THE ELASTIC INSTABILITY OF THIN CANTILEVER STRUTS ON ELASTIC SUPPORTS WITH AXIAL AND TRANSVERSE LOADS AT THE FREE END

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ABSTRACT

An analysis is made of the elastic instability of thin, tapered cantilever struts subjected to a general concentrated load acting in the plane of the strut at its tip. The strut is supported at its root on a structure permitting elastic rotations of the root section in the buckled mode. The influence of the support on the minimum buckling load is one of the main points of interest. It is shown that the general linearized problem can be formulated in one second order differential equation with variable coefficients, and two associated boundary conditions. This homogeneous eigenvalue system constitutes a simplified statement of the problem which permits the easy extension of exact linear theory to a wide class of taper functions, including the effect of elastic supports. The solution emerges in terms of a generalized deflection parameter, rather than of either the torsional or the bending components of the coupled buckling mode, which are governed respectively by third and fourth order differential equations.

Specific solutions are derived for some "natural" taper forms of the strut. The general solutions for the deflection mode are power series, which are rapidly convergent for certain limiting geometries. The problems of convergence of the series, some singular physical aspects associated with pointed tips, and the increasing numerical difficulty for large taper ratio are correlated with the behavior of the singular points of the equation. Numerical results showing the effects of the elastic supports on minimum buckling loads are presented for the uniform strut and for a simple case of the tapered strut. The series solutions for more general cases are given in a form which can be applied to digital computers.

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a	= numerical constant = $\frac{\bar{\ell}}{1 - \bar{\ell}}$
$\bar{a}$	= numerical constant = $1 + 2a = \frac{1 + \bar{\ell}}{1 - \bar{\ell}}$
a_n	= general coefficient in power series expansion
b	= flexural rigidity parameter = $\frac{B}{\ell^2}$
c	= torsional rigidity parameter = $\frac{C}{\ell^2}$
e_1, e_2	= spring constants, moment per radian, for strut root support in torsion and bending, respectively
$f = \frac{B}{B_o}$	= dimensionless flexural rigidity distribution
$g = \frac{C}{C_o}$	= dimensionless torsional rigidity distribution
h	= taper parameter (Figure 2)
ℓ	= length of strut
$\bar{\ell}$	= $1 - h\ell$
m	= exponent
n	= general index
$p = \frac{\tau}{\phi}$	= ratio of dimensionless load parameters
q	= $\frac{1}{p}$
r, s, t	= indices
s	= special independent variable = $\frac{\zeta}{\ell}$ (Figure 7)
u, u'	= deflection coordinates for strut neutral axis (Figure 2)
v	= $\frac{u(\ell) - u}{\ell}$ = dimensionless deflection
w	= special independent variable (Figure 7)
w^*	= special independent variable (Figure 7)
x, x'	= distances along strut neutral axis (Figure 2)
y, y'	= coordinates (Figure 2)

z	=	generalized deflection parameter
$B = B(x)$	=	variable bending rigidity
B_o	=	B at $x = 0$
$C = C(x)$	=	variable torsional rigidity
C_o	=	C at $x = 0$
E	=	Young's modulus
L	=	$\dfrac{2\bar{\ell}}{1 + \bar{\ell}}$
M	=	$\dfrac{2m}{1 + \bar{\ell}}$
M_x, $M_{x'}$; M_y, $M_{y'}$; M_u, $M_{u'}$		components of moment defined about x, x', y, etc. axes according to right-hand rule
P	=	transverse load applied at strut tip (Figure 3)
R, S	=	coefficients of general differential equation
R	=	index
T	=	axial load applied at strut tip (Figure 3)
α	=	exponent
β	=	angle of twist of strut cross-section
γ, δ, Δ, ∇	=	special coefficients
$\epsilon = \dfrac{B}{e_1 \ell}$	=	flexibility coefficient for strut support in bending about major principal axis
$\mu = \dfrac{C}{e_2 \ell}$	=	flexibility coefficient for strut support in torsion about elastic axis
θ	=	generalized twist deformation
κ	=	$\tau + \phi$
$\phi = \dfrac{P^2 \ell^4}{B_o C_o}$	=	dimensionless parameter for transverse load

ϕ^* $= \phi(\frac{L^2}{\bar{\ell}\, m})^2$

$\tau = \frac{T\ell^2}{B_o}$ = dimensionless parameter for axial load

τ^* $= \frac{\tau L^2}{\bar{\ell}\, m}$

λ $= \frac{1}{\tau}$

$\xi = \frac{\ell - x}{\ell}$ = dimensionless coordinate

$\zeta = (1 - hx)$ = dimensionless coordinate

ν $= (K - \frac{1}{4})^{1/2}$

$\bar{\nu}$ $= (\frac{1}{4} - K)^{1/2}$

PART A

I. INTRODUCTION

This report is divided into two parts. Part A is a brief résumé of the assumptions and important steps of the analysis, and collects the essential results. Readers who are interested in details, and in the less specific sidelights of the problem, will find them in Part B.

The elastic instability of cantilever struts under axial and/or lateral loads (Figure 1) has been treated by several investigators. References 1 through 4 have been standard works for the cases when either of the loads is applied alone. References 5, 7 and 8 have treated the problem for combinations of these loads. The particular case for thin tapered struts, such as are used in wind tunnels for model supports, has received attention in References 5 and 7. The investigations of combined loadings have all been limited either to very special strut geometries in the analysis by the exact linear equations, or to an approximate formulation by energy methods for the more general strut shapes. The strut has also been considered in these analyses to be rigidly supported at its root section.

This report considers the instability problem of the strut on elastic supports. Such supports for the strut are typical of the wind tunnel model rigging configurations which have evolved in recent years, as shown in Figure 1. In these configurations the strut root section can rotate elastically in the two degrees of freedom which are pertinent to the deflection mode of the buckled strut. In the process of reconsidering the stability analysis for these newer boundary conditions it has been possible also to formulate the problem completely and simply in the exact linear theory, with applications to perfectly general taper functions under combined loadings. Specific solutions are generated for a wide class of tapered struts in this report.

II. DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS

In the notation defined on Figures 2 and 3 and in the Table of Notation the following pair of differential equations express equilibrium of a section of the buckled strut at a distance x from the root, according to the elementary concepts of linear beam theory:

$$B(x)\frac{d^2u}{dx^2} = P(\ell - x)\beta + T(u_\ell - u)$$

$$C(x)\frac{d\beta}{dx} = P(u_\ell - u) - P(\ell - x)\frac{du}{dx} \tag{1.1}$$

The forces P and T act at the intersection of the elastic-neutral axis with the tip, in a plane parallel to the plane of the undeflected thin strut. In deriving these equations it is assumed that all deflections are small and that the strut sections are everywhere so thin that the flexural rigidity about the u-axis is very large compared with the rigidity about the y-axis. Then the deflection of the elastic-neutral axis out of the x-u plane is neglected.

The boundary conditions to be satisfied by the strut deformations u(x) and β (x) are

at x = 0: (1) $u = 0$

(2) $e_2\left(\frac{du}{dx}\right) = B\left(\frac{d^2u}{dx^2}\right)$ (1.2)

(3) $e_1\beta = C\left(\frac{d\beta}{dx}\right)$

at $x = \ell$: (4) $\dfrac{d^2u}{dx^2} = 0$

(5) $\dfrac{d\beta}{dx} = 0$

Conditions (2) and (3) introduce the assumed elastic spring rates of the strut support, in bending and torsion of the strut at its root.

It is convenient to introduce dimensionless variables

$\xi = 1 - \dfrac{x}{\ell}$ and $v = \dfrac{1}{\ell}(u_\ell - u)$;

the coupled equations 1.1 then become

$$-b(\xi)\frac{d^2v}{d\xi^2} = P\xi\beta + Tv$$

$$-c(\xi)\frac{d\beta}{d\xi} = Pv - P\xi\frac{dv}{d\xi} \qquad (1.3)$$

Retaining for the moment the variation of the flexural and torsional rigidities in the form of the parameters $b(\xi)$ and $c(\xi)$, equations 1.3 can be combined by eliminating either v or β. After some manipulation this results in the following alternate equations, in β and v respectively:

$$\xi\frac{d}{d\xi}\left[b\frac{d}{d\xi}\left(c\frac{d\beta}{d\xi}\right)\right] - 2b\frac{d}{d\xi}\left(c\frac{d\beta}{d\xi}\right) + \xi\left[P^2\xi^2\frac{d\beta}{d\xi} + Tc\frac{d\beta}{d\xi}\right] = 0 \qquad (1.4)$$

and

$$\frac{d^2}{d\xi^2}\left(b\frac{d^2v}{d\xi^2}\right) + T\frac{d^2v}{d\xi^2} + \frac{P^2}{\xi}\frac{d}{d\xi}\left[\frac{\xi^4}{c}\frac{d}{d\xi}\left(\frac{v}{\xi}\right)\right] = 0$$

Either of equations 1.4, with appropriate boundary conditions from equations 1.2, constitutes a homogeneous system for the solution of the eigenvalues of P and T. Since each system must yield the same eigenvalues, they are entirely equivalent systems for determining critical buckling loads. It follows that they are reducible to the same form. In fact, upon introducing the appropriate transformations

$$z = \xi\frac{dv}{d\xi} - v$$

and (1.5)

$$\theta = \frac{C_o}{P\ell^2}\, g(\xi)\, \frac{d\beta}{d\xi}$$

of the dependent variables, each of equations 1.4 can be reduced to the form

$$\xi^2 \frac{d}{d\xi}\left[\frac{f(\xi)}{\xi^2}\frac{dz}{d\xi}\right] + \left[\tau + \frac{\xi^2\phi}{g(\xi)}\right] z = 0 \qquad (1.6)$$

Here $f(\xi)$ and $g(\xi)$ are dimensionless representations of the rigidity functions, replacing $b(\xi)$ and $c(\xi)$; and τ and ϕ are the dimensionless load parameters representing T and P. In order to recognize that both of equations 1.4 reduce to equation 1.6, it is necessary to recognize that

$$z = \theta \tag{1.7}$$

as can be verified from the second of equations 1.3. The dependent variable $z(\xi)$ is a generalized deflection parameter, in terms of which the eigenvalue equation for the critical loads attains the relatively simple expression given in equation 1.6, which is only of second order.

The boundary conditions must now be expressed in terms of $z(\xi)$; this requires that the five conditions listed in equations 1.2 be reduced to two conditions in $z(\xi)$. The last two conditions 1.2 express the absence of applied moments at the strut tip. These are automatically satisfied by any functions $u(x)$ and $\beta(x)$ which satisfy equations 1.1, and thus also for any functions which satisfy equations 1.5 and 1.6. In the dimensionless variables v and ξ the first three conditions 1.2 become

$$\text{at } \xi = 0: \qquad (1)\ v(0) = 0$$

$$\text{at } \xi = 1: \qquad (2)\ \frac{dv}{d\xi}(1) = -\epsilon\frac{d^2v}{d\xi^2}(1) \tag{1.8}$$

$$(3)\ \beta(1) = -\mu\frac{d\beta}{d\xi}(1)$$

The dimensionless flexibility coefficients μ and ϵ, introduced here, replace the spring constants e_1 and e_2 in equations 1.2. With the help of equations 1.5 and 1.7, and the first of equations 1.1, the last two of conditions 1.8 may be rewritten and combined to form the last of conditions 1.9, which follow. The first condition 1.8 can be rewritten immediately as $z(0) = 0$.

Thus the boundary conditions

$$z(0) = 0$$

and (1.9)

$$\frac{dz}{d\xi}(1) = \frac{\tau + \mu\phi}{1 - \epsilon\tau} z(1)$$

plus the differential equation 1.6, are the final homogeneous system for the determination of the eigenvalues of τ and ϕ. The corresponding eigenfunctions $z(\xi)$ could then also be determined, from which the deflection functions $v(\xi)$ and $\beta(\xi)$ might be obtained with additional quadratures through equations 1.5.

III. GENERAL SOLUTIONS FOR A CLASS OF THIN, TAPERED STRUTS

Equation 1.6 can be written in a second canonical form:

$$\frac{d^2z}{d\xi^2} + R(\xi)\frac{dz}{d\xi} + S(\xi)z = 0$$

with (1.10)

$$R(\xi) = \frac{1}{f}\frac{df}{d\xi} - \frac{2}{\xi} \quad , \qquad S(\xi) = \frac{1}{f}\left(\tau + \frac{\xi^2\phi}{g}\right)$$

For thin sections it can be shown that the torsional and flexural rigidities vary in exactly the same manner. Choosing the taper function ζ^m (Figure 2) to represent a suitably wide class of strut geometries, then

$$f(\xi) = g(\xi) = \zeta^m$$

and

$$R(\xi) = m\frac{1 + a}{\xi + a} - \frac{2}{\xi} \tag{1.11}$$

$$S(\xi) = \left(\frac{1 + a}{\xi + a}\right)^m \left[\tau + \phi\xi^2 \left(\frac{1 + a}{\xi + a}\right)^m\right]$$

where a is a constant representing the location of the imaginary pointed tip. With reference to Figure 2 it will be noted that the parameter defined as

$$\bar{\ell} = 1 - hx$$

is a convenient index for the equivalent taper ratio of the strut. As $\bar{\ell}$ varies from 0 to 1 the strut varies in truncation from "complete" taper to zero taper, and the constant a varies from 0 to ∞. The numbers $\bar{\ell}$ and m completely specify the strut geometry for the purposes of this problem.

As will be seen, it is necessary to distinguish cases for $\bar{\ell} = 0$ from all others. When $\bar{\ell} \neq 0$ the problem can be solved in general by a power series expansion of z about the point $\xi = 0$, although certain degenerate cases may be expected to produce closed solutions (4). It will be observed from equations 1.11 that the differential equation has three singular points, namely the points at $\xi = 0, -a,$ and ∞, and that the singularity at $\xi = 0$ is always regular. The point at $\xi = -a$ is not a regular singularity, however, for $m > 1$ (and $\phi \neq 0$).

Regarding z and ξ as complex variables, for the moment, then it is apparent from the above that power series solutions can always be written about $\xi = 0$, when $\bar{\ell} \neq 0$. If the circle of convergence includes the point $\xi = 1$, then the series is valid in the interval $0 \leq \xi \leq 1$, as required for a solution of the eigenvalue problem. But when $m > 1$, the point $\xi = -a$ is an irregular singularity, such that the radius of convergence is limited by it. This singularity moves toward the origin as $\bar{\ell} \rightarrow 0$; for $\bar{\ell} < \frac{1}{2}$, $a < 1$ and consequently the power series solution does not extend to the strut root.

This difficulty may be circumvented by a suitable transformation of the independent variable. The transformation

$$\frac{\xi}{2a} = \frac{\bar{w}}{1 - \bar{w}}$$

$$\bar{w} = w\bar{a} \tag{1.12}$$

$$\bar{a} = 1 + 2a = \frac{1 + \bar{\ell}}{1 - \bar{\ell}}$$

serves the purpose of maintaining the maximum radius of convergence for a given $\bar{\ell}$. The singular points $\xi = -a$ and ∞ are transformed into the points $w = \pm \bar{a}$. The interval occupied by the strut remains $0 \leq w \leq 1$. Since $\bar{a} > 1$ for $\bar{\ell} > 0$, a series solution in w is convergent over the length of the strut for all $\bar{\ell} > 0$. The rate of convergence, however, can be expected to become less and less rapid as $\bar{\ell} \rightarrow 0$.

After transforming to the new independent variable, equation 1.10 takes the form, for $\bar{\ell} \neq 0$,

$$\frac{d^2z}{dw^2} + \left[\frac{M}{(1-\bar{w})^2} - \frac{2(1+\bar{w})}{w(1-\bar{w})}\right]\frac{dz}{dw} + \left[\tau^* \frac{(1-\bar{w})^{m-4}}{(1+\bar{w})^m} + \phi^* \frac{w^2(1-\bar{w})^{2m-6}}{(1+\bar{w})^{2m}}\right] z = 0 \tag{1.13}$$

and the boundary conditions 1.9 become .

$$\begin{aligned} &(1) \quad \text{at } w = 0: \quad z(0) = 0 \\ &(2) \quad \text{at } w = 1: \quad L\frac{dz}{dw}(1) = \frac{\tau + \mu\phi}{1 - \epsilon\tau} z(1) \end{aligned} \tag{1.14}$$

The parameters L, M, τ^* and ϕ^* are convenient recombinations of $\bar{\ell}$, m, τ and ϕ which become equal to the latter parameters, respectively, when $\bar{\ell} = 1$.

If the series expansion for z(w) is written as

$$z = \sum_{n=o}^{\infty} a_n w^{n+\alpha},$$

and substituted in equation 1.13, the indicial equation is

$$\alpha(\alpha - 3) = 0$$

Although the roots of this equation differ by an integer, the corresponding pair of solutions are linearly independent. The first boundary condition 1.14 eliminates the solution corresponding to $\alpha = 0$. The general solution is then

$$z = w^3 \sum_{n=o}^{\infty} a_n w^n \tag{1.15}$$

After substituting equation 1.15 into equation 1.13, clearing fractions and expanding polynomials, the following recurrence relation results for the coefficients a_n:

$$-n(n+3)a_n = \sum_{s=o}^{4}{}' \binom{4}{s}(-\bar{a})^{-s} \sum_{r=o}^{2m|n-s} \binom{2m}{r} \frac{(n+3-t)(n+2-t)}{\bar{a}^r} a_{n-t}$$

$$-2 \sum_{s=o}^{3}{}' \binom{3}{s}(-\bar{a})^{-s} \sum_{r=o}^{2m+1|n-s} \binom{2m+1}{r} \frac{(n+3-t)}{\bar{a}^r} a_{n-t} \tag{1.16}$$

$$+M \sum_{s=o}^{3} \binom{3}{s}(-\bar{a})^{-s} \sum_{r=o}^{2m-1|n-s-1} \binom{2m-1}{r} \frac{(n+2-t)}{\bar{a}^r} a_{n-t-1}$$

$$+\tau^* \sum_{s=o}^{m\left|\frac{n-2}{2}\right.} \binom{m}{s}(-\bar{a}^2)^{-s}\, a_{n-2s-2}$$

$$+\phi^* \sum_{s=o}^{2m-2\left|\frac{n-4}{2}\right.} \binom{2m-2}{s}(-\bar{a}^2)^{-s}\, a_{n-2s-4}$$

The coefficient a_o is arbitrary. In the above the following special notation is employed:

a) $t = r + s$

b) $\binom{m}{s}$ are the binomial coefficients of the expansion $(1 + x)^m$

c) The sign $\sum'$ denotes omission of the term for $t = 0$ from the summation

d) The summation index is limited to the lesser of the two upper limits whenever a double limit appears.

e) If the lesser upper limit is less than zero no contribution occurs from that index, except that in the last (or ϕ^*) summation $m = 0$ is to be treated as $m = 1$.

With the coefficients a_n determined the eigenvalues of τ^* and ϕ^* are found upon application of the second boundary condition 1.14. This can be done best by assuming a ratio

$$p = \frac{\phi^*}{\tau^*} \qquad (1.17)$$

and solving for the minimum eigenvalue of τ^* if p is small, and of ϕ^* if p is large. The ratio p becomes an additional arbitrary parameter.

The coefficients a_n will be polynomials in whichever load parameter, say τ^*, is selected. Upon application of the boundary condition the result is an infinite power series in τ^*, which may be written

$$\sum_{t=o}^{\infty} \Delta_t (\tau^*)^t = 0 \tag{1.18}$$

The coefficients Δ_t are functions of the arbitrary input data: m, $\bar{\ell}$, p, ϵ and μ. The minimum critical load corresponds with the minimum root of equation 1.18, which can be found by numerical methods.

In the general case the application of the recurrence relation 1.16 will be most efficiently handled by digital machine calculation. When the coefficients a_n are known for given m, $\bar{\ell}$ and p, they are the basis of the calculation for all flexibilities, ϵ and μ, of the strut support.

IV. SOLUTION FOR THE UNIFORM STRUT

When $\bar{\ell} = 1$ the taper function is a constant. This is also the case for m = 0. Then $w \rightarrow \xi$. The recurrence relation 1.16 is greatly simplified, reducing to

$$a_n = \frac{-\tau}{n(n+3)} (a_{n-2} + p\, a_{n-4}) \,, \quad n = 4, 6, 8, \ldots$$

$$a_2 = -\frac{\tau}{2 \cdot 5} a_o \qquad (1.19)$$

$$a_n = 0 \,, \quad n = 1, 3, 5, \ldots$$

These define the solution 1.15 for a uniform strut.

This series has an infinite radius of convergence since $\bar{a} = \infty$, and is so rapidly convergent at w = 1 that the infinite series 1.18 can be limited to its first four terms. The numerical calculations proceed fairly quickly on a desk calculator. They are outlined in greater detail in Part B, and are laid out as a sample for computer programming in Appendix 1. The results, illustrating the effects of elastic supports, are given in Figures 8 and 10.

