FLUID MECHANICS OF GAS - SOLID PARTICLE FLOW IN BOUNDARY LAYERS

Thesis by

Robert Edmund Singleton

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ABSTRACT

It is shown that the boundary layer approximation is applicable to two-phase flow over a semi-infinite flat plate and about a circular stationary cylinder, provided the particle density is of the order of ρ in the boundary layer. In the boundary layer equations, the importance of a new parameter, λ_m/x, which is the ratio of the distance required for particle velocity to reach that of the fluid to the length measured downstream from the stagnation point, is pointed out, and expansions are made in terms of this parameter in such a way that a similarity variable can be found for the semi-infinite flat plate. This analysis is carried out for both an incompressible and a compressible gas on the semi-infinite flat plate. New shear coefficients and heat transfer coefficients are derived from this analysis. Also, an integral method is applied to the semi-infinite flat plate when the gas is incompressible to compare with the numerical solution of the same problem. The boundary layer analysis of the stationary circular cylinder demonstrates effects of curvature on two-phase flow.

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I. INTRODUCTION

For many years engineers and scientists have been interested in gas - solid particle flows which arise in many industrial applications. To mention a few, gas - particle flow phenomena are important in sedimentation, pipe flows [12, 20], fluidized beds, gas purification, and transport processes [11, 21]. More recently, the fields of propulsion and combustion [15, 16] have stimulated new interest in the gas - solid particle flow phenomena.

The general problem considered in this thesis is the flow over a solid object of a viscous gas in which there is a distribution of spherical solid particles, all having the same radius. The radius of a particle is in the range of 0.1 to 5.0 microns. A sufficiently high Reynolds number is assumed for the gas flow so that a laminar boundary layer will form on the solid surface. The problem, then, is to analyze the effects of the particles on the boundary layer in regard to velocity profile, shear coefficient, and heat transfer.

Very few publications have been written on the subject of two-phase boundary layers. Both Soo [17] and Marble [2] have developed the conservation equations for two-phase flow in general, and Soo treats the problem of the turbulent two-phase boundary layer on a semi-infinite flat plate. Because of the complexity of turbulent two-phase boundary layers, Soo uses integral methods in his analysis. Chiu [18] treats the laminar two-phase boundary layer on a semi-infinite flat plate, but his boundary layer equations are incorrect since the conservation of particle momentum in the normal direction is neglected. Also, Chiu assumes that the particle density is constant, and his analysis is applicable only when the effects of the particles on the gas are of higher order than the gas inertia or viscous forces. Marble [2] also treats the laminar two-phase boundary layer on a semi-infinite flat plate, assuming that the normal particle velocity is equal to the normal gas velocity within the usual boundary layer assumptions. Also, Marble's analysis is not applicable for the entire length of the plate.

This thesis will treat the problem of a laminar two-phase boundary layer on a semi-infinite flat plate and on a stationary circular cylinder. The change in heat transfer to the wall and in shear at the wall caused by the presence of the particles in the flow will be computed. An integral method will also be applied to the flat plate problem, and the stationary circular cylinder problem will give the effects of curvature on two-phase flow.

In the boundary layer, the gas decelerates from its free stream velocity to zero velocity at the solid surface, but since the density of the particle material is much greater than the gas density, the particles cannot accommodate this rapid deceleration but tend to slip through the gas as they decelerate. The magnitude of the particle slip velocity depends on the region of the boundary layer being considered. Near the stagnation point on the surface, the particles have very large slip velocity, while on the other hand, far downstream of the stagnation point the slip velocity will be small, for in this region the particles have had time to adjust to the boundary layer.

The particles are assumed to be sufficiently dilute and to move at so nearly the same speed that they do not have collisions with each other and that the flow field about each individual particle does not interact with the flow field about any other particle. It is also assumed that there is no radiative heat transfer from one particle to another. The particles are regarded as a continuum since the individual motion of each particle is of no interest here, and it is assumed that the particles have no random motions, and hence the particle phase has no analog of pressure.

II. GENERAL THEORY OF TWO-DIMENSIONAL GAS - PARTICLE BOUNDARY LAYER

Because of the previously mentioned particle slip velocities, there will be a volume force acting on the gas and an equal and opposite force acting on the particle phase. This volume force is assumed to have the form of the Stokes drag law. For large particle Reynolds number, this assumption is erroneous, so it will be assumed that the particle Reynolds number is of order unity. Everywhere this restriction is not met, the results will still be qualitatively correct and quantitatively reasonable. Therefore

$$\underline{F}_p = n_p 6\pi\mu\sigma(\underline{v}_p - \underline{v})$$

where an underline denotes a vector quantity, and

$\underline{F}_p$ = force per unit volume of mixture of both phases acting on the gas ,

n_p = number of particles per unit volume of mixture of both phases ,

μ = coefficient of viscosity,

σ = radius of the particles ,

$\underline{v}_p$ = particle velocity ,

$\underline{v}$ = gas velocity ,

ρ_p = particle density = mass of particle phase per unit volume of mixture of the two phases ,

ρ_s = mass of particle material per unit volume of particle material ,

m = mass of one particle ,

$$n_p = \frac{\rho_p}{m} = \frac{\rho_p}{\frac{4}{3}\pi\sigma^3\rho_s} .$$

Then

$$\underline{F}_p = \frac{\rho_p}{\frac{4}{3}\pi\sigma^3\rho_s} 6\pi\mu\sigma(\underline{v}_p - \underline{v})$$

$$\underline{F}_p = \frac{\rho_p u_\infty(\underline{v}_p - \underline{v})}{\frac{2}{9}(\frac{\rho_s}{\rho})(\frac{\rho u_\infty \sigma}{\mu})\sigma}$$

$$\underline{F}_p = \frac{\rho_p u_\infty}{\lambda_m}(\underline{v}_p - \underline{v})$$

where u_∞ = free stream gas velocity,

$\lambda_m = \tau_m u_\infty = \frac{2}{9}(\frac{\rho_s}{\rho})(\frac{\rho u_\infty \sigma}{\mu})\sigma$ = momentum equilibration length,

$\tau_m = \frac{2}{9}(\frac{\rho_s \sigma^2}{\mu})$ = momentum equilibration time.

Now in the boundary layer the particles find themselves in a shear flow which causes them to rotate, thus giving rise to a lift force acting on the particles in addition to the Stokes drag forces. Unfortunately, the problem of the sphere in a shear flow has not been done; however, in reference 1 the lift force on a sphere which is spinning in a uniform rectilinear flow is discussed, and the following formula for the lift force is obtained:

$$\underline{F}_{pL} = \pi\sigma^3\rho\underline{\Omega}\times(\underline{v}_p - \underline{v})$$

where

$\underline{F}_{pL}$ = lift force

ρ = gas density

$\underline{\Omega}$ = angular velocity of the sphere.

These authors also tried to obtain an expression for the lift force on a sphere in a parabolic velocity profile, but the labor involved became prohibitive, although as far as they carried the analysis, the results for $\underline{F}_{pL}$ are the same as the above formula. Hence, the only form which can be discussed for the lift force per unit volume of mixture is

$$\underline{F}_{pL} = n_p \pi \sigma^3 \rho \tfrac{1}{2} (\nabla \times \underline{v}) \times (\underline{v}_p - \underline{v}) ,$$

and then in two dimensions,

$$\left| \frac{\underline{F}_{pL}}{\underline{F}_p} \right| = \frac{n_p \pi \sigma^3 \rho \frac{1}{2} (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) |\underline{v}_p - \underline{v}|}{n_p 6 \pi \mu \sigma |\underline{v}_p - \underline{v}|}$$

$$= \frac{1}{12} \left(\frac{\rho u_\infty \sigma}{\mu}\right) \left(\frac{\sigma}{\ell}\right) \left(\frac{1}{\sqrt{Re}} \frac{\partial v^*}{\partial x^*} + \sqrt{Re} \frac{\partial u^*}{\partial y^*} \right)$$

where

u, v = x- and y-components of $\underline{v}$

ℓ = some characteristic length

$x^* = x/\ell$

$Re = (\rho u_\infty \ell)/\mu$

$y^* = \frac{y}{\ell} \sqrt{Re}$

$u^* = u/u_\infty$

$v^* = \sqrt{Re} \frac{v}{u_\infty}$.

Then if $(\frac{\rho u_\infty \sigma}{\mu})(\frac{\sigma}{\ell})$ is small enough, we can neglect lift forces compared to Stokes drag forces. In the following work, lift forces are neglected.

In addition to the particle velocity slips, the particles will in general have a different temperature than that of the surrounding gas, and therefore there will be temperature defects as well. Because of these temperature defects, there will be heat transfer between the two phases. The heat transfer from a particle to the gas has the form

$$q_p = k(4\pi\sigma^2)\frac{(T_p - T)}{\sigma} Nu ,$$

where

Nu = Nusselt number ,

T = gas temperature ,

k = conductivity of the gas ,

T_p = temperature of the particle ,

q_p = heat transferred.

Under the same conditions as for the validity of the Stokes drag law, i. e., Reynolds number of order unity, the Nusselt number can be taken equal to one, provided the Prandtl number is also of order unity. Then taking

$$Nu = 1 ,$$

the total heat transfer to the gas per unit volume of mixture of the two phases is

$$Q_p = n_p k(4\pi\sigma^2)\left(\frac{T_p - T}{\sigma}\right)$$

$$Q_p = \rho_p C_s \frac{(T_p - T)}{\tau_T}$$

where

Q_p = total heat transferred ,

C_s = specific heat of the solid particles ,

C_p = specific heat at constant pressure for the gas,

$Pr = (C_p\mu)/k$ = Prandtl number,

$\lambda_T = \frac{1}{3}Pr(\frac{C_s}{C_p})(\frac{\rho_s}{\rho})(\frac{u_\infty \sigma^2}{\nu}) = \tau_T u_\infty$ = thermal equilibration length,

$\tau_T = \frac{3}{2}Pr\frac{C_s}{C_p}\tau_m$ = thermal equilibration time.

A discussion of the general problem of gas - particle flow systems is given in reference 2. In that paper, the conservation equations for the two phases are derived, and the importance of several new similarity parameters, among them being the parameter λ_m/x, is pointed out. As has been stated, there are regimes in the boundary layer which must be treated differently. Physically λ_m is the distance required for a particle to travel in order to reduce its initial slip velocity by e^{-1}. Hence, if

$$\lambda_m/x \gg 1 ,$$

then the particles have not had time to adjust to the gas flow and consequently take on large velocity slips. In this case, the particle motions are determined by their initial conditions, since they have not had time to be affected very much by the gas.

On the other hand, if

$$\lambda_m/x \ll 1 ,$$

then the particles have moved many times the required length to reduce their initial velocity, and hence the velocity slips in this regime are small. In this regime, the particle motions are determined by the gas and not their initial conditions. The following analysis will make use of these two regimes.

To write the two-dimensional conservation laws for two-phase flow over a semi-infinite flat plate, let x measure the distance along the wall from the stagnation point, and let y measure the distance normal to the wall. Then

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

is the equation of continuity for the gas.

The conservation of momentum equations for the gas are

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x} + \Lambda\nabla\cdot\underline{q}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x} + \mu\frac{\partial u}{\partial y}\right) + F_{px} \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y} + \Lambda\nabla\cdot\underline{q}\right) + \frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x} + \mu\frac{\partial u}{\partial y}\right) + F_{py}\,. \tag{3}$$

The conservation of energy equation for the gas is

$$\rho C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}$$
$$+ \Phi + \Phi_p + Q_p\,. \tag{4}$$

In these conservation equations,

$$F_{px} = \rho_p \frac{(u_p - u)}{\tau_m} \tag{5}$$

$$F_{py} = \rho_p \frac{(v_p - v)}{\tau_m} \tag{6}$$

$$Q_p = \rho_p C_s \frac{(T_p - T)}{\tau_T} \tag{7}$$

$$\Phi_p = \underline{F}_p \cdot (\underline{v}_p - \underline{v}) \tag{8}$$

Φ = viscous dissipation function

$$\nabla \cdot \underset{\sim}{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

μ = first coefficient of viscosity

Λ = second coefficient of viscosity

p = pressure

and u_p and v_p are the x- and y-components of the particle velocity. The equation of state is taken to be

$$p = \rho R T . \tag{9}$$

The continuity equation for the particle phase is

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v_p) = 0 . \tag{10}$$

The conservation of momentum equations for the particle phase are

$$\rho_p \left(\frac{\partial u_p}{\partial t} + u_p \frac{\partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y} \right) = - F_{px} , \tag{11}$$

$$\rho_p \left(\frac{\partial v_p}{\partial t} + u_p \frac{\partial v_p}{\partial x} + v_p \frac{\partial v_p}{\partial y} \right) = - F_{py} . \tag{12}$$

The conservation of energy for the particle phase is

$$\rho_p C_s \left(\frac{\partial T_p}{\partial t} + u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} \right) = - Q_p . \tag{13}$$

An essential point to make now is that the usual boundary layer assumptions are still valid, because equation 10 minus equation 1 gives

$$\frac{\partial}{\partial t}(\rho_p - \rho) + \frac{\partial}{\partial x}(\rho_p u_p - \rho u) + \frac{\partial}{\partial y}(\rho_p v_p - \rho v) = 0 .$$

But in the boundary layer

$$y \sim O(1/\sqrt{Re})$$

and hence

$$v_p - v \sim O(1/\sqrt{Re})$$

in the boundary layer, provided ρ_p is of the order of ρ in the boundary layer. Therefore, from 3,

$$\frac{\partial p}{\partial y} \sim O(1/\sqrt{Re}),$$

and consequently the variation of pressure across the boundary layer can be neglected for large Reynolds numbers of the gas flow. Thus

$$p = p(x,t)$$

and is computed from the flow external to the boundary layer. Now making the usual boundary layer approximation in equations 1 through 13 yields

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{14}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \rho_p\frac{(u_p-u)}{\tau_m}, \tag{15}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + \rho_p\frac{(u_p-u)^2}{\tau_m} + \rho_p C_s\frac{(T_p-T)}{\tau_T}, \tag{16}$$

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v_p) = 0, \tag{17}$$

$$\rho_p\left(\frac{\partial u_p}{\partial t} + u_p\frac{\partial u_p}{\partial x} + v_p\frac{\partial u_p}{\partial y}\right) = -\rho_p\frac{(u_p-u)}{\tau_m}, \tag{18}$$

$$\rho_p\left(\frac{\partial v_p}{\partial t} + u_p\frac{\partial v_p}{\partial x} + v_p\frac{\partial v_p}{\partial y}\right) = -\rho_p\frac{(v_p-v)}{\tau_m}, \tag{19}$$

$$\rho_p C_s\left(\frac{\partial T_p}{\partial t} + u_p\frac{\partial T_p}{\partial x} + v_p\frac{\partial T_p}{\partial y}\right) = -\rho_p C_s\frac{(T_p-T)}{\tau_T}. \tag{20}$$

The coefficient of viscosity, μ, is for the mixture of gas and solid particles and not for the gas phase alone. References 6 through

10 discuss somewhat how the viscosity of a gas with a suspension of small spherical particles changes. Although there is some disagreement as to whether the viscosity increases or decreases when particles are added to the gas, the viscosity, according to Einstein, can be written

$$\mu = \mu_o[1 + 2.5\,c + O(c^2)]$$

where μ_o is the viscosity of the gas phase alone and c is the volume of spheres in unit volume of the mixture. Then

$$c = \frac{\rho_p}{\rho_s} = \frac{\kappa\rho}{\rho_s} = O(10^{-3}) ,$$

and consequently μ will be taken as the gas phase viscosity throughout the rest of this thesis.

When the gas is incompressible, work done by compression is zero and all dissipation terms can be neglected. Also, μ and k can be assumed constant, and then for steady flow the boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{21}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\rho_p}{\rho}\frac{(u_p - u)}{\tau_m} , \tag{22}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \rho_p C_s\frac{(T_p - T)}{\tau_T} , \tag{23}$$

$$\frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v_p) = 0 , \tag{24}$$

$$\rho_p\left(u_p\frac{\partial u_p}{\partial x} + v_p\frac{\partial u_p}{\partial y}\right) = -\rho_p\frac{(u_p - u)}{\tau_m} , \tag{25}$$

$$\rho_p \left(u_p \frac{\partial v_p}{\partial x} + v_p \frac{\partial v_p}{\partial y} \right) = - \rho_p \frac{(v_p - v)}{\tau_m} , \tag{26}$$

$$\rho_p C_s \left(u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} \right) = - \rho_p C_s \frac{(T_p - T)}{\tau_T} . \tag{27}$$

The boundary conditions on the gas phase are:

(i) no mass transfer at the plate,

(ii) gas velocity on the plate vanishes,

(iii) gas velocity must approach free stream value as y approaches infinity,

(iv) temperature of the gas at the plate is that of the plate,

(v) temperature of the gas must approach free stream value as y approaches infinity.

The boundary conditions on the particle phase are

(i) no particle mass transfer at the wall,

(ii) particle velocities must approach their free stream values as y approaches infinity; if no pressure gradient exists, then particles and gas must be at equilibrium as y approaches infinity,

(iii) particle phase temperatures must approach their free stream values as y approaches infinity,

(iv) particle phase density must approach its free stream value as y approaches infinity.

When there is a pressure gradient, there will be temperature defects and velocity slips in the external flow as well as in the boundary layer. Consequently, before the boundary layer can be considered, the two-phase external flow must be solved, and this greatly complicates the problem. Therefore, throughout the rest of this thesis,

$$dp/dx = 0 ,$$

and

$$u = u_\infty , \quad \text{a constant,}$$

in the external flow. Since the particle phase and gas phase are in equilibrium in the external flow, the boundary conditions on the particles are greatly simplified.

Restrictions on Solutions

In the regime far downstream from the leading edge, there are two conditions which the flow must satisfy in order for the slip velocities to be small and at the same time for the Stokes' drag law on the particles to be valid. If the Stokes' drag law is valid, then

$$Re_\sigma = \frac{(u_p - u)\sigma}{\nu} \leq 1 ,$$

and if the particle slip velocities are small, then

$$(u_p - u)/u_\infty \ll 1 .$$

For a given flow field, one of these conditions is more restrictive on the flow than the other; hence, the extent of the regime of small slip velocities depends on the parameters of the flow field, i.e., ν, u_∞, ρ, σ.

III. SEMI-INFINITE FLAT PLATE WHEN THE GAS IS INCOMPRESSIBLE

For steady, incompressible flow over a semi-infinite flat plate, the boundary layer equations are, from Chapter II,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{28}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} + \frac{\rho_p u_\infty}{\lambda_m}(u_p - u) , \tag{29}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \frac{\rho_p C_s u_\infty}{\lambda_T}(T_p - T) , \tag{30}$$

$$\frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v_p) = 0 , \tag{31}$$

$$\rho_p\left(u_p\frac{\partial u_p}{\partial x} + v_p\frac{\partial u_p}{\partial y}\right) = -\frac{\rho_p u_\infty}{\lambda_m}(u_p - u) , \tag{32}$$

$$\rho_p\left(u_p\frac{\partial v_p}{\partial x} + v_p\frac{\partial v_p}{\partial y}\right) = -\frac{\rho_p u_\infty}{\lambda_m}(v_p - v) , \tag{33}$$

$$\rho_p C_s\left(u_p\frac{\partial T_p}{\partial x} + v_p\frac{\partial T_p}{\partial y}\right) = -\frac{\rho_p C_s u_\infty}{\lambda_T}(T_p - T) . \tag{34}$$

In view of equation 28 a gas stream function can be introduced.

$$\begin{aligned} u &= \partial\psi/\partial y , \\ v &= -\partial\psi/\partial x . \end{aligned} \tag{35}$$

A. Small Slip Approximation

For now the regime characterized by the statement

$$\lambda_m/x << 1$$

is considered, i.e., the regime of small particle slip velocities and small temperature defects. Therefore, taking advantage of how the slip quantities enter the equations 29 , 30 , 32 , 33 , and 34 , the equations for the behavior of the slip quantities to first order in λ_m/x can be immediately determined. Hence the introduction of a particle stream function, which is indicated in view of 31 , actually complicates the analysis. Since a small dimensionless parameter, λ_m/x , is involved in the boundary layer equations 28 through 34 , it would seem that a perturbational analysis is in order, and, therefore, the quantities ψ , $u_p - u$, $v_p - v$, $T_p - T$, T , and ρ_p will be expressed as expansions in λ_m/x . First of all, the equations 29 , 30 , 32 , 33 , and 34 are rearranged so that the slip quantities can be read off immediately to first order.

Dividing 29 by ρ and adding 32 , divided by ρ , yields

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \nu \frac{\partial^2 u}{\partial y^2} + \frac{\rho_p}{\rho} \left(u_p \frac{\partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y} \right) = 0 . \tag{36}$$

Dividing 32 by ρ_p and subtracting 29 , divided by ρ , yields

$$u_p \frac{\partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y} - \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \nu \frac{\partial^2 u}{\partial y^2} = -\left(1 + \frac{\rho_p}{\rho}\right) \frac{u_\infty}{\lambda_m} (u_p - u) . \tag{37}$$

Dividing 30 by ρC_p and adding 34 , divided by ρC_p , yields

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2} + \frac{\rho_p C_s}{\rho C_p} \left(u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} \right) = 0 . \tag{38}$$

Dividing 34 by $\rho_p C_s$ and subtracting 30 , divided by ρC_p , yields

$$u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} - \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) + \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2} = -\left(1 + \frac{\rho_p C_s}{\rho C_p}\right) \frac{u_\infty}{\lambda_T} (T_p - T) . \tag{39}$$

Expanding 31 yields

$$\rho_p \left(\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y} \right) + u_p \frac{\partial \rho_p}{\partial x} + v_p \frac{\partial \rho_p}{\partial y} = 0 . \tag{40}$$

Now the appropriate dependent variables except for the stream functions are

$$u_s^* = (u_p - u)/u_\infty$$

$$v_s^* = (v_p - v)/u_\infty$$

$$T_s^* = (T_p - T)/(T_\infty - T_w)$$

$$T^* = (T - T_w)/(T_\infty - T_w)$$

$$\rho_p^* = \rho_p/(\varkappa\rho)$$

where

$$\varkappa = (\rho_p^{(o)})/\rho \ ,$$

$\rho_p^{(o)}$ = density of particles far upstream of the plate ,

T_∞ = equilibrium temperature far upstream of the plate ,

T_w = temperature of the plate assumed constant.

Writing equations 40 , 36 , 33 , 37 , 38 , and 39 in terms of ψ , u_s^* , v_s^* , T_s^* , T^* , ρ_p^* yields the following equations.

$$\rho_p^*\left(\frac{\partial u_s^*}{\partial x} + \frac{\partial v_s^*}{\partial y}\right) + u_s^* \frac{\partial \rho_p^*}{\partial x} + v_s^* \frac{\partial \rho_p^*}{\partial y} + \frac{1}{u_\infty} \frac{\partial \psi}{\partial y} \frac{\partial \rho_p^*}{\partial x} - \frac{1}{u_\infty} \frac{\partial \psi}{\partial x} \frac{\partial \rho_p^*}{\partial y} = 0 \ . \tag{41}$$

$$\frac{1+\varkappa\rho_p^*}{u_\infty^2}\left(\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2}\right) - \frac{\nu}{u_\infty^2}\frac{\partial^3 \psi}{\partial y^3} + \varkappa\rho_p^*\Big(u_s^* \frac{\partial u_s^*}{\partial x} + v_s^* \frac{\partial u_s^*}{\partial y}$$

$$+ \frac{u_s^*}{u_\infty}\frac{\partial^2 \psi}{\partial x \partial y} + \frac{v_s^*}{u_\infty}\frac{\partial^2 \psi}{\partial y^2} + \frac{1}{u_\infty}\frac{\partial \psi}{\partial y}\frac{\partial u_s^*}{\partial x} - \frac{1}{u_\infty}\frac{\partial \psi}{\partial x}\frac{\partial u_s^*}{\partial y}\Big) = 0 \ . \tag{42}$$

$$u_s^* \frac{\partial v_s^*}{\partial x} + v_s^* \frac{\partial v_s^*}{\partial y} - \frac{u_s^*}{u_\infty}\frac{\partial^2 \psi}{\partial x^2} - \frac{v_s^*}{u_\infty}\frac{\partial^2 \psi}{\partial x \partial y} + \frac{1}{u_\infty}\frac{\partial \psi}{\partial y}\frac{\partial v_s^*}{\partial x} - \frac{1}{u_\infty}\frac{\partial \psi}{\partial x}\frac{\partial v_s^*}{\partial y} -$$

$$- \frac{1}{u_\infty^2} \frac{\partial\psi}{\partial y} \frac{\partial^2\psi}{\partial x^2} + \frac{1}{u_\infty^2} \frac{\partial\psi}{\partial x} \frac{\partial^2\psi}{\partial x \partial y} = - \frac{v_s^*}{\lambda_m} . \tag{43}$$

$$u_s^* \frac{\partial u_s^*}{\partial x} + v_s^* \frac{\partial u_s^*}{\partial y} + \frac{u_s^*}{u_\infty} \frac{\partial^2\psi}{\partial x \partial y} + \frac{v_s^*}{u_\infty} \frac{\partial^2\psi}{\partial y^2} + \frac{1}{u_\infty} \frac{\partial\psi}{\partial y} \frac{\partial u_s^*}{\partial x} - \frac{1}{u_\infty} \frac{\partial\psi}{\partial x} \frac{\partial u_s^*}{\partial y}$$

$$+ \frac{\nu}{u_\infty^2} \frac{\partial^3\psi}{\partial y^3} = - (1 + \varkappa\rho_p^*) \frac{u_s^*}{\lambda_m} . \tag{44}$$

$$\frac{1}{u_\infty} (1+\varkappa \frac{C_s}{C_p} \rho_p^*)(\frac{\partial\psi}{\partial y} \frac{\partial T^*}{\partial x} - \frac{\partial\psi}{\partial x} \frac{\partial T^*}{\partial y}) - \frac{\nu}{u_\infty Pr} \frac{\partial^2 T^*}{\partial y^2} + \frac{\varkappa C_s}{C_p} \rho_p^* \times$$

$$(u_s^* \frac{\partial T_s^*}{\partial x} + v_s^* \frac{\partial T_s^*}{\partial y} + u_s^* \frac{\partial T^*}{\partial x} + v_s^* \frac{\partial T^*}{\partial y} + \frac{1}{u_\infty} \frac{\partial\psi}{\partial y} \frac{\partial T_s^*}{\partial x}$$

$$- \frac{1}{u_\infty} \frac{\partial\psi}{\partial x} \frac{\partial T_s^*}{\partial y}) = 0 . \tag{45}$$

$$u_s^* \frac{\partial T_s^*}{\partial x} + v_s^* \frac{\partial T_s^*}{\partial y} + u_s^* \frac{\partial T^*}{\partial x} + v_s^* \frac{\partial T^*}{\partial y} + \frac{1}{u_\infty} \frac{\partial\psi}{\partial y} \frac{\partial T_s^*}{\partial x}$$

$$- \frac{1}{u_\infty} \frac{\partial\psi}{\partial x} \frac{\partial T_s^*}{\partial y} + \frac{\nu}{u_\infty Pr} \frac{\partial^2 T^*}{\partial y^2} = - (1+\varkappa \frac{C_s}{C_p} \rho_p^*) \frac{T_s^*}{\lambda_T} . \tag{46}$$

In these equations,

$\nu = \mu/\rho$ = kinematic coefficient of viscosity of the gas.

Since the slip quantities are small, the zeroth order approximation is taken to be that the gas and particle phases move as one phase in equilibrium. This zeroth order approximation can be considered as the limiting process where the particle radii approach zero, but at the same time the force exerted by the gas on the particles approaches a finite limit, and in addition the mass of the particles remains constant. Thus, in the zeroth order the particles and the gas are in equilibrium, but the particles still affect the gas flow through their mass. The first order solution will account for the slip quantities and modify slightly the zeroth order solution.

Now in defining the kinematic coefficient of viscosity for the mixture, one must take into account the mass of the particles. That is, let

$$\nu^* = \mu/(\rho + \rho_p) ,$$

where, in the zeroth order,

$$\rho_p = \varkappa\rho$$

and then

$$\nu^* = \nu/(1+\varkappa) .$$

This means that in the zeroth order approximation, one need consider only the flow of a gas phase with kinematic coefficient of viscosity, ν^*.

From what has been said,

$$u_p \sim u + O(\lambda_m/x)$$

$$v_p \sim v + O(\lambda_m/x)$$

and then, to zeroth order, 31 is

$$u\frac{\partial\rho_p}{\partial x} + v\frac{\partial\rho_p}{\partial y} = 0 ,$$

or, to zeroth order, ρ_p is constant; hence

$$\rho_p^* = 1 + O(\lambda_m/x) .$$

Likewise,

$$T_p \sim T + O(\lambda_m/x) .$$

Immediately then, it can be seen that to first order in λ_m/x, equations 41, 43, 44, and 46 become

$$\rho_p^*\left(\frac{\partial u_s^*}{\partial x}+\frac{\partial v_s^*}{\partial y}\right)+\frac{1}{u_\infty}\frac{\partial\psi}{\partial y}\frac{\partial\rho_p^*}{\partial x}-\frac{1}{u_\infty}\frac{\partial\psi}{\partial x}\frac{\partial\rho_p^*}{\partial y}=0 \quad , \tag{47}$$

$$\frac{\nu}{u_\infty^2}\frac{\partial^3\psi}{\partial y^3}=-(1+\varkappa)\frac{u_s^*}{\lambda_m} \quad , \tag{48}$$

$$-\frac{1}{u_\infty^2}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{u_\infty^2}\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}=-\frac{v_s^*}{\lambda_m} \quad , \tag{49}$$

$$\frac{\nu}{u_\infty Pr}\frac{\partial^2 T^*}{\partial y^2}=-\left(1+\varkappa\frac{C_s}{C_p}\right)\frac{T_s^*}{\lambda_T} \; . \tag{50}$$

In terms of the similarity variable

$$\eta=\sqrt{u_\infty/(\nu^* x)}\; y \; ,$$

it is found that

$$\left.\begin{aligned}
\frac{\partial}{\partial y} &= \sqrt{\frac{u_\infty}{\nu^* x}}\left(\frac{\partial}{\partial\eta}\right)_x\\
\frac{\partial^2}{\partial y^2} &= \frac{u_\infty}{\nu^* x}\left(\frac{\partial^2}{\partial\eta^2}\right)_x\\
\frac{\partial^3}{\partial y^3} &= \left(\frac{u_\infty}{\nu^* x}\right)^{3/2}\left(\frac{\partial^3}{\partial\eta^3}\right)_x\\
\left(\frac{\partial}{\partial x}\right)_y &= \left(\frac{\partial}{\partial x}\right)_\eta-\frac{\eta}{2x}\left(\frac{\partial}{\partial\eta}\right)_x\\
\frac{\partial^2}{\partial x\partial y} &= \sqrt{\frac{u_\infty}{\nu^* x}}\left[\frac{\partial^2}{\partial\eta\partial x}-\frac{\eta}{2x}\left(\frac{\partial^2}{\partial\eta^2}\right)_x-\frac{1}{2x}\left(\frac{\partial}{\partial\eta}\right)_x\right]\\
\left(\frac{\partial^2}{\partial x^2}\right)_y &= \left(\frac{\partial^2}{\partial x^2}\right)_\eta-\frac{\eta}{x}\frac{\partial^2}{\partial\eta\partial x}+\frac{3}{4}\frac{\eta}{x^2}\left(\frac{\partial}{\partial\eta}\right)_x+\frac{\eta^2}{4x^2}\left(\frac{\partial^2}{\partial\eta^2}\right)_x \; .
\end{aligned}\right\} \tag{51}$$

Using the transformation 51 , equations 47 , 42 , 48 , 49 , 50 , and 45 become

$$\rho_p^*\left(\frac{\partial u_s^*}{\partial x} - \frac{\eta}{2x}\frac{\partial u_s^*}{\partial \eta} + \sqrt{\frac{u_\infty}{\nu^* x}}\frac{\partial v_s^*}{\partial \eta}\right) + \frac{1}{\sqrt{u_\infty \nu^* x}}\frac{\partial \psi}{\partial \eta}\frac{\partial \rho_p^*}{\partial x}$$

$$- \frac{1}{\sqrt{u_\infty \nu^* x}}\frac{\partial \psi}{\partial x}\frac{\partial \rho_p^*}{\partial \eta} = 0 , \tag{52}$$

$$\frac{1+\kappa\rho_p^*}{u_\infty \nu^* x}\left[\frac{\partial \psi}{\partial \eta}\left(\frac{\partial^2 \psi}{\partial \eta \partial x} - \frac{1}{2x}\frac{\partial \psi}{\partial \eta}\right) - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial \eta^2}\right] - \frac{\nu}{u_\infty^2}\left(\frac{u_\infty}{\nu^* x}\right)^{\frac{3}{2}}\frac{\partial^3 \psi}{\partial \eta^3}$$

$$+ \kappa\rho_p^*\left[\frac{u_s^*}{\sqrt{u_\infty \nu^* x}}\left(\frac{\partial^2 \psi}{\partial \eta \partial x} - \frac{\eta}{2x}\frac{\partial^2 \psi}{\partial \eta^2} - \frac{1}{2x}\frac{\partial \psi}{\partial \eta}\right) + \frac{v_s^*}{\nu^* x}\frac{\partial^2 \psi}{\partial \eta^2}\right.$$

$$\left. + \frac{1}{\sqrt{u_\infty \nu^* x}}\frac{\partial \psi}{\partial \eta}\frac{\partial u_s^*}{\partial x} - \frac{1}{\sqrt{u_\infty \nu^* x}}\frac{\partial \psi}{\partial x}\frac{\partial u_s^*}{\partial \eta}\right] = 0 , \tag{53}$$

$$\frac{\nu}{u_\infty^2}\left(\frac{u_\infty}{\nu^* x}\right)^{3/2}\frac{\partial^3 \psi}{\partial \eta^3} = -(1+\kappa)\frac{u_s^*}{\lambda_m} , \tag{54}$$

$$-\frac{1}{u_\infty^2}\sqrt{\frac{u_\infty}{\nu^* x}}\frac{\partial \psi}{\partial \eta}\left[\frac{\partial^2 \psi}{\partial x^2} - \frac{\eta}{2x}\frac{\partial^2 \psi}{\partial \eta \partial x} + \frac{\eta}{2x^2}\frac{\partial \psi}{\partial \eta}\right]$$

$$+ \frac{1}{u_\infty^2}\sqrt{\frac{u_\infty}{\nu^* x}}\frac{\partial \psi}{\partial x}\left[\frac{\partial^2 \psi}{\partial \eta \partial x} - \frac{\eta}{2x}\frac{\partial^2 \psi}{\partial \eta^2} - \frac{1}{2x}\frac{\partial \psi}{\partial \eta}\right] = -\frac{v_s^*}{\lambda_m} , \tag{55}$$

$$\frac{(1+\kappa)}{\Pr x}\frac{\partial^2 T^*}{\partial \eta^2} = -\left(1+\frac{\kappa C_s}{C_p}\right)\frac{T_s^*}{\lambda_T} , \tag{56}$$

$$\frac{1}{\sqrt{u_\infty \nu^* x}}\left(1+\frac{\varkappa C_s}{C_p}\rho_p^*\right)\left(\frac{\partial\psi}{\partial\eta}\frac{\partial T^*}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T^*}{\partial\eta}\right)-\frac{(1+\varkappa)}{\Pr x}\frac{\partial^2 T^*}{\partial\eta^2}$$

$$+\frac{\varkappa C_s}{C_p}\rho_p^*\left[u_s^*\left(\frac{\partial T_s^*}{\partial x}-\frac{\eta}{2x}\frac{\partial T_s^*}{\partial\eta}\right)+\sqrt{\frac{u_\infty}{\nu^* x}}\,v_s^*\frac{\partial T_s^*}{\partial x}+u_s^*\left(\frac{\partial T^*}{\partial x}-\frac{\eta}{2x}\frac{\partial T^*}{\partial\eta}\right)\right.$$

$$+\sqrt{\frac{u_\infty}{\nu^* x}}\,v_s^*\frac{\partial T^*}{\partial\eta}+\frac{1}{\sqrt{u_\infty\nu^* x}}\frac{\partial\psi}{\partial\eta}\left(\frac{\partial T_s^*}{\partial x}-\frac{\eta}{2x}\frac{\partial T_s^*}{\partial\eta}\right)$$

$$\left.-\frac{1}{\sqrt{u_\infty\nu^* x}}\frac{\partial T_s^*}{\partial\eta}\left(\frac{\partial\psi}{\partial x}-\frac{\eta}{2x}\frac{\partial\psi}{\partial\eta}\right)\right]=0\,. \qquad (57)$$

The following expansions are assumed for the six variables ψ, u_s^*, v_s^*, ρ_p^*, T_s^*, T^*.

$$\left.\begin{aligned}
\psi &= \sqrt{u_\infty\nu^* x}\,[f_o(\eta)+\frac{\lambda_m}{x}f_1(\eta)+(\frac{\lambda_m}{x})^2 f_2(\eta)+\ldots]\\
u_s^* &= \frac{\lambda_m}{x}g_1(\eta)+(\frac{\lambda_m}{x})^2 g_2(\eta)+\ldots\\
v_s^* &= \left[\frac{\lambda_m}{x}h_1(\eta)+(\frac{\lambda_m}{x})^2 h_2(\eta)+\ldots\right]\sqrt{\frac{\nu^*}{u_\infty x}}\\
\rho_p^* &= 1+\frac{\lambda_m}{x}I_1(\eta)+(\frac{\lambda_m}{x})^2 I_2(\eta)+\ldots\\
T_s^* &= \frac{\lambda_m}{x}\theta_s^{(1)}(\eta)+(\frac{\lambda_m}{x})^2\theta_s^{(2)}(\eta)+\ldots\\
T^* &= \theta_o(\eta)+\frac{\lambda_m}{x}\theta_1(\eta)+(\frac{\lambda_m}{x})^2\theta_2(\eta)+\ldots
\end{aligned}\right\} \qquad (58)$$

Substituting the system 58 into equation 52 and equating the coefficient of λ_m/x to zero yields

$$\tfrac{1}{2}f_o I_1' + f_o' I_1 = h_1' - \frac{\eta}{2}g_1' - g_1\,, \qquad (59)$$

where the prime denotes differentiation with respect to η. Substituting 58 into 53 and equating coefficients of $(\lambda_m/x)^n$, $n = 0, 1$, to zero yields

$$f_o''' + \tfrac{1}{2} f_o f_o'' = 0 , \tag{60}$$

and

$$f_1''' + \tfrac{1}{2} f_o f_1'' + f_o' f_1' - \tfrac{1}{2} f_o'' f_1 = -\frac{\varkappa}{1+\varkappa}\left[\tfrac{1}{2} I_1 f_o f_o'' + g_1\left(\frac{\eta}{2} f_o'' + f_o'\right) - h_1 f_o'' + \tfrac{1}{2} f_o g_1'\right] . \tag{61}$$

Similarly, equation 54 yields

$$g_1 = - f_o''' = \tfrac{1}{2} f_o f_o'' , \tag{62}$$

and 55 yields

$$h_1 = \frac{\eta}{4} f_o f_o'' - \frac{1}{4} f_o f_o' + \frac{\eta}{4}(f_o')^2 , \tag{63}$$

and 57 yields

$$\theta_o'' + \left(\frac{1 + \dfrac{\varkappa C_s}{C_p}}{1 + \varkappa}\right)\frac{Pr}{2} f_o \theta_o' = 0 \tag{64}$$

to zeroth order, and

$$\theta_1'' + Pr\left(\frac{1 + \dfrac{\varkappa C_s}{C_p}}{1 + \varkappa}\right)\left(\frac{1}{2} f_o \theta_1' + f_o' \theta_1\right) = \frac{Pr}{2}\left(\frac{1 + \dfrac{\varkappa C_s}{C_p}}{1 + \varkappa}\right) f_1 \theta_o'$$

$$- \frac{Pr}{2}\left(\frac{\varkappa C_s/C_p}{1 + \varkappa}\right) f_o \theta_o' I_1 + \frac{Pr\ \varkappa C_s/C_p}{1 + \varkappa}\left[-\frac{\eta}{2} g_1 \theta_o' + h_1 \theta_o' - f_o' \theta_s^{(1)} + \frac{1}{2} f_o \theta_s'^{(1)}\right] \tag{65}$$

to first order. Equation 56 yields

$$\theta_s^{(1)} = -\frac{3}{2}\frac{C_s}{C_p}\left(\frac{1 + \varkappa}{1 + \dfrac{\varkappa C_s}{C_p}}\right)\theta_o'' = \frac{3}{2} Pr \frac{C_s}{C_p} \frac{1}{2} f_o \theta_o' . \tag{66}$$

Making use of equations 62 and 63 to simplify the other equations and grouping the equations into a zeroth order and first order problem, the results are the following equations.

