SOME RADIATION PROBLEMS IN ELASTODYNAMICS

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ABSTRACT

Three elastodynamic problems are studied. The first deals with waves generated by instantaneous and uniform closure of a semi-infinite crack, while in the second, a semi-infinite crack is suddenly initiated in a continuous medium initially subjected to uniform tension. The last of the three deals with a force moving at uniform velocity along a semi-infinite crack, starting from the edge. The problems are solved by means of the Wiener-Hopf integral methods. The characteristic wave patterns and stress singularities are discussed.

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INTRODUCTION

The present thesis deals with some problems of "radiation" of elastic waves. By "radiation" is meant a process taking place in problems of wave propagation where the medium contains a certain point or line (the source) endowed with some prescribed characteristic singularities. Thus the problems considered herein treat of phenomena associated with the initiation of a half-plane dislocation (Chapter 2), of a half-plane crack (Chapter 3) in an infinite elastic medium, and with the motion of a line source along a half-plane crack (Chapter 4), the medium being initially undisturbed in all these three cases.

For the solution of these problems, integral methods are employed. These methods have long been recognized as most suitable for problems of propagation of electromagnetic and acoustic waves (see Reference 1 for an extensive bibliography). The literature on elastodynamic problems treated by these methods is rather scarce. Maue (Reference 2) applied the Wiener-Hopf method as modified by Clemmow (Reference 3) to the problem of diffraction of plane harmonic elastic waves by a half-plane crack. Clemmow's version is different from the original Wiener-Hopf method in that his formulation is in terms of dual integral equations. Other formulations of elastodynamic problems leading to integral equations amenable to Wiener-Hopf techniques were given by Roseau (Reference 4) and De Hoop (Reference 5). In the present thesis, Clemmow's approach is followed. The wave functions, after being Laplace-transformed, are expressed as superpositions of plane waves, with the amplitude spectra as the unknowns. Application of the boundary conditions gives sets of dual integral equations which are solved by means of Cauchy's integral formula after splitting the kernel functions into factors containing prescribed portions of the singularities. The evaluation of the Laplace inversion integrals is finally carried out by the Cagniard method (Reference 6) as modified by De Hoop (Reference 5).

CHAPTER 1

GENERALITIES

I. THE ELASTIC WAVE EQUATIONS

In vectorial notation, the equation of elastic motion, in the absence of body forces, is:*

$$(\lambda + 2\mu)\nabla(\nabla . \vec{u}) - \mu \nabla \times (\nabla \times \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2} \qquad (1.1)$$

where

λ, μ	elastic constants
ρ	density
$\vec{u}$	displacement vector

Equation 1.1 can be decomposed into

$$\nabla^2 \Delta = v_1^2 \frac{\partial^2 \Delta}{\partial t^2} \qquad (1.2)$$

and

$$\nabla^2(2\vec{\omega}) = v_2^2 \frac{\partial^2(2\vec{\omega})}{\partial t^2} \qquad (1.3)$$

where

$$v_1^2 = \frac{\rho}{\lambda + 2\mu}$$

$$v_2^2 = \frac{\rho}{\mu}$$

$$\Delta = \nabla . \vec{u}$$

$$2\vec{\omega} = \nabla \times \vec{u}$$

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* See e.g. P. Morse and H. Feshbach, "Methods of Theoretical Physics", Vol. I, p. 142.

If we put

$$\vec{u} = -\nabla\phi + \nabla \times \vec{\Psi} \quad ^{(*)} \tag{1.4}$$

then, a solution of the system of equations 1.2 and 1.3 is

$$\nabla^2\phi = v_1^{\;2} \frac{\partial^2\phi}{\partial t^2} \tag{1.5}$$

$$\nabla^2\vec{\Psi} = v_2^{\;2} \frac{\partial^2\vec{\Psi}}{\partial t^2} \tag{1.6}$$

