NEAR MINIMUM ENERGY TRAJECTORIES IN THE TWO FIXED FORCE-CENTER PROBLEM

Thesis by

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ABSTRACT

The class of symmetric orbits with near minimum energy which originate very close to the earth and pass very close to a fixed moon of small mass are studied using asymptotic methods. An exact solution for the orbit is found using Bonnet's Theorem. This is an ellipse with the force centers as foci. Results obtained from the approximate solution are seen to agree exactly with the predictions of Bonnet's Theorem. The solutions thus obtained are the periodic solutions. A one dimensional study is undertaken as a guide to the planar problem.

TABLE OF CONTENTS

I. INTRODUCTION

The planar motion of a particle of negligible mass in a trajectory originating near a body of relatively large mass (the earth) and passing close to a body of relatively small mass (the moon) was discussed in references [1] and [3]. In [1] the earth and moon were assumed fixed in an inertial frame in order to illustrate the mathematical aspects of the development of the solution by uniformly valid asymptotic approximations. In [2] and [3] the more realistic case of motion within the restricted three-body framework was obtained for the case of elliptic initial orbits relative to the earth. The problem of near minimum energy trajectories in the restricted three-body problem required special treatment in [3] because of the distinguished behaviour of the solution before moon passage.

In this study near minimum energy trajectories are considered for the two fixed force-center problem where again the behaviour of the solution near the moon as predicted by the outer expansion changes type. Certain new aspects arise that were not present in references [1], [2] and [3]. For example Bonnet's theorem (cf. reference 4) guarantees the existence of periodic orbits which are ellipses with the force centers as foci. If in addition the assumption of small initial perigee distances is made as was done in references [2] and [3] it is found that these periodic orbits are near minimum energy elliptic trajectories. Explicit approximate formulas for the motion for this class of trajectories are developed. It is seen that during moon passage the effect of the earth must be retained in order to obtain a solution which matches with the prior motion. This distinctive feature is due to the relatively long time spent by the particle near the moon for the case of near minimum energy.

A study of the motion in one dimension is undertaken for the case of total energy equal to the minimum necessary to ensure escape from the earth. The study is largely academic as the particle is trapped at the equilibrium point after an infinite time and never reaches the moon. However it provides valuable insight and serves as a guide to the solution of the planar problem. The motion between the equilibrium point and the moon is solved as a separate problem.

II. ONE-DIMENSIONAL CASE

In this section motion along the line joining the two centers of attraction is considered with a total energy equal to the minimal value necessary to reach the moon. In [1] it was shown that the minimal value for the total energy, h, defined by:

$$h = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{1-\mu}{x} + \frac{\mu}{1-x} \tag{2.1}$$

is equal to

$$h_{min} = \frac{-(1-2\mu)(\sqrt{\mu}+\sqrt{1-\mu})}{\sqrt{1-\mu}-\sqrt{\mu}} = -1-2\mu^{1/2} + O(\mu^{3/2}) \tag{2.2}$$

Furthermore at the equilibrium point $x = x_s = 1 - \mu^{1/2} + O(\mu)$ the two forces of attraction exactly cancel. This is a saddle-point in the phase-plane of x and $\frac{dx}{dt}$ as can easily be verified from (2.1).

Determination of the trajectory in the form $t = t(x, \mu, h)$ reduces in this case to a quadrature with the use of (2.1). However the solution will be derived by constructing the limiting outer and inner expansions, and then exhibiting the details of the matching of the two solutions.

2.1 The Outer Expansion

An expansion procedure based on a limit process in which x is kept fixed while μ tends to zero will be valid for all x not close to unity (i.e. away from the moon). It is assumed that the outer expansion has the form:

$$t(x,\mu) = t_0(x) + \mu^{1/2} t_{1/2}(x) + \mu\, t_1(x) + O(\mu^{3/2}) \tag{2.3}$$

When the above expansion is used in (2.1) the following differential equations for the various terms in (2.3) are obtained.

$$\sqrt{2}\,\frac{dt_0}{dx} = \sqrt{\frac{x}{1-x}} \tag{2.4a}$$

$$\sqrt{2}\,\frac{dt_{1/2}}{dx} = \left(\frac{x}{1-x}\right)^{3/2} \tag{2.4b}$$

$$\sqrt{2}\,\frac{dt_1}{dx} = -\sqrt{\frac{x}{1-x}}\left\{\frac{2x-1}{2(x-1)^2} - \frac{3x^2}{2(1-x)^2}\right\} \tag{2.4c}$$

From [1], the differential equation governing motion written with x as independent variable, is

$$\frac{1}{(dt/dx)^3}\,\frac{d^2t}{dx^2} - \frac{(1-\mu)}{x^2} + \frac{\mu}{(1-x)^2} = 0 \tag{2.5}$$

It is noted that equations (2.4) are just the first integrals of (2.5) after (2.2) and (2.3) have been substituted. The existence of these first integrals is guaranteed by the existence of the exact integral (2.1). Conversely, the successive approximation scheme implied by (2.3) would still hold in the absence of such an integral.

