the term

$$- \frac{\rho_1^2 Q_1^2\left(1-e^{-2\alpha T}\right)e^{-2\alpha T(M+2)}}{\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}}\left[\delta_{mo}\delta_{\mu o} + \delta_{\mu M}\delta_{mM}\right]$$

Note that

$$- \frac{\rho_1^2 Q_1^2\left(1-e^{-2\alpha T}\right)e^{-2\alpha T(M+2)}}{\phi_D^2-\rho_1^2 e^{-2\alpha T(M+2)}} \to -\frac{2a}{K\Delta}(a-\beta)^2(a-\gamma)^2e^{-2aL}$$

$$\sum_{m=o}^{M}\delta_{mo}S_m = S(o) \quad \text{and} \quad \sum_{m=o}^{M}\delta_{mM}S_m = S(L)$$

Therefore the contribution of this term to $R^{-1}S$ is

$$- \frac{2a}{K\Delta} (a-\beta)^2(a-\gamma)^2 e^{-2aL}\Big[S(o)\delta_{\mu o} + S(L)\delta_{\mu M}\Big] \qquad (3.2.52)$$

See 3.2.32 and 3.2.33.

17.) Consider the term

$$\sum_{m=o}^{M} \frac{\rho_1 \phi_D Q_1^2 \left(1-e^{-2\alpha T}\right) e^{-\alpha T(M+2)}}{\phi_D^2 - \rho_1^2 e^{-\alpha T(M+2)}} \Big[\delta_{mM}\delta_{\mu o} + \delta_{\mu M}\delta_{mo}\Big] S_m$$

$$\rightarrow \frac{2\alpha}{K\Delta}\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right) e^{-aL}\Big[S(L)\delta_{\mu o} + S(o)\delta_{\mu M}\Big] \qquad (3.2.53)$$

See 3.2.32 and 3.2.33.

18.)* Consider $-Q_1^2\Big[\delta_{\mu M}\delta_{mM} + \delta_{\mu o}\delta_{mo}\Big]$

From 2.3.37

$$Q_1^2 = \frac{1}{k} e^{-2(\beta+\gamma-\alpha)T} \rightarrow \frac{1-2(\beta+\gamma-a)T}{KT[1-(\beta+\gamma-a)T]} \qquad \text{(Using 2.))}$$

$$\rightarrow \frac{1}{KT}\,[1-(\beta+\gamma-a)T]$$

Note that $\sum_{m=o}^{M} \delta_{mM} S_m = S(L)$ and $\sum_{m=o}^{M} \delta_{mo} S_m = S(o)$,

---

* In parts 18.) - 21.) the remainder of Kronecker delta terms will be determined. These terms arise for $\mu = o$ and $\mu = M$.

therefore

$$- Q_1^2 \sum_{m=o}^{M} \left[\delta_{\mu M}\delta_{mM}+\delta_{\mu o}\delta_{mo}\right] S_m \to - \frac{1}{KT} \, [1+(\beta+\gamma-\alpha)T] \left[S(L)\delta_{\mu M}+S(o)\delta_{\mu o}\right] \tag{3.2.54}$$

which is the contribution of this term to $R^{-1}\overline{S}$.

19.) Consider $\left[Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T}\right]\delta_{|m-\mu|,o}$ for $\mu = o,M$.

Note that $\sum_{m=o}^{M}\left[\delta_{mo}\delta_{\mu o}+\delta_{mM}\delta_{\mu M}\right]S_m = \left[S(o)\delta_{\mu o}+S(L)\delta_{\mu M}\right]$

From 5.)

$$Q_o^2 \to \frac{1}{k}\left[1-2(a-\beta)(a-\gamma)T^2\right]$$

$$Q_o\rho_1 \to \frac{1}{k}\left[1-(a-\beta)(a-\gamma)T^2\right]\left[(a-\beta)(a-\gamma)T^2\right] \to \frac{1}{k}(a-\beta)(a-\gamma)T^2$$

also $\quad Q_1\rho_1 e^{-\alpha T} \to -\ (a-\beta)(a-\gamma)\ \frac{T^2}{k}$

$$k \to KT$$

Therefore for $\mu = o,M$

$$\sum_{m=o}^{M}\left[Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T}\right]\delta_{|m-\mu|,o}S_m \to \left\{\frac{1}{k}[1+0T]+Q_1^2\right\}\left[S(o)\delta_{\mu o}+S(L)\delta_{\mu M}\right] \tag{3.2.55}$$

20.) Consider the term $\left[Q_1 Q_o + Q_1 \rho_1\right] \delta_{|m-\mu|,1}$ for

μ - o,M which is equivalent to considering the term

$$\left(Q_1 Q_o + Q_1 \rho_1\right)\left[\delta_{m1}\delta_{\mu o} + \delta_{m,M-1}\delta_{\mu M}\right]$$

Note that: $\sum_{m=o}^{M} \left[\delta_{m1}\delta_{\mu o}+\delta_{m,M-1}\delta_{\mu M}\right] S_m = \left[S(T)\delta_{\mu o}+S(L-T)\delta_{\mu M}\right]$

$$Q_1 Q_o = - k^{-\frac{1}{2}} e^{-(\beta+\gamma-\alpha)T} Q_o \rightarrow - \frac{1}{k}[1-(\beta+\gamma-a)T]$$

$$Q_1 \rho_1 \cong - k^{-1} T^2 (a-\beta)(a-\gamma) \qquad \text{(too high of an order)}$$

Therefore for $\mu = o,M$

$$\sum_{m=o}^{M} \left[Q_1 Q_o+Q_1\rho_1\right] \delta_{|m-\mu|,1} S_m \rightarrow - \frac{1}{k}[1-(\beta+\gamma-a)T]\left[S(T)\delta_{\mu o}+S(L-T)\delta_{\mu M}\right]$$

21.) Now consider

$$- Q_1^2\left[\delta_{\mu M}\delta_{mM}+\delta_{\mu o}\delta_{mo}\right]+\left[Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T}\right]\delta_{|m-\mu|,o}$$

$$+\left[Q_o Q_1+Q_1\rho_1\right]\delta_{|m-\mu|,1}$$

for $\mu = o,M$ and after summing with respect to the signal and m. From 18.), 19.) and 20.) (note that $k^{-1} \to \frac{1}{KT}[1+(\beta+\gamma-a)T]$)

$$\to \frac{1}{k}[1+0T]\left[S(o)\delta_{\mu o}+S(L)\delta_{\mu M}\right]-\frac{1}{k}[1-(\beta+\gamma-a)T]\left[S(T)\delta_{\mu o}+S(L-T)\delta_{\mu M}\right]$$

$$=\frac{1}{k}\left\{[S(o)-S(T)]\delta_{\mu o}+[S(L)-S(L-T)]\delta_{\mu M}+(\beta+\gamma-a)\left[S(T)\delta_{\mu o}+S(L-T)\delta_{\mu M}\right]T\right\}$$

$$\to \frac{1}{K}\left\{-\frac{S(T)-S(o)}{T}\delta_{\mu o}+\frac{S(L)-S(L-T)}{T}\delta_{\mu M}+(\beta+\gamma-a)\left[S(o)\delta_{\mu o}+S(L)\delta_{\mu M}\right]\right\}$$

$$\to \frac{1}{K}\left\{-\dot{S}(o)\delta_{\mu o}+\dot{S}(L)\delta_{\mu M}+(\beta+\gamma-a)\left[S(o)\delta_{\mu o}+S(L)\delta_{\mu M}\right]\right\} \qquad (3.2.56)$$

See the terms marked 21.) in 3.2.32 and 3.2.33.

This completes the determination of the delta function and the exponential components of the solution. The remainder of this part is devoted to the determination of the term $-\frac{1}{K}\left[\frac{d^2S}{dt^2}+\left(\beta^2+\gamma^2-a^2\right)S(t)\right]$.

22.) Consider $Q_1Q_o+Q_1\rho_1$ in more detail

$$Q_1\left(Q_o+\rho_1\right)=Q_1\left(k^{-\frac{1}{2}}-\rho_1+\rho_1\right)=k^{-\frac{1}{2}}Q_1=-k^{-1}e^{-(\beta+\gamma)T}e^{\alpha T}$$

$$=-\frac{1}{A_oe^{\alpha T}}e^{-(\beta+\gamma)T}e^{\alpha T}=-\frac{1}{A_oe^{(\beta+\gamma)T}} \qquad (3.2.57)$$

But

$$A_o e^{(\beta+\gamma)T} = \sigma_1^2\left(e^{\beta T}-e^{-\beta T}\right)+\sigma_2^2\left(e^{\gamma T}-e^{-\gamma T}\right) = 2\left(\sigma_1^2 \sinh \beta T + \sigma_2^2 \sinh \gamma T\right)$$

$$\cong 2\left[\sigma_1^2\beta T\left(1+\frac{\beta^2T^2}{6}\right)+\sigma_2^2\gamma T\left(1+\frac{\gamma^2T^2}{6}\right)\right]$$

$$\cong TK\left[1+\frac{T^2}{3K}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right]$$

$$\therefore -\frac{1}{A_o e^{(\beta+\gamma)T}} \cong -\frac{1}{TK}+\frac{T}{3K^2}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)$$

$$\left(\frac{1}{A_o e^{(\beta+\gamma)T}}\right)^2 \simeq \frac{1}{T^2K^2} - \frac{2}{3}\frac{1}{K^3}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right) \qquad (3.2.58)$$

23.) Consider $Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1e^{-\alpha T}$ in more detail

$$Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1e^{-\alpha T} = \left[Q_o^2+Q_1^2+\frac{Q_o}{k^{\frac{1}{2}}} - Q_o^2+Q_1\rho_1e^{-\alpha T}\right] = Q_1^2+\frac{Q_o}{k^{\frac{1}{2}}}+Q_1\rho_1e^{-\alpha T}$$

After using the definitions 2.3.37, the above equation takes the form

$$Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T} = \frac{1}{A_o e^{(\beta+\gamma)T}}\left(e^{\beta T}+e^{\gamma T}\right)+\frac{1}{A_o e^{(\beta+\gamma)T}}\left(e^{-\beta T}+e^{-\gamma T}\right)$$

$$-\frac{B_o e^{(\beta+\gamma)T}}{\left(A_o e^{(\beta+\gamma)T}\right)^2} \qquad (3.2.59)$$

But

$$B_o e^{(\beta+\gamma)T} = \sigma_1^2\left(e^{\beta T}-e^{-\beta T}\right)\left(e^{\gamma T}+e^{-\gamma T}\right)+\sigma_2^2\left(e^{\gamma T}-e^{-\gamma T}\right)\left(e^{\beta T}+e^{-\beta T}\right)$$

$$= 4\left[\sigma_1^2 \sinh \beta T \cosh \gamma T + \sigma_2^2 \sinh \gamma T \cosh \beta T\right]$$

$$\approx T\left[2K + \left(Ka^2 + \frac{2}{3}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right)T^2\right]$$

Upon noting the following limits

$$\left[A_o e^{(\beta+\gamma)T}\right]^{-1}\left(e^{\beta T}+e^{\gamma T}\right) \rightarrow \frac{1}{TK}\left[1-\frac{T^2}{3K}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right]\left[2+(\beta+\gamma)T+\tfrac{1}{2}\left(\beta^2+\gamma^2\right)T^2\right]$$

$$\rightarrow \frac{1}{TK}\left\{2+(\beta+\gamma)T+\left[\tfrac{1}{2}\left(\beta^2+\gamma^2\right)-\frac{2}{3K}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right]T^2\right\}$$

$$\left[A_o e^{(\beta+\gamma)T}\right]^{-1}\left(e^{-\beta T}+e^{-\gamma T}\right) \rightarrow \frac{1}{TK}\left[1-\frac{T^2}{3K}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right]\left[2-(\beta+\gamma)T+\tfrac{1}{2}\left(\beta^2+\gamma^2\right)T^2\right]$$

$$\rightarrow \frac{1}{TK}\left\{2-(\beta+\gamma)T+\left[\tfrac{1}{2}\left(\beta^2+\gamma^2\right)-\frac{2}{3K}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right]T^2\right\}$$

$$-\left[A_o e^{(\beta+\gamma)T}\right]^{-2} B_o e^{(\beta+\gamma)T} \rightarrow -\left[\frac{1}{TK^2}-\frac{2}{3}\frac{1}{K^2}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right]\Big[2K+$$

$$+\left(Ka^2+\frac{2}{3}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right)T^2\Big]T$$

$$\rightarrow -T\Big[\frac{2}{KT^2}-\frac{4}{3}\frac{1}{K^2}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)+\frac{a^2}{K}+$$

$$+\frac{2}{3}\frac{1}{K^2}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\Big] \qquad (3.2.60)$$

and from 3.2.59 and 3.2.60

$$Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T} \rightarrow \frac{2}{TK}\left[\left(\beta^2+\gamma^2-\alpha^2\right)-\frac{2}{3K}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right] \qquad (3.2.61)$$

24.) Consider the terms

$$\left[Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T}\right]\delta_{m\mu}+\left[Q_1 Q_o+Q_1\rho_1\right]\left(\delta_{m,\mu-1}+\delta_{m,\mu+1}\right)$$

for $\mu \neq 0, M$.

Note that $\sum_{m=o}^{M}\delta_{m\mu}S_m = S_\mu = S(t)$ and $\sum_{m=o}^{M}\left(\delta_{m,\mu-1}+\delta_{m,\mu+1}\right)S_m =$

$$= S(t-T) + S(t+T)$$

Therefore from the results of 22.) and 23.) the contribution of the above terms to $R^{-1}\bar{S}$ approaches

$$\rightarrow \frac{-1}{KT}\left[S(t-T)-2S(t)+S(t+T)\right]+\frac{T}{K}\left\{\left[\left(\beta^2+\gamma^2-a^2\right)-\frac{2}{3K}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\right]S(t)\right.$$

$$\left.+\frac{1}{3K}\left(\beta^3\sigma_1^2+\gamma^3\sigma_2^2\right)\left[S(t-T)+S(t+T)\right]\right\}$$

$$\rightarrow -\frac{T}{K}\frac{d^2S}{dt^2}+\frac{T}{K}\left(\beta^2+\gamma^2-a^2\right)S(t) \qquad (3.2.62)$$

which completes the demonstration of the equivalence between the limit of the discrete solution and the continuous solution in the case of the spectral density 3.2.1.

3.2.2.2 All Pole Second Order Noise

The limiting form of 3.2.22 is now going to be determined when the autocorrelation function is 3.2.24 and the condition 3.2.25, the all pole condition, is satisfied. Only the limiting form of $R^{-1}\bar{S}$ is necessary because of the reason mentioned in 3.2.2.1. For convenient reference the continuous filter is summarized below. The numbers on the terms are for identification with parts of the limiting discrete solution.

14.) 16.) 17.)

$$\frac{1}{\sigma_L^2} f(t) = \frac{\frac{1}{2}}{\gamma^2-\beta^2}\left\{\frac{d^4S}{dt^4} - \left(\beta^2+\gamma^2\right)\frac{d^2S}{dt^2} + \beta^2\gamma^2 S(t)\right.$$

27.) 23.) 22.)

$$+[\ddot{S}(o)+\left(\beta\gamma-(\beta+\gamma)^2\right)\dot{S}(o)+\beta\gamma(\beta+\gamma)S(o)]\delta(t)-$$

25.) 24.)

$$-[\dddot{S}(L)+\left(\beta\gamma-(\beta+\gamma)^2\right)\dot{S}(L)-\beta\gamma(\beta+\gamma)S(L)]\delta(t-L)+[\ddot{S}(o)-(\beta+\gamma)\dot{S}(o)+$$

20.)

$$\left. + \beta\gamma S(o)]\delta'(t)-[\ddot{S}(L)+(\beta+\gamma)\dot{S}(L)+\beta\gamma S(L)]\delta'(t-L)\right\} \qquad (3.2.63)$$

As in 3.2.2.1 the relations between the quantities β, γ, σ_1^2, σ_2^2, and K_p of 3.2.2 and 3.2.13 are not all independent. Without loss of generality the values for σ_1^2 and σ_2^2 which were chosen are

$$\sigma_1^2 = \frac{1}{\beta}$$

$$\sigma_2^2 = \frac{-1}{\gamma} \tag{3.2.64}$$

and hence

$$K_p = 2\left(\gamma^2-\beta^2\right) \tag{3.2.65}$$

It should be noted that the constraint 3.2.25 is satisfied.

The quantities A_o and B_o of 2.3.37 and some related functions will now be expanded.

1.) Using 3.2.64 the quantity A_o of 2.3.37 can be expressed

$$A_o = 2Te^{-(\beta+\gamma)T}\left[\frac{1}{\beta T}\sinh\beta T - \frac{1}{\gamma T}\sinh\gamma T\right]$$

$$\cong -\frac{T^3}{3}\left(\gamma^2-\beta^2\right)e^{-(\beta+\gamma)T}\left[1 + \frac{1}{20}\left(\beta^2+\gamma^2\right)T^2 + \frac{6}{7!}\left(\beta^4+\beta^2\gamma^2+\gamma^4\right)T^4+0\left(T^6\right)\right] \tag{3.2.66}$$

Similarly B_o can be expressed

$$B_o = 4e^{-(\beta+\gamma)T}\left[\frac{1}{\beta T}\sinh\beta T\cosh\gamma T - \frac{1}{\gamma T}\sinh\gamma T\cosh\beta T\right]T$$

$$\simeq \frac{4}{3}T^2e^{-(\beta+\gamma)T}\left\{1 + \frac{1}{10}\left(\beta^2+\gamma^2\right)T^2 + \frac{3}{6!}\left[\frac{6}{7}\left(\beta^4+\beta^2\gamma^2+\gamma^4\right)+2\beta^2\gamma^2\right]T^4 + 0\left(T^6\right)\right\}\left(\gamma^2-\beta^2\right) \tag{3.2.67}$$

2.) Using 3.2.66 and 3.2.67 gives

$$B_o^2 - 4A_o^2 \simeq \frac{4}{9}\left[\left(\gamma^2-\beta^2\right)T^3 e^{-(\beta+\gamma)T}\right]^2 \left\{3 + \frac{14}{20}\left(\beta^2+\gamma^2\right)T^2 + \left[\frac{132}{7!}\left(\beta^4+\beta^2\gamma^2+\gamma^4\right) + \right.\right.$$

$$\left.\left. + \frac{16}{6!}\beta^2\gamma^2 + \frac{15}{400}\left(\beta^2+\gamma^2\right)^2\right]T^4 + O\left(T^6\right)\right\} \tag{3.2.68}$$

which will be needed later.

3.) Again using 3.2.66 and 3.2.67, some other expansions of future interest are

$$\frac{1}{A_o} \cong \frac{-3}{T^3\left(\gamma^2-\beta^2\right)e^{-(\beta+\gamma)T}}\left\{1 - \frac{1}{20}\left(\beta^2+\gamma^2\right)T^2 + \left[\frac{1}{400}\left(\beta^2+\gamma^2\right)^2 - \right.\right.$$

$$\left.\left. - \frac{6}{7!}\left(\beta^4+\beta^2\gamma^2+\gamma^4\right)\right]T^4 + O\left(T^6\right)\right\} \tag{3.2.69}$$

$$\frac{1}{A_o^2} \cong \frac{9}{T^6\left(\gamma^2-\beta^2\right)^2 e^{-2(\beta+\gamma)T}}\left\{1 - \frac{1}{10}\left(\beta^2+\gamma^2\right)T^2 + \left[\frac{3}{400}\left(\beta^2+\gamma^2\right)^2 - \right.\right.$$

$$\left.\left. - \frac{12}{7!}\left(\beta^4+\beta^2\gamma^2+\gamma^4\right)\right]T^4 + O\left(T^6\right)\right\} \tag{3.2.70}$$

$$\frac{B_o}{A_o} \cong -4\left\{1 + \frac{1}{20}\left(\beta^2+\gamma^2\right)T^2 + \left[\frac{1}{5!}\beta^2\gamma^2 + \frac{12}{7!}\left(\beta^4+\beta^2\gamma^2+\gamma^4\right) - \right.\right.$$

$$\left.\left. - \frac{1}{400}\left(\beta^2+\gamma^2\right)^2\right]T^4 + O\left(T^6\right)\right\} \tag{3.2.71}$$

It should be noted that $\frac{B_o}{A_o}$ is important since $e^{\alpha T}$ and $e^{-\alpha T}$ are the roots of

$$0 = B_o - \left(z+z^{-1}\right)A_o$$

or

$$e^{\alpha T}+e^{-\alpha T} = \frac{B_o}{A_o} \tag{3.2.72}$$

4.) From 2.3.37

$$e^{-\alpha T} = \frac{1}{2A_o}\left(B_o - \sqrt{B_o^2-4A_o^2}\right) \quad ,$$

which by use of 3.2.62, 3.2.68 and 3.2.69 becomes

$$e^{-\alpha T} \cong -\left(2 - \sqrt{3}\right)\left[1 - \frac{\sqrt{3}}{30}\left(\beta^2+\gamma^2\right)T^2 + O\left(T^4\right)\right] \tag{3.2.73}$$

Equation 3.2.73 shows that α is a function of T, the time between samples.

In 5.) through 8.) some more algebraic manipulations and expansions are performed. The results are needed in the demonstration of the equivalence between the limit of the discrete solution and the continuous solution. It is recommended that the reader skip to 9.) and refer back to 5.) through 8.) when they are referenced.

5.) The quantity $\rho_1 \frac{\phi_D}{\phi_N}$ can be expressed

$$\rho_1 \frac{\phi_D}{\phi_N} = \rho_1\left[Q_o+Q_1e^{-\alpha T} + \frac{\rho_1}{1-e^{-2\alpha T}}\right]=$$

$$=\frac{\left(1-e^{-\beta T}e^{\alpha T}\right)\left(1-e^{-\gamma T}e^{\alpha T}\right)\left(1-e^{-\beta T}e^{-\alpha T}\right)\left(1-e^{-\gamma T}e^{-\alpha T}\right)}{A_oe^{\alpha T}\left(1-e^{-2\alpha T}\right)}$$

$$=\frac{\left[2\cosh\beta T-\frac{B_o}{A_o}\right]\left[2\cosh\gamma T-\frac{B_o}{A_o}\right]}{A_oe^{(\beta+\gamma)T}e^{\alpha T}\left(1-e^{-2\alpha T}\right)} \tag{3.2.74}$$

6.) A term which will be needed is

$$Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1e^{-\alpha T}+2Q_oQ_1+2Q_1\rho_1+\frac{1+e^{-\alpha T}}{1-e^{-\alpha T}}\rho_1\frac{\phi_D}{\phi_N}$$

$$=\frac{1}{A_oe^{(\beta+\gamma)T}}\left\{e^{\beta T}+e^{-\beta T}+e^{\gamma T}+e^{-\gamma T}-\frac{B_o}{A_o}-2+\right.$$

$$\left.+\left(\frac{1+e^{-\alpha T}}{1-e^{-\alpha T}}\right)\frac{\left[2\cosh\beta T-\frac{B_o}{A_o}\right]\left[2\cosh\gamma T-\frac{B_o}{A_o}\right]}{e^{\alpha T}\left(1+e^{-\alpha T}\right)\left(1-e^{-\alpha T}\right)}\right\}$$

$$=\frac{1}{A_oe^{(\beta+\gamma)T}}\left\{2\cosh\beta T+2\cosh\gamma T-2-\frac{B_o}{A_o}-\right.$$

$$\left.-\frac{1}{2-\frac{B_o}{A_o}}\left(2\cosh\beta T-\frac{B_o}{A_o}\right)\left(2\cosh\gamma T-\frac{B_o}{A_o}\right)\right\}$$

$$\equiv\frac{C_o}{A_oe^{(\beta+\gamma)T}}$$

where C_o is defined through the above equation. Equations 3.2.59

and 3.2.74 were used in deriving the final result.

After considerable expansion it can be shown that

$$C_o \cong -\frac{1}{6}\beta^2\gamma^2T^4\left[1 + O\left(T^2\right)\right] \tag{3.2.75}$$

and hence

$$\frac{C_o}{A_o} \cong \frac{1}{2}\frac{\beta^2\gamma^2}{\left(\gamma^2-\beta^2\right)} T\left[1 + O\left(T^2\right)\right] \tag{3.2.76}$$

7.) The following sums will be needed, where $\chi = e^{-\alpha T}$

a. $$\sum_0 = \sum_{k=o}^{\infty}\chi^k = \frac{1}{1-\chi}$$

b. $$\sum_1 = \sum_{k=o}^{\infty} k\chi^k = \chi(1-\chi)^{-2}$$

c. $$\sum_2 = \sum_{k=o}^{\infty}k^2\chi^k = \chi(1-\chi)^{-2} + 2\chi^2(1-\chi)^{-3} = \chi\frac{1+\chi}{(1-\chi)^3}$$

(3.2.77)

d. $$\sum_3 = \sum_{k=o}^{\infty}k^3\chi^k = \chi(1-\chi)^{-2}+6\chi^2(1-\chi)^{-3}+6\chi^3(1-\chi)^{-4}$$

e. $$\sum_4 = \sum_{k=o}^{\infty}k^4\chi^k = \chi(1-\chi)^{-2}+14\chi^2(1-\chi)^{-3}+36\chi^3(1-\chi)^{-4}+24\chi^4(1-\chi)^{-5}$$

8.) The following is an expansion involving the signal S(t)

$$S(t+T) \cong S(t) + \frac{dS}{dt} T + \frac{1}{2} \frac{d^2S}{dt^2} T^2 + \frac{1}{6} \frac{d^3S}{dt^3} T^3 + \frac{1}{24} \frac{d^4S}{dt^4} T^4 + 0\left(T^5\right)$$

$$S(t-T) \cong S(t) - \frac{dS}{dt} T + \frac{1}{2} \frac{d^2S}{dt^2} T^2 - \frac{1}{6} \frac{d^3S}{dt^3} T^3 + \frac{1}{24} \frac{d^4S}{dt^4} T^4 - 0\left(T^5\right)$$

and hence

$$S(t+T) + S(t-T) \approx 2S(t) + \frac{d^2S}{dt^2} T^2 + \frac{1}{12} \frac{d^4S}{dt^4} T^4 + 0\left(T^6\right) \qquad (3.2.78)$$

Inspection of 2.3.36 shows that in evaluating the limiting properties of $R^{-1}\overline{S}$ a term of the form $\sum_{0}^{M} S_m e^{-\alpha T|m-\mu|}$ will have to be investigated. It is the purpose of 9.) to investigate this term.