V. SOLUTIONS FOR $\bar{\ell} = 0$: "POINTED" STRUTS

As $\bar{\ell} \rightarrow 0$ there is a confluence of singularities of equation 1.10, since then $a \rightarrow 0$. The transformation 1.12 loses meaning at $\bar{\ell} = 0$ (the entire ξ-plane is mapped into the point $w = 1$), and the solution must be obtained in the ξ-plane. The confluent singularity at $\xi = 0$ remains regular for $m = 2$, and series solutions in powers of ξ exist for these values of m, convergent in the entire ξ-plane.

The case for $m = 0$ is the uniform strut of the preceding section. The case for $m = 1$ is, however, a nontrivial one whose solution is easily obtained and is of interest.

The indicial equation determined for the series solutions to equations 1.10 and 1.11, when $\bar{\ell} = 0$ and $m = 1$, is

$$\alpha(\alpha - 2) = 0 .$$

Here again the solution corresponding to the root $\alpha = 0$ is eliminated by the first boundary condition 1.9, and the general solution to the differential equation is

$$z = \xi^2 \sum_{n=0}^{\infty} a_n \xi^n \qquad (1.20)$$

The lower value of the second root ($\alpha = 2$) in this case corresponds to the fact that now, for $\bar{\ell} = 0$, $B(0) = C(0) = 0$; i.e. it is no longer required that $\frac{d^2u}{dx^2} = \frac{d\beta}{dx} = 0$ at the tip, as in the last pair of conditions 1.2.

The recurrence relations for the coefficients in equation 1.20 are

$$a_1 = -\frac{\tau}{3} a_o$$
$$a_n = \frac{-\tau}{n(n+2)} (a_{n-1} + p\, a_{n-2}) \qquad (1.21)$$

The details of the calculations for this case will be found in Part B and in Appendix 2. Typical results are plotted in Figures 9 and 11.

VI. EXPERIMENTAL VERIFICATION

A short experimental program was run to obtain a check on the analysis. The results of linear theory for struts on rigid supports have been verified many times by other investigators. In this case it was desired only to verify the effects of the elastic supports.

The test specimen is shown on Figure 12, and the method of loading is indicated in Figure 13. The specimen has the appearance of a carpenter's "L", the narrower leg of which represents the cantilever strut, while the wider leg constitutes the elastic support. As shown on Figure 13, by clamping the end of the wider leg at variable distance ℓ' from the narrow leg, the flexibility of the strut support could be varied.

It will be recognized that the boundary condition at the root of the strut (corner of the "L") differs in details from the ideal conditions assumed in the analysis. In particular it is necessary to determine an effective length for the strut. It is also necessary to correct the flexibility coefficient μ to account for the non-linear bending of the wider leg of the "L" under the action of the load P when the strut deflects.

When these effects are included in the analyses of the experimental data, the comparison between linear theory and experiment is as shown on Figure 14. The results are considered a satisfactory verification for the uniform strut. The solid curve on Figure 14 gives data obtained from the linear theory for the particular conditions of the experiment. The test model was loaded with equal axial and transverse loads, P = T. This is different in general from the condition p = 1, or $\tau = \phi$; and in fact over the range of the test points the condition P = T corresponds to variable p, in the range $2/3 < p < 3/2$. This variation has been taken into account in constructing the solid curve, to compare directly with the experimental points.

VII. CONCLUSION

It is felt that the possibilities for analysis of the problem by exact linear methods have been fully revealed and exploited in this investigation. In particular the association of the mathematical and numerical difficulties with large taper ratios ($\bar{\ell} \rightarrow 0$) is clarified by its connection with the confluence of singularities at $\xi = 0$. Thus while exact solutions become progressively more difficult as $\bar{\ell} \rightarrow 0$ (the associated differential equation has three singularities), the solutions become simplest at $\bar{\ell} = 0$ (the associated differential equation has only two singularities, at 0 and ∞).

This difficulty is removed for a strut with exponential taper, a solution for which is derived in Part B.

Computing experience with equation 1.16 will reveal practical lower limits of $\bar{\ell}$ for which to obtain answers from the exact theory. Approximate numerical methods may then prove to be more attractive, from this standpoint. For this purpose the reduction of the governing differential equation to its second order form, as in equation 1.6, will be a considerable help. In its given form equation 1.6 is directly applicable either to ordinary iteration methods or to the more powerful integral equation formulations and their approximate numerical solutions. (Ref. 9).

PART B

I. REVIEW OF THE PROBLEM

The analysis of the buckling problem for struts under loadings of the type considered here, and on rigid supports, is not new. Besides some of the more standard works, such as those described by Timoshenko (Ref. 1), Roark (Ref. 2) and others (Refs. 3, 4), an attempt at analysing the problem for a by now standard class of taper functions was made by Richardson in 1948 in a private work (Ref. 5). The class of taper functions which was considered represented the flexural and torsional rigidities as proportional to the function $(1 - hx)^m$, where x is the distance out from the strut root section, h is an arbitrary constant determined by the strut taper, and m = 0, 1, 2, ... (Figure 2). Richardson and Dinnik (Ref. 4) have obtained solutions in closed form for the eigenvalues corresponding to the Euler column problem (i.e. with zero lateral load), and Richardson attacked the Prandtl or lateral buckling problem (zero axial load) by an iteration procedure. He carried out numerical calculations on a desk calculator for the critical axial loads at m = 3 and 4, and for the critical lateral loads at m = 3, devoting considerable effort, in the formulation of the latter case, to attaining an accuracy to several figures. The results may be seen on Figure 4, for the complete range of taper, from pointed tip to uniform strut.

Richardson also considered the general problem of buckling under combined axial and lateral loads, choosing to make a variational formulation following the method of Rayleigh-Ritz. Here also much attention was devoted to detail, especially to achieving a plausible approximation to the buckled mode, in an attempt to hold accuracy to several figures, with the result that numerical calculations are tedious and beset by arithmetical difficulties. Richardson did not make any calculations. In the simplest case, for a uniform strut ($\bar{\ell} = 1$), the calculations have been carried out recently under the writer's direction on the CWT* electronic digital calculator, and the results are shown on Figure 5. It will be observed that they are indistinguishable on the figure from the curve of the exact solution due to Martin (Ref. 7). Richardson's expansion of the method to include taper involved a great deal more complication, requiring a mathematical restatement of the problem in terms of which the solution diverges if either of the limiting forms represented by a uniform strut and by a pointed tip is approached (i.e. if $\bar{\ell} \rightarrow 0, 1$). This seems to be a characteristic difficulty of this problem, and not necessarily a difficulty of method alone, as will be seen later. However in this case the attendant arithmetical divergences have been a serious problem and it has been possible at this point only to obtain the answers for $m = 3$, $\bar{\ell} = 0.5$ (which corresponds to a two-to-one taper in thickness, with strut chord held constant) shown on Figure 5.

In 1950 in his doctoral thesis Martin (Ref. 7) made exact analyses for the cases represented by the solid lines on Figure 5. He solved the fourth order linear differential equation for the deflection

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* So. Calif. Coop. W. T.

mode of the strut neutral axis, obtaining solutions in power series form for a strut of uniform cross section, and for what he calls complete taper, viz., taper to a pointed tip ($\bar{\ell} = 0$) for m = 1, which can be represented by linear chord variation at constant thickness.* The reduction of the solutions from the fourth order equation to eigenvalues was sufficiently complicated for these cases that Martin did not attempt a solution for more general taper, of which these two cases are degenerate, limit forms.

The only remaining work of importance in the area of this problem which has come to the writer's attention was published in 1952 by Di Maggio et al (Ref. 8), in Table 3 of which are listed critical buckling loads for a cantilevered rectangular beam on rigid support under several end loadings. The method of analysis again was approximate, based on variational and energy principles.

As is evident on Figure 5, the results developed by Richardson and Martin show a pronounced effect, on critical load, of the taper function exponent, m. For m = 1 the critical loads for Martin's taper ($\bar{\ell} = 0$) are more than half the critical loads of the uniform cantilever. For m = 3, on the other hand, the loads for $\bar{\ell} = 0.5$ are less than those for Martin's taper, and indeed are zero for the corresponding pointed tip ($\bar{\ell} = 0$; see Figure 4). It appears now

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* Actually this concept of the taper needs to be modified near the tip since it is a basic assumption used in the analysis that the strut cross sections are <u>thin</u>. Thus near the tip the thickness must also be reduced, as necessary, to maintain a thin section.

that for both axial and lateral loading the critical load near = 0 reduces abruptly between m = 1 and m = 2 (and not between m = 2 and m = 3 as tentatively suggested in Ref. 6). More work needs to be done on the theory for the pointed tip cases, under combined loads, to establish the correct solutions at $\bar{\ell} = 0$ for $m \geq 2$.

The work so far described has related to thin struts on rigid supports. In recent years structural configurations have appeared (for example, in wind tunnel mountings; see Figure 1) in which the thin model support strut can no longer be regarded as rigidly supported at its root section. However with good approximation it can be considered mounted on elastic supports which permit rotation of the root section in both of the pertinent degrees of freedom, i.e. about the strut neutral axis, and about the major axis of symmetry of the section.* This inclusion of elastic supports in the problem injected a need for a more straightforward approach even to the basic problem on rigid supports (now a limiting case of the newer, expanded project) than had previously been taken. Consequently a new start on the problem was made not only to incorporate the elastic supports but also to frame a more complete theory. The assumptions and notation used in the analysis are described in Section II. The development of the general equation and the statement of the boundary conditions is carred out in Section III. From the two coupled equations which express equilibrium of the

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* In another possible application, the pylons used to sling external stores beneath airplane wings are also supported elastically.

deflected strut in bending and torsion, elimination of either the bending or the torsional deflections yields respectively either a third order ordinary differential equation in the torsional deflection or a fourth order differential equation in the bending deflection. The same eigenvalues should result, of course, from a solution with either equation although the corresponding eigenfunctions would be different. It should be an essential characteristic of the eigenvalue problem per se, as a matter of fact, that it can be represented by a single mathematical formulation for both modes of deflection, i.e. that a single differential equation and set of boundary conditions can be obtained, deducible from each of the separate representations for the separate modes. In Section III this line of attack is followed and it is found that with suitable transformations of the dependent variables the eigenvalue problem is reducible to a second order equation, bearing marked resemblance to the second order equations of the degenerate single loading cases solved by Richardson, but with important differences. This constitutes a significant simplification in the problem, especially for the inclusion of elastic supports, the effect of which could now be expressed in just one of the two boundary conditions. The dependent variable of the problem is to be considered a generalized deflection, the eigenfunctions in which can be transformed through separate integrations into both the bending and the torsional deflection modes. Many characteristics of the problem, e.g. the critical shape effect, can be interpreted in the light of the singularities of this second order differential equation.

General solutions are derived in power series, for the generalized deflection, in Section IV. With the application of boundary conditions the eigenvalues are found as the roots of an infinite power series, by numerical methods. The series are convergent, so rapidly in some of the limit cases that the calculations are easily carried out manually on a desk calculator or with a slide rule. The results of such calculations are shown on Figures 8 through 11. These show the trend of the effects of elastic supports for the uniform strut and for Martin's taper.

Typically, as shown also by the experience of previous investigators, the solutions become numerically more formidable for the general taper function. In the present case the solutions become sufficiently less rapidly convergent to make them impracticable for manual calculations, but they are adaptable to programming on automatic calculators. They have the characteristic that by far the majority of the work lies in computing the series coefficients for the basic generalized deflection mode of the rigidly supported strut. The eigenvalues for any combination of the elastic supports can then be found without much additional work.

II. DESCRIPTION OF PROBLEM

1. Geometry and Notation

Typical wind-tunnel installations showing thin cantilever struts on elastic supports are sketched on Figure 1. The notation used to describe the strut geometry is defined on Figure 2, and in the Table of Notation.

The strut is required to be thin for aerodynamic applications in order to present the least frontal area practicable, for the required structural stiffness. To simplify the analysis it is assumed that the strut is so thin as to make its bending rigidity about the short principal axis of the cross sections very large compared with its rigidity about the long axis. The strut is also represented as having such symmetry properties in planform and cross section as may be necessary in order to possess an elastic axis, coincident with the neutral axis, with both axes straight and perpendicular to the root section.

The analysis then proceeds as a one-dimensional beam problem in which cross section changes and variations in elastic moduli (if any) are represented entirely in the variation of the bending and torsional rigidities, $B(x) = E(x)I_y(x)$ and $C(x)$ respectively. These variations are given by the dimensionless functions

$$f = \frac{B}{B_o}, \qquad g = \frac{C}{C_o}$$

where B_o, C_o are rigidities of the strut root section (Figure 2). The problem may be treated without additional difficulty for the function

$f = f(x)$ different from $g = g(x)$, and the separate identities of these functions is maintained in the analysis. However, as can easily be shown (see Refs. 5 or 7, for example), for thin sections the two functions may be considered identical. A wide class of practicable strut geometries can be represented by the functions

$$f(x) = g(x) = (1 - hx)^m = \xi^m , \quad m = 0, 1, 2 \ldots$$

The notation used here is defined in Figure 2. It is convenient to define the parameter

$$\bar{\ell} = 1 - h\ell$$

such that the "taper ratio" for the rigidity is given by

$$\frac{B_{tip}}{B_o} = \frac{C_{tip}}{C_o} = (\bar{\ell})^m$$

Also then $\bar{\ell} = 1$ for a uniform strut (any m), that is, for constant rigidity. For $\bar{\ell} = 0$ the strut is "pointed", i.e. the rigidity vanishes at the tip as ξ^m (see diagram, Figure 2). When m is an integer the strut assumes various "natural" taper rates, as summarized on Figure 2.

2. Formulation of Basic Equations

As mentioned in the previous section the problem is formulated as a one-dimensional elastic beam problem. It is also linearized, the justification for which, in the case of rigid strut support at least, has been given adequate attention by previous investigators. We are interested in the lateral buckling phenomenon involving both a bending and a twisting deformation out of the plane of the applied loads. The loads are applied to the strut tip in the major principal plane of the undeflected strut (Figures 1 and 3). An axial load T acts at the tip colinearly with the undeflected neutral axis, positive when it is a compressive load. A lateral load P is applied at the neutral axis of the tip. In the deflected configuration these loads are assumed to remain parallel to their initial directions, viz. to the x and y axes, respectively (Figure 3).

Since the loads are not applied eccentrically to the elastic-neutral axis, the deflections u and β of the buckled mode are not generated linearly or even continuously in accordance with increasing P and T, but occur suddenly* at some critical combination(s) of the load. The critical load then corresponds to the eigenvalues of the boundary value problem in terms of which the physical situation is approximated mathematically.

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* In practice it is almost impossible to avoid small eccentricities in applying the load so that the buckled mode grows rapidly but continuously, and not suddenly, at critical loading.

In the above interpretation the critical load is defined as well by infinitesimally small magnitudes of buckled deflections as it is by large ones. Indeed as is well known, in the linear buckling theory the magnitudes of the deflections are left undetermined. Hence the steps toward linearization which depend upon the assumption of small deflections can be justified philosophically. The results of linear theory for the problem of interest here, but with rigid supports, have also been checked experimentally by numerous investigators with excellent agreement (see list of references). The present problem is linearized by the assumption that the slope of the neutral axis is small, $du/dx << 1$, and hence that the curvature is given by

$$\frac{1}{\rho} = \frac{d^2u/dx^2}{\left[1 - (\frac{du}{dx})^2\right]^{3/2}} \doteq \frac{d^2u}{dx^2}$$

The result of including the complete expression for curvature, making the beam problem non-linear, is to remove the indeterminacy of the deflection magnitudes (see Ref. 1, Art. 13). The result obtained for the critical load remains the same in the non-linear problem as that obtained in the linear problem, however. These circumstances for the beam are in direct contrast to the situation for shells where not only is a linear theory inadequate and a non-linear theory necessary to an adequate description of the physical problem, but in many instances a satisfactory non-linear theory remains to be formulated.

It is further assumed that compared with the bending about the major principal axis, the bending about the minor axis of the narrow cross sections is negligibly small. This means that lateral deflections of the elastic-neutral axis occur entirely in the x - u plane, as shown in Figure 3. For thin sections and small deflections, as a matter of fact, it can be shown that the two bending deflections are not coupled in this problem, i.e. the critical loads are independent of bending about the minor axis (see, for example, Ref. 1, Art. 46).

In describing the twist of the beam the assumption is made that the cross sections are free to warp. If this condition is not met (as at the built-in root section, or perhaps at the tip if fastened to a pod, etc.) then a differential bending deformation stiffens the structure in torsion. The effect can be shown to be proportional to the third derivative, $d^3\beta / dx^3$. This additional stiffness is not of importance for thin cross sections, unless the ratio of strut length to chord is not large (Refs. 10, 11). Thus for thin sections and long struts, the assumption may be considered valid, as has been verified experimentally. It should be noted in this connection that, with other conditions given, the length of the strut is maximal for the buckling problem.

With these preliminary remarks it is possible to write down conditions for equilibrium of the deflected strut. At a station x along the strut the bending moment about the major principal axis is given by (Figure 3)

$$My' = \frac{B}{\rho} \doteq B \frac{d^2u}{dx^2}$$

The twisting moment about the elastic axis is given by

$$-Mx' = \frac{C}{\cos(x, x')} \frac{d\beta}{dx} \doteq C \frac{d\beta}{dx}$$

The system of axes x', y', u' moves with the cross section as it rotates and twists in the deflected mode. For small deflections the direction cosines of these moving axes with respect to the fixed axes x, y, u can be approximated by $\cos(x', x) = 1$, $\cos(x', u) = du/dx$, $\cos(y', u) = \beta$, etc. Then the moments are given by

$$My' = My \cos(y', y) + Mx \cos(y', x) + Mu \cos(y', u)$$

$$\doteq My + \beta Mu = T(u_{\ell} - u) + P(\ell - x)\beta$$

and

$$Mx' = Mx \cos(x', x) + My \cos(x', y) + Mu \cos(x', u)$$

$$\doteq Mx + Mu \frac{du}{dx}$$

$$= -P(u_{\ell} - u) + P(\ell - x)\frac{du}{dx}$$

The equilibrium conditions can now be stated in the fixed coordinate system x, y, u with the approximate expressions

$$B(x)\frac{d^2u}{dx^2} = P(\ell - x)\beta + T(u_\ell - u)$$

$$C(x)\frac{d\beta}{dx} = P(u_\ell - u) - P(\ell - x)\frac{du}{dx} \tag{2.1}$$

Equations 2.1 are a coupled pair of linear ordinary equations with variable coefficients. They are the basic equations for the analysis.

The boundary conditions to be satisfied by the system of equations may be summarized as follows, in the case of a rigidly built-in root section.

$$
\begin{aligned}
&(1)\quad u = 0 \quad \text{at} \quad x = 0 \\
&(2)\quad \frac{du}{dx} = 0 \quad \text{at} \quad x = 0 \\
&(3)\quad \beta = 0 \quad \text{at} \quad x = 0 \\
&(4)\quad \frac{d^2u}{dx^2} = 0 \quad \text{at} \quad x = \ell \\
&(5)\quad \frac{d\beta}{dx} = 0 \quad \text{at} \quad x = \ell
\end{aligned}
\tag{2.2}
$$

Conditions 1 through 3 are on displacements at the root section; conditions 4 and 5 express the fact that there are no couples (bending and torsion) applied to the tip.* Actually the latter are automatically satisfied for any functions which satisfy the basic equations 2.1.

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* Certain modifications in the case of pointed tips are discussed in Article III, 5.

For an elastically built-in root section conditions 2 and 3 above are changed to accommodate assumed linear spring rates between the rotational deflections of the root section and the corresponding restraining moments. The section is assumed rigid in rotation about its minor principal axis without loss of generality, in view of the assumption of effectively infinite flexural rigidities about the section minor axis. An elastic deformation at the root section about its minor axis then only reproportions the ratio between the load components P and T in a completely predictable way. In the case of the other two degrees of freedom let e_1 and e_2 be arbitrary constant spring rates such that

$$(2) \quad e_2 \left(\frac{du}{dx}\right)_o = My_o = B_o \left(\frac{d^2u}{dx^2}\right)_o$$

$$(2.3)$$

$$(3) \quad e_1 \beta_o = -Mx_o = C_o \left(\frac{d\beta}{dx}\right)_o$$

These conditions replace those which are correspondingly numbered in equations 2.2. For $e = \infty$ they are the same as for rigid supports; for $e = 0$ they correspond to infinitely weak or pinned supports for which, of course, the position of the cantilever is statically indeterminate.