Zeroth Order Problem:

$$f_o''' + \frac{1}{2} f_o f_o'' = 0 \tag{67}$$

$$\theta_o'' + \left(\frac{1 + (\varkappa C_s/C_p)}{1 + \varkappa}\right)\frac{Pr}{2} f_o \theta_o' = 0 \tag{68}$$

First Order Problem:

$$g_1 = \frac{1}{2} f_o f_o'' \tag{69}$$

$$h_1 = \frac{\eta}{4} f_o f_o'' - \frac{1}{4} f_o f_o' + \frac{\eta}{4}(f_o')^2 \tag{70}$$

$$\frac{1}{2} f_o I_1' + f_o' I_1 = \frac{f_o''}{2}(\eta f_o' - f_o) \tag{71}$$

$$F_1''' + \frac{1}{2} f_o F_1'' + f_o' F_1' - \frac{1}{2} f_o'' F_1 = - f_o''\left[\frac{1}{2} I_1 f_o + (f_o - \frac{\eta}{4} f_o') f_o' - \frac{1}{8} f_o^{\,3}\right] \tag{72}$$

$$\theta_s^{(1)} = \frac{3}{2} Pr \frac{C_s}{C_p} \frac{1}{2} f_o \theta_o' \tag{73}$$

$$\begin{aligned}
\theta_1'' + Pr\left(\frac{1 + (\varkappa C_s/C_p)}{1 + \varkappa}\right)\left(\frac{1}{2} f_o \theta_1' + f_o' \theta_1\right) &= Pr\left(\frac{1 + (\varkappa C_s/C_p)}{1 + \varkappa}\right)\frac{1}{2} f_1 \theta_o' \\
&- \frac{Pr(\varkappa C_s/C_p)}{1 + \varkappa}\frac{1}{2} f_o \theta_o' I_1 + \frac{Pr(\varkappa C_s/C_p)}{1 + \varkappa}\left[\theta_o'\left(\frac{\eta}{4} f_o'^{\,2} - \frac{1}{4} f_o f_o'\right)\right. \\
&\left. - \frac{9}{8}\frac{Pr\, C_s}{C_p} f_o f_o' \theta_o' + \frac{3}{16}\frac{Pr^2 C_s}{C_p} f_o^{\,3}\left(\frac{1 + (\varkappa C_s/C_p)}{1 + \varkappa}\right)\theta_o'\right]
\end{aligned} \tag{74}$$

where

$$F_1 = \left(\frac{1 + \varkappa}{\varkappa}\right) f_1 .$$

The boundary conditions on the gas phase as stated in Chapter II imply

$$\left.\begin{array}{lll} \text{(i)} & f_o(o) = 0\,, & f_1(o) = 0 \\ \text{(ii)} & f'_o(o) = 0\,, & f'_1(o) = 0 \\ \text{(iii)} & f'_o(\infty) = 1\,, & f'_1(\infty) = 0 \\ \text{(iv)} & \theta_o(o) = 0\,, & \theta_1(o) = 0 \\ \text{(v)} & \theta_o(\infty) = 1\,, & \theta_1(\infty) = 0\,. \end{array}\right\} \quad (75)$$

Now in the solution for the particle phase velocities and temperature, the leading derivatives of particle phase velocities and temperature were suppressed by the expansion procedure. This, of course, can be done, provided the derivatives of the particle phase velocities and temperature are not large in the region of interest. Consequently, it appears that the particle phase equations constitute a singular perturbation problem when the expansion procedure is attempted.

To see that the problem is indeed a singular perturbation problem, a one-dimensional problem will be considered. The equation for the particle phase is

$$u_p \frac{du_p}{dx} = -u_\infty \frac{(u_p - u)}{\lambda_m}\,.$$

Now it is assumed for simplicity that

$$u = u_\infty = \text{a constant},$$

and then letting

$$u_p - u_\infty = u_\infty u_s\,,$$

one gets

$$[u_s + 1]\frac{du_s}{dx} = -\frac{u_s}{\lambda_m}\,.$$

Now suppose that

$$u_s << 1 ,$$

then

$$\frac{du_s}{dx} = -\frac{u_s}{\lambda_m}$$

$$u_s = u_s^{(o)} e^{-x/\lambda_m}$$

where $u_s^{(o)} = u_s(x=0)$. Clearly, the rate of change of u_s is large only in the region

$$x \sim O(\lambda_m) .$$

For $x >> \lambda_m$, the approximation

$$u_s = 0$$

is a very good one. Hence, provided

$$\lambda_m / x << 1 ,$$

the derivatives of the particle velocities are not large and can be suppressed by the expansion procedure. Consequently, the particle phase boundary conditions, (i), (ii), and (iii), given in Chapter II, are likewise suppressed by the expansion in terms of λ_m/x. The condition (iv) given in Chapter II for the particle phase implies

$$I_1(\infty) = 0 . \tag{76}$$

Thus, the zeroth and first order problem compose a two-point boundary value problem over a semi-infinite domain which must, of course, be solved numerically.

The solution of 60 is the well-known Blasius solution for a gas with kinematic coefficient of viscosity, ν^*. From reference 3, the following information is given about $f_o(\eta)$:

$$f_o''(o) = .33206 .$$

For $\eta \ll 1$,

$$\left.\begin{aligned} f_o &= \frac{f_o''(o)}{2}\eta^2 , \\ f_o' &= f_o''(o)\eta . \end{aligned}\right\} \quad (77)$$

For η very large,

$$\left.\begin{aligned} f_o(\eta) &= \eta - \beta + \gamma \int_\infty^\eta d\eta \int_\infty^\eta e^{-\frac{1}{4}(\eta-\beta)^2} d\eta \\ f_o'(\eta) &= 1 + \gamma \int_\infty^\eta e^{-\frac{1}{4}(\eta-\beta)^2} d\eta \\ f_o''(\eta) &= \gamma\, e^{-\frac{1}{4}(\eta-\beta)^2} \\ \beta &= 1.73 \quad , \quad \gamma = 0.231 . \end{aligned}\right\} \quad (78)$$

Equation 71 can be written

$$\frac{d}{d\eta}(f_o^2 I_1) = f_o f_o''(\eta f_o' - f_o)$$

$$f_o^2 I_1 = \int_0^\eta f_o(x) f_o''(x)[x f_o'(x) - f_o(x)]dx$$

assuming $I_1(o)$ is finite. Then

$$I_1 = f_o^{-2} \int_0^\eta f_o(x) f_o''(x)[x f_o'(x) - f_o(x)]dx .$$

Let $\eta \ll 1$, then

$$I_1(\eta) = \frac{4}{[f_o''(o)]^2\eta^4} \int_0^\eta \frac{[f_o''(o)]^3}{2} \chi^2(\tfrac{1}{2}\chi^2)d\chi + O(\eta^2)$$

$$I_1(\eta) = \frac{f_o''(o)}{\eta^4} \int_0^\eta x^4\, dx + O(\eta^2)$$

$$I_1(\eta) = \frac{f_o''(o)}{5}\, \eta + O(\eta^2) \quad . \tag{79}$$

Hence,

$$I_1(o) = 0 \, , \tag{80}$$

and we can replace the boundary condition 76 with this one. Equation 79 shows how $I_1(\eta)$ goes away from $\eta = 0$.

The results of the numerical solution to the zeroth and first order problems are shown in figures 1, 2, 3, and 4. One might question the behavior of the curves for I_1 , F_1' , and θ_1 , since they approach zero so slowly as η approaches infinity. If one takes this feature literally, it would mean that the particle phase affects the gas flow well outside of the boundary layer, which is physically impossible. To explain why the curves behave as they do as η moves out of the boundary layer, it must be realized that the analysis began with the boundary layer equations. In particular, the boundary layer approximation gives rise to a vertical velocity, v , which does not vanish as η moves outside of the boundary layer. Because of this, there is a convection of particle quantities outside of the boundary layer where they cannot physically be. In short, the results could hardly be expected to be any better than the assumptions that were made originally. However, the equations should be a sufficiently close approximation to the actual physical situation near the plate such that quantities like the shear, heat transfer, and particle density on the plate computed from this analysis will be reasonably accurate.

When one tries to solve equation 72 numerically, the usual procedure is to guess a value of $F_1''(o)$ and integrate out along η to see if the solution satisfies $F_1'(\infty) = 0$. Using the behavior of the solution at ∞, one can usually get a better estimate for $F_1''(o)$; however, in this case, regardless of the value of $F_1''(o)$, the solution approaches zero at infinity, and hence there is no criterion for choosing the correct curve. Integrating equation 72 from zero to infinity will give a new criterion for choosing the correct curve.

$$\int_0^\infty [F_1''' + \frac{1}{2} f_o F_1'' + f_o' F_1' - \frac{1}{2} f_o'' F_1] d\eta = - \int_0^\infty G(\eta) d\eta$$

where

$$G(\eta) = f_o''[\frac{1}{2} f_o I_1 + (f_o - \frac{\eta}{4} f_o') f_o' - \frac{1}{8} f_o^{\,3}] ,$$

$$\int_0^\infty F_1''' d\eta = - F_1''(o) ,$$

$$\int_0^\infty \frac{1}{2} f_o F_1'' d\eta = - \frac{1}{2} \int_0^\infty f_o' F_1' d\eta ,$$

$$- \int_0^\infty \frac{1}{2} f_o'' F_1 d\eta = - \frac{1}{2} \int_0^\infty F_1' d\eta + \frac{1}{2} \int_0^\infty f_o' F_1' d\eta .$$

Then

$$F_1''(o) = \int_0^\infty [(f_o' - \frac{1}{2}) F_1' + G(\eta)] d\eta ,$$

and this will be a new criterion for picking out the correct curve. However, since the curve for $F_1'(\eta)$ does not converge properly, this integral will be incorrect and also will not converge fast enough. Physically, it is known that F_1' must approach zero like $e^{-\eta^2/4}$ for large η, and consequently, to get this kind of decay for large η, the decay of the homogeneous solution, which is responsible for the slow decay, must be subtracted out of the general solution for large η. Doing all this calculation gives the following estimate:

$$F_1''(o) = .159 .$$

The same thing can be said for the solution of 74. However, note that for $Pr = 1$ and $C_s/C_p = 1$, then

$$\theta_o = f_o'$$

but

$$\theta_1 \neq f_1'$$

since

$$\lambda_T \neq \lambda_m .$$

Consequently, similarity between the velocity and temperature profiles does not hold to first order. Also, for $Pr = 1$ and $C_s/C_p = 1$,

$$\theta_s^{(1)} = \frac{3}{2}\left(\frac{1}{2} f_o \theta_o'\right) = \frac{3}{2}\left(\frac{1}{2} f_o f_o''\right)$$

$$\theta_s^{(1)} = \frac{3}{2} g_1(\eta) .$$

When one examines the graphs for $F_1'(\eta)$ and $\theta_1(\eta)$, given in figures 1 and 4, it is seen that the curves become negative for $\eta \sim O(5)$. This behavior should not be taken literally, because this is the result of lack of technique in the numerical solution. This problem is due to the fact that $F_1'(\eta)$ and $\theta_1(\eta)$ do not approach zero as rapidly as they physically should. Nevertheless, the mathematics and formulation of the two-phase flow problem is correct, and the non-correspondence of the results with what one physically expects is due to the boundary layer assumption made originally. In particular, the troubles arise from the y-component momentum equation for the particle phase, which cannot be neglected in the boundary layer, whereas in the usual gas phase boundary layer, terms like those on the left side of 33 never arise. However, this formulation does provide some understanding of the general two-phase flow problem.

To understand that the positive nature of the curve for $F_1'(\eta)$ is correct requires an understanding of what is meant by the zeroth and first order approximations. In the zeroth order, the gas phase and the particle phase move as one phase, i. e., the particles are frozen to the surrounding gas. This means that the gas holds the particles with some finite force. However, to first order, the particles are allowed to slip through the gas, and it is not difficult to see that the particles will decelerate faster than the gas does as both phases move downstream since the particle slip velocities decrease as the particles move downstream. This means that the gas must exert a force on the particles even greater than that force required to keep the particles frozen to the gas in the zeroth order problem. Consequently, to first order, a force is exerted on the gas in the positive x-direction even greater than the zeroth order force. Thus the gas velocity increases above the zeroth order gas velocity.

The shear coefficient is calculated to be

$$C_\tau = \frac{\tau}{\rho u_\infty^2} = \frac{0.332}{\sqrt{Re_x}} \sqrt{1+\varkappa}\left[1 + 0.48\left(\frac{\varkappa}{1+\varkappa}\right)\frac{\lambda_m}{x} + \dots\right] \tag{81}$$

where

$$Re_x = \rho \frac{u_\infty x}{\mu} .$$

The heat transfer to the plate for $Pr = C_s/C_p = 1$ is

$$q(o) = -k \frac{(T_\infty - T_w)}{x} \sqrt{Re_x}\, 0.332 \sqrt{1+\varkappa}\left[1 + .504\left(\frac{\varkappa}{1+\varkappa}\right)\frac{\lambda_m}{x} + \dots\right]. \tag{82}$$

Hence the presence of particles means that the shear coefficient is modified by the factor

$$\sqrt{1+\varkappa}\left[1 + \left(\frac{\varkappa}{1+\varkappa}\right).48\left(\frac{\lambda_m}{x}\right)\right]$$

and, likewise, the heat transferred to the plate is modified by the factor

$$\sqrt{1+\varkappa}\left[1 + .504\left(\frac{\varkappa}{1+\varkappa}\right)\frac{\lambda_m}{x}\right]$$

in the region where $\lambda_m/x \ll 1$.

B. <u>Large Slip Approximation</u>

For the large slip approximation, the region of interest is characterized by the statement

$$\lambda_m/x \gg 1$$

which, of course, occurs near the leading edge of the plate. The governing equations for the boundary layer are, as before, 28, 29, 30, 31, 32, 33, and 34. As before, a stream function for the gas is introduced, but now the particle slip velocities are not small, and hence it is advantageous to introduce a particle stream function defined by

$$\rho_p u_p = \partial\psi_p/\partial y$$
$$\rho_p v_p = -\,\partial\psi_p/\partial x\,. \tag{83}$$

Now equations 28 and 31 are identically satisfied. Introducing the gas and particle stream functions in equations 29 , 30 , 32 , 33 , and 34 gives the following equations:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\,\frac{\partial^3\psi}{\partial y^3} + \frac{u_\infty}{\rho\lambda_m}\left(\frac{\partial\psi_p}{\partial y} - \rho_p\,\frac{\partial\psi}{\partial y}\right). \tag{84}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \frac{\nu}{Pr}\frac{\partial^2 T}{\partial y^2} + \frac{\rho_p}{\rho}\,\frac{C_s}{C_p}\,u_\infty\,\frac{(T_p - T)}{\lambda_T}\,. \tag{85}$$

$$\frac{\partial\psi_p}{\partial y}\left(\rho_p\,\frac{\partial^2\psi_p}{\partial x\partial y} - \frac{\partial\rho_p}{\partial x}\,\frac{\partial\psi_p}{\partial y}\right) - \frac{\partial\psi_p}{\partial x}\left(\rho_p\,\frac{\partial^2\psi_p}{\partial y^2} - \frac{\partial\rho_p}{\partial y}\,\frac{\partial\psi_p}{\partial y}\right)$$
$$= -\frac{u_\infty}{\lambda_m}\left(\rho_p^{\,2}\,\frac{\partial\psi_p}{\partial y} - \rho_p^{\,3}\,\frac{\partial\psi}{\partial y}\right). \tag{86}$$

$$\frac{\partial\psi_p}{\partial y}\left(\rho_p\,\frac{\partial^2\psi_p}{\partial x^2} - \frac{\partial\rho_p}{\partial x}\,\frac{\partial\psi_p}{\partial x}\right) - \frac{\partial\psi_p}{\partial x}\left(\rho_p\,\frac{\partial^2\psi_p}{\partial x\partial y} - \frac{\partial\rho_p}{\partial y}\,\frac{\partial\psi_p}{\partial x}\right)$$
$$= -\frac{u_\infty}{\lambda_m}\left(\rho_p^{\,2}\,\frac{\partial\psi_p}{\partial x} - \rho_p^{\,3}\,\frac{\partial\psi}{\partial x}\right). \tag{87}$$

$$\frac{\partial\psi_p}{\partial y}\,\frac{\partial T_p}{\partial x} - \frac{\partial\psi_p}{\partial x}\,\frac{\partial T_p}{\partial y} = -\,\rho_p u_\infty\,\frac{(T_p - T)}{\lambda_T}\,. \tag{88}$$

Now, since a particle stream function has been introduced, the particle phase will be treated as a perfectly respectable separate phase, which means that in the zeroth order, the gas flows independently of the particle phase and vice versa. This implies that there is no need to introduce ν^* as was done for the small slip approximation. With this thought in mind, the following transformation is made:

$$\eta = y / \sqrt{\nu x / u_\infty} \quad . \tag{89}$$

The derivatives for this transformation have already been computed in 51, provided ν^* is replaced by ν everywhere.

Making the transformation, 89, in equations 84 through 88 gives the following equations.

$$\frac{\partial\psi}{\partial\eta}\left(\frac{\partial^2\psi}{\partial x\partial\eta} - \frac{1}{2x}\frac{\partial\psi}{\partial\eta}\right) - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial\eta^2} = \left(\frac{\nu u_\infty}{x}\right)^{\frac{1}{2}}\frac{\partial^3\psi}{\partial\eta^3} + \frac{\sqrt{\nu u_\infty x}}{\rho\lambda_m}\left(\frac{\partial\psi_p}{\partial\eta} - \rho_p\frac{\partial\psi}{\partial\eta}\right). \tag{90}$$

$$\sqrt{\frac{u_\infty}{\nu x}}\left(\frac{\partial\psi}{\partial\eta}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial\eta}\right) = \frac{u_\infty}{x\Pr}\frac{\partial^2 T}{\partial\eta^2} + \frac{\rho_p}{\rho}\frac{u_\infty}{\lambda_T}\frac{C_s}{C_p}(T_p - T) . \tag{91}$$

$$\rho_p\frac{\partial\psi_p}{\partial\eta}\left(\frac{\partial^2\psi_p}{\partial x\partial\eta} - \frac{1}{2x}\frac{\partial\psi_p}{\partial\eta}\right) - \left(\frac{\partial\psi_p}{\partial\eta}\right)^2\frac{\partial\rho_p}{\partial x} - \frac{\partial\psi_p}{\partial x}\left(\rho_p\frac{\partial^2\psi_p}{\partial\eta^2} - \frac{\partial\rho_p}{\partial\eta}\frac{\partial\psi_p}{\partial\eta}\right)$$
$$= - \frac{\sqrt{\nu u_\infty x}}{\lambda_m}\left(\rho_p^2\frac{\partial\psi_p}{\partial\eta} - \rho_p^3\frac{\partial\psi}{\partial\eta}\right) \quad . \tag{92}$$

$$\rho_p\frac{\partial\psi_p}{\partial\eta}\left(\frac{\partial^2\psi_p}{\partial x^2} - \frac{\eta}{2x}\frac{\partial^2\psi_p}{\partial\eta\partial x} + \frac{\eta}{2x^2}\frac{\partial\psi_p}{\partial\eta}\right) - \frac{\partial\psi_p}{\partial\eta}\frac{\partial\rho_p}{\partial x}\left(\frac{\partial\psi_p}{\partial x} - \frac{\eta}{2x}\frac{\partial\psi_p}{\partial\eta}\right)$$
$$- \rho_p\frac{\partial\psi_p}{\partial x}\left(\frac{\partial^2\psi_p}{\partial x\partial\eta} - \frac{1}{2x}\frac{\partial\psi_p}{\partial\eta} - \frac{\eta}{2x}\frac{\partial^2\psi_p}{\partial\eta^2}\right) + \frac{\partial\psi_p}{\partial x}\frac{\partial\rho_p}{\partial\eta}\left(\frac{\partial\psi_p}{\partial x} - \frac{\eta}{2x}\frac{\partial\psi_p}{\partial\eta}\right)$$
$$= - \frac{\sqrt{\nu u_\infty x}}{\lambda_m}\rho_p^2\left(\frac{\partial\psi_p}{\partial x} - \frac{\eta}{2x}\frac{\partial\psi_p}{\partial\eta} - \rho_p\frac{\partial\psi}{\partial x} + \rho_p\frac{\eta}{2x}\frac{\partial\psi}{\partial\eta}\right). \tag{93}$$

$$\sqrt{\frac{u_\infty}{\nu x}}\left[\frac{\partial\psi_p}{\partial\eta}\frac{\partial T_p}{\partial x} - \frac{\partial\psi_p}{\partial x}\frac{\partial T_p}{\partial\eta}\right] = - \rho_p u_\infty\frac{(T_p - T)}{\lambda_T} . \tag{94}$$

Now since the particle phase flows independently of the gas phase to zeroth order, and since the particle phase has no way of feeling the presence of the flat plate except through its interaction with the gas, then to zeroth order $u_p = u_\infty$, $v_p = 0$, and therefore, by equation 31,

$$\rho_p = \text{constant}$$

to zeroth order. With this in mind, the following expansions are assumed for ψ, ψ_p, ρ_p, T, and T_p.

$$\left.\begin{aligned}
\psi &= \sqrt{\nu u_\infty x}\,[f_o(\eta) + (\tfrac{x}{\lambda_m}) f_1(\eta) + (\tfrac{x}{\lambda_m})^2 f_2(\eta) + \ldots] \\
\psi_p &= \varkappa\rho\sqrt{\nu u_\infty x}\,[f_p^{(o)}(\eta) + (\tfrac{x}{\lambda_m}) f_p^{(1)}(\eta) + (\tfrac{x}{\lambda_m})^2 f_p^{(2)}(\eta) + \ldots] \\
\rho_p^* &= \frac{\rho_p}{\varkappa\rho} = 1 + \frac{x}{\lambda_m} I_1(\eta) + (\tfrac{x}{\lambda_m})^2 I_2(\eta) + \ldots \\
T^* &= \frac{T - T_w}{T_\infty - T_w} = \theta_o(\eta) + (\tfrac{x}{\lambda_m})\theta_1(\eta) + (\tfrac{x}{\lambda_m})^2 \theta_2(\eta) + \ldots \\
T_p^* &= \frac{T_p - T_w}{T_\infty - T_w} = \theta_p^{(o)}(\eta) + (\tfrac{x}{\lambda_m})\theta_p^{(1)}(\eta) + (\tfrac{x}{\lambda_m})^2 \theta_p^{(2)}(\eta) + \ldots\ .
\end{aligned}\right\} \quad (95)$$

It is expected that $f_p^{(o)}(\eta) = \eta$ from what has already been said.

Now substituting these expansions into equations 90 through 94 and equating coefficients of $(x/\lambda_m)^n$, $n = 0, 1$, to zero gives the following. The zeroth order part of 93 gives

$$\eta f_p^{(o)} f_p''^{(o)} + \eta (f_p'^{(o)})^2 - f_p^{(o)} f_p'^{(o)} = 0$$

and the zeroth order part of 92 gives

$$\tfrac{1}{2} f_p^{(o)} f_p''^{(o)} = 0\,.$$

Hence

$$[\eta f_p'^{(o)} - f_p^{(o)}] f_p'^{(o)} = 0\,,$$

but since

$$f_p'^{(o)} \neq 0$$

$$\eta f_p'^{(o)} - f_p^{(o)} = 0$$

$$\frac{d}{d\eta}(f_p^{(o)}/\eta) = 0$$

$$f_p^{(o)} = A\eta$$

where A is a constant. The boundary conditions on the particle phase have been stated in Chapter II, and they are

$$\left.\begin{array}{ll} \text{(i)} \quad f_p^{(o)}(o) = 0 \;, & f_p^{(1)}(o) = 0 \\ \text{(ii)} \quad f_p'^{(o)}(\infty) = 1 \;, & f_p'^{(1)}(\infty) = 0 \\ \text{(iii)} \quad \theta_p^{(o)}(\infty) = 1 \;, & \theta_p^{(1)}(\infty) = 0 \\ \text{(iv)} \quad I_1(\infty) = 0 & \end{array}\right\} \quad (96)$$

to first order. Therefore

$$A = 1$$

and

$$f_p^{(o)}(\eta) = \eta \tag{97}$$

as was expected. Making use of 97, the first order part of 93 gives

$$\eta^2 f_p''^{(1)} - 3\eta f_p'^{(1)} + 3f_p^{(1)} = 2(f_o - \eta f_o') \;, \tag{98}$$

and the first order part of 92 gives

$$\tfrac{1}{2}\eta I_1' - I_1 = f_o' - 1 + \tfrac{1}{2}\eta f_p''^{(1)} - f_p'^{(1)} \,. \tag{99}$$

Equation 90 yields to zeroth order

$$f_o''' + \tfrac{1}{2} f_o f_o'' = 0 \;, \tag{100}$$

and to first order

$$F_1''' + \frac{1}{2} f_o F_1'' - f_o' F_1' + \frac{3}{2} f_o'' F_1 = (f_o' - 1) \;, \tag{101}$$

where 97 was used to simplify the algebra, and

$$F_1 = \frac{1}{\varkappa} f_1 .$$

Equation 91 yields to zeroth order

$$\theta_o'' + \frac{Pr}{2} f_o(\eta)\theta_o' = 0 \tag{102}$$

and to first order

$$\theta_1'' + \frac{Pr}{2} f_o\theta_1' - Pr f_o'\theta_1 = -\frac{3}{2} Pr f_1\theta_o' - \frac{2}{3}\varkappa(\theta_p^{(o)} - \theta_o) . \tag{103}$$

Equation 94 yields to zeroth order

$$\frac{1}{2} f_p^{(o)} \theta_p'^{(o)} = 0 ,$$

which implies

$$\theta_p^{(o)} = 1 \tag{104}$$

where the boundary condition given in 96 was used. Equation 94 yields to first order

$$\eta\theta_p'^{(1)} - 2\theta_p^{(1)} = \frac{4}{3\,Pr} \frac{C_p}{C_s} (1 - \theta_o) \tag{105}$$

where 97 and 104 were used to simplify the algebra.

The well known solution of the homogeneous part of 98 is

$$A\eta^3 + B\eta ,$$

and to obtain the particular solution, the method of variation of parameters is used, i.e., $A = A(\eta)$ and $B(\eta)$, which leads to

$$A(\eta) = -\frac{f_o(\eta)}{\eta^3} - 2\int_\infty^\eta \frac{f_o(x)}{x^4}\, dx + A_1$$

$$B(\eta) = \frac{f_o}{\eta} + B_1$$

where A_1 and B_1 are constants to be determined by the boundary conditions 96 . The complete solution is

$$f_p^{(1)}(\eta) = 2\eta^3 \int_\eta^\infty \frac{f_o(x)}{x^4} + A_1\eta^3 + B_1\eta .$$

Noting that

$$\lim_{\eta\to 0} 2\eta^3 \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx = \frac{2}{3} f_o(o) = 0 ,$$

it is seen that the boundary condition

$$f_p^{(1)}(o) = 0$$

is satisfied for arbitrary A and B . Applying the boundary condition

$$f_p'^{(1)}(\infty) = 0 ,$$

it is found that

$$\lim_{\eta\to\infty} f_p'^{(1)}(\eta) = \lim_{\eta\to\infty} \left[-\frac{2f_o(\eta)}{\eta} + 6\eta^2 \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx + 3A\eta^2 + B \right] = 0 ,$$

but

$$\lim_{\eta\to\infty} \frac{f_o(\eta)}{\eta} = f_o'(\infty) = 1 ,$$

$$\lim_{\eta\to\infty} 6\eta^2 \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx = \lim_{\eta\to\infty} \frac{3f_o(\eta)}{\eta} = 3 ,$$

and hence

$$\lim_{\eta\to\infty} [-2 + 3 + 3A\eta^2 + B] = 0 ,$$

which implies that

$$A = 0 \quad , \quad B = -1 .$$

Thus

$$f_p^{(1)}(\eta) = 2\eta^3 \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx - \eta \tag{106}$$

$$f_p'^{(1)}(\eta) = -\frac{2f_o(\eta)}{\eta} + 6\eta^2 \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx - 1 \tag{107}$$

$$f_p''^{(1)}(\eta) = \frac{2(f_o - \eta f_o')}{\eta^2} + 12\eta \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx - \frac{6f_o(\eta)}{\eta^2} \quad . \tag{108}$$

Note that

$$\lim_{\eta\to\infty} 2\eta^3 \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx = \frac{2}{3} f_o(\infty) ,$$

$$\lim_{\eta\to 0} 6\eta^2 \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx = 3f_o'(o) = 0 ,$$

$$\lim_{\eta\to 0} \eta \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx = \frac{f_o''(o)}{2} ,$$

$$\lim_{\eta\to\infty} \eta \int_\eta^\infty \frac{f_o(x)}{x^4}\, dx = 0 ,$$

so that the singular point of the integrals gives no problems.

Using the expressions obtained for $f_p'^{(1)}$ and $f_p''^{(1)}$ in equation 99 gives

$$\eta I_1' - 2I_1 = 0$$

$$\frac{d}{d\eta}(I_1/\eta^2) = 0$$

$$I_1 = C\,\eta^2$$

where C is a constant. But since

$$I_1(\infty) = 0 ,$$

it must be that

$$C = 0.$$

Hence,

$$I_1(\eta) \equiv 0 . \tag{109}$$

Also, note that

$$f_p'^{(1)}(o) = -1$$

and

$$f_p''^{(1)}(o) = -2f_o''(o)-2f_o''(o)+6f_o''(o) = 2f_o''(o) .$$

Thus there is a particle slip on the plate, and the $f_p'^{(1)}(o)$ should be negative since in the zeroth approximation, the particles move as though the plate were not there. Consequently, the first order correction should slow the particles down.

Equation 105 can be written

$$\frac{d}{d\eta}\left(\frac{\theta_p^{(1)}}{\eta^2}\right) = \frac{4}{3\,Pr}\left(\frac{C_p}{C_s}\right)\frac{(1-\theta_o)}{\eta^3}$$

$$\theta_p^{(1)} = \frac{4}{3\,Pr}\left(\frac{C_p}{C_s}\right)\eta^2 \int_{\infty}^{\eta} \frac{(1-\theta_o(x))}{x^3}\,dx$$

$$\theta_p^{(1)} = \frac{4}{3\,Pr}\left(\frac{C_p}{C_s}\right)\eta^2\left[-\frac{1}{2\eta^2} - \int_{\infty}^{\eta}\frac{\theta_o(x)}{x^3}\,dx\right]$$

$$\theta_p^{(1)} = -\frac{2}{3\,Pr}\,\frac{C_p}{C_s} - \frac{4}{3\,Pr}\,\frac{C_p}{C_s}\,\eta^2\int_{\infty}^{\eta}\frac{\theta_o(x)}{x^3}\,dx . \tag{110}$$

The boundary conditions for the gas phase are the same as those given in 75. Thus the solution for equation 100 is just the Blasius solution for the gas phase alone. Equation 101, together with its boundary conditions, compose a two-point boundary-value problem with none of the problems that were present in the small-slip approximation. The solution curves for $f_o'(\eta)$ and $F_1'(\eta)$ are given in figure 5. In a similar way, the solutions for 102 and 103 are found for Pr = .10, 1, and 10, and are plotted in figures 6, 7, and 8. It is seen that the particle influence on the gas temperature decreases with increasing Prandtl number. The curves for the solutions given by equations 107 and 110 are given in figures 9a and 9b. It is clear that the first order profiles have the right sign in view of what the zeroth order approximation is.

The shear coefficient for the regime of large slip is

$$c_\tau = \frac{\tau}{\rho u_\infty^2} = \frac{0.332}{\sqrt{Re_x}}[1 + 3.454\,\kappa\,\frac{x}{\lambda_m} + \ldots] \qquad (111)$$

and the heat transfer to the plate is

$$q(o) = -k\,\frac{(T_\infty - T_w)}{x}\,\sqrt{Re_x}\,0.332[1 + 2.575\,\kappa\,(\frac{x}{\lambda_m}) + \ldots] \qquad (112)$$

for Pr = 1.0,

$$q(o) = -k\,\frac{(T_\infty - T_w)}{x}\,\sqrt{Re_x}\,0.14003[1 + 10.406\,\kappa\,(\frac{x}{\lambda_m}) + \ldots] \qquad (113)$$

for Pr = 0.1, and

$$q(o) = -k\,\frac{(T_\infty - T_w)}{x}\,\sqrt{Re_x}\,0.7281[1 + 1.480\,\kappa\,(\frac{x}{\lambda_m}) + \ldots] \qquad (114)$$

for Pr = 10.0.

In figure 10 a plot of the shear coefficients for both the large and small slip approximations is shown. The dotted line represents how the shear coefficient behaves when x is of the order of λ_m. It is seen that the maximum shear coefficient is shifted toward the leading edge as $\varkappa$ increases.

The perturbation method is valid in Section A only if

$$Re_\sigma = \frac{u_p - u}{u_\infty} \frac{u_\infty \sigma}{\nu} \leq 1$$

and $(u_p - u)/u_\infty << 1$. In figure 11, the curve corresponding to $Re_\sigma = 1$ is plotted as the dotted curve. The two above conditions are then

$$\frac{\lambda_m}{x} g_{1_{max}}(\eta) \frac{u_\infty \sigma}{\nu} = \frac{\overline{x}}{x} \leq 1$$

$$\lambda_m / x << 1/(g_{1_{max}}(\eta)) = 8.84$$

where $\overline{x} = \lambda_m (.113) \frac{u_\infty \sigma}{\nu}$. If, for a given flow, the second condition isn't satisfied unless $x > \overline{x}$, then the second condition governs the regime of validity for the expansion.

For air flowing at 1000 ft/sec and $\sigma = 5.4$ microns, then $\overline{x} = 11.3\, \lambda_m$, and then for $x \geq \overline{x}$, the perturbation terms account for something less than 8 per cent of the total solution in the small-slip approximation.

IV. APPLICATION OF INTEGRAL METHODS TO TWO-PHASE FLOW OVER A SEMI-INFINITE FLAT PLATE WHEN THE GAS IS INCOMPRESSIBLE

Just as in the case of a pure gas flow over a semi-infinite flat plate, it is interesting to apply the Karman-Pohlhausen technique to the two-phase flow over the semi-infinite flat plate, and to compare the results with the numerical solutions presented in Chapter III. The equations governing the boundary layer flow are the same as equations 28 through 34 . However, by using the continuity equations for both phases, these equations can be put in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{115}$$

$$\frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y}(uv) = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\rho_p}{\rho} \frac{u_\infty}{\lambda_m}(u_p - u) \tag{116}$$

$$\frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v_p) = 0 \tag{117}$$

$$\frac{\partial}{\partial x}\left(\frac{\rho_p}{\rho} u_p^2\right) + \frac{\partial}{\partial y}\left(\frac{\rho_p}{\rho} u_p v_p\right) = -\frac{\rho_p}{\rho} \frac{u_\infty}{\lambda_m}(u_p - u) \tag{118}$$

$$\frac{\partial}{\partial x}\left(\frac{\rho_p}{\rho} u_p v_p\right) + \frac{\partial}{\partial y}\left(\frac{\rho_p}{\rho} v_p^2\right) = -\frac{\rho_p}{\rho} \frac{u_\infty}{\lambda_m}(v_p - v) \tag{119}$$

where the energy equations have not been considered.