For the problems considered, we choose a system of Cartesian co-ordinates x, y, z, such that there is no variation in the z-direction. Moreover, the z-component of the displacement can be taken equal to zero since the medium extends indefinitely in the z-direction.** For this reason, we can also take the vector $\vec{\Psi}$ as pointing in the z-direction, and in what follows, we shall consider it as a scalar. If u and v are the displacements in the x- and y-directions respectively, then relation 1.4 can be written as:

$$u = -\frac{\partial\phi}{\partial x} + \frac{\partial\Psi}{\partial y} \tag{1.7}$$

$$v = -\frac{\partial\phi}{\partial y} - \frac{\partial\Psi}{\partial x} \tag{1.8}$$

Denoting by σ_y, σ_x and τ_{xy}, the usual stress components, the stress-displacement relations give:

$$\sigma_y = \rho\left\{\left(\frac{1}{v_1^{\;2}} - \frac{2}{v_2^{\;2}}\right)\frac{\partial u}{\partial x} + \frac{1}{v_1^{\;2}}\frac{\partial v}{\partial y}\right\} \tag{1.9}$$

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* The arrow indicates a vectorial quantity.

** This amounts to considering the problem as one in plane strain.

$$\sigma_x = \rho\left\{\left(\frac{1}{v_1^2} - \frac{2}{v_2^2}\right)\frac{\partial v}{\partial y} + \frac{1}{v_1^2}\frac{\partial u}{\partial x}\right\} \tag{1.10}$$

$$\tau_{xy} = \frac{\rho}{v_2^2}\left\{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right\} \tag{1.11}$$

and in terms of the wave functions:

$$\frac{\sigma_y}{\rho} = -\frac{1}{v_1^2}\nabla^2\phi - \frac{2}{v_2^2}\left(\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x^2}\right) \tag{1.12}$$

$$\frac{\sigma_x}{\rho} = -\frac{1}{v_1^2}\nabla^2\phi + \frac{2}{v_2^2}\left(\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2}\right) \tag{1.13}$$

$$\frac{\tau_{xy}}{\rho} = -\frac{2}{v_2^2}\frac{\partial^2\phi}{\partial x\partial y} - \frac{1}{v_2^2}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{v_2^2}\frac{\partial^2\psi}{\partial y^2} \tag{1.14}$$

II. STRESS SINGULARITIES AND CONDITIONS AT INFINITY *

In addition to regular boundary conditions depending on each individual problem, it will be necessary to specify certain subsidiary conditions based on physical reasoning. We have to distinguish an actual source corresponding to an actual energy input, from a virtual source giving no energy input. For an actual source, we have a nonzero resultant external force; therefore the stress singularities should be of the order

$$\frac{1}{r} \quad \text{as} \quad r \to 0$$

r being the distance from the source.

For a virtual source, Maue (Reference 2) specified the singularities to be of the order $\frac{1}{\sqrt{r}}$. This specification does satisfy the requirements that there be no net external force at the source, and that the energy density be an integrable function of r in any neighborhood containing the source. Maue's specification, strictly speaking, has no logical grounds, since a priori there is no way of knowing what the stress singularities are exactly. In what follows, the stresses are specified to be of the order $\frac{1}{r^{\alpha}}$, $r \to 0$, where $0 \leq \alpha < 1$. The actual solution, however, comes out to be of the order $\frac{1}{\sqrt{r}}$, $r \to 0$, as predicted by Maue. As to the conditions at infinity, we only require that the waves be diverging waves.

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* The present discussion of stress singularities is valid only for two-dimensional problems.

CHAPTER 2

RADIATION FROM A DISLOCATION

I. FORMULATION OF THE PROBLEM

Consider an infinite elastic medium with a semi-infinite cut as shown in Figure 1. The cut is assumed infinite in the z-direction and symmetric with respect to the x-axis. The system being initially free from stresses, suppose that the faces of the cut are suddenly, at $t = 0$, drawn into contact and welded together. It is proposed to study the wave patterns generated by such a process. The physical phenomenon involved is called by Love a dislocation. The static and pseudo-dynamic aspects of dislocations have been studied by various authors— see e.g. Rongved (Reference 7), Eshelby (Reference 8). The present problem is however a purely dynamic problem whose characteristics are essentially different from pseudo-dynamic phenomena.