III. DEVELOPMENT OF GENERALIZED EQUATION

1. Derivations

The basic equations 2.1 for strut equilibrium in buckled mode have been obtained in the preceding section, with boundary conditions 2.2 and 2.3. These equations are coupled if $P \neq 0$, and must be solved simultaneously. It is convenient to introduce the change of independent variable from dimensional x to dimensionless ξ as (Figure 3)

$$\xi = \frac{\ell - x}{\ell}$$

$$\frac{d}{dx} = -\frac{1}{\ell}\frac{d}{d\xi} \tag{3.1}$$

$$\frac{d^2}{dx^2} = \frac{1}{\ell^2}\frac{d^2}{d\xi^2}$$

and to change the dependent variable u to dimensionless form

$$v = \frac{u_\ell - u}{\ell} \tag{3.2}$$

then

$$\frac{du}{dx} = \frac{dv}{d\xi}$$

$$\frac{d^2u}{dx^2} = -\frac{1}{\ell}\frac{d^2v}{d\xi^2} \tag{3.3}$$

and the equations become

$$\text{(a)} \quad -\frac{B(\xi)}{\ell^2}\frac{d^2v}{d\xi^2} = P\xi\beta + Tv = -b\frac{d^2v}{d\xi^2}$$

$$(3.4)$$

$$\text{(b)} \quad -\frac{C(\xi)}{\ell^2}\frac{d\beta}{d\xi} = Pv - P\xi\frac{dv}{d\xi} = -c\frac{d\beta}{d\xi}$$

where $b(\xi) = \frac{1}{\ell^2}B(\xi)$ and $c(\xi) = \frac{1}{\ell^2}C(\xi)$ are introduced for brevity. With the identity

$$v - \xi\frac{dv}{d\xi} = -\xi^2\frac{d}{d\xi}\left(\frac{v}{\xi}\right)$$

equation 3.4b becomes

$$\text{(c)} \qquad c\frac{d\beta}{d\xi} = P\xi^2\frac{d}{d\xi}\left(\frac{v}{\xi}\right) \qquad (3.4)$$

A single differential equation can be determined from the coupled equations 3.4a, b either by eliminating v, or by elimination of β. In the former case equation 3.4b, differentiated once, yields, with substitution from equation 3.4a:

$$\frac{d}{d\xi}\left(c\frac{d\beta}{d\xi}\right) = P\xi\frac{d^2v}{d\xi^2}$$

$$(3.5)$$

$$= -\frac{P\xi}{b}(P\xi\beta + Tv)$$

After multiplying by $b(\xi)$ and differentiating once more

$$\text{(a)} \qquad -\frac{d}{d\xi}\left[b\frac{d}{d\xi}\left(c\frac{d\beta}{d\xi}\right)\right] = \left[P^2\frac{d}{d\xi}(\xi^2\beta) + PT\frac{d}{d\xi}(\xi v)\right] \qquad (3.6)$$

From equation 3.4c there is

$$\text{(b)} \qquad Tc\frac{d\beta}{d\xi} = PT\xi^2\frac{d}{d\xi}\left(\frac{v}{\xi}\right) = TP\left[\frac{d}{d\xi}(\xi v) - 2v\right] \qquad (3.6)$$

Solving for v from equations 3.6a, b, as

$$-PTv = \frac{1}{2}\left\{\frac{d}{d\xi}\left[b\frac{d}{d\xi}\left(c\frac{d\beta}{d\xi}\right)\right] + P^2\frac{d}{d\xi}(\xi^2\beta) + Tc\frac{d\beta}{d\xi}\right\},$$

and substituting into equation 3.5, gives

$$\frac{d}{d\xi}\left(c\frac{d\beta}{d\xi}\right) + \frac{P^2\xi^2\beta}{b} - \frac{\xi}{2b}\left\{\frac{d}{d\xi}\left[b\frac{d}{d\xi}\left(c\frac{d\beta}{d\xi}\right)\right] + P^2\frac{d}{d\xi}(\xi^2\beta) + Tc\frac{d\beta}{d\xi}\right\} = 0$$

which reduces to the third order equation in β:

$$\xi\frac{d}{d\xi}\left[b\frac{d}{d\xi}\left(c\frac{d\beta}{d\xi}\right)\right] - 2b\frac{d}{d\xi}\left(c\frac{d\beta}{d\xi}\right) + \xi\left[P^2\xi^2\frac{d\beta}{d\xi} + Tc\frac{d\beta}{d\xi}\right] = 0 \quad (3.7)$$

On the other hand β can be eliminated from the set 3.4 by differentiating equation 3.4a twice and substituting for β from equation 3.4c:

$$-\frac{d}{d\xi}\left(b\frac{d^2v}{d\xi^2}\right) = P\left(\xi\frac{d\beta}{d\xi} + \beta\right) + T\frac{dv}{d\xi}$$

$$-\frac{d^2}{d\xi^2}\left(b\frac{d^2v}{d\xi^2}\right) = P\left(\xi\frac{d^2\beta}{d\xi^2} + 2\frac{d\beta}{d\xi}\right) + T\frac{d^2v}{d\xi^2}$$

$$\frac{d^2}{d\xi^2}\left(b\frac{d^2v}{d\xi^2}\right) + T\frac{d^2v}{d\xi^2} + P^2\left\{\xi\frac{d}{d\xi}\left[\frac{\xi^2}{c}\frac{d}{d\xi}\left(\frac{v}{\xi}\right)\right] + 2\frac{\xi^2}{c}\frac{d}{d\xi}\left(\frac{v}{\xi}\right)\right\} = 0$$

The quantity in curly brackets is of the form

$$\xi \frac{d\gamma}{d\xi} + 2\gamma = \frac{1}{\xi}\frac{d}{d\xi}(\xi^2 \gamma)$$

and so the last equation can be written more compactly as

$$\frac{d^2}{d\xi^2}\left(b\frac{d^2 v}{d\xi^2}\right) + T\frac{d^2 v}{d\xi^2} + \frac{P^2}{\xi}\frac{d}{d\xi}\left[\frac{\xi^4}{c}\frac{d}{d\xi}\left(\frac{v}{\xi}\right)\right] = 0 \qquad (3.8)$$

Equation 3.8 is a fourth order equation in the bending deflection, in its most general form, for arbitrary $b(\xi)$ and $c(\xi)$. It reduces to the particular fourth order equations used in the analysis by Martin when Martin's taper functions are substituted, viz. for $b = B_o/\ell^2$ and $c = C_o/\ell^2$ (i.e. uniform strut)

$$\frac{d^4 v}{d\xi^4} + \left(\frac{T\ell^2}{B_o} + \frac{P^2\ell^4}{B_o C_o}\xi^2\right)\frac{d^2 v}{d\xi^2} + 2\frac{P^2\ell^4}{B_o C_o}\left(\xi\frac{dv}{d\xi} - v\right) = 0$$

(see Ref. 7, p. 25) and for $\frac{b\ell^2}{B_o} = \frac{c\ell^2}{C_o} = \xi$ (Martin's pointed taper)

$$\xi^2\frac{d^4 v}{d\xi^4} + 2\xi\frac{d^3 v}{d\xi^3} + \left(\frac{T\ell^2}{B_o}\xi + \frac{P^2\ell^4}{B_o C_o}\xi^2\right)\frac{d^2 v}{d\xi^2} + \frac{P^2\ell^4}{B_o C_o}\left(\xi\frac{dv}{d\xi} - v\right) = 0$$

(see Ref. 7, p. 65)

At this point* it is convenient to introduce the dimensionless parameters for the loads

$$\tau = \frac{T\ell^2}{B_o}, \quad \phi = \frac{P^2\ell^4}{B_o C_o} \qquad (3.9)$$

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* The convenience of definition of ϕ is not at once evident in equations 3.4.

and the dimensionless functions for the taper functions (Figure 2)

$$\frac{B(\xi)}{B(1)} = \frac{\ell^2 b(\xi)}{B_o} = f(\xi) \,, \quad \frac{C(\xi)}{C(1)} = \frac{\ell^2 c(\xi)}{C_o} = g(\xi) \qquad (3.10)$$

Equations 3.7 and 3.8 then become

$$\xi \frac{d}{d\xi}\left[f \frac{d}{d\xi}(g \frac{d\beta}{d\xi})\right] - 2f \frac{d}{d\xi}(g \frac{d\beta}{d\xi}) + \xi\left[\phi \xi^2 \frac{d\beta}{d\xi} + \tau g \frac{d\beta}{d\xi}\right] = 0 \qquad (3.11)$$

$$\frac{d^2}{d\xi^2}(f \frac{d^2 v}{d\xi^2}) + \tau \frac{d^2 v}{d\xi^2} + \frac{\phi}{\xi} \frac{d}{d\xi}\left[\frac{\xi^4}{g} \frac{d}{d\xi}(\frac{v}{\xi})\right] = 0 \qquad (3.12)$$

The critical loads could now be found as eigenvalues of either of these equations in conjunction with boundary conditions 2.2 or 2.3; for example the eigenvalues of τ for a selected value of the ratio $p = \phi / \tau$. The corresponding solutions $\beta(\xi)$ and $v(\xi)$ are the eigenfunctions of the problem, i.e. the deflection patterns of the buckled modes. There is no immediate interest in these functions; the linear theory does not reveal their magnitudes within an unknown multiplicative constant, as has been remarked earlier, and the eigenvalues are always determined first in any event.

If left in the form of a pair of "independent" equations such as equations 3.11 and 3.12, the statement of the problem lacks an inherent unification. It remains to be shown that both of equations 3.11 and 3.12 can be transformed into identical statements, which happen also to effect a considerable simplification of the problem. To this end we make a change of dependent variable, writing

$$\text{(a)} \qquad z \equiv \xi^2 \frac{d}{d\xi}(\frac{v}{\xi}) = \xi \frac{dv}{d\xi} - v \qquad (3.13)$$

and

(b) $$\frac{dz}{d\xi} = \xi \frac{d^2 v}{d\xi^2} \qquad (3.13)$$

(c) $$\frac{d^2 z}{d\xi^2} = \xi \frac{d^3 v}{d\xi^3} + \frac{d^2 v}{d\xi^2}$$

Reference to equation 3.13c will be made later. With the transformations 3.13a, b equation 3.12 is reduced from fourth to third order:

$$\xi \frac{d^2}{d\xi^2}\left(\frac{f}{\xi}\frac{dz}{d\xi}\right) + \tau \frac{dz}{d\xi} + \phi \frac{d}{d\xi}\left(\frac{\xi^2}{g} z\right) = 0 \qquad (3.14)$$

Now with the identity

$$\xi \frac{d^2 \gamma}{d\xi^2} \equiv \frac{d}{d\xi}\left[\xi^2 \frac{d}{d\xi}\left(\frac{\gamma}{\xi}\right)\right]$$

and with

$$\gamma = \frac{f}{\xi}\frac{dz}{d\xi}$$

the first term of equation 3.14 is integrable. Since both v and ξ vanish at the tip, then z and $\frac{dz}{d\xi}$ are also zero there. With these conditions, equation 3.14 can be integrated at once to give

$$\xi^2 \frac{d}{d\xi}\left(\frac{f}{\xi^2}\frac{dz}{d\xi}\right) + \left(\tau + \frac{\xi^2 \phi}{g}\right) z = 0 \qquad (3.15)$$

which is the final form desired.

On the other hand the reduction to the form of equation 3.15 can be started from equation 3.11, in . Let the change in dependent variable be*

$$\theta = \frac{C_o}{P\ell^2}\, g(\xi)\frac{d\beta}{d\xi} \qquad (3.16)$$

Then equation 3.11 becomes

$$\xi\frac{d}{d\xi}(f\frac{d\theta}{d\xi}) - 2f\frac{d\theta}{d\xi} + \xi(\tau + \frac{\xi^2\phi}{g})\theta = 0$$

and with the identity

$$\gamma' - 2\frac{\gamma}{\xi} = \xi^2\frac{d}{d\xi}(\frac{\gamma}{\xi^2})$$

$$\gamma = f\frac{d\theta}{d\xi}$$

it is

$$\xi^2\frac{d}{d\xi}(\frac{f}{\xi^2}\,\frac{d\theta}{d\xi}) + (\tau + \frac{\xi^2\phi}{g})\theta = 0 \qquad (3.17)$$

The last equation, developed from 3.11, is identical with equation 3.15, which was deduced from 3.12, provided that

$$\theta = z \qquad (3.18)$$

Upon substituting into this last equation from the definitions 3.13 a and 3.16, it is seen that

$$\frac{C_o}{P\ell^2}\, g(\xi)\frac{d\beta}{d\xi} = \xi^2\frac{d}{d\xi}(\frac{v}{\xi}) ,$$

---

* The multiplier $\frac{C_o}{P\ell^2}$ is for later convenience, in equation 3.18.

or

$$\frac{C(\xi)}{\ell^2}\frac{d\beta}{d\xi} = P\xi^2\frac{d}{d\xi}\left(\frac{v}{\xi}\right) ,$$

and this is the same as equation 3.4c for the equilibrium in torsion.

Thus condition 3.18 is an identity, expressing a required relation between the torsional and bending deflections, and representing the desired unification of the mathematical statement for the eigenvalues. The variable z (or θ) could be called a generalized deflection parameter for the problem, for which there has occurred a reduction to a second order differential equation. With a solution of the eigenvalues for $\mathcal{T}$ and ϕ, and hence of the eigenfunctions $z(\xi)$, the deflections $v(\xi)$ and $\beta(\xi)$ can be obtained from additional quadratures through equations 3.13a and 3.16 respectively.

With the differential equation in hand, the boundary conditions must now be modified to the new variables and to accomodate the reduced order of the equation.

2. Boundary Conditions

The boundary conditions stated in equations 2.2 and 2.3 were

at $x = 0$: (root)

(1) $u = 0$

(2) $e_2\left(\frac{du}{dx}\right)_o = B_o\left(\frac{d^2u}{dx^2}\right)_o$

(3) $e_1(\beta)_o = C_o\left(\frac{d\beta}{dx}\right)_o$

at $x = \ell$: (tip)

(4) $\left(\frac{d^2u}{dx^2}\right)_\ell = 0$

(5) $\left(\frac{d\beta}{dx}\right)_\ell = 0$

With the new variables ξ and v, given by the definitions 3.1 and 3.2, and introducing the dimensionless flexibility coefficients

$$\mu = \frac{C_o}{\ell e_1}$$
$$\epsilon = \frac{B_o}{\ell e_2} \qquad (3.19)$$

the boundary conditions are restated as follows:

at $\xi = 0$: (tip)

(a) $\frac{d\beta}{d\xi}(0) = 0$

(b) $\frac{d^2 v}{d\xi^2}(0) = 0$

(c) $v(0) = 0$ (3.20)

at $\xi = 1$: (root)

(d) $\beta(1) = -\mu \frac{d\beta}{d\xi}(1)$

(e) $\frac{dv}{d\xi}(1) = -\epsilon \frac{d^2 v}{d\xi^2}(1)$

It has been pointed out previously that the first two conditions as stated in equations 3.20, expressing the absence of applied couples at the tip, are inherently satisfied by functions which satisfy the differential equation. The eigenvalue problem is therefore governed by the last three of conditions 3.20

In terms of the new dependent variable, z, given by equation 3.13a, boundary condition (c) becomes

$$z(0) = 0 \qquad (3.21)$$

The condition (e) gives

$$z(1) = -\epsilon \frac{d^2v}{d\xi^2}(1) - v(1)$$

or, from 3.13b .

$$z(1) = -\epsilon \frac{dz}{d\xi}(1) - v(1) \tag{3.22}$$

With condition (d), and using 3.13b, 3.4a, 3.9, 3.16 and 3.18, in that order:

$$\begin{aligned} \frac{dz}{d\xi}(1) &= \frac{d^2v}{d\xi^2}(1) = \frac{-\ell^2}{B_o}\left[P\beta(1) + Tv(1)\right] \\ &= \frac{\ell^2}{B_o}\left[P\mu\frac{d\beta}{d\xi}(1) - Tv(1)\right] \\ &= \frac{P^2\ell^4}{B_oC_o}\mu\theta(1) - \frac{T\ell^2}{B_o}v(1) \\ &= \mu\phi z(1) - \tau v(1) \end{aligned} \tag{3.23}$$

From the last two numbered equations, eliminating z:

$$\begin{aligned} -\frac{dz}{d\xi}(1) &= \mu\phi\left[\epsilon\frac{dz}{d\xi}(1) + v(1)\right] + \tau v(1) \\ \frac{1}{v(1)}\frac{dz}{d\xi}(1) &= -\left(\frac{\tau + \mu\phi}{1 + \epsilon\mu\phi}\right) \end{aligned} \tag{3.24}$$

Then, upon substituting back into 3.22:

$$\begin{aligned} \frac{z(1)}{v(1)} &= -1 - \frac{\epsilon}{v(1)}\frac{dz}{d\xi}(1) \\ &= -\left(\frac{1 - \epsilon\tau}{1 + \epsilon\mu\phi}\right) \end{aligned} \tag{3.25}$$

Equations 3.21, 3.24 and 3.25 replace conditions 3.20c, d, e as boundary conditions, applicable to the differential equation 3.15 in z. The latter is a second order equation. We can divide equation 3.24 by equation 3.25 and obtain the single homogeneous boundary condition at the root as given in equations 3.26 which follow.

The final reduced formulation of the boundary value problem can now be stated in the following homogeneous system of the generalized deflection function z (or θ) — in summary from equations 3.18, 3.21, 3.24 and 3.25:

$$\xi^2 \frac{d}{d\xi}\left[\frac{f(\xi)}{\xi^2} \frac{dz}{d\xi}\right] + \left[\tau + \frac{\xi^2 \phi}{g(\xi)}\right] z = 0$$

$$z(0) = 0 \qquad (3.26)$$

$$\frac{dz}{d\xi}(1) = \frac{\tau + \mu\phi}{1 - \epsilon\tau} z(1)$$

Of the two generalized parameters, z and θ, z is the more convenient to use, in general, since θ is defined explicitly in terms of one of the taper functions, $g(\xi)$. The normal range of the flexibility coefficients ϵ and μ lies between 0 (for rigid strut support) to ∞ (for a weak support approaching a pinned or swivel joint). The effect of ϵ or μ vanishes in the problem as the associated load vanishes, as is to be expected. The constant μ represents the twisting spring rate, effective only if the lateral load is present, i.e. if $\phi \neq 0$; while the constant ϵ represents the bending spring rate, effective only if there is an axial load, $\tau \neq 0$. These relations are slightly more complex in the individual conditions 3.24 and 3.25. The loading parameter ϕ is proportional to P^2 and consequently is non-negative; this corresponds to the fact that the problem is identical for either direction of application of the lateral load. The axial loading parameter τ, on the other hand may be positive or negative. The problem is defined for negative as well as positive spring rates, ϵ^{-1} and μ^{-1}, if an application can be found for the former.

It will presently be seen that the boundary conditions of the system 3.26 represent all five of the conditions 3.20 in compact form, while the differential equation is a compact statement of many problems heretofore stated and solved individually. In deriving the equation it is perhaps more direct to proceed from the equation in β than from the equation in v. However the boundary conditions are conceptually less difficult to work with in z than in θ.

3. Some Limiting Cases of the Reduced Equation

It is of interest to apply equation 3.26 to some of the problems which have already been solved and which are limiting or degenerate cases of the general equation, in order to establish its consistency with previous theory.

(a) Martin's solution

The nearest approach to the general coupled problem has been made by Martin in Ref. 7. It has been shown previously how equation 3.8 in the bending deflection, v, can be put into agreement with Ref. 7, for the cases of uniform and linear rigidity distribution considered there.

Equation 3.8 would be regenerated from equation 3.26 by differentiation, and substitution from 3.13a. However it will be shown in Section IV that numerical calculations from the system 3.26 agree with those of Ref. 7.

(b) Prandtl problem ($\tau = 0$)

Equation 3.26 bears in some respects a resemblance to its limiting form for the case of zero axial load (compare equations 3.27 and 3.28 below). Of course if left in terms of z, equation 3.26 becomes, for $\tau = 0$

$$\frac{d}{d\xi}\left(\frac{f}{\xi^2}\frac{dz}{d\xi}\right) + \frac{\phi}{g} z = 0$$

$$z(0) = 0 \tag{3.27}$$

$$\frac{dz}{d\xi}(1) = \mu\phi z(1)$$

But from equations 3.16 and 3.18 z may be replaced by $g\frac{d\beta}{d\xi}$, and then after one integration the above is

$$\frac{f}{\xi^2}\frac{d}{d\xi}\left(g\frac{d\beta}{d\xi}\right) + \phi\beta = \text{const.}$$

However, the integration constant is zero as can be verified by substitution from equation 3.4, and consequently there is obtained the final form in β:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(g\frac{d\beta}{d\xi}\right) + \frac{\phi}{f}\beta = 0 \tag{3.28}$$

It is worthwhile also to trace the retrotransformation of the boundary conditions as a further check of system 3.26. The first of these becomes immediately (the special case of the "pointed" tip, $g(0) = 0$, excluded)

$$\frac{d\beta}{d\xi}(0) = 0 \tag{3.29}$$

The second can be written easily with the aid of equation 3.28, for since $f(1) = g(1) = 1$, then

$$\frac{dz}{d\xi}(1) = \mu\phi z(1)$$

becomes

$$\frac{d}{d\xi}\left[g\frac{d\beta}{d\xi}\right](1) = \mu\phi\frac{d\beta}{d\xi}(1)$$

or

$$\beta(1) = -\mu\frac{d\beta}{d\xi}(1) \tag{3.30}$$

Equation 3.28 is identical with the equation which can be derived from the set 3.4a, b for $\tau = 0$, as has been done, for example, by Richardson in Ref. 5, p. 4. The boundary conditions 3.29 and 3.30 are the same as conditions 3.20a, d.