A. Small Slip Approximation

As before, the equations are rewritten so that the particle slip velocities can be easily calculated. Thus, adding 116 and 118 gives

$$\frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial x}\left(\frac{\rho_p}{\rho} u_p^2\right) + \frac{\partial}{\partial y}\left(\frac{\rho_p}{\rho} u_p v_p\right) - \nu \frac{\partial^2 u}{\partial y^2} = 0 , \tag{120}$$

and equation 116 minus 118 gives

$$\frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y}(uv) - \frac{\partial}{\partial x}(\frac{\rho_p}{\rho} u_p^2) - \frac{\partial}{\partial y}(\frac{\rho_p}{\rho} u_p v_p) - \nu \frac{\partial^2 u}{\partial y^2} = \frac{2\rho_p}{\rho} \frac{u_\infty}{\lambda_m}(u_p - u). \quad (121)$$

Now integrating equation 115 from $y = 0$ to $y = \delta(x)$, where $\delta(x)$ is the boundary layer thickness, yields

$$v(y=\delta) = -\frac{\partial}{\partial x}\int_0^{\delta(x)} u\,dy + u_\infty \frac{d\delta}{dx} . \quad (122)$$

Similarly, integrating 117 across the boundary layer gives

$$\varkappa v_p(y=\delta) = -\frac{\partial}{\partial x}\int_0^{\delta(x)} \frac{\rho_p}{\rho} u_p\,dy + \varkappa u_\infty \frac{d\delta}{dx} . \quad (123)$$

In arriving at 122 and 123, the conditions

$$v(o) = 0, \qquad u(y=\delta) = u_\infty$$

$$v_p(o) = 0, \qquad u_p(y=\delta) = u_\infty$$

$$\rho_p(y=\delta) = \varkappa\rho$$

were used. Integrating equation 120 across the boundary layer gives

$$\frac{\partial}{\partial x}\int_0^{\delta(x)} u(u-u_\infty)dy + \frac{\partial}{\partial x}\int_0^{\delta(x)} \frac{\rho_p}{\rho} u_p(u_p - u_\infty)dy + \nu(\frac{\partial u}{\partial y})_{y=0} = 0 \quad (124)$$

where 122 was used to simplify the equation, and the condition $(\partial u/\partial y)_{y=\delta} = 0$ was used. Integrating equation 121 across the boundary layer gives

$$\frac{\partial}{\partial x}\int_0^{\delta(x)} \frac{\rho_p}{\rho} u_p(u_p - u_\infty)dy = -\frac{u_\infty}{\lambda_m}\int_0^{\delta(x)} \frac{\rho_p}{\rho}(u_p - u)dy \quad (125)$$

where equation 123 was used to simplify the equation. Integrating equation 119 across the boundary layer gives

$$\frac{\partial}{\partial x}\int_0^{\delta(x)}\frac{\rho_p}{\rho}u_p v_p dy+\left[\frac{\partial}{\partial x}\int_0^{\delta(x)} u\,dy\right]\left[\frac{\partial}{\partial x}\int_0^{\delta(x)}\frac{\rho_p}{\rho}u_p dy\right]$$

$$-u_\infty\frac{d\delta}{dx}\frac{\partial}{\partial x}\int_0^{\delta(x)}\frac{\rho_p}{\rho}u_p dy=-\frac{u_\infty}{\lambda_m}\int_0^{\delta(x)}\frac{\rho_p}{\rho}(v_p-v)dy \quad . \tag{126}$$

where 123 was used. Now changing dependent variables by letting $u_s = u_p - u$, $v_s = v_p - v$, equations 124, 125, and 126 become

$$\frac{\partial}{\partial x}\int_0^{\delta(x)} u(u-u_\infty)dy+\varkappa\frac{\partial}{\partial x}\int_0^{\delta(x)}\frac{\rho_p}{\varkappa\rho}(u_s+u)(u_s+u-u_\infty)dy+\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}=0, \tag{127}$$

$$\frac{\partial}{\partial x}\int_0^{\delta(x)}\frac{\rho_p}{\varkappa\rho}(u_s+u)(u_s+u-u_\infty)dy=-\frac{u_\infty}{\lambda_m}\int_0^{\delta(x)}\frac{\rho_p}{\varkappa\rho}u_s dy\ , \tag{128}$$

and

$$\frac{\partial}{\partial x}\int_0^{\delta(x)}\frac{\rho_p}{\varkappa\rho}(u_s+u)(v_s+v)dy+\left[\frac{\partial}{\partial x}\int_0^{\delta(x)} u\,dy\right]\left[\frac{\partial}{\partial x}\int_0^{\delta(x)}\frac{\rho_p}{\varkappa\rho}(u_s+u)dy\right]$$

$$-u_\infty\frac{d\delta}{dx}\frac{\partial}{\partial x}\int_0^{\delta(x)}\frac{\rho_p}{\varkappa\rho}(u_s+u)dy=-\frac{u_\infty}{\lambda_m}\int_0^{\delta(x)}\frac{\rho_p}{\varkappa\rho}v_s dy\ . \tag{129}$$

Since equation 129 involves v_s, and since v_s is not involved in 127 and 128, equations 127 and 128 can be considered alone. Letting $\eta=\frac{y}{\delta(x)}$, 127 becomes

$$\frac{\partial}{\partial x}\left[\delta(x)\int_0^1\frac{u}{u_\infty}\left(\frac{u}{u_\infty}-1\right)d\eta\right]+\varkappa\frac{\partial}{\partial x}\left[\delta(x)\int_0^1\frac{\rho_p}{\varkappa\rho}\left(\frac{u_s}{u_\infty}+\frac{u}{u_\infty}\right)\left(\frac{u_s}{u_\infty}+\frac{u}{u_\infty}-1\right)d\eta\right]$$

$$+\frac{\nu}{u_\infty\delta}\left[\frac{\partial}{\partial\eta}\left(\frac{u}{u_\infty}\right)\right]_{\eta=0}=0\ , \tag{130}$$

and 128 becomes

$$\frac{\partial}{\partial x}\left[\delta(x)\int_0^1\frac{\rho_p}{\varkappa\rho}\left(\frac{u_s}{u_\infty}+\frac{u}{u_\infty}\right)\left(\frac{u_s}{u_\infty}+\frac{u}{u_\infty}-1\right)d\eta\right]=-\frac{\delta(x)}{\lambda_m}\int_0^1\frac{\rho_p}{\varkappa\rho}\frac{u_s}{u_\infty}d\eta\ . \tag{131}$$

Now the functions under the integrals must be expressed in terms of the similarity variable, η, but it is known that u/u_∞,

$\rho_p/\varkappa\rho$, and u_s/u_∞ are not functions of η alone, i.e., they are not completely similar, but have the form

$$\left.\begin{aligned} u/u_\infty &= u^{(o)}(\eta) + (\lambda_m/x)u^{(1)}(\eta) + \ldots \\ u_s/u_\infty &= u_s^{(1)}(\eta)(\lambda_m/x) + \ldots \\ \rho_p/\varkappa\rho &= 1 + (\lambda_m/x)\rho_p^{(1)}(\eta) + \ldots \end{aligned}\right\} \qquad (132)$$

Putting these expansions in 130 and equating coefficients of $(\lambda_m/x)^n$, $n = 0, 1$, to zero yields

$$\frac{1}{2}\left[\int_0^1 u^{(o)}(\eta)(u^{(o)}(\eta)-1)d\eta\right]\frac{d\delta^2}{dx} + \frac{\nu^*}{u_\infty}\left(\frac{\partial u^{(o)}}{\partial\eta}\right)_{\eta=0} = 0 \qquad (133)$$

and

$$\left[\frac{1}{2}\frac{d\delta^2}{dx} - \frac{\delta^2(x)}{x}\right]\int_0^1 \{u^{(1)}(2u^{(o)}-1)+\varkappa[u_s^{(1)}(u^{(o)}-1)+u^{(1)}(2u^{(o)}-1)$$

$$+u^{(o)}u_s^{(1)}+\rho_p^{(1)}u^{(o)}(u^{(o)}-1)]\}d\eta + (1+\varkappa)\frac{\nu^*}{u_\infty}\left(\frac{\partial u^{(1)}}{\partial\eta}\right)_{\eta=0} = 0 . \qquad (134)$$

Putting 132 into 131 and taking only the zeroth order part gives

$$\left[\int_0^1 u^{(o)}(u^{(o)}-1)d\eta\right]\frac{d\delta}{dx} = -\frac{\delta(x)}{x}\int_0^1 u_s^{(1)}d\eta . \qquad (135)$$

Now there are three equations, 133, 134, and 135, to solve; consequently, there must be three unknowns, of which $\delta(x)$ is one. This means that an unknown can be included in the specification of $u^{(1)}$ and $u_s^{(1)}$, but $u^{(o)}$ and $\rho_p^{(1)}$ must be completely specified. Note that ν^* is used instead of ν, as it should be, since both phases flow as one phase with kinematic coefficient of viscosity, ν^*, in the zeroth order.

Let

$$A_1 = \int_0^1 u^{(o)}(u^{(o)}-1)d\eta \qquad (136)$$

$$a_1 = \left(\frac{\partial u^{(o)}}{\partial \eta}\right)_{\eta=0} \tag{137}$$

then 133 gives

$$\delta^2 = -\frac{2\nu^*}{u_\infty}\frac{a_1}{A_1}x \tag{138}$$

where we have applied the condition that $\delta(o) = 0$. Let

$$B_1 = \int_0^1 \{2u^{(o)}u^{(1)} - u^{(1)} + \frac{\varkappa}{1+\varkappa}[u_s^{(1)}(2u^{(o)}-1) + \rho_p^{(1)}u^{(o)}(u^{(o)}-1)]\}d\eta \tag{139}$$

$$b_1 = \left(\frac{\partial u^{(1)}}{\partial \eta}\right)_{\eta=0} \quad ;$$

then, since

$$\frac{d\delta^2}{dx} = \frac{\delta^2}{x} ,$$

equation 134 gives

$$-\tfrac{1}{2}\frac{\delta^2}{x}(1+\varkappa)B_1 + \frac{(1+\varkappa)\nu^*}{u_\infty}b_1 = 0 . \tag{140}$$

Or, using 138,

$$B_1/A_1 = -b_1/a_1 . \tag{141}$$

Equation 135 gives

$$\int_0^1 u_s^{(1)}d\eta = -\tfrac{1}{2}A_1 . \tag{142}$$

To evaluate the functions A_1, B_1, a_1, b_1, polynomial profiles are specified for $u^{(o)}$, $u^{(1)}$, $\rho_p^{(1)}$, and $u_s^{(1)}$. Thus, let

$$u^{(o)}(\eta) = c_o + c_1\eta + c_2\eta^2 + c_3\eta^3 ,$$

where the c_i's, $i = 0, 1, 2, 3$, are constants. The boundary conditions for $u^{(o)}$ are

$u^{(o)}(o) = 0$, no slip on the plate,

$u''^{(o)}(o) = 0$, zero pressure gradient,

$u^{(o)}(1) = 1$, gas velocity reaches free stream value at $\eta = 1$,

$u'^{(o)}(1) = 0$, smooth joining condition.

Then

$$u^{(o)}(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3 . \tag{143}$$

Let

$$u^{(1)}(\eta) = d_o + d_1\eta + d_2\eta^2 + d_3\eta^3 ,$$

where the d_i's , $i = 0, 1, 2, 3$, are constants. The boundary conditions for $u^{(1)}$ are

$u^{(1)}(o) = 0$, no slip on the plate,

$u^{(1)}(1) = 0$, gas velocity reaches free stream value at $\eta = 1$,

$u'^{(1)}(1) = 0$, smooth joining condition.

Thus

$$u^{(1)}(\eta) = \frac{\varkappa}{1+\varkappa} C(\eta - 2\eta^2 + \eta^3) \tag{144}$$

where C is an unknown constant. Let

$$\rho_p^{(1)} = e_o + e_1\eta + e_2\eta^2 + e_3\eta^3$$

where the e_i's , $i = 0, 1, 2, 3$, are constants. The boundary conditions for $\rho_p^{(1)}$ are

$\rho_p^{(1)}(o) = 0$, $\rho_p = \varkappa\rho$ on plate,

$\rho_p^{(1)}(1) = 0$, $\rho_p = \varkappa\rho$ at $\eta = 1$,

$\rho_p'^{(1)}(1) = 0$, smooth joining condition.

Thus

$$\rho_p^{(1)}(\eta) = D(\eta - 2\eta^2 + \eta^3) \tag{145}$$

where D is a known constant. Let

$$u_s^{(1)} = f_o + f_1\eta + f_2\eta^2 + f_3\eta^3 + f_4\eta^4 ,$$

where the f_i's , $i = 0, 1, 2, 3, 4$, are constants. The boundary conditions for $u_s^{(1)}$ are

$u_s^{(1)}(o) = 0$,	no slip on plate,
$u_s'^{(1)}(o) = 0$,	zero slope at plate,
$u_s^{(1)}(1) = 0$,	no slip at $\eta = 1$,
$u_s'^{(1)}(1) = 0$,	smooth joining condition.

Then

$$u_s^{(1)} = E(\eta^2 - 2\eta^3 + \eta^4) , \tag{146}$$

where E is an unknown constant. Using 143, 144, 145, and 146,

$$\left.\begin{aligned} A_1 &= -\frac{39}{280} , \\ a_1 &= 3/2 , \\ B_1 &= \frac{\varkappa}{1+\varkappa}\left[\frac{1}{140} C + \frac{3}{280} E - \frac{23}{1440} D\right] , \\ b_1 &= \frac{\varkappa}{1+\varkappa} C . \end{aligned}\right\} \tag{147}$$

Using 147 in 138, 142, and 141 yields

$$\delta^2 = \frac{280}{13} \frac{\nu^* x}{u_\infty} , \tag{148}$$

$$E = \frac{117}{56} , \tag{149}$$

$$C = \frac{351}{1344} - \frac{161}{864} D . \tag{150}$$

Thus, when D is specified, C is determined and the profiles are determined. From Chapter III, small slip approximation,

$$D = .308 ,$$

where the fact that $y/\delta(x)$ is not the same as the η used in Chapter III, Section A, is taken into account. Then

$$C = .204$$

and

$$u^{(1)} = \frac{\varkappa}{1+\varkappa}(.204)\left[\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^2 + \left(\frac{y}{\delta}\right)^3\right]$$

$$u^{(1)} = \frac{\varkappa}{1+\varkappa}(.204)\left[\frac{\eta_o}{4.64} - \frac{2}{(4.64)^2}\eta_o^2 + \frac{1}{(4.64)^3}\eta_o^3\right] \tag{151}$$

where η_o is a new symbol for the η used in Chapter III, Section A. Then

$$\left[\frac{du^{(1)}}{d\eta_o}\right]_{\eta_o=0} = \frac{\varkappa}{1+\varkappa}.0440 ,$$

and this is to be compared with the numerically computed value,

$$\left[\frac{du^{(1)}}{d\eta_o}\right]_{\eta_o=0} = \frac{\varkappa}{1+\varkappa}.159 .$$

Consequently, the quantitative results of this integral method are very poor, although the qualitative results are correct.

To understand why the integral method gives such poor quantitative results, the profiles given by

$$u_s^{(1)} = \frac{117}{56}\left[\frac{1}{(4.64)^2}\eta_o^2 - \frac{2}{(4.64)^3}\eta_o^3 + \frac{1}{(4.64)^4}\eta_o^4\right]$$

$$\rho_p^{(1)} = .308\left[\frac{1}{4.64}\eta_o - \frac{2}{(4.64)^2}\eta_o^2 + \frac{1}{(4.64)^3}\eta_o^3\right]$$

and 151 are plotted and compared with the numerically computed curves in figures 12, 13 , and 14. It is seen that the assumed profiles are very much different from the numerically computed profiles. In particular, the approximate curve for $\rho_p^{(1)}$, which has to be completely specified before the integral method can proceed, is very much in error. If the cubic for $u^{(1)}$ is plotted with C determined from the numerically computed solution in Chapter III, Section A, then the curve given by the dotted line in figure 14 is obtained. Consequently, one has to conclude that the curvature of the first order solutions changes too rapidly to be closely approximated by cubic polynomials.

B. Large Slip Approximation

Starting with equations 115, 116, 117, and 118 and integrating each one from $y = 0$ to $y = \delta(x)$ yields

$$v(y=\delta) = -\frac{\partial}{\partial x}\int_0^{\delta(x)} u\,dy + u_\infty \frac{d\delta}{dx}, \tag{152}$$

$$\varkappa v_p(y=\delta) = -\frac{\partial}{\partial x}\int_0^{\delta(x)} \frac{\rho_p}{\rho} u_p dy + \varkappa u_\infty \frac{d\delta}{dx}, \tag{153}$$

$$\frac{\partial}{\partial x}\left[\int_0^{\delta(x)} \frac{u}{u_\infty}\left(\frac{u}{u_\infty} - 1\right)dy\right] + \frac{\nu}{u_\infty}\left[\frac{\partial}{\partial y}\left(\frac{u}{u_\infty}\right)\right]_{y=0} = \frac{\varkappa}{\lambda_m}\int_0^{\delta(x)} \frac{\rho_p}{\varkappa\rho}\left(\frac{u_p}{u_\infty} - \frac{u}{u_\infty}\right)dy, \tag{154}$$

$$\frac{\partial}{\partial x}\left[\int_0^{\delta(x)} \frac{\rho_p}{\varkappa\rho}\frac{u_p}{u_\infty}\left(\frac{u_p}{u_\infty} - 1\right)dy\right] = -\frac{1}{\lambda_m}\int_0^{\delta(x)} \frac{\rho_p}{\varkappa\rho}\left(\frac{u_p}{u_\infty} - \frac{u}{u_\infty}\right)dy, \tag{155}$$

where equations 152 and 153 were used to get 154 and 155, and the conditions

$$v(o) = 0\,, \qquad u(y=\delta) = u_\infty$$

$$v_p(o) = 0\,, \qquad u_p(y=\delta) = u_\infty$$

$$\rho_p(y=\delta) = \varkappa\rho\,, \qquad \left(\frac{\partial u}{\partial y}\right)_{y=\delta} = 0\,,$$

also were used. Letting $\eta = \frac{y}{\delta(x)}$, then 154 and 155 become

$$\frac{\partial}{\partial x}\left[\delta(x)\int_0^1 \frac{u}{u_\infty}\left(\frac{u}{u_\infty}-1\right)d\eta\right] + \frac{\nu}{u_\infty\delta}\left[\frac{\partial}{\partial\eta}\left(\frac{u}{u_\infty}\right)\right]_{\eta=0}$$

$$= \frac{\varkappa\delta}{\lambda_m}\int_0^1 \frac{\rho_p}{\varkappa\rho}\left(\frac{u_p}{u_\infty}-\frac{u}{u_\infty}\right)d\eta\;, \tag{156}$$

$$\frac{\partial}{\partial x}\left[\delta(x)\int_0^1 \frac{\rho_p}{\varkappa\rho}\frac{u_p}{u_\infty}\left(\frac{u_p}{u_\infty}-1\right)d\eta\right] = -\frac{\delta}{\lambda_m}\int_0^1 \frac{\rho_p}{\varkappa\rho}\left(\frac{u_p}{u_\infty}-\frac{u}{u_\infty}\right)d\eta\,. \tag{157}$$

In the regime of large particle slip velocities, the particles and gas phase flow independently of each other in the zeroth order, as was explained in Chapter III, Section B. Therefore, the following expansions are assumed for u/u_∞, u_p/u_∞, and $\rho_p/\varkappa\rho$.

$$\left.\begin{aligned}
\frac{u}{u_\infty} &= u^{(o)}(\eta) + \frac{x}{\lambda_m}u^{(1)}(\eta) + \ldots \\
\frac{u_p}{u_\infty} &= 1 + \frac{x}{\lambda_m}u_p^{(1)}(\eta) + \ldots \\
\frac{\rho_p}{\varkappa\rho} &= 1 + \frac{x}{\lambda_m}\rho_p^{(1)}(\eta) + \ldots\;.
\end{aligned}\right\} \tag{158}$$

Substituting 158 into 156 gives to zeroth order

$$\delta^2 = -\frac{2\nu}{u_\infty}\frac{a_1}{A_1}x\,, \tag{159}$$

where

$$A_1 = \int_0^1 u^{(o)}(u^{(o)}-1)d\eta\;,$$

$$a_1 = \left(\frac{\partial u^{(o)}}{\partial \eta}\right)_{\eta=0} ,$$

$$\delta(o) = 0 ,$$

and to first order

$$3B_1 - A_1 \frac{b_1}{a_1} = 2\kappa C_1 , \tag{160}$$

where

$$B_1 = \int_0^1 u^{(1)}(2u^{(o)}-1)d\eta ,$$

$$b_1 = \left(\frac{\partial u^{(1)}}{\partial \eta}\right)_{\eta=0} ,$$

$$C_1 = \int_0^1 (1 - u^{(o)})d\eta .$$

Substituting 158 into 157 gives to first order

$$\frac{3}{2} \int_0^1 u_p^{(1)} d\eta = - C_1 . \tag{161}$$

Equation 159 implies that $u^{(o)}$ must be completely specified. Equation 160 implies that the assumed form for $u^{(1)}$ must contain an unknown constant, and likewise, 161 implies that the assumed form for $u_p^{(1)}$ must contain an unknown constant. Let $u^{(o)}(\eta)$ be represented by a cubic polynomial, and then with the conditions

$$u^{(o)}(o) = 0 , \quad u'^{(o)}(1) = 0$$

$$u^{(o)}(1) = 1 , \quad u''^{(o)}(o) = 0 ,$$

$u^{(o)}(\eta)$ becomes

$$u^{(o)}(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3 . \tag{162}$$

Likewise, let $u^{(1)}(\eta)$ be represented by a cubic polynomial, and then with the conditions

$$u^{(1)}(o) = 0 \ , \quad u^{(1)}(1) = 0 \ , \quad u'^{(1)}(1) = 0 \ ,$$

$u^{(1)}(\eta)$ becomes

$$u^{(1)}(\eta) = \varkappa G(\eta - 2\eta^2 + \eta^3) \tag{163}$$

where G is an unknown constant. Let $u_p^{(1)}$ be represented by a quadratic polynomial with the conditions that

$$u_p^{(1)}(1) = 0 \ , \qquad u_p'^{(1)}(1) = 0 \ .$$

Then

$$u_p^{(1)} = E\,(1 - 2\eta + \eta^2) \tag{164}$$

where E is an unknown constant.

Then

$$A_1 = -39/280 \ ,$$

$$a_1 = 3/2 \ ,$$

$$C_1 = 3/8 \ ,$$

$$B_1 = \varkappa G/140 \ ,$$

$$b_1 = \varkappa G \ ,$$

and 159 becomes

$$\delta^2(x) = \frac{280}{13} \frac{\nu x}{u_\infty} \ ,$$

160 gives

$$G = 105/16 \ ,$$

and 161 gives

$$E = -3/4 \ .$$

Therefore,

$$u^{(1)} = \varkappa \frac{105}{16} \left[\frac{1}{4.64} \eta_o - \frac{2}{(4.64)^2} \eta_o^2 + \frac{1}{(4.64)^3} \eta_o^3 \right] \ , \tag{165}$$

$$u_p^{(1)} = -\frac{3}{4}\left[1 - \frac{2}{4.64}\eta_o + \frac{1}{(4.64)^2}\eta_o^2\right] , \qquad (166)$$

where η_o is a new symbol representing the symbol , η , in Chapter III, Section B , since the η used in that chapter is not equal to y/δ , i.e.,

$$\frac{\eta_o}{4.64} = \frac{y}{\delta} = \eta .$$

Therefore,

$$\left[\frac{du^{(1)}}{d\eta_o}\right]_{\eta_o=0} = \varkappa\, 1.412 ,$$

and this is to be compared with the numerically computed value

$$\left[\frac{du^{(1)}}{d\eta_o}\right]_{\eta_o=0} = \varkappa\, 1.146 .$$

Thus, there is good quantitative agreement in the large slip regime. Equations 165 and 166 are plotted in figures 15 and 16 together with their corresponding numerically computed curves for comparison.

V. CURVATURE EFFECTS ON TWO-PHASE BOUNDARY LAYER WHEN THE GAS IS INCOMPRESSIBLE

In studying the two-phase boundary layer on a curved surface, the particle density distribution is one of the important effects to be considered. The velocity normal to the boundary layer is crucial in determining this density distribution, and consequently, the normal velocity must be computed correctly. However, it has already been pointed out in Chapter III, Section A, that the usual boundary layer approximation does not give the correct normal velocity, and thus it is necessary to treat the curved wall boundary layer in a somewhat different manner than in the usual way.

A solid circular cylinder, radius R, is considered to be at rest in a gas containing a distribution of the usual particles. At time $t = 0$, the mixture is set into a vortex motion while the cylinder remains fixed. The governing equations for the system written in two-dimensional, cylindrical coordinates, where

u = gas radial velocity

u_p = particle radial velocity

v = gas tangential velocity

v_p = particle tangential velocity

r, ϕ = cylindrical coordinates

t = time,

are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \phi} = 0 , \tag{167}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \phi} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \phi}\right) + \frac{1}{\rho}F_{pr}, \quad (168)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \phi} + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \phi} + \nu\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \phi} - \frac{v}{r^2}\right) + \frac{1}{\rho}F_{p\theta}, \quad (169)$$

$$\rho C_p\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \phi}\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{k}{r}\frac{\partial T}{\partial \phi}\right) + Q_p, \quad (170)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r},$$

$$\frac{\partial \rho_p}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_p u_p) + \frac{1}{r}\frac{\partial}{\partial \phi}(\rho_p v_p) = 0, \quad (171)$$

$$\rho_p\left(\frac{\partial u_p}{\partial t} + u_p\frac{\partial u_p}{\partial r} + \frac{v_p}{r}\frac{\partial u_p}{\partial \phi} - \frac{v_p^2}{r}\right) = -F_{pr}, \quad (172)$$

$$\rho_p\left(\frac{\partial v_p}{\partial t} + u_p\frac{\partial v_p}{\partial r} + \frac{v_p}{r}\frac{\partial v_p}{\partial \phi} + \frac{u_p v_p}{r}\right) = -F_{p\theta}, \quad (173)$$

$$\rho_p C_s\left(\frac{\partial T_p}{\partial t} + u_p\frac{\partial T_p}{\partial r} + \frac{v_p}{r}\frac{\partial T_p}{\partial \phi}\right) = -Q_p, \quad (174)$$

$$F_{pr} = \rho_p\frac{(u_p - u)}{\tau_m},$$

$$F_{p\theta} = \rho_p\frac{(v_p - v)}{\tau_m},$$

$$Q_p = \rho_p C_s\frac{(T_p - T)}{\tau_T}.$$

For incompressible flow, the work done by compression and all dissipation terms can be neglected in the energy equation for the gas. For this problem, there is no angular dependence, and hence from 167

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) = 0$$

$$ru = B(t)$$

$$u = \frac{B(t)}{r} \quad .$$

But

$$u(r=R) \equiv 0 \ ,$$

hence

$$B(t) \equiv 0 \ ,$$

and

$$u \equiv 0 \ .$$

Therefore, letting

$$r = R + y \ ,$$

equations 168 through 174 become

$$-\frac{v^2}{R+y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\rho_p}{\rho}\frac{u_p}{\tau_m} \ , \tag{175}$$

$$\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial y^2} + \frac{1}{R+y}\frac{\partial y}{\partial y} - \frac{v}{(R+y)^2}\right) + \frac{\rho_p}{\rho}\frac{(v_p - v)}{\tau_m} \ , \tag{176}$$

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial y^2} + \frac{k}{R+y}\frac{\partial T}{\partial y} + \rho_p C_s \frac{(T_p - T)}{\tau_T} \ , \tag{177}$$

$$\frac{\partial \rho_p}{\partial t} + \rho_p \frac{\partial u_p}{\partial y} + u_p \frac{\partial \rho_p}{\partial y} + \frac{\rho_p u_p}{R+y} = 0 \ , \tag{178}$$

$$\frac{\partial u_p}{\partial t} + u_p \frac{\partial u_p}{\partial y} - \frac{v_p^2}{R+y} = -\frac{u_p}{\tau_m} \ , \tag{179}$$

$$\frac{\partial v_p}{\partial t} + u_p \frac{\partial v_p}{\partial y} + \frac{u_p v_p}{R+y} = -\frac{(v_p - v)}{\tau_m} \ , \tag{180}$$

$$\rho_p C_s\left(\frac{\partial T_p}{\partial t} + u_p \frac{\partial T_p}{\partial y}\right) = -\rho_p C_s \frac{(T_p - T)}{\tau_T} \ . \tag{181}$$

Now, making the usual boundary layer approximation by letting

$$y = \frac{1}{\sqrt{Re}}\, y^* \ ,$$

and then from 178

$$u_p = \frac{1}{\sqrt{Re}} u_p^* ,$$

provided $R \gg y$, i.e., $R \sim O(\sqrt{Re})$, where Re = Reynolds number. Then the resulting boundary layer equations from 175 through 181 are

$$\frac{\partial p}{\partial y} \sim O(\frac{1}{\sqrt{Re}}) ,$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2} + \frac{\rho_p}{\rho} \frac{(v_p - v)}{\tau_m} , \tag{182}$$

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2} + \rho_p C_s \frac{(T_p - T)}{\tau_T} , \tag{183}$$

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial y} (\rho_p u_p) = 0 , \tag{184}$$

$$\frac{\partial u_p}{\partial t} + u_p \frac{\partial u_p}{\partial y} - \frac{v_p^2}{R} = - \frac{u_p}{\tau_m} , \tag{185}$$

$$\frac{\partial v_p}{\partial t} + u_p \frac{\partial v_p}{\partial y} = - \frac{(v_p - v)}{\tau_m} , \tag{186}$$

$$\rho_p C_s (\frac{\partial T_p}{\partial t} + u_p \frac{\partial T_p}{\partial y}) = - \rho_p C_s \frac{(T_p - T)}{\tau_T} . \tag{187}$$

Thus, whereas pure gas phase boundary layers are not affected by curvature, a gas with particles is affected by curvature since curvature does affect the particle motions.

This chapter will deal with the regime of small particle slip velocities and small temperature defects which may be characterized by the statement

$$\tau_m / t \ll 1 ,$$

which has the same interpretation as the statement $\lambda_m/x << 1$ in Chapter II. With an expansion in powers of τ_m/t in mind, the equations are rewritten in the following way so that the first-order slip quantities can be determined immediately.

Multiplying equation 186 by ρ_p/ρ and adding to 182 gives

$$(1+\frac{\rho_p}{\rho})\frac{\partial v}{\partial t} - \nu\frac{\partial^2 v}{\partial y^2} + \frac{\rho_p}{\rho}\left[\frac{\partial v_s}{\partial t} + u_p\frac{\partial v_s}{\partial y} + u_p\frac{\partial v}{\partial y}\right] = 0 \tag{188}$$

where $v_s = v_p - v$. Equation 182 minus 186 gives

$$-\nu\frac{\partial^2 v}{\partial y^2} - \left[\frac{\partial v_s}{\partial t} + u_p\frac{\partial v_s}{\partial y} + u_p\frac{\partial v}{\partial y}\right] = (1+\frac{\rho_p}{\rho})\frac{v_s}{\tau_m} . \tag{189}$$

Adding equations 183 and 187 gives

$$\frac{\partial T}{\partial t} - \frac{\nu}{Pr}\frac{\partial^2 T}{\partial y^2} + \frac{\rho_p}{\rho}\frac{C_s}{C_p}(\frac{\partial T}{\partial t} + u_p\frac{\partial T}{\partial y} + \frac{\partial T_s}{\partial y} + u_p\frac{\partial T_s}{\partial y}) = 0 \tag{190}$$

where $T_s = T_p - T$.

Equation 183 divided by ρC_p minus equation 187 divided by $\rho_p C_s$ gives

$$-\frac{\nu}{Pr}\frac{\partial^2 T}{\partial y^2} - \left[\frac{\partial T_s}{\partial t} + u_p\frac{\partial T_s}{\partial y} + u_p\frac{\partial T}{\partial y}\right] = (1+\frac{\rho_p}{\rho}\frac{C_s}{C_p})\frac{T_s}{\tau_T} . \tag{191}$$

Equation 185 becomes

$$\frac{\partial u_p}{\partial t} + u_p\frac{\partial u_p}{\partial y} - \frac{1}{R}[v_s^2 + 2vv_s + v^2] = -\frac{u_p}{\tau_m} . \tag{192}$$

Now since u_p is of first order, i.e., $O(\tau_m/t)$, then from equation 184,

$$\rho_p = \varkappa\rho + O(\tau_m/t) .$$

Hence the following expansions are made for the dependent variables v, u_p, v_s, $\rho_p/\varkappa\rho$, T, and T_s.

$$\left.\begin{aligned}
v &= V_o(y,t) + \left(\frac{\tau_m}{t}\right)\frac{\varkappa}{1+\varkappa} V_1(y,t) + \ldots \\
u_p &= u_s^{(1)}(y,t)\left(\frac{\tau_m}{t}\right) + \ldots \\
v_s &= V_s^{(1)}(y,t)\left(\frac{\tau_m}{t}\right) + \ldots \\
\frac{\rho_p}{\varkappa\rho} &= 1 + I_1(y,t)\left(\frac{\tau_m}{t}\right) + \ldots \\
\frac{T-T_w}{T_\infty - T_w} &= \theta_o(y,t) + \left(\frac{\tau_m}{t}\right)\left(\frac{\varkappa}{1+\varkappa}\right)\theta_1(y,t) \\
\frac{T_p - T}{T_\infty - T_w} &= \theta_s^{(1)}(y,t)\left(\frac{\tau_m}{t}\right) + \ldots
\end{aligned}\right\} \tag{193}$$

Substituting the expansion 193 into equation 188 gives to zeroth order

$$\frac{\partial V_o}{\partial t} = \nu^* \frac{\partial^2 V_o}{\partial y^2} , \tag{194}$$

and to first order

$$\nu^* \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial V_1}{\partial t} + \frac{V_1}{t} = I_1 \frac{\partial V_o}{\partial t} + \frac{\partial V_s^{(1)}}{\partial t} - \frac{1}{t} V_s^{(1)} + u_s^{(1)} \frac{\partial V_o}{\partial y} . \tag{195}$$

Likewise, equation 184 gives to first order

$$\frac{\partial I_1}{\partial t} - \frac{1}{t} I_1 + \frac{\partial u_s^{(1)}}{\partial y} = 0 ; \tag{196}$$

equation 189 gives to zeroth order

$$V_s^{(1)} = -\nu^* t \frac{\partial^2 V_o}{\partial y^2} ; \tag{197}$$

equation 192 gives to zeroth order

$$u_s^{(1)} = \frac{t}{R} V_o^2 ; \tag{198}$$

equation 190 gives to zeroth order

$$\left(\frac{1+\varkappa(C_s/C_p)}{1+\varkappa}\right)\frac{\partial \theta_o}{\partial t} - \frac{\nu^*}{Pr} \frac{\partial^2 \theta_o}{\partial y^2} = 0 ; \tag{199}$$

and to first order

$$\frac{\nu^*}{Pr}\frac{\partial^2\theta_1}{\partial y^2} - \left(\frac{1+\varkappa(C_s/C_p)}{1+\varkappa}\right)\left(\frac{\partial\theta_1}{\partial t} - \frac{\theta_1}{t}\right) = \frac{C_s}{C_p}\left[I_1\frac{\partial\theta_o}{\partial t} + \frac{\partial\theta_s^{(1)}}{\partial t} - \frac{1}{t}\theta_s^{(1)} + u_s^{(1)}\frac{\partial\theta_o}{\partial y}\right] \;; \tag{200}$$

equation 191 gives to zeroth order

$$\theta_s^{(1)} = -\left(\frac{1+\varkappa}{1+\varkappa(C_s/C_p)}\right)\left(\frac{\tau_T}{\tau_m}\right)\left(\frac{\nu^*}{Pr}\right)t\,\frac{\partial^2\theta_o}{\partial y^2} \,. \tag{201}$$

The well-known general solution to 194 is

$$V_o = A\int_0^{y/\sqrt{\nu^* t}} e^{-x^2/4}\,dx + B \,.$$

The boundary conditions are that

$$V_o(0,t) = 0$$

and

$$V_o(r,t) \to \frac{\Gamma}{2\pi r} \quad \text{as} \quad r \to \infty \,,$$

which in the boundary layer approximation becomes

$$V_o(\infty,t) = \frac{\Gamma}{2\pi(R+y)} \approx \frac{\Gamma}{2\pi R} = V \,, \text{ a constant.}$$

Hence, $B = 0$, $A = V/\sqrt{\pi}$ and

$$V_o = \frac{V}{\sqrt{\pi}}\int_0^{y/\sqrt{\nu^* t}} e^{-x^2/4}\,dx = V\,\mathrm{erf}\left(\frac{y}{\sqrt{4\nu^* t}}\right) \tag{202}$$

where

$$\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\xi^2}d\xi \,.$$

Then

$$\pi_s^{(1)} = \frac{t}{R}V^2\,\mathrm{erf}^2\left(\frac{y}{\sqrt{4\nu^* t}}\right) \tag{203}$$

and

$$V_s^{(1)} = \frac{V}{2\sqrt{\pi\nu^* t}}\, y\, e^{-\frac{1}{4}\frac{y^2}{\nu^* t}} . \tag{204}$$

By multiplying equation 196 by $1/t$, it can be put in the form

$$\frac{\partial}{\partial t}\left(\frac{I_1}{t}\right) = -\left(\frac{4V^2}{\pi R\sqrt{2\nu^*}}\right)\frac{1}{\sqrt{t}}\, e^{-\frac{1}{4}(y^2/\nu^* t)} \int_0^{y/\sqrt{2\nu^* t}} e^{-x^2/2}\, dx ,$$

which, upon integration and a change of integration variable, yields

$$I_1 = -\frac{V^2}{\sqrt{\pi}\, R}\,\frac{ty}{\nu^*}\int_{y^2/4\nu^* t}^{\infty} \xi^{-3/2}\, e^{-\xi}\, \mathrm{erf}(\sqrt{\xi})\, d\xi + A(y)t .$$

The term $A(y)t$ in the formula for I_1 is a solution of the homogeneous differential equation which must vanish as $t \to \infty$, since the particular solution is the only one that is wanted for large times. Hence, $A(y)$ is taken to be zero, and then

$$I_1(y, t) = -\left(\frac{V^2}{\sqrt{\pi}\, R}\,\frac{ty}{\nu^*}\right)\int_{y^2/4\nu^* t}^{\infty} \xi^{-3/2}\, e^{-\xi}\, \mathrm{erf}(\sqrt{\xi})\, d\xi . \tag{205}$$

Now using equations 202, 203, 204, and 205, equation 195 becomes

$$\nu^* \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial V_1}{\partial t} + \frac{V_1}{t} = \left(\frac{V^3}{2\pi R\nu^*}\right)\frac{y^2}{\sqrt{\nu^* t}}\, e^{-\frac{1}{4}(y^2/\nu^* t)}$$

$$\times \int_{\frac{y^2}{4\nu^* t}}^{\infty} \xi^{-3/2}\, e^{-\xi}\operatorname{erf}(\sqrt{\xi})\,d\xi - \frac{3}{4}\left(\frac{V}{\sqrt{\pi\nu^*}}\right)\frac{y}{t^{3/2}}\, e^{-\frac{1}{4}(y^2/\nu^* t)} \tag{206}$$

$$+ \frac{V}{8\nu^*\sqrt{\pi\nu^*}}\frac{y^3}{t^{5/2}}\, e^{-\frac{1}{4}(y^2/\nu^* t)} + \left(\frac{V^3}{R\sqrt{\nu^*\pi}}\right)\sqrt{t}\, e^{-\frac{1}{4}\frac{y^2}{\nu^* t}}\operatorname{erf}^2\left(\frac{y}{\sqrt{4\nu^* t}}\right) .$$

Multiplying equation 206 by $1/t$ and defining $\widetilde{V}_1 = \frac{1}{t} V_1$, then

$$\nu^* \frac{\partial^2 \widetilde{V}_1}{\partial y^2} - \frac{\partial \widetilde{V}_1}{\partial t} = F(y, t) \tag{207}$$

where

$$F(y,t) = \left(\frac{V^3}{2\pi R\nu^*\sqrt{\nu^*}}\right)\frac{y^2}{t^{3/2}}\, e^{-\frac{1}{4}\frac{y^2}{\nu^* t}} \int_{\frac{y^2}{4\nu^* t}}^{\infty} \xi^{-3/2} e^{-\xi}\operatorname{erf}(\sqrt{\xi})\,d\xi$$

$$- \frac{3}{4}\left(\frac{V}{\sqrt{\pi\nu^*}}\right)\frac{y}{t^{5/2}}\, e^{-\frac{1}{4}\frac{y^2}{\nu^* t}} + \frac{V}{8\nu^*\sqrt{\pi\nu^*}}\frac{y^3}{t^{7/2}}\, e^{-\frac{1}{4}\frac{y^2}{\nu^* t}}$$

$$+ \left(\frac{V^3}{R\sqrt{\nu^*\pi}}\right)\frac{1}{\sqrt{t}}\, e^{-\frac{1}{4}\frac{y^2}{\nu^* t}}\operatorname{erf}^2\left(\frac{y}{\sqrt{4\nu^* t}}\right) .$$

For $Pr = 1$, $C_s/C_p = 1$, equation 199 becomes

$$\frac{\partial\theta_o}{\partial t} - \nu^* \frac{\partial^2\theta_o}{\partial y^2} = 0 \tag{208}$$

which, with the boundary condition (v) for the gas phase, implies that

$$\theta_o = \frac{1}{V} V_o . \tag{209}$$

Then since $\tau_T/\tau_m = 3/2$,

$$\theta_s^{(1)} = -\frac{3}{2}\frac{\nu^* t}{V}\frac{\partial^2 V_o}{\partial y^2} = \frac{3}{2}\frac{V_s^{(1)}}{V} . \tag{210}$$

Then equation 200 becomes, with $\tilde{\theta}_1 = \frac{1}{t}\theta_1$,

$$\nu^* \frac{\partial^2 \tilde{\theta}_1}{\partial y^2} - \frac{\partial \tilde{\theta}_1}{\partial t} = \frac{1}{V} G(y, t) \tag{211}$$

where

$$G(y, t) = \left(\frac{V^3}{\pi 2R\nu^* \sqrt{\nu^*}}\right)\frac{y^2}{t^{3/2}} e^{-\frac{1}{4}\frac{y^2}{\nu^* t}} \int_{\frac{y^2}{4\nu^* t}}^{\infty} \xi^{-3/2} e^{-\xi} \operatorname{erf}(\sqrt{\xi})d\xi$$

$$- \frac{9}{8}\left(\frac{V}{\sqrt{\pi\nu^*}}\right)\frac{y}{t^{5/2}} e^{-\frac{1}{4}\frac{y^2}{\nu^* t}} + \frac{3}{16}\frac{V}{\nu^*\sqrt{\pi\nu^*}}\frac{y^3}{t^{7/2}} e^{-\frac{1}{4}\frac{y^2}{\nu^* t}}$$

$$+ \frac{V^3}{R\sqrt{\nu^*\pi}}\frac{1}{\sqrt{t}} e^{-\frac{1}{4}\frac{y^2}{\nu^* t}} \operatorname{erf}^2\left(\frac{y}{\sqrt{4\nu^* t}}\right) .$$

To get a complete solution to equation 207 would be a very difficult task indeed; hence a solution will be obtained for very large t and then for very small y . For very large t ,

$$\operatorname{erf}\left(\frac{y}{\sqrt{4\nu^* t}}\right) \approx \frac{2}{\sqrt{\pi}}\frac{y}{\sqrt{4\nu^* t}}$$

and

$$\frac{1}{t^{3/2}} \int_{\frac{y^2}{4\nu^* t}}^{\infty} \xi^{-3/2} e^{-\xi} \operatorname{erf}(\sqrt{\xi})d\xi \approx \frac{2}{3}\frac{1}{\sqrt{\pi}}\frac{1}{t^{3/2}} e^{-y^2/(4\nu^* t)}$$

and hence

$$F(y,t) \approx \frac{V^3}{\pi R\nu^* \sqrt{\nu^*\pi}} \frac{y^2}{t^{3/2}} \left(\frac{2}{3} e^{-(y^2/2\nu^* t)} + e^{-\frac{1}{4}(y^2/\nu^* t)} \right)$$
$$+ O\left(\frac{e^{-\frac{1}{4}(y^2/\nu^* t)}}{t^{5/2}} \right) . \qquad (212)$$

Now let

$$\tilde{V}_1 = f(y) \frac{\partial^2 V_o}{\partial y^2}$$

and substitute in equation 207 where $F(y,t)$ is given by 212. Then

$$\frac{df}{dy}\left(2 - \frac{y^2}{\nu^* t}\right) + y \frac{d^2 f}{dy^2} = -\left(\frac{2V^2}{\pi R\nu^*}\right) y^2 \left(\frac{2}{3} e^{-y^2/4\nu^* t} + 1\right) + O\left(\frac{1}{t}\right) .$$

For very large t, the approximations

$$y^2/\nu^* t \ll 1$$
$$e^{-y^2/4\nu^* t} \approx 1$$

are made. Then

$$y^2 \frac{d^2 f}{dy^2} + 2y \frac{df}{dy} = -\left(\frac{10V^2}{3\pi R\nu^*}\right) y^3$$

$$\frac{d}{dy}\left(y^2 \frac{df}{dy}\right) = -\left(\frac{10V^2}{3\pi R\nu^*}\right) \frac{1}{4} \frac{d}{dy}\left(y^4\right)$$

$$\frac{df}{dy} = -\left(\frac{10V^2}{3\pi R\nu^*}\right) \frac{1}{4} y^2 + \frac{A}{y^2}$$

$$f(y) = -\left(\frac{10V^2}{3\pi R\nu^*}\right) \frac{1}{12} y^3 - \frac{A}{y} + B$$

where A and B are integration constants. Then

$$V_1 = -\frac{1}{2} \frac{V}{\sqrt{\pi\nu^*}\ \nu^*} \frac{y}{t^{1/2}} e^{-\frac{1}{4}(y^2/\nu^* t)} \left[-\left(\frac{10V^2}{3\pi R\nu^*}\right) \frac{y^3}{12} - \frac{A}{y} + B \right] .$$

The boundary conditions, (ii) and (iii), given in Chapter II for the gas phase imply that

$$V_1(0,t) = V_1(\infty, t) = 0 , \tag{213}$$

and hence $A = 0$. Both boundary conditions are satisfied for arbitrary B, but B corresponds to a solution of the homogeneous differential equation and hence is taken equal to zero since only a particular solution is wanted. Then

$$V_1 = \frac{5}{36} \frac{V^3}{\pi \sqrt{\pi\nu^*}\, \nu^{*2} R} \frac{y^4}{t^{17/2}} e^{-\frac{1}{4}(y^2/\nu^* t)} \tag{214}$$

and since for large t

$$G(y,t) = F(y,t) ,$$

then

$$\theta_1 = \frac{1}{V} V_1 . \tag{215}$$

For very small y, i.e., y close to the boundary,

$$F(y,t) \approx -\frac{3}{4} \left(\frac{V}{\sqrt{\pi\nu^*}} \right) \frac{y}{t^{5/2}} e^{-\frac{1}{4}(y^2/\nu^* t)} + O(y^2)$$

and then

$$\nu^* \frac{\partial^2 \tilde{V}_1}{\partial y^2} - \frac{\partial \tilde{V}_1}{\partial t} = -\frac{3}{4} \left(\frac{V}{\sqrt{\pi\nu^*}} \right) \frac{y}{t^{5/2}} e^{-\frac{1}{4}(y^2/\nu^* t)} . \tag{216}$$

Letting

$$\tilde{V}_1 = g(t) \frac{\partial^2 V_o}{\partial y^2}$$

and substituting into 216 gives

$$-\frac{1}{2} \frac{dg}{dt} = \frac{3}{4} \frac{\nu^*}{t}$$

or

$$\frac{dg}{d\left(\frac{tV^2}{\nu^*}\right)} = -\frac{3}{2}\,\frac{\nu^*}{\left(\frac{tV^2}{\nu^*}\right)}$$

$$g = -\frac{3}{2}\nu^*\log\left(\frac{tV^2}{\nu^*}\right) + C$$

where C is an integration constant. Hence

$$V_1 = \frac{3}{4}\frac{V}{\sqrt{\pi\nu^*}}\left[\frac{\log}{t^{1/2}}\left(\frac{tV^2}{\nu^*}\right)\right] y\, e^{-\frac{1}{4}(y^2/\nu^* t)} + C\frac{\partial^2 V_o}{\partial y^2}. \qquad (217)$$

Both boundary conditions given in 213 are satisfied by 217 for arbitrary C , but C corresponds to the solution of the homogeneous equation and hence is set equal to zero. Since, for small y ,

$$G(y,t) = \frac{3}{2}F(y,t),$$

then

$$\theta_1 = \frac{9}{8}\frac{1}{\sqrt{\pi\nu^*}}\left[\frac{\log}{t^{1/2}}\left(\frac{tV^2}{\nu^*}\right)\right] y\, e^{-\frac{1}{4}(y^2/\nu^* t)}. \qquad (218)$$

The shear at the wall is

$$\tau = \frac{\mu V}{\sqrt{\pi\nu^* t}}\left[1 + \frac{\varkappa}{1+\varkappa}\frac{3}{4}\log\left(\frac{tV^2}{\nu^*}\right)\frac{\tau_m}{t} + \dots\right] \qquad (219)$$

and the heat transfer to the wall is

$$q(o) = \frac{k(T_\infty - T_w)}{\sqrt{\pi\nu^* t}}\left[1 + \frac{\varkappa}{1+\varkappa}\frac{9}{8}\log\left(\frac{tV^2}{\nu^*}\right)\frac{\tau_m}{t} + \dots\right]. \qquad (220)$$

It is interesting to note that neither 219 nor 220 depend on the curvature, R , and that both 219 and 220 decrease with time as they should since the boundary layer becomes thicker as time increases.