Mathematically, the problem reduces to solving the two elastic wave equations 1.5 and 1.6 subjected to the following boundary conditions at $y = 0$:

$$\begin{aligned} v^+ - v^- &= -H(t)^{(*)} \qquad & x < 0 \\ &= 0 & x > 0 \end{aligned} \tag{2.1}$$

where the $\pm$ signs refer to the values obtained as the x-axis is approached from above and from below respectively. In relations 2.1, $H(t)$ is the Heaviside step function defined as

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* The minus sign is necessary to ensure that the σ_y stress be a tensile one, at $t = 0^+$, along the cut, as shown in the final result.

$$H(t) = 1 \qquad t > 0$$
$$= 0 \qquad t < 0 .$$

Conditions 2.1 mean that for positive values of t, there is a discontinuity in the v-displacement of unit magnitude along the negative x -axis. Together with the conditions 2.1, the u-displacement and all the stresses are specified to be continuous at $y = 0$ except possibly at $x = 0$. Furthermore, because of symmetry,

$$\tau^{+}_{xy} = \tau^{-}_{xy} = 0 \quad \text{at } y = 0 \tag{2.2}$$

II. METHOD OF SOLUTION

Under a Laplace transformation with respect to the time t defined in the usual manner, the wave equations 1.5 and 1.6 become:

$$\nabla^2\overline{\phi} - p^2 v_1{}^2 \overline{\phi} = 0 \tag{2.3}$$

$$\nabla^2\overline{\Psi} - p^2 v_2{}^2 \overline{\Psi} = 0 \tag{2.4}$$

where the bars indicate Laplace-transformed quantities, and p the parameter of the transformation. In terms of $\overline{\phi}$ and $\overline{\Psi}$, the conditions 2.1 and 2.2 can, in view of relations 1.8 and 1.14, be written:

$$\frac{\partial}{\partial y}(\overline{\phi}^- - \overline{\phi}^+) + \frac{\partial}{\partial x}(\overline{\Psi}^- - \overline{\Psi}^+) = -\frac{1}{p} \qquad (x < 0,\ y = 0)$$

$$= 0 \qquad (x > 0,\ y = 0) \tag{2.5}$$

$$-2\frac{\partial^2\overline{\phi}^+}{\partial x \partial y} + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\overline{\Psi}^+$$

$$= -2\frac{\partial^2\overline{\phi}^-}{\partial x \partial y} + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\overline{\Psi}^- = 0 \tag{2.6}$$

$$\text{all } x,\ y = 0$$

where $\frac{1}{p}$ was obtained as the Laplace transform of H(t), and the $\pm$ signs again refer to the values obtained by approaching $y = 0$ from above and from below. The continuity of the displacements and stresses at $y = 0$, $x > 0$, will be ensured by the continuity of the derivatives of $\overline{\phi}$ and $\overline{\Psi}$ up to and including the second order. At $y = 0$, $x < 0$, however, since there is a jump in ϕ and Ψ due to the jump in the v-displacement, the derivatives of ϕ and Ψ do not exist. The continuity of u and of the stresses at $x < 0$, $y = 0$, therefore, should be understood as meaning

$$\left.\begin{aligned} u^+ &= u^- \\ \sigma_y^+ &= \sigma_y^- \end{aligned}\right\} \quad y = 0 \quad x < 0 \tag{2.7}$$