9.) The term $\sum_{0}^{M} S_m e^{-\alpha T|m-\mu|}$ can be expressed

$$\sum_{0}^{M} S_m e^{-\alpha T|m-\mu|} = \sum_{0}^{M} S_m \left[e^{\alpha T(m-\mu)} - e^{-\alpha T(m-\mu)}\right] + e^{\alpha T\mu} \sum_{0}^{M} S_m e^{-\alpha Tm} \qquad (3.2.79)$$

Let $k = m-\mu$, then

$$\sum_{0}^{M} S_m e^{-\alpha T|m-\mu|} = \sum_{-\mu}^{M-k} S_{\mu+k} e^{-\alpha T|k|}$$

$$= \sum_{-\mu}^{0} S_{\mu+k} e^{-\alpha T|k|} + \sum_{0}^{M-\mu} S_{\mu+k} e^{-\alpha Tk} - S_{\mu}$$

$$= \sum_{0}^{\mu} S_{\mu-k} e^{-\alpha Tk} + \sum_{0}^{M-\mu} S_{\mu+k} e^{-\alpha Tk} - S_{\mu} \qquad (3.2.80)$$

Expanding $S_{\mu-k}$ and $S_{\mu+k}$ about $S_{\mu} = S(t)$ and assuming $\mu \gg 0$ and $M-\mu \gg 0$ gives

$$\sum_{0}^{M} S_m e^{-\alpha T|m-\mu|} \cong S(t)\left[2\sum_{0} -1\right] + T^2 \frac{d^2S}{dt^2} \sum_{2} + \frac{T^4}{12} \frac{d^4S}{dt^4} \sum_{4} \qquad (3.2.81)$$

The behavior of $\sum_{0}^{M} S_m e^{-\alpha T|m-\mu|}$ will now be investigated for small μ. The following approximate sums are obtained by expanding S_m about $S_o = S(0)$.

$$\sum_{0}^{M} S_m e^{-\alpha T|m-\mu|} = e^{-\alpha T\mu} \sum_{0}^{\mu} S_m e^{m\alpha T} - e^{\alpha T\mu} \sum_{0}^{\mu} S_m e^{-m\alpha T} + e^{\alpha T\mu} \sum_{0}^{M} S_m e^{-\alpha Tm} \qquad (3.2.82)$$

$$\sum_{0}^{\mu} S_m e^{m\alpha T} \cong S(o)\sum_{0}^{\mu} e^{m\alpha T} + T\dot{S}(o)\sum_{0}^{\mu} m e^{m\alpha T} + \frac{1}{2} T^2 \ddot{S}(o)\sum_{0}^{\mu} m^2 e^{m\alpha T}$$

$$+ \frac{T^3}{6} \dddot{S}(o)\sum_{0}^{\mu} m^3 e^{m\alpha T} \qquad (3.2.83)$$

$$\sum_{0}^{\mu} S_m e^{-m\alpha T} \cong S(o)\sum_{0}^{\mu} e^{-m\alpha T} + T\dot{S}(o)\sum_{0}^{\mu} m e^{-m\alpha T} + \frac{T^2}{2} \ddot{S}(o)\sum_{0}^{\mu} m^2 e^{-m\alpha T}$$

$$+ \frac{T^3}{6} \dddot{S}(o)\sum_{0}^{\mu} m^3\, e^{-m\alpha T} \qquad (3.2.84)$$

$$\sum_{0}^{M} S_m e^{-\alpha T m} \cong S(o)\sum_{0} + T\dot{S}(o)\sum_{1} + \frac{1}{2} T^2 \ddot{S}(o)\sum_{2} + \frac{1}{6} T^3 \dddot{S}(o)\sum_{3} \qquad (3.2.85)$$

Now from (3.2.81) note that for $\mu \gg 0$, and $M-\mu \gg 0$, which loosely speaking is the "middle" of the range $0 \le t \le L$,

$$\sum_{o}^{M}{}' S_m e^{-\alpha T|m-\mu|} \cong S(t)\left[2\sum_{o} -1\right] + T^2\ddot{S}(t)\sum_{2} + \cdots$$

$$\approx \left(2\sum_{o} -1\right)\left[S(o) + \mu T\dot{S}(o) + \cdots\right] +$$

$$+ \sum_{2}\left[T^2\ddot{S}(o) + \mu T^3\dddot{S}(o) + \cdots\right] + \cdots$$

$$\cong \left(2\sum_{o} -1\right)S(o) + T^2\sum_{2}\ddot{S}(o) + \text{ terms in } \mu$$

$$\cong \frac{1+e^{-\alpha T}}{1-e^{-\alpha T}}S(o) + T^2\ddot{S}(o)e^{-\alpha T}\frac{1+e^{-\alpha T}}{\left(1-e^{-\alpha T}\right)^3} + \text{ terms in } \mu \tag{3.2.86}$$

where 3.2.77 was used.

The behavior of the sums 3.2.83, 3.2.84 and 3.2.85 for small μ will now be found and hence the behavior of 3.2.82 determined. The method of attack will be to examine separately the functions of μ which multiply $S(o)$, $\dot{S}(o)$, $\ddot{S}(o)$, and $\dddot{S}(o)$.

<u>Coefficient of $S(o)$</u>

From 3.2.82 through 3.2.85

$$F_o = \text{coefficient of } S(o) = \left[e^{-\alpha T\mu} \sum_o^{\mu} e^{m\alpha T} - e^{\alpha T\mu} \sum_o^{\mu} e^{-m\alpha T} + e^{-\alpha T\mu} \sum_o \right]$$

which can be expressed

$$F_o = a_o e^{-\alpha T\mu} + C_o$$

where a_o and C_o are at present unknown.

Now it is important to note that as $T \to 0$, the quantity $e^{-\alpha T}$ approaches a number with magnitude less than unity (see 3.2.73), therefore $e^{-\alpha T\mu}$ approaches zero "faster" than any power of T as $\mu \to \infty$. This means that

$$F_o \to C_o \quad \text{for} \quad \mu \gg 0$$

but from 3.2.86 the coefficient of $S(o)$ is $\frac{1+e^{-\alpha T}}{1-e^{-\alpha T}}$, therefore

$$C_o = \frac{1+e^{-\alpha T}}{1-e^{-\alpha T}}$$

Now note that for $\mu = o$,

$$F_o = \sum_o = a_o + C_o \qquad \text{or} \qquad a_o = \frac{-e^{-\alpha T}}{1-e^{-\alpha T}} = \frac{1}{1-e^{\alpha T}}$$

Therefore

$$\text{coef of} \quad S(o) = \frac{1}{1-e^{\alpha T}} e^{-\alpha T \mu} + \frac{1+e^{-\alpha T}}{1-e^{-\alpha T}} \tag{3.2.87}$$

<u>Coefficient of $\dot{S}(o)T$</u>

From 3.2.82 through 3.2.85

$$F_1 = \text{coefficient of} \quad \dot{S}(o)T = e^{-\alpha T \mu} \sum_{o}^{\mu} m e^{m\alpha T} - e^{+\alpha T \mu} \sum_{o}^{\mu} m e^{-m\alpha T}$$

$$+ e^{-\alpha T \mu} \sum_1$$

which can be expressed

$$F_1 = a_1 e^{-\alpha T \mu} + b_1 \mu + c_1$$

where a_1, b_1, and c_1 are at present unknown. As was noted in the determination of the coefficient of $S(o)$

$$F_1 \rightarrow b_1 \mu + c_1 \qquad \text{as} \qquad \mu \rightarrow \infty$$

Comparing the above with 3.2.86 shows that $c_1 = o$ since 3.2.86 contains no constant term in $\dot{S}(o)$. Now note that for $\mu = o$

$$F_1 = a_1 + c_1 = a_1 = \sum_1 = \frac{e^{-\alpha T}}{\left(1-e^{-\alpha T}\right)^2}$$

Therefore

$$\text{coef of } \dot{S}(o)T = \frac{e^{-\alpha T}}{\left(1-e^{-\alpha T}\right)^2} e^{-\alpha T\mu} + \text{terms of 3.2.86} \tag{3.2.88}$$

Coefficient of $\frac{1}{2}\ddot{S}(o)T^2$

Application of the technique just used shows that

$$\text{coef of } \tfrac{1}{2}\ddot{S}(o)T^2 = -\sum\nolimits_2 e^{-\alpha T\mu} + \text{terms of 3.2.86} \tag{3.2.89}$$

Coefficient of $\frac{1}{6}\dddot{S}(o)T^3$

Application of the technique just used shows that

$$\text{coef of } \tfrac{1}{6}\dddot{S}(o)T^3 = \sum\nolimits_3 e^{-\alpha T\mu} + \text{terms of 3.2.86} \tag{3.2.90}$$

Therefore to summarize, a more accurate expansion of the term $\sum_{o}^{M} S_m e^{-\alpha T|m-\mu|}$ has been obtained. The result is

$$\sum_{o}^{M} S_m e^{-\alpha T|m-\mu|} \cong \left[\frac{1}{1-e^{\alpha T}} S(o) + \sum\nolimits_1 \dot{S}(o)T - \sum\nolimits_2 \ddot{S}(o)\frac{T^2}{2} + \sum\nolimits_3 \dddot{S}(o)\frac{T^3}{6} + \cdots\right] e^{-\alpha T\mu} + \left[2\sum\nolimits_6 -1\right] S(t) + T^2\frac{d^2S}{dt^2}\sum\nolimits_2 + \frac{T^4}{12}\frac{d^4S}{dt^4}\sum\nolimits_4 \tag{3.2.91}$$

10.) From 3.2.73 it can be seen that

$$|e^{-\alpha T}| < 1$$

therefore

$$\lim_{M\to\infty} T^k e^{-(M+p)T} \to 0 \qquad \text{for } p,k \text{ finite}$$

This means that for large M, all pole noise, the inverse covariance matrix 2.3.36 becomes

$$\begin{aligned} W_\mu(mT) \cong & - Q_1^2\left[\delta_{\mu M}\delta_{mM}+\delta_{\mu o}\delta_{mo}\right] - \rho_1 Q_1\left[\delta_{\mu M}e^{-\alpha T(M+1-m)}+\delta_{mM}e^{-\alpha T(M+1-\mu)}\right. \\ & \left. + \delta_{\mu o}e^{-\alpha T(1+m)}+\delta_{mo}e^{-\alpha T(1+\mu)}\right] + \left[Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T}\right]\delta_{|m-\mu|,o} \\ & + \left[Q_1Q_o+Q_1\rho_1\right]\delta_{|m-\mu|,1}+\rho_1\left[Q_o+Q_1e^{-\alpha T}+\frac{\rho_1}{1-e^{-2\alpha T}}\right]e^{-\alpha T|m-\mu|} \\ & - \rho_1^2\,\frac{e^{-2\alpha T}}{1-e^{-2\alpha T}}\left[e^{-\alpha T[(M-m)+(M-\mu)]} + e^{-\alpha T(m+\mu)}\right] \qquad (3.2.92) \end{aligned}$$

11.) From 10.) the μ^{th} component of $R^{-1}\overline{S}$ is

$$\sum_{m=0}^{M} W_\mu(mT)S_m = - Q_1^2[S(L)\delta_{\mu M}+S(o)\delta_{\mu o}]$$

$$- \rho_1 Q_1\left[\delta_{\mu M}e^{-\alpha T(M+1)}\sum_{o}^{M} e^{\alpha Tm}S_m+S(L)e^{-\alpha T(M+1-\mu)}\right.$$

$$\left.+ \delta_{\mu o}e^{-\alpha T}\sum_{o}^{M} e^{-\alpha Tm}S_m+S(o)e^{-\alpha T(1+\mu)}\right]+$$

$$+\left[Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T}\right]S(t) +\left[Q_1 Q_o+Q_1\rho_1\right]\left[S(t+T)+S(t-T)\right]$$

$$+ \rho_1\left[Q_o+Q_1 e^{-\alpha T}+ \frac{\rho_1}{1-e^{-2\alpha T}}\right]\sum_{o}^{M} S_m e^{-\alpha T|m-\mu|}$$

$$- \rho_1^2 \frac{e^{-2\alpha T}}{1-e^{-2\alpha T}}\left[e^{-\alpha T(M-\mu)}\sum_{o}^{M} S_m e^{-\alpha T(M-m)}+e^{-\alpha T\mu}\sum_{o}^{M} S_m e^{-\alpha Tm}\right] \tag{3.2.93}$$

It now remains to determine the limiting behavior of 3.2.93. The behavior for $\mu \gg 0$, $M-\mu \gg 0$ will be determined first.

12.) Consider the term $\left[Q_1Q_o+Q_1\rho_1\right]\left[S(t+T)+S(t-T)\right]$. From 3.2.57 and 3.2.69

$$Q_1Q_o+Q_1\rho_1 = \frac{-1}{A_oe^{(\beta+\gamma)T}} \rightarrow \frac{3}{T^3\left(\gamma^2-\beta^2\right)}\left\{1-\frac{1}{20}\left(\beta^2+\gamma^2\right)T^2\right\}$$

Therefore from 3.2.78

$$\left[Q_1Q_o+Q_1\rho_1\right]\left[S(t+T)+S(t-T)\right] \rightarrow 2\left(Q_1Q_o+Q_1\rho_1\right)S(t)$$

$$+\frac{3}{T\left(\gamma^2-\beta^2\right)}\left[1-\frac{1}{20}\left(\beta^2+\gamma^2\right)T^2\right]\frac{d^2S}{dt^2}$$

$$+\frac{T}{4\left(\gamma^2-\beta^2\right)}\frac{d^4S}{dt^4} \qquad (3.2.94)$$

The term

$$\rho_1\left[Q_o+Q_1e^{-\alpha T}+\frac{\rho_1}{1-e^{-2\alpha T}}\right]\sum_{m=o}^{M} e^{-\alpha T|m-\mu|}S_m=\rho_1\frac{\phi_D}{\phi_N}\sum_{m=o}^{M}e^{-\alpha T|M-\mu|}S_m$$

$$\cong\left\{\left[2\sum\nolimits_o-1\right]S(t)+T^2\frac{d^2S}{dt^2}\sum\nolimits_2+\frac{T^4}{12}\frac{d^4S}{dt^4}\sum\nolimits_4\right\}\frac{\rho_1\phi_D}{\phi_N}\quad \mu\gg 0,\ M-\mu\gg 0 \qquad (3.2.95)$$

will now be discussed.

13.) Consider the component of 3.2.95 (see 3.2.74)

$$\rho_1 \frac{\phi_D}{\phi_N} \sum_4 \frac{d^4S}{dt^4}\frac{T^2}{12} = \frac{\left[2\cosh\beta T - \frac{B_o}{A_o}\right]\left[2\cosh\gamma T - \frac{B_o}{A_o}\right]}{A_o e^{(\beta+\gamma)} e^{\alpha T}\left(1-e^{-2\alpha T}\right)} \sum_4{}' \frac{T^2}{12}\frac{d^4S}{dt^4}$$

Holding only the term of order T (which is the lowest order term) gives

$$\rho_1 \frac{\phi_D}{\phi_N}\left(\frac{T^4}{12}\frac{d^4S}{dt^4}\sum_4\right) \rightarrow -\frac{(3)(6)(6)}{T^3\left(\gamma^2-\beta^2\right)}\left[\frac{-1}{2+\sqrt{3}-2+\sqrt{3}}\right]\left(\frac{\sqrt{3}}{18}\right)\frac{T^4}{12}\frac{d^4S}{dt^4}$$

$$\rightarrow \frac{T}{4\left(\gamma^2-\beta^2\right)}\frac{d^4S}{dt^4} \qquad (3.2.96)$$

Equations 3.2.69, 3.2.71, 3.2.73 and the fact that

$$\sum_4 - \sum_{m=o}^{\infty} m^4 e^{-\alpha T m} \rightarrow \frac{\sqrt{3}}{18}$$

were used in deriving 3.2.96.

14.) Combining the results of 12.) and 13.) for the term in $\frac{d^4S}{dt^4}$ gives

$$\left(\frac{1}{4}+\frac{1}{4}\right)\frac{T}{\gamma^2-\beta^2}\frac{d^4S}{dt^4} = \frac{1}{2}\frac{T}{\gamma^2-\beta^2}\frac{d^4S}{dt^4}$$

where T is similar to dt.

15.) Consider the component of 3.2.95 (see 3.2.77)

$$\rho_1 \frac{\phi_D}{\phi_N}\left(T^2 \frac{d^2S}{dt^2} \sum_2\right) = \frac{1}{A_o e^{(\beta+\gamma)T}}\left(2\cosh\beta T - \frac{B_o}{A_o}\right)\left(2\cosh\gamma T - \frac{B_o}{A_o}\right)\left[\frac{e^{-2\alpha T}}{\left(1-e^{-\alpha T}\right)^4}\right]T^2 \frac{d^2S}{dt^2}$$

$$\rightarrow \frac{-3}{T}\left[1 + \frac{7}{60}\left(\beta^2+\gamma^2\right)T^2\right]\frac{d^2S}{dt^2}\frac{1}{\gamma^2-\beta^2} \qquad (3.2.97)$$

Equations 3.2.69, 3.2.71, and 3.2.73 were used in deriving the above result.

16.) From 3.2.94 and 3.2.97 the term in $\frac{d^2S}{dt^2}$ is

$$\frac{3}{T\left(\gamma^2-\beta^2\right)}\left[1 - \frac{1}{20}\left(\beta^2+\gamma^2\right)T^2\right]\frac{d^2S}{dt^2} - \frac{3}{T\left(\gamma^2-\beta^2\right)}\left[1 + \frac{7}{60}\left(\beta^2+\gamma^2\right)T^2\right]\frac{d^2S}{dt^2}$$

$$= \frac{1}{2}\,\frac{1}{\gamma^2-\beta^2}\left(\beta^2+\gamma^2\right)\frac{d^2S}{dt^2}\,T .$$

17.) From 3.2.95, 3.2.94, and 3.2.93 the term in $S(t)$ is

$$\left[Q_o^2+Q_1^2+Q_o\rho_1+Q_1\rho_1 e^{-\alpha T} + 2Q_oQ_1+2Q_1\rho_1 + \frac{1+e^{-\alpha T}}{1-e^{-\alpha T}}\,\rho_1\,\frac{\phi_D}{\phi_N}\right]S(t)$$

$$\rightarrow \frac{1}{2}\,\frac{\beta^2\gamma^2}{\left(\gamma^2-\beta^2\right)}\,S(t)T$$

where the results of 6.) were used.

Thus far the non-delta function components of the solution have been determined. The coefficients of $\delta(t)$ and $\delta'(t)$ will now be determined.

18.) From 3.2.93, for small μ, it is seen that only the following terms need be considered in the determination of the coefficients of $\delta(t)$ and $\delta'(t)$:

$$\rho_\mu = -S(o)Q_1^2\delta_{\mu o}-\rho_1 Q_1 e^{-\alpha T}\left[\delta_{\mu o}\sum_{o}^{M} S_m e^{-\alpha Tm}+S(o)e^{-\alpha T\mu}\right]$$

$$+ Q_1\left(Q_o+\rho_1\right)S(T)\delta_{\mu o} \qquad (3.2.98)$$

$$+ \rho_1 \frac{\phi_D}{\phi_N}\sum_{o}^{M} e^{-\alpha T|m-\mu|}S_m-\rho_1^2\frac{e^{-2\alpha T}}{1-e^{-2\alpha T}}e^{-\alpha T\mu}\sum_{o}^{M} S_m e^{-\alpha Tm}$$

From 1.2.2 the minimum variance estimate is

$$\hat{\theta} = \sigma_\theta^2\left(R^{-1}\overline{S}\right)'\overline{y} = \sigma_\theta^2\sum_{\mu=o}^{M}\left[\sum_{m=o}^{M} W_\mu(mT)S_m\right]y_\mu \qquad (3.2.99)$$

Therefore to evaluate the component of the estimate associated with the terms 3.2.98 it is necessary to evaluate the sum $\sum_{\mu=o}^{M}\rho_\mu y_\mu$.

To this end it is useful to note the following

$$S(T) \simeq S(o) + \dot{S}(o)T + \frac{1}{2}\ddot{S}(o)T^2 + \frac{1}{6}\dddot{S}(o)T^3 + O\left(T^4\right) \qquad (3.2.100)$$

$$\lim_{M\to\infty} \sum_{0}^{M} S_m e^{-\alpha Tm} \cong S(o)\sum_{0}^{\infty} e^{-\alpha Tm} + T\dot{S}(o)\sum_{0}^{\infty} m e^{-\alpha Tm} + \frac{T^2}{2}\ddot{S}(o)\sum_{0}^{\infty} m^2 e^{-\alpha Tm}$$

$$+ \frac{T^3}{6}\dddot{S}(o) \sum_{0}^{\infty} m^3 e^{-\alpha Tm} + O\left(T^4\right)$$

$$\cong S(o)\sum\nolimits_0 + T\dot{S}(o)\sum\nolimits_1 + \frac{1}{2}T^2\ddot{S}(o)\sum\nolimits_2 + \frac{1}{6}T^3\dddot{S}(o)\sum\nolimits_3 + O\left(T^4\right)$$

(3.2.101)

$$\lim_{M\to\infty} \sum_{0}^{M} y_\mu e^{-\alpha T\mu} \approx y(o)\sum\nolimits_0 + \dot{y}(o)\sum\nolimits_1 T + O_r\left(T^3\right) \; * \qquad (3.2.102)$$

where it should be noted that $y(o)$ and $\dot{y}(o)$ are random variables.

19.) From 3.2.91, 3.2.98, 3.2.100, 3.2.101 and 3.2.102 the coefficient of $S(o)$ in $\sum_{\mu=0}^{M} \rho_\mu y_\mu$ is

$$\overline{C}_o = -\left[Q_1^2 + \rho_1 Q_1 e^{-\alpha T}\sum\nolimits_0 - Q_1\left(Q_o+\rho_1\right)\right] y(o)$$

$$+\left[y(o)\sum\nolimits_0 + \dot{y}(o)T + \sum\nolimits_1\right]\left[-\rho_1 Q_1 e^{-\alpha T} + \rho_1 \frac{\phi_D}{\phi_N}\frac{1}{1-e^{\alpha T}} - \rho_1^2 \frac{e^{-2\alpha T}\sum_0}{1-e^{-2\alpha T}}\right]$$

(3.2.103)

---

* The notation $O_r\left(T^k\right)$ denotes a random variable whose variance approaches zero like T^k as T approaches zero and hence $k=3$ gives too high an order to warrent consideration in the analysis which follows. See Appendix A for a further discussion of 3.2.102.

20.) Consider the term of 3.2.103 in $\dot{y}(o)$ and $S(o)$

$$\text{coef} = T\sum_1 \left\{ -\rho_1 Q_1 e^{-\alpha T} + \rho_1 \left(Q_o + Q_1 e^{-\alpha T} + \frac{\rho_1}{1-e^{-2\alpha T}} \right) \frac{1}{1-e^{\alpha T}} \right.$$

$$\left. - \rho_1^2 \frac{c^{-2\alpha T}}{1-e^{-2\alpha T}} \frac{1}{1-e^{-\alpha T}} \right\} \qquad (3.2.104)$$

$$= T\sum_1 \left\{ \frac{-\rho_1 Q_1 e^{-\alpha T} + \rho_1 Q_1 + Q_o \rho_1 + Q_1 \rho_1 e^{-\alpha T}}{1-e^{\alpha T}} - \rho_1^2 \frac{1+e^{-\alpha T}}{\left(1-e^{-2\alpha T}\right)\left(1-e^{-\alpha T}\right)} e^{-\alpha T} \right\}$$

which after substituting for ρ_1, Q_1, Q_o, $\sum_1$, can be put in the form

$$\text{coef} = -\frac{T}{A_o e^{\alpha T}} \frac{\left(1-e^{\alpha T} e^{-\beta T}\right)\left(1-e^{\alpha T} e^{-\gamma T}\right) e^{-2\alpha T}}{\left(1-e^{\alpha T}\right)^4} \left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right) \qquad (3.2.105)$$

$$\text{coef} \to \frac{3}{\left(\gamma^2-\beta^2\right)e^{\alpha T}} \frac{\beta\gamma}{\left(1-e^{-\alpha T}\right)^2} \frac{T}{T} [1+O(T)] \to \frac{-3}{\left(\gamma^2-\beta^2\right)\left(2+\sqrt{3}\right)} \frac{\beta\gamma}{6\left(2-\sqrt{3}\right)}$$

$$= -\frac{1}{2} \frac{\beta\gamma}{\gamma^2-\beta^2}$$

Therefore this part of the estimate is

$$-\frac{1}{2} \frac{\beta\gamma}{\gamma^2-\beta^2} S(o)\dot{y}(o) = \frac{1}{2} \frac{\beta\gamma}{\gamma^2-\beta^2} S(o) \int_o^L \delta'(t) y(t) dt$$

21.) It is now important to make the following observation.

$$Q_1^2+\rho_1 Q_1 \frac{e^{-\alpha T}}{1-e^{-\alpha T}} = -\frac{1}{A_o e^{(\beta+\gamma)T}\left(1-e^{-\alpha T}\right)}\left[\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)-\left(1-e^{-\alpha T}\right)\right]$$

$$Q_1\left(Q_o+\rho_1\right) = -\frac{1}{A_o e^{(\beta+\gamma)T}}$$

$$\therefore \left[-\ Q_1^2+\rho_1 Q_1 e^{-\alpha T}\sum\nolimits_o -\ Q_1\left(Q_o+\rho_1\right)\right] = \frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{A_o e^{(\beta+\gamma)T}\left(1-e^{-\alpha T}\right)} - \frac{2}{A_o e^{(\beta+\gamma)T}}$$

But the term $-\dfrac{2}{A_o e^{(\beta+\gamma)T}}$ is part of the non-delta function solution - it is equal $2Q_1\left(Q_o+\rho_1\right)$ (see 12.)). It will be dropped in the remainder of this analysis when it is associated with $S(o)$ and $\ddot{S}(o)$. (See 26.))