A graph of equation 202 is given in figure 17 showing how the boundary layer increases as time increases. A graph of equations

203 and 204 is given in figures 18 and 19 showing that the particle slip velocities move away from the boundary as time increases in accordance with the thickening of the boundary layer. Figure 18 also shows that the radial slip velocity approaches a constant outside the boundary layer. Furthermore, equation 204 shows that the tangential particle slip velocity decreases like $t^{3/2}$, while in the zeroth order, the particles decelerate like $t^{1/2}$. This indicates that the force exerted on the particles to first order is greater than the force exerted on the particles to zeroth order. Thus, the first-order gas velocity should be positive as shown in figure 21 . Figure 21 also shows that as time increases, the first-order gas velocity moves away from the boundary. Figure 20 is a graph of equation 205, and indicates that, as time increases, the particles move out of the boundary layer and into the external flow.

VI. SEMI-INFINITE FLAT PLATE WHEN THE GAS IS COMPRESSIBLE

The governing equations are obtained by taking equations 14 through 20 , setting all time derivatives to zero and setting all pressure derivatives to zero. Then

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 , \tag{221}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \rho_p\left(\frac{u_p - u}{\tau_m}\right) , \tag{222}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 + \rho_p\frac{(u_p - u)^2}{\tau_m} + \rho_p C_s\left(\frac{T_p - T}{\tau_T}\right) , \tag{223}$$

$$\frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v_p) = 0 , \tag{224}$$

$$\rho_p\left(u_p\frac{\partial u_p}{\partial x} + v_p\frac{\partial u_p}{\partial y}\right) = -\rho_p\left(\frac{u_p - u}{\tau_m}\right) , \tag{225}$$

$$\rho_p\left(u_p\frac{\partial v_p}{\partial x} + v_p\frac{\partial v_p}{\partial y}\right) = -\rho_p\left(\frac{v_p - v}{\tau_m}\right) , \tag{226}$$

$$\rho_p C_s\left(u_p\frac{\partial T_p}{\partial x} + v_p\frac{\partial T_p}{\partial y}\right) = -\rho_p C_s\left(\frac{T_p - T}{\tau_T}\right) , \tag{227}$$

Since ρ is now a variable, the equation of state

$$p = \rho R T \tag{228}$$

must be included. Consequently, there are eight equations to solve for the eight unknowns, $u, v, T, \rho, u_p, v_p, T_p$, and ρ_p. Since the gas is compressible, μ and k are no longer constants but vary with gas temperature. Consequently, it is convenient to let

$$\mu/\mu_{\infty} = (T/T_{\infty})^{\frac{1}{2}} \tag{229}$$

$$k/k_{\infty} = (T/T_{\infty})^{\frac{1}{2}} \tag{230}$$

$$\lambda_m = \tau_m a \tag{231}$$

$$\lambda_T = \tau_T a \tag{232}$$

where a is the local gaseous velocity of sound given by

$$a = \sqrt{\gamma RT} \; . \tag{233}$$

As a consequence of 229, 230, and 233, λ_m, λ_T and Prandtl number are constants. It is assumed that C_p and C_v are also constants.

Of the several methods used in the past on compressible boundary layers of a pure gas, only one is applicable to two-phase flow. The famed Howarth method does not work in this case, since there is a y-component momentum equation, 226, and hence the integral transformation does not drop out as it does in one-phase flow. The von Mises method changes independent variables from x, y to x, ψ, where ψ is the gas stream function, and requires

$$\mu/\mu_{\infty} = C\,(T/T_{\infty}) \,,$$

where C is a constant. Crocco's method requires a change of independent variables from x, y to x, u, which is not convenient for two-phase flow. Emmons and Brainard in reference 5 use a method adopting the equations of motion for a numerical solution. An extension of the Emmons and Brainard method will be applied to equations 221 through 228.

A. Small Slip Approximation

As for the case of an incompressible gas, the first region of interest is the regime of small particle slip velocities. With this in mind, the equations of motion are rewritten somewhat so that the slip quantities can be easily determined. Equation 222 plus 225 gives

$$\rho_p(u_p \frac{\partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y}) + \rho(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) - \frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) = 0 . \tag{234}$$

Equation 222, divided by ρ, subtracted from 225, divided by ρ_p, gives

$$u_p \frac{\partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y} - (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) + \frac{1}{\rho}\frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) = -(1 + \frac{\rho_p}{\rho})\frac{a}{\lambda_m}(u_p - u) . \tag{235}$$

Equation 226 becomes

$$u_p \frac{\partial v_p}{\partial x} + v_p \frac{\partial v_p}{\partial y} = -\frac{a}{\lambda_m}(v_p - v) . \tag{236}$$

Equation 223 added to 227 gives

$$\rho C_p(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}) + \rho_p C_s(u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y}) - \frac{\partial}{\partial y}(k \frac{\partial T}{\partial y})$$

$$- \mu (\frac{\partial u}{\partial y})^2 - \rho_p \frac{a}{\lambda_m}(u_p - u)^2 = 0 , \tag{237}$$

and 227, divided by $\rho_p C_s$, minus 223, divided by ρC_p, gives

$$u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} - (u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}) + \frac{1}{\rho C_p}\frac{\partial}{\partial y}(k \frac{\partial T}{\partial y})$$

$$+ \frac{\mu}{\rho C_p}(\frac{\partial u}{\partial y})^2 + \frac{\rho_p a}{\rho C_p}\frac{(u_p - u)^2}{\lambda_m} = -(1 + \frac{C_s \rho_p}{C_p \rho})\frac{a}{\lambda_T}(T_p - T) . \tag{238}$$

A transformation of dependent variables is now made by letting

$$T_s = T_p - T , \qquad v_s = v_p - v . \tag{239}$$
$$u_s = u_p - u ,$$

In the regime of small particle slip velocities, T_s, u_s, and v_s are first order quantities with respect to λ_m/x, therefore making the transformation 239 in equations 221, 224, 234, 235, 236, 237 and 238, using the equation of state to simplify 221, and keeping terms up to first order only yields

$$T\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}, \tag{240}$$

$$\rho_p\left(\frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u_s\frac{\partial \rho_p}{\partial x} + v_s\frac{\partial \rho_p}{\partial y} + u\frac{\partial \rho_p}{\partial x} + v\frac{\partial \rho_p}{\partial y} = 0, \tag{241}$$

$$\frac{\rho_p}{\rho}\left(u_s\frac{\partial u}{\partial x} + v_s\frac{\partial u}{\partial y} + u\frac{\partial u_s}{\partial x} + v\frac{\partial u_s}{\partial y}\right) + \left(1+\frac{\rho_p}{\rho}\right)\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)$$
$$- \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = 0, \tag{242}$$

$$u_s\frac{\partial u}{\partial x} + v_s\frac{\partial u}{\partial y} + u\frac{\partial u_s}{\partial x} + v\frac{\partial u_s}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = -\left(1+\frac{\rho_p}{\rho}\right)\frac{a}{\lambda_m}u_s, \tag{243}$$

$$u_s\frac{\partial v}{\partial x} + v_s\frac{\partial v}{\partial y} + u\frac{\partial v_s}{\partial x} + v\frac{\partial v_s}{\partial y} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{a}{\lambda_m}v_s, \tag{244}$$

$$\frac{\rho_p C_s}{\rho C_p}\left(u_s\frac{\partial T}{\partial x} + v_s\frac{\partial T}{\partial y} + u\frac{\partial T_s}{\partial x} + v\frac{\partial T_s}{\partial y}\right) + \left(1+\frac{\rho_p C_s}{\rho C_p}\right)\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$$
$$- \frac{1}{\rho C_p}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) - \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\rho_p a}{\rho C_p}\frac{u_s^2}{\lambda_m} = 0, \tag{245}$$

$$u\frac{\partial T_s}{\partial x} + v\frac{\partial T_s}{\partial y} + u_s\frac{\partial T}{\partial x} + v_s\frac{\partial T}{\partial y} + \frac{1}{\rho C_p}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\rho_p a}{\rho C_p}\frac{u_s^2}{\lambda_m}$$
$$+ \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2 = -\left(1+\frac{\rho_p C_s}{\rho C_p}\right)\frac{a}{\lambda_T}T_s. \tag{246}$$

Let

$$M_\infty = u_\infty / a_\infty ,$$

$$\varkappa = (\rho_p/\rho)_\infty = \text{constant},$$

$$\nu_\infty^* = \nu_\infty/(1+\varkappa) ,$$

$$u_s^* = u_s/u_\infty ,$$

$$v_s^* = v_s \sqrt{x/(\nu_\infty^* u_\infty)} ,$$

$$u^* = u/u_\infty ,$$

$$v^* = v \sqrt{x/(\nu_\infty^* \; u_\infty)} ,$$

$$T^* = T/T_\infty ,$$

$$T_s^* = T_s/T_\infty ,$$

$$\rho_p^* = \rho_p/(\varkappa\rho_\infty) ,$$

and then making the transformation

$$\eta = \sqrt{\frac{u_\infty}{\nu_\infty^* x}} \; y ,$$

i.e., the transformation 51 with ν^* replaced by ν_∞^*, with

$$\Psi = \eta u^* - 2v^* \tag{247}$$

in equations 240 through 246 yields

$$T^* \frac{\partial u^*}{\partial x} + \frac{1}{2x} T^* u^* - \frac{1}{2x} T^* \frac{\partial \Psi}{\partial \eta} - u^* \frac{\partial T^*}{\partial x} + \frac{1}{2x} \Psi \frac{\partial T^*}{\partial \eta} = 0 , \tag{248}$$

$$\rho_p^* \left[\frac{\partial u_s^*}{\partial x} - \frac{\eta}{2x} \frac{\partial u_s^*}{\partial \eta} + \frac{1}{x} \frac{\partial v_s^*}{\partial \eta} + \frac{\partial u^*}{\partial x} + \frac{1}{2x} (u^* - \frac{\partial \Psi}{\partial \eta}) \right] + u_s^* \frac{\partial \rho_p^*}{\partial x}$$

$$- \frac{\eta}{2x} u_s^* \frac{\partial \rho_p^*}{\partial \eta} + \frac{v_s^*}{x} \frac{\partial \rho_p^*}{\partial \eta} + u^* \frac{\partial \rho_p^*}{\partial x} - \frac{\Psi}{2x} \frac{\partial \rho_p^*}{\partial \eta} = 0 , \tag{249}$$

$$\varkappa\rho_p^{\;*}T^*\left(u_s^{\;*}\frac{\partial u^*}{\partial x} - \frac{\eta}{2x}u_s^{\;*}\frac{\partial u^*}{\partial \eta} + \frac{1}{x}v_s^{\;*}\frac{\partial u^*}{\partial \eta} + u^*\frac{\partial u_s^{\;*}}{\partial x} - \frac{\psi}{2x}\frac{\partial u_s^{\;*}}{\partial \eta}\right)$$

$$+ (1+\varkappa\rho_p^{\;*}T^*)\left(u^*\frac{\partial u^*}{\partial x} - \frac{\psi}{2x}\frac{\partial u^*}{\partial \eta}\right) - \frac{(1+\varkappa)T^*}{x}\frac{\partial}{\partial \eta}\left(\sqrt{T^*}\frac{\partial u^*}{\partial \eta}\right) = 0, \tag{250}$$

$$M_\infty\left[u_s^{\;*}\frac{\partial u^*}{\partial x} - \frac{\eta}{2x}u_s^{\;*}\frac{\partial u^*}{\partial \eta} + \frac{1}{x}v_s^{\;*}\frac{\partial u^*}{\partial \eta} + u^*\frac{\partial u_s^{\;*}}{\partial x} - \frac{\psi}{2x}\frac{\partial u_s^{\;*}}{\partial \eta}\right.$$

$$\left. + \left(\frac{1+\varkappa}{x}\right)T^*\frac{\partial}{\partial \eta}\left(\sqrt{T^*}\frac{\partial u^*}{\partial \eta}\right)\right] = -(1+\varkappa\rho_p^{\;*}T^*)\frac{\sqrt{T^*}}{\lambda_m}u_s^{\;*}, \tag{251}$$

$$M_\infty\left[\frac{1}{2}\eta u^*\frac{\partial u^*}{\partial x} - \frac{1}{2}u^*\frac{\partial \psi}{\partial x} - \frac{1}{4x}\eta u^{*2} - \frac{\psi}{4x}\eta\frac{\partial u^*}{\partial \eta} + \frac{\psi}{4x}\frac{\partial \psi}{\partial \eta}\right]$$

$$= \frac{-\sqrt{T^*}}{\lambda_m}v_s^{\;*}, \tag{252}$$

$$\varkappa T^*\rho_p^{\;*}\frac{C_s}{C_p}\left[u^*\frac{\partial T_s^{\;*}}{\partial x} - \frac{1}{2x}\psi\frac{\partial T_s^{\;*}}{\partial \eta} + u_s^{\;*}\frac{\partial T^*}{\partial x} - \frac{\eta}{2x}u_s^{\;*}\frac{\partial T^*}{\partial \eta} + \frac{v_s^{\;*}}{x}\frac{\partial T^*}{\partial \eta}\right]$$

$$+ \left(1+\varkappa T^*\rho_p^{\;*}\frac{C_s}{C_p}\right)\left[u^*\frac{\partial T^*}{\partial x} - \frac{1}{2x}\psi\frac{\partial T^*}{\partial \eta}\right] - \left(\frac{1+\varkappa}{x}\right)\frac{T^*}{Pr}\frac{\partial}{\partial \eta}\left(\sqrt{T^*}\frac{\partial T^*}{\partial \eta}\right)$$

$$-\left(\frac{1+\varkappa}{x}\right)T^*\sqrt{T^*}(\gamma-1)M_\infty^{\;2}\left(\frac{\partial u^*}{\partial \eta}\right)^2 - \varkappa\rho_p^{\;*}T^*\sqrt{T^*}(\gamma-1)M_\infty\frac{u_s^{\;*2}}{\lambda_m} = 0, \tag{253}$$

$$M_\infty\left[u^*\frac{\partial T_s^{\;*}}{\partial x} - \frac{1}{2x}\psi\frac{\partial T_s^{\;*}}{\partial \eta} + u_s^{\;*}\frac{\partial T^*}{\partial x} - \frac{\eta}{2x}u_s^{\;*}\frac{\partial T^*}{\partial \eta} + \frac{v_s^{\;*}}{x}\frac{\partial T^*}{\partial \eta}\right.$$

$$+ \varkappa\rho_p^{\;*}T^*\sqrt{T^*}(\gamma-1)\frac{M_\infty u_s^{\;*2}}{\lambda_m} + \left(\frac{1+\varkappa}{x}\right)\frac{T^*}{Pr}\frac{\partial}{\partial \eta}\left(\sqrt{T^*}\frac{\partial T^*}{\partial \eta}\right)$$

$$\left. + \left(\frac{1+\varkappa}{x}\right)\sqrt{T^*}T^*(\gamma-1)M_\infty^{\;2}\left(\frac{\partial u^*}{\partial \eta}\right)^2\right] = -\left(1+\frac{\varkappa C_s}{C_p}T^*\rho_p^{\;*}\right)\frac{\sqrt{T^*}}{\lambda_T}T_s^{\;*}. \tag{254}$$

Now in the regime of small particle slip velocities, the following expansions are assumed.

$$\left.\begin{aligned}
u^* &= u_o(\eta) + \frac{\lambda_m}{x}\, u_1(\eta) + \ldots \\
\Psi &= \psi_o(\eta) + \frac{\lambda_m}{x}\, \psi_1(\eta) + \ldots \\
T^* &= \theta_o(\eta) + \frac{\lambda_m}{x}\, \theta_1(\eta) + \ldots \\
u_s^* &= \frac{\lambda_m}{x}\, g_1(\eta) + \ldots \\
v_s^* &= \frac{\lambda_m}{x}\, h_1(\eta) + \ldots \\
\rho_p^* &= I_o(\eta) + \frac{\lambda_m}{x}\, I_1(\eta) + \ldots \\
T_s^* &= \theta_s^{(1)}(\eta)\, \frac{\lambda_m}{x} + \ldots
\end{aligned}\right\} \tag{255}$$

Putting 255 into 248 and equating coefficients of $(\lambda_m/x)^n$, $n = 0, 1$, gives

$$\theta_o(\psi_o' - u_o) = \theta_o'\psi_o \tag{256}$$

and

$$\psi_1' - \frac{\theta_o'}{\theta_o}\psi_1 = -\frac{\theta_o'}{\theta_o^2}\psi_o\theta_1 + 2\frac{u_o}{\theta_o}\theta_1 + \frac{\psi_o}{\theta_o}\theta_1' - u_1 . \tag{257}$$

Similarly, equation 249 gives

$$I_o' + I_o\frac{\theta_o'}{\theta_o} = 0 \tag{258}$$

and

$$\begin{aligned}
\psi_o I_1' + (2u_o + \frac{\theta_o'}{\theta_o}\psi_o)I_1 = M_\infty\Bigg[& -\frac{\psi_o u_o'}{\theta_o^{3/2}} + \eta\frac{u_o u_o'}{\theta_o^{3/2}} - \frac{3}{4}\frac{\theta_o'\psi_o u_o}{\theta_o^{5/2}} \\
& + \frac{1}{2}\frac{\psi_o^2}{\theta_o^{5/2}}\left((\frac{1+\varkappa C_s/C_p}{1+\varkappa})\frac{Pr}{2}\psi_o\frac{\theta_o'}{\theta_o^{3/2}} + (\gamma-1)M_\infty^2 Pr\, u_o'^2 \right) \\
& - \frac{3}{4}\frac{\theta_o' u_o^2\eta}{\theta_o^{5/2}} + \frac{\theta_o'\psi_o}{M_\infty\theta_o^3}\theta_1 - \frac{2u_o}{M_\infty\theta_o^2}\theta_1 - \frac{\theta_1'\psi_o}{M_\infty\theta_o^2} + \frac{1}{2}\frac{\theta_o'^2\psi_o^2}{\theta_o^3\sqrt{\theta_o}}\Bigg] ;
\end{aligned} \tag{259}$$

equation 250 gives

$$\psi_o + \frac{2(1+\varkappa)}{(1+\varkappa I_o \theta_o)} \frac{\theta_o}{u_o'} \frac{d}{d\eta} \left(\sqrt{\theta_o}\, u_o' \right) = 0 \tag{260}$$

and

$$u_1'' + \left(\frac{\frac{1}{2}\psi_o + \frac{1}{2}\sqrt{\theta_o}\,\theta_o'}{\theta_o \sqrt{\theta_o}} \right) u_1' + \frac{u_o}{\theta_o\sqrt{\theta_o}} u_1 = -\frac{1}{2} \frac{u_o'}{\theta_o\sqrt{\theta_o}} \psi_1$$

$$+ \left(\frac{\varkappa}{1+\varkappa}\right) \frac{1}{\theta_o\sqrt{\theta_o}} \left[\frac{M_\infty}{4} \eta \frac{u_o' u_o^2}{\sqrt{\theta_o}} - \frac{M_\infty}{4} \frac{\theta_o' u_o' \psi_o^2}{\theta_o \sqrt{\theta_o}} - \frac{M_\infty \psi_o u_o u_o'}{\sqrt{\theta_o}} \right.$$

$$\left. + \frac{M_\infty}{8} \frac{\psi_o^3 u_o'}{\theta_o^2} \right] - \left(\frac{\varkappa}{1+\varkappa}\right) \frac{1}{\theta_o^{3/2}} \left(\frac{1}{2} \frac{\psi_o u_o'}{\theta_o} \theta_1 + \frac{1}{2} \psi_o \theta_o u_o' I_1 \right)$$

$$+ \frac{1}{2} \frac{\theta_o' u_o'}{\theta_o^2} \theta_1 - \frac{1}{2} \frac{u_o'}{\theta_o} \theta_1' + \frac{3}{4} \frac{\psi_o u_o'}{\theta_o^{5/2}} \theta_1 \quad ; \tag{261}$$

equation 251 gives to first order

$$g_1(\eta) = \frac{M_\infty}{2} \frac{\psi_o u_o'}{\sqrt{\theta_o}} \quad ; \tag{262}$$

equation 252 gives to first order

$$h_1(\eta) = \frac{M_\infty}{\sqrt{\theta_o}} \left(\frac{\eta}{4} u_o^2 + \eta \frac{\psi_o}{4} u_o' - \frac{\theta_o'}{4\theta_o} \psi_o^2 - \frac{\psi_o}{4} u_o \right) \quad ; \tag{263}$$

equation 253 gives

$$\theta_o'' + \frac{(\theta_o')^2}{2\theta_o} + \frac{Pr}{2} \left(\frac{1 + \varkappa C_s/C_p}{1+\varkappa} \right) \frac{\psi_o \theta_o'}{\theta_o^{3/2}} + (\gamma - 1) M_\infty^2 Pr (u_o')^2 = 0 \tag{264}$$

and

$$\theta_1'' + \left[\frac{\frac{Pr}{2}\left(\frac{1+\varkappa C_s/C_p}{1+\varkappa}\right)\psi_o + \sqrt{\theta_o}\,\theta_o'}{\theta_o\sqrt{\theta_o}}\right]\theta_1' + \left[Pr\left(\frac{1+\varkappa C_s/C_p}{1+\varkappa}\right)u_o\right.$$

$$\left. - \frac{1}{4}\left(\frac{\varkappa}{1+\varkappa}\right)Pr\frac{C_s}{C_p}\psi_o\frac{\theta_o'}{\theta_o} - \frac{3}{4}\left(\frac{Pr}{1+\varkappa}\right)\frac{\psi_o\theta_o'}{\theta_o} - \frac{1}{2}\frac{\theta_o'^{\,2}}{\sqrt{\theta_o}}\right]\frac{\theta_1}{\theta_o^{3/2}} =$$

$$\frac{\frac{\varkappa C_s}{C_p}Pr}{1+\varkappa}\;\frac{M_\infty}{\theta_o\sqrt{\theta_o}}\left[-\frac{3}{4}\frac{C_s}{C_p}Pr\frac{\psi_o u_o\theta_o'}{\sqrt{\theta_o}} + \frac{3}{16}\frac{C_s}{C_p}Pr^2\left(\frac{1+\varkappa C_s/C_p}{1+\varkappa}\right)\frac{\psi_o^{\,3}\theta_o'}{\theta_o^{\,2}}\right.$$

$$+ \frac{3}{8}\frac{C_s}{C_p}Pr^2(\gamma-1)\frac{M_\infty^{\,2}\psi_o^{\,2}u_o'^{\,2}}{\sqrt{\theta_o}} - \frac{3}{8}\frac{C_s}{C_p}Pr\,\psi_o\frac{u_o\theta_o'}{\sqrt{\theta_o}} + \frac{\eta}{4}\frac{\theta_o' u_o^{\,2}}{\sqrt{\theta_o}}$$

$$\left. - \frac{\theta_o'^{\,2}\psi_o^{\,2}}{4\theta_o\sqrt{\theta_o}} - \frac{\psi_o u_o\theta_o'}{4\sqrt{\theta_o}}\right] - \frac{Pr}{2}\left(\frac{1+\varkappa C_s/C_p}{1+\varkappa}\right)\frac{\theta_o'}{\theta_o^{3/2}}\psi_1 \tag{265}$$

$$- \frac{Pr}{2}\left(\frac{\varkappa C_s/C_p}{1+\varkappa}\right)\frac{\psi_o\theta_o'}{\sqrt{\theta_o}}I_1 - 2Pr(\gamma-1)M_\infty^{\,2}u_o'u_1' - \frac{\varkappa}{1+\varkappa}Pr(\gamma-1)\frac{M_\infty^{\,3}}{4}\frac{\psi_o^{\,2}u_o'^{\,2}}{\theta_o^{\,2}},$$

and equation 254 gives to first order

$$\theta_s^{(1)}(\eta) = \frac{3}{2}\frac{C_s}{C_p}Pr\left(\frac{M_\infty}{2}\frac{\psi_o\theta_o'}{\sqrt{\theta_o}}\right). \tag{266}$$

Equation 258 can be written

$$\frac{d}{d\eta}(I_o\theta_o) = 0$$

$$I_o\theta_o = \text{const.}$$

From the boundary conditions stated in Chapter II,

$$I_o(\infty) = 1, \quad \theta_o(\infty) = 1;$$

hence

$$I_o\theta_o = 1. \tag{267}$$

Equations 267, 262, 263, and 266 were used in obtaining 257, 259, 261, and 265. Using 267, the zeroth order problem reduces to

$$\psi_o' - \frac{\theta_o'}{\theta_o}\psi_o - u_o = 0 , \tag{268}$$

$$u_o'' + \left(\frac{1}{2}\frac{\theta_o'}{\theta_o} + \frac{\psi_o}{2}\,\theta_o^{-3/2} \right) u_o' = 0 , \tag{269}$$

$$\theta_o'' + \frac{(\theta_o')^2}{2\theta_o} + \frac{Pr}{2}\left(\frac{1 + \kappa C_s/C_p}{1 + \kappa} \right)\frac{\psi_o \theta_o'}{\theta_o^{3/2}} + (\gamma - 1)M_\infty^{\,2} Pr(u_o')^2 = 0. \tag{270}$$

For $Pr = 1$, $C_s/C_p = 1$, the Crocco integral of 270 is

$$\theta_o = \theta_o(o) + [1 - \theta_o(o)]u_o + \left(\frac{\gamma-1}{2}\right)M_\infty^{\,2} u_o(1 - u_o) . \tag{271}$$

The first order problem is given by equations 257, 259, 261, 262, 263, and 265.

As stated in the introduction, the boundary conditions for the gas are

$$\left.\begin{array}{ll}
\text{(i)} & \psi_o(o) = 0 , \quad \psi_1(o) = 0 \\
\text{(ii)} & u_o(o) = 0 , \quad u_1(o) = 0 \\
\text{(iii)} & u_o(\infty) = 1 , \quad u_1(\infty) = 0 \\
\text{(iv)} & \theta_o(o) = \alpha , \text{ a given constant} \\
 & \theta_1(o) = 0 \\
\text{(v)} & \theta_o(\infty) = 1 , \quad \theta_1(\infty) = 0 .
\end{array}\right\} \tag{272}$$

As in the case for an incompressible gas, the boundary conditions for the particle phase velocities and temperatures have been suppressed by the expansion procedure. The particle density has the boundary conditions

$$I_o(\infty) = 1 , \quad I_1(\infty) = 0 .$$

Equation 259 has a singular point at $\eta = 0$, hence it is necessary to find how I_1 goes away from $\eta = 0$ before a numerical solution can be obtained. Let

$$b = (\gamma-1)M_\infty^{\;2}$$

$$\alpha = \theta_o(o)$$

$$\beta = \theta_o'(o)$$

$$\gamma = u_o'(o)$$

$$\epsilon = \theta_1'(o)$$

$$\delta = u_1'(o) \ .$$

Then with the boundary conditions 272 and with the equations 268, 269, 270, and 265, the following series can be found for $\eta \ll 1$.

$$\psi_o = \frac{\gamma}{2}\eta^2 + \frac{1}{12}\frac{\beta\gamma}{\alpha}\eta^3 + \dots$$

$$\theta_o = \alpha + \beta\eta - \frac{1}{2}\left[\frac{\beta^2}{2\alpha} + bPr\gamma^2\right]\eta^2 + \frac{1}{6}\left[\frac{\beta^3}{\alpha^2} + \frac{2\beta\gamma^2}{\alpha}\,bPr\right]\eta^3 + \dots$$

$$u_o = \gamma\eta - \frac{1}{4}\left(\frac{\beta\gamma}{\alpha}\right)\eta^2 + \frac{1}{6}\left[\left(\frac{\beta}{\alpha}\right)^2\gamma + \frac{bPr\gamma^3}{2\alpha}\right]\eta^3 + \dots$$

$$\theta_1 = \epsilon\eta - \frac{1}{2}\left[\frac{\beta\epsilon}{\alpha} + 2bPr\gamma\delta\right]\eta^2 + \dots$$

Then keeping only the largest terms in 259

$$I_1' + \frac{4}{\eta}I_1 = M_\infty\left(\frac{\gamma}{\alpha^{3/2}} - \frac{5\epsilon}{M_\infty\alpha^2}\right) + 0(\eta)$$

$$\eta^4 I_1 = \frac{M_\infty}{5}\left(\frac{\gamma}{\alpha^{3/2}} - \frac{5\epsilon}{M_\infty\alpha^2}\right)\eta^5 + \dots$$

$$I_1 = \frac{M_\infty}{5}\left(\frac{\gamma}{\alpha^{3/2}} - \frac{5\epsilon}{M_\infty\alpha^2}\right)\eta + \dots$$

for $\eta \ll 1$. Therefore,

$$I_1(\eta) = \frac{M_\infty}{5}\left(\frac{u_o'(o)}{\alpha^{3/2}} - \frac{5\theta_1'(o)}{M_\infty \alpha^2}\right)\eta + \ldots .$$

The solution of the zeroth order problem is plotted in figures 22 , 23 , and 24 along with the incompressible solution curves for comparison. Care must be taken in making this comparison, since different reference values were used in the two cases. The solution for the first order problem is difficult to obtain because of the high coupling between energy and momentum equations. In addition, regardless of the choice of $u_1'(o)$ and $\theta_1'(o)$, $u_1(\eta)$ and $\theta_1(\eta)$ approach zero as η approaches infinity, making it impossible to pick out the correct solutions. Therefore, only the particle slip velocities can be plotted and compared with the corresponding incompressible curves in figures 25 , 26 , and 27 .