Putting $f(\xi) = g(\xi) = 1$, the equation for the uniform strut is

$$\frac{d^2\beta}{d\xi^2} + \xi^2\phi\beta = 0 \tag{3.31}$$

(compare Ref. 1, p. 247).

(c) Euler problem ($\phi = 0$)

Now, putting $\phi = 0$, equation 3.26 becomes for purely axial load

$$\frac{d}{d\xi}\left(\frac{f}{\xi^2}\frac{dz}{d\xi}\right) + \frac{\tau}{\xi^2} z = 0$$

$$z(0) = 0 \qquad (3.32)$$

$$\frac{dz}{d\xi}(1) = \frac{\tau z(1)}{1 - \epsilon\tau}$$

Substituting for z from 3.13a, b, and integrating:

$$\frac{f}{\xi}\frac{d^2v}{d\xi^2} + \frac{\tau v}{\xi} = \text{const.}$$

But the integration constant is zero since $f(0)\dfrac{d^2v}{d\xi^2}(0) = 0$ and $v(0) = 0$; consequently the final form in v is

$$f\frac{d^2v}{d\xi^2} + \tau v = 0 \qquad (3.33)$$

The first boundary condition becomes, since $f(1) = g(1) = 1$:

$$v(0) = 0 \qquad (3.34)$$

The second boundary condition takes the form

$$\tau\left[\frac{dv}{d\xi}(1) - v(1)\right] = (1 - \epsilon\tau)\frac{d^2v}{d\xi^2}(1)$$

or

$$\tau\left[\frac{dv}{d\xi}(1) + \epsilon\frac{d^2v}{d\xi^2}(1)\right] = \frac{d^2v}{d\xi^2}(1) + \tau\frac{dv}{d\xi}(1) \qquad (3.35)$$

$$\frac{dv}{d\xi}(1) + \epsilon\frac{d^2v}{d\xi^2}(1) = 0$$

where the right side is zero, from equation 3.33.

The statements in the last three numbered equations are identical with equations 3.4a and 3.20c, e (see also Richardson, Ref. 5, p. 27; Timoshenko, Ref. 1, p. 65; Dinnik, Ref. 4), for the Euler column problem.

4. Discussion of the Differential Equation

The characteristics of the differential equation 3.26 can be examined by identifying its singular points. After expanding the first term in the equation, and dividing through by f(ξ), it assumes the second canonical form

$$\frac{d^2 z}{d\xi^2} + \left(\frac{1}{f}\frac{df}{d\xi} - \frac{2}{\xi}\right)\frac{dz}{d\xi} + \frac{1}{f}\left(\tau + \frac{\xi^2 \phi}{g}\right) z = 0$$

or

$$\frac{d^2 z}{d\xi^2} + R(\xi)\frac{dz}{d\xi} + S(\xi) z = 0 \qquad (3.36)$$

$$R(\xi) = \frac{1}{f}\frac{df}{d\xi} - \frac{2}{\xi}\ , \qquad S(\xi) = \frac{1}{f}\left(\tau + \frac{\xi^3 \phi}{g}\right)$$

A specific class of taper functions will now be introduced for which it is intended to obtain solutions (see Figure 2, and the discussion in Article II, 1), viz.

$$f(\xi) = g(\xi) = \zeta^m\ , \quad m = 0, 1, 2 \ldots, \qquad (3.37)$$

where

$$\zeta = (1 + hx) = \frac{\xi + a}{1 + a} = (1 - \bar{\ell})(\xi + a)\ ,$$

with

$$a = \frac{\bar{\ell}}{1 - \bar{\ell}}$$

The strut shapes to be considered range from the uniform strut, denoted by $\zeta = 1$ or $a = \infty$, to the "pointed strut", for which $\zeta = \xi$ or $a = 0$. For these shapes the derivative in the coefficient $R(\xi)$ can be written

$$\frac{1}{f}\frac{df}{d\xi} = \frac{m}{\zeta} = m\frac{1 + a}{\xi + a}$$

Then the coefficients themselves become

$$R(\xi) = m\frac{1 + a}{\xi + a} - \frac{2}{\xi}$$

(3.38)

$$S(\xi) = \left(\frac{1 + a}{\xi + a}\right)^m \left[\tau + \phi\xi^2\left(\frac{1 + a}{\xi + a}\right)^m\right]$$

A general scheme for the solution* of the problem consists in finding a power series solution to the differential equation about the point $\xi = 0$. The coefficients of the series will be polynomials in , for an assumed fixed ratio, $p = \phi/\tau$. This series will satisfy the first boundary conditions in 3.26 for any value of τ. The result of applying the second boundary condition will be a power series in τ. The minimum eigenvalue for τ will be the least root of the series. These

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* The form of $R(\xi)$ suggests immediately the transformation $z = \bar{z}\, f/\xi^{1/2}$ which will eliminate the first derivative from the differential equation. This does not alter the singularities, however; it tends to complicate the boundary conditions; and anyhow on investigation it has not proved worth while.

calculations are carried out in Section IV. The mechanics of the solution involve identifying the roots of the indicial equation, and writing recurrence relations for the coefficients of the power series in z. Prior to this it is necessary to determine, however, that the series solution exists and that it will converge throughout the region of interest. It is also of interest to the success of numerical calculations that the series be as rapidly convergent as possible. These latter questions will now be examined by looking at the singularities of equation 3.36 with the coefficients 3.38, in the light of the theory of linear differential equations.

(a) Singularities of the differential equation

The general taper case of equations 3.36 and 3.38 will be defined as corresponding to the parameter ranges $m \neq 0$ and $0 < \tilde{\ell} < 1$. Certain other cases will be identified individually.

It will be observed that the differential equation has in the general taper case three distinct singular points*, which are $\xi = 0$, $\xi = -a$ and the point at infinity. These are shown on Figure 7a. It will be convenient in what follows to consider both z and ξ to be complex quantities, as necessary. Then while the problem at hand is defined on the axis of reals in the complex ξ-plane, it will be possible also to discuss the circle of convergence of the complex power series in complex ξ. For the general case of the functions 3.38 as just defined,

---

* The points of $R(\xi)$ and $S(\xi)$ at which they are regular functions are called ordinary points of the differential equation. The singular points are points for which $R(\xi)$ and $S(\xi)$ are not regular. The singular point $\xi = 0$ is itself said to be regular, for example, if $\xi R(\xi)$ and $\xi^2 S(\xi)$ are regular at $\xi = 0$. Otherwise $\xi = 0$ is said to be an irregular singularity (see Ref. 12, Chapter 10).

every point in the ξ-plane is ordinary except the three singular points on the real axis. A power series solution can be written about any such ordinary point, convergent within that circle which extends to the nearest singularity of the solution. The same can be done about any regular singular point of the equation. The procedure breaks down if the singularity about which the expansion is attempted is irregular.

In the present problem any solution, to be useful, must be defined over the closed interval, $0 \leq \xi \leq 1$. Hence a solution expanded about any point $\xi \neq 0$ might not be useful since it would not necessarily be defined at $\xi = 0$. Fortunately the point $\xi = 0$ is always regular for the general taper case as defined above ($\xi = 0$ is always regular as long as $\bar{\ell} > 0$). This means that a general series solution is always possible in powers of ξ, convergent within that circle which just extends to the nearest singularity. It follows that an indicial equation can be written at $\xi = 0$ for the differential equation in the general taper case $m \neq 0$, $0 < \bar{\ell} < 1$ (other cases for which this is true are to be discussed later), and in fact the indicial equation for the general case is always

$$\alpha(\alpha - 1) - 2 = \alpha(\alpha - 3) = 0, \tag{3.39}$$

with roots $\alpha = 0, 3$, as can be verified by inspection of the differential equation (3.36). Although the roots differ by an integer, the two solutions are linearly independent. Then the general solution is

$$z = a'_o \sum_{n=o}^{\infty} \left(\frac{a'_n}{a'_o}\right) \xi^n + a_o \sum_{n=o}^{\infty} \left(\frac{a_n}{a_o}\right) \xi^{n+3}$$

where a_o, a'_o are arbitrary constants to be determined. Since one of the boundary conditions 3.26 requires $z(0) = 0$, the constant $a'_o = 0$, and the general solution reduces to

$$z = \xi^3 \sum_{n=o}^{\infty} a_n \xi^n \tag{3.40}$$

There will be a recurrence relation giving the a_n for $n \geq 1$ in terms of a_o (see Section IV), and these will be polynomials in τ, as has been said, of ascending degree as n increases. The remaining boundary condition is to be satisfied at $\xi = 1$, giving a power series in τ which is equated to zero. It is necessary then that the series 3.40 converge, and converge as rapidly as possible in the closed interval $0 \leq \xi \leq 1$, in order to apply the second boundary condition without numerical difficulty.

Thus in the general taper case a solution has been found, provided the series does not diverge in the interval $0 \leq \xi \leq 1$. Actually it is not difficult to show that the series is not necessarily convergent for all values of $\bar{\ell}$. More precisely, for $\bar{\ell} \leq 1/2$ the radius of convergence is ≤ 1, so that the series 3.40 does not extend to the point $\xi = 1$. This is easily seen on Figure 13a, if it is remembered that $a = \frac{\bar{\ell}}{1 - \bar{\ell}}$, so that $0 \leq a \leq \infty$ correspondingly as $0 \leq \bar{\ell} \leq 1$. In particular $a = 1$ for $\bar{\ell} = 1/2$, and for $\bar{\ell} > 1/2$, $a > 1$ while for $\bar{\ell} < 1/2$, $a < 1$. Since the radius of convergence is equal to a (the point $\xi = -a$ is the singularity nearer to $\xi = 0$)*, the truth of the above follows at once.

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* The point $\xi = -a$, when $m > 1$ and $\phi \neq 0$, is an irregular singularity of the equation. Otherwise, when $\xi = -a$ is a regular singularity, it is not at once evident whether or not the radius of convergence is limited to a.

This requires that in the general case, in order to include $\bar{\ell} \leq 1/2$, the solution must be redefined. This will be done in paragraph f below, where it is accomplished by a transformation of the independent variable ξ, with corresponding changes in the locations of singularities. It is convenient, meanwhile, to discuss first some special cases for which the form of equation 3.36 remains applicable.

(b) $\bar{\ell} = 1$ (m = 0) ... a uniform strut

If $\bar{\ell} = 1$ (any m) the rigidities B and C are constant along the strut; the same is true if m = 0. In either event the differential equation (3.36) has the coefficients

$$R(\xi) = -\frac{2}{\xi}$$
$$S(\xi) = \tau + \phi\xi^2 \qquad (3.41)$$

since now $f(\xi) = g(\xi) = 1$. There are only two singularities of this differential equation, which is a special, confluent form of the general equation 3.38. That is as $\bar{\ell} \to 1$, $a \to \infty$, so that in the limit $\bar{\ell} = 1$ there is a confluence between two singularities as the one at $\xi = -a$ coincides with the one at infinity.

However the singularity at $\xi = 0$ remains regular, and the indicial equation 3.39 and hence the general solution 3.40 are applicable. In fact the radius of convergence of the latter is now infinite, and as one consequence of this it turns out that the series 3.40 is very rapidly convergent over the interval of interest, $0 \leq \xi \leq 1$. It is therefore quite easy to obtain solutions for the uniform beam (see Section IV).

(c) $\bar{\ell} = 0$, $m = 1$ ... Martin's taper

It has been shown in the preceding paragraph that the equation for the uniform strut ($\bar{\ell} = 1$) is a confluent form of the general differential equation 3.36, independent of the taper function exponent m. It will now be shown that the equations for "pointed struts", i.e. $\bar{\ell} = 0$, are another confluent form, in which the exponent m plays a decisive role. These will be considered separately for the several integer values of m.

For $m = 1$ and $\bar{\ell} = 0$ the taper function corresponds to the linear rigidity distributions, $f(\xi) = g(\xi) = \xi$, for which the differential equation has the coefficients

$$R(\xi) = -\frac{1}{\xi}$$
$$S(\xi) = \frac{\tau}{\xi} + \phi \qquad (3.42)$$

This is the tapered strut considered by Martin (Ref. 7). Since for $\bar{\ell} \to 0$, $a \to 0$ then at $\bar{\ell} = 0$ there is a confluence of the moveable singularity $\xi = -a$ with the fixed singularity at the origin (cf. Figure 7a). The singularity at $\xi = 0$ remains regular (for $m = 1$), however, so that an indicial equation and general power series solution can be written about the point $\xi = 0$. The remaining singularity is the point at infinity, so that in this case, just as for the uniform strut, the radius of convergence is infinite and the rate of convergence is extremely rapid at $\xi = 1$.

The indicial equation is now somewhat different, however, being

$$\alpha(\alpha - 2) = 0 \qquad (3.43)$$

with roots $\alpha = 0, 2$. Since, by the first of the boundary conditions 3.26, $z(0) = 0$, the general solution in this case is the Taylor series

$$z(\xi) = \xi^2 \sum_{n=o}^{\infty} a_n \xi^n \tag{3.44}$$

The details of this solution are worked out in Section IV.

(d) $\bar{\ell} \neq 0$ ... the general taper case

In paragraph a it was found that the solution to the general equation 3.36 for the taper functions 3.37, written as a power series in ξ, was not convergent at $\xi = 1$ for $\bar{\ell} \leq 1/2$. This is directly related to the movement of the point $\xi = -a$ within the unit circle (cf. Figure 7a) when $\bar{\ell} \leq 1/2$.

The difficulty can be removed by a homographic transformation such as that illustrated by the new independent variable w^* on Figure 7b, where

$$\frac{\xi}{a} = \frac{\bar{w}^*}{1 - \bar{w}^*}$$

$$\bar{w}^* = w^*(1 - \bar{\ell}) \tag{3.45}$$

In this transformation the points $\xi = 0, 1$ are mapped on $w^* = 0, 1$ respectively, i.e. the strut appears over the same range of variables in the two planes. However the singularity $\xi = -a$ is moved to the point at infinity in the w^*-plane, while the singularity at infinity in the ξ-plane becomes the moving singularity $w^* = \frac{1}{1 - \bar{\ell}}$ in the w^*-plane. These corresponding points are listed in the following table

	1	2	3	4
ξ	0	$-a$	$\pm\infty$	1
$\bar{w}^*$	0	$\pm\infty$	1	$(1-\bar{\ell})$
w^*	0	$\pm\infty$	$(1-\bar{\ell})^{-1}$	1

Over the range $0 < \bar{\ell} \leq 1$ the unit circle in w^* is now free of singularities and the solution to the differential equation in powers of w^* will be convergent at the strut root, $w^* = 1$. However for small values of $\bar{\ell}$ the moveable singularity moves up close to the point $w^* = 1$ from the right. Consequently the series can be expected to be slowly convergent for $\bar{\ell}$ near zero.

In fact as $\bar{\ell} \rightarrow 0$, $a \rightarrow 0$ and the entire ξ-plane collapses into the point $w^* = 1$. As $\bar{\ell} \rightarrow 1$, on the other hand, the moveable singularity moves out to $+\infty$, $w^* \rightarrow \xi$ and the solution approaches that of paragraph b.

The w^* transformation can be replaced by a somewhat better one, but the w^* transformation is of interest in comparison with others. Such another is the variable s selected by Richardson in his solution to the coupled problem in which he used variational methods (Ref. 5). Since this solution exhibits very strong tendency to numerical difficulty as $\xi \rightarrow 0$ and as $\xi \rightarrow 1$, the transformation to s will be discussed briefly. By definition

$$s = \frac{\zeta}{\bar{\ell}} = \frac{1 - hx}{\bar{\ell}} = \frac{\bar{\ell} + (1-\bar{\ell})\xi}{\bar{\ell}} = 1 + \frac{\xi}{a} \qquad (3.46)$$

(also $s = \frac{1}{1 - \overline{w}^*}$), and the following points correspond:

	1	2	3	4
ξ	0	-a	$\pm\infty$	1
s	1	0	$\pm\infty$	$\bar{\ell}^{-1}$

These are shown on Figure 7c. The interval representing the strut is $1 \leq s \leq 1/\bar{\ell}$, becoming vanishingly small for $\bar{\ell} \rightarrow 1$, and infinitely long for $\bar{\ell} \rightarrow 0$. However the characteristics of consequence, closely related to the arithmetical problems in Richardson's solution, are that for both $\bar{\ell} = 1$ and $\bar{\ell} = 0$ the transformation collapses, mapping the ξ-plane into $s = 1$ at $\bar{\ell} = 1$, and into $s = \infty$ at $\bar{\ell} = 0$.

This behavior is unavoidable near $\bar{\ell} = 0$, as long as the points $\xi = -a$ and $\xi = 0$ are both singular points of the differential equation. It will be noted in this connection that while the transformation used by Dinnik (see paragraph e above) is somewhat similar, viz. $\bar{\ell}$ s, it is straightforward at $\bar{\ell} = 0$. The success of the latter transformation is related to the fact that for purely axial loading (Dinnik's case) the differential equation is not singular at $\xi = 0$.

In the case of combined loads, on the other hand, both $\xi = 0$ and $\xi = -a$ are singular and the purpose of the transformation must be to develop an independent variable that holds the tip separate from the nearest singularity by at least the length of the interval to the root. This procedure will always break down as $\bar{\ell} \rightarrow 0$. However for $\bar{\ell} = 1$ the mapping should revert to an identical mapping, as is the case for w^*. This suggests that the difficulties in Ref. 5 near $\bar{\ell} = 1$ could be removed by a change from s to either w^* or w (see below).

It was seen that the transformation to w^* removed the confluence of singularities at the tip point $\xi = 0$, for $\bar{\ell} = 0$, and introduced a confluence from the right at the root point, $w^* = 1$. While it is not possible to avoid the latter, it is possible to devise a more uniform convergence of the singularities toward the unit circle. In the ξ-plane and in the w^*-plane one of the three singularities remains fixed at infinity. In the w-plane, which is now introduced, the two external singularities are made to approach the unit circle along the positive and negative real axes as $\bar{\ell} \to 0$. The transformation is given by mapping the point $\xi = -2a$ into the point at infinity in the w-plane:

$$\frac{\xi}{2a} = \frac{\bar{w}}{1 - \bar{w}}$$

$$\bar{w} = w\bar{a} \tag{3.47}$$

$$\bar{a} = 1 + 2a = \frac{1 + \bar{\ell}}{1 - \bar{\ell}}$$

The following points correspond

	1	2	3	4
ξ	0	-a	$\pm\infty$	1
$\bar{w}$	0	-1	1	$\bar{a}^{-1}$
w	0	$-\bar{a}$	$\bar{a}$	1

as shown on Figure 7d.

The strut lies in the interval $0 \leq w \leq 1$. The transformation becomes $w \to \xi$ as $\bar{\ell} \to 1$. While it is subject to the same type of difficulties at $\bar{\ell} = 0$ as is w^*, the singularity does approach the strut root point, $w = w^* = 1$, at a slower rate in the w-plane than in the w^*-plane, as $\tilde{\ell} \rightarrow 0$. The general series solution to the combined load problem is derived in powers of w in Section IV.