Therefore the analysis of 21.) gives

$$\left[-\ Q_1^2+\rho_1 Q_1 e^{-\alpha T}\sum\nolimits_o - Q_1\left(Q_o+\rho_1\right)\right] \rightarrow \frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{A_o e^{(\beta+\gamma)T}\left(1-e^{-\alpha T}\right)} \qquad (3.2.106)$$

22.) Consider the term of 3.2.103 in $y(o)$ and $S(o)$. From 3.2.103, 3.2.106, 3.2.104, 3.2.106 and the observation that

$$\frac{\sum_o}{\sum_1} = \frac{1-e^{-\alpha T}}{e^{-\alpha T}}$$

the coefficient of the term in y(o) and S(o) becomes

$$\text{coef} = \frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{A\, e^{(\beta+\gamma)T}\left(1-e^{-\alpha T}\right)}\left[1 - \frac{\left(e^{\beta T}-e^{\alpha T}\right)\left(e^{\gamma T}-e^{\alpha T}\right)}{e^{2\alpha T}\left(1-e^{-\alpha T}\right)^2}\right]$$

which after expansion

$$\text{coef} \rightarrow \frac{1}{2}\,\frac{\beta\gamma(\beta+\gamma)}{\gamma^2-\beta^2}$$

Therefore this part of the estimate is

$$\frac{1}{2}\,\frac{\beta\gamma(\beta+\gamma)}{\gamma^2-\beta^2}\,y(o)S(o) = \frac{1}{2}\,\frac{\beta\gamma(\beta+\gamma)}{\gamma^2-\beta^2}\,S(o)\int_o^L \delta(t)y(t)dt$$

23.) From 3.2.98, 3.2.101, 3.2.91, and 3.2.102 the coefficient of $y(o)\dot{S}(o)T$ in $\sum_{\mu=o}^{M} \rho_\mu y_\mu$ is

$$\text{coef} = -\ \rho_1 Q_1 e^{-\alpha T}\sum\nolimits_1 + Q_1\left(Q_o+\rho_1\right) + \sum\nolimits_o\left[\rho_1\,\frac{\phi_D}{\phi_N}\,\frac{e^{-\alpha T}}{\left(1-e^{-\alpha T}\right)^2} - \rho_1^2\,\frac{e^{-2\alpha T}\quad e^{-\alpha T}}{1-e^{-2\alpha T}\left(1-e^{-\alpha T}\right)^2}\right]$$

$$= \frac{1}{A_o e^{(\beta+\gamma)T}}\left\{\frac{\left(e^{\beta T}-e^{\alpha T}\right)\left(e^{\gamma T}-e^{\alpha T}\right)e^{-\alpha T}}{\left(1-e^{-\alpha T}\right)^2 e^{\alpha T}}\left[e^{-(\beta+\gamma)T} + \frac{1-e^{-(\beta+\gamma)T}}{1-e^{-\alpha T}}\right] - 1\right\}$$

Upon expansion the above coefficient becomes (see 3.2.69 and 3.2.73)

$$\text{coef} \to \frac{1}{2}\,\frac{1}{\gamma^2-\beta^2}\left[\beta\gamma-(\beta+\gamma)^2\right]\frac{1}{T}$$

or the term in $y(o)\dot{S}(o)T$ is

$$\frac{1}{2}\,\frac{1}{\gamma^2-\beta^2}\left[\beta\gamma-(\beta+\gamma)^2\right]y(o)\dot{S}(o) = \frac{1}{2}\,\frac{1}{\gamma^2-\beta^2}\,(\beta\gamma-(\beta+\gamma)^2)\dot{S}(o)\int_o^L \delta(t)y(t)dt$$

24.) From 3.2.98, 3.2.101, 3.2.91, and 3.2.102 the coefficient of $\dot{y}(o)\dot{S}(o)$ in $\sum_{\mu=o}^{M} p_\mu y_\mu$ is

$$\text{coef} = T^2\sum_1\left[\rho_1\,\frac{\phi_D}{\phi_N}\,\frac{e^{-\alpha T}}{\left(1-e^{-\alpha T}\right)^2} - \rho_1^2\,\frac{e^{-2\alpha T}}{1-e^{-2\alpha T}}\,\frac{e^{-\alpha T}}{\left(1-e^{-\alpha T}\right)^2}\right] =$$

$$\frac{T^2e^{-2\alpha T}}{\left(1-e^{-\alpha T}\right)^4}\,\rho_1\left[Q_o+Q_1e^{-\alpha T} + \frac{\rho_1}{1-e^{-2\alpha T}} - \frac{\rho_1 e^{-2\alpha T}}{1-e^{-2\alpha T}}\right]=$$

$$= k^{-\frac{1}{2}}\,\frac{T^2e^{-2\alpha T}}{\left(1-e^{-\alpha T}\right)^4}\,\rho_1\left[1-e^{-(\beta+\gamma)T}\right]=\frac{T^2e^{-2\alpha T}}{\left(1-e^{-\alpha T}\right)^4}\,\frac{1}{A_oe^{(\beta+\gamma)T}e^{\alpha T}}\left(e^{\beta T}-e^{\alpha T}\right)x$$

$$\left(e^{\gamma T}-e^{\alpha T}\right)\left(1-e^{-(\beta+\gamma)T}\right) \to -(\beta+\gamma)\,\frac{e^{-2\alpha T}}{\left(1-e^{-\alpha T}\right)^4}\,\frac{\left(1-e^{\alpha T}\right)^2}{e^{\alpha T}}\,\frac{3}{\gamma^2-\beta^2} =$$

$$= -\,3\,\frac{\beta+\gamma}{\gamma^2-\beta^2}\,\frac{1}{e^{\alpha T}\left(1-e^{-\alpha T}\right)^2} \to \frac{3\,\frac{\beta+\gamma}{\gamma^2-\beta^2}}{(2+\sqrt{3})(6)(2-\sqrt{3})} \to \frac{1}{2}\,\frac{\beta+\gamma}{\gamma^2-\beta^2}$$

Therefore the term in $\dot{y}(o)\dot{S}(o)$ is

$$\frac{1}{2}\,\frac{\beta+\gamma}{\gamma^2-\beta^2}\,\dot{S}(o)\dot{y}(o) = -\frac{1}{2}\,\frac{\beta+\gamma}{\gamma^2-\beta^2}\,\dot{S}(o)\int_o^L \delta'(t)y(t)dt$$

25.) From 3.2.98, 3.2.101, 3.2.91 and 3.2.103 the coefficient of $\frac{1}{2}\ddot{S}(o)\dot{y}(o)$ in $\sum_{\mu=o}^{M}\rho_\mu y_\mu$ is

$$\text{coef} = -\left[\rho_1\,\frac{\phi_D}{\phi_N} + \rho_1^2\,\frac{e^{-2\alpha T}}{1-e^{-2\alpha T}}\right]\left[\sum_2\sum_1 T^3\right]$$

$$= -\sum_1\sum_2 \frac{e^{-\alpha T}}{A_o e^{(\beta+\gamma)T}}\left[\frac{\left(2\cosh\beta T-\frac{B_o}{A_o}\right)\left(2\cosh\gamma T-\frac{B_o}{A_o}\right)}{\left(1-e^{-2\alpha T}\right)}\right.$$

$$\left.+\frac{\left(e^{\beta T}-e^{\alpha T}\right)^2\left(e^{\gamma T}-e^{\alpha T}\right)^2}{\left(1-e^{-2\alpha T}\right)}\,e^{-2\alpha T}e^{-(\beta+\gamma)T}\right]$$

$$\rightarrow \frac{3}{\left(\gamma^2-\beta^2\right)}\,\frac{e^{-2\alpha T}}{\left(1-e^{-\alpha T}\right)^4}\left[\left(-\frac{1}{6}\right)6.6+\frac{\left(1-e^{-\alpha T}\right)^4}{e^{-2\alpha T}}\right] \rightarrow -\frac{1}{2}\,\frac{1}{\gamma^2-\beta^2}\,\frac{1}{36}\,(36+36)$$

$$= -\frac{1}{\gamma^2-\beta^2}$$

where 3.2.69, 3.2.73, 3.2.74, and 3.2.77 were used in obtaining the final result. Therefore the term in $\frac{1}{2}\ddot{S}(o)\dot{y}(o)$ is

$$-\frac{\frac{1}{2}}{\gamma^2-\beta^2} S(o)\dot{y}(o) = \frac{\frac{1}{2}}{\gamma^2-\beta^2} S(o) \int_o^L y(t)\delta'(t)dt$$

26.) Now consider the term in $\frac{1}{2}\ddot{S}(o)y(o)T^2$

$$\text{coef} = -\rho_1 Q_1 e^{-\alpha T}\sum\nolimits_2 + \frac{1}{A_o e^{(\beta+\gamma)T}} - \sum\nolimits_2 \sum\nolimits_b \left[\rho_1 \frac{\phi_D}{\phi_N} + \rho_1^2 \frac{e^{-2\alpha T}}{1-e^{-2\alpha T}}\right] -$$

$$-\frac{2}{A_o^{(\beta+\gamma)T}}$$

As explained in 21.) the term $-\frac{2}{A_o e^{(\beta+\gamma)T}}$ is part of the non-delta function component of the solution and will be dropped. Therefore, upon dropping this term and rewriting, the above equation can be put in the form

$$\text{coef} = e^{-\alpha T}\frac{1+e^{-\alpha T}}{\left(1-e^{-\alpha T}\right)^3}\;\frac{e^{-(\beta+\gamma)T}}{A_o e^{(\beta+\gamma)T} e^{\alpha T}}\left(e^{\beta T}-e^{\alpha T}\right)\left(e^{\gamma T}-e^{\alpha T}\right) + \frac{1}{A_o e^{(\beta+\gamma)T}}$$

$$-\sum\nolimits_b \sum\nolimits_2 \frac{e^{-\alpha T}}{A_o e^{(\beta+\gamma)T}}\left[\frac{\left(2\cosh\beta T - \frac{B_o}{A_o}\right)\left(2\cosh\gamma T - \frac{B_o}{A_o}\right)}{1-e^{-2\alpha T}} + \right.$$

$$\left. + e^{-(\beta+\gamma)T}\,\frac{\left(e^{\beta T}-e^{\alpha T}\right)^2\left(e^{\gamma T}-e^{\alpha T}\right)^2}{1-e^{-2\alpha T}}\quad e^{-2\alpha T}\right.$$

$$\rightarrow \frac{1}{A_o e^{(\beta+\gamma)T}}\left\{\frac{e^{-2\alpha T}}{\left(1-e^{-\alpha T}\right)^4}\left(1-e^{-2\alpha T}\right)\left[1-(\beta+\gamma)T\right]\left[\left(1-e^{\alpha T}\right)^2+(\beta+\gamma)T\left(1-e^{\alpha T}\right)T\right]+1\right.$$

$$\left.-\ e^{-2\alpha T}\frac{1+e^{-\alpha T}}{\left(1-e^{-\alpha T}\right)^5}\left[36+e^{-2\alpha}\left[\left(1-e^{-\alpha T}\right)^2+(\beta+\gamma)\left(1-e^{\alpha T}\right)T\right]^2\left[1-(\beta+\gamma)T\right]\right]\right\}$$

$$\rightarrow \frac{1}{A_o e^{(\beta+\gamma)T}}\left\{\frac{1}{36}\left(1-e^{-2\alpha T}\right)\left(1-e^{\alpha T}\right)^2+1-\frac{2}{1-e^{-\alpha T}}+\frac{(\beta+\gamma)T}{36}\left[\left(1-e^{-2\alpha T}\right)\left(1-e^{\alpha T}\right)-\right.\right.$$

$$\left.\left.-\left(1-e^{-2\alpha T}\right)\left(1-e^{\alpha T}\right)^2-2e^{-2\alpha T}\frac{\left(1-e^{\alpha T}\right)^3}{1-e^{-\alpha T}}+\frac{36}{1-e^{-\alpha T}}\right]\right\}$$

$$\rightarrow \frac{1}{A_o e^{(\beta+\gamma)T}}\left\{0+\frac{1}{36}(\beta+\gamma)T\left[-6-12+18+\left(6-12+6\right)\sqrt{3}\right]\right\}\rightarrow \frac{1}{A_o e^{(\beta+\gamma)T}}\left\{0+0T+0\left(T^2\right)\right\}$$

$$\rightarrow -\frac{3}{T^3\left(\gamma^2-\beta^2\right)}\left\{0+0T+0\left(T^2\right)\right\}$$

where 3.2.69, 3.2.73, 3.2.74 and 3.2.77 were used in obtaining the final result. From the above equation

$$\text{coef of } \frac{1}{2}\ddot{S}(o)y(o) \rightarrow (\text{const})\ T \rightarrow 0 \quad \text{as} \quad T \rightarrow o$$

Therefore there should be no term of the form $\ddot{S}(o)\delta(t)$ in the continuous filter.

27.) Finally consider the term in $\frac{1}{6}\dddot{S}(o)y(o)$

$$\text{coef} = \left\{ -\rho_1 Q_1 e^{-\alpha T} \sum_3 + Q_1\left(Q_o+\rho_1\right) + \sum_3 \sum_o \left[\rho_1 \frac{\phi_D}{\phi_N} - \rho_1^2 \frac{e^{-2\alpha T}}{1-e^{-2\alpha T}} \right] \right\} T^3$$

It can be shown that $\sum_3$, which is defined in 3.2.77, has the limiting form

$$\sum_3 = 0 + 0\left(T^2\right)$$

Therefore

$$\text{coef} \to Q_1\left(Q_o+\rho_1\right)T^3 = -\frac{T^3}{Ae^{(\beta+\gamma)T}} \to \frac{3}{\gamma^2-\beta^2}$$

which gives as a contribution to $R^{-1}S$

$$\frac{3}{\gamma^2-\beta^2}\left[\frac{1}{6}\dddot{S}(o)y(o)\right] = \frac{1}{2}\,\frac{1}{\gamma^2-\beta^2}\,\dddot{S}(o) \int_o^L \delta(t)y(t)dt$$

This completes the determination of the coefficients of $\delta(t)$ and $\delta'(t)$. The coefficients of $\delta(t-L)$ and $\delta'(t-L)$ are determined in precisely the same manner as above, in fact, with the exception of sign changes, the algebra is identical. It should be noted that if the term of 3.2.102 given by $Q_r\left(T^3\right)$ had been carried through in the above analysis it would have given terms whose means and variances would have approached zero as $T \to 0$. Therefore, in the limit as $T \to 0$, they vanish.

CHAPTER IV

ASYMPTOTIC PROPERTIES OF THE VARIANCE OF THE DISCRETE ESTIMATE

4.1 First Order Noise Case for a General Signal

In part 3.1.2 of this thesis it was shown that if the process

$$y(t) = \theta S(t) + n(t) \qquad 0 \le t \le L \tag{3.1.1}$$

is observed at equally spaced points in time which are T seconds apart, the variance of the minimum variance estimate is defined by

$$\frac{\sigma^2}{\sigma_\theta^2} = \frac{1}{1-e^{-2\beta T}}\left[S_o^2+S_M^2-2e^{-\beta T}\sum_{\mu=o}^{M-1} S_\mu S_{\mu+1} + \left(1+e^{-2\beta T}\right)\sum_{\mu=1}^{M-1} S_\mu^2\right] \tag{3.1.9}$$

where the autocorrelation function of $n(t)$ is

$$\phi(t) = \sigma^2 e^{-\beta|t|} \tag{3.1.5}$$

It is the purpose of this section to expand 3.1.9 to zero order and first order in T. The first order term is defined as the asymptotic term. It should be noted that the zero order term has already been calculated and the result is given in 3.1.10.

Rewriting 3.1.9 in terms of hyperbolic functions gives

$$\frac{2\sigma^2}{\sigma_\theta^2} = \left(S_o^2+S_M^2\right)e^{\beta T}\operatorname{csch}\beta T - 2\operatorname{csch}\beta T\sum_{j=o}^{M-1} S_j S_{j+1} + 2\coth\beta T\sum_{j=1}^{M-1} S_j^2 \tag{4.1.1}$$

and upon using the series

$$e^{\beta T} \cong 1 + \beta T + \tfrac{1}{2}\beta^2 T^2$$

$$\operatorname{csch} \beta T = \frac{1}{\beta T}\left[1 - \frac{(\beta T)^2}{3!} + \frac{14(\beta T)^4}{6!} \cdot \cdot \cdot\right] \tag{4.1.2}$$

$$\tanh \beta T = \frac{1}{\beta T}\left[1 + \frac{(\beta T)^2}{3} - \frac{(\beta T)^4}{45} + \cdot \cdot \cdot\right]$$

4.1.1 can be expressed

$$\frac{2\sigma^2}{\sigma_\theta^2} \cong \left(S_o^2+S_M^2\right)\left(\frac{1}{\beta T} + 1 + \frac{1}{3}\beta T + O\left(T^3\right)\right) - \frac{2}{\beta T}\left[1 - \frac{1}{6}(\beta T)^2 + O\left(T^4\right)\right]\sum_{j=o}^{M-1} S_j S_{j+1}$$

$$+ \frac{2}{\beta T}\left[1 + \frac{1}{3}(\beta T)^2 + O\left(T^4\right)\right]\sum_{j=1}^{M-1} S_j^2 \tag{4.1.3}$$

Therefore to first order in T

$$\frac{2\sigma^2}{\sigma_\theta^2} \cong \frac{1}{\beta T}\left[S_o^2+S_M^2 - 2\sum_{j=o}^{M-1} S_j S_{j+1} + 2\sum_{j=1}^{M-1} S_j^2\right] + \left[S_o^2+S_M^2\right] +$$

$$+ \frac{1}{3}\beta T\left[S_o^2+S_M^2 + \sum_{j=o}^{M-1} S_j S_{j+1} + 2\sum_{j=1}^{M-1} S_j^2\right] \tag{4.1.4}$$

but

$$S_o^2+S_M^2- 2 \sum_{j=o}^{M-1} S_jS_{j+1}+ 2 \sum_{j=1}^{M-1} S_j^2 = T^2 \sum_{j=o}^{M-1}\left(\frac{S_{j+1}-S_j}{T}\right)^2$$

hence

$$\frac{2\sigma^2}{\sigma_\theta^2} = \frac{1}{\beta T} \sum_{j=o}^{M-1}\left(\frac{S_{j+1}-S_j}{T}\right)^2 + \left[S_o^2+S_M^2\right]+ \frac{1}{3}\beta T\left[T^2 \sum_{j=o}^{M-1}\left(\frac{S_{j+1}-S_j}{T}\right)^2 + 3 \sum_{j=o}^{M-1} S_jS_{j+1}\right] \tag{4.1.5}$$

If the signal is sufficiently smooth so that

$$S_{j+1} \cong S_j + \dot{S}_jT + \frac{1}{2}\ddot{S}_jT^2 \tag{4.1.6}$$

where

$$\dot{S}_j = \frac{dS}{dt}\Big|_{t=jT}$$

$$\ddot{S}_j = \frac{d^2S}{dt^2}\Big|_{t=jT}$$

then upon using 4.1.6 in 4.1.5 and retaining up to first order terms

$$\frac{2\sigma^2}{\sigma_\theta^2} \cong \frac{T}{\beta}\left[\sum_{j=o}^{M-1} \dot{S}_j^2 + T\sum_{j=o}^{M-1} \dot{S}_j\ddot{S}_j\right] + S_o^2+S_M^2+\beta T\left[\sum_{j=o}^{M-1} S_j^2 + T\sum_{j=o}^{M-1} \dot{S}_j\dot{S}_j\right] \qquad (4.1.7)$$

The trapesoidal approximation to an integral gives

$$\int_o^L \chi(t)dt \cong T\sum_{j=o}^{M-1} \chi(iT) + \frac{T}{2}[\chi(L)-\chi(o)] \qquad (4.1.8)$$

as a first order expression for the integral. Therefore using 4.1.8 in 4.1.7 and again retaining only up to first order terms

$$\frac{2\sigma^2}{\sigma_\theta^2} \rightarrow \frac{1}{\beta}\int_o^L \dot{S}^2dt - \frac{T}{2\beta}\left[\dot{S}_M^2-\dot{S}_o^2\right] + \frac{T}{\beta}\int_o^L \dot{S}\ddot{S}dt + S_o^2+S_M^2+\beta\int_o^L S^2dt - \frac{\beta T}{2}\left[S_M^2-S_o^2\right]$$

$$+ \beta T\int_o^T S\dot{S}dt$$

or

$$\frac{\sigma^2}{\sigma_\theta^2} \rightarrow \frac{1}{2}\left[\frac{1}{\beta}\int_o^L \dot{S}^2dt + \left(S_o^2+S_M^2\right) + \beta\int_o^L S^2dt\right] + \frac{T}{2}\left[\frac{1}{\beta}\int_o^L \dot{S}\ddot{S}dt - \frac{1}{2\beta}\left(\dot{S}_M^2-\dot{S}_o^2\right)\right.$$

$$\left. + \beta\int_o^L S\dot{S}dt - \frac{\beta}{2}\left(S_M^2-S_o^2\right)\right] \qquad (4.1.9)$$

But the integrals in the first order term can be expressed

$$\frac{1}{\beta}\int_0^L \dot{S}\ddot{S}dt = \frac{1}{\beta}\int_0^L \dot{S}d\dot{S} = \frac{1}{2\beta}\left[\dot{S}_M^2-\dot{S}_o^2\right]$$

and

$$\beta\int_0^L S\dot{S}dt = \beta\int_0^L SdS = \frac{1}{2}\,\beta\left[S_M^2-S_o^2\right]$$

hence the coefficient of the first order term is zero. In conclusion the variance of the estimate in first order noise can be expressed

$$\sigma_\theta^2 = \sigma_L^2\left(1 + O\left(T^2\right)\right) \tag{4.1.10}$$

where σ_L^2 is defined in 3.1.18.

4.1.1 Asymptotic Variance when the Signal is a Constant

The variance of the estimate of a constant is expanded in terms of T, the time between samples, in this section. Let the observed process be

$$y(t) = \theta + n(t) \qquad 0 \le t \le L$$

hence

$$S(t) = 1 \tag{4.1.11}$$

$$S_\mu = 1 \qquad 0 \le \mu \le M \tag{4.1.12}$$

From 3.1.9 and a little algebra

$$\sigma_\theta^2 = \sigma^2 \frac{1+e^{-\beta T}}{M\left(1-e^{-\beta T}\right)+\left(1+e^{-\beta T}\right)} = \frac{\sigma^2}{1 + \frac{L}{T} \tanh \frac{1}{2} \beta T} \qquad (4.1.13)$$

Therefore

$$\frac{\sigma^2}{\sigma_\theta^2} = 1 + \frac{L}{T} \tanh \frac{1}{2} \beta T \cong 1 + \frac{L}{T}\left[\frac{1}{2}\beta T - \frac{1}{8\cdot 3} (\beta T)^2 + \frac{1}{15\cdot 16} (\beta T)^4 - \cdots\right]$$

$$\cong 1 + \frac{\beta L}{2} - \frac{L}{24} (\beta T)^2\left[1 - \frac{1}{10} (\beta T)^2 + \cdots +\right]$$

and

$$\frac{\frac{\sigma^2}{1 + \frac{\beta L}{2}}}{\sigma_\theta^2} \cong 1 - \frac{1}{12} \frac{\frac{\beta L}{2}}{1 + \frac{\beta L}{2}} (\beta T)^2\left[1 - \frac{1}{10} (\beta T)^2 + \cdots +\right]$$

But $\frac{\sigma^2}{1 + \frac{\beta L}{2}} = \sigma_L^2 =$ variance for continuous "sampling",

therefore

$$\frac{\sigma_L^2}{\sigma_\theta^2} \cong 1 - \frac{1}{12} \frac{\frac{\beta L}{2}}{1 + \frac{\beta L}{2}} (\beta T)^2\left[1 - \frac{1}{10}(\beta T)^2 + \cdots\right] \qquad (4.1.14)$$

It is important to note that $\frac{\sigma_L^2}{\sigma_\theta^2}$ does not have a first order term and hence $\frac{\sigma_\theta^2}{\sigma_L^2}$ does not have this term either (as predicted in 4.1.10). It is also important to observe that the series 4.1.14 is an alternating series and hence the error is less than the last term used. Therefore the improvement in variance gained by going to continuous sampling is defined by (for $\beta T < 1$)

$$\frac{\sigma_\theta^2}{\sigma_L^2} < \frac{1}{1 - \frac{1}{12} \frac{\frac{\beta L}{2}}{1 + \frac{\beta L}{2}} (\beta T)^2} \tag{4.1.15}$$

Since

$$\frac{\frac{\beta L}{2}}{1 + \frac{\beta L}{2}} < 1 \quad ,$$

if the time between samples is chosen such that

$$\beta T = \frac{1}{3} \tag{4.1.16}$$

then $\frac{\sigma_\theta^2}{\sigma_L^2} < 1.01$ which gives the following useful criterion:

If the samples are spaced at one-third the correlation time $\left(\frac{1}{\beta}\right)$ apart, then the variance of the estimate of the unknown constant θ is within one percent of the variance that would be obtained at an infinite sampling rate.