The solution of equations (2.4) subject to the initial conditions $t(o)=o$ gives

$$\sqrt{2}\,t_0(x) = \sin^{-1}\sqrt{x} - \sqrt{x(1-x)} \tag{2.6a}$$

$$\sqrt{2}\,t_{1/2}(x) = \frac{-x^{3/2}}{\sqrt{1-x}} + 3\sqrt{\frac{x}{1-x}} - 3\sin^{-1}\sqrt{x} \tag{2.6b}$$

$$\sqrt{2}\,t_1(x) = -\frac{11}{2}\left\{\sqrt{\frac{x}{1-x}} - \sin^{-1}\sqrt{x}\right\} + \left(\frac{x}{1-x}\right)^{3/2}\left\{\frac{13}{6} - \frac{3x}{2}\right\} \tag{2.6c}$$

It is noted that $t_{1/2}(x)$ has a square root singularity and that $t_1(x)$ has in addition a three-halves singularity at $x = 1$, a reflection of the fact that the above outer expansion is not valid near the moon.

2.2 The Inner Expansion

In order to study the motion in the neighbourhood of the equilibrium point and the moon, the coordinates are translated to the moon location and are enlarged. The inner variables are then defined in the form:

$$\bar{x} = \frac{1-x}{\mu^{\alpha}} \qquad \bar{t} = \frac{t-\tau}{\mu^{\beta}}$$

where α and β are positive quantities and τ represents the time to reach an appropriate point near lunar encounter. Since for the present case the motion approaches a saddle-point and is dominated by both centers of attraction in that phase one must choose $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$ in order to exhibit this balance of forces. This domain is larger than the one chosen in [1] where the class of trajectories considered had a non-zero velocity throughout the interval. The differential equation of motion in the above limit then reduces to:

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \frac{1}{\bar{x}^2} - 1 = 0 \tag{2.7}$$

where the effect of the earth is exhibited to this order by the unit attraction. Furthermore, the domain of validity of equation (2.7) includes both the moon and the equilibrium point which is located at $\bar{x} = 1$.

Equation (2.7) has a first integral

$$\sqrt{2}\ \frac{d\bar{t}}{d\bar{x}} = -\sqrt{\frac{\bar{x}}{(1-\bar{x})^2}} \tag{2.8}$$

which can also be obtained by evaluating the inner limit of the energy integral (2.1). The singularity at the saddle-point $\bar{x} = 1$ is exhibited explicitly in (2.8) and the integration must be performed in two steps.

Let τ be the time taken to reach an arbitrarily chosen point $\bar{x} = 1 + \epsilon$, near the singular point. Then a simple integration yields

$$\sqrt{2}\ \bar{t}(\bar{x}) = -2\bar{x}^{1/2} + \log \frac{\bar{x}^{1/2}+1}{\bar{x}^{1/2}-1} + K \tag{2.9a}$$

where $K = -\bar{t}(o) = 2\sqrt{1+\epsilon} - \log \frac{4+\epsilon}{\epsilon}$ for $\epsilon << 1$ (2.9b)

The motion between the equilibrium point and the moon is in the present case a completely separate problem since it takes an infinite time to reach or leave the saddle-point. If for this phase the origin of time is taken to be the moon location, $\bar{x} = 0$, integration of (2.8) yields

$$\sqrt{2}\ \bar{t}(\bar{x}) = 2\bar{x}^{1/2} - \log \frac{1+\bar{x}^{1/2}}{1-\bar{x}^{1/2}}, \quad 0 \le \bar{x} \le 1 \tag{2.10}$$

2.3 Matching of the Inner and Outer Expansions

As in [1], it is sufficient for matching to require the outer solution evaluated in the inner region to agree with the inner solution for large values of the inner variable. The inner limit of the outer expansion (2.6) is easily determined:

$$\sqrt{2}\ t(\bar{x},\mu) = \frac{\Pi}{2} + \mu^{1/4}\left\{-2\bar{x}^{1/2} + 2\bar{x}^{-1/2} + \frac{2}{3}\bar{x}^{-3/2}\right\} - \frac{3\Pi}{2}\mu^{1/2} + O(\mu^{3/4}) \tag{2.11}$$

The above should match with the limit as $\bar{x} \to \infty$ of (2.8), the appropriate overlapping branch of the inner solution. Taking this limit gives:

$$\sqrt{2}\, t(\bar{x},\mu) = \sqrt{2}\,\tau + \mu^{1/4}\left\{-2\bar{x}^{1/2} + 2\bar{x}^{-1/2} + \frac{2}{3}\bar{x}^{-3/2} + K\right\} \tag{2.12}$$

Comparing equations (2.11) and (2.12) it is clear that the inner and outer expansions match to order $\mu^{1/2}$ provided

$$\sqrt{2}\,\tau = \frac{\pi}{2} - K\mu^{1/4} - \frac{3\pi}{2}\mu^{1/2} \tag{2.13}$$

Thus the composite solution which represents the motion uniformly to order $\mu^{1/2}$ in the interval $0 \le x \le 1 - \mu^{1/2} + O(\mu)$ is

$$\begin{aligned}\sqrt{2}\, t(x,\mu) = {} & \sin^{-1}\sqrt{x} - \sqrt{x(1-x)} + \mu^{1/2}\left\{\frac{-x^{3/2}}{\sqrt{1-x}} + 3\sqrt{\frac{x}{1-x}} - 3\sin^{-1}\sqrt{x}\right\} \\ & + \mu\left\{\frac{-11}{2}\left[\sqrt{\frac{x}{1-x}} - \sin^{-1}\sqrt{x}\right] + \left(\frac{x}{1-x}\right)^{3/2}\left[\frac{13}{6} - \frac{3x}{2}\right]\right\} \\ & + \mu^{1/4}\left\{\log\frac{1+\bar{x}^{1/2}}{1-\bar{x}^{1/2}} - 2\bar{x}^{-1/2} - \frac{2}{3}\bar{x}^{-3/2}\right\}\end{aligned} \tag{2.14}$$

where the last term is a "boundary-layer correction."