IV. SOLUTIONS

1. Power Series for Uniform Strut; m = 0 ($\tilde{\ell} = 1$)

The general equation 3.26 becomes, for constant torsional and flexural rigidities:

$$\frac{d^2z}{d\xi^2} - \frac{2}{\xi}\frac{dz}{d\xi} + (\tau + \phi\xi^2) = 0$$

$$z(0) = 0 \qquad (4.1)$$

$$\frac{dz}{d\xi}(1) = \frac{\tau + \mu\phi}{1 - \epsilon\tau}\, z(1)$$

(cf. paragraph b, Article III, 4). Putting

$$z = \sum_{o}^{\infty} a_n \xi^{n+\alpha}$$

$$\frac{dz}{d\xi} = \sum_{o}^{\infty} (n+\alpha)\, a_n \xi^{n+\alpha-1} \qquad (4.2)$$

$$\frac{d^2z}{d\xi^2} = \sum_{o}^{\infty} (n+\alpha)(n+\alpha-1) a_n \xi^{n+\alpha-2}$$

then the equation gives (after dividing out ξ^{α})

$$\sum_{o}^{\infty} (n+\alpha)(n+\alpha-3) a_n \xi^{n-2} + \sum_{o}^{\infty} \tau\, a_n \xi^n + \sum_{o}^{\infty} \phi a_n \xi^{n+2} = 0$$

or

$$\alpha(\alpha-3)a_o\xi^{-2}+(\alpha+1)(\alpha-2)a_1\xi^{-1}+\left[(\alpha+2)(\alpha-1)a_2+\tau a_o\right]+$$
$$\left[(\alpha+3)\alpha a_3+\tau a_1\right]\xi+\sum_{n=2}^{\infty}\left[(n+\alpha+2)(n+\alpha-1)a_{n+2}+\tau a_n+\phi a_{n-2}\right]\xi^n = 0$$

The roots of the indicial equation are $\alpha = 0, 3$; these give independent solutions although they differ by an integer. Corresponding to each root the constant a_o is arbitrary, and the succeeding even-numbered coefficients will be multiples of a_o. Having taken $a_o \neq 0$, then $a_1 = 0$, and thus all odd-numbered coefficients are also zero. In order to satisfy the first boundary condition, $z(0) = 0$, the general solution cannot contain the expansion from the root $\alpha = 0$. Therefore the general solution is

$$z = \xi^3 \sum_{o}^{\infty} a_n \xi^n \tag{4.3}$$

in which $a_n = 0$ when $n = 1, 3, 5, \ldots$ (4.4)

and

$$a_2 = \frac{-\tau}{2\cdot 5}a_o$$

$$a_{n+2} = \frac{-1}{(n+2)(n+5)}(\tau a_n + \phi a_{n-2}) \tag{4.5}$$
$$n = 2, 4, 6, \ldots$$

Thus for the Euler column ($\phi = 0$) problem, for example, letting $a_o = 1$, the coefficients are

$$a_2 = \frac{-\tau}{2\cdot 5}, \quad a_4 = \frac{\tau^2}{2\cdot 4\cdot 5\cdot 7}, \quad a_6 = \frac{-\tau^3}{2\cdot 4\cdot 6\cdot 5\cdot 7\cdot 9}, \quad \ldots$$

and

$$z = \xi^3 - \frac{\tau}{2\cdot 5}\xi^5 + \frac{\tau^2}{2\cdot 4\cdot 5\cdot 7}\xi^7 - \frac{\tau^3}{2\cdot 4\cdot 6\cdot 5\cdot 7\cdot 9}\xi^9 + \ldots$$

Applying the second boundary condition (4.1), for $\epsilon = 0$:

$$\frac{dz}{d\xi}(1) = 3 - \frac{5\tau}{2\cdot 5} + \frac{7\tau^2}{2\cdot 4\cdot 5\cdot 7} - \frac{9\tau^3}{2\cdot 4\cdot 6\cdot 5\cdot 7\cdot 9} + \ldots$$

$$+ \tau z(1) = \tau - \frac{\tau^2}{2\cdot 5} + \frac{\tau^3}{2\cdot 4\cdot 5\cdot 7} - \ldots$$

$$\frac{dz}{d\xi}(1) - \tau z(1) = 3(1 - \frac{\tau}{2!} + \frac{\tau^2}{4!} - \frac{\tau^3}{6!} + \ldots) = 0$$

or

$$\cos \tau^{1/2} = 0$$

or

$$\tau = \frac{\pi^2}{4} \qquad (4.6)$$

This is the result originally due to Euler for the uniform column (cf. Ref. 1, p. 66) on rigid support.

Similarly, for the Prandtl buckling problem ($\tau = 0$), with $a_o = 1$, then in addition to the results 4.4, $a_n = 0$ for n = 2, 6, 10, . . . , and

$$a_4 = -\frac{\phi}{4\cdot 7}, \quad a_8 = \frac{\phi^2}{4\cdot 8\cdot 7\cdot 11}, \quad a_{12} = \frac{-\phi^3}{4\cdot 8\cdot 10\cdot 7\cdot 11\cdot 15}, \ldots$$

Therefore

$$z = \xi^3 - \frac{\phi}{4\cdot 7}\xi^7 + \frac{\phi^2}{4\cdot 8\cdot 7\cdot 11}\xi^{11} - \frac{\phi^3}{4\cdot 8\cdot 10\cdot 7\cdot 11\cdot 15}\xi^{15} \ldots$$

Applying the boundary condition 4.1 at the strut root for this case:

$$\frac{dz}{d\xi}(1) = 3\left(1 - \frac{\phi}{3\cdot 4} + \frac{\phi^2}{3\cdot 4\cdot 7\cdot 8} - \frac{\phi^3}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12} + \ldots\right) \tag{4.7}$$

$$= 0$$

This agrees with Prandtl's result (Ref. 1, p. 247) and could of course also be written in Bessel's functions.

Turning back now to the combined loads, the numerical solutions can be carried out with only slightly more complication, including also the effects of elastic supports (i.e. $\epsilon \neq 0$, $\mu \neq 0$). Putting

$$p = \frac{\phi}{\tau} \tag{4.8}$$

and solving* for the critical load τ at constant p, it is evident from the recurrence relation 4.5 that the a_n are polynomials in τ. Thus we can rewrite this relation as

$$a_s = \frac{-\tau}{s(s+3)}(pa_{s-4} + a_{s-2}) \tag{4.9}$$

which will produce the polynomials

$$a_s = \sum \gamma_r \tau^r \tag{4.10}$$

Putting the boundary condition at $\xi = 1$ to use, for given values of ϵ and μ, there is then obtained a power series in τ which is equated to zero, viz.

$$\sum_{t=o}^{\infty} \Delta_t \tau^t = 0\ , \tag{4.11}$$

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* Otherwise, at large p, it will be advisable to put $q = \frac{\tau}{\phi}$, and solve for critical ϕ at constant q.

This series is so rapidly convergent that the first few terms are sufficient to guarantee four figure accuracy in the result. Equating the first four terms to zero and writing them as a third order polynomial in τ^{-1},

$$f(\lambda) = \sum_{t=o}^{3} \Delta_t \tau^{3-t} = 0 , \qquad \lambda = \frac{1}{\tau} \tag{4.12}$$

The greatest root in τ^{-1}, and hence the least eigenvalue of τ, can be obtained by any one of several numerical methods (see, for example, Ref. 13, p. 191 and ff.). The method to be used here consists of squaring the roots (Graeffe's method): the result of the first squaring yields a first approximation to the root, in the present case, accurate within less than 1 %; the accuracy can be improved to any desired degree by repeated squaring. The first squaring gives

$$-f(\lambda) f(-\lambda) = \sum_{o}^{3} \nabla_t (\lambda^2)^{3-t} = 0 \tag{4.13}$$

where

$$\nabla_o = \Delta_o^2$$
$$\nabla_1 = 2\Delta_o\Delta_2 - \Delta_1^2$$
$$\nabla_2 = \Delta_2^2 - 2\Delta_1\Delta_3$$
$$\nabla_3 = -\Delta_3^2$$

The first approximation is then

$$(\tau_1^{-1})^2 = -\nabla_1/\nabla_o \tag{4.14}$$

The second squaring gives

$$-f(\lambda^2)\, f(-\lambda^2) = \sum_{o}^{3} \delta_t (\lambda^4)^{3-t} = 0 \qquad (4.15)$$

where

$$\delta_o = \nabla_o^{\,2}$$
$$\delta_1 = 2\nabla_o \nabla_2 - \nabla_1^{\,2}$$
$$\delta_2 = \nabla_2^{\,2} - 2\nabla_1 \nabla_3$$
$$\delta_3 = -\nabla_3^{\,2}$$

and the second approximation is

$$(\tau_2^{-1})^4 = -\delta_1 / \delta_o \qquad (4.16)$$

The details of numerical calculations for this case are tabulated in Appendix 1. These are shown broken down into the literal step-by-step procedure that would be followed in programming the calculations to an electronic computer. These same steps would be followed in the rather more complicated case for the general taper, the solution to which is derived in Article V, 3. It will be noted that the computation consists of two steps. The calculation of the coefficients γ_r, of the a_s, is carried out once for an assumed value of p, independent of ϵ and μ. The same a_s are then used in the second step, for all ϵ and μ, in applying the boundary condition at $\xi = 1$ to calculate the coefficients Δ_t. It is convenient in the present case to write, for the second step,

$$\sum_{t=o}^{\infty} \Delta_t \tau^t = \frac{1}{3} \sum_{s=o}^{\infty} \left\{ (s+3) - \left[(1+\mu p) + \epsilon (s+3) \right] \tau \right\} a_s \qquad (4.17)$$

so that $\Delta_o = 1$. The particular cases calculated through the second approximation in Appendix 1 are the following: (1) $p = 1$, $\epsilon = \mu = 0$; (2) $p = 1$, $\epsilon = 0.1$, $\mu = 0.2$; $p = 0$, $\epsilon = \mu = 0$; (3) $p = \infty$, $\epsilon = \mu = 0$. As a result of the rapid convergence of the series, the effect of $\epsilon \neq 0$, $\mu \neq 0$ in all cases can be handled to good approximation in the following simple way. The values of Δ_t are first calculated for $\epsilon = \mu = 0$. Then, since only a_o contributes a term in ϵ and μ to Δ_1 and a_2 and a_4 make the only contributions of ϵ and μ to Δ_2, the following results are obtained

$$\begin{aligned} \Delta_1 &= {}_o\Delta_1 + a_{11}\mu + a_{12}\epsilon \\ \Delta_2 &= {}_o\Delta_2 + a_{21}\mu + a_{22}\epsilon \\ \Delta_3 &= {}_o\Delta_3 + a_{31}\mu + a_{32}\epsilon \end{aligned} \qquad (4.18)$$

where the ${}_o\Delta$ are values of the Δ for $\epsilon = \mu = 0$, and the a_{ij} are functions of p. With equations 4.14 or 4.16 the critical loads can be obtained, and for given p the ratios τ/τ_o and $(\phi/\phi_o)^{1/2} = (\tau/\tau_o)$ tabulated or plotted against the two parameters ϵ and μ. In these ratios τ_o and ϕ_o represent values for $\epsilon = \mu = 0$. First approximation calculations are listed in Appendix 1 and plotted in the figures at the end of this report for representative cases showing the trend of the effects of elastic supports.

2. Power Series for Martin's Taper; $m = 1$, $\bar{\ell} = 0$

Referring back to paragraph c of Article III, 4, the general equation becomes, for linear rigidity variations and "pointed" tip:

$$\frac{d^2z}{d\xi^2} - \frac{1}{\xi}\frac{dz}{d\xi} + \left(-\frac{\tau}{\xi} + \phi\right)z = 0$$

$$z(0) = 0 \qquad (4.19)$$

$$\frac{dz}{d\xi}(1) = \frac{\tau + \mu\phi}{1 - \epsilon\tau} z(1)$$

The development proceeds identically to the preceding article. Substituting the series 4.2 into equation 4.19 gives, after dividing through by ξ^{α}:

$$\alpha(\alpha - 2)a_o\xi^{-2} + \left[(\alpha^2 - 1)a_1 + \tau a_o\right]\xi^{-1} + \sum_{o}^{\infty}\left\{(n+\alpha)(n+\alpha+2)a_{n+2} + \tau a_{n+1} + \phi a_n\right\} = 0$$

The solution which satisfied the first boundary condition $z(0) = 0$, for $\alpha = 2$, is

$$z = \xi^2 \sum_{o}^{\infty} a_n \xi^n \qquad (4.20)$$

in which

$$a_1 = -\frac{\tau}{3} a_o$$

$$a_{n+2} = \frac{-1}{(n+2)(n+4)}(\tau a_{n+1} + \phi a_n)$$

The procedure outlined in equations 4.8 through 4.18 is now applicable, provided that equation 4.9 is replaced with

$$a_s = \frac{-\tau}{s(s+2)}(pa_{s-2} + a_{s-1}), \qquad (4.21)$$

and if equation 4.17 is changed to

$$\sum_{t=o}^{\infty} \Delta_t \tau^t = \frac{1}{2} \sum_{s=o}^{\infty} \left\{ (s+2) - \left[(1+\mu p) + \epsilon (s+2) \right] \tau \right\} a_s \tag{4.22}$$

Numerical calculations for this case are listed in Appendix 2 and presented in the figures at the back of the report.

3. Power Series for the General Taper Function; $\bar{\ell} \neq 0$

In the articles just preceding, two simple cases of the general equation 3.36 have been reduced to rapidly convergent series solutions in powers of ξ. A discussion of these and the more general case when $0 < \bar{\ell} < 1$, $m = 1, 2, 3, \ldots$ is given in Article III, 4, where it is pointed out that the series solution for the general case is divergent for $\bar{\ell} \leq 1/2$, when written in powers of ξ. The linear transformation

$$\frac{\xi}{2a} = \frac{w}{\bar{a} - w} = \frac{\bar{w}}{1 - \bar{w}} \tag{4.23}$$

guarantees convergence of a series solution in w, but with convergence occurring the more slowly the more closely $\bar{\ell} \rightarrow 0$.

Writing the terms of the differential equation in the new variable w, there are

$$\frac{d}{d\xi} = \frac{(1 - \bar{w})^2}{L} \frac{d}{dw} \; ; \quad L = \frac{2\bar{\ell}}{1 + \bar{\ell}}$$

$$\frac{d^2}{d\xi^2} = \frac{1}{L^2} (1 - \bar{w})^4 \frac{d^2}{dw^2} - \frac{2}{a}(1 - \bar{w})^3 \frac{d}{dw} \tag{4.24}$$

$$f = g = \zeta^m$$

$$\zeta = (1 - \bar{\ell})(\xi + a) = \bar{\ell}\,(\frac{1 + w}{1 - \bar{w}})$$

$$R = \frac{f'}{f} - \frac{2}{\xi} = \frac{m}{\zeta} - \frac{2}{\xi} = \frac{m}{\bar{\ell}}\,(\frac{1 - \bar{w}}{1 + \bar{w}}) - (\frac{1 - \bar{w}}{a\bar{w}})$$

$$S = \frac{\tau}{f} + \frac{\xi^2 \phi}{fg} = \frac{\tau}{\bar{\ell}^m}\,(\frac{1 - \bar{w}}{1 + \bar{w}})^m + \frac{\phi L^2}{\bar{\ell}^m}\,\frac{w^2(1 - \bar{w})^{2m-2}}{(1 + \bar{w})^{2m}}$$

After combining terms in $dz/d\xi$ the differential equation becomes, for the taper functions defined by equation 3.37: (4.25)

$$\frac{d^2 z}{dw^2} + \left[\frac{M}{(1-\bar{w})^2} - \frac{2(1+\bar{w})}{w(1-\bar{w})}\right]\frac{dz}{dw} + \left[\tau^* \frac{(1-\bar{w})^{m-4}}{(1+\bar{w})^m} + \phi^* \frac{w^2(1-\bar{w})^{2m-6}}{(1+\bar{w})^{2m}}\right] z = 0$$

where

$$M = \frac{2m}{1 + \bar{\ell}}$$

$$\tau^* = \frac{\tau L^2}{\bar{\ell}^m} \qquad (4.26)$$

$$\phi^* = \phi(\frac{L^2}{\bar{\ell}^m})^2$$

The transformation of the independent variable does not alter the indicial equation; moreover in this transformation the points $\xi = w = 0$ represent the tip so that the tip boundary condition is $w(0) = 0$. The solution must then be of the form

$$z = w^3 \sum_{o}^{\infty} a_n w^n \qquad (4.27)$$

At $\xi = w = 1$,

$$\frac{dz}{d\xi}(1) = L\frac{dz}{dw}(1)$$

and the second of the boundary conditions 3.26 becomes

$$L\frac{dz}{dw}(1) = \frac{\tau + \mu\phi}{1 - \epsilon\tau} z(1) \tag{4.28}$$

It will be noted that as $\bar{\ell} \rightarrow 1$, $M \rightarrow m \rightarrow 0$, $\bar{a} \rightarrow \infty$, $\bar{w} \rightarrow 0$ and equation 4.25 becomes identical to equation 3.41 for the uniform strut.

The recurrence relations for the coefficients in the series 4.27 may be derived just as in the two preceding paragraphs. To do so, equation 4.25 is cleared of fractions

$$(1+\bar{w})^{2m}(1-\bar{w})^4 \frac{d^2z}{dw^2} + \left[(1-\bar{w})^3 \; M(1+\bar{w})^{2m-1} - \frac{2(1+\bar{w})^{2m+1}}{w}\right]\frac{dz}{dw}$$
$$+ \left[\tau^*(1-\bar{w}^2)^m + \phi^*(1-\bar{w})^{2m-2}\, w^2\right] z = 0 \tag{4.29}$$

and substitutions are made from equation 4.27. The expansions of the coefficients of the derivatives involve the product of two polynomials and an infinite series. It is convenient to write, for example,

$$(1 + \bar{w})^{2m} = \sum_{r=o}^{2m} \binom{2m}{r}\left(\frac{w}{\bar{a}}\right)^4 \quad .$$

Then from the general term in w^{R+1} of equation (4.29), the coefficient a_R can be obtained from the following recurrence relation for any m and $\bar{\ell}$:

$$\begin{aligned}
-R(R+3)a_R = {} & \sum_{s=o}^{4}{}' \binom{4}{s}(-\bar{a})^{-s} \sum_{r=o}^{2m\,|\,R-s} \binom{2m}{r}\frac{(R+3-t)(R+2-t)}{\bar{a}^{r}}\, a_{R-t} \\
& -2\sum_{s=o}^{3}{}' \binom{3}{s}(-\bar{a})^{-s} \sum_{r=o}^{2m+1\,|\,R-s} \binom{2m+1}{r}\frac{(R+3-t)}{\bar{a}^{r}}\, a_{R-t} \\
& +M\sum_{s=o}^{3} \binom{3}{s}(-\bar{a})^{-s} \sum_{r=o}^{2m-1\,|\,R-s-1} \binom{2m-1}{r}\frac{(R+2-t)}{\bar{a}^{r}}\, a_{R-t-1} \\
& +\tau^{*}\sum_{s=o}^{m\,|\,\frac{R-2}{2}} \binom{m}{s}(-\bar{a}^{2})^{-s}\, a_{R-2s-2} \\
& +\phi^{*}\sum_{s=o}^{2m-2\,|\,\frac{R-4}{2}} \binom{2m-2}{s}(-\bar{a}^{2})^{-s}\, a_{R-2s-4}
\end{aligned} \tag{4.30}$$

The coefficient a_o is arbitrary. In the above the following special notation has been employed, in addition to $t = r + s$:

(a) The sign $\sum{}'$ denotes omission of the term for $t = r + s = 0$ from the double summation.

(b) The summation index is limited to the lesser of the two upper limits whenever a double upper limit appears.

(c) If the lesser upper limit is less than zero, no contribution occurs from that index, except that in the last (or ϕ^{*}) summation $m = 0$ is to be treated as $m = 1$.

It is easily seen that in the general case equation 4.30 is a subject for an average-speed digital computer, the programming of which would follow the lines outlined by the simpler problems of Appendix 1 and 2. With the coefficients γ_r determined for the a_s, the calculation will proceed exactly as outlined in equations 4.10 through 4.18, if it is remembered to modify equation 4.17 as necessary from 4.28.

4. Exponential Taper

The singularities of the mathematical problem associated with the class of taper functions $f = g = \zeta^m$ have been treated in detail in this report. This class of functions includes several natural taper forms as shown on Figure 2. However it is obvious that these mathematical difficulties can be side-stepped by using an exponential taper function, viz.

$$f = g = e^{c(\xi - 1)} \tag{4.31}$$

where here $c > 0$ is an arbitrary constant playing much the same role as both $\bar{\ell}$ and m in the previous class. It will be recognized at once that now the differential equation has no singularity at $\xi = -a$, and hence no difficulty associated with confluence of the latter with the point $\xi = 0$ at $\bar{\ell} = 0$. Indeed the exponential taper does not provide the pointed tip case.

In order to make the presentation of solutions in this report more complete, therefore, the equations for the case of exponential taper will be summarized here without detailed development. The differential equation becomes

$$\frac{d^2 z}{d\xi^2} + \left(c - \frac{2}{\xi}\right)\frac{dz}{d\xi} + \left(\bar{\tau}^* e^{-c\xi} + p\xi^2 e^{-2c\xi}\right) z = 0 \tag{4.32}$$

where

$$p = \frac{\bar{\phi}^*}{\bar{\tau}^*} = pe^c$$

$$\bar{\phi}^* = \phi e^{2c}$$

$$\bar{\tau}^* = \tau e^c$$

The solution has the form

$$z = \xi^3 \sum_{n=o}^{\infty} a_n \xi^n$$

with

$$a_1 = -\frac{3}{4} c$$

$$-r(r+3)a_n = c(r+2)a_{n-1} + \bar{\tau}^* \sum_{s=o}^{n-2} \frac{(-c)^s}{s!} a_{n-(s+2)} \qquad (4.33)$$

$$+ \bar{\phi}^* \sum_{s=o}^{n-4} \frac{(-2c)^s}{s!} a_{n-(s+4)}$$

satisfying z(0) = 0, and with the statement of the second boundary condition remaining

$$\frac{dz}{d\xi}(1) = \frac{\tau + \mu\phi}{1 - \epsilon\tau} z(1)$$

V. EXPERIMENTAL RESULTS

A short series of experiments was run to obtain experimental verification of the theoretical results for the case of the uniform strut under equal axial and transverse loads. The strut was the narrower leg of an "L" cut as shown on Figure 12. The wider leg acted as the elastic support for the strut; its flexibility could be varied according to the length ℓ' which was left unclamped, as indicated in Figure 13. The latter also indicates the method used for loading the strut.

A loading hook was suspended from the pin in the free end of the strut. Over this hook was strung a thin piano wire with horizontal and vertical runs, as shown. The wire was loaded by dead weight to produce equal horizontal and vertical force components at the pin, when the test specimen and the horizontal wire were leveled. The horizontal wire was approximately fifteen feet long, or as long as possible in the space available, in order to minimize angular errors. Of special importance here was the necessity to reduce side load on the strut produced by the load in this wire when the strut deflected in buckled mode. Even so these extraneous loads often proved troublesome during the tests.