B. Large Slip Approximation

The governing equations are 221 through 228, which become

$$p = \rho R T , \tag{273}$$

$$T\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} , \tag{274}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{RT}{p}\frac{\mu_\infty}{\sqrt{T_\infty}}\frac{\partial}{\partial y}\left(\sqrt{T}\frac{\partial u}{\partial y}\right) + \frac{\rho_p a}{\lambda_m}\frac{RT}{p}(u_p - u), \tag{275}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{RT}{pC_p}\frac{k_\infty}{\sqrt{T_\infty}}\frac{\partial}{\partial y}\left(\sqrt{T}\frac{\partial T}{\partial y}\right) + \frac{RT^{3/2}}{p}\frac{\mu_\infty}{C_p\sqrt{T_\infty}}\left(\frac{\partial u}{\partial y}\right)^2$$

$$+ \frac{RT}{pC_p}\frac{\rho_p a}{\lambda_m}(u_p - u)^2 + \frac{C_s}{C_p}\frac{RT}{p}\frac{\rho_p a}{\lambda_T}(T_p - T) , \tag{276}$$

$$\frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v_p) = 0 , \qquad (277)$$

$$u_p \frac{\partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y} = -\frac{a}{\lambda_m}(u_p - u) , \qquad (278)$$

$$u_p \frac{\partial v_p}{\partial x} + v_p \frac{\partial v_p}{\partial y} = -\frac{a}{\lambda_m}(v_p - v) , \qquad (279)$$

$$u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} = -\frac{a}{\lambda_T}(T_p - T) . \qquad (280)$$

Being a regime of large particle slip velocity means that in the zeroth order, both phases flow independently of each other, just as in the incompressible case. Therefore, let

$$u^* = u/u_\infty ,$$

$$v^* = v \sqrt{x/(\nu_\infty u_\infty)} ,$$

$$T^* = T/T_\infty ,$$

$$\rho_p^* = \rho_p/(\kappa\rho_\infty) ,$$

$$T_p^* = T_p/T_\infty ,$$

$$\rho_p u_p = \kappa\rho_\infty (\partial\psi_p/\partial y) ,$$

$$\rho_p v_p = -\kappa\rho_\infty (\partial\psi_p/\partial x) ,$$

and then 277 is identically satisfied, and the rest of the governing equations become

$$T^* \left(\frac{\partial u^*}{\partial x} + \sqrt{\frac{\nu_\infty}{u_\infty x}} \frac{\partial v^*}{\partial y} \right) = u^* \frac{\partial T^*}{\partial x} + \sqrt{\frac{\nu_\infty}{u_\infty x}}\, v^* \frac{\partial T^*}{\partial y} , \qquad (281)$$

$$u_\infty u^* \frac{\partial u^*}{\partial x} + \sqrt{\frac{\nu_\infty u_\infty}{x}}\, v^* \frac{\partial u^*}{\partial y} = \nu_\infty T^* \frac{\partial}{\partial y}\left(\sqrt{T^*}\frac{\partial u^*}{\partial y} \right)$$

$$+ \frac{\varkappa \rho_p^* T^{*3/2}}{M_\infty \lambda_m}\left(\frac{1}{\rho_p^*}\frac{\partial \psi_p}{\partial y} - u_\infty u^* \right), \tag{282}$$

$$T^*\left(u_\infty \frac{\partial u^*}{\partial x} + \sqrt{\frac{\nu_\infty u_\infty}{x}}\frac{\partial v^*}{\partial y}\right) = \frac{\nu_\infty}{Pr} T^* \frac{\partial}{\partial y}\left(\sqrt{T^*}\frac{\partial T^*}{\partial y} \right)$$

$$+ \nu_\infty M_\infty^2 T^{*3/2}(\gamma-1)\left(\frac{\partial u^*}{\partial y}\right)^2 + \frac{\varkappa(\gamma-1)\rho_p^* T^{*3/2}}{\lambda_m a_\infty} \times$$

$$\left[\frac{1}{\rho_p^{*2}}\left(\frac{\partial \psi_p}{\partial y}\right)^2 - \frac{2}{\rho_p^*}\frac{\partial \psi_p}{\partial y} u_\infty u^* + u_\infty^2 u^{*2}\right] + \frac{\varkappa C_s}{C_p} a_\infty \rho_p^* \frac{T^{*3/2}}{\lambda_T}(T_p^* - T^*), \tag{283}$$

$$\frac{\partial \psi_p}{\partial y}\left[\rho_p^* \frac{\partial^2 \psi_p}{\partial x \partial y} - \frac{\partial \rho_p^*}{\partial x}\frac{\partial \psi_p}{\partial y}\right] - \frac{\partial \psi_p}{\partial x}\left[\rho_p^* \frac{\partial^2 \psi_p}{\partial y^2} - \frac{\partial \rho_p^*}{\partial y}\frac{\partial \psi_p}{\partial y}\right] =$$

$$- \frac{a_\infty \sqrt{T^*}}{\lambda_m}\left[\frac{\partial \psi_p}{\partial y} - u_\infty \rho_p^* u^*\right]\rho_p^{*2}, \tag{284}$$

$$- \frac{\partial \psi_p}{\partial y}\left[\rho_p^* \frac{\partial^2 \psi_p}{\partial x^2} - \frac{\partial \rho_p^*}{\partial x}\frac{\partial \psi_p}{\partial x}\right] + \frac{\partial \psi_p}{\partial x}\left[\rho_p^* \frac{\partial^2 \psi_p}{\partial x \partial y} - \frac{\partial \rho_p^*}{\partial y}\frac{\partial \psi_p}{\partial x}\right] =$$

$$\frac{a_\infty \sqrt{T^*}}{\lambda_m}\rho_p^{*2}\left(\frac{\partial \psi_p}{\partial x} + \rho_p^* \sqrt{\frac{\nu_\infty u_\infty}{x}}\, v^*\right), \tag{285}$$

$$\frac{\partial \psi_p}{\partial y}\frac{\partial T_p^*}{\partial x} - \frac{\partial \psi_p}{\partial x}\frac{\partial T_p^*}{\partial y} = - \frac{a_\infty \rho_p^* \sqrt{T^*}}{\lambda_T}(T_p^* - T^*) . \tag{286}$$

Now let

$$\eta = \sqrt{u_\infty/(\nu_\infty x)}\, y \qquad \Psi = \eta u^* - 2v^*$$

$$\psi_p = \sqrt{u_\infty \nu_\infty x}\, f_p(x, \eta) \tag{287}$$

and then with the transformation given by 51 with ν^* replaced everywhere by ν_∞, the equations 281 through 286 become

$$T^*\left(\frac{\partial u^*}{\partial x} + \frac{1}{2x}u^* - \frac{1}{2x}\frac{\partial \psi}{\partial \eta}\right) = u^*\frac{\partial T^*}{\partial x} - \frac{\psi}{2x}\frac{\partial T^*}{\partial \eta}, \tag{288}$$

$$u^*\frac{\partial u^*}{\partial x} - \frac{1}{2x}\psi\frac{\partial u^*}{\partial \eta} = \frac{T^*}{x}\frac{\partial}{\partial \eta}\left(\sqrt{T^*}\frac{\partial u^*}{\partial \eta}\right) + \frac{\varkappa T^{*3/2}}{M_\infty \lambda_m}\left(\frac{\partial f_p}{\partial \eta} - \rho_p^* u^*\right), \tag{289}$$

$$\begin{aligned}
\rho_p^*\left(\frac{\partial u^*}{\partial x} + \frac{u^*}{2x} - \frac{1}{2x}\frac{\partial \psi}{\partial \eta}\right) = {} & \frac{\rho_p^*}{x\mathrm{Pr}}\frac{\partial}{\partial \eta}\left(\sqrt{T^*}\frac{\partial T^*}{\partial \eta}\right) + \rho_p^*(\gamma-1)\frac{M_\infty^2 T^{*1/2}}{x} \\
& \times \left(\frac{\partial u^*}{\partial \eta}\right)^2 + \frac{\varkappa M_\infty(\gamma-1)\sqrt{T^*}}{\lambda_m}\left[\left(\frac{\partial f_p}{\partial \eta}\right)^2 - 2\rho_p^* u^*\frac{\partial f_p}{\partial \eta} + \rho_p^{*2}u^{*2}\right] \\
& + \frac{C_s}{C_p}\frac{\varkappa \rho_p^{*2}\sqrt{T^*}}{M_\infty \lambda_T}\left(T_p^* - T^*\right),
\end{aligned} \tag{290}$$

$$\begin{aligned}
& \frac{\partial f_p}{\partial \eta}\left[\rho_p^*\left(\frac{\partial^2 f_p}{\partial \eta \partial x} - \frac{\eta}{2x}\frac{\partial^2 f_p}{\partial \eta^2}\right) - \frac{\partial f_p}{\partial \eta}\left(\frac{\partial \rho_p^*}{\partial x} - \frac{\eta}{2x}\frac{\partial \rho_p^*}{\partial \eta}\right)\right] - \left(\frac{\partial f_p}{\partial x} + \frac{1}{2x}f_p\right) \\
& \times \left(\rho_p^*\frac{\partial^2 f_p}{\partial \eta^2} - \frac{\partial \rho_p^*}{\partial \eta}\frac{\partial f_p}{\partial \eta}\right) + \frac{\eta}{2x}\frac{\partial f_p}{\partial \eta}\left(\rho_p^*\frac{\partial^2 f_p}{\partial \eta^2} - \frac{\partial \rho_p^*}{\partial \eta}\frac{\partial f_p}{\partial \eta}\right) = \\
& \qquad - \frac{\sqrt{T^*}\rho_p^{*2}}{M_\infty \lambda_m}\left(\frac{\partial f_p}{\partial \eta} - \rho_p^* u^*\right),
\end{aligned} \tag{291}$$

$$\begin{aligned}
& -\rho_p^*\frac{\partial f_p}{\partial \eta}\left(\frac{\partial^2 f_p}{\partial x^2} + \frac{1}{x}\frac{\partial f_p}{\partial x} - \frac{1}{4x^2}f_p - \frac{\eta}{2x}\frac{\partial^2 f_p}{\partial x \partial \eta} + \frac{\eta}{4x^2}\frac{\partial f_p}{\partial \eta}\right) + \frac{\partial f_p}{\partial \eta}\frac{\partial \rho_p^*}{\partial x} \\
& \times \left(\frac{\partial f_p}{\partial x} + \frac{1}{2x}f_p - \frac{\eta}{2x}\frac{\partial f_p}{\partial \eta}\right) + \rho_p^*\frac{\partial f_p}{\partial x}\left(\frac{\partial^2 f_p}{\partial \eta \partial x} + \frac{1}{2x}\frac{\partial f_p}{\partial \eta} - \frac{\eta}{2x}\frac{\partial^2 f_p}{\partial \eta^2} - \frac{1}{2x}\frac{\partial f_p}{\partial \eta}\right) \\
& - \frac{\partial f_p}{\partial x}\frac{\partial \rho_p^*}{\partial \eta}\left(\frac{\partial f_p}{\partial x} + \frac{1}{2x}f_p - \frac{\eta}{2x}\frac{\partial f_p}{\partial \eta}\right) + \frac{1}{2x}\rho_p^* f_p\left(\frac{\partial^2 f_p}{\partial \eta \partial x} + \frac{1}{2x}\frac{\partial f_p}{\partial \eta} - \frac{\eta}{2x}\frac{\partial^2 f_p}{\partial \eta^2}\right. \\
& \left. - \frac{1}{2x}\frac{\partial f_p}{\partial \eta}\right) - \frac{f_p}{2x}\frac{\partial \rho_p^*}{\partial \eta}\left(\frac{\partial f_p}{\partial x} + \frac{1}{2x}f_p - \frac{\eta}{2x}\frac{\partial f_p}{\partial \eta}\right) =
\end{aligned}$$

$$= \rho_p^{*2} \frac{\sqrt{T^*}}{M_\infty \lambda_m} \left(\frac{\partial f_p}{\partial x} + \frac{1}{2x} f_p - \frac{\eta}{2x} \frac{\partial f_p}{\partial \eta} + \frac{\eta}{2x} \rho_p^* u^* - \rho_p^* \frac{\Psi}{2x} \right), \tag{292}$$

$$\frac{\partial f_p}{\partial \eta} \frac{\partial T_p^*}{\partial x} - \frac{\partial T_p^*}{\partial \eta} \left(\frac{\partial f_p}{\partial x} + \frac{1}{2x} f_p \right) = - \rho_p^* \frac{\sqrt{T^*}}{M_\infty \lambda_T} (T_p^* - T^*) . \tag{293}$$

As in the case of large particle slip velocity for an incompressible gas phase, the following expansions are made in terms of x/λ_m.

$$\left.\begin{aligned}
u^* &= u_o(\eta) + \frac{x}{\lambda_m} u_1(\eta) + \ldots \\
\Psi &= \psi_o(\eta) + \frac{x}{\lambda_m} \psi_1(\eta) + \ldots \\
T^* &= \theta_o(\eta) + \frac{x}{\lambda_m} \theta_1(\eta) + \ldots \\
\psi_p &= \sqrt{u_\infty \nu_\infty x}\, [f_p^{(o)}(\eta) + \frac{x}{\lambda_m} f_p^{(1)}(\eta) + \ldots] \\
\rho_p^* &= I_o(\eta) + \frac{x}{\lambda_m} I_1(\eta) + \ldots \\
T_p^* &= \theta_p^{(o)}(\eta) + \frac{x}{\lambda_m} \theta_p^{(1)}(\eta) + \ldots
\end{aligned}\right\} \tag{294}$$

Putting 294 in 288 yields to zeroth order

$$\theta_o u_o - \theta_o \psi_o' + \psi_o \theta_o' = 0 \tag{295}$$

and to first order

$$-\theta_o \psi_1' + \psi_1 \theta_o' + 3\theta_o u_1 - \theta_1 \psi_o' - u_o \theta_1 + \psi_o \theta_1' = 0 . \tag{296}$$

Putting 294 into 289 gives to zeroth order

$$\theta_o^{3/2} u_o'' + \left(\frac{1}{2}\psi_o + \frac{1}{2}\theta_o^{1/2} \theta_o'\right) u_o' = 0 \tag{297}$$

and to first order

$$\theta_o^{3/2} u_1'' + \frac{1}{2}\theta_o^{1/2} \theta_o' u_1' + \frac{1}{2}\psi_o u_1' - u_o u_1 + \frac{3}{2}\theta_o^{1/2} u_o'' \theta_1 + \frac{1}{2}\theta_o^{1/2} u_o' \theta_1' +$$

$$+\frac{1}{4}\theta_o^{-1/2}\theta_o' u_o' \theta_1 + \frac{1}{2}u_o'\psi_1 + \frac{\varkappa}{M_\infty}\theta_o^{3/2}(f_p'^{(o)} - I_o u_o) = 0 . \tag{298}$$

Putting 294 into 290 gives to zeroth order

$$\theta_o'' + \frac{1}{2}\frac{(\theta_o')^2}{\theta_o} - \frac{1}{2}\frac{Pr}{\sqrt{\theta_o}}(u_o - \psi_o') = -(\gamma-1)M_\infty^2 Pr(u_o')^2 \tag{299}$$

and to first order

$$\theta_1'' + \frac{\theta_o'}{\theta_o}\theta_1' + \left[\frac{1}{2}\frac{\theta_o''}{\theta_o} - \frac{1}{4}\left(\frac{\theta_o'}{\theta_o}\right)^2 + \frac{(\gamma-1)}{2}M_\infty^2 Pr\frac{(u_o')^2}{\theta_o}\right]\theta_1 + (\gamma-1)M_\infty^2 2Pr u_o' u_1'$$

$$-\frac{3}{2}\frac{Pr}{\theta_o^{1/2}}u_1 + \frac{Pr}{2}\frac{\psi_1'}{\sqrt{\theta_o}} = -\frac{Pr}{I_o}\varkappa M_\infty(\gamma-1)\left[(f_p'^{(o)})^2 - 2I_o u_o f_p'^{(o)} + I_o^2 u_o^2\right]$$

$$-\frac{2}{3}\frac{\varkappa}{M_\infty}I_o(\theta_p^{(o)} - \theta_o) . \tag{300}$$

Similarly, 291 gives to zeroth order

$$f_p^{(o)}(I_o f_p''^{(o)} - I_o' f_p'^{(o)}) = 0, \tag{301}$$

which implies

$$I_o f_p''^{(o)} - I_o' f_p'^{(o)} = 0$$

since $f_p^{(o)} \neq 0$. Thus

$$\frac{d}{d\eta}\left(\frac{f_p'^{(o)}}{I_o}\right) = 0$$

$$f_p'^{(o)} = AI_o$$

where A is an integration constant. Boundary conditions (ii) and (iv) given in Chapter II for the particle phase imply that

$$f_p'^{(o)}(\infty) = I_o(\infty) = 1 .$$

Hence

$$A = 1 \qquad \text{and}$$

$$f_p'^{(o)} = I_o \, . \tag{302}$$

Using 302, then 291 gives to first order

$$-\frac{1}{2} f_p^{(o)} I_o f_p''^{(1)} + (I_o f_p'^{(o)} + \frac{1}{2} f_p^{(o)} I_o') f_p'^{(1)} - \left[(f_p'^{(o)})^2 + \frac{1}{2} f_p^{(o)} f_p''^{(o)} \right] I_1$$

$$+ \frac{1}{2} f_p^{(o)} f_p'^{(o)} I_1' = -\frac{1}{M_\infty} \theta_o^{1/2} I_o^{2} [f_p'^{(o)} - I_o u_o] \, . \tag{303}$$

Putting 294 into 292 gives to zeroth order

$$f_p'^{(o)} [-f_p'^{(o)} f_p^{(o)} + \eta f_p'^{(o)} f_p'^{(o)}] + f_p''^{(o)} (f_p^{(o)})^2 = 0 \tag{304}$$

and to first order

$$-\frac{3}{4} I_o f_p'^{(o)} f_p^{(1)} + \frac{3}{4} I_o f_p^{(o)} f_p'^{(1)} + \frac{3}{4} f_p^{(o)} f_p'^{(o)} I_1 - \frac{3}{4} \eta I_1 (f_p'^{(o)})^2$$

$$-\frac{3}{2} I_o' f_p^{(o)} f_p^{(1)} - \frac{\eta}{4} I_o f_p^{(o)} f_p''^{(1)} - \frac{\eta}{4} f_p^{(o)} f_p''^{(o)} I_1 + \frac{\eta}{4} f_p^{(o)} I_o' f_p'^{(1)} - \frac{1}{4} (f_p^{(o)})^2 I_1'$$

$$+ \frac{\eta}{4} f_p^{(o)} f_p'^{(o)} I_1' = \frac{I_o^2 \theta_o^{1/2}}{M_\infty} \left(\frac{1}{2} f_p^{(o)} - \frac{\eta}{2} f_p'^{(o)} + \frac{\eta}{2} I_o u_o - \frac{1}{2} I_o \psi_o \right) \tag{305}$$

where 302 is used for simplification.

Likewise, 293 gives to zeroth order

$$f_p^{(o)} \theta_p'^{(o)} = 0 \, ,$$

or

$$\theta_p^{(o)} = 1 \, , \tag{306}$$

applying boundary conditions (iii) for the particle phase given in Chapter II, and to first order

$$f_p^{(o)} \theta_p'^{(1)} - 2 \theta_p^{(1)} f_p'^{(o)} = -\frac{4 I_o \sqrt{\theta_o}}{3 \frac{C_s}{C_p} Pr \, M_\infty} (\theta_o - 1) \tag{307}$$

where 306 is used.

A solution of 304 is

$$f_p^{(o)} = \text{const} \times \eta \ ,$$

but since $f_p'^{(o)}(\infty) = 1$, then

$$f_p^{(o)} = \eta \ . \tag{308}$$

Then rewriting the zeroth order equations in standard form yields

$$\psi_o' - \frac{\theta_o'}{\theta_o}\psi_o = u_o \ , \tag{309}$$

$$u_o'' + \left(\frac{\frac{1}{2}\psi_o + \frac{1}{2}\theta_o^{1/2}\theta_o'}{\theta_o^{3/2}}\right)u_o' = 0 \ , \tag{310}$$

$$\theta_o'' + \left(\frac{\frac{1}{2}\Pr\psi_o + \frac{1}{2}\theta_o^{1/2}\theta_o'}{\theta_o^{3/2}}\right)\theta_o' + (\gamma-1)M_\infty^2 \Pr(u_o')^2 = 0 \ , \tag{311}$$

$$I_o = 1 \ , \tag{312}$$

$$f_p'^{(o)} = 1 \ , \tag{313}$$

$$\theta_p^{(o)} = 1 \ . \tag{314}$$

For $\Pr = 1$, the Crocco integral of 311 gives

$$\theta_o = \theta_o(o) + [1 - \theta_o(o)]u_o + \frac{\gamma-1}{2} M_\infty^2 u_o(1-u_o) \ . \tag{315}$$

Rewriting the first order equations in standard form gives

$$\psi_1' - \frac{\theta_o'}{\theta_o}\psi_1 - 3u_1 + \left(2u_o + \frac{\theta_o'}{\theta_o}\psi_o\right)\frac{\theta_1}{\theta_o} - \frac{\psi_o}{\theta_o}\theta_1' = 0 \ , \tag{316}$$

$$u_1'' + \frac{1}{2}\left[\frac{\sqrt{\theta_o}\,\theta_o' + \psi_o}{\theta_o\sqrt{\theta_o}}\right]u_1' - \frac{u_o}{\theta_o^{3/2}}u_1 - \left(\frac{3}{4}\frac{u_o'\psi_o}{\theta_o^{5/2}} + \frac{1}{2}\frac{\theta_o' u_o'}{\theta_o^2}\right)\theta_1$$
$$+ \frac{1}{2}\frac{u_o'}{\theta_o}\theta_1' + \frac{1}{2}\frac{u_o'}{\theta_o^{3/2}}\psi_1 = -\frac{\varkappa}{M_\infty}(1-u_o) \ , \tag{317}$$

$$\theta_1'' + \left[\frac{\theta_o'}{\theta_o} + \frac{Pr}{2}\frac{\psi_o}{\theta_o^{3/2}}\right]\theta_1' + \left[-\frac{1}{4}\frac{Pr\psi_o\theta_o'}{\theta_o^{5/2}} - \frac{1}{2}\left(\frac{\theta_o'}{\theta_o}\right)^2 + \frac{Pr}{2}(\gamma-1)M_\infty^2\frac{(u_o')^2}{\theta_o}\right.$$

$$\left.-\frac{Pr\,u_o}{\theta_o^{3/2}} - \frac{Pr}{2}\frac{\psi_o\theta_o'}{\theta_o^{5/2}}\right]\theta_1 + 2Pr(\gamma-1)M_\infty^2 u_o' u_1' + \frac{Pr}{2}\frac{\theta_o'}{\theta_o^{3/2}}\psi_1 =$$

$$- Pr\varkappa M_\infty(\gamma-1)(1-2u_o+u_o^2) - \frac{2}{3}\frac{\varkappa}{M_\infty}(1-\theta_o)\,, \tag{318}$$

$$\eta I_1' - 2I_1 = -\frac{2\sqrt{\theta_o}}{M_\infty}\left(1-\frac{\psi_o}{\eta}\right) + f_p'^{(1)} - \frac{3}{\eta}f_p^{(1)}\,, \tag{319}$$

$$\eta^2 f_p''^{(1)} - 3\eta f_p'^{(1)} + 3f_p^{(1)} = -\frac{2\sqrt{\theta_o}}{M_\infty}(\eta u_o - \psi_o)\,, \tag{320}$$

$$\eta\theta_p'^{(1)} - 2\theta_p^{(1)} = -\frac{4\sqrt{\theta_o}}{3\frac{C_s}{C_p}PrM_\infty}(\theta_o - 1)\,. \tag{321}$$

The boundary conditions for the gas phase as stated in Chapter II are

$$\left.\begin{array}{ll}
\text{(i)} & \psi_o(o) = 0\,, \quad \psi_1(o) = 0 \\
\text{(ii)} & u_o(o) = 0\,, \quad u_1(o) = 0 \\
\text{(iii)} & u_o(\infty) = 1\,, \quad u_1(\infty) = 0 \\
\text{(iv)} & \theta_o(o) = \frac{T_w}{T_\infty} = \alpha\,, \quad \theta_1(o) = 0 \\
\text{(v)} & \theta_o(\infty) = 1\,, \quad \theta_1(\infty) = 0\,.
\end{array}\right\} \tag{322}$$

The boundary conditions for the particle phase as stated in Chapter II are:

$$\begin{array}{ll}
\text{(i)} & f_{p(o)}^{(o)} = 0\,, \quad f_{p(o)}^{(1)} = 0 \\
\text{(ii)} & f_p'^{(o)}(\infty) = 1\,, \quad f_p'^{(1)}(\infty) = 0
\end{array} \tag{323a}$$

$$\text{(iii)}\quad \theta_p^{(o)}(\infty) = 1 \ , \quad \theta_p^{(1)}(\infty) = 0$$
$$\text{(iv)}\quad I_o(\infty) = 1 \ , \quad I_1(\infty) = 0 \ . \tag{323b}$$

The well-known solution of the homogeneous part of 320 is

$$A\eta^3 + B\eta$$

so by variation of parameters, i.e.,

$$f_p^{(1)} = A(\eta)\eta^3 + B(\eta)\eta \ ,$$

the following equations are obtained:

$$A = -\frac{1}{M_\infty}\int_\infty^\eta \frac{\theta_o^{1/2}(x)}{x^4}[xu_o(x) - \psi_o(x)]dx + A_o \tag{324}$$

$$B = \frac{1}{M_\infty}\int_\infty^\eta \frac{\theta_o^{1/2}(x)}{x^2}[xu_o(x) - \psi_o(x)]dx + B_o \tag{325}$$

where A_o and B_o are constants. Therefore

$$f_p^{(1)} = -\frac{\eta^3}{M_\infty}\int_\infty^\eta \frac{\theta_o^{1/2}(x)}{x^4}[xu_o(x)-\psi_o(x)]dx + \frac{\eta}{M_\infty}\int_\infty^\eta \frac{\theta_o^{1/2}(x)}{x^2} \times$$
$$[xu_o(x)-\psi_o(x)]dx + A_o\eta^3 + B_o\eta \ , \tag{326}$$

and the boundary conditions 323 are used to determine A_o and B_o. Applying the conditions

$$f_p^{(1)}(o) = 0$$
$$f_p'^{(1)}(\infty) = 0$$

yields

$$A_o = B_o = 0 \ .$$

Thus

$$f_p^{(1)}(\eta) = -\frac{\eta^3}{M_\infty}\int_\infty^\eta \frac{\theta_o^{\frac{1}{2}}(x)}{x^4}\,[xu_o(x)-\psi_o(x)]dx$$

$$+\frac{\eta}{M_\infty}\int_\infty^\eta \frac{\theta_o^{\frac{1}{2}}(x)}{x^2}\,[xu_o(x)-\psi_o(x)]dx\ , \tag{327}$$

and

$$f_p'^{(1)}(\eta) = -\frac{3\eta^2}{M_\infty}\int_\infty^\eta \frac{\theta_o^{\frac{1}{2}}(x)}{x^4}\,[xu_o(x)-\psi_o(x)]dx$$

$$+\frac{1}{M_\infty}\int_\infty^\eta \frac{\theta_o^{\frac{1}{2}}(x)}{x^2}\,[xu_o(x)-\psi_o(x)]dx\ , \tag{328}$$

$$f_p'^{(1)}(o) = -\frac{1}{M_\infty}\int_0^\infty \frac{\theta_o^{\frac{1}{2}}(x)}{x^2}\,[xu_o(x)-\psi_o(x)]dx\ . \tag{329}$$

Note that since $xu_o(x)-\psi_o(x)$ approaches a constant as x approaches infinity, $f_p'^{(1)}(o)$ is a well defined number, and also $f_p'^{(1)}(\eta)$ exists for η approaching zero as well as η approaching infinity.

The equation 321 can be written

$$\frac{d}{d\eta}\left(\frac{1}{\eta^2}\,\theta_p^{(1)}\right) = -\frac{4\theta_o^{\frac{1}{2}}}{3\,\frac{C_s}{C_p}\,PrM_\infty}\;\frac{(\theta_o-1)}{\eta^3}\ , \tag{330}$$

and hence

$$\theta_p^{(1)} = -\frac{4}{3\,\frac{C_s}{C_p}\,PrM_\infty}\,\eta^2\int_\infty^\eta \theta_o^{\frac{1}{2}}(x)\,\frac{[\theta_o(x)-1]}{x^3}\,dx\ , \tag{331}$$

$$\theta_p^{(1)}(o) = \frac{2}{3\,\frac{C_s}{C_p}\,Pr\,M_\infty}\,\sqrt{\theta_o(o)}\,[\theta_o(o)-1] \tag{332}$$

where the boundary condition,

$$\theta_p^{(1)}(\infty) = 0 ,$$

is applied. Equation 319 can be written

$$\frac{d}{d\eta}\left(\frac{I_1}{\eta^2}\right) = -\frac{2\theta_o^{\frac{1}{2}}}{M_\infty}\frac{(\eta-\psi_o)}{\eta^4} - \frac{2}{M_\infty}\frac{1}{\eta^3}\int_\infty^\eta \frac{\theta_o^{\frac{1}{2}}}{x^2}[xu_o(x)-\psi_o(x)]dx \qquad (333)$$

where 327 and 328 have been used. Then integrating yields

$$I_1 = -\frac{\eta^2}{M_\infty}\int_\infty^\eta 2\theta_o^{\frac{1}{2}}(x)\frac{[x-\psi_o(x)]}{x^4}dx$$

$$-\frac{2}{M_\infty}\eta^2\int_\infty^\eta \frac{1}{t^3}\left[\int_\infty^t \frac{\theta_o^{\frac{1}{2}}(x)}{x^2}[xu_o(x)-\psi_o(x)]dx\right]dt . \qquad (334)$$

where the boundary condition,

$$I_1(\infty) = 0 ,$$

has been used. Then integrating by parts and simplifying the resulting expression gives

$$I_1(\eta) = -\frac{2\eta^2}{M_\infty}\int_\infty^\eta \frac{\theta_o^{\frac{1}{2}}}{x^4}[x-\psi_o(x)]dx - \frac{\eta^2}{M_\infty}\int_\infty^\eta \frac{\theta_o^{\frac{1}{2}}(x)[xu_o(x)-\psi_o(x)]}{x^4}dx$$

$$+\frac{1}{M_\infty}\int_\infty^\eta \frac{\theta_o^{\frac{1}{2}}(x)}{x^2}[xu_o(x)-\psi_o(x)]dx \qquad (335)$$

and

$$I_1(o) = \frac{1}{M_\infty}\theta_o^{\frac{1}{2}}(o) - \frac{1}{M_\infty}\int_0^\infty \frac{\theta_o^{\frac{1}{2}}(x)}{x^2}[xu_o(x)-\psi_o(x)]dx . \qquad (336)$$

The solutions for equations 309 , 310 , 311 are just those for the flow of a compressible gas over a semi-infinite flat plate and are shown plotted in figures 22 and 24. Since in the large slip case the gas flows independently of the particles in the zeroth order, the kinematic coefficient of viscosity is unchanged, and no statement needs to be made about C_s/C_p. As in the previous case, whenever curves are drawn for comparing the incompressible gas phase case with the compressible gas phase case, comparable quantities must be made dimensionless in the same manner.

The solutions for equations 316 and 317 when

$$Pr = 1 \qquad (\gamma-1)M_\infty^2 = 1$$

$$\theta_o(o) = \alpha = 1/2$$

are shown in figure 28. The solution curve for 318 is shown in figure 29, and the solution curves for 335, 328, and 331 are shown in figures 30, 31, and 32. The curves showing the comparison between the compressible gas solutions and the corresponding incompressible gas solutions are given in figures 33, 34, 35, and 36. The nature of all the solution curves is fairly easy to understand except for figure 30 which gives $I_1(\eta)$. To understand figure 30, one must realize that the gas is being cooled by the presence of the flat plate at a temperature, T_w, which is less than T_∞. Now, in the first order problem, the thermal interaction between the gas phase and the particle phase is taken into account. To first order then, the particle phase is subjected to a cooling gas, and consequently heat is transferred from the particles to the gas as indicated by figure 29. Therefore, a gas with particles does not cool as fast as a gas without particles, and thus the gas density does not increase as fast as it would without particles. This means that there is a first order decrease in gas density due to the particles, and this causes a first order decrease in particle density. Therefore, the first-order particle density should be negative near the plate, as indeed it is.

The formulas for the shear coefficient and heat transfer to the wall are given by

$$\frac{\tau}{\rho_\infty u_\infty^2} = \frac{\sqrt{\alpha}}{\sqrt{Re_x}} (.4903) \left[1 + \frac{x}{\lambda_m} 2.197 \frac{\varkappa}{M_\infty} + \ldots\right] , \tag{337}$$

$$q(o) = -k_\infty \frac{T_\infty}{x} \sqrt{\alpha} \sqrt{R_x} (.4903) \left[1 + \frac{x}{\lambda_m} 2.663 \frac{\varkappa}{M_\infty} + \ldots\right] , \tag{338}$$

provided

$$\alpha = 1/2 ,$$

$$Pr = 1 ,$$

$$(\gamma - 1) M_\infty^2 = 1 .$$

REFERENCES

1. Rubinow, S. I. and Keller, Joseph B., "The Transverse Force on a Spinning Sphere Moving in a Viscous Fluid," Journal of Fluid Mechanics, Vol. II (1961), pp. 447-459.

2. Marble, Frank E., "Dynamics of a Gas Containing Small Solid Particles," Presented at the Fifth AGARD Combustion and Propulsion Colloquium, Braunschweig (April, 1962).

3. Schlichting, Hermann, Boundary Layer Theory, McGraw-Hill Book Co., Inc., New York (1960).

4. von Mises, Richard and von Karman, Theodore, Advances in Applied Mechanics, Vol. II (1951), pp. 23-91.

5. Emmons, H. W. and Brainerd, G. J., "Temperature Effects in a Laminar Compressible-Fluid Boundary Layer Along a Flat Plate," Journal of Applied Mechanics, Vol. 8 (1941), A-105.

6. Sproull, Wayne T., "Viscosity of Dusty Gases," Nature, Vol. 190 (1961), pp. 976-978.

7. Saffman, P. G., "Flow of a Dusty Gas Between Rotating Cylinders," Nature, Vol. 193 (1962), pp. 463-464.

8. Maude, A. D., "The Viscosity of a Suspension of Spheres," Journal of Fluid Mechanics, Vol. 7 (1959), pp. 230-236.

9. Einstein, A., Ann. Phys., Lpz., Vol. 19 (1906), p. 289.

10. Einstein, A., Ann. Phys., Lpz., Vol. 34 (1911), p. 591.

11. Soo, S. L. and Tien, C. L., "Effect of the Wall on Two-Phase Turbulent Motion," Journal of Fluid Mechanics, Vol. 27 (1960); Trans. ASME, Vol. 82, Series E, pp. 5-11.

12. Tien, C. L., "Fluid-Solids Mixture in a Pipe," Journal of Heat Transfer, Vol. 83 (1961), pp. 183-188.

13. Segre, G. and Silbersky, A., "Behavior of Macroscopic Rigid Spheres in Poiseville Flow," Journal of Fluid Mechanics, Vol. 14 (1962), pp. 115-157.

14. Proudman, I., and Pearson, J. R. A., "Expansions at Small Reynolds Numbers for the Flow Past a Sphere and a Circular Cylinder," Journal of Fluid Mechanics, Vol. 2 (1957), pp. 237-262.

15. Rannie, W. D., "A Perturbation Analysis of One-Dimensional Heterogeneous Flow in Rocket Nozzles," Progress in Astronautics and Rocketry, Volume 6, Edited by S. S. Penner and F. A. Williams, Academic Press (1962).

16. Hoglund, R. F., "Recent Advances in Gas-Particle Nozzle Flows," ARS Journal, Vol. 32 (1962), pp. 662-671.

17. Soo, S. L., "Boundary Layer Motion of a Gas-Solid Suspension," University of Illinois, Project Squid Report ILL-3P (October 1961).

18. Chiu, H. H., "Boundary Layer Flow with Suspended Particles," Princeton University, Report 620 (August 1962).

19. Carrier, G. F., "Shock Waves in a Dusty Gas," Journal of Fluid Mechanics, Vol. 4 (1955), Part 4, pp. 376-382.

20. Tien, C. L., "Heat Transfer by a Turbulently Flowing Fluid-Solids Mixture in a Pipe," Journal of Fluid Mechanics, Vol. 10 (1961), pp. 183-188.

21. Soo, S. L., "Gas Dynamic Processes Involving Suspended Solids," A. I. Ch. E. Journal (September 1961).

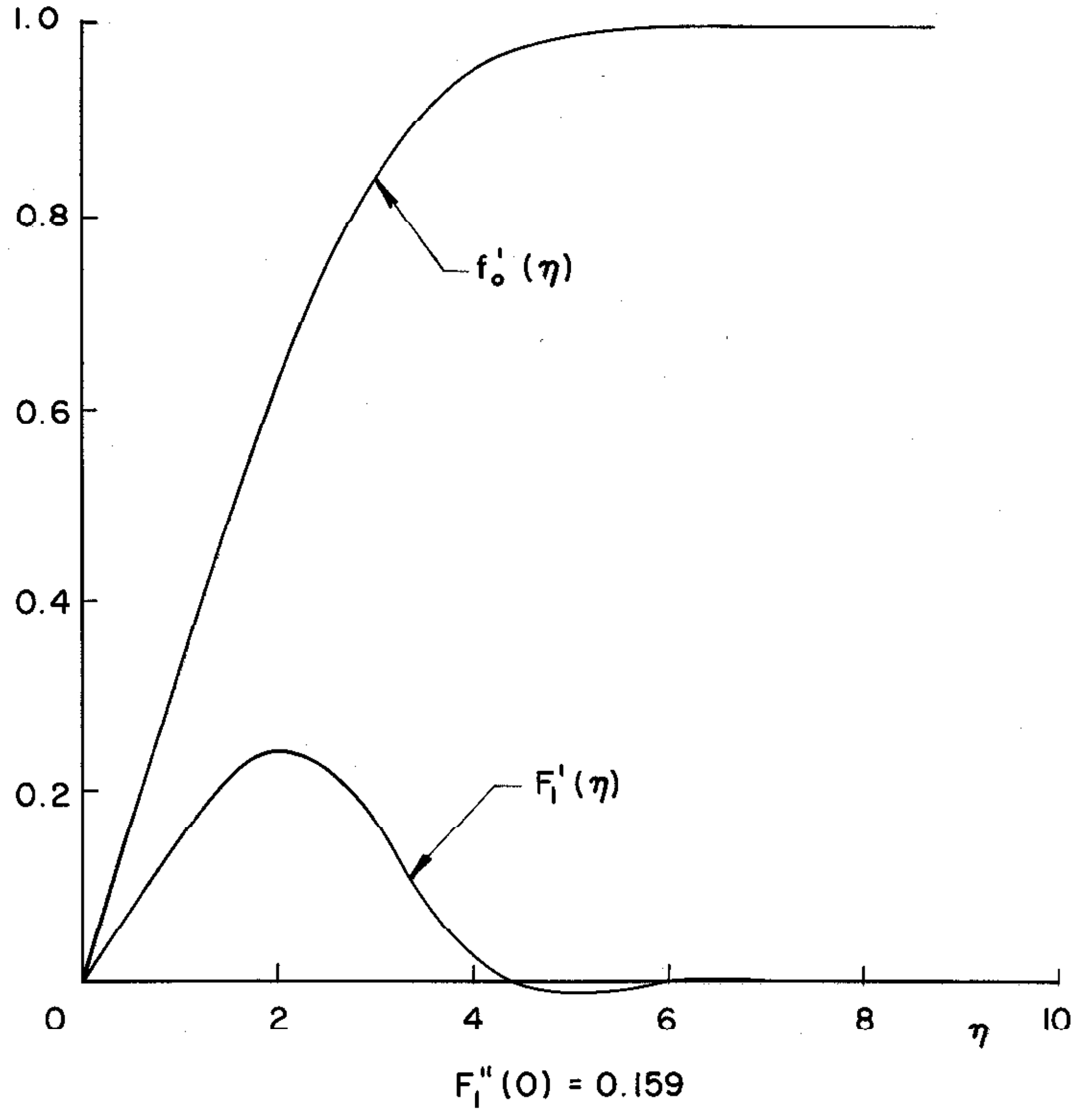


Fig. 1 Zeroth and First Order Gas Velocities.

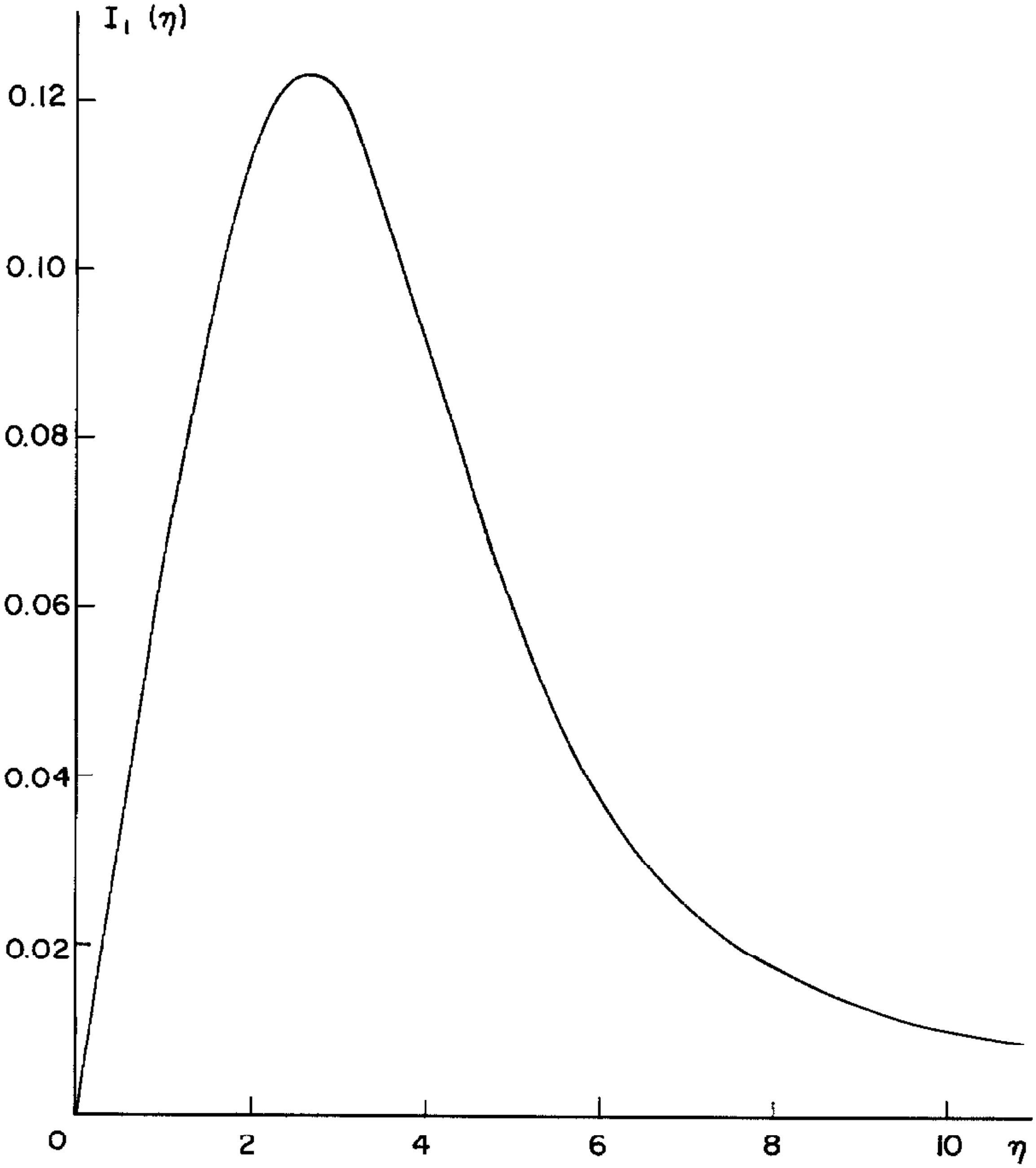


Fig. 2 First Order Particle Density.