It is observed that the square-root and three-halves singularities of the outer expansion are cancelled by the boundary-layer correction, which in turn introduces the real logarithmic singularity at $\bar{x} = 1$. The singularity of order $\mu(1-x)^{-1/2}$ in $\mu\, t_1(x)$ which is uncancelled will match with a comparable higher order term if the inner solution is carried out to the next order. This is the reason the above expansion is only uniform to order $\mu^{1/2}$.

III. TWO-DIMENSIONAL CASE

In this section planar motion between two fixed centers (the earth with mass $(1-\mu)$ located at the origin of coordinates and the moon with mass μ located at the point (1, 0) is studied. The particular class of trajectories considered are initially symmetric with respect to the earth-moon axis and originate from a neighbourhood of order μ of the earth (i.e., have a perigee distance of order μ).

In [1] the two integrals of motion were exhibited, and as a consequence the motion is in principle soluble by quadrature. However, the explicit representation of such an exact solution is most cumbersome and asymptotic approximate formulas are both useful and interesting.

In [4] a detailed discussion of Bonnet's theorem for the general n-body case is given. For the present study a particular choice of initial conditions ensures the applicability of this theorem which states: "If the orbits due to n centers of attraction considered one at a time are identical, then the motion under the collective influence of these n centers will proceed along this orbit with a local speed equal to the square root of the sum of the squares of the individual speeds."

Clearly, for two fixed Newtonian centers of gravitation any ellipse with foci at the centers satisfies the requirements of an exact orbit as long as the perigee speed, say, is precisely equal to the value appropriate to the given perigee distance. In addition to the necessary symmetry condition for the applicability of this powerful result it will be shown in Section V that for perigee distances of order μ the initial velocity requirement is equivalent to a choice of total energy which is minimal to order unity. Use of this theorem allows exact calculation of the energy and angular momentum for the motion during moon passage and provides a useful check for the asymptotic results.

3.1 Formulation of the Problem

The notation and point of view of references [2] and [3], where the restricted three-body formulation was considered, will be adopted. The equations of motion in the Cartesian earth-centered coordinates are (cf. Figure I for the geometry):

$$\frac{d^2x}{dt^2} + (1-\mu)\,\frac{x}{r^3} = \frac{(1-x)\mu}{r_m^3} \tag{3.1a}$$

$$\frac{d^2y}{dt^2} + (1-\mu)\,\frac{y}{r^3} = \frac{-\mu\, y}{r_m^3} \tag{3.1b}$$

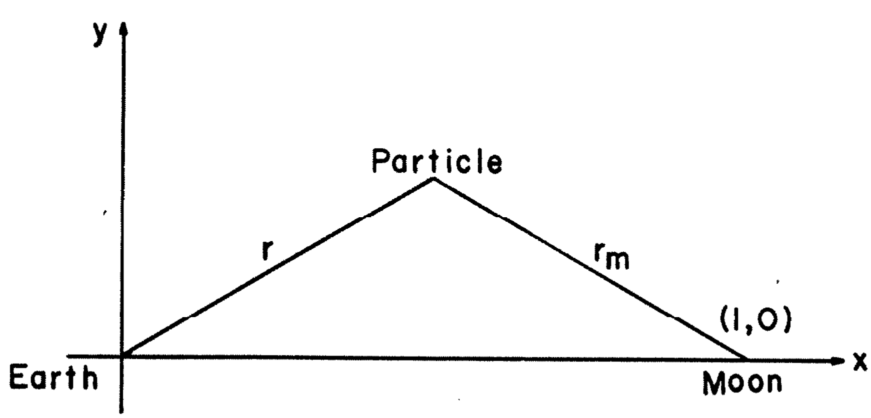

Fig. I The Earth-Centered Cartesian Coordinate System

As in [2], the initial conditions are prescribed by specifying the Keplerian integrals at x = 0.

$$h_e = \frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} - \frac{1-\mu}{r} = -1 + \mu\rho_1$$

$$l_e = x\frac{dy}{dt} - y\frac{dx}{dt} \qquad = \mu^{1/2}\lambda$$

$$p_e = l_e\frac{dx}{dt} + (1-\mu)\frac{y}{r} \qquad = 0$$

$$q_e = -l_e\frac{dy}{dt} + (1-\mu)\frac{x}{r} \qquad > 0$$

$$t = 0$$

With $p_e = 0$, the Keplerian starting trajectory is an ellipse whose major axis lies along the x-axis. The condition $l_e = \text{order } \mu^{1/2}$ implies that perigee distance is order μ while $h_e = -1 + O(\mu)$ implies minimal energy to reach the moon.