The specimen itself was clamped to a rigid frame of heavy steel channel and angle iron construction. This same frame supported a simple fulcrum arrangement* which sensed the lateral deflection of the

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* A light, elastic wire feeler on one end of a lever completed the connection through a penlite battery to light a bulb when it was brought into contact with the strut. The other end of the lever actuated the dial gage.

strut tip and actuated a dial gage. There was thus no side load produced on the strut by the dial gage. The frame also served as a deflection limiter for the side motion of the weight pan, and prevented the catastrophic failure of the specimen if the buckling load was accidentally reached during loading.

The method of determining the critical load was the one commonly described as Southwell's method. Due to unavoidable eccentricities the strut deflected elastically (but not linearly) at subcritical loads. The load P was increased, the deflections δ measured, and the ratio P/δ plotted against P. At buckling $P/\delta \rightarrow 0$ and the corresponding P is the critical load. However the loading can be stopped at a reasonable margin short of critical load and the curve extrapolated linearly to zero. The method is illustrated in the experimental loading data of Figure 15.

It will be recognized that the root boundary conditions for the strut, i.e. the elbow of the test specimen, are difficult to relate, in detail, to the ideal boundary conditions assumed in the analysis. Runs 1 and 2 were used to determine that an effective length for the strut could be taken as $\ell_{eff} = 7.7''$, which is a not unreasonable value from the point of view of engineering approximations. Using this value and denoting the support geometry by primed symbols, then it is easily seen that the support flexibility coefficients can be written

$$\epsilon = \frac{\ell'}{\ell_{eff}} \frac{B}{C'}$$

(5.01)

$$\mu = \frac{\ell'}{\ell_{eff}} \frac{C}{B'}$$

where

$$B = E\frac{ct^3}{12}, \quad C = G\frac{ct^3}{16}(5.333 - 3.36\frac{t}{c})$$

c = width of strut

t = thickness of strut

The coefficient μ must be modified, however, to account for the fact that the wide leg of the specimen, under action of load P, has applied to it both an end moment and an axial load. The effect is non-linear, and μ is a function of the load P; the effective value of μ can be written

$$\mu_{eff} = \mu \frac{\tanh \ell'\sqrt{\frac{P}{B'}}}{\ell'\sqrt{\frac{P}{B'}}} \tag{5.02}$$

A series of tests were made in which the unclamped length, ℓ', had the values 0, 2, 4, 6 and 8 inches, and the critical loads P = $\mathcal{T}$ were determined as above. The critical load ratio, expressing the effect of the elastic support (see, for example, Figure 8), was then computed as

$$\frac{\tau}{\tau_o} = \frac{P}{P_o} \tag{5.03}$$

where P_o was the load for $\ell' = 0$. When expressed as a ratio in this manner, the errors introduced by the uncertain boundary conditions tend to be minimized. The final results are shown by the experimental points on Figure 14. The original data are shown on Figure 15.

In order to compare the experimental points with the theory, equations 5.01 and 5.02 were used to determine μ and ϵ for each point. In addition since, when $P = T$, the parameter p is a variable, it was necessary to identify p for each point. However the value was never much different from $p = 1$, to which the chart on Figure 8 applies. Actually over the range of the tests $2/3 < p < 3/2$, but as it turns out the critical loads are relatively insensitive to p in this range. The solid curve on Figure 14 represents the theoretical results of Figure 8 with small corrections for the corresponding deviations in value of p. From the comparison on Figure 14 it is concluded that the test series confirms the trends predicted by the theory for elastic supports on uniform struts.

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APPENDIX I

SAMPLE CALCULATIONS FOR UNIFORM STRUT

(a) Determination of a_s for $m = 0$ ($\bar{\ell} = 1$); $p = 1$:

$$a_s = \frac{-\tau}{s(s+3)}(a_{s-4} + a_{s-2})$$

s	γ_o	γ_1	γ_2	γ_3	γ_4
0	1.00000				
2		-0.10000			
4		-0.03572			
			0.003572		
6			0.002513		
				-0.000066	
8			0.000406		
				-0.000069	
					0.000001
10				-0.000022	
					0.000001
12				-0.000002	
					0.0000005

APPENDIX I (cont'd)

SAMPLE CALCULATIONS FOR UNIFORM STRUT

(b) Determination of critical load, $m = 0$ ($\bar{\ell} = 1$); $p = 1$; $\epsilon = \mu = 0$:

$$\sum \Delta_t \tau^t = -\frac{1}{3} \sum \left\{ \left[1 + \mu p + \epsilon (s+3) \right] \tau a_s - (s+3) a_s \right\}$$

$$= \sum \left(\frac{s+3}{3} - \frac{\tau}{3} \right) a_s$$

s	Δ_o	Δ_1	Δ_2	Δ_3	Δ_4
0	1.00000	-0.33333			
2		-0.16667	0.03333		
4		-0.08333	0.01191		
			0.00833	-0.00119	
6			0.00754	-0.00084	
				-0.00020	0.000022
8			0.00149	-0.00014	
				-0.00025	0.000023
					0.000004
10				-0.00010	0.000007
					0.000004
12				-0.00001	0.0000007
					0.0000025
	1.00000	-0.58333	0.06260	-0.00273	0.00006

APPENDIX 1 (cont'd)

SAMPLE CALCULATIONS FOR UNIFORM STRUT

1st approx. $\Delta_1^{\,2} = 0.34030 \qquad 2\Delta_2 = 0.12520$

$(\tau_1^{-1})^2 = 0.21510$

$\tau_1 = 2.156, \qquad \phi_1^{1/2} = 1.468$

2nd approx. $(\tau_2^{-1})^2 = 0.21166$

$\tau_2 = 2.173, \qquad \phi_2^{1/2} = 1.474$

(c) Approximate effect of elastic supports, $m = 0$ ($\bar{\ell} = 1$); $p = 1$:

$$\Delta_o = 1.000$$

$$\Delta_1 = -0.58333 - \frac{\mu + 3\epsilon}{3}$$

$$\Delta_2 = 0.06260 + 0.04524\mu + 0.25000\epsilon$$

$$\tau_1 / \tau_o \quad [\tau_o \backsim \epsilon = \mu = 0]$$

μ	$\epsilon = 0$	0.2	0.4	0.6
0	1.000	0.744	0.579	0.470
0.4	0.781	0.610	0.498	0.411
0.8	0.640	0.517	0.430	0.366
1.2	0.542	0.449	0.381	0.330

APPENDIX II

SUMMARY OF CALCULATIONS FOR POINTED STRUT

$m = 1, \ \bar{\ell} = 0; \ p = 1$

(a) Critical load, $\epsilon = \mu = 0$:

1st approx.	$\Delta_o = 1.0000$	
	$\Delta_1 = -1.2500$	$\Delta_1^2 = 1.5625$
	$\Delta_2 = 0.4045$	$2\Delta_2 = 0.8090$
	$\Delta_3 = -0.0587$	
		$(\tau_1^{-1})^2 = 0.7535$
	$\tau_1 = 1.152$,	$\phi_1^{1/2} = 1.073$
2nd approx.	$\tau_2 = 1.170$,	$\phi_2^{1/2} = 1.082$

(b) Approximate effect of elastic supports:

$$\tau_1/\tau_o \quad [\tau_o \backsim \epsilon = \mu = 0]$$

μ	$\epsilon = 0$	0.2	0.4	0.6
0	1.00	0.88	0.76	0.67
0.4	0.83	0.73	0.64	0.56
0.8	0.70	0.63	0.56	0.50
1.2	0.61	0.55	0.50	0.45

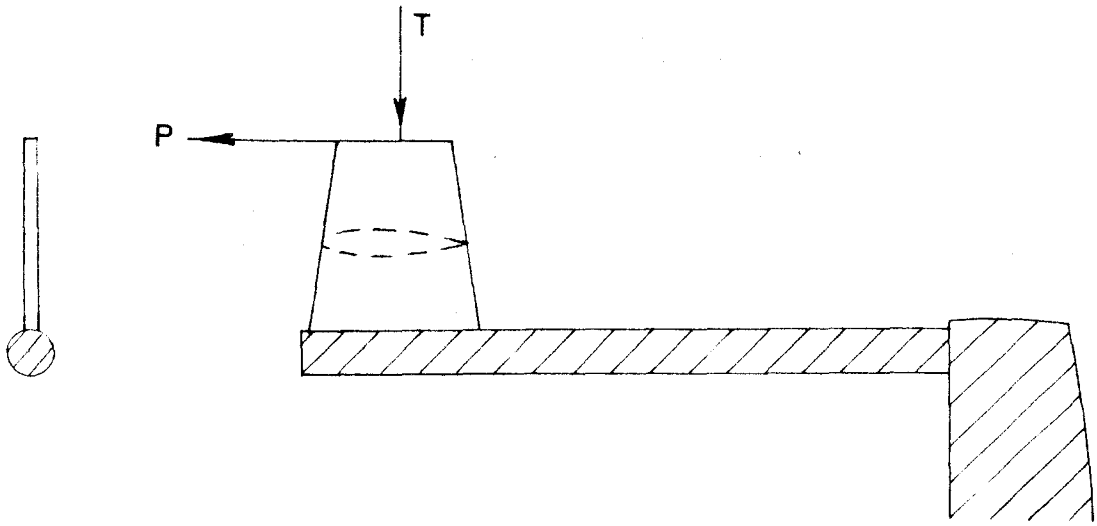


FIG. 1a THIN STRUT ON THE END OF A STING

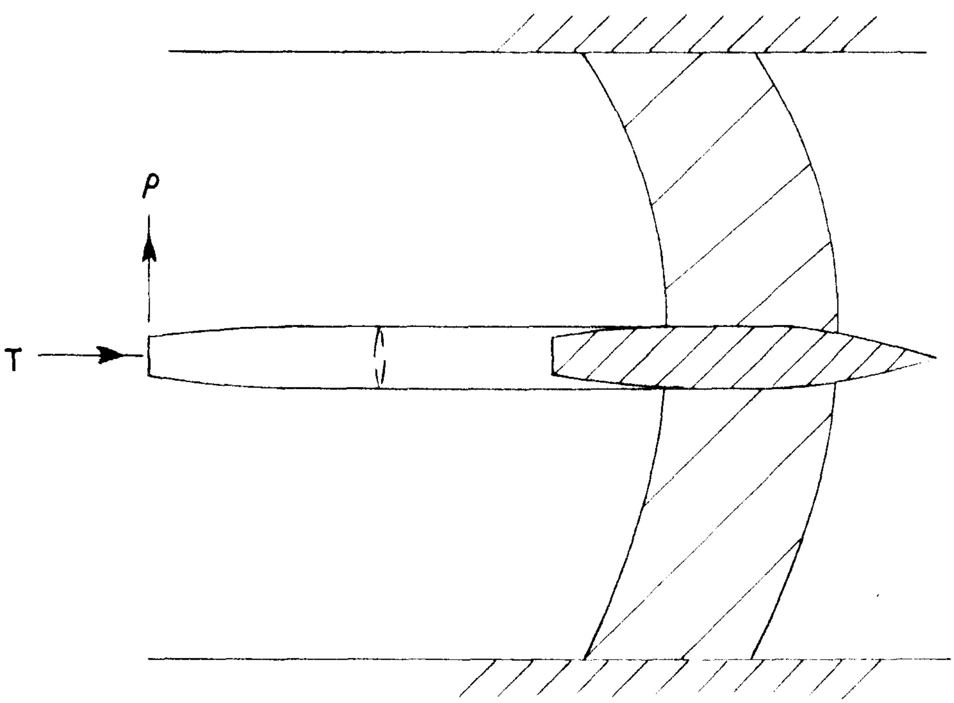
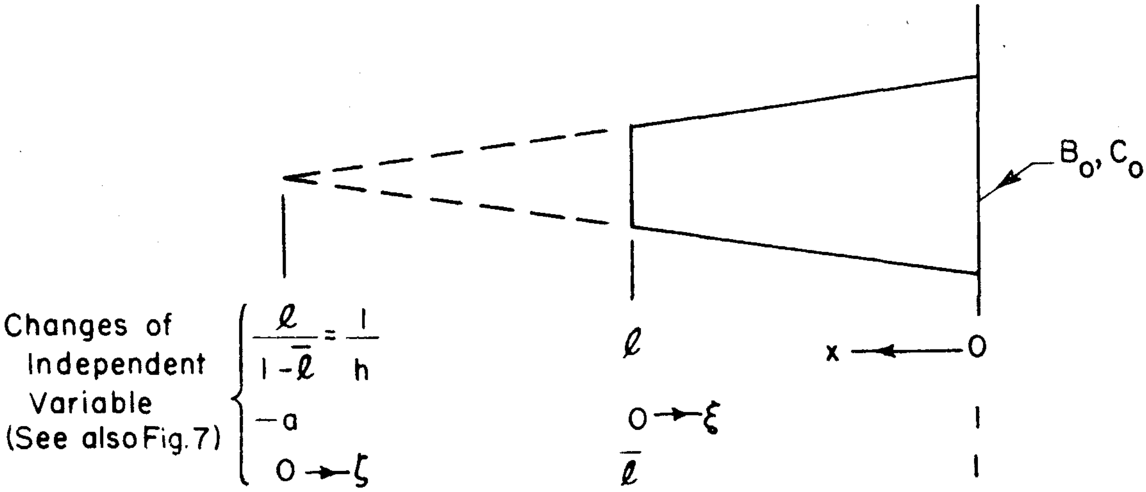


FIG. 1b THIN STRUT MOUNTED FROM PITCH-ANGLE SECTOR



Definitions: $\bar{\ell} = 1 - h\ell$, $\xi = \dfrac{\ell - x}{\ell}$, $\zeta = (1 - hx)$

Rigidity Functions: $f = \dfrac{B(x)}{B_0} = (1 - hx)^m = \zeta^m$

$$g = \frac{C(x)}{C_0} = (1 - hx)^m = \zeta^m$$

Misc. Relations: $a = \dfrac{\bar{\ell}}{1 - \bar{\ell}}$, $\bar{\ell} = \dfrac{a}{1 + a}$

$$\xi = (a+1)\zeta - a = \frac{\zeta - \bar{\ell}}{1 - \bar{\ell}} \; ; \; \zeta = \frac{\xi + a}{1 + a} = \bar{\ell} + (1 - \bar{\ell})\xi$$

Taper Forms: $\bar{\ell} = 0 \sim a = 0 \sim$ "pointed" strut

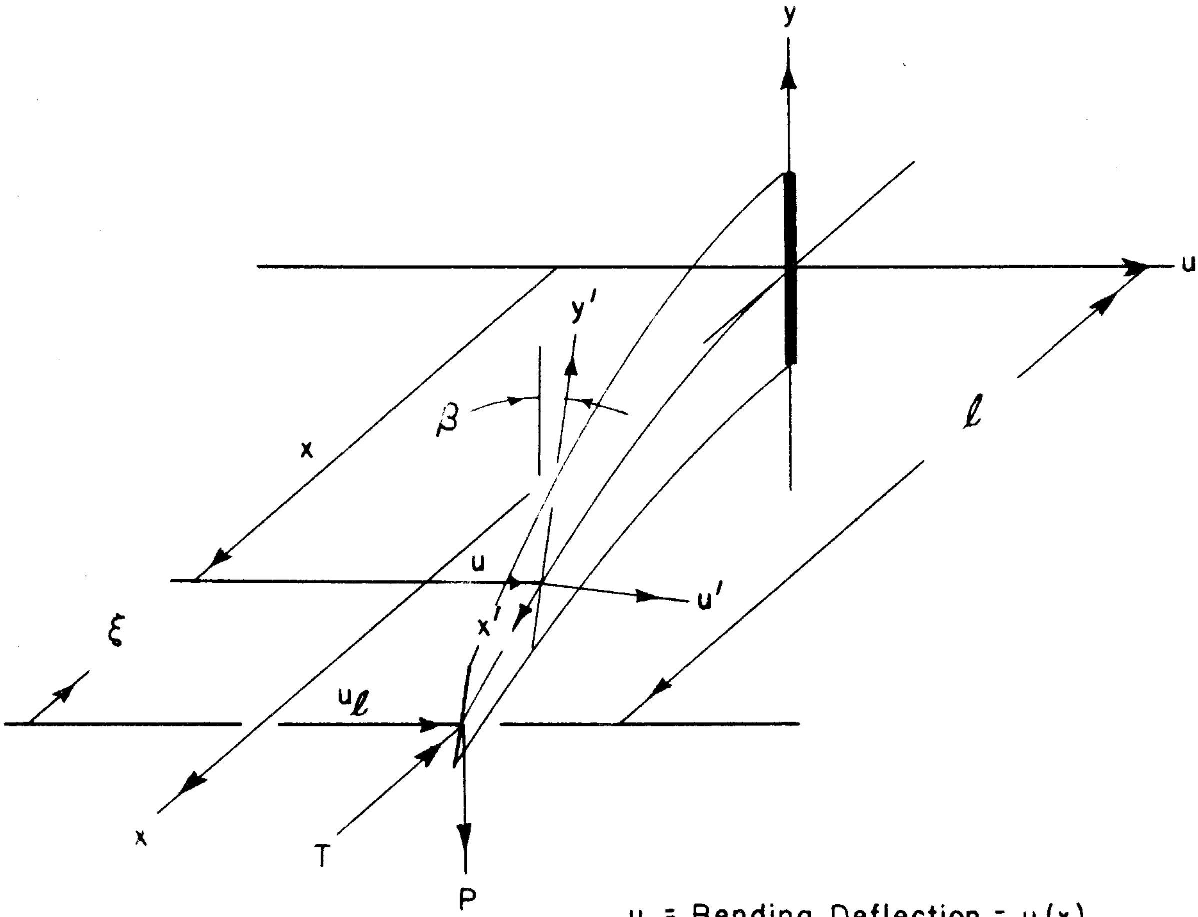
$m = 0$ and/or $\bar{\ell} = 1 \sim a = \infty \sim$ uniform strut

$m = 1 \sim$ linear chord variation at constant thickness (cf. footnote, p. 22)

$m = 3 \sim$ linear thickness variation at constant chord

$m = 4 \sim$ linear variation of similar cross-sections

FIG. 2 SKETCH DESCRIBING NOTATION FOR THE TAPER FUNCTION ζ^m



u = Bending Deflection = u(x)
β = Torsional Deflection = β(x)

x - y = Plane Occupied by Undeflected Strut

y - u = Plane of Root Section of Undeflected Strut — with Elastic Support the Root Section Deflects as a Spring in Rotation about the x and y Axes with Different, Arbitrary Spring Rates