III. THE INTEGRAL EQUATIONS

Let the solutions be of the forms:

$$\overline{\Phi} = \int_{\Gamma} P(\xi)\, e^{-p\left\{\pm\sqrt{v_1^{\,2}+\xi^2}\,y - i\xi x\right\}} d\xi \qquad (2.8a)$$

$$\overline{\Psi} = \mp\int_{\Gamma} Q(\xi)\, e^{-p\left\{\pm\sqrt{v_2^{\,2}+\xi^2}\,y - i\xi x\right\}} d\xi \qquad (2.8b)$$

where p is a positive quantity, $P(\xi)$ and $Q(\xi)$ are two unknown functions, the upper and lower signs refer to the upper and lower half-spaces respectively (Figure 1), and Γ the path of integration to be specified later on. In relations 2.8, the quantities $\sqrt{v_{1,2}^{\,2} + \xi^2}$ are given positive real parts along the path of integration. For further use, we specify their real parts to be positive throughout the ξ complex plane. This is achieved by introducing branch cuts from $\pm iv_1$ to $\pm i\infty$ and from $\pm iv_2$ to $\pm i\infty$. We shall return to this point later.

The representations 2.8 clearly satisfy the conditions at infinity, since the wave functions decay exponentially with increasing $|y|$.

By means of formulae 1.7, 1.12 and 1.13, one can easily verify that the representations 2.8 satisfy the conditions

$$\begin{aligned} u^+ &= u^- \\ \sigma_y^+ &= \sigma_y^- \qquad\qquad y = 0, \text{ all } x \\ \sigma_x^+ &= \sigma_x^- \end{aligned}$$

Condition 2.6 gives:

$$\int_{\Gamma} \left\{ 2i\xi \sqrt{v_1^2 + \xi^2}\, P(\xi) - (2\xi^2 + v_2^2) Q(\xi) \right\} e^{ip\xi x} d\xi = 0 \quad (2.9)$$

for all x

which is satisfied by

$$P(\xi) = (2\xi^2 + v_2^2) R(\xi) \quad (2.10a)$$

$$Q(\xi) = 2i\xi \sqrt{v_1^2 + \xi^2}\, R(\xi) \quad (2.10b)$$

where $R(\xi)$ is a new unknown function. Condition 2.5 combined with relations 2.8 and 2.10 gives:

$$\int_{\Gamma} \sqrt{v_1^2 + \xi^2}\, R(\xi)\, e^{ip\xi x} d\xi = \frac{-1}{2 v_2^2 p^2} \qquad x < 0 \quad (2.11a)$$

$$= 0 \qquad x > 0 \quad (2.11b)$$

But condition 2.11b also guarantees that $\overline{\Phi}$ and $\overline{\Psi}$ and their derivatives of the second order, are continuous at $y = 0$, $x > 0$. The relations 2.11 are the integral equations of the problem.

IV. SOLUTION OF THE INTEGRAL EQUATIONS

Define the path of integration Γ as in Figure 2. Then the system 2.11 can be solved immediately

$$R(\xi) = +\frac{1}{4\pi i} \frac{1}{v_2^2 p^2} \frac{1}{\xi\sqrt{v_1^2+\xi^2}} \tag{2.12}$$

by applying Cauchy's integral formula. It is noted that the method of solution used here will be applied again in the subsequent chapters, with the only difference that the equations involved will be more complex so that Cauchy's formula can be applied only after the kernel functions have been split into factors having singularities in prescribed portions of the ξ-plane.

V. THE STRESS WAVE PATTERNS

Because of the symmetry with respect to the x-axis, it is sufficient to consider the upper half-space ($y > 0$) only. Combining relations 2.8, 2.10 and 2.12 yields:

$$\bar{\phi} = + \frac{1}{2\pi i} \frac{1}{v_2^2 p^2} \int_{\Gamma} \frac{\xi^2 + \frac{v_2^2}{2}}{\xi \sqrt{v_1^2 + \xi^2}} e^{-p\left\{\sqrt{v_1^2+\xi^2}\, y - i\xi x\right\}} d\xi \qquad (2.13a)$$

$$\bar{\Psi} = - \frac{1}{2\pi i} \frac{1}{v_2^2 p^2} \int_{\Gamma} e^{-p\left\{\sqrt{v_2^2+\xi^2}\, y - i\xi x\right\}} d\xi \qquad (2.13b)$$