The Keplerian solutions obtained by setting the right hand side of equations (3.1) equal to zero are:

$$y(x,\mu) = -a\sqrt{1-e^2}\left\{1-\left(e-\frac{x}{a}\right)^2\right\}^{1/2} \tag{3.2a}$$

$$t(x,\mu) = a^{3/2}\left\{\sin^{-1}\left[1-\left(e-\frac{x}{a}\right)^2\right]^{1/2} - e\left[1-\left(e-\frac{x}{a}\right)^2\right]^{1/2}\right\} \tag{3.2b}$$

where a = semi-major axis = $\dfrac{1-\mu(1-\rho_1)}{2}$

e = eccentricity = $1 - \dfrac{2\mu\lambda^2(1-\mu\rho_1)}{(1-\mu)^2}$

3.2 The Outer Expansion

In equations (3.1) the right hand sides are clearly perturbations of order μ so long as r_m is not small. Consequently if these terms are zero the solutions for t(x) and y(x) are Keplerian relative to earth, as has been shown. Clearly then, an expansion procedure based on a limit process in which x is kept fixed while μ tends to zero will be valid for all r_m bounded away from zero. Considering y(x) only, equation (3.1) suggests that the leading term is of the form $\mu^{1/2} y_{1/2}(x,\mu)$. This form of a leading term which depends both on μ and x is chosen to avoid the trivial non-uniformities that would otherwise occur near the earth. These non-uniformities are exhibited by expanding (3.1) in the form

$$y(x,\mu) = -\mu^{1/2} \lambda \sqrt{2x(1-x)} \left\{ 1 + O(\mu) \right\} \tag{3.2c}$$

Thus, y is of order $\mu^{1/2}$ for $x, (1-x) = O(1)$ and of order μ for $x, (1-x) = O(\mu)$. Then for x not small and r_m bounded away from zero an expansion of the form of (3.2c) is uniformly valid. Inspection of (3.2b) shows that exactly the same considerations apply to $t(x,\mu)$. Hence the outer expansion for both y(x) and t(x) takes the form

$$y(x,\mu) = \mu^{1/2} y_{1/2}(x,\mu) + \mu y_1(x) + O(\mu^{3/2}) \tag{3.3a}$$

$$t(x,\mu) = t_0(x,\mu) + \mu t_1(x) + O(\mu^2) \tag{3.3b}$$

The terms $y_1(x)$ and $t_1(x)$ are independent of μ and are first corrections to $\mu^{1/2} y_{1/2}$ and t_0 respectively. These corrections are necessary for the matching processes which follow later. Furthermore, equations (3.3) are not valid near the moon which necessitates a different form of expansion in this region.

Away from the earth (3.3b) may be written:

$$t(x,\mu) = t_{0o}(x) + \mu\left[t_{01}(x) + t_1(x)\right] + O(\mu^2) \tag{3.4}$$

where $t_0(x,\mu) = t_{00}(x) + \mu t_{01}(x)$ and is given by (3.2b)

Direct expansion of equation (3.2b) gives:

$$\sqrt{2}\, t_{00}(x) = \sin^{-1}\sqrt{x} - \sqrt{x(1-x)} \tag{3.5}$$

$$\sqrt{2}\, t_{01}(x) = \rho_1\left\{\frac{3}{2}\sin^{-1}\sqrt{x} - \frac{\sqrt{x}(3-x)}{2\sqrt{1-x}}\right\} - \sin^{-1}\sqrt{x} + \sqrt{x(1-x)} + \frac{2x^2+\lambda^2 x(3-2x)}{2\sqrt{x(1-x)}} \tag{3.6}$$

From equation (3.5), it is seen that $\sqrt{2}\, t'_{00}(x) = \sqrt{\frac{x}{1-x}}$ (3.7)

As in [1] the equations of motion (3.1) are written with x as independent variable

$$\frac{-t''}{t'^3} + (1-\mu)\frac{x}{r^3} = \frac{\mu(1-x)}{r_m{}^3} \tag{3.8a}$$

$$\frac{y''}{t'^2} - \frac{t''y'}{t'^3} + (1-\mu)\frac{y}{r^3} = \frac{-\mu y}{r_m^3} \tag{3.8b}$$

Also from [1] an equivalent form of the above which exhibits the role of h_e and l_e are

$$h'_e = \frac{dh_e}{dx} \equiv \frac{d}{dx}\left\{\frac{1+y'^2}{2t'^2} - \frac{(1-\mu)}{r}\right\} = \frac{\mu}{r_m^3}\left\{1-x-yy'\right\} \tag{3.9a}$$

$$l'_e = \frac{dl_e}{dx} \equiv \frac{d}{dx}\left\{\frac{xy'-y}{t'}\right\} = \frac{-\mu t' y}{r_m^3} \tag{3.9b}$$

As in [2] and [3] insertion of the expansions (3.3) in the above, integrating from x = 0 and using the fact that t_0 and $\mu^{1/2} y_{1/2}$ give the exact initial conditions on h_e and l_e at x = 0, the perturbations t_1 and y_1 are obtained: *

$$\sqrt{2}\, t_1(x) = -\sin^{-1}\sqrt{x} + \frac{3-4x}{3}\,\frac{\sqrt{x}}{(1-x)^{3/2}} \tag{3.10a}$$

$$y_1(x) \equiv 0 \tag{3.10b}$$

The outer expansions for y(x) and t(x) away from the earth are thus:

$$y(x,\mu) = -\mu^{1/2}\lambda\sqrt{2x(1-x)}\left\{1+\mu\left[\rho_1 + \frac{1-2x}{4x(1-x)}\left\{\lambda^2 + 2x(1-\rho_1)\right\}\right]\right\} \tag{3.11a}$$

$$\sqrt{2}\, t(x,\mu) = \sin^{-1}\sqrt{x} - \sqrt{x(1-x)} + \mu\left\{\rho_1\left[\frac{3}{2}\sin^{-1}\sqrt{x} - \frac{\sqrt{x}\,(3-x)}{2\sqrt{1-x}}\right]\right.$$

$$\left. -2\sin^{-1}\sqrt{x} + 2\sqrt{\frac{x}{1-x}} - \frac{1}{3}\left(\frac{x}{1-x}\right)^{3/2} + \frac{\lambda^2\, x(3-2x)}{2\sqrt{x(1-x)}}\right\} \tag{3.11b}$$