4.2 Second Order Noise Case for a Constant Signal

In this part of the thesis as in part 4.1.1 it is assumed that the process

$$y(t) = \theta + n(t) \qquad\qquad 0 \le t \le L$$

is observed at equally spaced times and a minimum variance estimate of θ is made. From 1.2.3 the variance of the estimate is defined by

$$\frac{1}{\sigma_\theta^2} = \bar{S}'R^{-1}S \tag{4.2.1}$$

where

$$\bar{S}' = [1\ 1\ \cdots\ 1] \tag{4.2.2}$$

$$R^{-1} = \left(W_\mu(mT)\right) \quad . \tag{4.2.3}$$

Therefore, upon substituting 4.2.2 and 4.2.3 into 4.2.1 the result

$$\frac{1}{\sigma_\theta^2} = \sum_{\mu=0}^{M} \sum_{m=0}^{M} W_\mu(mT) \tag{4.2.4}$$

is obtained. In this part of the thesis 4.2.4 will be expanded in T up to and including the first order term, which is defined as the asymptotic term, for the case of second order noise. As in Chapter III it will be convenient to treat the all pole noise and the noise with two zeros in its spectral density separately. However, before undertaking this task, 4.2.4 will be expressed in closed form by a rather efficient method which is now described.

From 2.1.5 and 2.1.13

$$\overline{W}_{\mu}^{*}(Z) = Z^{-\mu}\phi^{*-1}(Z)+Z^{-(M+1)}\frac{\phi_D(Z)P_{\mu}\left(Z^{-1}\right)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} + Z\,\frac{\phi_D\left(Z^{-1}\right)P_{\mu}'(Z)}{\phi_N(Z)\phi_N\left(Z^{-1}\right)} =$$

$$= \sum_{m=o}^{M} W_{\mu}(mT)Z^{-m} \qquad (4.2.5)$$

therefore by letting $Z = 1$

$$\sum_{m-o}^{M} W_{\mu}(mT) = \phi^{*-1}(1) + \frac{\phi_D(1)P_{\mu}(1)}{\phi_N(1)\phi_N(1)} + \frac{\phi_D(1)P_{\mu}'(1)}{\phi_N(1)\phi_N(1)} \qquad (4.2.6)$$

and

$$\sum_{\mu=o}^{M}\sum_{m=o}^{M} W_{\mu}(mT) = (M+1)\ \phi^{*-1}(1) + \frac{\phi_D(1)}{\phi_N(1)}\sum_{\mu=o}^{M}\left[\frac{P_{\mu}(1)}{\phi_N(1)} + \frac{P_{\mu}'(1)}{\phi_N(1)}\right] \qquad (4.2.7)$$

It now remains to evaluate 4.2.7 in closed form. The autocorrelation function of 2.3.1 is assumed with $\beta_1 = \beta$, $\beta_2 = \gamma$.

From 2.3.2,

$$\phi^{*-1}(1) = \frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{\sigma_1^2\left(1+e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)+\sigma_2^2\left(1+e^{-\gamma T}\right)\left(1-e^{-\beta T}\right)} \quad ; \tag{4.2.8}$$

2.1.35, 2.1.45 and 2.3.14,

$$\frac{P_\mu(1)}{\phi_N(1)} = -\, Q_1\delta_{\mu M} + \frac{\rho_{\mu 1}}{1-e^{-\alpha T}} \quad ; \tag{4.2.9}$$

2.1.53, 2.1.63 and 2.3.15,

$$\frac{P'_\mu(1)}{\phi_N(1)} = -\, Q_1\delta_{\mu o} + \frac{\rho'_{\mu 1}}{1-e^{-\alpha T}} \quad ; \tag{4.2.10}$$

2.3.8 and 2.3.9

$$\frac{\phi_D(1)}{\phi_N(1)} = \frac{1}{\sqrt{k}}\,\frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{\left(1-e^{-\alpha T}\right)} \quad ; \tag{4.2.11}$$

and finally from 2.3.22 and 2.3.23

$$\rho_{\mu 1}+\rho'_{\mu 1} = -\,\frac{\rho_1\phi_D e^{-\alpha T}}{\phi_D+\rho_1 e^{-\alpha T(M+2)}}\left\{e^{-\alpha MT}e^{\alpha\mu T}+e^{-\alpha\mu T}\right\}+$$

$$\frac{\rho_1 Q_1\left(1-e^{-2\alpha T}\right)e^{-\alpha T(M+2)}}{\phi_D+\rho_1 e^{-\alpha T(M+2)}}\left(\delta_{\mu o}+\delta_{\mu M}\right) \tag{4.2.12}$$

Upon substituting 4.2.8 through 4.2.12 into 4.2.7, the result

$$\frac{1}{\sigma_\theta^2} = \frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)(M+1)}{\sigma_1^2\left(1+e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)+\sigma_2^2\left(1+e^{-\gamma T}\right)\left(1-e^{-\beta T}\right)}$$

$$+\frac{1}{\sqrt{k}}\frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{\left(1-e^{-\alpha T}\right)}\left[-2Q_1+\frac{\sum_{\mu=o}^{M}\left(\rho_{\mu 1}+\rho'_{\mu 1}\right)}{1-e^{-\alpha T}}\right] \tag{4.2.13}$$

where

$$\sum_{\mu=o}^{M}\left(\rho_{\mu 1}+\rho'_{\mu 1}\right) = -\frac{2\rho_1\phi_D}{\phi_D+\rho_1 e^{-\alpha T(M+2)}}\left[\frac{1-e^{-\alpha T(M+1)}}{e^{\alpha T}-1}\right]$$

$$+\frac{2\rho_1 Q_1\left(1-e^{-2\alpha T}\right)e^{-\alpha T(M+2)}}{\phi_D+\rho_1 e^{-\alpha T(M+2)}} \tag{4.2.14}$$

is obtained which is the final result. It should be noted that the quantities of 4.2.13 and 4.2.14 are defined in 2.3.37.

The asymptotic term of 4.2.13 will now be determined for second order noise when the condition 3.2.25 does not hold. In so doing it will be necessary to make many expansions similar to those of Chapter III, hence only the final results and occasionally an intermediate step will be given. The definitions of the quantities ρ_1, ϕ_D, etc. are identical to those of Chapter II and Chapter III.

Second Order Noise When $\beta\sigma_1^2 + \gamma\sigma_2^2 \neq 0$

The components of 4.2.13 and 4.2.14 are expanded below

1.) $$\frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)(M+1)}{\sigma_1^2\left(1+e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)+\sigma_2^2\left(1+e^{-\gamma T}\right)\left(1-e^{-\beta T}\right)} =$$

$$= \frac{M+1}{\sigma_1^2 \coth\frac{\beta T}{2} + \sigma_2^2 \coth\frac{\gamma T}{2}}$$

$$\rightarrow \left(\frac{\beta\gamma}{a}\right)^2 \frac{L}{K} + \left[\frac{1}{K}\left(\frac{\beta\gamma}{a}\right)^2\right]T \qquad (4.2.15)$$

2.) $$\frac{1}{k}\frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{\left(1-e^{-\alpha T}\right)} \rightarrow$$

$$\rightarrow \frac{1}{KT[1+(a-\beta-\gamma)T]}\left\{\frac{\left[1-\left(1-\beta T+\frac{1}{2}\beta^2T^2\right)\right]\left[1-\left(1-\gamma T+\frac{1}{2}\gamma^2T^2\right)\right]}{\left[1-\left(1-aT+\frac{1}{2}a^2T^2\right)\right]}\right.$$

$$\rightarrow \frac{\beta\gamma}{Ka}\left[1+\tfrac{1}{2}(\beta+\gamma-a)T\right]^* \qquad (4.2.16)$$

3.) $$-2k^{\frac{1}{2}}Q_1 = e^{-(\beta+\gamma)T}e^{\alpha T} \rightarrow 2[1-(\beta+\gamma-a)T] \qquad (4.2.17)$$

---

* By expanding 3.2.36 for one more term it can be shown that $e^{-\alpha T} = 1-aT+\frac{1}{2}a^2T^2+O\left(T^3\right)$.

4.) $$k^{\frac{1}{2}}\phi_D \to (a+\beta)(a+\gamma)T^2[1-\tfrac{1}{2}(2a+\beta+\gamma)T]$$

$$k^{\frac{1}{2}}\rho_1 \to (a-\beta)(a-\gamma)T^2[1+\tfrac{1}{2}(2a-\beta-\gamma)T]$$

$$e^{-\alpha T(M+2)} \to e^{-aL}[1-2aT]$$

$$k^{\frac{1}{2}}\rho_1 e^{-\alpha T(M+2)} \to T^2(a-\beta)(a-\gamma)e^{-aL}[1-\tfrac{1}{2}(2a+\beta+\gamma)T]$$

$$\therefore\ k^{\frac{1}{2}}\left[\phi_D+\rho_1 e^{-\alpha T(M+2)}\right] \to T^2\left[1-\tfrac{1}{2}(2a+\beta+\gamma)T\right]\Big[(a+\beta)(a+\gamma) + (a-\beta)(a-\gamma)e^{-aL}\Big] \tag{4.2.18}$$

$$k\rho_1\phi_D \to \left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)T^4[1-(\beta+\gamma)T] \tag{4.2.19}$$

5.) $$\frac{2}{1-e^{-\alpha T}}\left[\frac{1-e^{-\alpha T(M+1)}}{e^{\alpha T}-1}\right] \to \frac{2}{a^2T^2}\left(1-e^{-aL}\right)\left[1+aT\left(\frac{e^{-aL}}{1-e^{-aL}}\right)\right] \tag{4.2.20}$$

6.) $$\frac{-1}{\sqrt{k}}\left[\frac{\rho_1\phi_D}{\phi_D+\rho_1 e^{-\alpha T(M+2)}}\right]\frac{2}{1-e^{-\alpha T}}\left[\frac{1-e^{-\alpha T(M+1)}}{e^{\alpha T}-1}\right]$$

$$\to \frac{-2}{Ka^2}\ \frac{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\left(1-e^{-aL}\right)}{(a+\beta)(a+\gamma)+(a-\beta)(a-\gamma)e^{-aL}}\left[1-\tfrac{1}{2}\left(\beta+\gamma-\frac{2a}{1-e^{-aL}}\right)T\right] \tag{4.2.21}$$

7.) $kQ_1\rho_1 = -k^{\frac{1}{2}}e^{-(\beta+\gamma)T}e^{\alpha T}\rho_1 \rightarrow$

$$\rightarrow -(a-\beta)(a-\gamma)T^2\left[1+\tfrac{1}{2}(4a-3\beta-3\gamma)T\right] \qquad (4.2.22)$$

8.) $$\frac{1-e^{-2\alpha T}}{1-e^{-\alpha T}}\, e^{-\alpha T(M+2)} \rightarrow 2aT\left(1-\frac{5}{2}aT\right)e^{-aL} \qquad (4.2.23)$$

9.) $$\frac{2k^{\frac{1}{2}}}{1-e^{-\alpha T}}\,\frac{\rho_1 Q_1\left(1-e^{-2\alpha T}\right)c^{-\alpha T(M+2)}}{\phi_D+\rho_1 e^{-\alpha T(M+2)}} \rightarrow$$

$$\rightarrow -4\,\frac{(a-\beta)(a-\gamma)e^{-aL}}{(a+\beta)(a+\gamma)+(a-\beta)(a-\gamma)e^{-aL}}\left[1+\left(\frac{a}{2}-\beta-\gamma\right)T\right] \qquad (4.2.24)$$

The preliminary calculations have now been completed. The zero order term of 4.2.13 will now be determined. Upon substituting 4.2.15, 4.2.16, 4.2.17, 4.4.21 and 4.2.24 into 4.2.13 and 4.2.14 and collecting terms of zero order, the result

$$\frac{1}{\sigma_\theta^2} \rightarrow \frac{1}{K}\left\{\left(\frac{\gamma\beta}{a}\right)^2 L - \left(\frac{2\beta\gamma}{a^3}\right)\frac{\left(\beta^2-a^2\right)\left(\gamma^2-a^2\right)\left(1-e^{-aL}\right)}{(\beta+a)(\gamma+a)+(\beta-a)(\beta-a)e^{-aL}} + \right.$$

$$\left. +\left(\frac{2\beta\gamma}{a}\right)\frac{(\beta+a)(\gamma+a)-(\beta-a)(\gamma-a)e^{-aL}}{(\beta+\gamma)(\gamma+a)+(\beta-a)(\gamma-a)e^{-aL}}\right\} = \frac{1}{\sigma_L^2} \qquad (4.2.25)$$

is obtained.

Repeating the above substitutions, but this time collecting terms of order T gives,

$$\frac{1}{K}\left(\frac{\beta\gamma}{a}\right)^2 + \frac{\beta\gamma}{Ka}\left[2 - \frac{2}{a^2}\,\frac{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\left(1-e^{-aL}\right)}{(a+\beta)(a+\gamma)+(a-\beta)(a-\gamma)e^{-aL}} - \right.$$

$$\left. - 4\,\frac{(a-\beta)(a-\gamma)e^{-aL}}{(a+\beta)(a+\gamma)+(a-\beta)(a-\gamma)e^{-aL}}\right]\frac{\beta+\gamma-a}{2}$$

$$+ \frac{\beta\gamma}{Ka}\left[-\,2(\beta+\gamma-a) + \frac{1}{a^2}\,\frac{\left(a^2-\beta^2\right)\left(a^2-\gamma^2\right)\left(1-e^{-aL}\right)}{(a+\beta)(a+\gamma)+(a-\beta)(a-\gamma)e^{-aL}}\left(\beta+\gamma - \frac{2a}{1-e^{-aL}}\right)\right.$$

$$\left. - 4\,\frac{(a-\beta)(a-\gamma)e^{-aL}}{(a+\beta)(a+\gamma)+(a-\beta)(a-\gamma)e^{-aL}}\left(\frac{a}{2} - \beta - \gamma\right)\right]$$

$$= \frac{\beta\gamma}{Ka}\left[\frac{(a-\beta)(a-\gamma)}{a}\right]\left\{1 - \frac{(a+\beta)(a+\gamma)\left(1+e^{-aL}\right) - 2a(\beta+\gamma)e^{-aL}}{(a+\beta)(a+\gamma)+(a-\beta)(a-\gamma)e^{-aL}}\right\}$$

$$= \frac{\beta\gamma}{Ka^2}\,(a-\beta)(a-\gamma)\left\{1 - \frac{(a+\beta)(a+\gamma)+(a-\beta)(a-\beta)e^{-aL}}{(a+\beta)(a+\gamma)+(a-\beta)(a-\gamma)e^{-aL}}\right\}$$

$$= 0 \tag{4.2.26}$$

which gives the interesting result that the asymptotic term is zero. It will turn out, as is next shown, that the asymptotic term is not zero when $\beta\sigma_1^2 + \gamma\sigma_2^2 = 0$ (the condition for all pole noise).

Second Order Noise When $\beta\sigma_1^2 + \gamma\sigma_2^2 = 0$

The asymptotic term of $\frac{1}{\sigma_\theta^2}$ is now determined for second order noise when the condition 3.2.25 holds, that is, the noise is "all pole." Again it is necessary to make many expansions similar to those of Chapter III, hence only the final results and occasionally an intermediate step are shown. Without loss of generality the conditions

$$\sigma_1^2 = \frac{1}{\beta} \qquad (3.2.64)$$

$$\sigma_2^2 = -\frac{1}{\gamma}$$

are used again.

The components of 4.2.13 and 4.2.14 are expanded below

1.) $$\frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)(M+1)}{\frac{1}{\beta}\left(1+e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)-\frac{1}{\gamma}\left(1+e^{-\gamma T}\right)\left(1-e^{-\beta T}\right)} = \frac{M+1}{\frac{1}{\beta}\coth\frac{\beta T}{2} - \frac{1}{\gamma}\coth\frac{\gamma T}{2}}$$

$$\rightarrow \frac{\gamma^2\beta^2}{2\left(\gamma^2-\beta^2\right)}(L+T) \qquad (4.2.27)$$

2.) $$-\frac{1}{\sqrt{k}}\frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{1-e^{-\alpha T}}Q_1 = \frac{1}{Ae^{(\beta+\gamma)T}}\frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{\left(1-e^{-\alpha T}\right)}$$

$$\rightarrow \frac{-\frac{6}{3-\sqrt{3}}\beta\gamma}{T\left(\gamma^2-\beta^2\right)}\left\{1 - \frac{1}{2}(\beta+\gamma)T + \left[\left(\beta^2+\gamma^2\right)\left(\frac{1}{6} - \frac{15-7\sqrt{3}}{60(3-\sqrt{3})}\right) + \frac{1}{4}\beta\gamma\right]T^2\right\} \tag{4.2.28}$$

3.) As mentioned in Chapter III, $e^{-\alpha T(M+1)} \rightarrow 0$ faster than any finite power of T when $\beta\sigma_1^2 + \gamma\sigma_2^2 = 0$. Therefore

$$\sum_{\mu=0}^{M}\left(\rho_{\mu 1}+\rho'_{\mu 1}\right) \rightarrow -2\rho_1 e^{-\alpha T}\frac{1}{1-e^{-\alpha T}} \tag{4.2.29}$$

4.) $$-\frac{1}{k^{\frac{1}{2}}}\frac{\left(1-e^{-\beta T}\right)\left(1-e^{-\gamma T}\right)}{\left(1-e^{-\alpha T}\right)^2}\rho_1 e^{-\alpha T}$$

$$\rightarrow \frac{3\beta\gamma}{T\left(\gamma^2-\beta^2\right)}\left\{1 - \frac{1}{2}\left[\frac{\sqrt{3}-1}{3-\sqrt{3}}\right](\beta+\gamma)T + \frac{7}{60}\left(\beta^2+\gamma^2\right)T^2 + \frac{1}{12}T^2\right\} \tag{4.2.30}$$

5.) $$\frac{1}{1-e^{-\alpha T}} \rightarrow \frac{1}{3-\sqrt{3}}\left[1 - \frac{3-2\sqrt{3}}{30\left(3-\sqrt{3}\right)}\left(\beta^2+\gamma^2\right)T^2\right] \tag{4.2.31}$$

The preliminary results have now been given. Upon substituting 4.2.27, 4.2.28, 4.2.30, and 4.2.31 and collecting terms of zero and first order, the result

$$\frac{1}{\sigma_\theta^2} \to \frac{\gamma\beta}{\gamma-\beta}\left[\frac{1}{2}\,\frac{\gamma\beta}{\gamma+\beta}\,L + 1 - \frac{1}{2}\,\frac{\gamma\beta}{\gamma+\beta}\left(\frac{\sqrt{3}-1}{3-\sqrt{3}}\right)T\right] \tag{4.2.32}$$

is obtained. The above result completes the analysis of this section.

4.3 Numerical Results

Numerical values for the variance of the estimate of a constant in second order noise are easily calculated from the closed form expressions 4.2.13 and 4.2.14. If σ_1^2 and σ_2^2 are chosen to be

$$\sigma_1^2 = \gamma\,\frac{a^2-\beta^2}{(\gamma-\beta)\left(a^2+\gamma\beta\right)} \tag{4.3.1}$$

$$\sigma_2^2 = \beta\,\frac{\gamma^2-a^2}{(\gamma-\beta)\left(a^2+\gamma\beta\right)} \tag{4.3.2}$$

then the random process has the spectral density

$$G(\omega) = \frac{2\beta\gamma\left(\gamma^2-\beta^2\right)}{(\gamma-\beta)\left(a^2+\gamma\beta\right)}\;\frac{\omega^2+a^2}{\left(\omega^2+\beta^2\right)\left(\omega^2+\gamma^2\right)} \tag{4.3.3}$$

and

$$\sigma^2 = \text{variance of process}$$

$$= \int_{-\infty}^{\infty} G(\omega)\,\frac{d\omega}{2\pi} = 1 \tag{4.3.4}$$

The curves shown in Figure 4.3.1 give the variance of the estimate of a constant (when the spectral density is that given in 4.3.3) versus M. It should be remembered that $M + 1$ is the number of equally spaced data points. An observation time of 10 seconds was assumed for each of the three cases: $a = 1$, $\beta = 2$, $\gamma = 3$; $a = 2$, $\beta = 3$, $\gamma = 1$; $a = 3$, $\beta = 1$, $\gamma = 2$. Values for M were: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000 and the total 7094 execution time was 40 seconds.

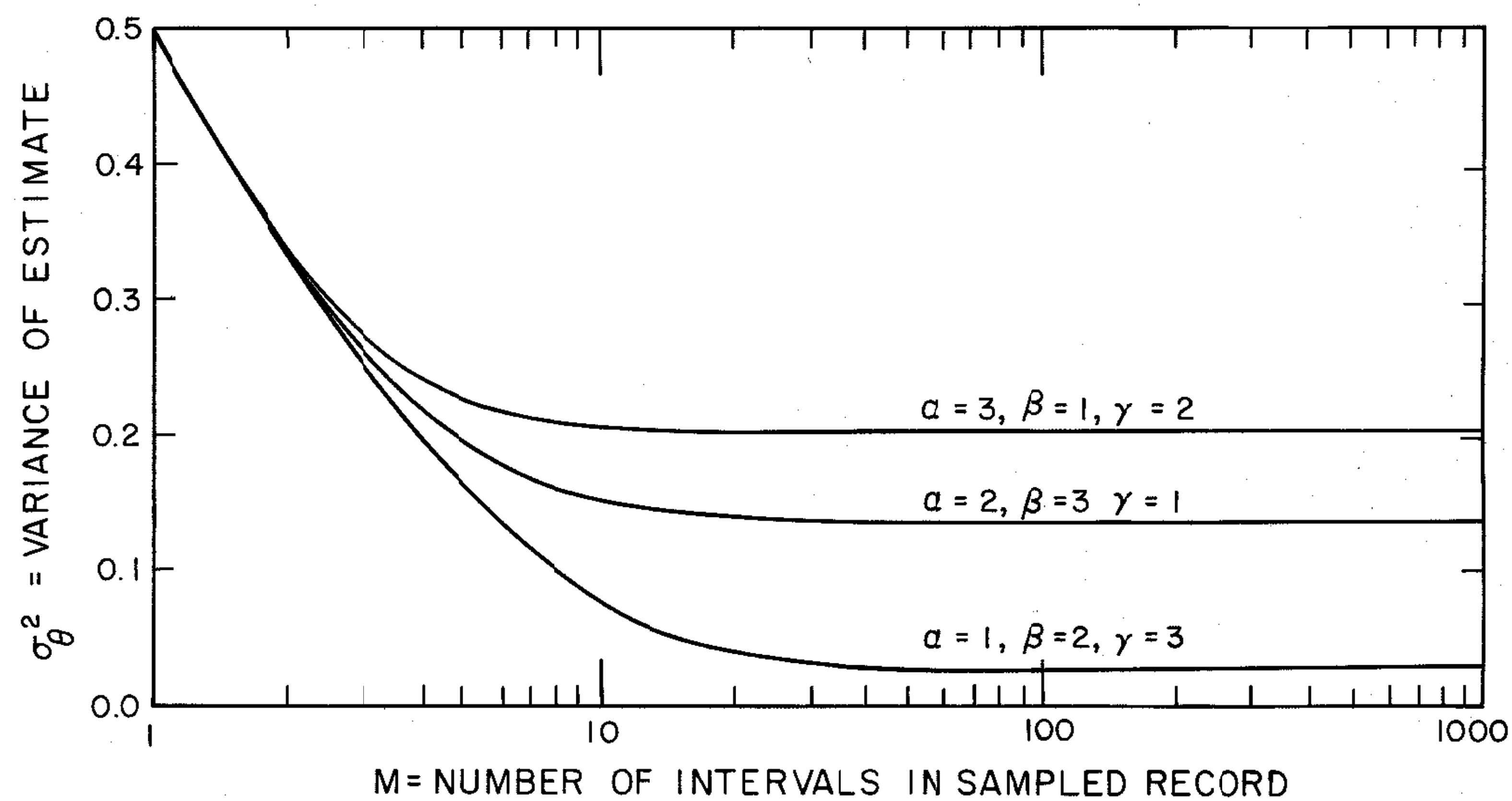


Figure 4.3.1 -- Variance of the Estimate of a Constant

CHAPTER V

DEGENERATE SAMPLING RATE

5.1 Advantage of Sampling at the Degenerate Rate

In Section 2.4 it was shown that if the two-sided Z-transform of the sampled autocorrelation function had no finite zeros, then the inverse covariance matrix had nonzero elements only on the main diagonal and the 2D adjacent diagonals. A sampling rate T for which this phenomena occurs is termed a degenerate rate in this thesis.