After some rearrangement, the stresses -- in the Laplace plane -- are given, in view of formulae 1.12, 1.14, 2.3 and 2.4, by:

$$\frac{v_2^2 \bar{\sigma}_y}{2\rho} = \left(\frac{\partial^2}{\partial x^2} - p^2 \frac{v_2^2}{2}\right)\bar{\phi} - \frac{\partial^2 \bar{\Psi}}{\partial x \partial y} \qquad (2.14a)$$

$$\frac{v_2^2 \bar{\sigma}_x}{2\rho} = \left(\frac{\partial^2}{\partial y^2} - p^2 \frac{v_2^2}{2}\right)\bar{\phi} + \frac{\partial^2 \bar{\Psi}}{\partial x \partial y} \qquad (2.14b)$$

$$\frac{v_2^2 \bar{\tau}_{xy}}{2\rho} = \frac{\partial^2 \bar{\phi}}{\partial x \partial y} + \left(p^2 \frac{v_2^2}{2} - \frac{\partial^2}{\partial x^2}\right)\bar{\Psi} \qquad (2.14c)$$

In terms of expressions 2.13, the relations 2.14 give:

$$\frac{v_2^2 \bar{\sigma}_y}{2\rho} = - \frac{1}{2\pi i} \frac{1}{v_2^2} \int_{\Gamma} \frac{\left(\xi^2 + \frac{v_2^2}{2}\right)^2}{\xi \sqrt{v_1^2+\xi^2}} e^{-p\left\{\sqrt{v_1^2+\xi^2}\, y - i\xi x\right\}} d\xi + \frac{1}{2\pi i} \frac{1}{v_2^2} \int_{\Gamma} \xi \sqrt{v_2^2+\xi^2}\, e^{-p\left\{\sqrt{v_2^2+\xi^2}\, y - i\xi x\right\}} d\xi \qquad (2.15a)$$

$$\frac{v_2^2\overline{\sigma}_x}{2\rho} = +\frac{1}{2\pi i}\frac{1}{v_2^2}\int_{\Gamma} \frac{(v_1^2-\frac{v_2^2}{2}+\xi^2)(\xi^2+\frac{v_2^2}{2})}{\xi\sqrt{v_1^2+\xi^2}}\, e^{-p\left\{\sqrt{v_1^2+\xi^2}\,y-i\xi x\right\}}d\xi$$

$$-\frac{1}{2\pi i}\frac{1}{v_2^2}\int_{\Gamma}\xi\sqrt{v_2^2+\xi^2}\; e^{-p\left\{\sqrt{v_2^2+\xi^2}\,y-i\xi x\right\}}d\xi \qquad (2.15b)$$

$$\frac{v_2^2\overline{\tau}_{xy}}{2\rho} = +\frac{1}{2\pi}\frac{1}{v_x^2}\int_{\Gamma}(\xi^2+\frac{v_2^2}{2})e^{-p\left\{\sqrt{v_1^2+\xi^2}\,y-i\xi x\right\}}d\xi$$

$$-\frac{1}{2\pi}\frac{1}{v_2^2}\int_{\Gamma}(\xi^2+\frac{v_2^2}{2})e^{-p\left\{\sqrt{v_2^2+\xi^2}\,y-i\xi x\right\}}d\xi \qquad (2.15c)$$

To obtain the stresses in the physical plane, it would, in general, be necessary to evaluate directly the inversion integrals. However, Cagniard (Reference 6) pointed out that for functions of the form shown in expressions 2.15, it is possible to transform the variables in such a way that they can be recognized as the Laplace transforms of certain functions of the time t, thus making the direct evaluation of the inversion integrals unnecessary. This method has been applied rather widely — see e.g. Pekeris (Reference 9), Garvin (Reference 10). It was recently modified slightly by De Hoop (Reference 5), whose version will be followed here.