In preparation for matching the following substitution is made:

$$w = 1 - x \tag{3.12}$$

Then

$$y(w,\mu) = -\mu^{1/2}\lambda\sqrt{2w(1-w)}\left\{1+\mu\left[\rho_1 + \frac{2w-1}{4w(1-w)}\left\{\lambda^2 + 2(1-\rho_1)(1-w)\right\}\right]\right\} \tag{3.13a}$$

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* This result also follows directly from eq. (5.2b) of [1] in the limit $\rho \to 1$. It is interesting to note that the logarithmic singularity at x = 1 disappears and is replaced by the stronger three-halves singularity exhibited.

$$\sqrt{2}\; t(w,\mu) = \sin^{-1}\sqrt{1-w} - \sqrt{w(1-w)} + \mu\Big\{\rho_1\Big[\frac{3}{2}\sin^{-1}\sqrt{1-w} - \frac{\sqrt{1-w}\,(2+w)}{2\sqrt{w}}\Big] - 2\sin^{-1}\sqrt{1-w} + \frac{2\sqrt{1-w}}{\sqrt{w}} - \frac{1}{3}\Big(\frac{1-w}{w}\Big)^{3/2} + \frac{\lambda^2(1-w)(2w+1)}{2\sqrt{w(1-w)}}\Big\} \tag{3.13b}$$

3.3 The Inner Expansion

The essential difference between the present study and the cases considered in [1],[2],[3] is the fact that with near minimum energy in the fixed center problem, the velocity of the particle as predicted by the leading term of the outer expansion is vanishingly small near the position of the moon, equation (3.7). As the one-dimensional study showed, matching such a trajectory with a Keplerian orbit relative to the moon is not possible because prior to the onset of lunar dominance, the particle spends a relatively long period in a region where both forces of attraction are important. However, with $y \neq 0$ and $h_e > h_{min}$ the particle is not "trapped" at the saddle-point, and will eventually encounter a strictly lunar phase after a finite time. In anticipation of this, two inner regions will exist, one defined as before by the relations

$$\bar{x} = \frac{x-1}{\mu^{1/2}} \qquad \bar{y} = \frac{y}{\mu^{1/2}} \qquad \bar{t} = \frac{t-\tau}{\mu^{1/4}} \tag{3.14}$$

where. τ is the time taken to reach the moon starting from the earth, and an innermost region where the motion is Keplerian relative to the moon.

Actually the region defined by (3.14) includes the Keplerian innermost region. However, unlike the simple one-dimensional case it will be difficult to express the time as a function of x throughout the entire inner region. It will be shown that if a neighbourhood of order μ of the moon is excluded (where the trajectory has a sharp curvature) the effect of two-dimensionality is negligible as far as the computation of t is concerned, and that the resulting expression for t matches with the easily computed Keplerian solution very close to the moon as well as the outer representation. The form of the orbit will be parabolic in the entire region, and can be immediately computed. The limiting differential equations are easily determined and show that the effect of the earth appears to first order as a uniform gravitational field. These are shown below:

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \frac{\bar{x}}{\bar{r}^3} + 1 = 0 \tag{3.15a}$$

$$\frac{d^2\bar{y}}{d\bar{t}^2} + \frac{\bar{y}}{\bar{r}^3} = 0 \tag{3.15b}$$

Two integrals of motion for the above can be computed either by taking the appropriate limits of the exact integrals h and δ of [1] or by direct construction from the above. These integrals are:

$$\bar{h} = \frac{1 + \bar{y}'^2}{2\,\bar{t}'^2} - \frac{1}{\bar{r}} + \bar{x} \tag{3.16}$$

$$\bar{\delta} = \frac{\bar{y}'(\bar{y}'\bar{x} - \bar{y})}{\bar{t}'^2} - 1 - \frac{\bar{y}^2}{2} - \frac{\bar{x}}{\bar{r}} \tag{3.17}$$

The inner expansion with the use of these integrals can be found in principle by quadrature, and the matching would then define $\bar{h}$ and $\bar{\delta}$ and the two added constants that are introduced by integrating for $\bar{t}$ and $\bar{y}$ as functions of $\bar{x}$.