It has already been noted that the discrete minimum variance estimate of the parameter θ in the process

$$y(t) = \theta S(t) + n(t) \qquad 0 \le t \le L$$

is computed from the vector of observed values, $\bar{y}$, by the relation

$$\hat{\theta} = \bar{f}'\bar{y}$$

where

$$\bar{f} = \sigma_\theta^2 \, R^{-1}\bar{S}$$

$$\sigma_\theta^2 = \left(\bar{S}'R^{-1}\bar{S}\right)^{-1}$$

R^{-1} = inverse covariance matrix of the sampled noise.

The prominent role played by the inverse covariance matrix in the above relations should be noted. It can be seen from the above relations that if the inverse covariance matrix has the simple form mentioned in the first paragraph, then the calculation of σ_θ^2 and $\overline{F}$ will be relatively simple. Therefore simplicity is the main advantage of sampling at this rate.

In order to be able to take advantage of the simplicity gained when the samples are taken at the degenerate rate it is necessary that the variance of the estimate be below an acceptable maximum. A special case will now be worked out which will show that the variance obtained when sampling occurs at the degenerate rate can be acceptable under all but the most stringent criterions.

Consider the problem of estimating the constant θ in the process

$$y(t) = \theta + n(t) \qquad\qquad 0 \le t \le L$$

where the noise has the spectral density

$$G(\omega) = \left[\frac{2\beta\gamma\left(\gamma^2-\beta^2\right)}{(\gamma-\beta)\left(a^2+\gamma\beta\right)}\right]\frac{\left(\omega^2+a^2\right)}{\left(\omega^2+\beta^2\right)\left(\omega^2+\gamma^2\right)} \qquad (5.1.1)$$

and the corresponding autocorrelation function

$$\phi(t) = \frac{1}{(\gamma-\beta)\left(a^2+\gamma\beta\right)}\left[\gamma\left(a^2-\beta^2\right)e^{-\beta|t|} + \beta\left(\gamma^2-a^2\right)e^{-\gamma|t|}\right] \tag{5.1.2}$$

It should be noted that the $\phi(o) = 1$, hence the noise has unit variance.

In the notation of Section 2.4

$$\phi(t) = \sigma_1^2 e^{-\beta|t|} + \sigma_2^2 e^{-\gamma|t|} \tag{5.1.3}$$

where

$$\sigma_1^2 = \frac{\gamma\left(a^2-\beta^2\right)}{(\gamma-\beta)\left(a^2+\gamma\beta\right)} \tag{5.1.4}$$

and

$$\sigma_2^2 = \frac{\beta\left(\gamma^2-a^2\right)}{(\gamma-\beta)\left(a^2+\gamma\beta\right)} \tag{5.1.5}$$

From 2.3.2, for $\phi^*(Z)$, the Z transform of the sampled autocorrelation function, to have no finite Z-plane zeros it is necessary and sufficient that the quantity A_o defined by

$$A_o = \sigma_1^2\left(1-e^{-2\beta T}\right)e^{-\gamma T} + \sigma_2^2\left(1-e^{-2\gamma T}\right)e^{-\beta T} \tag{5.1.6}$$

be equal zero or equivalently

$$-\frac{\sigma_2^2}{\sigma_1^2} = \frac{\beta}{\gamma}\,\frac{a^2-\gamma^2}{a^2-\beta^2} = \frac{\sinh \beta T_o}{\sinh \gamma T_o} \tag{5.1.7}$$

where T_o is the time between the samples taken at the degenerate rate. Under the assumption that the sampling rate has been chosen such that 5.1.7 holds and from the results of Section 2.4, the inverse covariance matrix of the noise takes the form

$$\begin{bmatrix}
Q_o^2 & Q_1Q_o & Q_2Q_o & & & & & \\
Q_1Q_o & Q_o^2+Q_1^2 & Q_1Q_o+Q_1Q_2 & Q_2Q_o & & & \bigcirc & \\
Q_2Q_o & Q_1Q_o+Q_1Q_2 & Q_o^2+Q_1^2+Q_2^2 & Q_1Q_o+Q_1Q_2 & Q_2Q_o & & & \\
\ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & Q_2Q_o & Q_1Q_o+Q_1Q_2 & Q_o^2+Q_1^2+Q_2^2 & Q_1Q_o+Q_1Q_2 & Q_2Q_o & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & Q_2Q_o & Q_1Q_o+Q_1Q_2 & Q_o^2+Q_1^2+Q_2^2 & Q_1Q_o+Q_1Q_2 & Q_2Q_o & \\
 & \bigcirc & & Q_2Q_o & Q_1Q_o+Q_1Q_2 & Q_o^2+Q_1^2 & Q_1Q_o & \\
 & & & & Q_2Q_o & Q_1Q_o & Q_o^2 &
\end{bmatrix}$$

where

$$k = \sigma_1^2\left(1-e^{-2\beta T_o}\right)\left[1+e^{-2\gamma T_o} - e^{+(\beta-\gamma)T_o} - e^{-(\beta+\gamma)T}\right] \tag{5.1.8}$$

$$Q_o - k^{-\frac{1}{2}} \tag{5.1.9}$$

$$Q_1 = -k^{-\frac{1}{2}}\left(e^{-\beta T_o} + e^{-\gamma T_o}\right) \tag{5.1.10}$$

$$Q_2 = k^{-\frac{1}{2}} e^{-(\beta+\gamma)T_o} \tag{5.1.11}$$

The optimum weighting vector $\bar{f}$ and its associated variance will now be computed. Upon noting that $\bar{S}' = (1\ 1\ \cdots\ 1\ 1)$ and the form of the inverse covariance matrix given above, σ_θ^2 can be expressed

$$\sigma_\theta^2 = (\bar{S}' R^{-1} \bar{S})^{-1} = \Big[(M+1)\left(Q_o^2+Q_1^2+Q_2^2\right) + 2M\left(Q_oQ_1+Q_1Q_2\right) + 2(M-1)Q_oQ_2 - 4Q_1Q_2 - 4Q_2^2 - 2Q_1^2\Big]^{-1} \tag{5.1.12}$$

and the elements of the weighting vector $\bar{f} = \left(f_\mu\right) = \sigma_0^2 R^{-1}\bar{S}$ are

$$f_o = \sigma_\theta^2\left(Q_o^2+Q_1Q_o+Q_2Q_o\right)$$

$$f_1 = \sigma_\theta^2\left(Q_o^2+Q_1^2+2Q_1Q_o+Q_1Q_2+Q_1Q_o\right)$$

$$f_2 = \sigma_\theta^2\left(Q_o^2+Q_1^2+Q_2^2+2Q_1Q_o+2Q_1Q_2+2Q_2Q_o\right)$$

$$\vdots$$

$$f_k = \sigma_\theta^2\left(Q_o^2+Q_1^2+Q_2^2+2Q_1Q_o+2Q_1Q_2+2Q_2Q_o\right) \qquad (5.1.13)$$

$$\vdots$$

$$f_{M-2} = \sigma_\theta^2\left(Q_o^2+Q_1^2+Q_2^2+2Q_1Q_o+2Q_1Q_2+2Q_2Q_o\right)$$

$$f_{M-1} = \sigma_\theta^2\left(Q_o^2+Q_1^2+2Q_1Q_o+Q_1Q_2+Q_1Q_o\right)$$

$$f_M = \sigma_0^2\left(Q_o^2+Q_1Q_o+Q_2Q_o\right)$$

The simplicity of 5.1.12 and 5.1.13 should be noted.

A numerical example will be worked out now. Let $a = 3$, $\beta = 1$, and $\gamma = 2$, then 5.1.7 has only one solution for T_o and the result is $T_o \cong 1$. Inspection of Figure 4.3.1 shows that $M = 10$ corresponds to $T_o = 1$ and the variance for an eleven point estimate is $\sigma_\theta^2 \cong .205$. Also from this figure it can be seen that the variance of an infinite point estimate is $\sigma_\theta^2 \simeq .20$. Therefore, if the units are seconds, it can be concluded that sampling at one point per second gives a variance within 3 percent of that obtained at an infinite rate.

5.2 Pole-Zero Configuration for the Degenerate Rate to Exist for Second Order Noise

In this section it will be shown that a real T_o cannot be found as a solution to 5.1.7 for arbitrary a, β, γ. It will be shown that a, β, γ must satisfy the relation

$$a > \gamma > \beta \tag{5.2.1}$$

where it has been assumed, without loss of generality, that

$$\gamma > \beta \tag{5.2.2}$$

The proof of 5.2.1 is as follows. From 5.1.7

$$\frac{\sinh \beta T_o}{\sinh \gamma T_o} = \frac{\beta}{\gamma} \frac{a^2-\gamma^2}{a^2-\beta^2} \geqq 0 \tag{5.2.3}$$

therefore

$$\text{sign}\left(a^2-\gamma^2\right) = \text{sign}\left(a^2-\beta^2\right) \tag{5.2.4}$$

and hence the relation

$$\gamma > \alpha > \beta$$

<u>cannot</u> hold. Equation 5.2.4 allows 5.2.3 to be written in the form

$$\frac{\frac{\sinh \beta T_o}{\beta T_o}}{\frac{\sinh \gamma T_o}{\gamma T_o}} = \frac{|a^2-\gamma^2|}{|a^2-\beta^2|} \leqq 1 \tag{5.2.5}$$

where the fact that the function $\frac{\sinh X}{X}$ is monotonically increasing from unity was used. Rearrangement of the above equation gives

$$|a^2-\gamma^2| \leq |a^2-\beta^2| \tag{5.2.6}$$

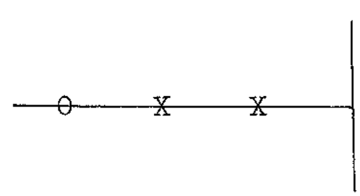
which implies that

$$a > \gamma > \beta \tag{5.2.7}$$

and that the relation

$$\gamma > \beta > a$$

cannot hold. Hence it has been shown that for a degenerate rate to exist, the pole-zero configuration of the filter, which will generate noise with the spectral density 5.1.1 when white noise is the input, must be as shown below.



It remains to show that the relation

$$-\frac{\sigma_2^2}{\sigma_1^2} = \frac{\sinh \beta T_o}{\sinh \gamma T_o} \qquad (5.1.8)$$

cannot be satisfied for nonzero, real T_o if the noise is all pole second order noise. Without loss of generality, let $\sigma_1^2 = \frac{1}{\beta}$ and $\sigma_2^2 = -\frac{1}{\gamma}$ where $\gamma > \beta$, which forces the noise to be all pole. From 5.1.8

$$\frac{\dfrac{\sinh \beta T_o}{\beta T_o}}{\dfrac{\sinh \gamma T_o}{\gamma T_o}} = 1 \qquad (5.1.9)$$

which only has the solution $T_o = 0$. Therefore a degenerate rate does not exist for all pole noise.

5.3 Connection Between Autoregressive Noise and the Degenerate Sampling Rate

In this section the connection between the autoregressive noise discussed in Section 2.4 and samples of continuous noise taken at the degenerate rate, when it exists, will be explored for second order noise. In Reference 9 it is shown that a discrete random process defined in the following manner

$$Q_o n_t + Q_1 n_{t-T} + Q_2 n_{t-2T} = W_t \quad , \qquad (5.3.1)$$

where the W_t are independent gaussian random variables with zero mean and unit variance, has an autocorrelation function

$$\phi[(i-j)T] = E\{n_{t-iT}n_{t-jT}\} \qquad (5.3.2)$$

and an inverse covariance matrix which is identical in form to that shown in Section 5.1. Because of the uniqueness of an inverse matrix, the covariance matrix of samples taken at the degenerate rate must be identical in form to the covariance matrix of the autoregressive samples, and hence they must have the same sampled autocorrelation function. In conclusion, it can be said that when the degenerate rate exists, and samples are taken at that rate, then the linear combination 5.3.1 gives an independent gaussian random variable where the Q's are defined in 5.1.8 through 5.1.11. Stated in another way, it can be said that samples taken at the degenerate rate can be simulated by obtaining a new sample from a linear combination of the past two samples and an independent gaussian random variable.

It should be noted that if samples of a continuous random process are to conform to the model 5.3.1 then it is necessary that

$$En_{t-T}W_t = 0 \qquad (5.3.3)$$

$$En_{t-2T}W_t = 0 \qquad (5.3.4)$$

Conversely, 5.3.3 and 5.3.4 can be used to derive a relation between the parameters of the noise and the sampling rate which must be satisfied in order for the sampled noise to be autoregressive and hence degenerate. Of course this relation has to be 5.1.7. That this is true will now be shown directly after some preliminary concepts, which are important in themselves, are discussed.

The simulation of samples of random processes with the spectral density

$$G(\omega) = K^2 \frac{\omega^2+a^2}{\left(\omega^2+\beta^2\right)\left(\omega^2+\gamma^2\right)} \tag{5.3.5}$$

at equally spaced intervals in time will now be discussed. Noise with the spectral density 5.3.5 can be generated by passing gaussian white noise of unit spectral density amplitude through the system of Figure 5.3.1.

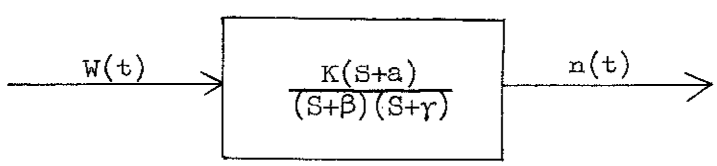


Figure 5.3.1
System for the generation of noise with the spectral density 5.3.5.

where $W(t)$ is the input white noise, $n(t)$ is the output correlated noise, and S is the Laplace variable. Upon noting that

$$K \frac{S+a}{(S+\beta)(S+\gamma)} = \frac{K}{\gamma-\beta}\left[\frac{a-\beta}{S+\beta} - \frac{a-\gamma}{S+\gamma}\right] \tag{5.3.6}$$

it is seen that the system of Figure 5.3.1 is equivalent to

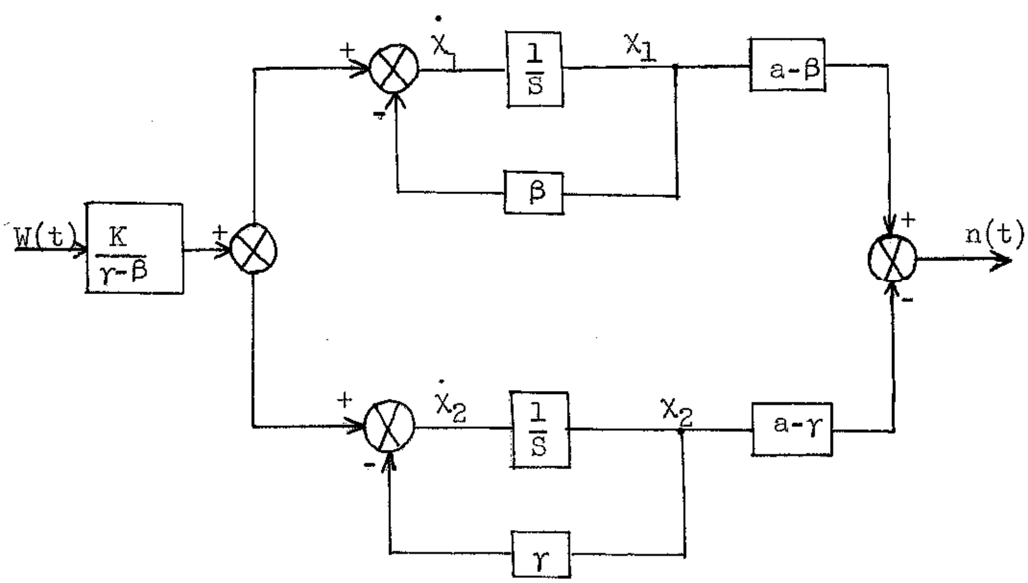


Figure 5.3.2

Equivalent system for the generation of noise

with the spectral density 5.3.5.

where $x_1(t)$ and $x_2(t)$ have been introduced for mathematical convenience. It should be noted that

$$n(t) = (a-\beta)x_1(t) - (a-\gamma)x_2(t) \tag{5.3.7}$$

In matrix notation the differential equation describing the system of Figure 5.3.2 is

$$\dot{\underline{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\beta & 0 \\ 0 & -\gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{K}{\gamma-\beta} \begin{bmatrix} W(t) \\ W(t) \end{bmatrix} \tag{5.3.8}$$

The solution of 5.3.8 is (where t_o is an arbitrary initial time, $t_o < t$)

$$\overline{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\beta(t-t_o)} & 0 \\ 0 & e^{-\gamma(t-t_o)} \end{bmatrix} \begin{bmatrix} x_1(t_o) \\ x_2(t_o) \end{bmatrix}$$

$$+ \frac{K}{\gamma-\beta} \int_{t_o}^{t} \begin{bmatrix} e^{-\beta(t-\tau)} & 0 \\ 0 & e^{-\gamma(t-\tau)} \end{bmatrix} \begin{bmatrix} W(\tau) \\ W(\tau) \end{bmatrix} d\tau \tag{5.3.9}$$

which can be verified by substitution into 5.3.8. Now suppose the process started at $t_o = -\infty$ and it is desired to obtain the covariance matrix at some finite time $t = t_i$. Putting $t = t_i$ and

$t_o = -\infty$ in 5.3.9 gives

$$\overline{\mathbf{X}}(t_i) = \frac{K}{\gamma-\beta} \int_{-\infty}^{t_i} \begin{bmatrix} e^{-\beta(t_i-\tau)} & 0 \\ 0 & e^{-\gamma(t_i-\tau)} \end{bmatrix} \begin{bmatrix} W(\tau) \\ W(\tau) \end{bmatrix} d\tau$$

$$= \frac{K}{\gamma-\beta} \int_{-\infty}^{0} \begin{bmatrix} e^{\beta\tau} & 0 \\ 0 & e^{\gamma\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} W(\tau+t_i)d\tau \quad (5.3.10)$$

and hence

$$E\{\overline{X}(t_i)\overline{X}'(t_i)\} = \frac{K^2}{(\gamma-\beta)^2} \int_{-\infty}^{0}\int_{-\infty}^{0} \begin{bmatrix} e^{\beta\tau} & 0 \\ 0 & e^{\gamma\tau} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\beta n} & 0 \\ 0 & e^{\gamma n} \end{bmatrix} \delta(\tau-n)d\tau dn \quad (5.3.11)$$

$$= \frac{K^2}{(\gamma-\beta)^2} \begin{bmatrix} \frac{1}{2\beta} & \frac{1}{\beta+\gamma} \\ \frac{1}{\beta+\gamma} & \frac{1}{2\gamma} \end{bmatrix} \quad (5.3.12)$$

where the orders of integration and expectation were interchanged and the white noise property of $W(t)$

$$E\{W(t_i+\tau)W(t_i+n)\} = \delta(\tau-n) \quad (5.3.13)$$

was used in obtaining 5.3.11. Equation 5.3.12 illustrates the stationarity of the vector $\overline{X}(t_i)$ in that its covariance matrix does not depend on t_i. Before progressing to the technique of simulating discrete samples of the random process it is necessary to calculate another covariance matrix. Define the vector $\overline{Z}(t,t_o)$ by

$$\overline{Z}(t,t_o) = \frac{K}{\gamma-\beta} \int_{t_o}^{t} \begin{bmatrix} e^{-\gamma(t-\tau)} & 0 \\ 0 & e^{-\beta(t-\tau)} \end{bmatrix} \begin{bmatrix} W(\tau) \\ W(\tau) \end{bmatrix} d\tau \qquad (5.3.14)$$

then in a manner similar to that used in obtaining 5.3.12, the relation

$$E\{\overline{Z}(t,t_o)\overline{Z}'(t,t_o)\} = \frac{K^2}{(\gamma-\beta)^2} \begin{bmatrix} \frac{1}{2\beta}\left(1-e^{-2\beta(t-t_o)}\right) & \frac{1}{\beta+\gamma}\left(1-e^{-(\beta+\gamma)(t-t_o)}\right) \\ \frac{1}{\beta+\gamma}\left(1-e^{-(\beta+\gamma)(t-t_o)}\right) & \frac{1}{2\gamma}\left(1-e^{-2\gamma(t-t_o)}\right) \end{bmatrix} \qquad (5.3.15)$$

is obtained. It should be noted that $\overline{Z}(t,t_o)$ is independent of $\overline{X}(t_o)$ since $\overline{Z}(t,t_o)$ depends on white noise which occurs after t_o.

Equation 5.3.9 can now be written in the form

$$\overline{X}(t) = \phi(t-t_o)\overline{X}(t_o) + \overline{Z}(t,t_o) \qquad (5.3.16)$$

where

$$\phi(\tau) = \begin{bmatrix} e^{-\beta\tau} & 0 \\ 0 & e^{-\gamma\tau} \end{bmatrix} \qquad (5.3.17)$$

and the only restrictions on t and t_o is that $t > t_o$. The technique of simulating discrete samples of the process $n(t)$ at the times $0,\ T,\ 2T,\ \cdots,\ (N-1)T,\ NT,\ \cdots$ will now be described. First a random vector with the covariance matrix 5.3.12 is generated and denoted $\overline{X}(0)$

$$\overline{X}(0) = \begin{bmatrix} x_1(o) \\ x_2(o) \end{bmatrix} \qquad (5.3.18)$$

and the linear combination (see 5.3.7)

$$n(o) = (a-\beta)x_1(o) - (a-\gamma)x_2(o) \qquad (5.3.19)$$

is computed. The quantity $n(o)$ is the first sample of the process with spectral density 5.3.5. To get the sample at $t = T$ a random vector

$$\overline{Z}(T,0) = \begin{bmatrix} z_1(T) \\ z_2(T) \end{bmatrix} \qquad (5.3.20)$$

is generated such that its components have the covariance matrix 5.3.15 and then the calculations

$$\overline{X}(T) = \phi(T)\overline{X}(0) + \overline{Z}(T,0) \tag{5.3.21}$$

$$n(T) = (a-\beta)\chi_1(T) - (a-\gamma)\chi_2(T) \tag{5.3.22}$$

are made. The quantity $n(T)$ is the second sample of the process. In general, the N^{th} sample is determined from the $(N-1)^{th}$ sample from

$$\overline{X}(NT) = \phi(T)\overline{X}((N-1)T) + \overline{Z}(NT,(N-1)T) \tag{5.3.23}$$

$$n(NT) = (a-\beta)\chi_1(NT) - (a-\gamma)\chi_2(NT) \tag{5.3.24}$$

$$Z(NT,(N-1)T) \equiv \begin{bmatrix} z_1(NT) \\ z_2(NT) \end{bmatrix} \tag{5.3.25}$$

where the $\overline{Z}'s$ are independent of one another and have the covariance matrix

$$E\{\overline{Z}\overline{Z}'\} = \frac{K^2}{(\gamma-\beta)^2} \begin{bmatrix} \frac{1}{2\beta}\left(1-e^{-2\beta T}\right) & \frac{1}{\beta+\gamma}\left(1-e^{-(\beta+\gamma)T}\right) \\ \frac{1}{\beta+\gamma}\left(1-e^{-(\beta+\gamma)T}\right) & \frac{1}{2\gamma}\left(1-e^{-2\gamma T}\right) \end{bmatrix} \tag{5.3.26}$$

Now that the preliminaries are completed, attention is turned to deriving 5.1.7 by a direct method.