Consider the first integral of relation 2.15a

$$\int_{\Gamma} \frac{\left(\xi^2+\frac{v_2^2}{2}\right)^2}{\xi\sqrt{v_1^2+\xi^2}} e^{-p\left\{\sqrt{v_1^2+\xi^2}\, y - i\,\xi x\right\}} d\xi = \bar{I} \quad \text{(say)} \tag{2.16}$$

Let us define

$$t = \sqrt{v_1^2+\xi^2}\; y - i\xi x \tag{2.17a}$$

and choose a new path of integration such that t is real and positive on it. Solving for ξ, we get:

$$\xi = \pm\sqrt{\frac{t^2}{r^2} - v_1^2}\; \sin\theta + i\frac{t}{r}\cos\theta \tag{2.17b}$$

where we have defined the polar coordinates r, θ as:

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x} \qquad 0 \leq \theta \leq \pi \tag{2.18}$$

Then, for $\frac{t}{r} \geq v_1$, the new path of integration represented by relation 2.17b is a hyperbola whose asymptotes make an angle θ with the imaginary ξ-axis (Figure 3). In order to establish the equivalence or non-equivalence of the two paths of integration, we remark that the integrand of $\bar{I}$ in equation 2.16 is regular everywhere in the ξ-plane, except at the pole $\xi = 0$ and along the branch cuts from $\pm iv_1$ to $\pm i\infty$, which we introduced earlier. However, the branch cuts give no difficulties since the hyperbola never crosses them. Therefore, for $0 \leq \theta \leq \frac{\pi}{2}$, there are no singularities between Γ and the hyperbola, and we can write:*

$$\bar{I} = \int_{v_1 r}^{\infty} \left\{ \frac{(\xi_+^{(1)2}+\frac{v_2^2}{2})^2}{\xi_+^{(1)}\sqrt{v_1^2+\xi_+^{(1)2}}} \frac{\partial\xi_+^{(1)}}{\partial t} - \frac{(\xi_-^{(1)2}+\frac{v_2^2}{2})^2}{\xi_-^{(1)}\sqrt{v_1^2+\xi_-^{(1)2}}} \frac{\partial\xi_-^{(1)}}{\partial t} \right\} e^{-pt}\, dt \qquad (2.19a)$$

where

$$\xi_\pm^{(1)} = \pm\sqrt{\frac{t^2}{r^2} - v_1^2}\,\sin\theta + i\frac{t}{r}\cos\theta \qquad (2.19b)$$

Therefore, I, the inverse Laplace transform of $\bar{I}$, can be obtained by inspection:

$$I(r, \theta, t) = H(t-v_1 r) \left\{ \frac{(\xi_+^{(1)2}+\frac{v_2^2}{2})^2}{\xi_+^{(1)}\sqrt{v_1^2+\xi_+^{(1)2}}} \frac{\partial\xi_+^{(1)}}{\partial t} - \frac{(\xi_-^{(1)2}+\frac{v_2^2}{2})^2}{\xi_-^{(1)}\sqrt{v_1^2+\xi^{(1)2}}} \frac{\partial\xi^{(1)}}{\partial t} \right\} \qquad (2.20)$$

with $\xi_\pm^{(1)}$ given by relation 2.19b. The inverse Laplace transform of the second integral of relation 2.15a can be deduced in exactly the same manner by introducing the new variable

$$t = \sqrt{v_2^2 + \xi^2}\; y - i\xi x$$

We therefore get:

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* The two circular arcs at infinity give no contribution because of the exponential decay of the integrand for $\theta > 0$. For $\theta = 0$ ($y = 0$), both integrals of equation 2.15a have to be considered together. It is shown that as $|\xi| \rightarrow \infty$, the two integrands combined behave like $e^{ip\xi x}$. The integrals over Γ therefore do not converge in the strict mathematical sense. We nevertheless attribute them a meaning by considering them as the limit of $\bar{\delta}_y$ as $y \rightarrow 0$.