Knowledge of the exact solution for the orbit provides considerable simplification. As will be shown in Section 5 the elliptic orbit given by the outer expansion is an exact solution for y everywhere as long as the speed is suitably modified. This being the case the ellipse in the neighbourhood of the moon would appear as a parabola in the present $\bar{x}, \bar{y}$ variables. In fact, this parabola at its minimum approach to the moon is at a distance of order $\mu^{1/2}$ in the present inner variables. With this observation it is easy to show that

$$\bar{y} = \mu^{1/4} \bar{\lambda} \left\{ \mu^{1/2} \bar{\lambda}^2 - 2\bar{x} \right\}^{1/2}, \quad \bar{\lambda} = O(1) \tag{3.18}$$

which is the general formula for the class of parabolas that have angular momentum of order $\mu^{1/4}$ and minimum approach distances of order $\mu^{1/2}$, satisfies equation (3.15), and (3.17) to order unity. It is easy to show that for this class of trajectories the integrals of motion $\bar{h}$ and $\bar{\delta}$ are both equal to zero to first order. From [1], the following are quoted:

$$h = \frac{1}{2} \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right) - \frac{1-\mu}{r} - \frac{\mu}{r_m} \tag{3.19}$$

$$\delta = l_e l_m + (1-\mu) \frac{x}{r} - \frac{\mu(x-1)}{r_m} \tag{3.20}$$

From the initial conditions, $h = -1 + O(\mu^{1/2})$, and the right hand side of (3.19) evaluated in inner variables is equal to $-1+O(\mu^{1/2})$. Similarly with $\bar{\delta}$ the right hand side is $1+O(\mu)$ while from initial data $\bar{\delta}$ is computed as unity. It is also easy to show that (3.18) holds when $\bar{x}$ is of order unity and of order $\mu^{1/2}$ i.e., in the innermost region.

When $\bar{x}$ is of order unity, $\bar{y}$ is of order $\mu^{1/4}$, from (3.18). This provides considerable simplification in evaluating $\bar{t}$ since in this case (3.16) reduces to

$$\sqrt{2}\ \frac{d\bar{t}}{d\bar{w}} = -\sqrt{\frac{\bar{w}}{1+\bar{w}^2}} \qquad \text{with } \bar{w} = -\bar{x} \tag{3.21}$$

where the negative sign is used for lunar approach.

Equation (3.21) is valid for all $\bar{w} > 0$. The lower limit of integration ($\bar{t} = 0$) is chosen as $\bar{w} = \bar{w}_1$, slightly greater than zero. This constant is evaluated in the limit $\bar{w}_1 \to 0$. Integrating directly gives:

$$\sqrt{2}\ \bar{t} = F(\phi,k) - 2E(\phi,k) - \frac{2\bar{w}^{1/2}\sqrt{1+\bar{w}^2}}{1+\bar{w}^2} - K_1(\bar{w}_1) \tag{3.22a}$$

$$\text{with } K_1(\bar{w}_1) = F(\phi_1,k) - 2E(\phi_1,k) - \frac{2\bar{w}_1^{1/2}\sqrt{1+\bar{w}_1^2}}{1+\bar{w}_1} \tag{3.22b}$$

where F and E are elliptic functions of the first and second kind respectively, and ϕ and k are the amplitude and modulus respectively.

Then limit of $K_1(\bar{w}_1)$ as $\bar{w}_1$ approaches zero = -1.692.

The equations describing the motion in the inner region are:

$$\sqrt{2}\,\bar{t} = F(\phi, k) - 2E(\phi, k) - \frac{2\,\bar{w}^{1/2}\sqrt{1+\bar{w}^2}}{1+\bar{w}} + 1.692 \qquad (3.23a)$$

$$\bar{y} = \mu^{1/4}\,\bar{\lambda}\left\{\mu^{1/2}\bar{\lambda}^2 + 2\bar{w}\right\}^{1/2} \qquad (3.23b)$$

3.4 The Innermost Expansion

As was stated in Section 3.3, the solution for the time-history in the inner region was determined by excluding a region of order μ in outer variables around the moon. This implies that equation (3.21) is valid so long as $\bar{x}$ (or $-\bar{w}$) is of order unity and less than zero. When $\bar{x}$ is positive, it becomes of order $\mu^{1/2}$, $\bar{y}$ of order $\mu^{1/2}$, as can be seen from equation (3.18). Clearly then (3.21) cannot be used for $\bar{x} > 0$. Innermost variables are accordingly defined:

$$\tilde{x} = \frac{\bar{x}}{\mu^{1/2}} \qquad \tilde{t} = \frac{\bar{t} - \beta}{\mu^{3/4}} \qquad (3.24)$$

where the choice of the powers of μ are clearly consistent with the above and the anticipation that the motion in this region is strictly Keplerian relative to the moon. β is defined as the time taken for the particle to travel from the moon to moon perigee and will be evaluated as a result of matching. The innermost limit of equation (3.16) then becomes:

$$\frac{d\tilde{t}}{d\tilde{x}} = \frac{\bar{\lambda}^2 - \tilde{x}}{\sqrt{\bar{\lambda}^2 - 2\tilde{x}}} + O(\mu) \qquad (3.25)$$

The above equation represents Keplerian motion relative to the moon as can be seen by expressing equation (3.15a) in innermost variables.