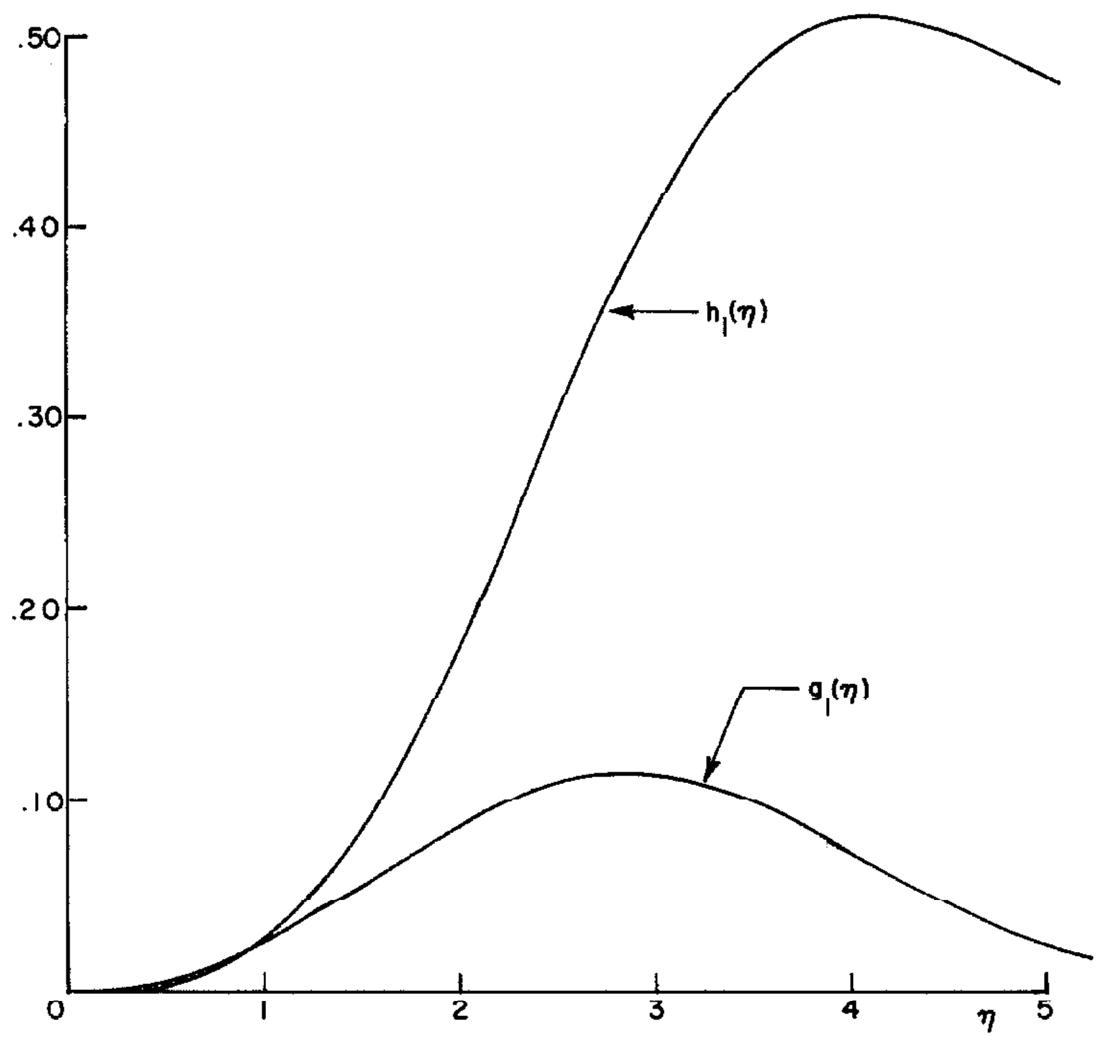


Fig. 3 Particle Slip Velocities.

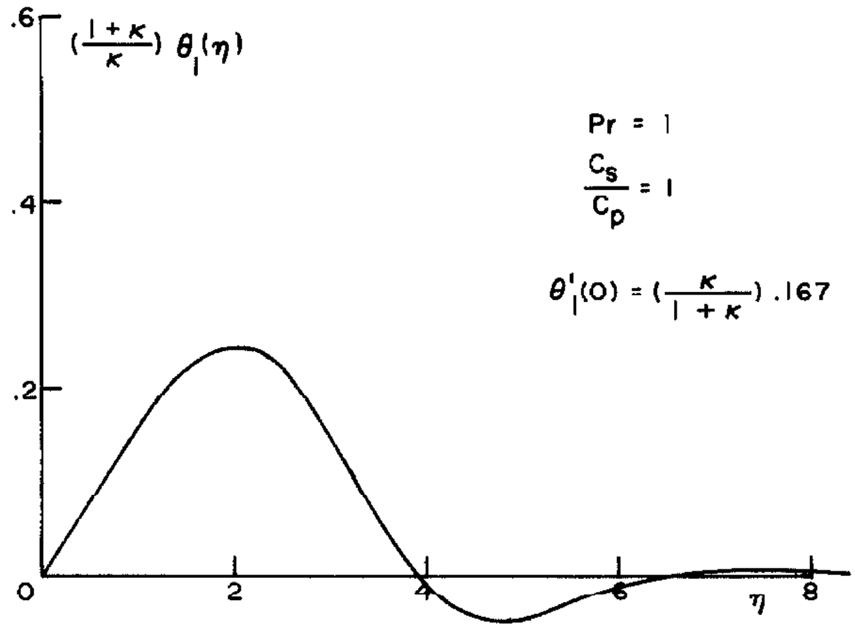


Fig. 4 First-Order Gas Temperature.

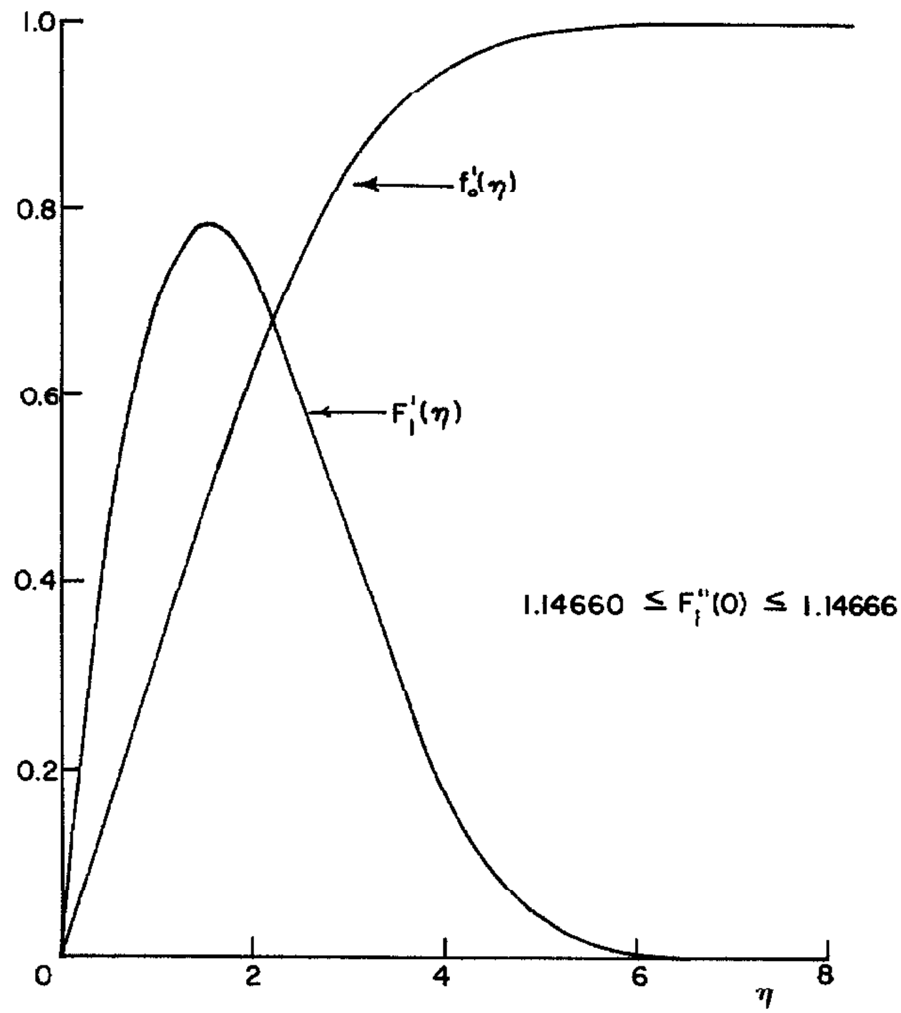


Fig. 5 Zeroth- and First-Order Gas Velocities.

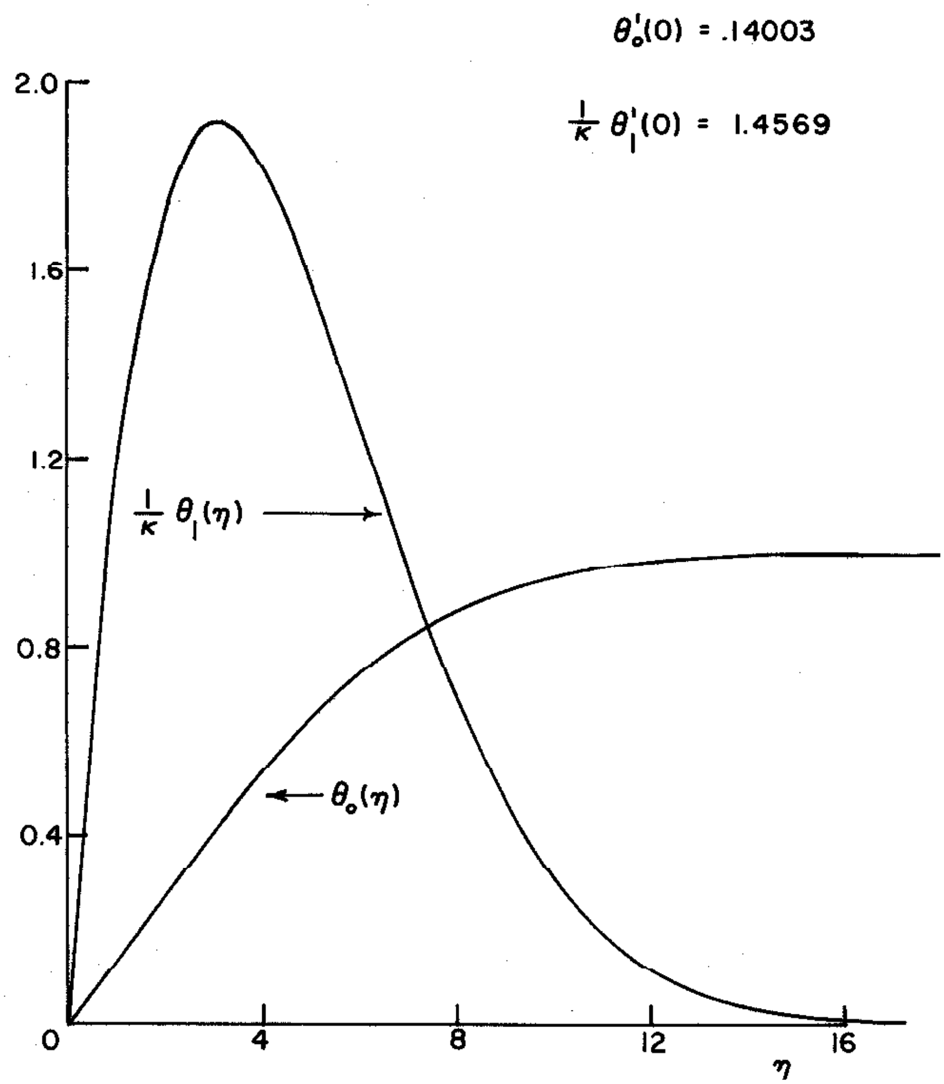


Fig. 6. Zeroth- and First-Order Gas Temperatures for Pr = .10.

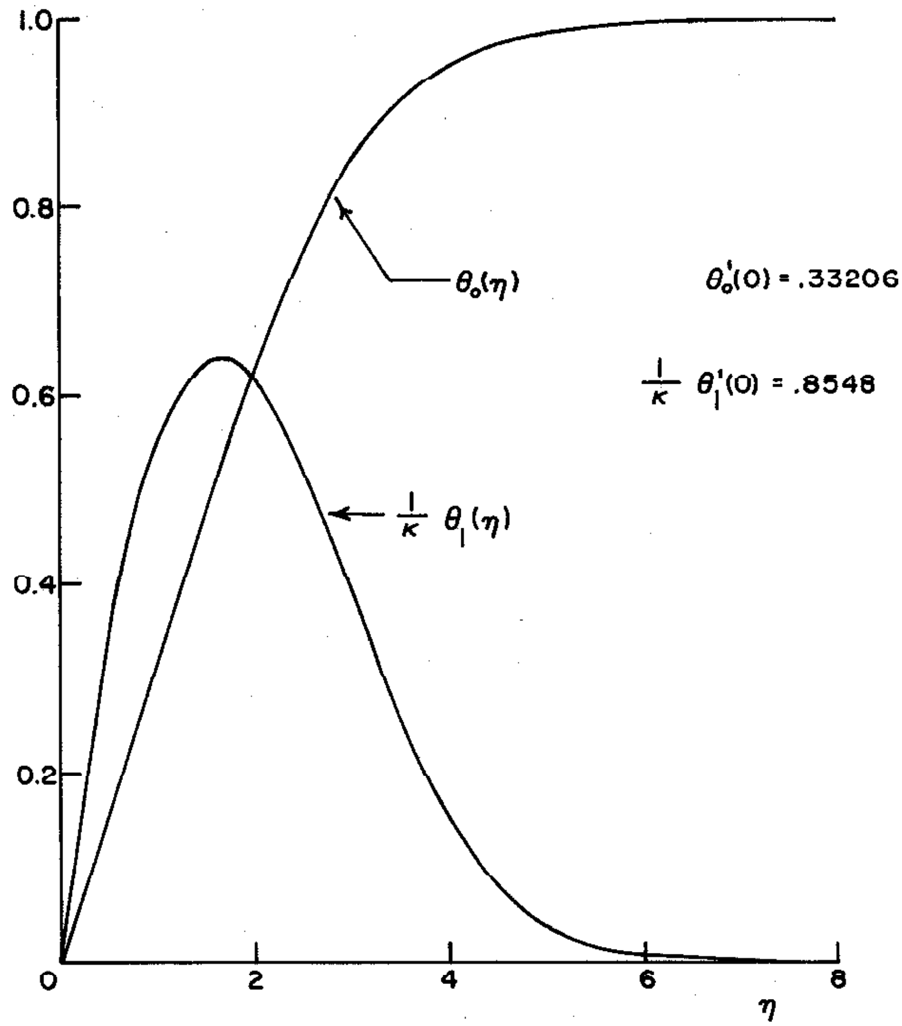


Fig. 7 Zeroth- and First-Order Gas Temperatures for Pr = 1.0.

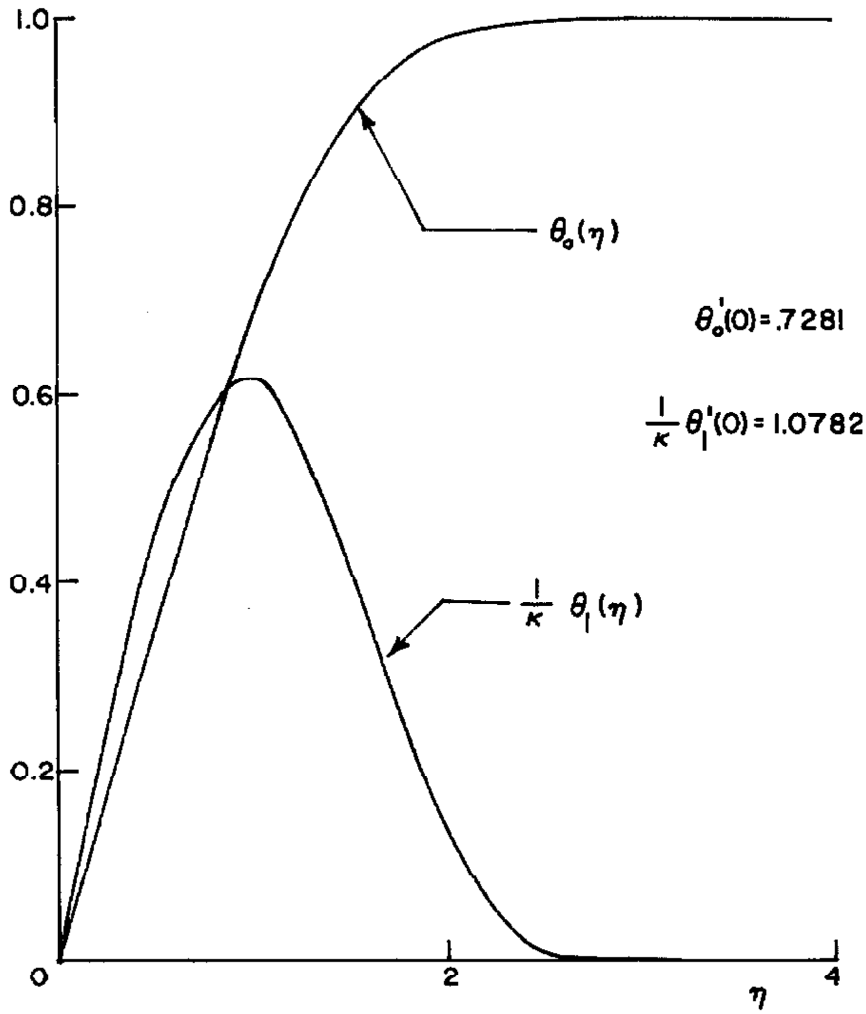


Fig. 8 Zeroth- and First-Order Gas Temperatures for Pr = 10.0.

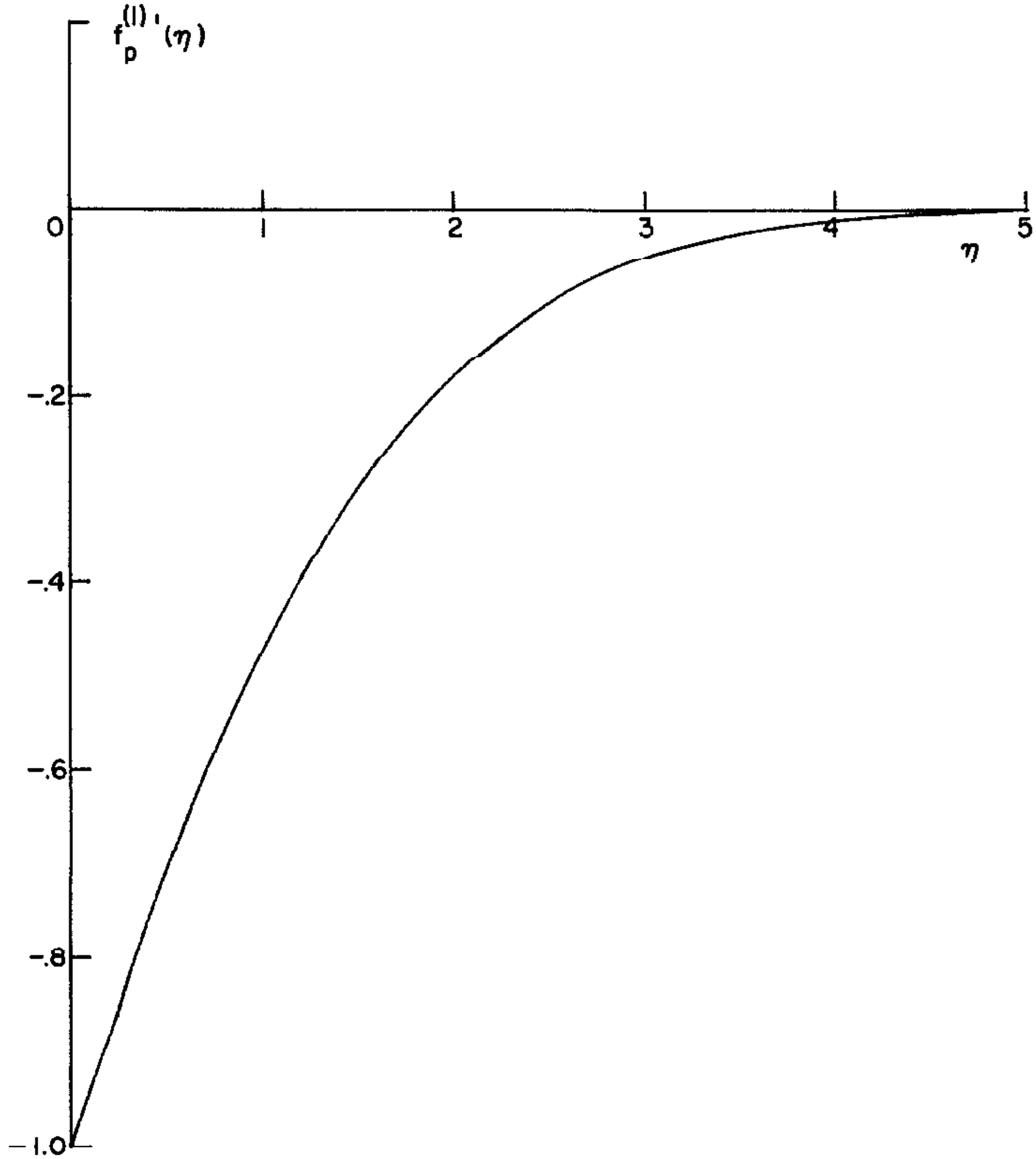


Fig. 9a First-Order Particle Velocity.

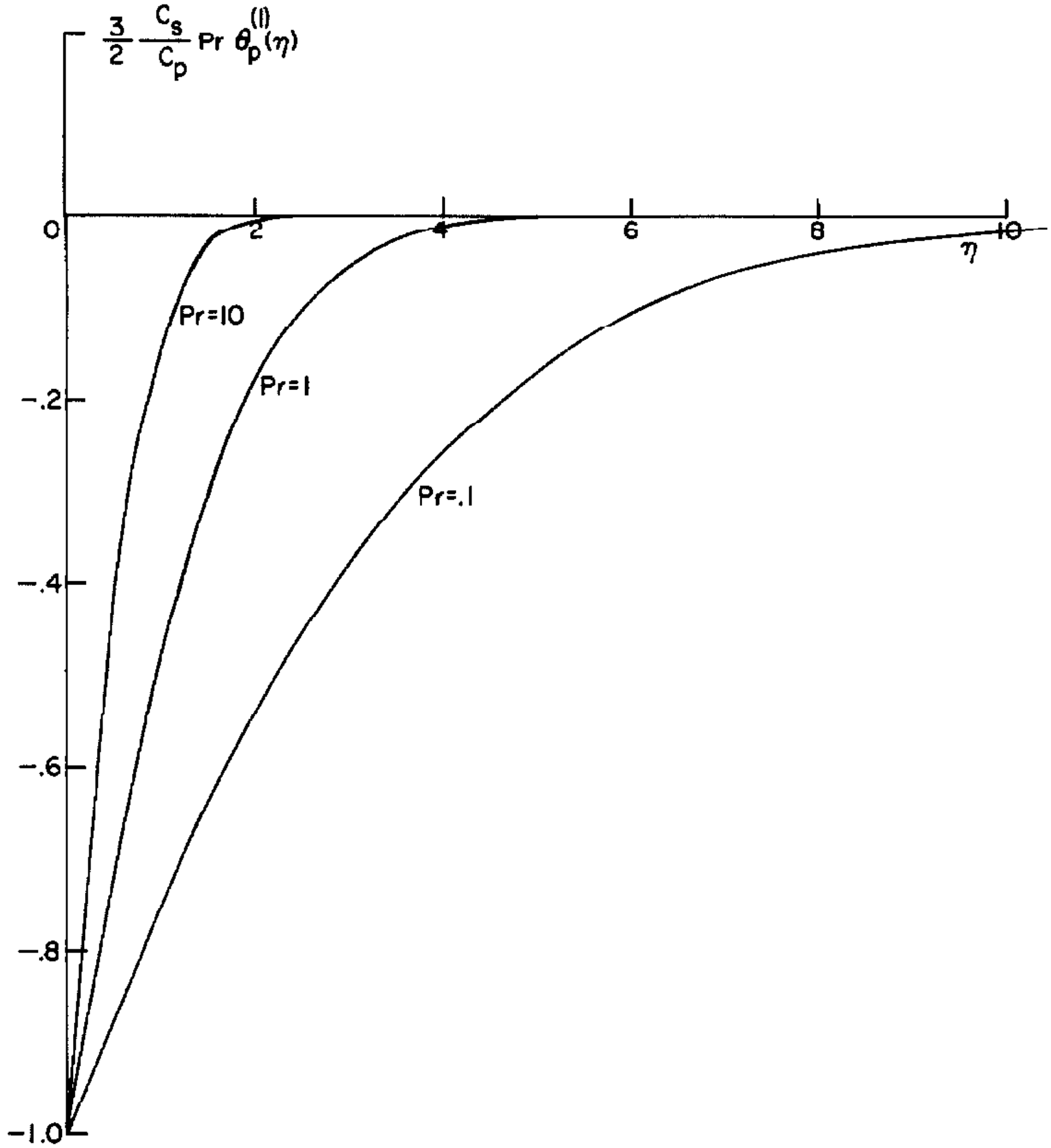


Fig. 9b First-Order Particle Temperatures.

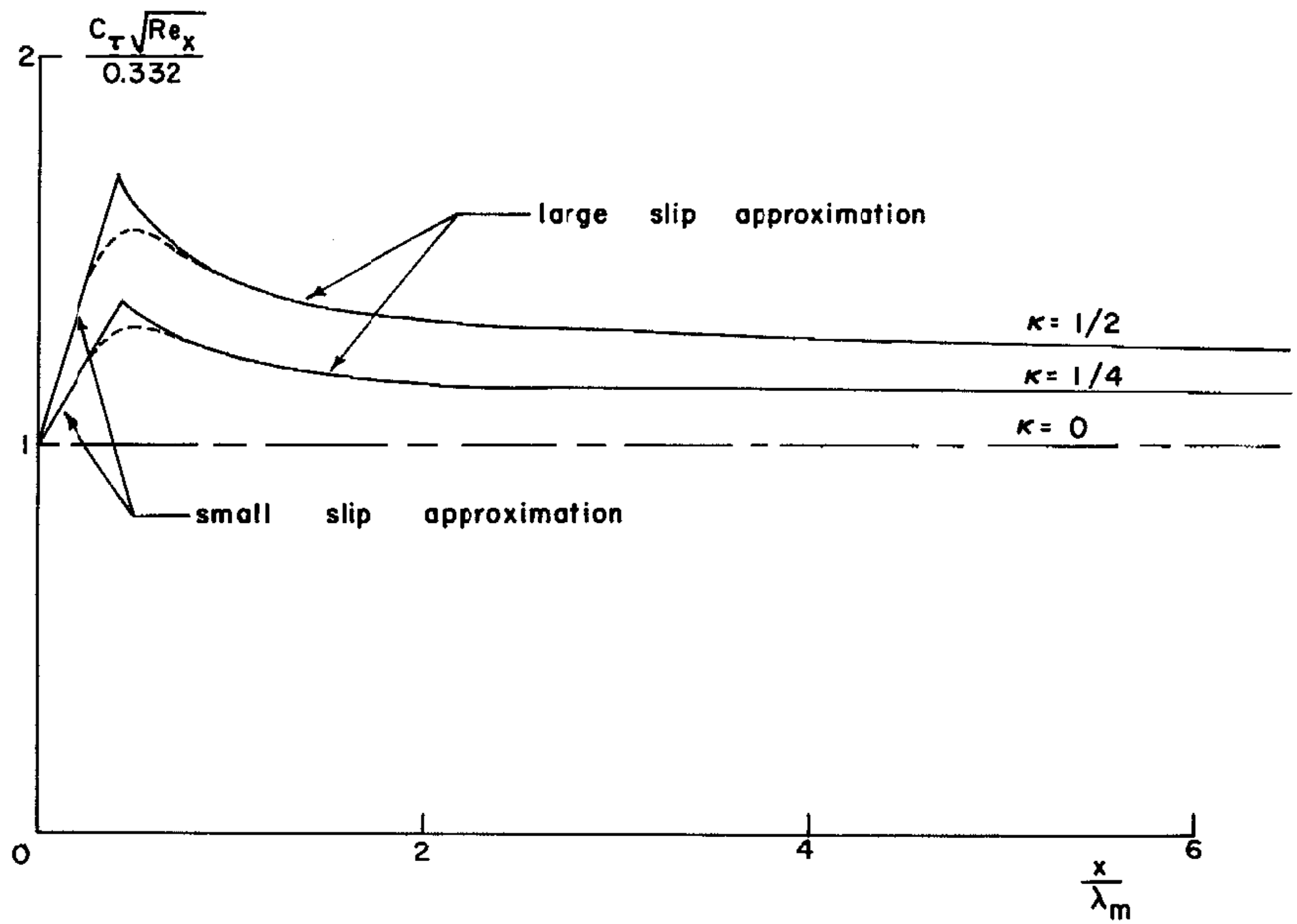


Fig. 10 Shear Coefficient for $\kappa = 0, 1/4, 1/2$.

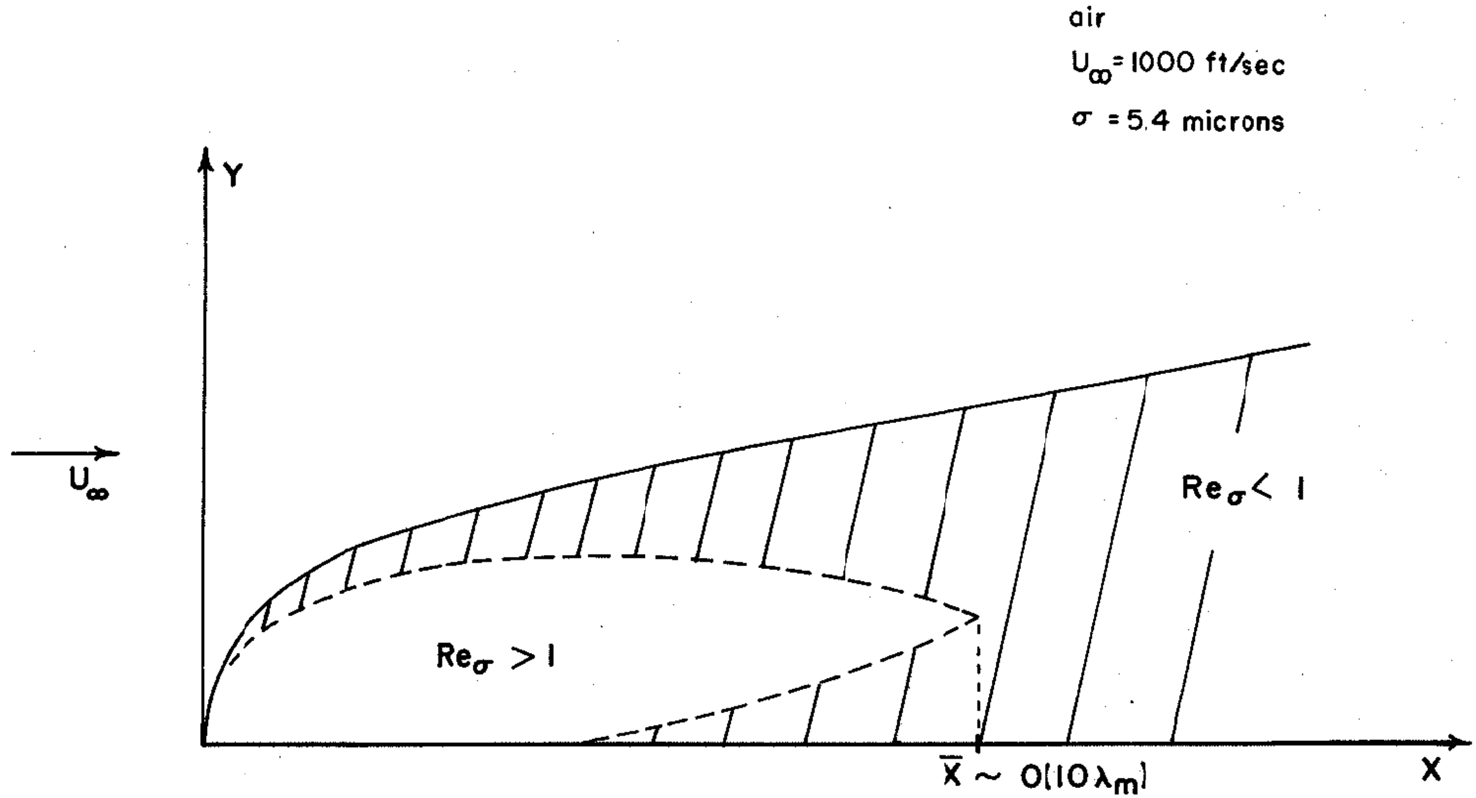


Fig. 11 Regime of Valid Stokes' Drag Law for Semi-Infinite Flat Plate.

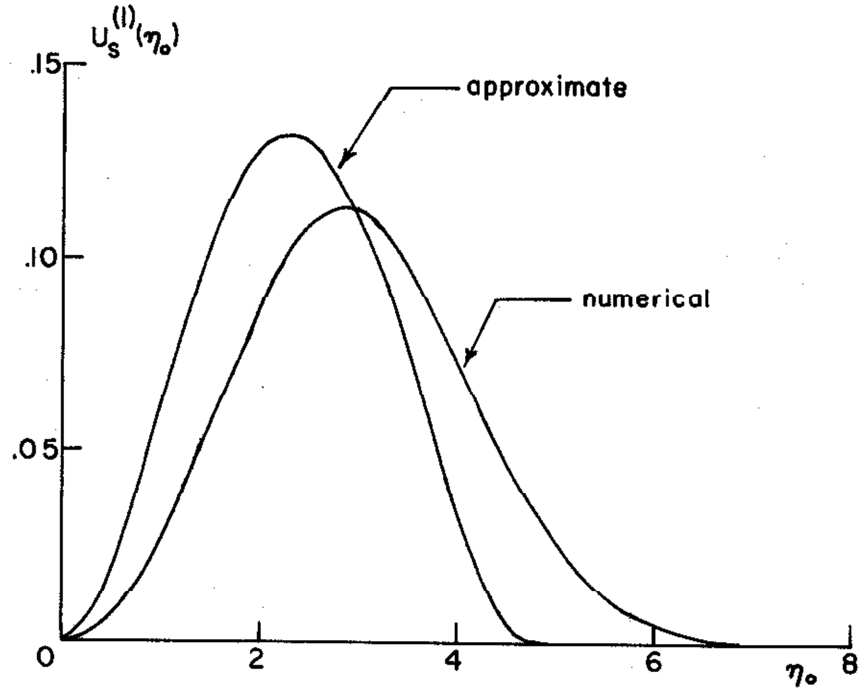


Fig. 12 Comparison of Approximate and Numerical Slip Velocities.

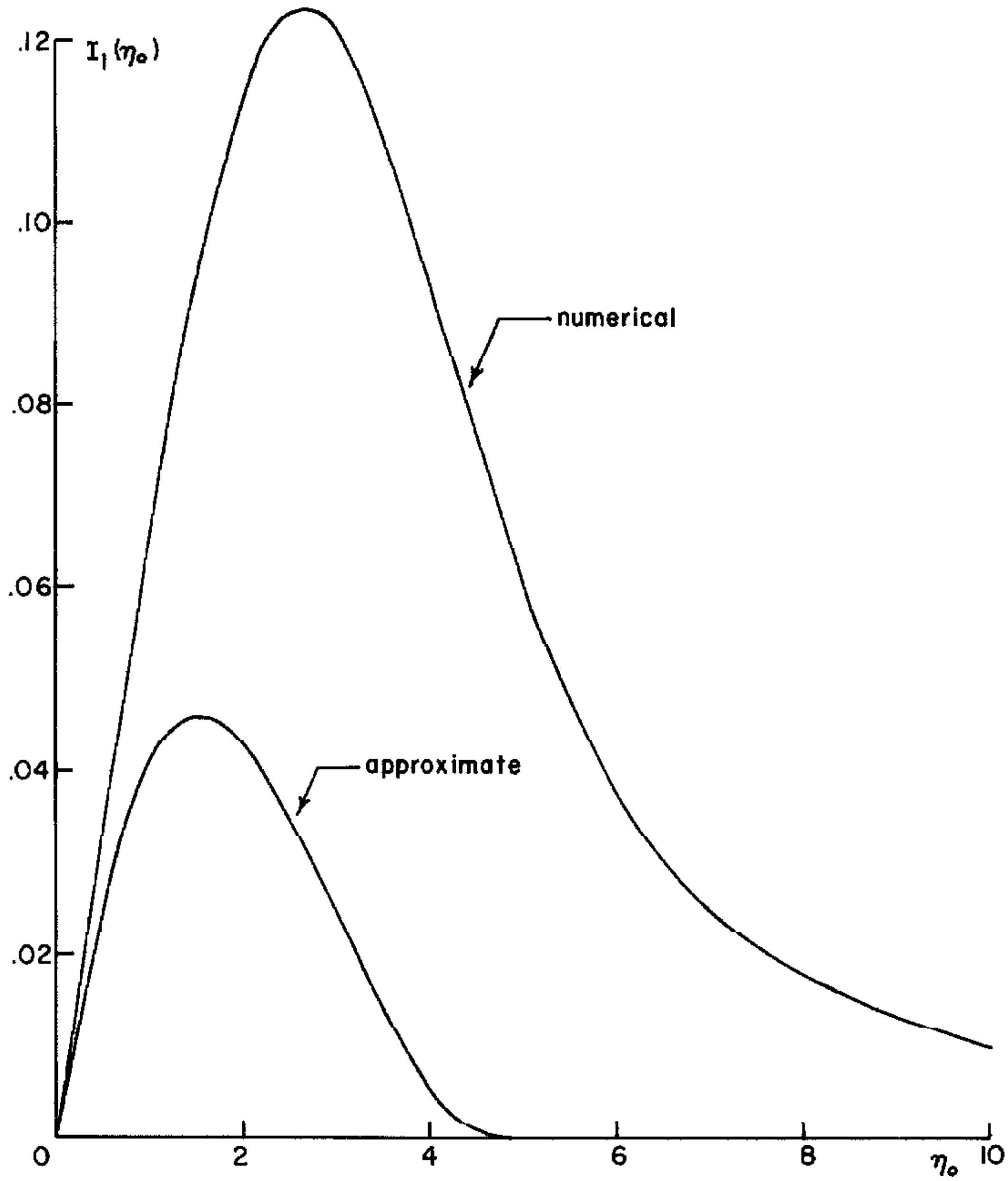


Fig. 13 Comparison of Approximate and Numerical First-Order Particle Densities.

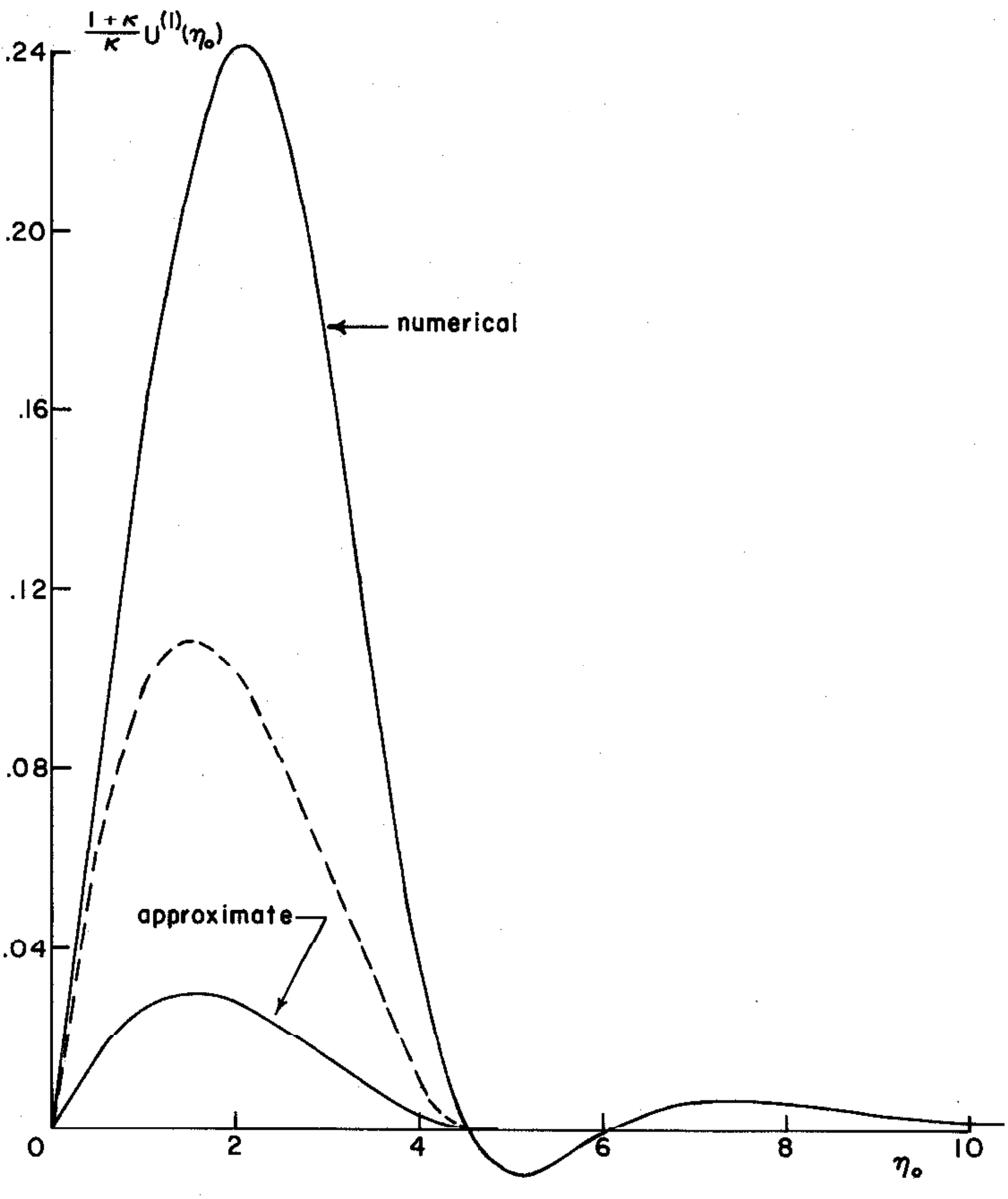


Fig. 14 Comparison of Approximate and Numerical First-Order Gas Velocities.