$$\frac{v_2^4 \delta_y}{2\rho} = \frac{-H(t-v_1 r)}{2\pi i} \left\{ \frac{\left(\xi_+^{(1)2} + \frac{v_2^2}{2}\right)^2}{\xi_+^{(1)} \sqrt{v_1^2 + \xi_+^{(1)2}}} \frac{\partial \xi_+^{(1)}}{\partial t} - \frac{\left(\xi_-^{(1)2} + \frac{v_2^2}{2}\right)^2}{\xi_-^{(1)} \sqrt{v_1^2 + \xi^{(1)2}}} \frac{\partial \xi_-^{(1)}}{\partial t} \right\} \tag{2.21}$$

$$+ \frac{H(t-v_2 r)}{2\pi i} \left\{ \xi_+^{(2)} \sqrt{v_2^2 + \xi_+^{(2)2}} \frac{\partial \xi_+^{(2)}}{\partial t} - \xi_-^{(2)} \sqrt{v_2^2 + \xi_-^{(2)}} \frac{\partial \xi_-^{(2)}}{\partial t} \right\}$$

with $\xi_{\pm}^{(1)}$ given by formula 2.19b and $\xi^{(2)}$ given by:

$$\xi_{\pm}^{(2)} = \pm \sqrt{\frac{t^2}{r^2} - v_2^2} \sin \theta + i \frac{t}{r} \cos \theta \tag{2.22}$$

For $\frac{\pi}{2} < \theta \leq \pi$, the contribution of the pole $\xi = 0$ should be taken into account. Hence:

$$\frac{v_2^4 \delta_y}{2\rho} = \frac{-H(t-v_1 r)}{2\pi i} \left\{ \frac{\left(\xi_+^{(1)2} + \frac{v_2^2}{2}\right)^2}{\xi_+^{(1)} \sqrt{v_1^2 + \xi_+^{(1)2}}} \frac{\partial \xi_+^{(1)}}{\partial t} - \frac{\left(\xi_-^{(1)2} + \frac{v_2^2}{2}\right)^2}{\xi_-^{(1)} \sqrt{v_1^2 + \xi_-^{(1)2}}} \frac{\partial \xi_+^{(1)}}{\partial t} \right\}$$

$$+ \frac{v_2^4}{4 v_1} H(-\cos \theta)\, \delta(t - v_1 y) \tag{2.23}$$

$$+ \frac{H(t-v_2 r)}{2\pi i} \left\{ \xi_+^{(2)} \sqrt{v_2^2 + \xi_+^{(2)2}} \frac{\partial \xi_+^{(2)}}{\partial t} - \xi_-^{(2)} \sqrt{v_2^2 + \xi_-^{(2)2}} \frac{\partial \xi_-^{(2)}}{\partial t} \right\}$$

$$0 \leq \theta \leq \pi$$

with $\xi_{\pm}^{(1)}$ and $\xi_{\pm}^{(2)}$ given by formulae 2.19b and 2.22, and $\delta(t-v_1 y)$ being a Dirac delta function. The representation 2.23 is valid for the whole range $0 \leq \theta \leq \pi$ (i.e. including $\theta = \frac{\pi}{2}$), provided that the step function $H(-\cos \theta)$ be interpreted as follows:

$$H(x) = 1 \qquad x > 0$$
$$= \frac{1}{2} \qquad x = 0$$
$$= 0 \qquad x < 0$$

This is the case since at $\theta = \frac{\pi}{2}$ $(x = 0)$ the hyperbola degenerates into the real ξ-axis, and we can let the integral share half the contribution of the pole $\xi = 0$.