x - u = Plane of Undeflected and Deflected Neutral Axis for Thin Cross Sections

FIG. 3 SKETCH SHOWING DEFLECTED STRUT AND COORDINATE AXES

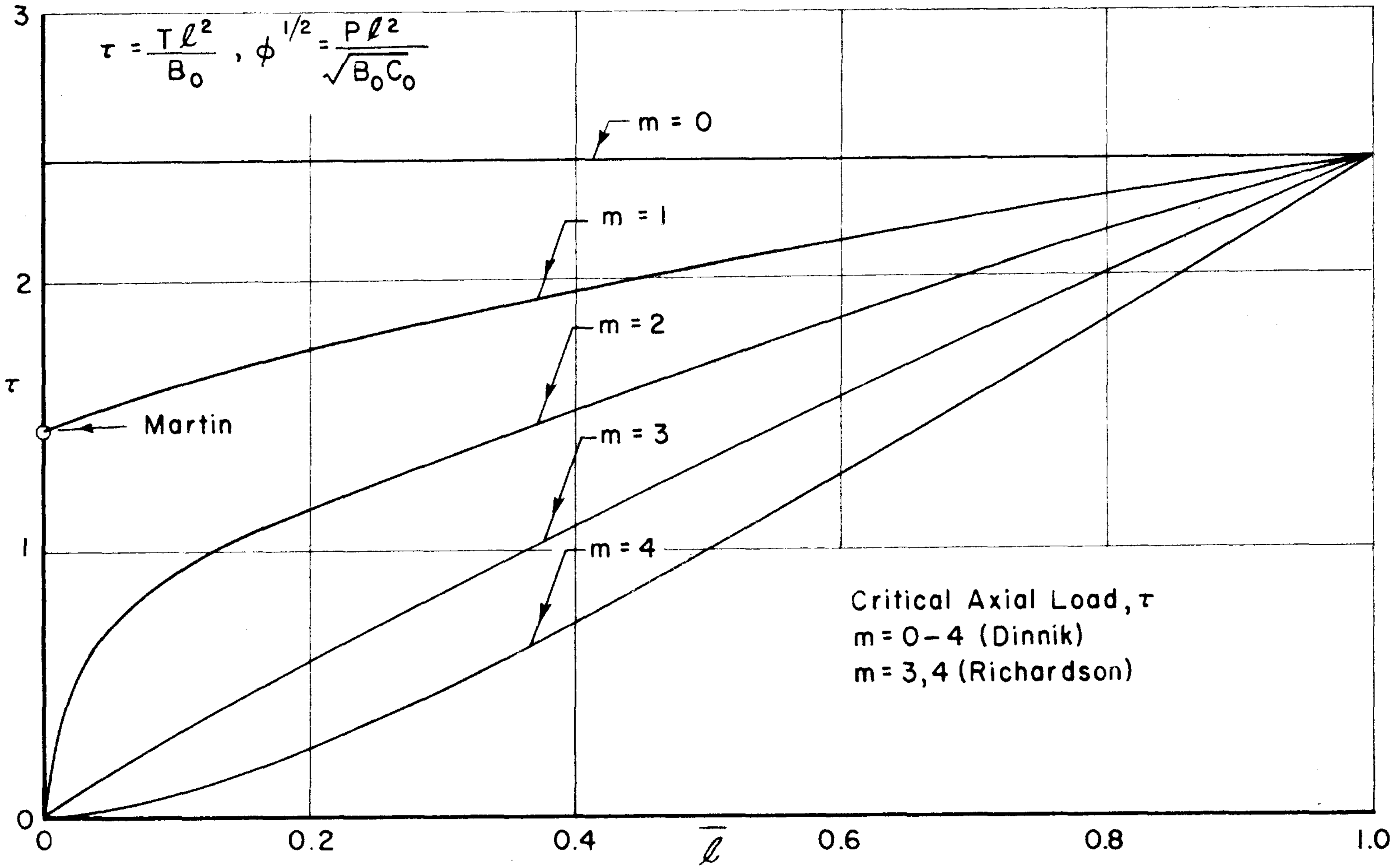


FIG. 4 CRITICAL AXIAL LOADS FOR THE RIGIDLY SUPPORTED, TAPERED STRUT--- RICHARDSON AND DINNIK RESULTS

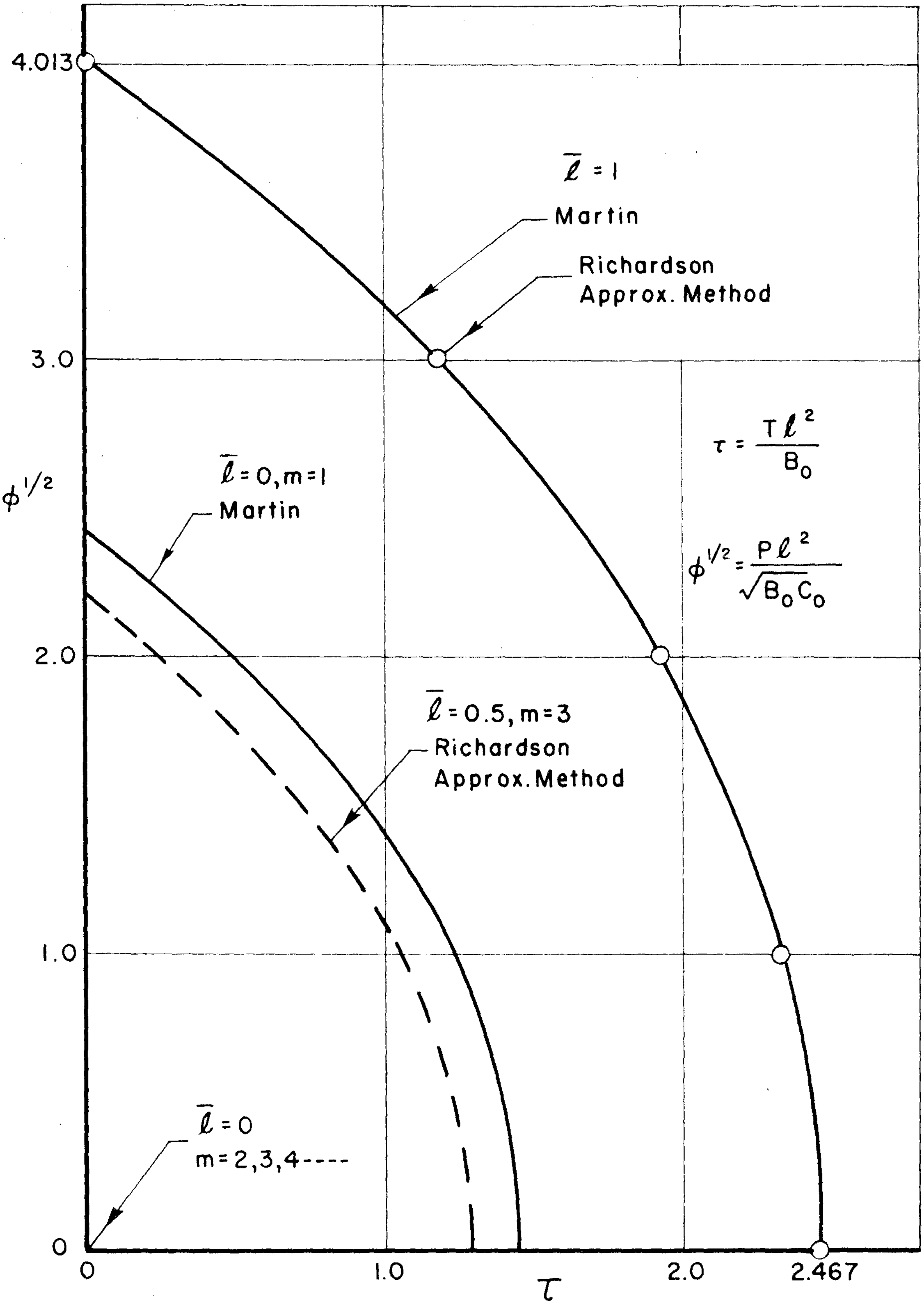


FIG. 5 CRITICAL COMBINED LOADS FOR THE RIGIDLY SUPPORTED, TAPERED STRUT --- MARTIN AND RICHARDSON RESULTS

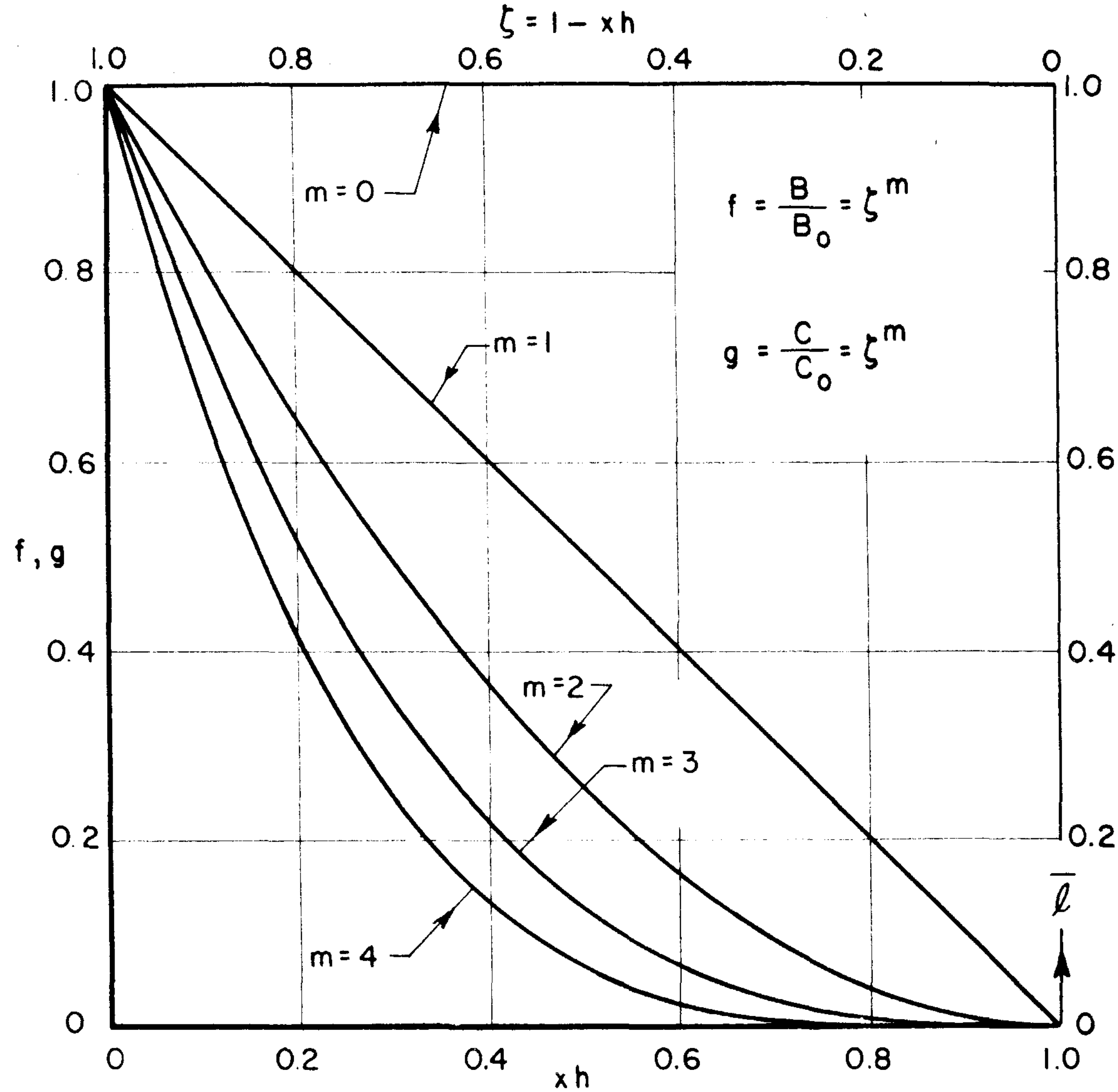


FIG. 6 DISTRIBUTIONS OF RIGIDITIES REPRESENTED BY THE TAPER FUNCTION ζ^m

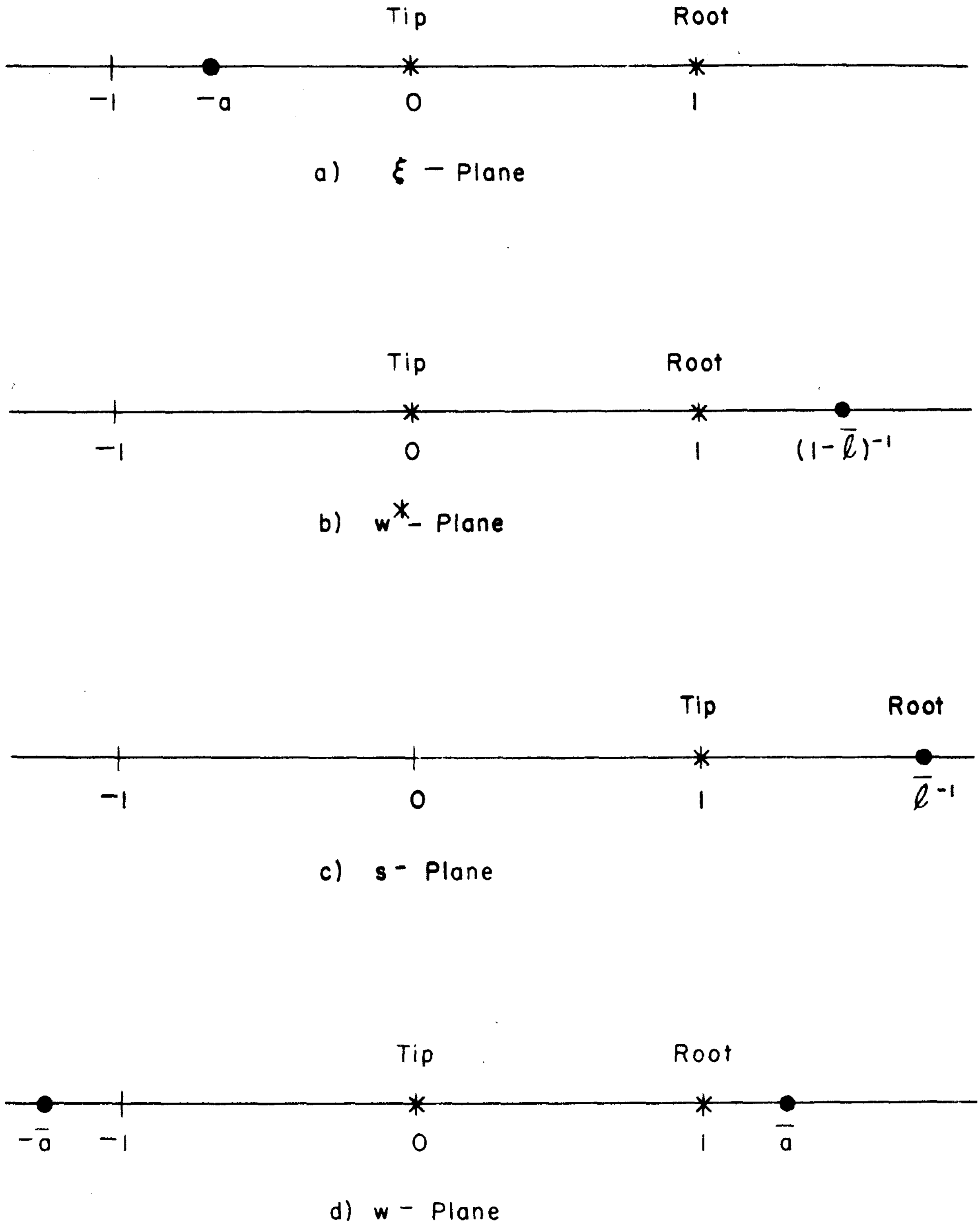


FIG.7 TRANSFORMATIONS OF THE INDEPENDENT VARIABLES: ξ, s, w^*, w

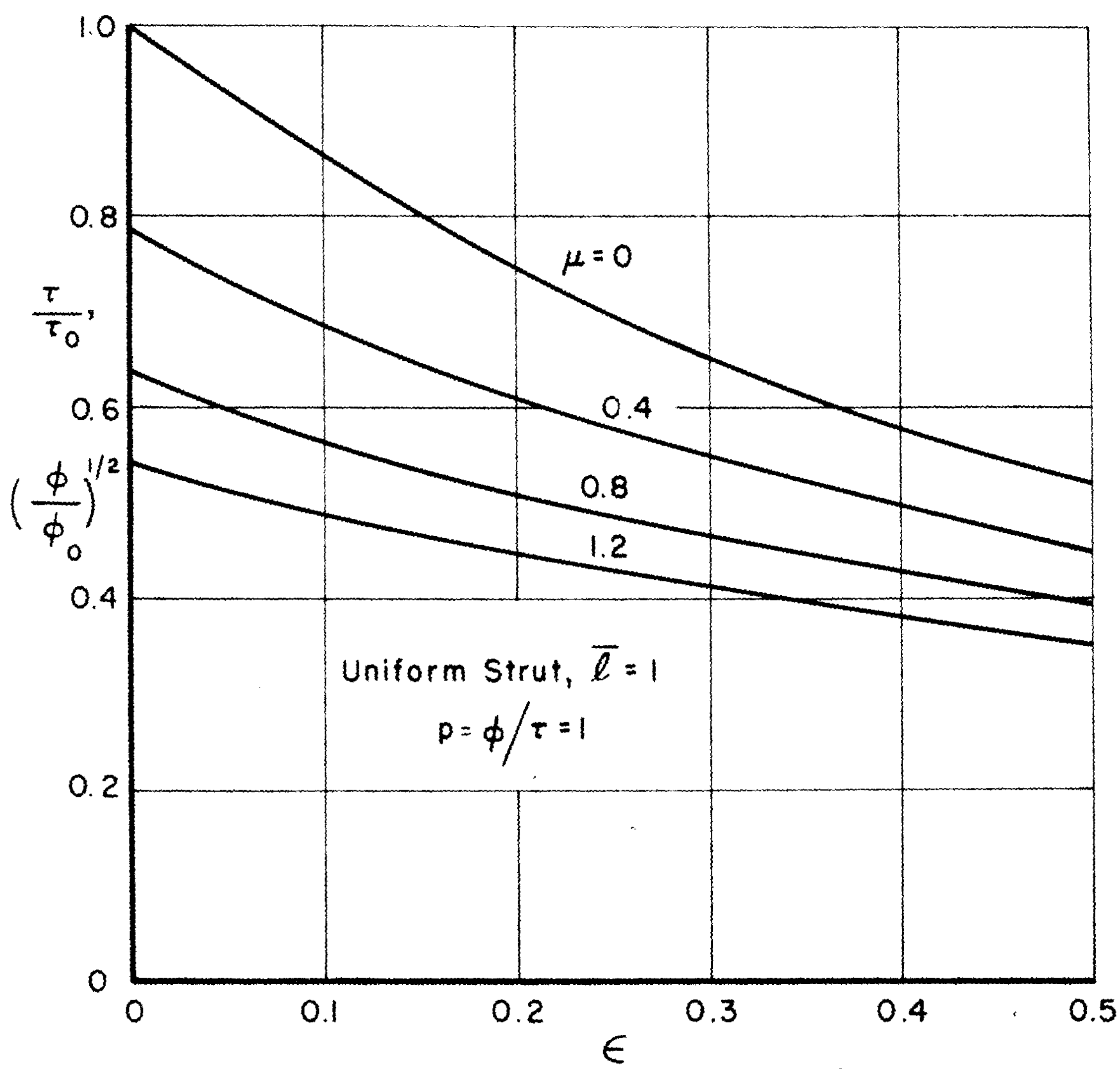


FIG. 8 EFFECT OF ELASTIC SUPPORTS IN COMBINED LOADING---
$p = 1, \overline{\ell} = 1$