To restate the problem, it is desired to derive a necessary condition such that

$$n(2T) + a_1 n(T) + a_2 n(0) = W(2T) \tag{5.3.27}$$

where

$$a_1 = \frac{Q_1}{Q_o} = e^{-\beta T} + e^{-\gamma T} \tag{5.3.28}$$

$$a_2 = \frac{Q_2}{Q_o} = e^{-(\beta+\gamma)T} \tag{5.3.29}$$

$W(2T)$ = finite variance discrete white noise

and hence

$$E\{n(T)W(2T)\} = 0 \tag{5.3.30}$$

$$E\{n(0)W(2T)\} = 0 \tag{5.3.31}$$

In the analysis which follows it is convenient to define the matrix

$$M = [(a-\beta) \quad -(a-\gamma)]$$

so that 5.3.24 can be written in matrix notation

$$n(NT) = (a-\beta)\chi_1(NT) - (a-\gamma)\chi_2(NT)$$

$$= M\bar{X}(NT) \tag{5.3.32}$$

The samples $n(0)$ and $n(T)$ will now be expressed in terms of the components of $\overline{X}(0)$ and $\overline{Z}(T,0)$. From 5.3.32

$$n(0) = M\overline{X}(0) = (a-\beta)\chi_1(o) - (a-\gamma)\chi_2(o) \tag{5.3.33}$$

and from 5.3.23 and 5.3.32

$$n(T) = M\overline{X}(T) = M[\phi(T)X(o)+\overline{Z}(T,o)]$$

$$= (a-\beta)e^{-\beta T}\chi_1(o)-(a-\gamma)e^{-\gamma T}\chi_2(o)+(a-\beta)z_1(T)-(a-\gamma)z_2(T) \tag{5.3.34}$$

Solving 5.3.33 and 5.3.34 for $\chi_1(o)$ and $\chi_2(o)$ gives

$$\chi_1(o) = (a-\beta)^{-1}\left(e^{-\beta T}-e^{-\gamma T}\right)^{-1}\left[n(T)-e^{-\gamma T}n(o)-(a-\beta)z_1(T)+(a-\gamma)z_2(T)\right] \tag{5.3.35}$$

$$\chi_2(o) = (a-\gamma)^{-1}\left(e^{-\beta T}-e^{-\gamma T}\right)^{-1}\left[n(T)-e^{-\beta T}n(o)-(a-\beta)z_1(T)+(a-\gamma)z_2(T)\right]$$

But

$$n(2T) = M\overline{X}(2T) = M\{\phi(T)\overline{X}(T)+\overline{Z}(2T,T)\}$$

$$= M\{\phi(T)[\phi(T)X(o)+\overline{Z}(T,0)] + \overline{Z}(2T,T)\}$$

$$= M\phi^2(T)\overline{X}(o) + M\phi(T)\overline{Z}(T,o) + M\overline{Z}(2T,T) \quad , \tag{5.3.36}$$

therefore upon substituting 5.3.35 in 5.3.36, using 5.3.25, and some algebraic manipulation

$$n(2T)-\left(e^{-\beta T}+e^{-\gamma T}\right)n(T)+e^{-(\beta+\gamma)T}n(o) = e^{-\beta T}(a-\gamma)z_2(T)-e^{-\gamma T}(a-\beta)z_1(T)$$

$$+ (a-\beta)z_1(2T)-(a-\gamma)z_2(2T) \tag{5.3.37}$$

Comparing 5.3.27, 5.3.28, 5.3.29 and 5.3.37 shows that the left hand sides of 5.3.27 and 5.3.37 agree. The quantity $W(2T)$ is defined as

$$W(2T) = e^{-\beta T}(a-\gamma)z_2(T)-e^{-\gamma T}(a-\beta)z_1(T)+(a-\beta)z_1(2T)-(a-\gamma)z_2(2T) \tag{5.3.38}$$

and it remains to derive a condition for

$$E\{n(T)W(2T)\} = 0$$

$$E\{n(o)W(2T)\} = 0$$

It should be noted that $n(o)$ is uncorrelated with $\overline{Z}(T,o)$ and $\overline{Z}(2T,T)$ since they are generated independent of $n(o)$ and similarly $n(T)$ is uncorrelated with $\overline{Z}(2T,T)$. Therefore it remains to find the condition for $n(T)$ to be uncorrelated with

$$e^{-\beta T}(a-\gamma)z_2(T)-e^{-\gamma T}(a-\beta)z_1(T) = M \begin{bmatrix} -e^{-\gamma T} & 0 \\ 0 & -e^{-\beta T} \end{bmatrix} \overline{Z}(T,o) \tag{5.3.39}$$

Now

$$n(T) = M\overline{X}(T) = M[\phi(T)\overline{X}(o)+\overline{Z}(T,o)] = M\phi(T)\overline{X}(o)+M\overline{Z}(T,o) \quad ,$$

therefore since $\overline{X}(o)$ and $\overline{Z}(T,o)$ are independent

$$E\left\{n(T)M\begin{bmatrix}-e^{-\gamma T} & 0\\ 0 & -e^{-\beta T}\end{bmatrix}\overline{Z}(T,o)\right\} = E\left\{M\overline{Z}(T,o)M\begin{bmatrix}-e^{-\gamma T} & 0\\ 0 & -e^{-\beta T}\end{bmatrix}\overline{Z}(T,o)\right\} \tag{5.3.40}$$

$$= E\left\{(a-\beta)^2e^{-\gamma T}z_1^2(T)-(a-\beta)(a-\gamma)z_1(T)z_2(T)\left(e^{-\beta T}+e^{-\gamma T}\right)+(a-\gamma)^2e^{-\beta T}z_2^2(T)\right\}$$

$$= \frac{1}{2\beta}(a-\beta)^2e^{-\gamma T}\left[1-e^{-\beta T}\right]-\frac{(a-\beta)(a-\gamma)}{\beta+\gamma}\left[1-e^{-(\beta+\gamma)T}\right]\left[e^{-\beta T}+e^{-\gamma T}\right]+$$

$$+\frac{1}{2\gamma}(a-\gamma)^2e^{-\beta T}\left[1-e^{-2\gamma T}\right] \tag{5.3.41}$$

$$= [\beta\gamma(\beta+\gamma)]^{-1}(\beta-\gamma)e^{-(\beta+\gamma)T}\left\{\gamma\left(a^2-\beta^2\right)\sinh\ \beta T-\beta\left(a^2-\gamma^2\right)\sinh\ \gamma T\right\} \tag{5.3.42}$$

where 5.3.26 was used in obtaining 5.3.41. Therefore from 5.3.42 for the correlation between $n(T)$ and $W(2T)$ to be zero, T must be chosen such that

$$\frac{\sinh\ \beta T_o}{\sinh\ \gamma T_o} = \frac{\beta}{\gamma}\,\frac{a^2-\gamma^2}{a^2-\beta^2} \tag{5.1.7}$$

This completes the analysis of this chapter.

CHAPTER VI

ESTIMATION OF A CONSTANT IN D-th ORDER BUTTERWORTH NOISE

6.1 Derivation of the Autocorrelation Function of Butterworth Noise

The problem of estimating the constant θ in the process

$$y(t) = \theta + n(t) \qquad 0 \le t \le L \tag{6.1.1}$$

when the noise has the spectral density

$$G(\omega) = \frac{K}{1+\left(\frac{\omega}{\omega_o}\right)^{2D}} = \frac{K\omega_o^{2D}}{\omega^{2D}+\omega_o^{2D}} \tag{6.1.2}$$

is considered in this chapter. Noise with the spectral density 6.1.2 is usually called "Butterworth" noise since it can be generated by passing white noise through the "maximally flat" or Butterworth filter. The integer D is referred to as the order of the noise and ω_o and K are constants.

In the next section formulas will be derived for the variance of the estimate of a constant in D-th Order Butterworth noise when an arbitrary number of equally spaced data points are used in making the estimate. In order to apply the general formulas of Chapter II it is necessary to determine the autocorrelation function corresponding to the spectral density 6.1.2 and to place it in the form

$$\phi(t) = \sum_{k=1}^{D} \sigma_k^2 e^{-\beta_k |t|} \tag{6.1.3}$$

It is the purpose of this section to compute the autocorrelation function of Butterworth noise and to place it in the form 6.1.3.

The first step in obtaining the autocorrelation function is to make a partial fraction expansion of 6.1.2

$$G(\omega) = \frac{K\omega_o^{2D}}{\omega^{2D}+\omega_o^{2D}} = K \sum_{k=1}^{2D} \frac{R_k}{\omega-\omega_o\bar{\omega}_k} \tag{6.1.4}$$

where the $\bar{\omega}_k$ are the roots of

$$X^{2D} + 1 = 0 \tag{6.1.5}$$

and the R_k are the residues of $\frac{\omega_o^{2D}}{\omega^{2D}+\omega_o^{2D}}$ at the poles $\omega_o\bar{\omega}_k$.

Equation 6.1.5 is simple to solve for the roots $\bar{\omega}_k$ and the result is

$$\bar{\omega}_k = \cos\theta_k + i \sin\theta_k \qquad k = 1, 2, \cdots, 2D \tag{6.1.6}$$

where

$$\theta_k = \frac{\pi}{2D} + \frac{2\pi}{2D}(k-1) = \frac{90}{D} + \frac{180}{D}(k-1) \quad k = 1, 2, \cdots, 2D \tag{6.1.7}$$

$$= \angle\bar{\omega}_k = \text{angle of } \bar{\omega}_k$$

The residue of $\dfrac{\omega_o^{2D}}{\omega^{2D}+\omega_o^{2D}}$ at the pole $\omega_o\bar{\omega}_k$ can be obtained from*

$$R_k = \left.\frac{\omega_o^{2D}}{2D\omega^{2D-1}}\right|_{\omega=\omega_o\bar{\omega}_k} = \frac{\omega_o}{2D\bar{\omega}_k^{2D-1}} \tag{6.1.8}$$

and therefore the magnitude of R_k is

$$|R_k| = \frac{\omega_o}{2D}\,\frac{1}{|\bar{\omega}_k^{2D-1}|} = \frac{\omega_o}{2D} \tag{6.1.9}$$

and the angle of R_k is

$$\angle R_k = -(2D-1)\angle\bar{\omega}_k = -2D\angle\bar{\omega}_k + \angle\bar{\omega}_k$$

$$= -2D\left[\frac{90}{D} + (k-1)\,\frac{180}{D}\right] + \theta_k$$

$$= -180 + 360 - k360 + \theta_k \tag{6.1.10}$$

---

* In general, if $Z = \dfrac{p(\omega)}{q(\omega)}$ is the ratio of two polynomials in ω, where the order of p is lower than the order of q, and Z has only simple poles, then the residue of the function at the pole ω_i is $r_i = \left.\dfrac{p(\omega)}{\frac{dq}{d\omega}}\right|_{\omega=\omega_i}$

which is equivalent to

$$\angle R_k = 180 + \theta_k \tag{6.1.11}$$

where θ_k is defined in 6.1.7. Using the above results, $G(\omega)$ can be written in the form

$$G(\omega) = \frac{K\omega_o^{2D}}{\omega^{2D}+\omega_o^{2D}} = \frac{K\omega_o}{2D} \sum_{k=1}^{2D} \frac{e^{i180} e^{i\theta_k}}{\omega-\omega_o\bar{\omega}_k} \tag{6.1.12}$$

Before proceeding further in obtaining the autocorrelation function of the noise, the constant K will be chosen such that the noise has unit variance. To accomplish this let K be chosen such that

$$1 = \int_{-\infty}^{\infty} S(\omega)\, \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} \frac{K}{1+\left(\frac{\omega}{\omega_o}\right)^{2D}} \frac{d\omega}{2\pi} = \frac{K\omega_o}{\pi} \int_{0}^{\infty} \frac{dx}{1+X^{2D}} = \frac{K\omega_o}{2D} \frac{1}{\sin \frac{\pi}{2D}}$$

or

$$K = \frac{2D}{\omega_o} \sin \frac{\pi}{2D} \tag{6.1.13}$$

Substituting 6.1.13 in 6.1.12 gives

$$G(\omega) = - \sin \frac{90}{D} \sum_{k=1}^{2D} \frac{e^{i\theta_k}}{\omega-\omega_o\bar{\omega}_k} \tag{6.1.14}$$

Since the autocorrelation function and the spectral density are Fourier Transform pairs

$$\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) c^{i\omega t} d\omega$$

$$= - i \sin \frac{90}{D} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sum_{k=1}^{2D} \frac{e^{-i\omega t} e^{i\theta_k}}{\omega - \omega_o \overline{\omega}_k} d\omega$$

$$= \begin{cases} - i \sin \frac{90}{D} \sum_{k=1}^{D} e^{i\theta_k} e^{-i\omega_o \overline{\omega}_k t} & t < 0 \\ + i \sin \frac{90}{D} \sum_{k=D+1}^{2D} e^{i\theta_k} e^{-i\omega_o \overline{\omega}_k t} & t > 0 \end{cases} \qquad (6.1.15)$$

$$= \begin{cases} - i \sin \frac{90}{D} \sum_{k=1}^{D} e^{i\theta_k} e^{-i\omega_o \overline{\omega}_k t} & t < 0 \\ i \sin \frac{90}{D} \sum_{k=1}^{D} e^{i\left(\theta_k + \pi\right)} e^{-i\omega_o \left(-\overline{\omega}_k\right) t} & t > 0 \end{cases}$$

$$= \begin{cases} - i \sin \frac{90}{D} \sum_{k=1}^{D} e^{i\theta_k} e^{-i\omega_o \overline{\omega}_k t} & t < 0 \\ - i \sin \frac{90}{D} \sum_{k=1}^{D} e^{i\theta_k} e^{i\omega_o \overline{\omega}_k t} & t > 0 \end{cases} \qquad (6.1.16)$$

$$= - i \sin \frac{90}{D} \sum_{k=1}^{D} e^{i\theta_k} e^{i\omega_o \overline{\omega}_k |t|} \qquad (6.1.17)$$

where the symmetry of the $\bar{\omega}_k$ about the origin in the ω-plane was used in obtaining 6.1.16. Since

$$e^{i\theta_k} = \cos\theta_k + i\sin\theta_k$$

and

$$\bar{\omega}_k = \cos\theta_k + i\sin\theta_k$$

then

$$\phi(t) = \sum_{k=1}^{D} \sin\frac{90}{D}\left(\sin\theta_k - i\cos\theta_k\right)e^{-\omega_o\left(\sin\theta_k - i\cos\theta_k\right)|t|} \tag{6.1.18}$$

If the definitions

$$\sigma_k^2 = \sin\frac{90}{D}\left(\sin\theta_k - i\cos\theta_k\right) \tag{6.1.19}$$

$$\beta_k = \omega_o\left(\sin\theta_k - i\cos\theta_k\right) \tag{6.1.20}$$

are used in 6.1.18 then

$$\phi(t) = \sum_{k=1}^{D} \sigma_k^2 e^{-\beta_k|t|} \tag{6.1.21}$$

Equations 6.1.21, 6.1.20, 6.1.21, and the relation

$$\theta_k = \frac{90}{D} + \frac{180}{D}(k-1) \qquad k = 1, \cdots, D \tag{6.1.22}$$

complete the analysis of this section.

6.2 Analytic Formulas for the Variance of the Estimate of a Constant in Butterworth Noise

In this section analytic formulas for the variance of the estimate of a constant in Butterworth noise are derived. It will be seen that the formulas can be used for all noises that have autocorrelation functions of the form

$$\phi(t) = \sum_{j=1}^{D} \sigma_j^2 e^{-\beta_j |t|} \tag{6.2.1}$$

and hence sampled autocorrelations of the form

$$\phi^*(Z) = \sum_{j=1}^{D} \frac{\sigma_j^2\left(1-e^{-2\beta_j T}\right)}{\left(1-e^{-\beta_j T}Z\right)\left(1-e^{-\beta_j T}Z^{-1}\right)} \tag{6.2.2}$$

As was shown in the last section

$$\sigma_j^2 = \sin\frac{90}{D}\left(\sin\theta_j - i\cos\theta_j\right) \tag{6.2.3}$$

$$\beta_j = \omega_o\left(\sin\theta_j - i\cos\theta_k\right) \tag{6.2.4}$$

$$\theta_j = \frac{1}{D}(180j - 90) \tag{6.2.5}$$

for Butterworth noise.

The formulas for the variance of the estimate of a constant in a noise with the autocorrelation function of the form 6.2.1 will now be derived. Upon making the definitions

$$a_j = \sigma_j^2\left(1-e^{-2\beta_j T}\right) \tag{6.2.6}$$

$$b_j = 1 + e^{-2\beta_j T} \tag{6.2.7}$$

$$e_j = e^{-\beta_j T} \tag{6.2.8}$$

$$\chi = Z + Z^{-1} \tag{6.2.9}$$

6.2.2 can be put in the forms

$$\phi^*(Z) = \sum_{j=1}^{D} \frac{a_j}{b_j - e_j\chi} = \frac{\sum\limits_{j=0}^{D-1} A_j \chi^i}{\prod\limits_{j=1}^{D}\left(b_j - e_j\chi\right)} \tag{6.2.10}$$

where the A_j are new constants and are defined through 6.2.10. The quantity $\phi^*(Z)$ will now be expressed in factored form (see 2.1.5).

Working on the numerator of 6.2.10 gives

$$\sum_{j=0}^{D-1} A_j \chi^j = A_{D-1}\left[\sum_{j=0}^{D-1} \frac{A_j}{A_{D-1}} \chi^j\right] = A_{D-1} \prod_{j=1}^{D-1} (\chi - \chi_j) \tag{6.2.11}$$

$$= A_{D-1} Z^{-(D-1)} \prod_{j=1}^{D-1} \left(Z^2 - \chi_j Z + 1\right) \tag{6.2.12}$$

where the χ_j are the roots of the polynomial on the left hand side of 6.2.11 and 6.2.9 was used in obtaining the above equation. After solving the (D-1) quadratic equations

$$Z^2 - \chi_j Z+1 = \left(Z-e^{-\alpha_j T}\right)\left(Z-e^{\alpha_j T}\right) \tag{6.2.13}$$

where $e^{\alpha_j T}$ and $e^{-\alpha_j T}$ are the roots with the largest and smallest magnitudes, respectively, 6.2.12 can be expressed*

$$\sum_{j=0}^{D-1} A_j \chi^j = \left[(-1)^{D-1} A_{D-1} \prod_{j=1}^{D-1} e^{\alpha_j T}\right] \prod_{j=1}^{D-1}\left(1-e^{-\alpha_j T} Z^{-1}\right)\left(1-e^{-\alpha_j T} Z\right) \tag{6.2.14}$$

Using the above factorization allows $\phi^*(Z)$ to be written in the form

$$\phi^*(Z) = \frac{\prod_{j=1}^{D-1}\left(1-e^{-\alpha_j T} Z^{-1}\right)}{k \prod_{j=1}^{D}\left(1-e^{-\beta_j T} Z^{-1}\right)} \cdot \frac{\prod_{j=1}^{D-1}\left(1-e^{-\alpha_j T} Z\right)}{k \prod_{j=1}^{D}\left(1-e^{-\beta_j T} Z\right)} \cdot \frac{\phi_N\left(Z^{-1}\right)}{\phi_D\left(Z^{-1}\right)} \cdot \frac{\phi_N(Z)}{\phi_D(Z)} \tag{6.2.15}$$

where

$$k^2 = \left[(-1)^{D-1} A_{D-1} \prod_{j=1}^{D-1} e^{\alpha_j T}\right]^{-1} \tag{6.2.16}$$

---

* Note that it has been assumed that the polynomial is of order D-1. It is possible that for certain noises and sampling rates this assumption would not be valid. The derivation is easily modified to handle such cases.

$$\phi_N(Z) = \prod_{j=1}^{D-1} \left(1-e^{-\alpha_j T} Z\right) \qquad (6.2.17)$$

$$\phi_D(Z) = \prod_{j=1}^{D} \left(1-e^{-\beta_j T} Z\right) \qquad (6.2.18)$$

The operations described in the preceeding paragraphs are easily programmed on a digital computer.

The method of attack which is used in the remainder of the derivation will now be described. In Chapter IV it was shown that the variance of the estimate of a constant could be expressed

$$\sigma_\theta^2 = \left\{(M+1)\phi^{*-1}(1) + \frac{\phi_D(1)}{\phi_N(1)} \sum_{\mu=0}^{M} \left[\frac{P_\mu(1)}{\phi_N(1)} + \frac{P_\mu'(1)}{\phi_N(1)}\right]\right\}^{-1} \qquad (6.2.19)$$

where from 2.1.35 and 2.1.45 (where $N = D-1$)

$$\frac{P_\mu(1)}{\phi_N(1)} = Q_\mu(1) + \sum_{\ell=1}^{D-1} \frac{\rho_{\mu\ell}}{1-e^{-\alpha_\ell T}} \qquad (6.2.20)$$

and from 2.1.53 and 2.1.63

$$\frac{P_\mu'(1)}{\phi_N(1)} = Q_\mu'(1) + \sum_{\ell=1}^{D-1} \frac{\rho_{\mu\ell}'}{1-e^{-\alpha_\ell T}} \qquad (6.2.21)$$

It should be noted that $\phi^*(1)$, $\phi_N(1)$, and $\phi_D(1)$ can be calculated from 6.2.2, 6.2.17, and 6.2.18 respectively.

From 2.1.75 and 2.1.78, and in view of the assumption that $N = D-1$

$$Q_{\mu o} = - Q_1 \delta_{\mu M} \qquad (6.2.22)$$

$$Q'_{\mu o} = - Q_1 \delta_{\mu o} \qquad (6.2.23)$$

Therefore from 6.2.20 through 6.2.23, σ_θ^2 can be expressed

$$\sigma_\theta^2 = \left\{ (M+1)\phi^{*-1}(1) + \frac{\phi_D(1)}{\phi_N(1)} \left[\sum_{\ell=1}^{D-1} \frac{1}{1-e^{-\alpha_\ell T}} \sum_{\mu=o}^{M} \left(\rho_{\mu\ell} + \rho'_{\mu\ell} \right) - 2Q_1 \right] \right\} \qquad (6.2.24)$$

which shows that it is only necessary to calculate Q_1 and $\sum_{\mu=o}^{M} \left(\rho_{\mu\ell} + \rho'_{\mu\ell} \right)$ to complete the analysis.

The system of equations defining the $\rho_{\mu\ell}$, $\rho'_{\mu\ell}$ (see 2.1.85, 2.1.86, and 2.1.87) is

$$A\vec{\rho}_\mu + B\vec{\rho}\,'_\mu = \vec{\Gamma}_\mu \qquad (6.2.25)$$

$$B\vec{\rho}_\mu + A\vec{\rho}\,'_\mu = \vec{\Gamma}_\mu \qquad (6.2.26)$$

Adding the above equations and summing over μ gives

$$(A+B) \left[\sum_{\mu=o}^{M} \left(\vec{\rho}_\mu + \vec{\rho}\,'_\mu \right) \right] = \sum_{\mu=o}^{M} \left(\vec{\Gamma}_\mu + \vec{\Gamma}'_\mu \right) \qquad (6.2.27)$$

From 2.1.85, 6.2.22, 6.2.23

$$\sum_{\mu=0}^{M}\left(\Gamma_{\mu k}+\Gamma'_{\mu k}\right)= 2Q_o\rho_k e^{-(M+1)\alpha_k T} - 2\,\frac{1-e^{-(M+1)\alpha_k T}}{1-e^{-\alpha_k T}}\left[Q_o+Q_1 e^{-\alpha_k T}\right]\rho_k$$

$$-2\,\frac{1-e^{-(M+1)\alpha_k T}}{1-e^{-\alpha_k T}}\,\rho_k\sum_{r=1}^{D-1}\frac{\rho_r}{1-e^{-(\alpha_k+\alpha_r)T}} \qquad (6.2.28)$$

which is the k^{th} element of the right hand side of 6.2.27. The definitions

$$\vec{\xi} = \left(\xi_k\right) \qquad (6.2.29)$$

$$\xi_k = \sum_{\mu=0}^{M}\left(\rho_{\mu k}+\rho'_{\mu k}\right) \qquad (6.2.30)$$

$$\vec{V} = \left(V_k\right) \qquad (6.2.31)$$

$$V_k = \sum_{\mu=0}^{M}\left(\Gamma_{\mu k}+\Gamma'_{\mu k}\right) \qquad (6.2.32)$$

allow 6.2.27 to be written in the form

$$(A+B)\vec{\xi} = \vec{V} \qquad (6.2.33)$$

or

$$\vec{\xi} = (A+B)^{-1}\,\vec{V} \qquad (6.2.34)$$

Using the solutions of 6.2.34 in 6.2.24 gives

$$\sigma_\theta^2 = \left\{(M+1)\phi^{*-1}(1) + \frac{\phi_D(1)}{\phi_N(1)}\left[\sum_{\ell=1}^{D-1} \frac{\xi_\ell}{1-e^{-\alpha_\ell T}} - 2Q_1\right]\right\}^{-1} \tag{6.2.35}$$

It now remains to determine formulas for Q_o, Q_1, and the ρ's to completely specify σ_θ^2 .

Expanding 6.2.17 and 6.2.18 gives

$$\phi_N(Z) = \prod_{j=1}^{D-1}\left(1-e^{-\alpha_j T} Z\right) = 1 + \phi_{N1} Z + \cdots + \phi_{N,D-1} Z^{D-1} \tag{6.2.36}$$

and

$$\phi_D(Z) = k \prod_{j=1}^{D}\left(1-e^{-\beta_j T} Z\right) = k\left[1+\phi_{D1} Z + \cdots + \phi_{D,D} Z^D\right] \tag{6.2.37}$$

where the ϕ_N's and ϕ_D's are new constants defined through 6.2.36 and 6.2.37. Dividing $\phi_D(z)$ by $\phi_N(z)$ by long division gives

$$\frac{\phi_D(Z)}{\phi_N(Z)} = Q_1 Z + Q_o + k \frac{\sum_{j=2}^{D} r_{D-j} Z^{D-j}}{\prod_{i=1}^{D-1}\left(1-e^{-\alpha_i T} Z\right)} \tag{6.2.38}$$

where

$$Q_1 = k \frac{\phi_{DD}}{\phi_{N,D-1}} \tag{6.2.39}$$

$$Q_o = k \frac{\gamma_{D-1}}{\phi_{N,D-1}} \tag{6.2.40}$$

$$\gamma_{D-j} = \phi_{D,D-j} - \frac{\phi_{DD}\phi_{N,D-j-1}}{\phi_{N,D-1}} \qquad j=1,\cdots,D-1 \tag{6.2.41}$$

$$r_{D-j} = \gamma_{D-j} - \frac{\phi_{N,D-j}\gamma_{D-1}}{\phi_{N,D-1}} \qquad j=2,\cdots,D \tag{6.2.42}$$

$$\phi_{D,o} = 1 \tag{6.2.43}$$

$$\phi_{N,o} = 1 \tag{6.2.44}$$

$$\gamma_o = 1 \tag{6.2.45}$$

A partial fraction expansion of the last term on the RHS of 6.2.38 gives

$$k \frac{\sum_{j=2}^{D} r_{D-j} Z^{D-1}}{\prod_{i=1}^{D-1}\left(1-e^{-\alpha_i T} Z\right)} = \sum_{i=1}^{D-1} \frac{\rho_i}{1-e^{-\alpha_i T} Z} \tag{6.2.46}$$

where the ρ's are calculated from

$$\rho_i = \frac{k \sum_{j=2}^{D-1} r_{D-j}\left(e^{\alpha_i T}\right)^{D-j}}{\prod_{\ell \neq i}^{D-1}\left(1-e^{-\alpha_\ell T} e^{\alpha_i T}\right)} \tag{6.2.47}$$

Equations (6.2.39) through (6.2.47) define the Q's and ρ's and hence complete the analysis of this section.