This gives:

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} + \frac{\tilde{x}}{\tilde{r}^3} = 0 \tag{3.26}$$

with $\tilde{r} = (\bar{\lambda}^2 - \tilde{x})$. Direct substitution of (3.25) satisfies (3.26) identically. Integration of equation (3.25) from $\tilde{x} = \bar{\lambda}^2/2$ gives, since the integrand is improper at the lower limit:

$$\tilde{t} = \lim_{\epsilon \to 0} \int_{\bar{\lambda}^2/2 - \epsilon}^{\tilde{x}} \frac{\bar{\lambda}^2 - \tilde{x}}{\sqrt{\bar{\lambda}^2 - 2\tilde{x}}} \, d\tilde{x} \tag{3.27}$$

This is evaluated as:

$$\tilde{t}(\tilde{x}) = \frac{-\sqrt{\bar{\lambda}^2 - 2\tilde{x}}}{3} \left\{ 3 - \bar{\lambda}^2 - \tilde{x} \right\} \tag{3.28}$$

or substituting $\tilde{w} = -\tilde{x}$:

$$\tilde{t}(\tilde{w}) = \frac{-\sqrt{\bar{\lambda}^2 + 2\tilde{w}}}{3} \left\{ 3 - \bar{\lambda}^2 + \tilde{w} \right\} \tag{3.29}$$

IV. SOLUTION MATCHING

4.1 Innermost - Inner Matching

The simplified matching criterion discussed earlier holds throughout and will be used in the subsequent computation. The innermost limit is easily determined from (3.23a) by substituting $\bar{w} = \mu^{1/2} \tilde{w}$ and taking the limit $\tilde{w}$ approaching zero. This gives:

$$\bar{t}(\tilde{w}) = -\frac{\sqrt{2}}{3} \mu^{3/4} \tilde{w}^{3/2} + O(\mu^{5/4}) + 1.196 \tag{4.1}$$

The above should match with the limit of (3.29) as $\tilde{w}$ approaches infinity. This gives

$$\bar{t}(\tilde{w}) = \beta + \mu^{3/4} \left\{ -\frac{\sqrt{2}}{3} \tilde{w}^{3/2} - \frac{4-\bar{\lambda}^2}{2\sqrt{2}} \tilde{w}^{1/2} + O(\tilde{w}^{-1/2}) \right\} \tag{4.2}$$

The singularities at the location of the moon are evident. Comparing (4.1) and (4.2) it is seen that the leading singularities (of order $\tilde{w}^{3/2}$) match identically. The singularity of order $\tilde{w}^{1/2}$ remains uncancelled. Since the inner parabolic orbit satisfies the differential equation to order unity only and the inner time-history was determined by neglecting the contribution of $\bar{y}'^2$ it is not possible to recover a term of order $\tilde{w}^{1/2}$ in the limit as $\tilde{w}$ approaches zero unless higher order corrections are determined for $\bar{y}(x)$ and $\bar{t}(x)$. Under these conditions the solutions match to order $\mu^{1/2}$ provided $\beta = 1.196$.

4.2 Inner-Outer Matching

The inner limit of the outer expansion, equation (3.13a), is determined by the substitution $w = \mu^{1/2} \bar{w}$ and taking the limit as $\bar{w}$

approaches zero. This gives:

$$y(\bar{w},\mu) = -\mu^{3/4}\lambda\left\{\sqrt{2\bar{w}} + O(\mu^{1/2})\right\} + O(\mu^{3/2}) \qquad (4.3)$$

The limit of (3.23b) as $\bar{w}$ approaches infinity is

$$y(\bar{w},\mu) = +\mu^{3/4}\bar{\lambda}\left\{\sqrt{2\bar{w}} + O(\mu^{1/2})\right\} \qquad (4.4)$$

Comparison of (4.3) and (4.4) shows that for matching it is necessary that $\lambda = -\bar{\lambda}$ (4.5)

and hence if $\bar{l}$ = angular momentum of the inner parabola

$$\bar{l} = +\mu^{1/4}\lambda \qquad (4.6)$$

Similarly, for $t(w,\mu)$, the inner limit of the outer expansion, equation (3.13b) is

$$\sqrt{2}\, t(\bar{w},\mu) = \frac{\pi}{2} - \mu^{1/4}\left\{2\bar{w}^{1/2} + \frac{1}{3}\bar{w}^{-3/2}\right\} + \mu\left\{\frac{3\pi\rho_1}{4} - \pi\right\} + O(\mu^{3/4}) \qquad (4.7)$$

The limit as $\bar{w}$ approaches infinity of equation (3.23a) is:

$$\sqrt{2}\, t(\bar{w},\mu) = \sqrt{2}\,\tau - \mu^{1/4}\left\{2\bar{w}^{1/4} + \frac{1}{3}\bar{w}^{-3/2}\right\} + 1.692\,\mu^{1/4} \qquad (4.8)$$

Comparison of (4.7) and (4.8) shows that for matching

$$\tau + 1.196\,\mu^{1/4} = \frac{\tilde{\pi}}{2\sqrt{2}} + \frac{\mu}{\sqrt{2}}\left(\frac{3\pi\rho_1}{4} - \tilde{\pi}\right) \qquad (4.9)$$

Then if τ is expanded in the form

$$\tau = \tau_0 + \mu\tau_1 \qquad (4.10)$$

$$\tau_0 = \frac{\bar{\pi}}{2\sqrt{2}} \qquad \tau_1 = \frac{3\pi\rho_1}{4} - \tilde{\pi} \qquad (4.11)$$

Hence the half period of the motion is

$$\tau + 1.196\ \mu^{1/4} \tag{4.12}$$

V. GENERAL DISCUSSION

In the one-dimensional case the particle is constrained to follow a trajectory in space-time right into the equilibrium point with an infinite period. The time history for the motion between the equilibrium point and the moon is determined as a separate problem. The earth's influence is felt as a uniform attraction near the singular point and the moon.