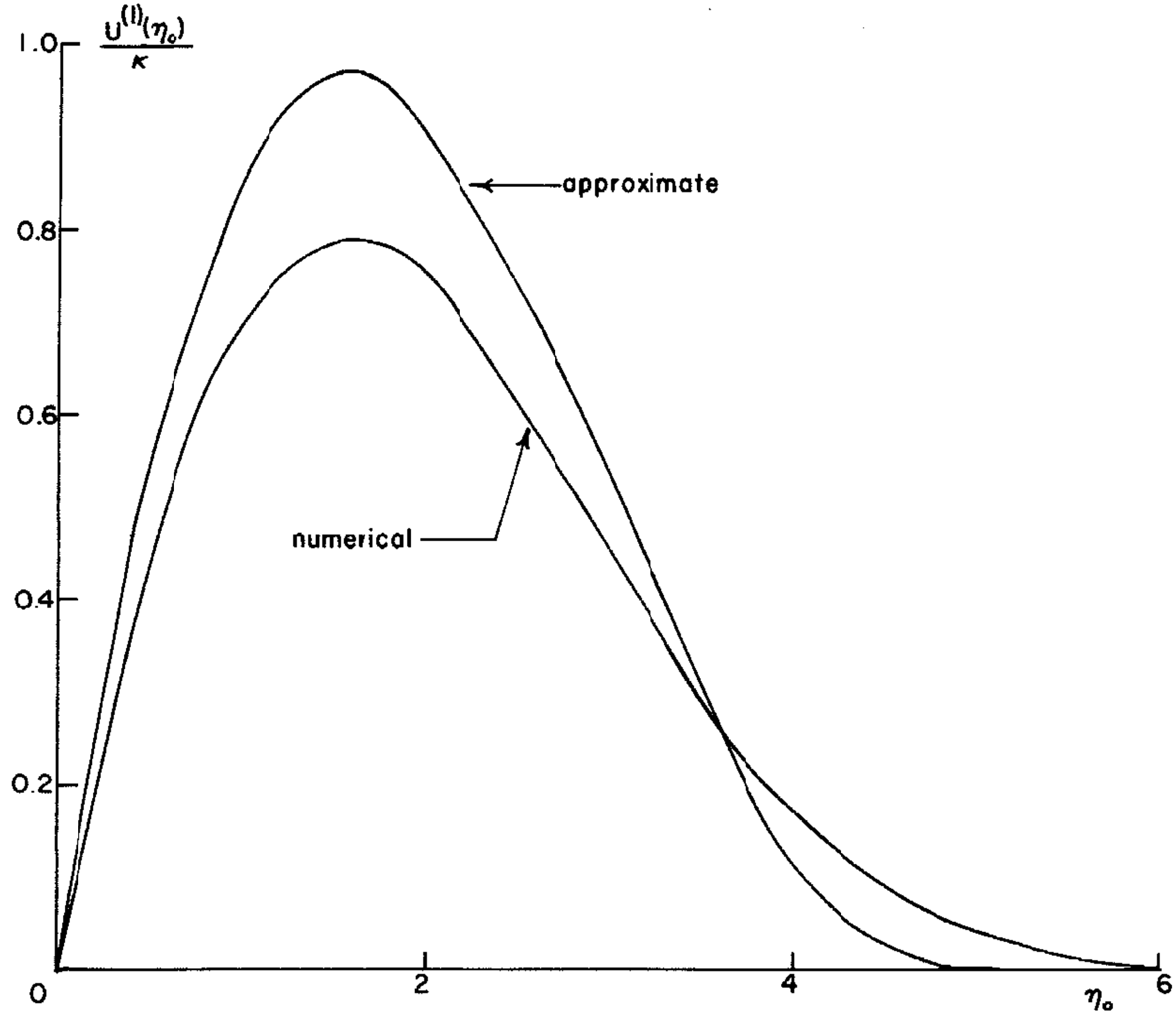


Fig. 15 Comparison of Approximate and Numerical First-Order Gas Velocities.

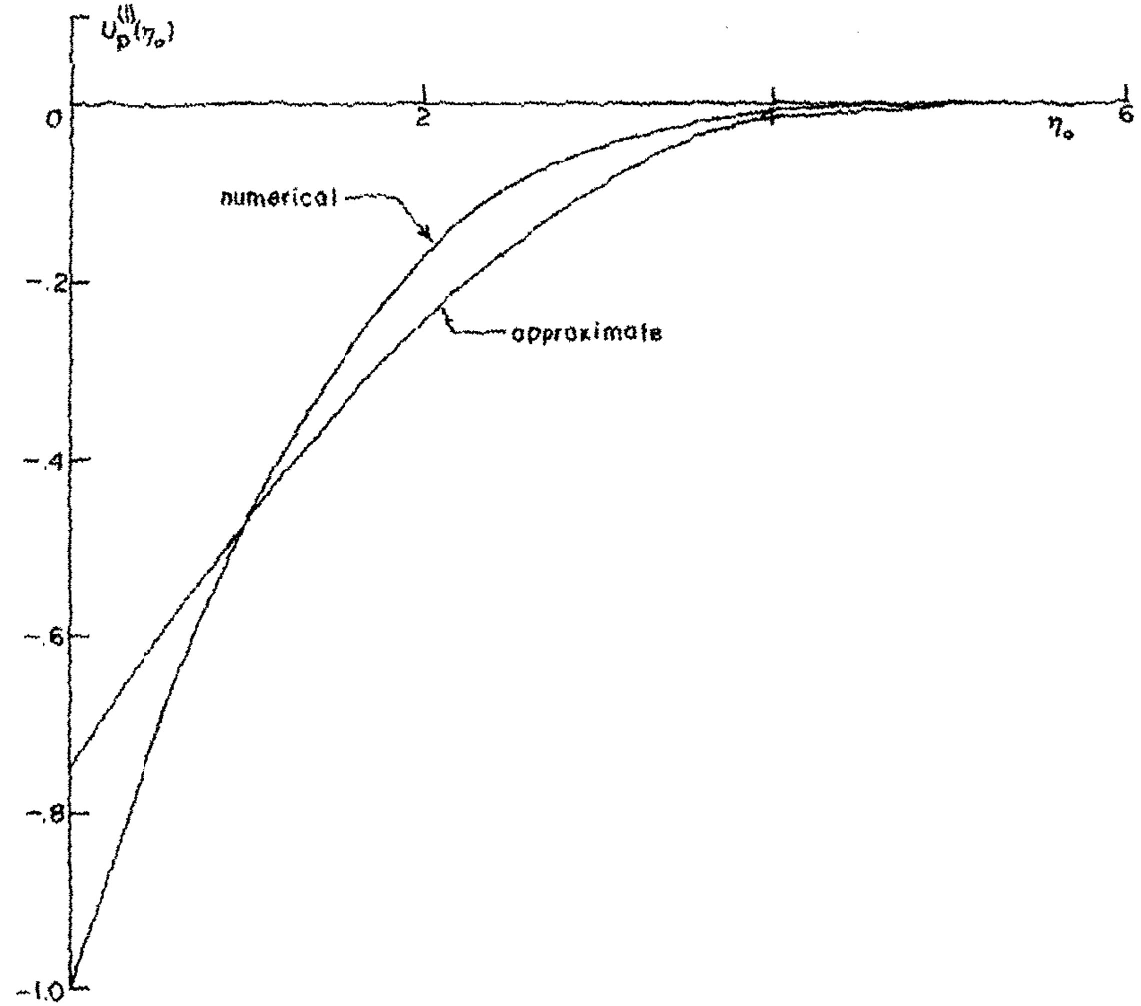


Fig. 16 Comparison of Approximate and Numerical First-Order Particle Velocities.

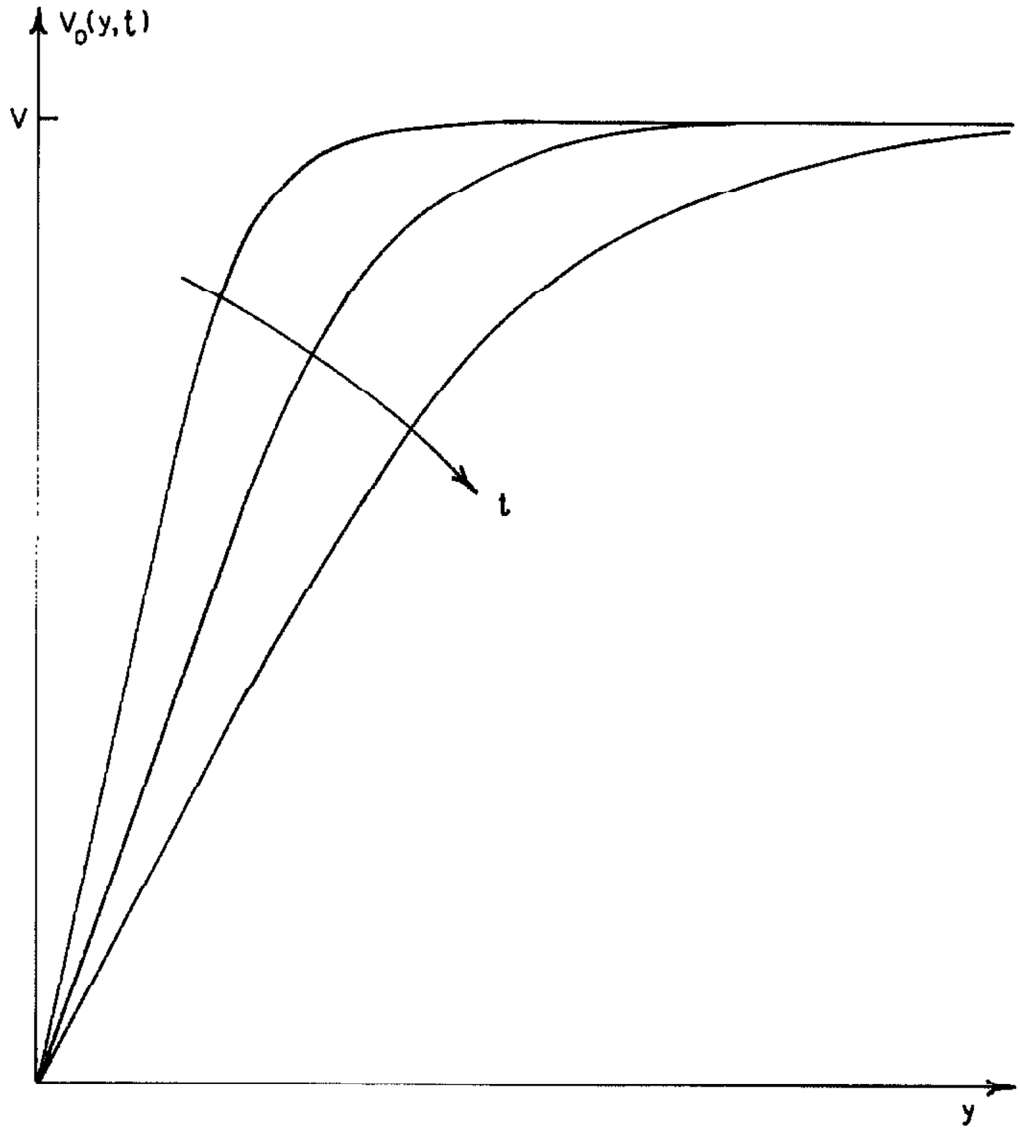


Fig. 17 Zeroth-Order Gas Velocity for Curved Wall.

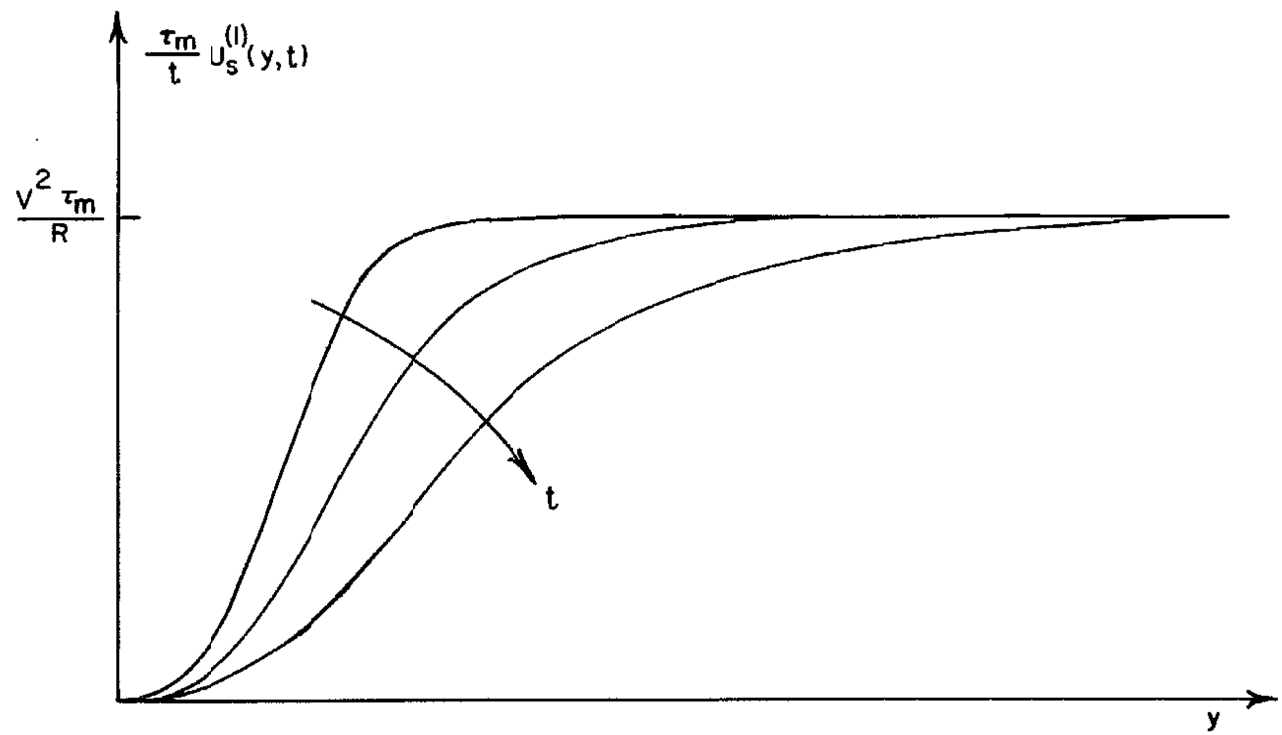


Fig. 18 Radial Particle-Slip Velocity for Curved Wall.

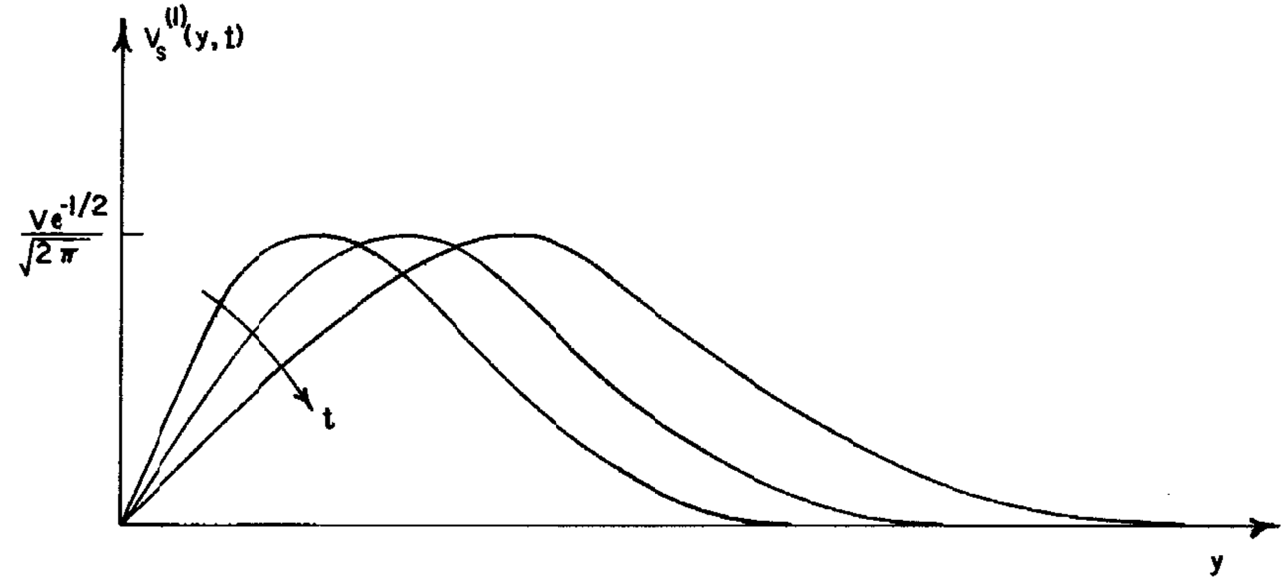


Fig. 19 Tangential Particle-Slip Velocity for Curved Wall.

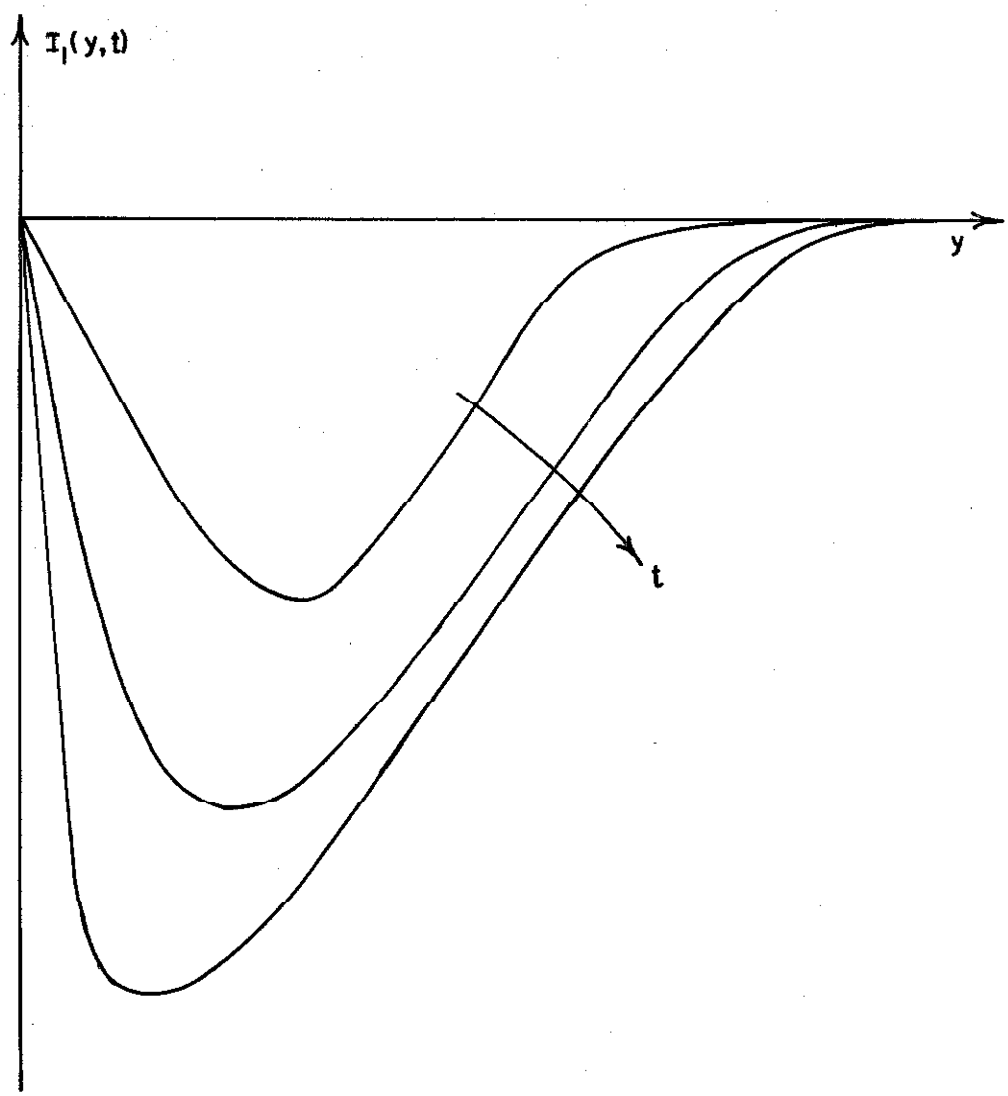
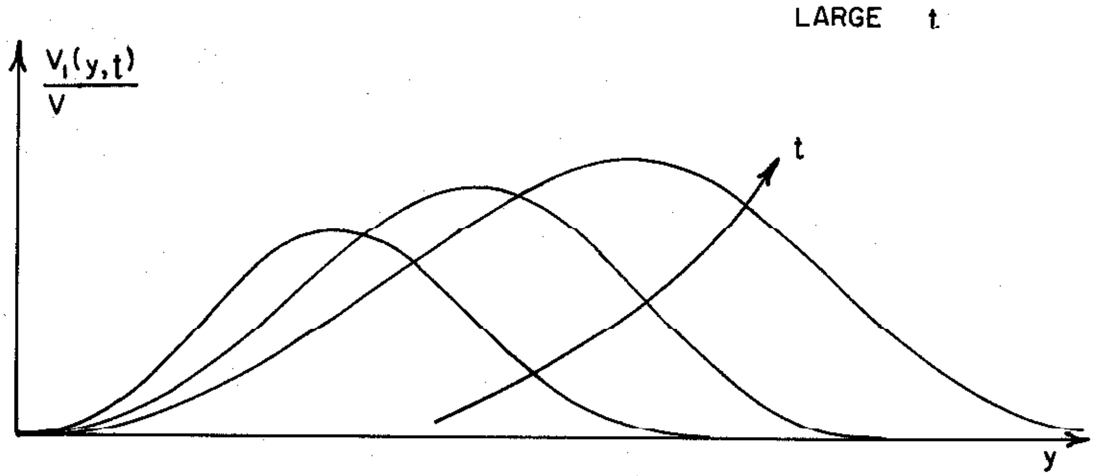


Fig. 20 First-Order Particle-Density Perturbation for Curved Wall.



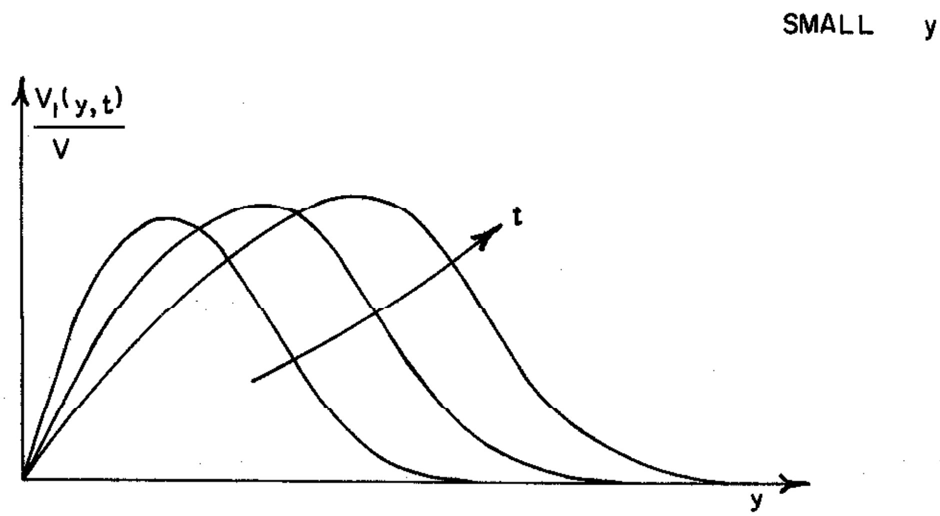


Fig. 21 First-Order Gas Velocity for Curved Wall.

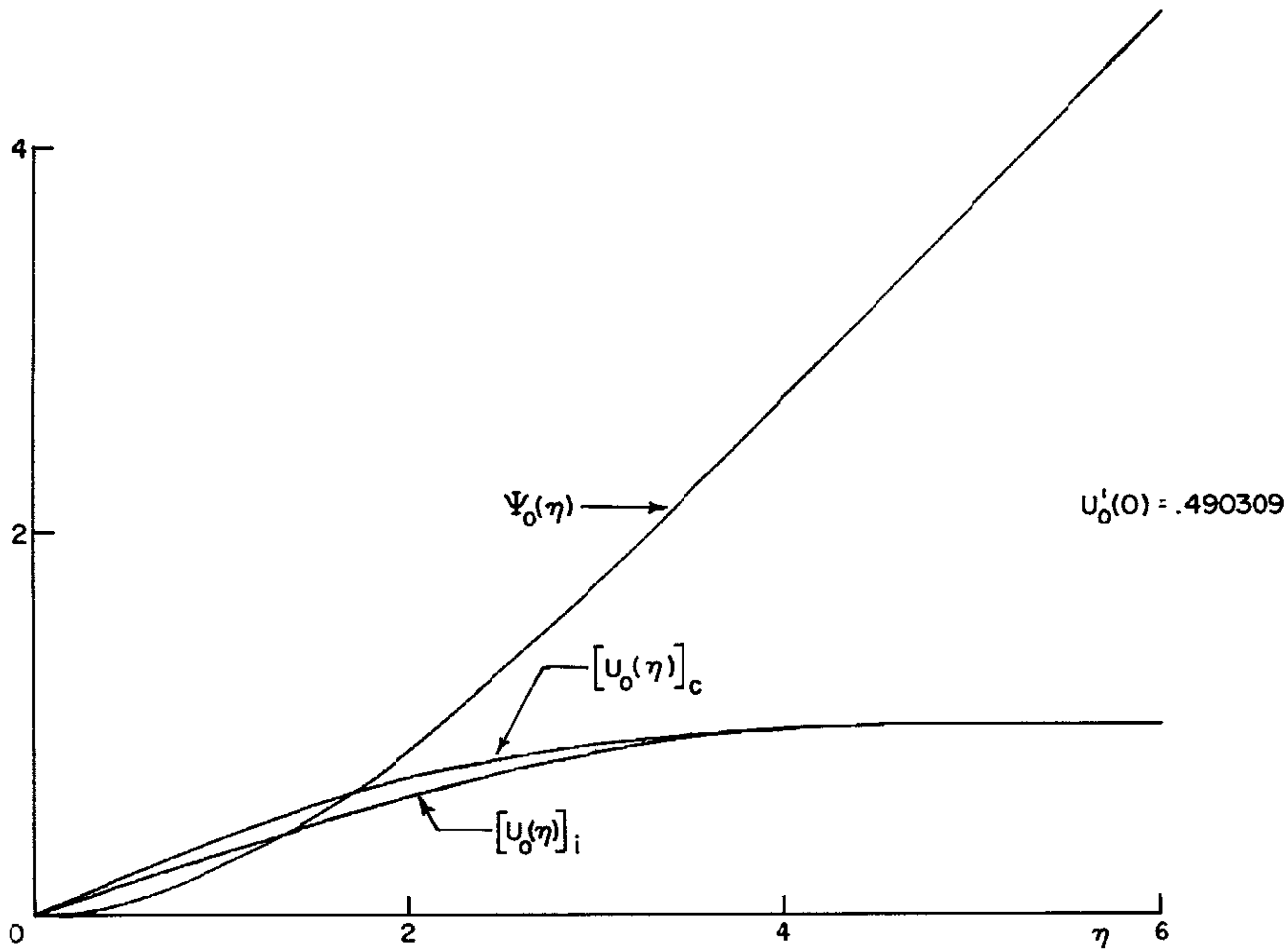


Fig. 22 Zeroth-Order Curves for Compressible Gas.

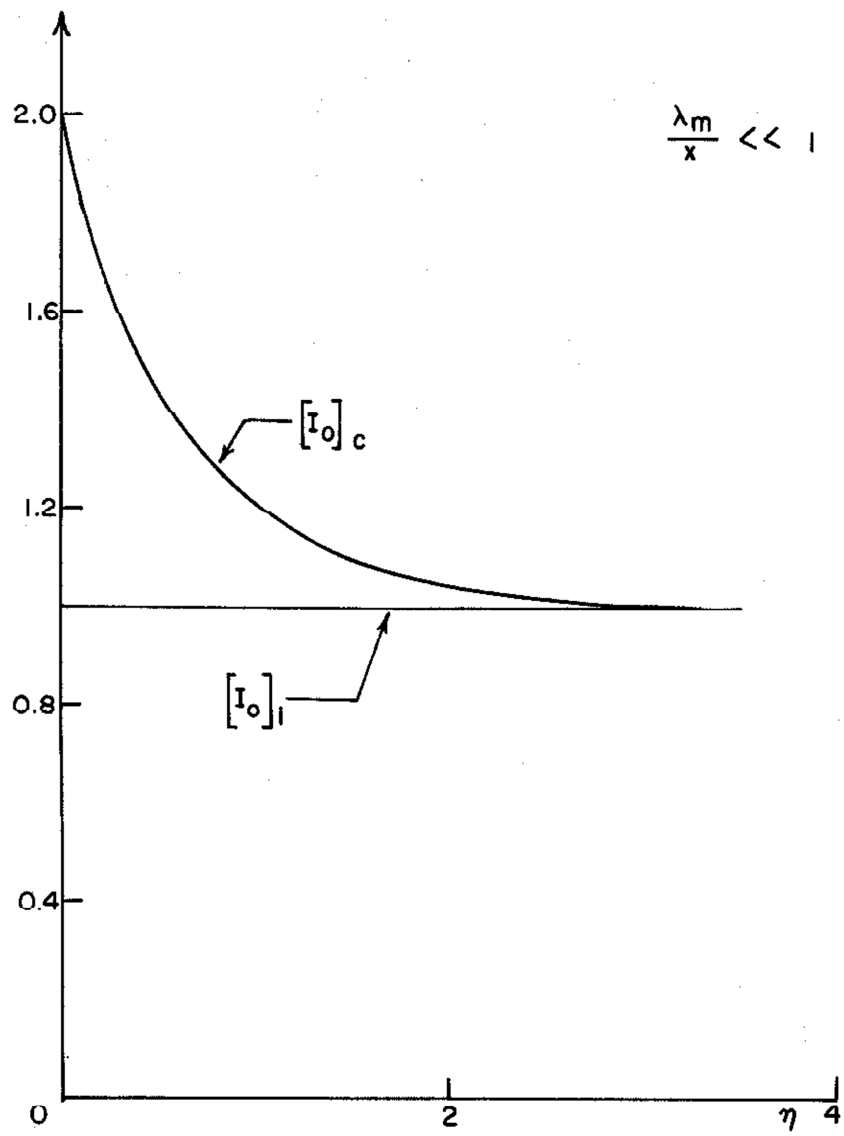


Fig. 23 Comparison Between Compressible and Incompressible Zeroth-Order Particle Density.

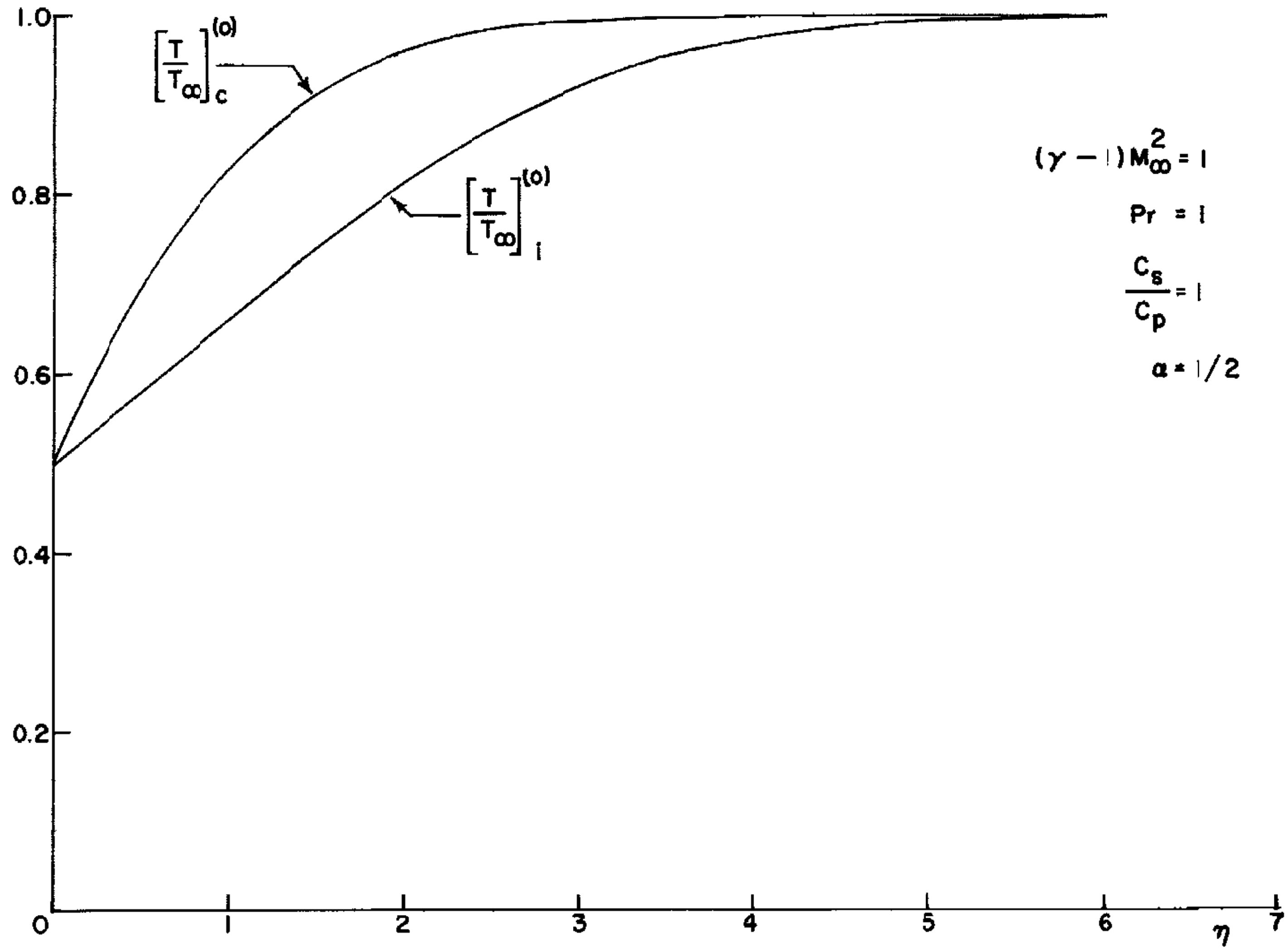


Fig. 24 Comparison of Zeroth-Order Temperatures for Compressible and Incompressible Gas.

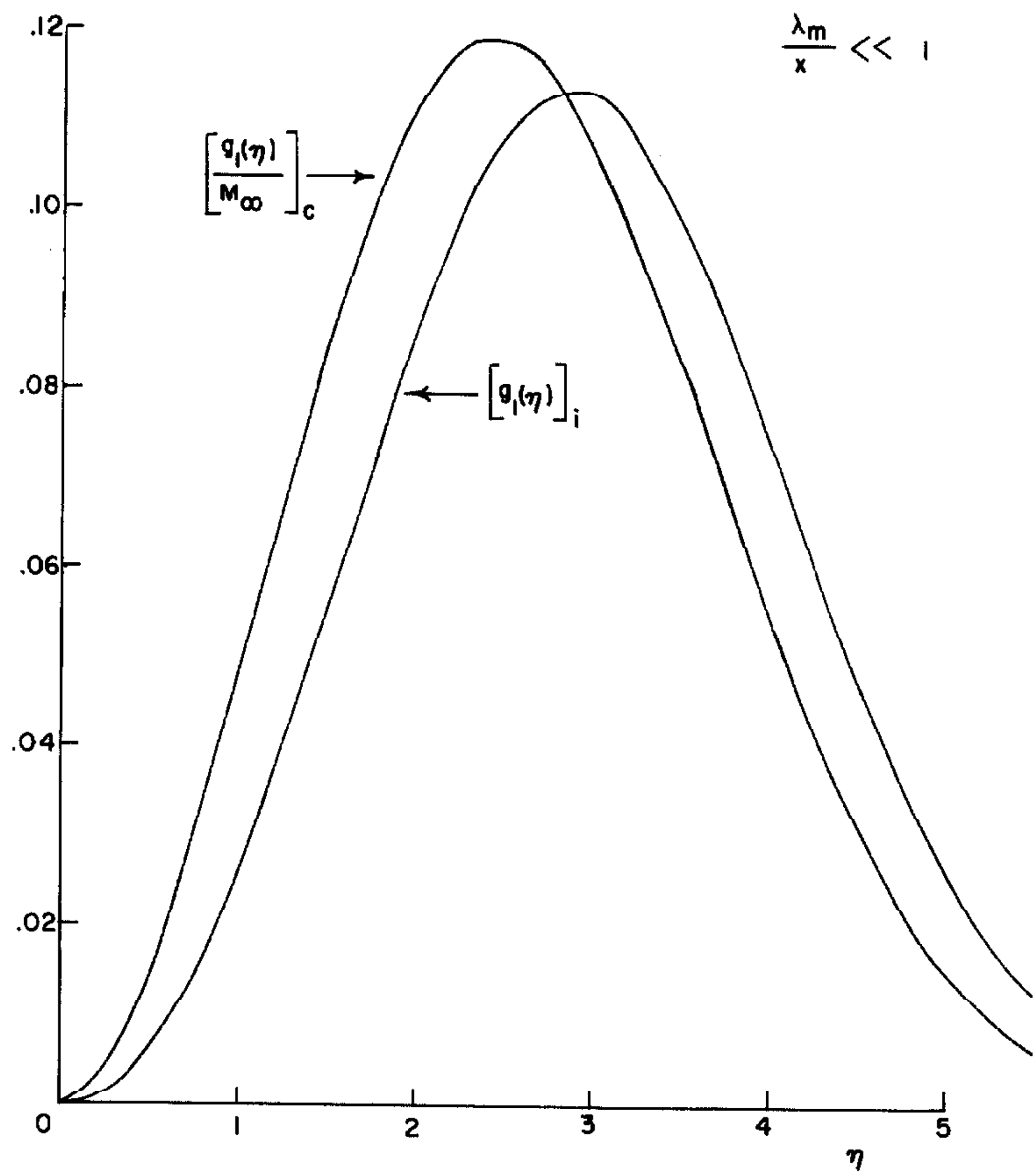


Fig. 25 Comparison of Compressible and Incompressible Slip Velocities.