The equations just derived were programmed on the 7094. Figure 6.2.1 shows curves which were plotted from results of the computer program.

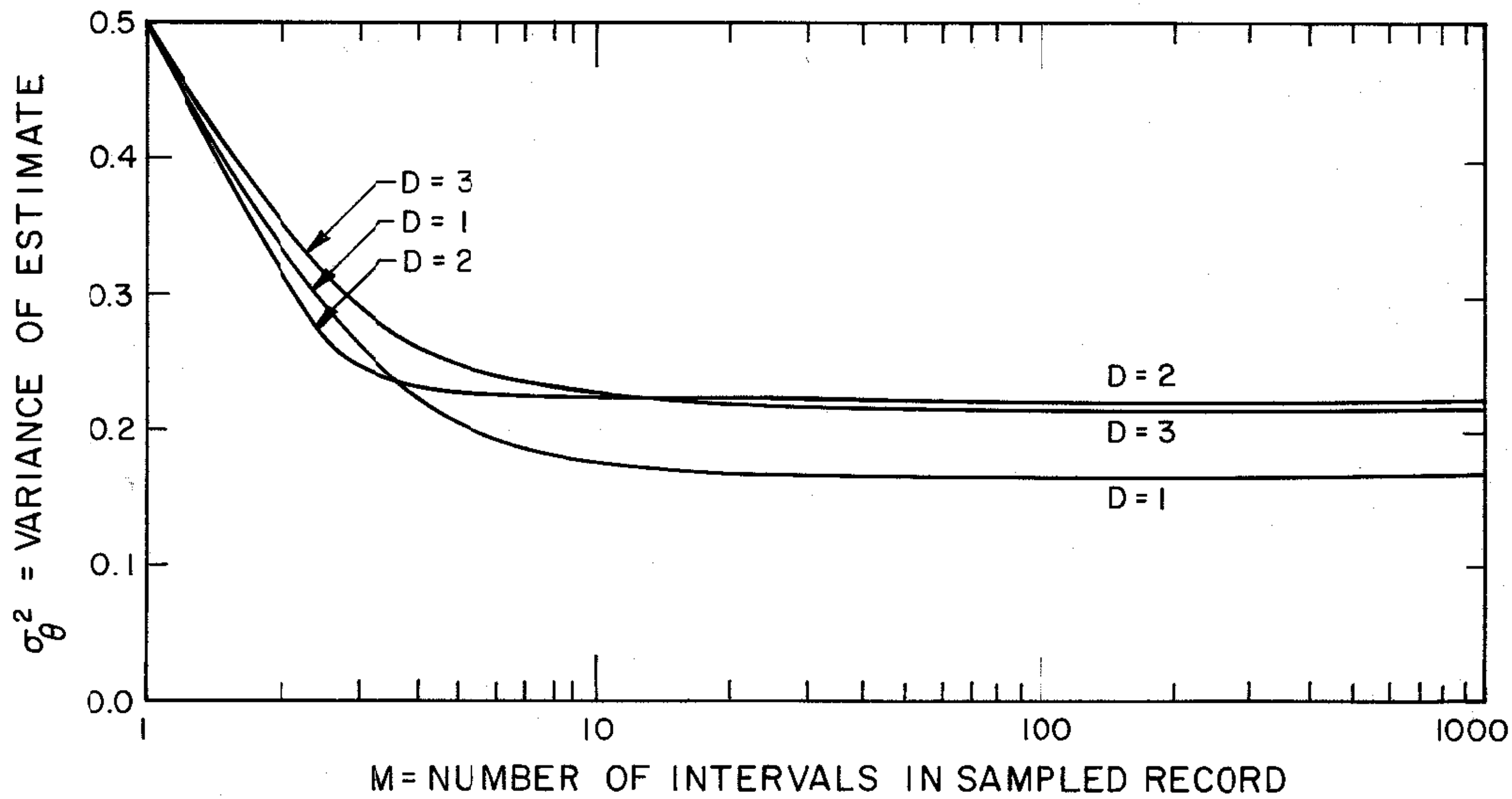


Figure 6.2.1 - Variance of the Estimate of a Constant in Butterworth Noise

CHAPTER VII

MULTIPLE UNCORRELATED DATA SOURCES

7.1 Derivation of the Matrix Integral Equation Satisfied by the Optimum Filter

As an extension of the one dimensional process

$$y(t) = \theta S(t) + n(t) \qquad 0 \le t \le L \tag{1.2.1}$$

consider the vector random process

$$\underset{(\ell x1)}{\bar{y}(t)} = \underset{(\ell x p)}{\phi(t)}\underset{(px1)}{\bar{\gamma}} + \underset{(\ell x1)}{\bar{n}(t)} \tag{7.1.1}$$

where $\bar{\gamma}$ is a (px1) vector of unknown parameters, $\phi(t)$ is a (ℓxp) matrix, and $\bar{n}(t)$ is the $(\ell x1)$ noise vector of the observed $(\ell x1)$ vector process $\bar{y}(t)$. It is desired to find the minimum variance, unbiased, linear estimate of $\bar{\gamma}$ of the form

$$\underset{(px1)}{\hat{\bar{\gamma}}} = \int_0^L \underset{(px\ell)}{Z(t)}\underset{(\ell x1)}{\bar{y}(t)}dt \tag{7.1.2}$$

where $Z(t)$ is a $px\ell$ matrix which is defined as the optimum filter. Substituting 7.1.1 into 7.1.2 gives

$$\hat{\bar{\gamma}} = \left[\int_0^L Z(t)\phi(t)dt\right]\bar{\gamma} + \int_0^L Z(t)n(t)dt \quad , \tag{7.1.3}$$

hence for the estimate to be unbiased, the constraint

$$\int_0^L Z(t)\phi(t)dt = I = \text{ the identity matrix} \tag{7.1.4}$$

must be satisfied. Using the constraint 7.1.4 in 7.1.3 gives

$$\hat{\bar{\gamma}} = \bar{\gamma} + \int_0^L Z(t)n(t)dt \tag{7.1.5}$$

The covariance matrix is defined as $E\{(\hat{\bar{\gamma}}-\bar{\gamma})(\hat{\bar{\gamma}}-\bar{\gamma})'\}$ where "prime" denotes transpose. Therefore from 7.1.5

$$\sum_{\bar{\gamma}} = E\{(\hat{\bar{\gamma}}-\bar{\gamma})(\hat{\bar{\gamma}}-\bar{\gamma})'\} = \int_0^L \int_0^L Z(t)E\{\bar{n}(t)\bar{n}'(\tau)\}Z'(\tau)dtd\tau$$

$$= \int_0^L \int_0^L Z(t)R(t-\tau)Z'(\tau)dtd\tau \tag{7.1.6}$$

where $R(t)$ is the autocorrelation matrix of the noise and is defined by

$$R(t-\tau) = E\{\bar{n}(t)\bar{n}'(\tau)\}$$

The minimum variance estimate is defined as the estimate which minimizes the quadratic form $\bar{\xi}'\sum_{\bar{\gamma}}\bar{\xi}$ for all (pxl) vectors $\bar{\xi}$. Because of the constraint, minimizing $\bar{\xi}'\sum_{\bar{\gamma}}\bar{\xi}$ is equivalent to minimizing

$$Q = \bar{\xi}'\sum_{\gamma}\bar{\xi} + \bar{\xi}'\left\{\left[\int_0^L Z(t)\phi(t)dt - I\right]\lambda + \lambda'\left[\int_0^L \phi'(t)Z'(t)dt - I\right]\right\}\bar{\xi} \quad (7.1.7)$$

where the transpose of the constraint has been added to facilitate determining the final integral equation and the matrix of Lagrange parameters λ is used in this natural generalization of the method of Lagrange parameters.

Using 7.1.6 in 7.1.7 gives

$$Q = \bar{\xi}'\left\{\int_0^L\int_0^L Z(t)R(t-\tau)Z'(\tau)dtd\tau + \left[\int_0^L Z(t)\phi(t)dt - I\right]\lambda + \right.$$

$$\left. + \lambda'\left[\int_0^L \phi'(t)Z'(t)dt - I\right]\right\}\bar{\xi} \quad (7.1.8)$$

Taking variations in 7.1.8 gives

$$\delta Q = \bar{\xi}'\left\{\int_0^L\int_0^L [\delta Z(t)R(t-\tau)Z'(\tau)dtd\tau + Z(t)R(t-\tau)\delta Z'(\tau)]dtd\tau \right.$$

$$\left. + \int_0^L \delta Z(t)\phi(t)dt\lambda + \lambda'\int_0^L \phi'(t)\delta Z'(t)dt\right\}\bar{\xi}$$

$$= \bar{\xi}'\left\{\int_0^L \delta Z(t)\left[\int_0^L R(t-\tau)Z'(\tau)d\tau + \phi(t)\lambda\right]dt + \right.$$

$$\left. + \int_0^L\left[\int_0^L Z(\tau)R(t-\tau)d\tau + \lambda'\phi'(t)\right]\delta Z'(t)dt\right\}\bar{\xi}$$

$$= \bar{\xi}'\left\{\int_0^L \delta Z(t)\left[\int_0^L R(t-\tau)Z'(\tau)d\tau + \phi(t)\lambda\right]dt + \right.$$

$$\left. + \left[\int_0^L \delta Z(t)\left(\int_0^L R(t-\tau)Z'(\tau)d\tau + \phi(t)\lambda\right)dt\right]'\right\}\bar{\xi} \quad (7.1.9)$$

For Q to be stationary, $\delta Q = 0$. A necessary and sufficient condition for $\delta Q = 0$ for all $\delta Z(t)$ is

$$\int_0^L R(t-\tau)Z'(\tau)d\tau = -\phi(t)\lambda \qquad (7.1.10)$$

From 7.1.4, 7.1.6 and the above equation

$$\int_0^L \int_0^L Z(t)R(t-\tau)Z'(\tau)dtd\tau = -\int_0^L Z(t)\phi(t)dt\lambda = -\lambda = \sum_{\overline{\gamma}} \qquad (7.1.11)$$

when $Z(t)$ is the optimum filter.

Therefore the integral equation which must be solved is

$$\int_0^L R(t-\tau)Z'(\tau)d\tau = \phi(t)\sum_{\overline{\gamma}} \qquad (7.1.12)$$

or equivalently

$$\int_0^L Z(\tau)R(t-\tau)d\tau = \sum_{\overline{\gamma}} \phi'(t) \qquad (7.1.13)$$

under the constraint 7.1.4.

Under the assumption that the components of the noise vector are uncorrelated, the matrix $R(t-\tau)$ takes the form

$$R(t-\tau) = \begin{bmatrix} R_1(t-\tau) & & & \\ & R_2(t-\tau) & & \Large 0 \\ & & \ddots & \\ \Large 0 & & & R_\ell(t-\tau) \end{bmatrix} \tag{7.1.14}$$

and a typical equation of the set 7.1.13 is

$$\int_0^L Z_{ij} R_j(t-\tau)\,d\tau = \sum_k \sigma_{ik} \phi_{jk}(t) \tag{7.1.15}$$

For the noises considered in this thesis, Z_{ij} is determined from a linear operation on the right hand side of the integral equation. That is

$$Z_{ij} = L_j \left\{ \sum_k \sigma_{ik} \phi_{jk}(t) \right\} = \sum_k \sigma_{ik} L_j \phi_{jk}(t) \tag{7.1.16}$$

where L_j is the linear operation associated with the autocorrelation function $R_j(t-\tau)$. Equation 7.1.16 expresses the superposition property of integral equations of the form 7.1.15. In matrix notation 7.1.16 becomes

$$Z = \Sigma_{T}\, \Phi' L \tag{7.1.17}$$

where L is a matrix operator which operates on Φ' and is defined in the following manner

$$L = \begin{bmatrix} L_1 & & & \bigcirc \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ \bigcirc & & & L_\ell \end{bmatrix} \qquad (7.1.18)$$

Equation 7.1.17 can be put in a more standard operator form since $L' = L$:

$$Z = \Sigma_{\bar{\gamma}}\, \phi' L = \Sigma_{\bar{\gamma}}\, (L\phi)' \qquad (7.1.19)$$

The covariance matrix $\Sigma_{\bar{\gamma}}$ can be computed from the constraint

$$I = \int_0^L Z(t)\phi(t)dt = \Sigma_{\bar{\gamma}} \int_0^L (L\phi)'\phi dt \qquad (7.1.20)$$

or

$$\Sigma_{\bar{\gamma}}^{-1} = \int_0^L (L\phi)'\phi dt \qquad (7.1.21)$$

Therefore when the components of the noise vector are uncorrelated, the matrix filter and covariance matrix of the estimate can be determined directly from the solution of the one-dimensional integral equation. An example of this is given in the next section.

7.2 Explicit Solution of the Matrix Integral Equation for all Pole Noise

The solution of the integral equation 7.1.13, for the case of uncorrelated components of the noise vector, is specified by 7.1.19 and 7.1.21. This solution can be given explicitly in terms of the elements of the matrix $\sum_{\bar{\gamma}}$ when the Fourier transforms of the individual autocorrelation functions of 7.1.14 have only a denominator polynomial.

The solution of 7.1.13 for all pole noise and uncorrelated components of the noise vector involves a straight forward extension of the results obtained in Reference 8 for the one dimensional case. Therefore, in preparation for the multiple dimensional solution, a brief discription of the results of Reference 8 will be given now. Consider the one-dimensional integral equation

$$\int_0^L Z(\tau)R(t-\tau)d\tau = S(t) \qquad 0 \le t \le L \qquad (7.2.1)$$

where $R(t)$ is the autocorrelation function of a noise whose spectral density is a rational function of frequency having only poles and $S(t)$ is an arbitrary known signal. In Reference 8 it is shown that the solution of 7.2.1 is

$$Z(t) = \sum_{k=0}^{D} a_{2k} S^{(2k)} + \sum_{i=0}^{D-1} \left[f_i \delta^{(i)}(t) + g_i \delta^{(i)}(t-L) \right]^* \qquad (7.2.2)$$

* The superscript (n) indicates n-fold differentiation with respect to time and the quantity $\delta^{(i)}(t)$ is the i^{th} derivative of a delta function.

where

$$f_i = \sum_{k=i}^{D-1} b_{k+1} U_2^{(k-i)}(0) \tag{7.2.3}$$

$$i=0,1,\cdots,D-1$$

$$g_i = \sum_{k=i}^{D-1} (-1)^k b_{k+1} U_1^{(k-i)}(L) \tag{7.2.4}$$

$$U_1(t) = \sum_{k=o}^{D} b_k S^{(k)}(t) \tag{7.2.5}$$

and

$$U_2(t) = \sum_{k=o}^{D} (-1)^k b_k S^{(k)}(t) \tag{7.2.6}$$

The quantities a_{2k} and b_k are defined in terms of the spectral density and will now be determined. If the spectral density $G(\omega)$ is rational in frequency and contains only poles ($2D$ in number), it can be expressed in the form

$$G(\omega) = \frac{1}{a_o - a_2\omega^2 + a_4\omega^4 - \cdots \pm a_{2D}\omega^{2D}} \tag{7.2.7}$$

Such a noise could have been produced by passing white noise of unit spectral density through a filter whose transfer function $H(S)$ has D poles,

$$H(S) = \frac{1}{b_o+b_1S+\cdots+b_DS^D} \tag{7.2.8}$$

The a's and b's are related through the well known relation

$$G(\omega) = |H(j\omega)|^2 = H(j\omega)H(-j\omega) \tag{7.2.9}$$

Therefore from 7.2.2 - 7.2.6 it can be seen that the solution of 7.2.1 can be expressed

$$Z(t) = LS(t) \tag{7.2.10}$$

where L is a linear operator.

To illustrate the use of the above equations a simple example will be worked.

Example 1: Consider 7.1.13 in the one-dimensional case,

$$\int_0^L Z(\tau)R(t-\tau) = \sigma_\gamma^2 S(t) \quad , \tag{7.2.11}$$

where

$$R(t) = \sigma^2 e^{-\beta|t|} \tag{7.2.12}$$

The spectral density corresponding to the autocorrelation function 7.2.12 is

$$G(\omega) = \int_{-\infty}^{\infty} \sigma^2 e^{-\beta|t|} e^{-j\omega t} dt$$

$$= \sigma^2 \frac{2\beta}{\omega^2+\beta^2} = \frac{1}{\frac{1}{2\sigma^2}\left(\beta+\frac{\omega^2}{\beta}\right)} \tag{7.2.13}$$

$$= \frac{1}{\frac{1}{\sqrt{2}\sigma}\left(\sqrt{\beta}+\frac{j\omega}{\sqrt{\beta}}\right)} \frac{1}{\frac{1}{\sqrt{2}\sigma}\left(\beta-\frac{j\omega}{\sqrt{\beta}}\right)} \tag{7.2.14}$$

and therefore from 7.2.9

$$H(s) = \frac{1}{\frac{1}{\sqrt{2}\sigma}\left(\sqrt{\beta}+\frac{s}{\sqrt{\beta}}\right)} \qquad (D=1) \tag{7.2.15}$$

From 7.2.7, 7.2.8, 7.2.13, and 7.2.15 the a's and b's can be determined and the results are

$$a_o = \frac{\beta}{2\sigma^2}$$

$$a_2 = -\frac{1}{2\sigma^2\beta} \tag{7.2.16}$$

$$b_o = \sqrt{\frac{\beta}{2}}\ \frac{1}{\sigma}$$

$$b_1 = \frac{1}{\sqrt{2\beta}\sigma}$$

From 7.2.5, 7.2.6, 7.2.3, and 7.2.24

$$U_1(t) = \sqrt{\frac{\beta}{2}}\,\frac{1}{\sigma}\,S(t) + \frac{1}{\sqrt{2\beta}\sigma}\,\frac{dS}{dt} \qquad (7.2.17)$$

$$U_2(t) = \sqrt{\frac{\beta}{2}}\,\frac{1}{\sigma}\,S(t) - \frac{1}{\sqrt{2\beta}\sigma}\,\frac{dS}{dt} \qquad (7.2.18)$$

$$f_o = b_1 U_2(o) = \frac{1}{2\sigma^2}\left[S(o) - \frac{1}{\beta}\,S'(o)\right]$$

$$= \frac{1}{2\sigma^2}\left[1 - \frac{1}{\beta}\,\frac{d}{dt}\right] S(o) \qquad (7.2.19)$$

$$g_o = b_1 U_1(L) = \frac{1}{2\sigma^2}\left[1 + \frac{1}{\beta}\,\frac{d}{dt}\right] S(L) \qquad (7.2.20)$$

Therefore from 7.2.2 the solution for $Z(t)$ is

$$\frac{1}{\sigma_\gamma^2}\,Z(t) = \frac{1}{2\sigma^2}\left[\beta - \frac{1}{\beta}\,\frac{d^2}{dt^2}\right] S(t) + \frac{1}{2\sigma^2}\,\delta(t)\left[1 - \frac{1}{\beta}\,\frac{d}{dt}\right] S(o) +$$

$$+ \frac{1}{2\sigma^2}\,\delta(t-L)\left[1 + \frac{1}{\beta}\,\frac{d}{dt}\right] S(L) \qquad (7.2.21)$$

where σ_γ^2 is determined from the constraint

$$\int_o^L Z(t)S(t)dt = 1 \qquad (7.2.22)$$

To illustrate the multiple-dimension solution specified by 7.1.19 and 7.1.21 a two dimensional example will now be worked.

Example 2. Consider the random process

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \chi_0 \\ \dot{\chi}_1 \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix} \qquad 0 \le t \le L \tag{7.2.23}$$

or in the notation of 7.1.1

$$\overline{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} ; \quad \phi = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} ; \quad \overline{n}(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix} \tag{7.2.24}$$

It is desired to estimate the quantities χ_0 and $\dot{\chi}_0$ which can be thought of as an initial position and rate of an unaccelerated particle. In accordance with 7.1.14 let

$$R_1(t) = \sigma_1^2 e^{-\beta_1|t|} \tag{7.2.25}$$

$$R_2(t) = \sigma_2^2 e^{-\beta_2|t|} \tag{7.2.26}$$

and therefore from the results of Example 1 and 7.1.19

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$= \sum_{\gamma} \left\{ \begin{bmatrix} \frac{1}{2\sigma_1^2}\left(\beta_1 - \frac{1}{\beta_1}\frac{d^2}{dt^2}\right) & 0 \\ 0 & \frac{1}{2\sigma_2^2}\left(\beta_2 - \frac{1}{\beta_2}\frac{d^2}{dt^2}\right) \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} + \right.$$

$$\begin{bmatrix} \frac{\delta(t)}{2\sigma_1^2}\left(1 - \frac{1}{\beta_1}\frac{d}{dt}\right)_{t=o} & 0 \\ 0 & \frac{\delta(t)}{2\sigma_2^2}\left(1 - \frac{1}{\beta_2}\frac{d}{dt}\right)_{t=o} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\left. + \begin{bmatrix} \frac{\delta(t-L)}{2\sigma_1^2}\left(1 + \frac{1}{\beta_1}\frac{d}{dt}\right)_{t=L} & 0 \\ 0 & \frac{\delta(t-L)}{2\sigma_2^2}\left(1 + \frac{1}{\beta_2}\frac{d}{dt}\right)_{t=L} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right\}' \qquad (7.2.27)$$

It should be noted that the operators defined in 7.1.18 are

$$L_i = \frac{1}{2\sigma_i^2}\left(\beta_i - \frac{1}{\beta_i}\frac{d^2}{dt^2}\right) + \frac{\delta(t)}{2\sigma_i^2}\left(1 - \frac{1}{\beta_i}\frac{d}{dt}\right)_{t=o} + \frac{\delta(t-L)}{2\sigma_i^2}\left(1 + \frac{1}{\beta_i}\frac{d}{dt}\right)_{t=L} \qquad i=1,2$$

(7.2.28)

After performing the indicated differentiations 7.2.27 becomes

$$Z = \frac{1}{2}\sum_{\bar{\gamma}}\left\{\begin{bmatrix} \frac{\beta_1}{\sigma_1^2} & \frac{\beta_1 t}{\sigma_1^2} \\ 0 & \frac{\beta_2}{\sigma_2^2} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma_1^2} & -\frac{1}{\beta_1\sigma_1^2} \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix}\delta(t) + \begin{bmatrix} \frac{1}{2\sigma_1^2} & \frac{L+\frac{1}{\beta_1}}{2\sigma_1^2} \\ 0 & \frac{1}{2\sigma_2^2} \end{bmatrix}\delta(t-L)\right\}'$$

(7.2.29)

It is now necessary to compute $\sum_{\bar{\gamma}}$. From the constraint,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \int_o^L Z(t)\phi(t)dt$$

or equivalently

$$I = \int_o^L \phi'(t)Z'(t)dt \qquad (7.2.30)$$

From 7.2.24 and 7.2.29

$$I = \frac{1}{2}\int_0^L \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \left\{ \begin{bmatrix} \frac{\beta_1}{\sigma_1^2} & \frac{\beta_1 t}{\sigma_1^2} \\ 0 & \frac{\beta_2}{\sigma_2^2} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-1}{\beta\sigma_1^2} \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix} \delta(t) + \begin{bmatrix} \frac{1}{2\sigma_1^2} & \frac{L+\frac{1}{\beta_1}}{2\sigma_1^2} \\ 0 & \frac{1}{2\sigma_2^2} \end{bmatrix} \delta(t-L) \right\} \Sigma_{\bar{\gamma}}$$

$$= \frac{1}{2}\left\{ \begin{bmatrix} \frac{1}{\sigma_1^2}\left(\beta_1 L+2\right) & \frac{L}{2\sigma_1^2}\left(2+\beta_1 L\right) \\ \frac{L}{2\sigma_1^2}\left(\beta_1 L+2\right) & \frac{L}{\sigma_1^2}\left(\frac{1}{3}\beta_1 L^2+L+\frac{1}{\beta_1}\right) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sigma_2^2}\left(\beta_2 T+2\right) \end{bmatrix} \right\} \Sigma_{\bar{\gamma}}$$

or

$$\Sigma_{\bar{\gamma}} = 2 \begin{bmatrix} \frac{1}{\sigma_1^2}\left(2+\beta_1 L\right) & 0 \\ \frac{L}{2\sigma_1^2}\left(2+\beta_1 L\right) & \frac{L}{\sigma_1^2}\left(\frac{1}{3}\beta_1 L^2+L+\frac{1}{\beta_1}\right)+\frac{1}{\sigma_2^2}\left(\beta_2 T+2\right) \end{bmatrix}^{-1} \qquad (7.2.31)$$

7.3 Discrete Minimum Variance Estimation

Again consider the vector random process

$$\underset{(\ell x1)}{\bar{y}(t)} = \underset{(\ell xp)}{\phi(t)}\underset{(px1)}{\bar{\gamma}} + \underset{(\ell x1)}{\bar{n}(t)} \qquad 0 \le t \le L \qquad (7.3.1)$$

or

$$y_i(t) = \sum_{j=1}^{p} \phi_{ij}(t)\gamma_j + n_i(t) \quad i=1,\cdots,\ell \qquad (7.3.2)$$

If this process is sampled at the equally spaced times $t_1, t_2, \cdots, t_N$, then the relations between the observations, unknown parameters, and noise points are

$$y_i(t_k) = \sum_{j=1}^{p} \phi_{ij}(t_k)\gamma_j + n_i(t_k) \qquad \begin{matrix} i=1,\cdots,\ell \\ k=1,\cdots,N \end{matrix} \qquad (7.3.3)$$

Equation 7.3.3 can be expressed in matrix form in the following manner.

$$
\begin{bmatrix} y_1(t_1) \\ \vdots \\ y_1(t_N) \\ \hdashline \vdots \\ \hdashline y_\ell(t_1) \\ \vdots \\ y_\ell(t_N) \end{bmatrix}
=
\begin{bmatrix} \phi_{11}(t_1) & \phi_{12}(t_1) & \cdots & \phi_{1p}(t_1) \\ \vdots & \vdots & & \vdots \\ \phi_{11}(t_N) & \phi_{12}(t_N) & \cdots & \phi_{1p}(t_N) \\ \hdashline \vdots & \vdots & & \vdots \\ \hdashline \phi_{\ell 1}(t_1) & \phi_{\ell 2}(t_1) & \cdots & \phi_{\ell p}(t_1) \\ \vdots & \vdots & & \vdots \\ \phi_{\ell 1}(t_N) & \phi_{\ell 2}(t_N) & & \phi_{\ell p}(t_N) \end{bmatrix}
\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{bmatrix}
+
\begin{bmatrix} n_1(t_1) \\ \vdots \\ n_1(t_N) \\ \hdashline \vdots \\ \hdashline n_\ell(t_1) \\ \vdots \\ n_\ell(t_N) \end{bmatrix}
\tag{7.3.4}
$$

where the partitions are made relative to the noise points which are correlated. Upon making the definitions

$$
\overline{Y}_i = \begin{bmatrix} y_i(t_1) \\ \vdots \\ y_i(t_N) \end{bmatrix} ; \quad \Phi_k = \begin{bmatrix} \phi_{k1}(t_1) \phi_{k2}(t_1) & \cdots & \phi_{kp}(t_1) \\ \vdots \quad \vdots & & \vdots \\ \phi_{k1}(t_N) \phi_{k2}(t_N) & \cdots & \phi_{kp}(t_N) \end{bmatrix} ; \quad \overline{N}_r = \begin{bmatrix} n_r(t_1) \\ \vdots \\ n_r(t_N) \end{bmatrix}
\tag{7.3.5}
$$

$$
\overline{Y} = \begin{bmatrix} \overline{Y}_1 \\ \vdots \\ \overline{Y}_\ell \end{bmatrix} ; \quad A = \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_\ell \end{bmatrix} ; \quad \overline{N} = \begin{bmatrix} N_1 \\ \vdots \\ N_\ell \end{bmatrix} , \tag{7.3.6}
$$

the relations 7.3.4 can be written

$$\bar{Y} = A\bar{\gamma} + \bar{N} \tag{7.3.7}$$

The minimum variance estimate of the vector $\bar{\gamma}$ is (7)

$$\bar{\gamma} = (A'R^{-1}A)^{-1} A'R^{-1}\bar{Y} \tag{7.3.8}$$

where

$$R = E\{\bar{N}\bar{N}'\} = \begin{bmatrix} R_{11} & & \bigcirc \\ & \ddots & \\ \bigcirc & & R_{\ell\ell} \end{bmatrix} \text{ and } R^{-1} = \begin{bmatrix} R_{11}^{-1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & R_{\ell\ell}^{-1} \end{bmatrix} \tag{7.3.9}$$

and the quantity $(A'R^{-1}A)^{-1} A'R^{-1}$ is the matrix weighting function.