In the planar problem the energy is greater than the absolute minimum; motion between the two centers is thus possible. Prior to entering a strictly lunar phase the particle traverses a region where earth and lunar effects are equally important. In these two regions parabolic orbits with angular momentum of order $\mu^{1/4}$ and minimum approach distances of order $\mu^{1/2}$ in inner variables define the motion to lowest order. The effects of two dimensionality on the time history is negligible to lowest order. The relation matches with the innermost Keplerian motion as well as the outer motion.

The main purpose is to reproduce the exact results given by Bonnet's theorem through the use of asymptotic methods. This exact solution is an ellipse with the force centers as foci.

Consider an ellipse with foci at the centers of attraction. If the semi-major axis is denoted by a, the eccentricity by e, the energy relative to the earth by h_e and the angular momentum relative to the earth by l_e , the following formulas relate the Keplerian elements to the integrals:

$$a = \frac{-(1-\mu)}{2 h_e} \tag{5.1}$$

$$e^2 = 1 + \frac{2 h_e l_e^2}{(1-\mu)^2} \tag{5.2}$$

Furthermore, the fact that the earth-moon distance is equal to unity implies:

$$2 a e = 1 \tag{5.3}$$

If the earth is considered alone, then a simple calculation shows that the velocity at earth perigee, i.e., at $x = -a(1-e), y = 0$, is

$$V_p^{(e)} = \left\{(1+e)(1-\mu)/a(1-e\right\}^{1/2} \tag{5.4}$$

and the velocity at earth apogee, i.e., at $x = a(1+e), y = 0$ is

$$V_a^{(e)} = \left\{(1-e)(1-\mu)/a(1+e)\right\}^{1/2} \tag{5.5}$$

Next, if the moon is considered alone (located at x = 1, y = 0) the velocities at the points $x = a(1+e), y = 0$, moon perigee, and $x = -a(1-e), y = 0$, moon apogee, are given respectively by

$$V_p^{(m)} = \left\{(1+e)\mu/a(1-e)\right\}^{1/2} \tag{5.6}$$

$$V_a^{(m)} = \left\{(1-e)\mu/a(1+e)\right\}^{1/2} \tag{5.7}$$

Then according to Bonnet's theorem the velocity at earth perigee (or moon apogee) for the motion under the influence of both centers is:

$$V_1 = \sqrt{V_P^{(e)2} + V_a^{(m)2}} = \frac{(1+e)^2 - 4\mu e}{a(1-e^2)} \tag{5.8}$$

and the velocity at moon perigee (or earth apogee) is:

$$V_2 = \sqrt{V_a^{(e)2} + V_P^{(m)2}} = \frac{(1-e)^2 + 4\mu e}{a(1-e)^2} \tag{5.9}$$

Evaluation of the two integrals of motion at the two turning points gives:

$$h = -\frac{1}{2a} \tag{5.10}$$

$$\delta = \frac{1+e^2}{2e} \tag{5.11}$$

If the exact perigee distance (i.e., $a(1-e)$) equals $\alpha\mu$ where α is order unity, all pertinent quantities are easily computed as functions of α and μ :

$$a = \frac{1}{2} + \alpha\mu \tag{5.12}$$

$$e = 1 - 2\alpha\mu + 0(\mu^2) \tag{5.13}$$

$$V_1 = \sqrt{\frac{2}{\alpha\mu}} \left\{ 1 - \frac{\mu(\alpha+1)}{2} + 0(\mu^2) \right\} \tag{5.14}$$

$$V_2 = \sqrt{\frac{2}{\alpha}} \left\{ 1 + \frac{\mu(\alpha-1)}{2} + 0(\mu^2) \right\} \tag{5.15}$$

$$h = -1 - 2\alpha\mu + 0(\mu^2) \tag{5.16}$$

$$\delta = 1 \tag{5.17}$$

$$h_e = -1 + \mu(1+2\alpha) + O(\mu^2) \tag{5.18}$$

$$l_e = \sqrt{2\alpha\mu}\left\{1 - \frac{\mu(\alpha+1)}{2} + O(\mu^2)\right\} \tag{5.19}$$

The assumed initial conditions for the asymptotic solutions were:

$$h_e = -1 + \mu\rho_1 \qquad \text{Thus} \qquad \rho_1 = 1 + 2\alpha$$

$$l_e = \mu^{1/2}\lambda \qquad \qquad \lambda = \sqrt{2\alpha}$$

Viewed in inner variables the ellipse appears as a parabola near the moon. With the use of equations (5.12) through (5.19) the following constants for the inner parabolic motion are easily determined in inner variables as:

Approach distance $= \alpha\,\mu^{1/2}$ (5.20)

Angular momentum $\bar{l} = \mu^{1/4}\sqrt{2\alpha}$ (5.21)

Energy $\bar{h} = \mu^{1/2}\alpha + O(\mu^{3/2})$ (5.22)

From the matching (cf. Section IV) the angular momentum was shown to be $\bar{l} = \mu^{1/4}\lambda$ and the minimum approach distance as $\mu^{1/2}\lambda^2/2$. Clearly these results agree exactly with the predicted results, equations (5.20) and (5.21). However the matching does not provide any more information than that the order unity value of the energy is zero. The effect of ρ_1 is seen to occur only in the determination of the half-period of the motion $\tau + \beta\,\mu^{1/4}$, cf. equation (4.12). The solutions found are the periodic solutions.

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