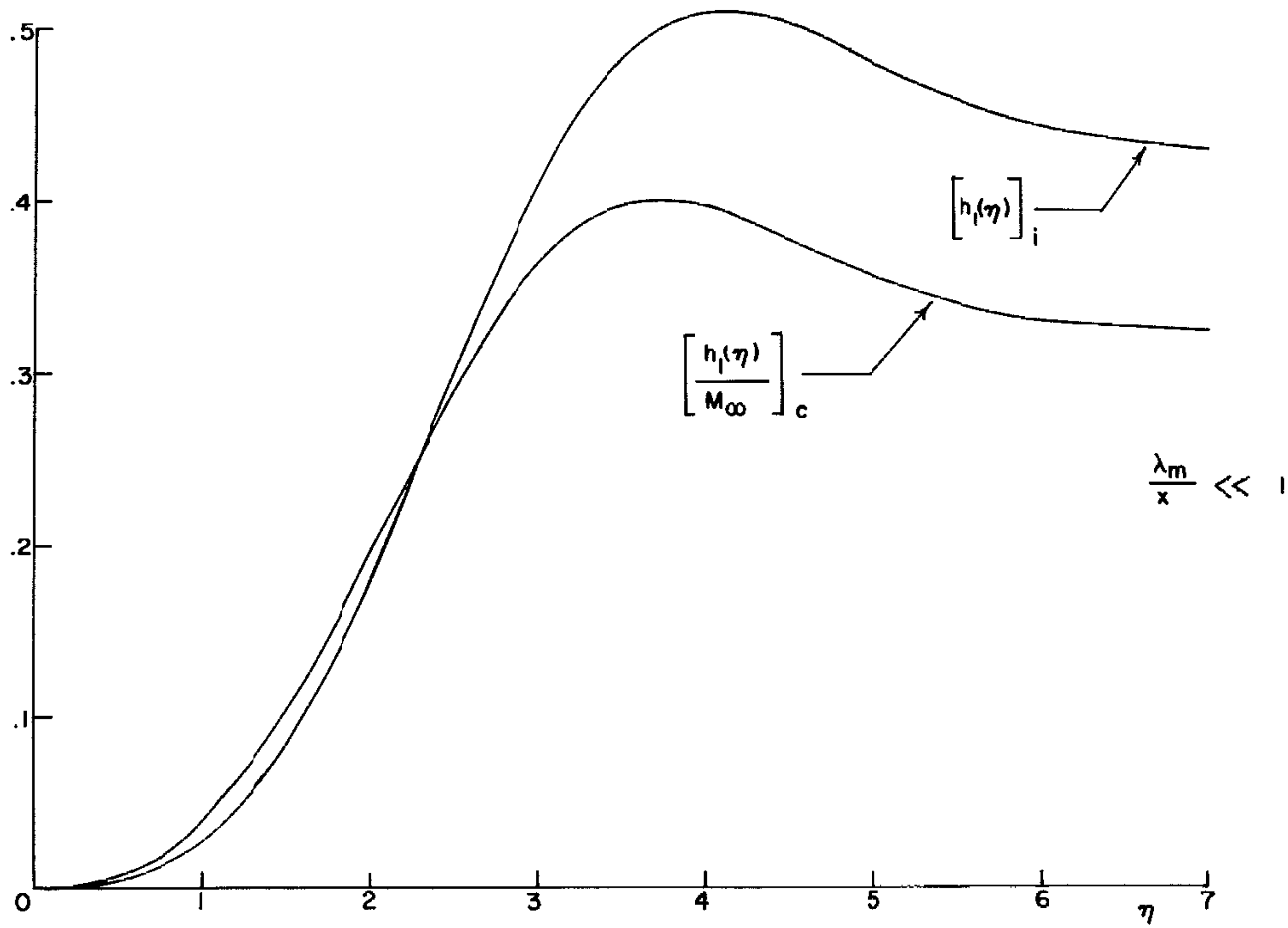


Fig. 26 Comparison of Compressible and Incompressible Slip Velocities.

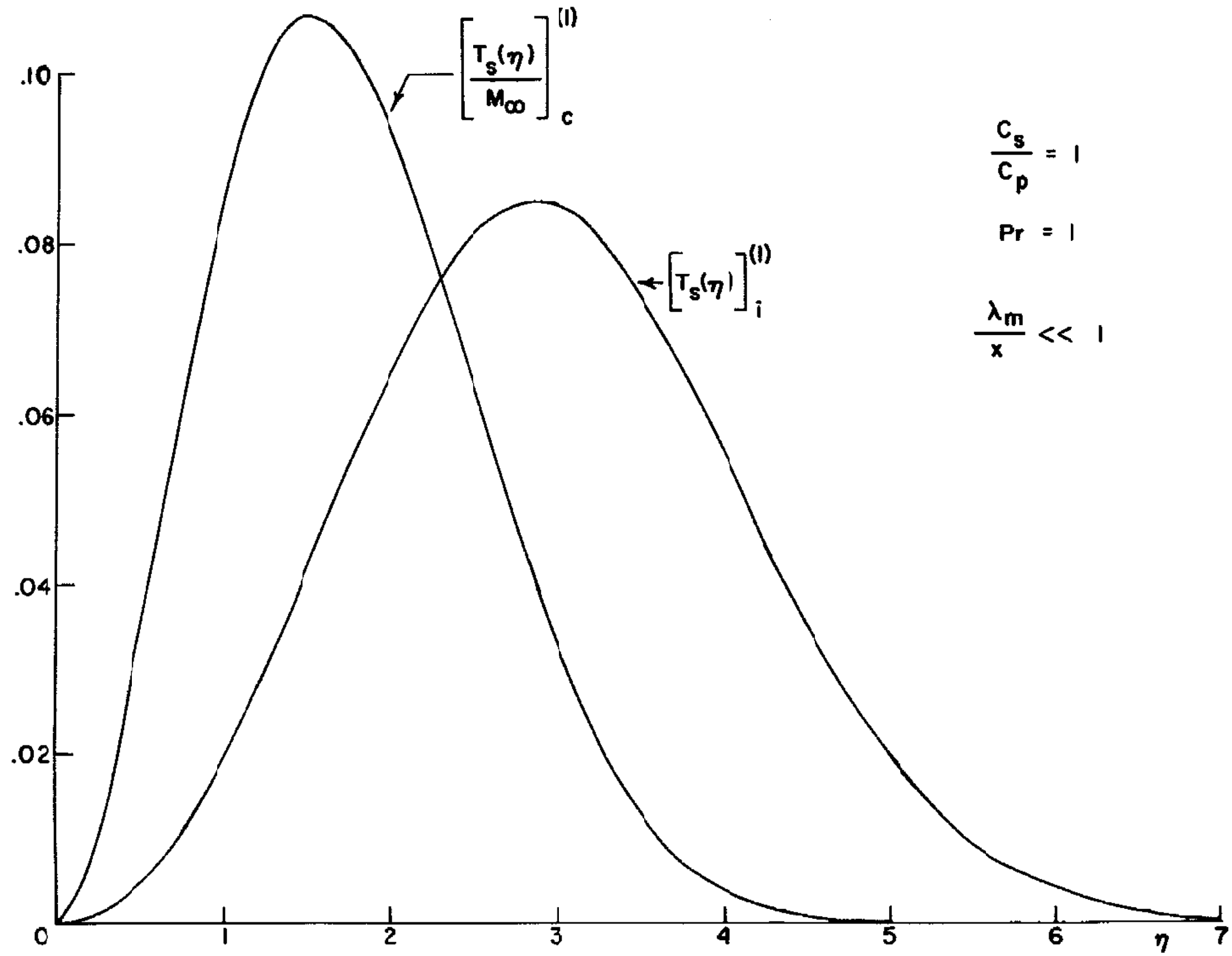


Fig. 27 Comparison of Compressible and Incompressible Temperature Defects.

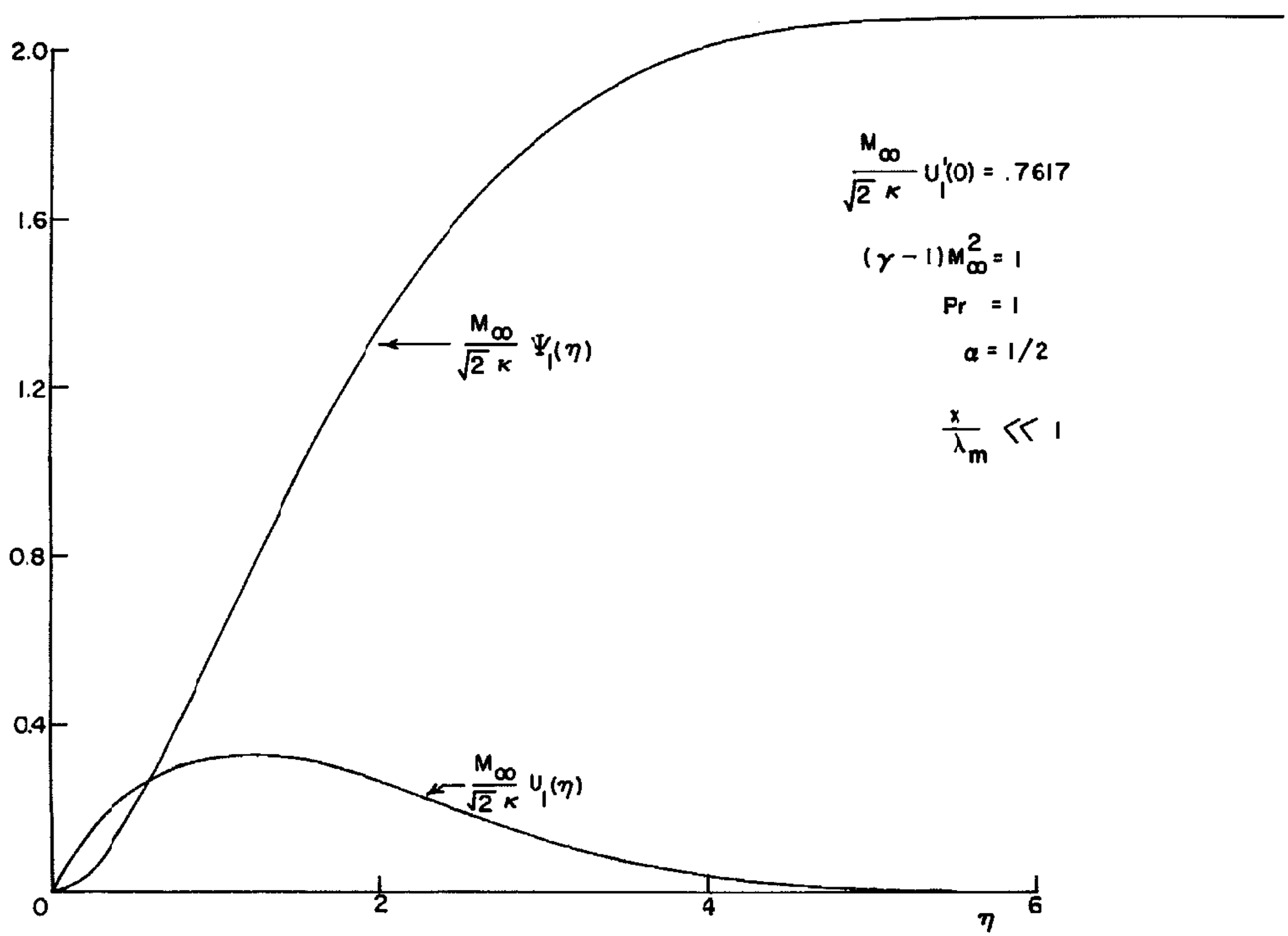


Fig. 28 First-Order Gas Velocities for a Compressible Gas.

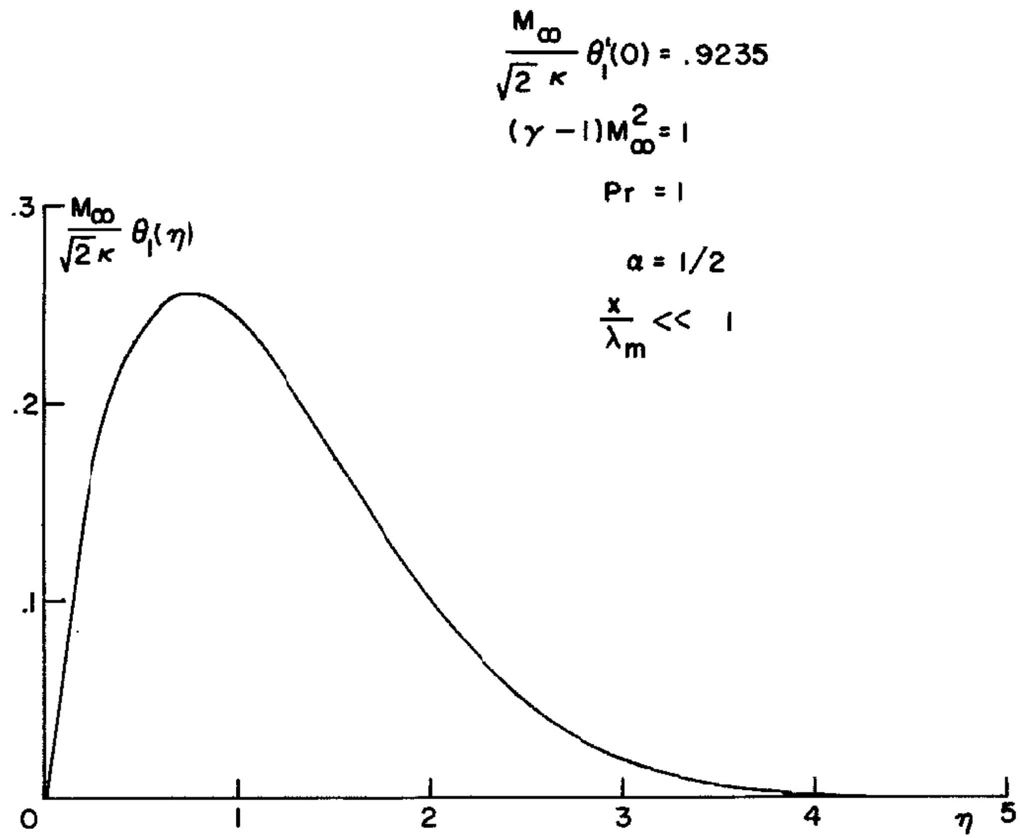


Fig. 29 First-Order Compressible Gas Temperature.

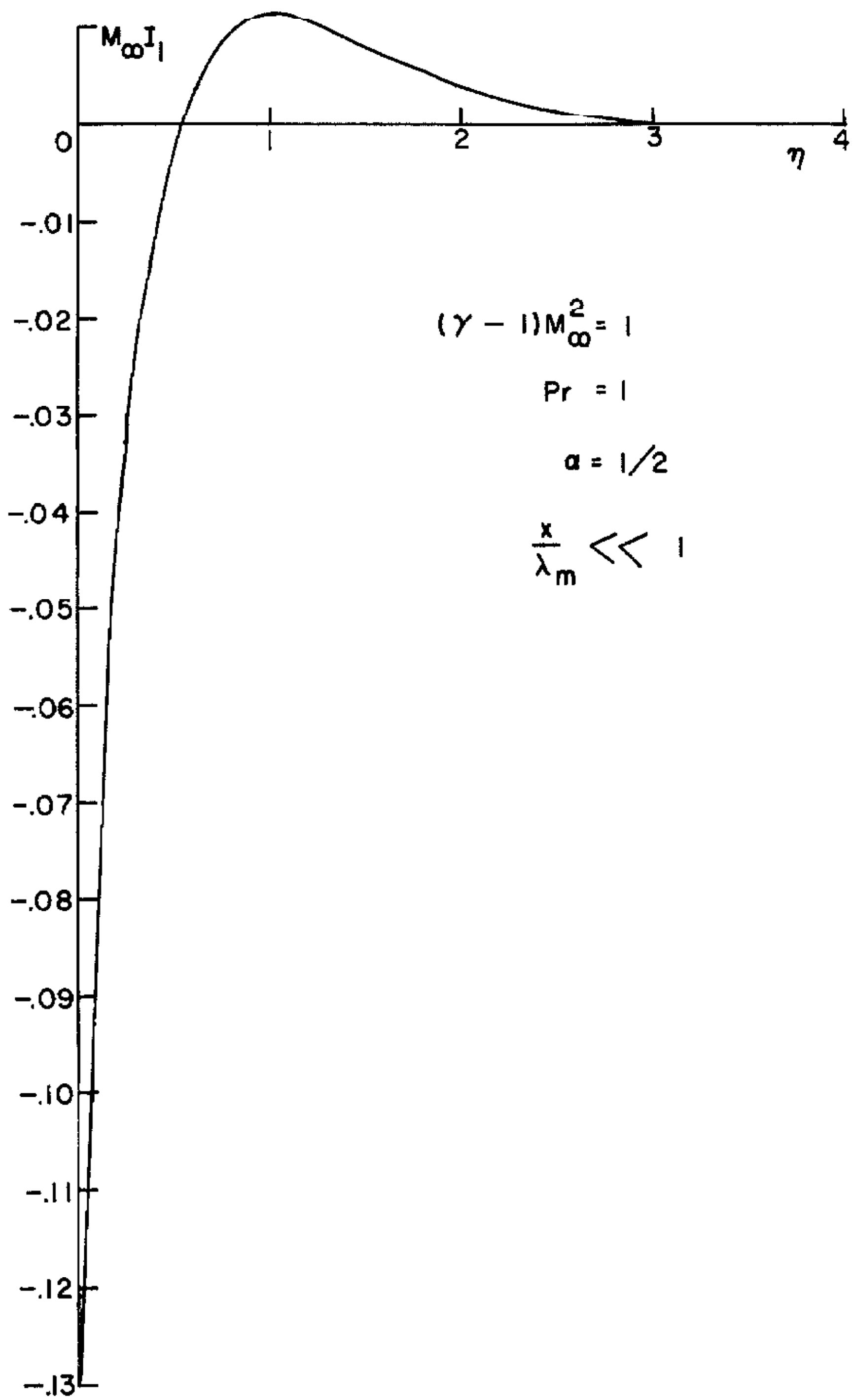


Fig. 30 First-Order Particle Density when Gas Is Compressible.

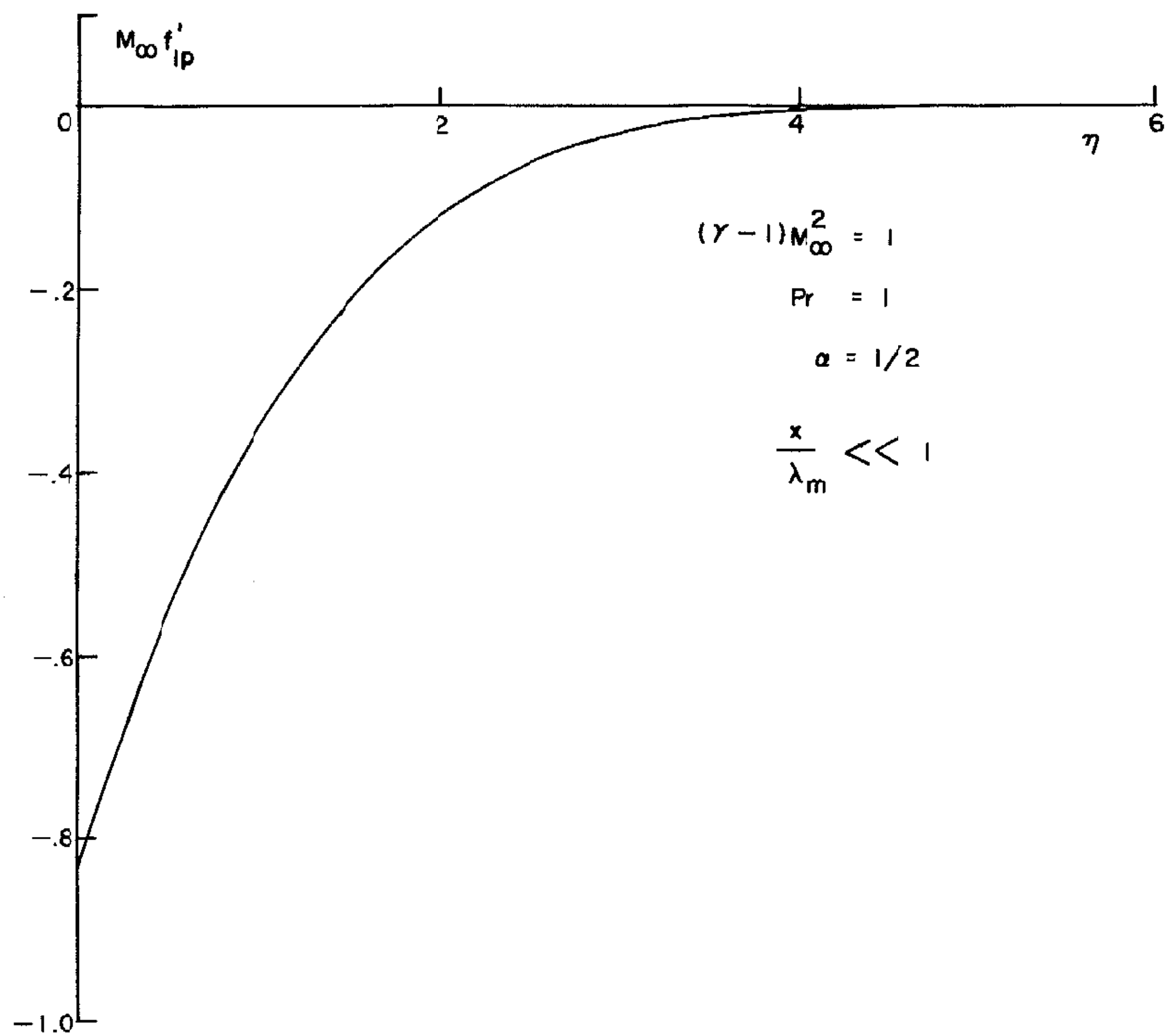


Fig. 31 First-Order Particle Velocity When Gas Is Compressible.

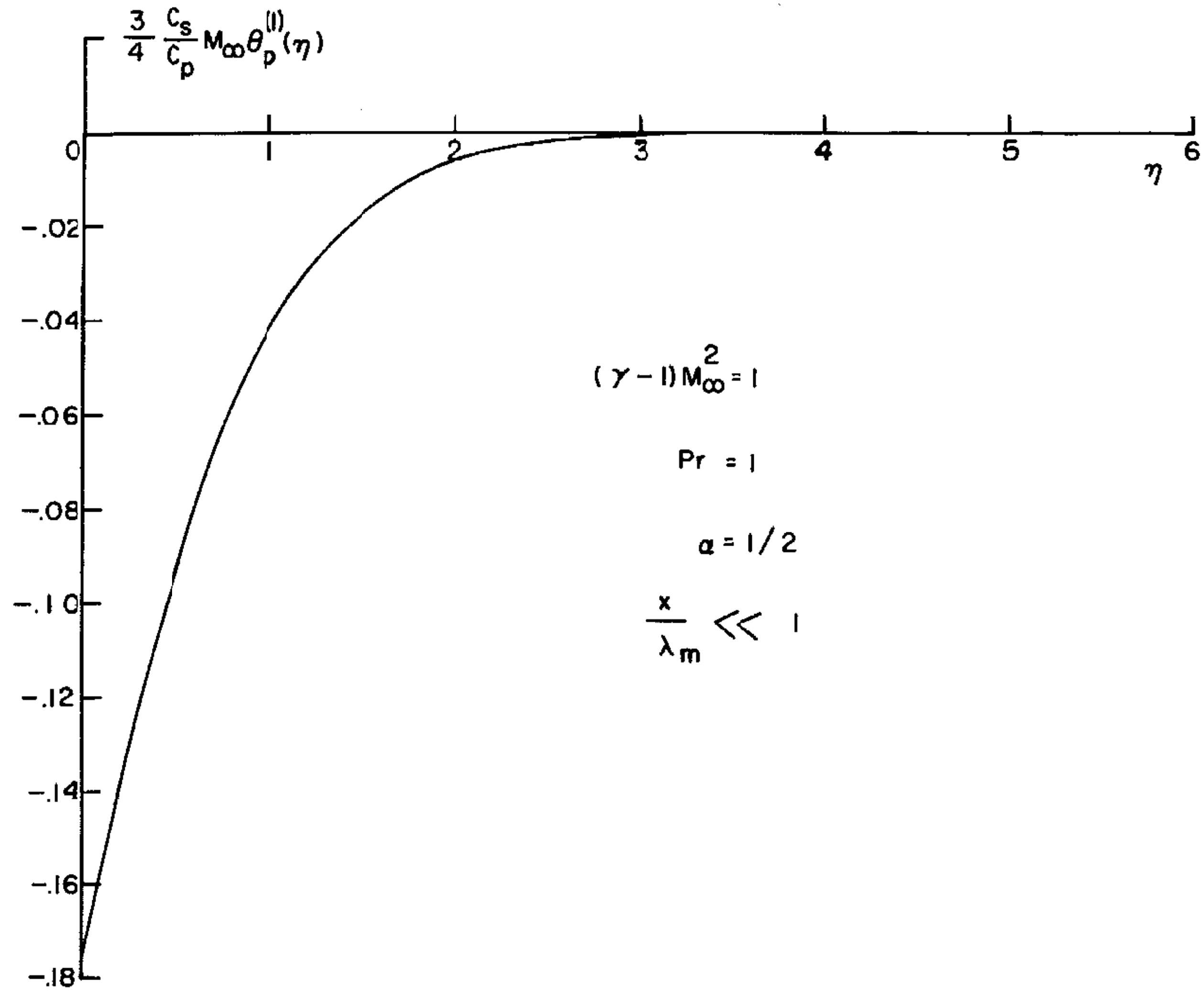


Fig. 32 First-Order Particle Temperature When Gas Is Compressible.

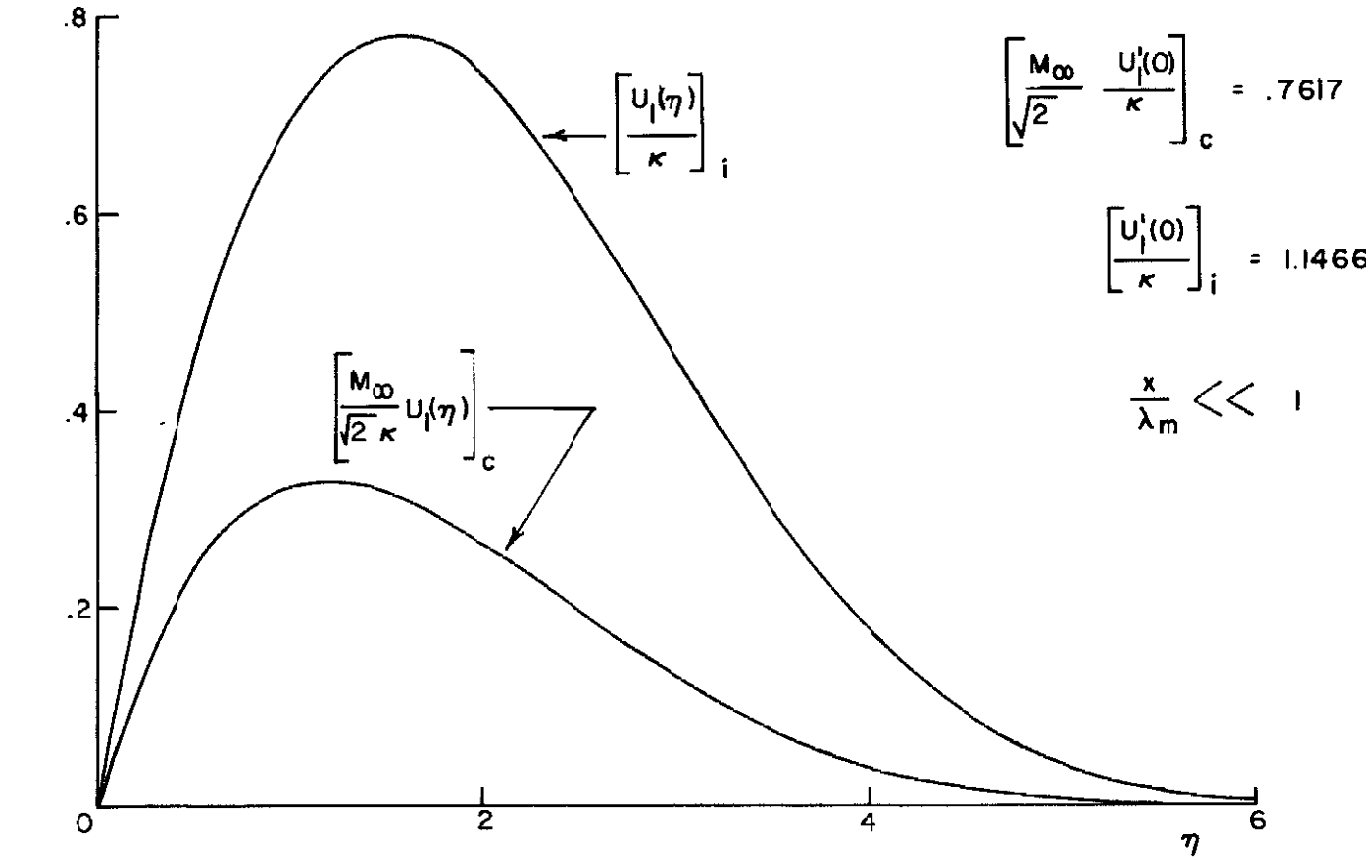


Fig. 33 Comparison of Compressible and Incompressible First-Order Gas Velocity.

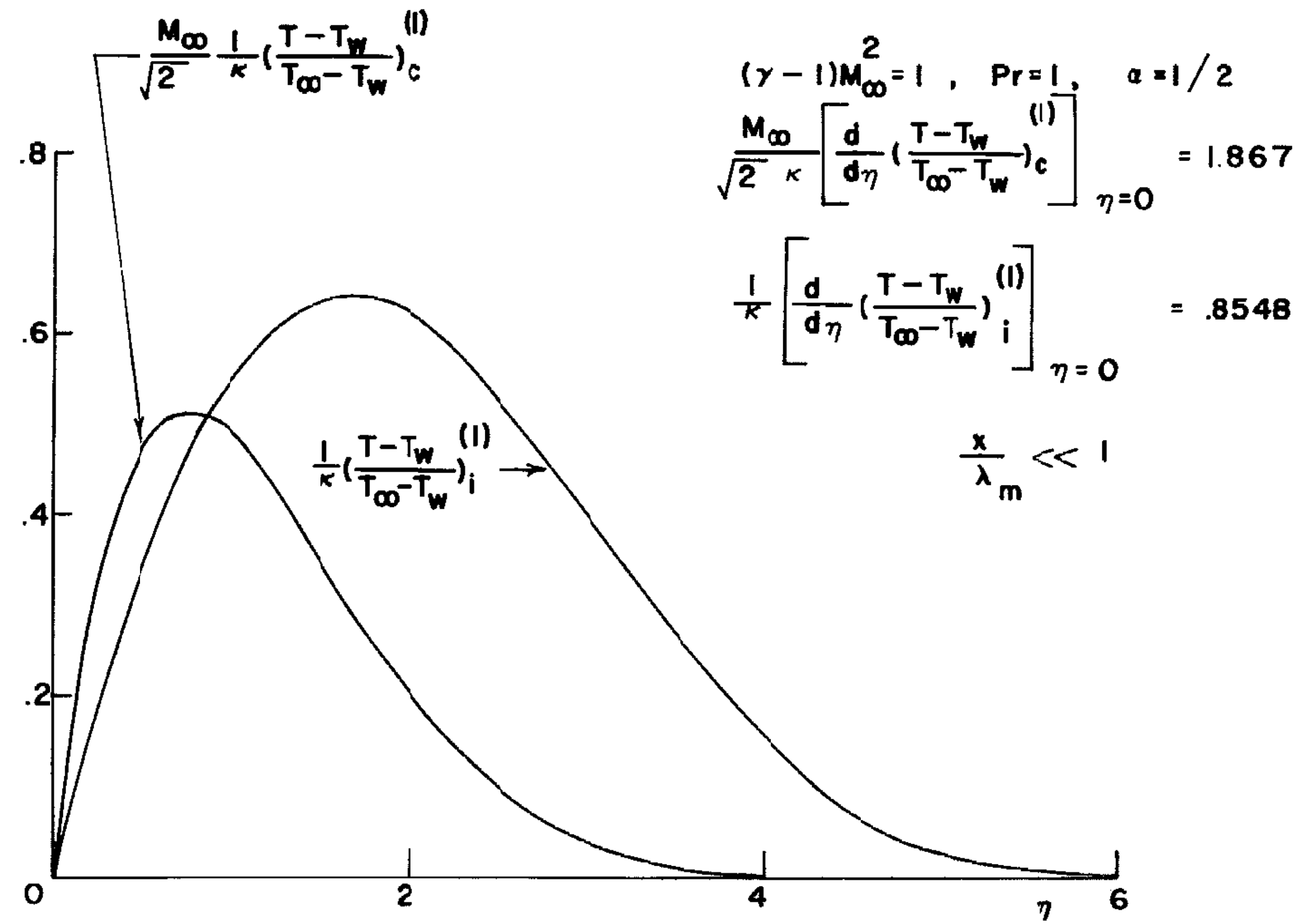


Fig. 34 Comparison of Compressible and Incompressible First-Order Gas Temperature.

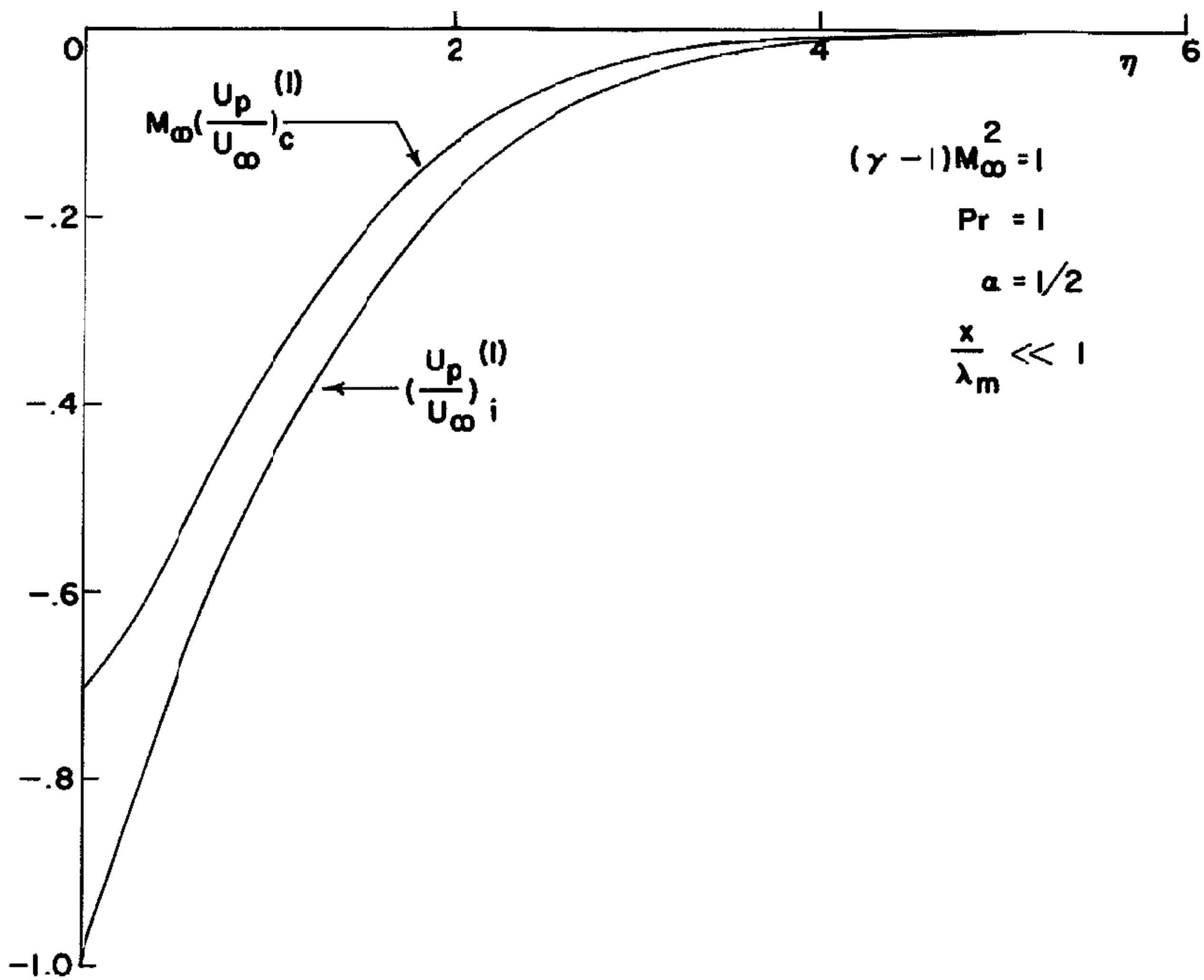


Fig. 35 Comparison of Incompressible and Compressible First-Order Particle Velocities.

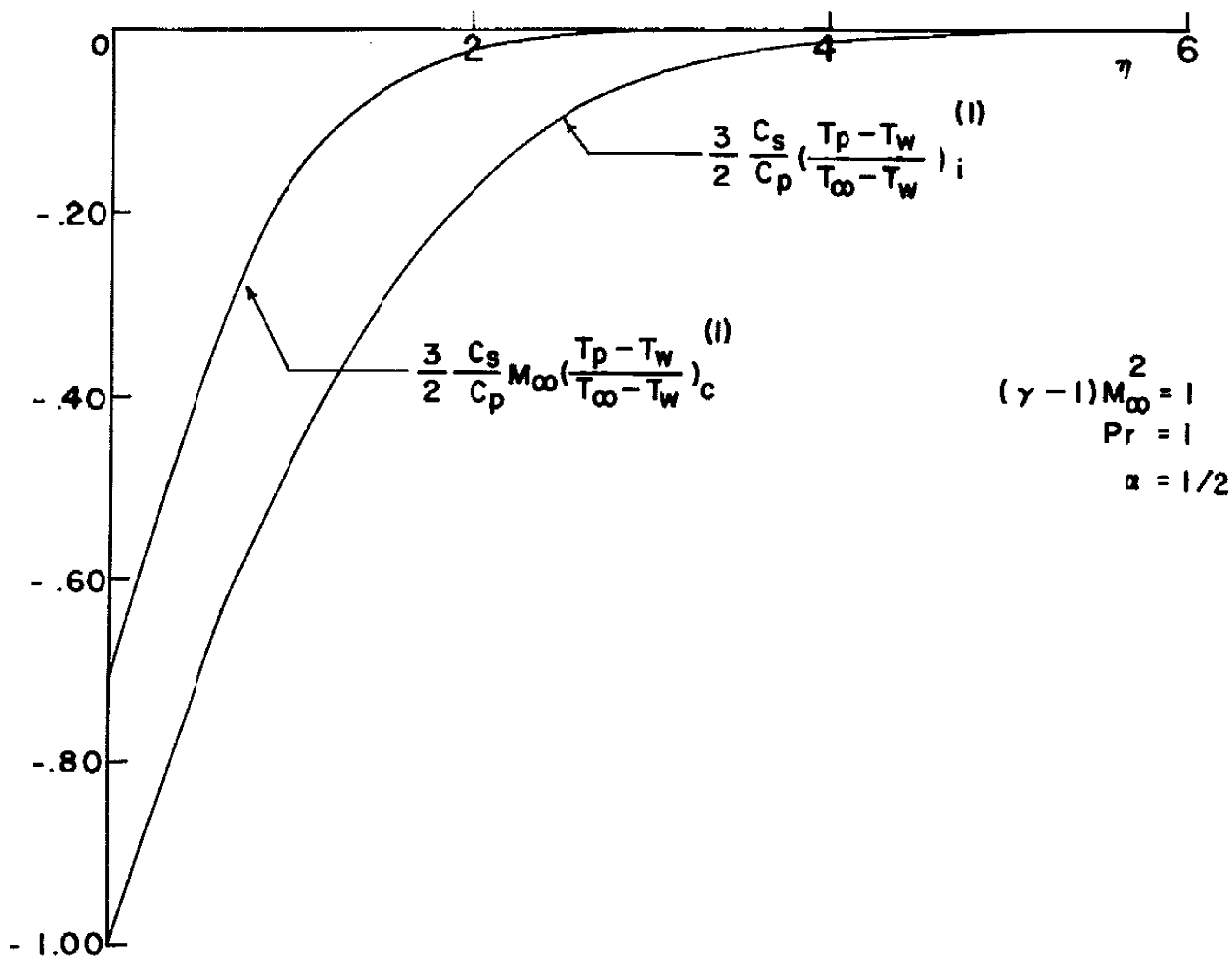


Fig. 36 Comparison of Compressible and Incompressible First-Order Particle Temperature.