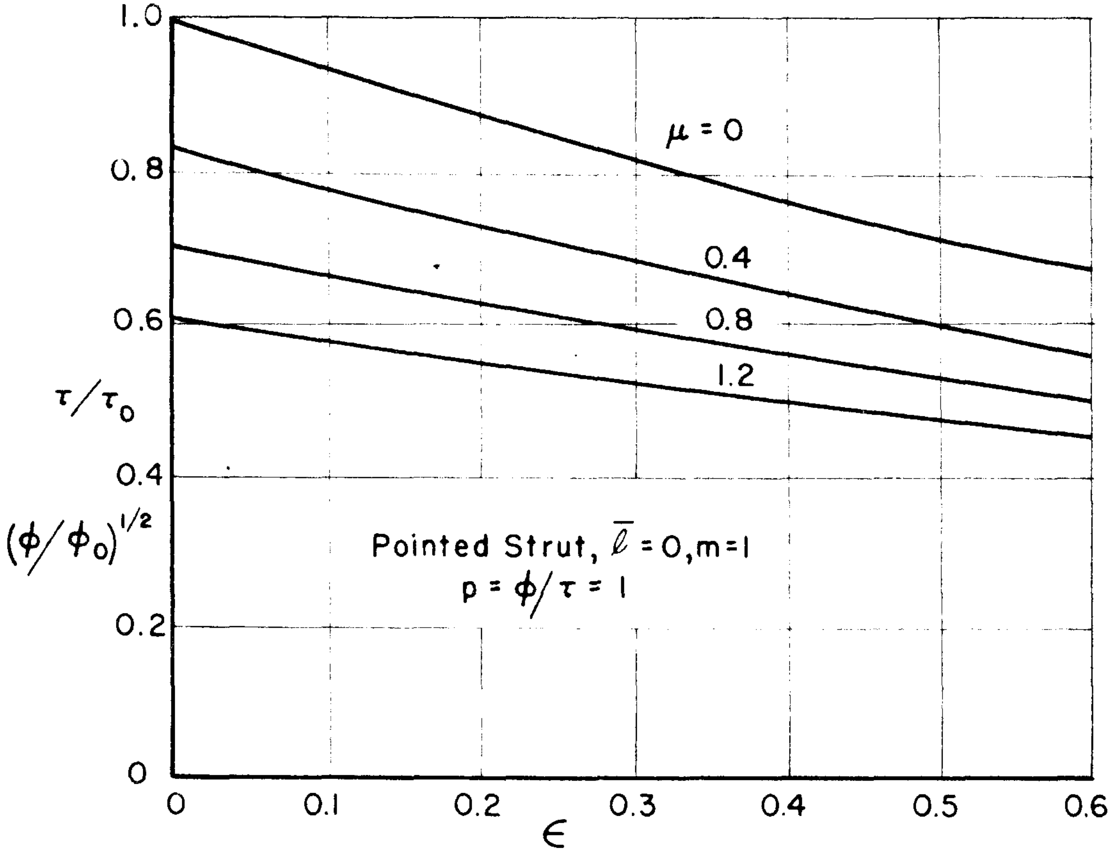


FIG. 9 EFFECT OF ELASTIC SUPPORTS IN COMBINED LOADING---
$p=1,\ \bar{\ell}=0,\ m=1$

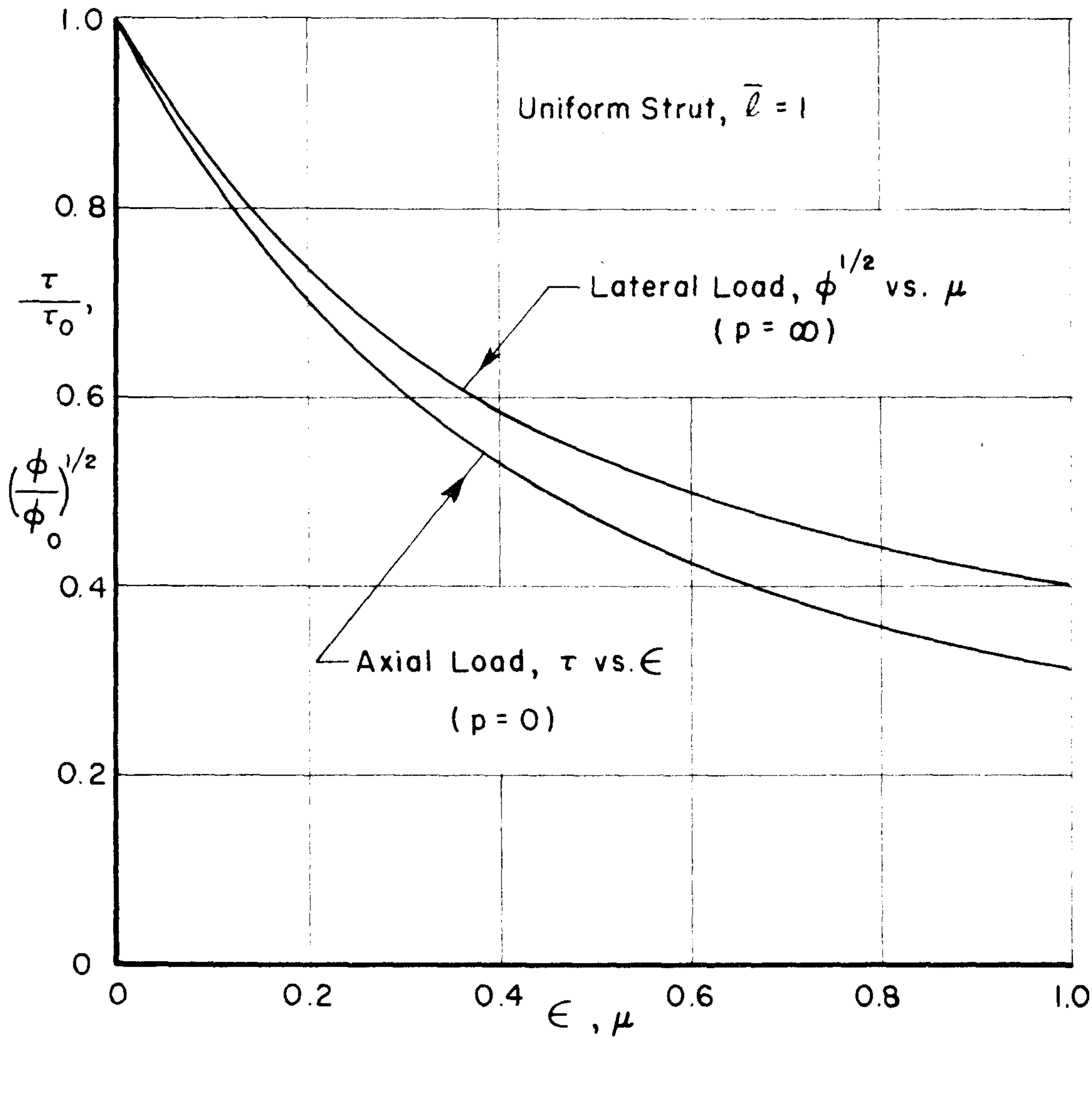


FIG. 10 EFFECT OF ELASTIC SUPPORTS FOR UNCOMBINED LOADS---
$\bar{\ell} = 1$

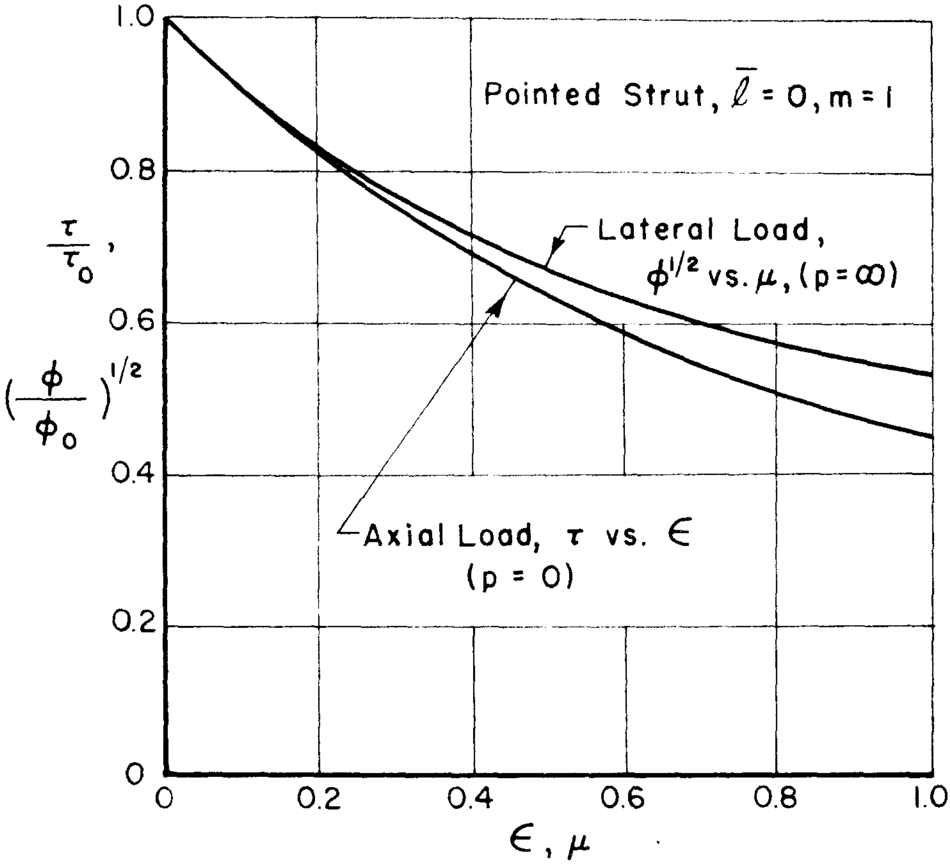


FIG. 11 EFFECT OF ELASTIC SUPPORTS FOR UNCOMBINED LOADS --- $\bar{\ell}$ = 0, m = 1

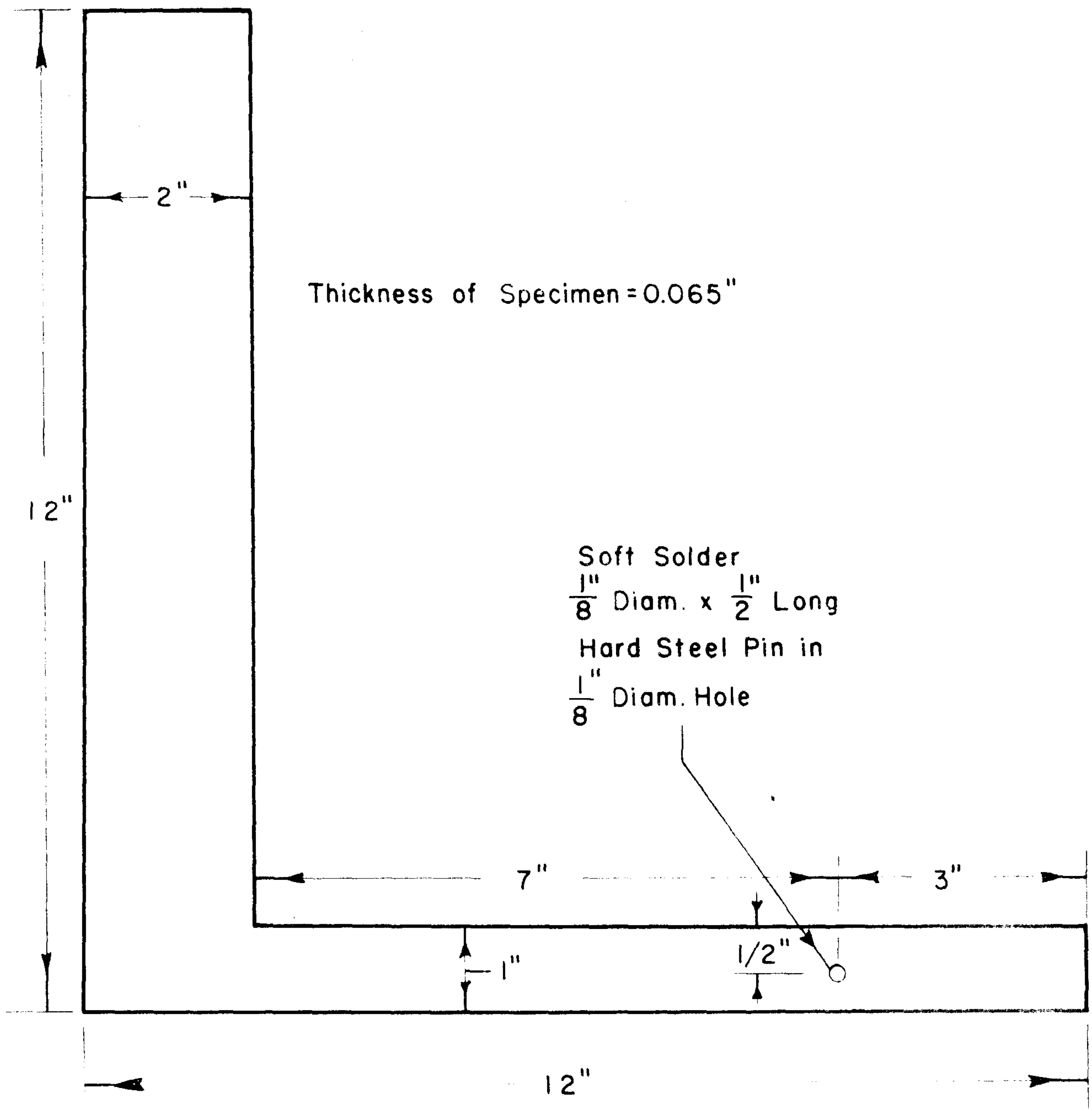


FIG. 12 LINE DRAWING OF TEST SPECIMEN

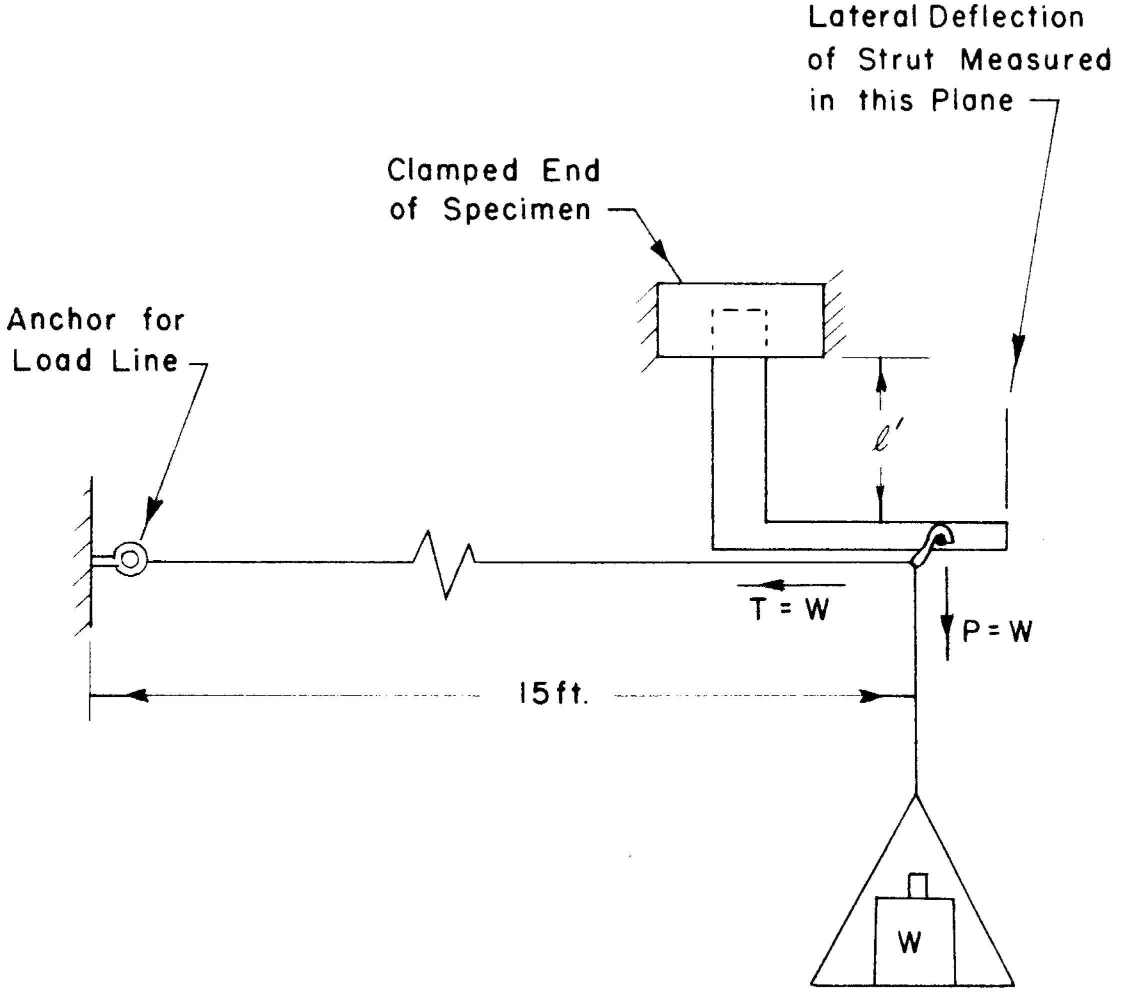


FIG. 13 SCHEMATIC DRAWING OF EXPERIMENTAL SETUP

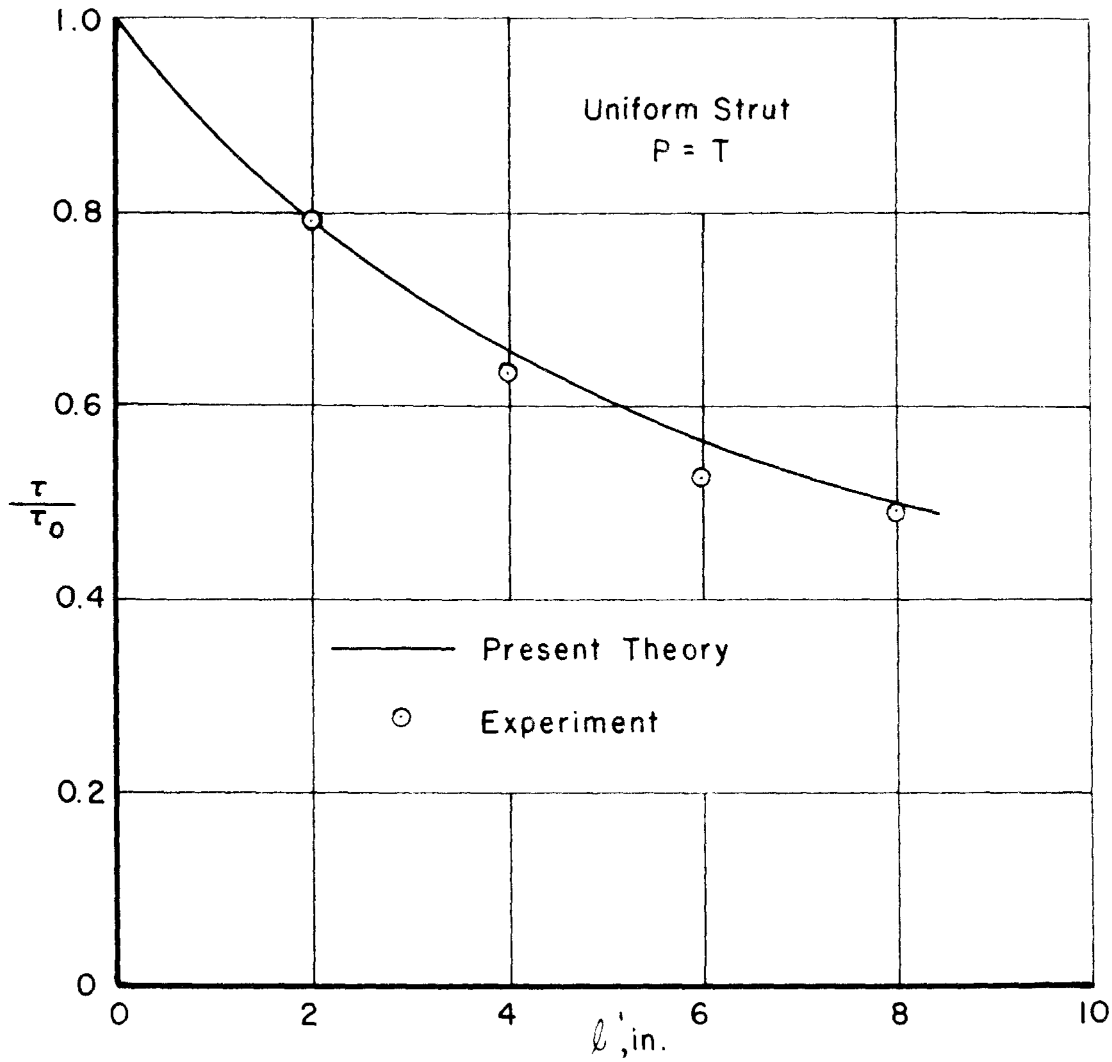


FIG. 14 COMPARISON BETWEEN THEORY AND EXPERIMENT FOR UNIFORM STRUT

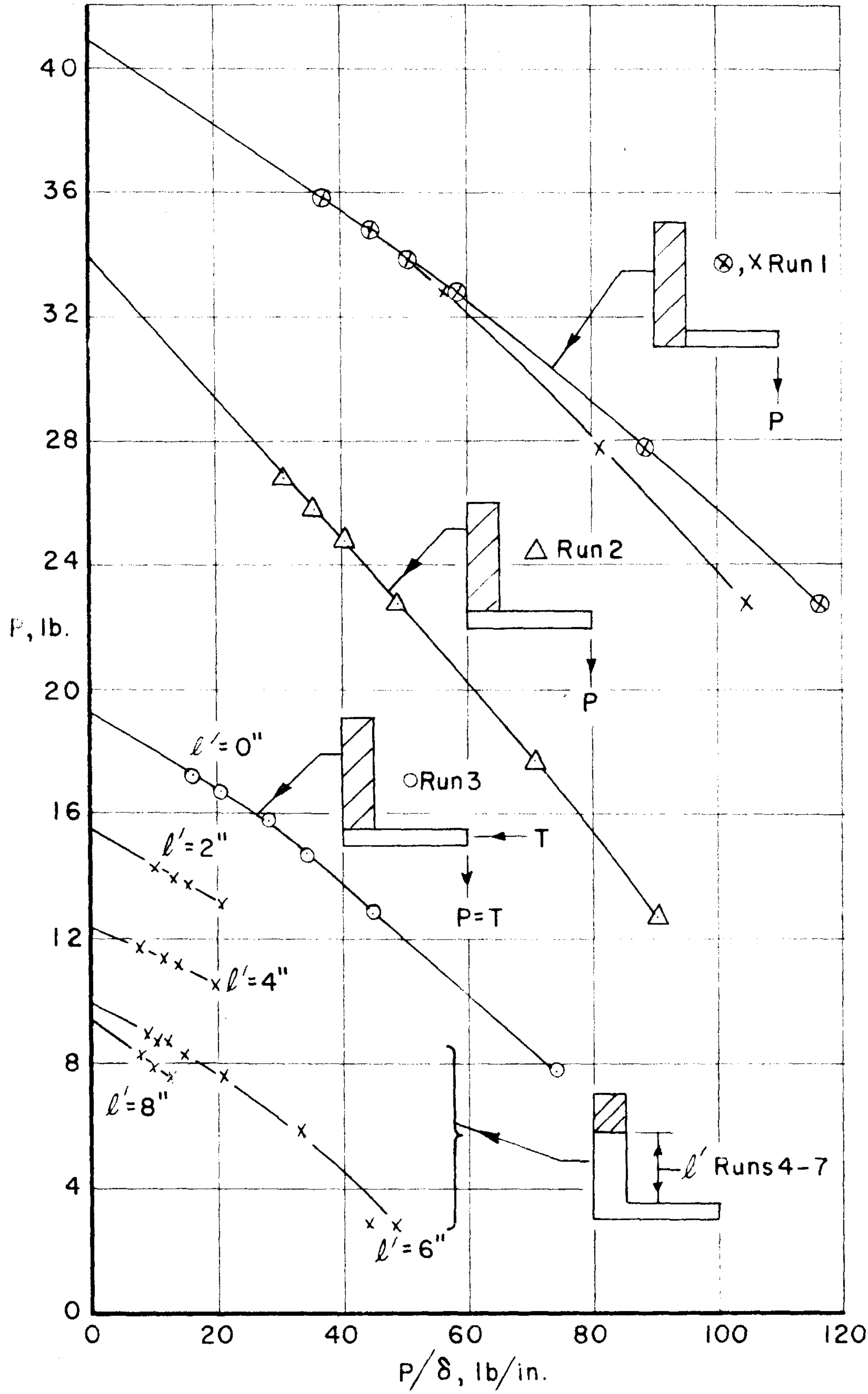


FIG. 15 EXPERIMENTAL DETERMINATION OF CRITICAL LOADS