The inverse covariance matrix of the estimate is

$$\Sigma_{\bar{\gamma}}^{-1} = A'R^{-1}A \tag{7.3.10}$$

or from 7.3.6 and 7.3.9

$$\Sigma_{\bar{\gamma}}^{-1} = \phi_1'R_{11}^{-1}\phi_1 + \cdots + \phi_\ell'R_{\ell\ell}^{-1}\phi_\ell = \sum_{k=1}^{\ell} \phi_k'R_{kk}^{-1}\phi_k \tag{7.3.11}$$

The matrix weighting function is

$$Z = \sum\nolimits_{\gamma} A' R^{-1} \qquad (7.3.12)$$

$$= \sum\nolimits_{\gamma} \left[\phi_1' R_{11}^{-1} \,\vdots\, \phi_2' R_{22}^{-1} \,\vdots\, \cdots \,\vdots\, \phi_r' R_{rr}^{-1} \right] \qquad (7.3.13)$$

Therefore as can be seen from the form of 7.3.11 and 7.3.13, the formulas which have been derived in this thesis for the inverse covariance matrix can be applied directly to the multiple dimension minimum variance problem.

The limiting form of the partitioned components of 7.3.13 can be deduced, in the case of second order noise, from the results of Chapter III. That this is so can be seen by expanding a typical component of 7.3.13. Consider

$$\left(\phi_k' R_{kk}^{-1}\right)_{ij} = \sum_r \phi_{ki}(t_r) \left(R_{kk}^{-1}\right)_{rj} \qquad (7.3.14)$$

which is the ij^{th} element of the product $\phi_k' R_{kk}^{-1}$. Equation 7.3.14 is of the form

$$\sum_j \left(R^{-1}\right)_{ij} S(t_i)$$

which is the form of the sum whose limit was found in Chapter III. Therefore the limiting form of 7.3.13 can be deduced directly from the results of Chapter III.

The limiting form of 7.3.11 can also be evaluated from the results of Chapter III. A typical component of 7.3.11 is

$\phi_k' R_{kk}^{-1} \phi_k$, or expanded

$$\left(\phi_k' R_{kk}^{-1} \phi_k\right)_{ij} = \sum_q \sum_r \phi_{ki}(t_r)\left(R_{kk}^{-1}\right)_{rq} \phi_{kj}(t_q)$$

$$= \sum_q \phi_{kj}(t_q)\left\{\sum_r \phi_{ki}(t_r)\left(R_{kk}^{-1}\right)_{rq}\right\} \tag{7.3.15}$$

But the limiting form of the sum in curly brackets is known (see Chapter III) and inspection of the form shows that the limiting form of 7.3.15 can be expressed

$$\sum_q \phi_{kj}(t_q) L_k \phi_{ki}(t_q) T \rightarrow \int_0^L \phi_{kj}(t) L_k \phi_{ki}(t) dt \tag{7.3.16}$$

where T is the time between samples and L_k is the linear operator relative to the k^{th} noise (see Section 7.1).

From 7.3.16 the ij term of 7.3.11 is $\sum_k \int_0^L \phi_{kj}(t) L_k \phi_{ki}(t) dt$, but this is precisely the ij term of 7.1.21, therefore the limiting form of the inverse covariance matrix of the discrete estimate approaches that of the continuous estimate.

CHAPTER VIII

FINIS

8.1 Summary and Conclusions

In this thesis analytic formulas were derived for the elements of the inverse covariance matrix of sampled rational noise. It was shown that the number of terms composing these formulas was dependent only on the order of the noise and not on the dimension of the covariance matrix. It was seen that the calculation of numerical results with these formulas involves at most the solution of a $(D-1)^{th}$ order polynomial, $D-1$ quadratic equations, and the inversion of a matrix of dimension $D-1$, where D is the order of the noise. As examples of the use of the analytic formulas, inverse covariance matrices were derived for noises with the autocorrelation functions: $\sigma^2 e^{-\beta|t|}$ and $\sigma_1^2 e^{-\beta_1|t|} + \sigma_2^2 e^{-\beta_2|t|}$. The inverse covariance matrix corresponding to the autocorrelation function $e^{-a|t|}\cos bt$, which is a special case of $\sigma_1^2 e^{-\beta_1|t|} + \sigma_2^2 e^{-\beta_2|t|}$, was also discussed.

The estimation of the parameter θ in the process

$$y(t) = \theta S(t) + n(t) \qquad 0 \le t \le L \qquad (8.1.1)$$

was considered in detail for first and second order noise. A minimum variance continuous filter, $f(t)$, which gives an estimate of θ through the integral relation

$$\hat{\theta} = \int_{0}^{L} f(t)y(t)dt \qquad (8.1.2)$$

and its associated variance were computed. Also computed was a discrete minimum variance estimate of the form

$$\hat{\theta} = \sum_{\mu=0}^{M} f_d(\mu T)y(\mu T) \qquad (8.1.3)$$

where the $f_d(\cdot)$ are the optimum "weights" of the sampled data. It was then shown that the discrete weighting function and its associated variance approached the continuous weighting function and its variance when the density of observations approached infinity. The second order, all pole noise solution exposed the fact that in general the discrete weighting function does not create the equivalent of a delta function and its derivatives by a simple differencing operation through the use of Kronecker deltas. To identify the equivalent of a delta function and its derivative it was necessary to expand 8.1.3 in a power series and collect the terms that were of zero order in μT. It was seen that the function of μT,

$$e^{-\alpha T\mu} = \left\{-(2-\sqrt{3})[1+0(T^2)]\right\}^{\mu} \qquad 0 \le \mu \le M$$

became the equivalent of a delta function in much the same way that

$$\lim_{\alpha\to\infty} e^{-\alpha|t|}\cos \alpha t$$

becomes a delta function.

The asymptotic properties of the variance of estimates of the form 8.1.3 were considered next. The asymptotic term was defined as the first order term in the power series expansion

$$\sigma_\theta^2 = \sigma_L^2 \left(1 + a_1 T + a_2 T^2 + \cdots \right) \tag{8.1.4}$$

where σ_L^2 is the variance that would be obtained with an infinite density of observations. It was seen that for a smooth $S(t)$ and first order noise, the coefficient, a_1 , was zero. This indicates that the variance of the discrete estimate approaches the variance of the continuous estimate very rapidly as the time between samples approaches zero. In the special case of a constant signal and second order noise it was shown that a_1 was zero if the noise had zeros in its spectral density and nonzero if the noise was all pole.

The connection between the historically significant autoregressive noise and rational noise was considered in detail for second order noise. It was shown that rational noise will have autoregressive properties only for a special pole-zero configuration and a particular sampling rate. This rate was designated a degenerate rate. The proposal was made that if the conditions were such that a degenerate rate exists, then samples should be taken at that rate because of the major simplification of the filters that occurs when the rate is degenerate. It was seen, in a special case, that the variance obtained at the degenerate rate was within a few percent of the variance obtained at an infinite rate.

As a further example of the usefulness of the analytic formulas for the elements of the inverse covariance matrix, formulas were derived for the variance of the estimate of a constant in Butterworth noise. Some numerical results were presented.

Finally, it was shown that the results obtained for one parameter, one signal, and one noise could be easily extended to a vector of parameters, a matrix of signals, and a vector of noises. The only restriction was that the components of the noise vector be uncorrelated.

8.2 Suggestions for Further Study

An area of investigation which was exposed in this thesis was that of exploring the connection between degenerate rational noise and autoregressive noise. This exploration was considered only in the special case of second order processes. What is needed is a general method of determining the pole-zero locations of the rational filter which, with white noise as an input and a properly chosen sampling rate, will give an autoregressive process. As was pointed out, the numerical aspects of the estimation problem are greatly simplified when sampling takes place at a degenerate rate. Formulas can be derived which express the ratio of the variance obtained when samples are taken at the degenerate rate to the variance obtained with an infinite rate. Any statements that could be made concerning the conditions under which this ratio is close to unity would be very useful. Of course, in any particular case, numerical results can be obtained.

In Chapter II it was shown that the two-sided Z-transform of the sampled autocorrelation function could be expressed

$$\phi^*(Z) = \frac{\phi_N(Z)\phi_N\left(Z^{-1}\right)}{\phi_D(Z)\phi_D\left(Z^{-1}\right)} \tag{8.2.1}$$

where N is the order of the polynomial $\phi_N(Z)$, and $N \leq D-1$. It is of interest to determine the pole-zero configurations for which sampling rates exist such that N takes on values less than D-1. This knowledge would be useful in the programming of the formulas for the elements of the inverse covariance matrix on a digital computer.

It has been noted that in order to compute numerical values for the elements of the inverse covariance matrix it is usually necessary to invert a matrix of order D-1 and solve a $(D-1)^{th}$ order polynomial. An explicit solution for the roots of the polynomial and/or the elements of the $(D-1)^{th}$ order inverse matrix would simplify numerical calculations for large D.

Finally, analytic formulas for the elements of the inverse covariance matrix of sampled, band-limited noise would be useful in theoretical studies. It has been noted in Reference 8 that a continuous filter gives an estimate which has zero variance when the noise is band-limited. Therefore it is of theoretical interest to determine the manner in which the discrete filter and its variance approaches the continuous filter and its variance.

APPENDIX A

AN EXPANSION OF ALL-POLE SECOND ORDER NOISE

An expansion of all-pole second order noise is given in this appendix. This expansion will then be used to derive Equation 3.2.102 of Chapter III:

$$\lim_{M\to\infty} \sum_{0}^{M} y_\mu e^{-\alpha T\mu} \approx y(o)\sum\nolimits_0 + \dot{y}(o)\sum\nolimits_1 T + 0_r\left(T^3\right) \qquad (A.1)$$

where $0_r(\cdot)$ is a random variable whose variance decreases as its argument. The noise considered in this section can be generated by passing zero mean, white noise through a linear filter with the transfer function (the gain constant has been chosen as unity).

$$\frac{1}{(S+\gamma)(S+\beta)} = \frac{1}{\gamma-\beta}\left[\frac{1}{S+\beta} - \frac{1}{S+\gamma}\right] \qquad (A.2)$$

and hence the impulse response, $h(t)$, is

$$h(t) = \frac{1}{\gamma-\beta}\left[e^{-\beta t} - e^{-\gamma t}\right] \qquad (A.3)$$

Since the filter is linear, the output noise, $n(t)$, can be expressed in terms of the impulse response by the equation

$$n(t) = \frac{1}{\gamma-\beta}\left\{[n'(o)+\gamma n(o)]e^{-\beta t} - [n'(o)+\beta n(o)]e^{-\gamma t}\right\} + \int_0^t h(t-\tau)W(\tau)d\tau$$

$$n(t) \cong n(o) + \dot{n}(o)t + 0\left(t^2\right) + \int_0^t h(t-\tau)W(\tau)d\tau \qquad (A.4)$$

where $W(t)$ is the white noise.

The last term on the right hand side of A.4 will now be examined. Let

$$z(t) = \int_0^t h(t-\tau)W(\tau)d\tau \quad , \tag{A.5}$$

then

$$E\{z(t)\} = \int_0^t h(t-\tau)E\{W(\tau)\}d\tau = 0 \tag{A.6}$$

and

$$\sigma_z^2 = E\{z^2(t)\} = E\left\{\int_0^t\int_0^t h(t-\tau)h(t-n)W(\tau)W(n)dnd\tau\right\}$$

$$= \int_0^t\int_0^t h(t-\tau)h(t-n)\delta(\tau-n)dnd\tau = \int_0^t h^2(t-\tau)d\tau$$

$$= \frac{1}{(\gamma-\beta)^2}\int_0^t\left[e^{-2\beta t}-2e^{-(\beta+\gamma)t}+e^{-2\gamma t}\right]dt$$

$$= \frac{1}{(\gamma-\beta)^2}\left\{\frac{1}{2\beta}\left[1-e^{-2\beta t}\right]-\frac{2}{\beta+\gamma}\left[1-e^{-(\beta+\gamma)t}\right]+\frac{1}{2\gamma}\left[1-e^{-2\gamma t}\right]\right\} \tag{A.7}$$

The fact that the autocorrelation function of white noise is a delta function and A.3 were used in deriving A.7.

Expanding A.7 gives

$$\sigma_z^2 \cong \frac{1}{(\gamma-\beta)^2}\left\{\frac{1}{2\beta}\left[1 - \left(1-2\beta t+2\beta^2 t^2 - \frac{8}{6}\beta^3 t^3 + \cdots\right)\right]\right.$$

$$+ \frac{1}{2\gamma}\left[1 - \left(1-2\gamma t+2\gamma^2 t^2 - \frac{8}{6}\gamma^3 t^3 + \cdots\right)\right]$$

$$\left. - \frac{2}{\beta+\gamma}\left[1 - \left[1-(\beta+\gamma)t + \frac{1}{2}(\beta+\gamma)^2 t^2 - \frac{1}{6}(\beta+\gamma)^3 t^3 + \cdots\right]\right]\right\}$$

$$= \frac{1}{3}(\beta-\gamma)t^3 + 0\left(t^4\right) + \cdots \qquad (A.8)$$

From A.8 and the definition of $0_r(\cdot)$, A.4 can be written

$$n_\mu = n(\mu T) \cong n(o) + \mu T\dot{n}(o) + \mu^2 0_r\left(T^3\right) + \mu^3 0_r\left(T^4\right) + \cdots \qquad (A.9)$$

and hence

$$\lim_{M\to\infty} \sum_{\mu=o}^{M} n_\mu e^{-\alpha T\mu} = n(o)\sum\nolimits_o + \dot{n}(o)\sum\nolimits_1 T + 0_r\left(T^3\right)\sum\nolimits_2 + 0_r\left(T^4\right)\sum\nolimits_3 + \cdots \qquad (A.10)$$

where

$$\sum\nolimits_i = \sum_{\mu=o}^{\infty} \mu^i e^{-\alpha T\mu} \qquad (A.11)$$

$$e^{-\alpha T} = -(2-\sqrt{3})\left[1 + 0\left(T^2\right)\right]$$

Since

$$y(t) = \theta S(t) + n(t) \tag{A.12}$$

and

$$S(t) = S(o) + \dot{S}(o)t + \cdots$$

then

$$\lim_{M\to\infty} \sum_{o}^{M} y_{\mu} e^{-\alpha T\mu} = y(o) + \dot{y}(o) \sum\nolimits_{1} T + O_{r}\left(T^{3}\right) \tag{A.13}$$

where $y_{\mu} \equiv y(\mu T)$. Equation A.13 is the desired expansion.

APPENDIX B

DISCUSSION OF AN EQUIVALENT DETECTION THEORY PROBLEM

In this appendix a brief discussion of maximum likelihood detection is given for the case of additive gaussian noise. This discussion follows the discussion of Reference 8 very closely. It will be shown that the results of this thesis apply directly to the solution of the problem defined in Reference 8.

Let the random process

$$y(t) = \begin{cases} n(t) \\ \\ S(t) + n(t) \end{cases} \qquad 0 \le t \le L \qquad \text{(B.1)}$$

be observed at the $M+1$ equally spaced instants of time $0, T, 2T, \cdots, MT$. It is desired to determine in an optimum manner (which will be defined shortly) from the sampled observations if the noise, $n(t)$, alone is present or if the signal, $S(t)$, plus the noise is present. The assumption is usually made that the noise is gaussian with zero mean and that it has a rational power spectral density.

Because of the assumption of a rational power spectral density the noise has an autocorrelation function of the form

$$\phi(t) = \sum_{k=1}^{D} \sigma_k^2 e^{-\beta_k |t|} \qquad \text{(2.1.1)}$$

and hence

$$\phi_{\mu m} \equiv E\{n(\mu T)n(mT)\} = \phi[(m-\mu)T] \tag{B.2}$$

where $\phi_{\mu m}$ is the μ,m^{th} element of the covariance matrix of the sampled noise. The notation $W_{\mu}(mT)$ will be used to denote the elements of the inverse covariance matrix of the sampled noise. From the assumption of zero mean gaussian noise the density function of the samples of $y(t)$, when the process is composed of noise alone, can be expressed through

$$f_{yn}(y_o, \cdots, y_M) = f_n(y_o, \cdots, y_M) \tag{B.3}$$

and

$$f_n(N_o, \cdots, n_M) = (2\pi)^{-(M+1)}|W|^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\sum_{\mu,m} W_{\mu}(mT)n_{\mu}n_m\right\} \tag{B.4}$$

where $|W|$ is the determinant of the inverse covariance matrix and $n_{\mu} = n(\mu T)$. The density function of the samples of $y(t)$ when both the signal and the noise are present is

$$f_{ysn}\left(y_o, \cdots, y_M\right) = (2\pi)^{-(M+1)}|W|^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\sum_{\mu,m} W_{\mu}(mT)\left(y_{\mu}-S_{\mu}\right)\left(y_m-S_m\right)\right\} \tag{B.5}$$

where $S_{\mu} = S(\mu T)$.

The likelihood ratio $L\left(y_o , \cdots , y_M\right)$ is defined as

$$L\left(y_o, \cdots , y_M\right) = \frac{f_{ysn}\left(y_o,\cdots,y_M\right)}{f_{yn}\left(y_o,\cdots,y_M\right)}$$

$$= \exp\left\{- \frac{1}{2} \sum_{\mu,m} W_\mu(mT)S_\mu S_m\right\}\exp\left\{\sum_{\mu,m} W_\mu(mT)S_\mu y_m\right\} \qquad (B.6)$$

As mentioned in Reference 8, a maximum likelihood detector, which is defined as the optimum detector, says that the signal is present if the test statistic $L\left(y_o,\cdots,y_M\right)$ is greater than some threshold α and will maximize the conditional probability of detecting a signal when it is present for a given conditional probability of indicating signal for noise alone. Since $L\left(y_o,\cdots,y_M\right)$ is a monotonic function of the statistic ψ,

$$\psi = \sum_{\mu,m} W_\mu(mT)S_\mu y_m \quad , \qquad (B.7)$$

an equally good test is $\psi > \alpha_c$, where α_c is an equivalent threshold. The quantity ψ is characterized by two density functions, one if the random process is noise alone, the other if the process is signal plus noise. For noise alone, ψ_n (the subscript "n" designates noise alone, "sn" signal plus noise) is gaussian with zero mean and variance,

$$E\{\psi_n^2\} = \sum_{\mu,m} W_\mu(mT)S_\mu S_m \qquad (B.8)$$

If signal and noise are present, ψ_{sn} is also gaussian with the variance given by B.8 but with mean

$$E\{\psi_{sn}\} = \sum_{\mu,m} W_{\mu}(mT)S_{\mu}S_{m} \qquad (B.9)$$

As discussed in Reference 8 the effectiveness of the maximum likelihood detector can be characterized by the detectability d, defined by

$$d = \sum_{\mu,m} W_{\mu}(mT)S_{\mu}S_{m} \qquad (B.10)$$

Comparing the above equation with Equation 1.2.3 shows that the reciprocal of the variance of the minimum variance estimate and the detectability are analogous. Similarly, a comparison of B.7 with the minimum variance estimate of a parameter θ (see 3.1.1)

$$\hat{\theta} = \sigma_{\theta}^{2} \sum_{\mu,m} W_{\mu}(mT)S_{\mu}y_{m}$$

shows that the test statistic ψ and θ differ only by a constant or equivalently $\frac{1}{\sigma_{\theta}^{2}}\hat{\theta}$ is analogous to ψ. The preceeding comparisons show that the results of the previous chapters apply not only to minimum variance estimators but also to maximum likelihood detectors of signals in gaussian noise. In particular, it has been explicitly demonstrated in the case of second order noise that the limiting form of B.7 is

$$\psi = \int_0^L f(t)y(t)dt \tag{B.11}$$

where $f(t)$ satisfies the integral equation

$$\int_0^L \phi(t-u)f(u)du = S(t) \qquad 0 \leq t < L \tag{B.12}$$

Also it has been demonstrated that in the limit of $M \to \infty$, $T \to 0$ that the minimum variance approaches

$$\frac{1}{\sigma_\theta^2} \to \int_0^L f(t)S(t)dt \tag{B.13}$$

and hence because of the analogy between d and $\frac{1}{\sigma_\theta^2}$

$$d \to \int_0^L f(t)S(t)dt \quad . \tag{B.14}$$

This concludes the discussion of detection problems.

REFERENCES

1. Wiener, N.: Extrapolation, Interpolation, and Smoothing of Stationary Time Series, John Wiley, New York, 1949.

2. Kolmogorov, A. N.: "Interpolation and Extrapolation", Bull. Acad. Sci. U.S.S.R., ser. math., 5: 3-14 (1944).

3. Kalman, R. E.: "A New Approach to Linear Filtering and Prediction Problems," Journal of Basic Engineering, vol. 82, no. 1, March 1960.

4. Kalman, R. E., and R. S. Bucy: "New Results in Linear Filtering and Prediction Theory," Journal of Basic Engineering, vol. 83, no. 1, March 1961.

5. Zadeh, L. A., and J. R. Ragazzini: "An Extension of Wiener's Theory of Prediction," Journal of Applied Physics, 21, pp. 645-655, July 1950.

6. Davenport, W. B., and W. L. Root, An Introduction to the Theory of Random Signals and Noise, McGraw-Hill Book Company, Inc., New York, 1958, Chapt. 14.

7. Grenander, U., and M. Rosenblatt: Statistical Analysis of Stationary Time Series, Wiley, New York, 1957.

8. Martel, H. C., and M. V. Mathews: "Further Results on the Detectability of Known Signals in Gaussian Noise," The Bell System Technical Journal, Vol. XL, no. 2, March 1961.

9. Siddiqui, M. M.: "On the Inversion of the Sample Covariance Matrix in a Stationary Autoregressive Process," Annals of Math. Statistics, vol. 129, pp. 585-588, June 1958.

10. Swerling, P.: "Optimum Linear Estimation for Random Processes as the Limit of Estimates Based on Sampled Data," 1958 IRE WESCON Convention Record, pt. 4; pp. 158-163.

11. Janos, W. A.: "Analytic Inversion of a Class of Covariance Matrices," IRE Transactions on Information Theory, September 1960.