We get similar expressions for σ_x and τ_{xy} from formulae 2.15b and 2.15c in exactly the same manner:

$$\frac{v_2^4 \sigma_x}{2\rho} = + \frac{H(t-v_1 r)}{2\pi i} \left\{ \frac{(v_1^2 - \frac{v_2^2}{2} + \xi_+^{(1)2})(\frac{v_2^2}{2} + \xi_+^{(1)2})}{\xi_+^{(1)} \sqrt{v_1^2 + \xi_+^{(1)2}}} \frac{\partial \xi_+^{(1)}}{\partial t} - \frac{(v_1^2 - \frac{v_2^2}{2} + \xi_-^{(1)2})(\frac{v_2^2}{2} + \xi_-^{(1)2})}{\xi_-^{(1)} \sqrt{v_1^2 + \xi_-^{(1)2}}} \frac{\partial \xi_-^{(1)}}{\partial t} \right\}$$
$$- \frac{(2v_1^2 - v_2^2) v_2^2}{4 v_1} H(-\cos\theta)\, \delta(t - v_1 y)$$
$$- \frac{H(t-v_2 r)}{2\pi i} \left\{ \xi_+^{(2)} \sqrt{v_2^2 + \xi_+^{(2)2}} \frac{\partial \xi_+^{(2)}}{\partial t} - \xi_-^{(2)} \sqrt{v_2^2 + \xi_-^{(2)2}} \frac{\partial \xi_-^{(2)}}{\partial t} \right\} \qquad (2.24)$$

$$0 \leq \theta \leq \pi$$

$$\frac{v_2^{\,4}\,\tau_{xy}}{2\rho} = + \frac{H(t-v_1 r)}{2\pi}\left\{(\xi_+^{(1)2}+\frac{v_2^{\,2}}{2})\frac{\partial\xi_+^{(1)}}{\partial t} - (\xi_-^{(1)2}+\frac{v_2^{\,2}}{2})\frac{\partial\xi_-^{(1)}}{\partial t}\right\}$$

$$- \frac{H(t-v_2 r)}{2\pi}\left\{(\xi_+^{(2)2}+\frac{v_2^{\,2}}{2})\frac{\partial\xi_+^{(2)}}{\partial t} - (\xi_-^{(2)2}+\frac{v_2^{\,2}}{2})\frac{\partial\xi_-^{(2)}}{\partial t}\right\}$$

$$0 \lesssim \theta \lesssim \pi \qquad (2.25)$$

with $\xi_\pm^{(1)}$ and $\xi_\pm^{(2)}$ given by formulae 2.19b and 2.22.

It is clear from the foregoing expressions that the radiated waves consist of three types. First, there is a sort of plane wave* emanating from the surface of the dislocation — this is represented by the second terms in equations 2.23 and 2.24. Next there are the two cylindrical waves radiating from the edge of the dislocation with the shear and dilatation wave velocities, respectively. Let us, first, focus our attention on the disturbances generated by the plane wave. By itself, it gives rise to no shear deformation. However, since it influences only the left half of the elastic space, it generates a discontinuity across the plane $X = 0$. The discontinuity involved here is one in the v-displacement. This fact is in agreement with our intuitive view that, since the two faces of the cut ($y = 0$, $X < 0$) are given an infinite vertical velocity at $t = 0$, the particles belonging to the right half space ($X > 0$) do not "have enough time" to follow the disturbance, and hence a discontinuity should exist along the plane $X = 0$. Thus a dislocation suddenly generated along the half plane

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* For convenience, we shall call it "plane wave" although this is clearly a misnomer.

$y = 0$, $x < 0$, gives rise to another one propagating with the velocity of the dilatation wave velocity in the plane $x = 0$. The existence of the plane wave in the left half space alone cannot satisfy the equilibrium condition throughout the whole medium. Thus the two cylindrical waves come into the picture. It is remarkable that these waves do not interact with the discontinuous surface $y = 0$, $x < 0$, or with the propagating dislocation along $x = 0$. This fact is to be contrasted with phenomena to be found in the subsequent chapters when a dilatation cylindrical wave interacts with a free boundary to produce shear waves. The complete wave patterns for this dislocation problem are represented in Figure 4.

VI. THE STRESS SINGULARITIES

To study the stresses more closely, it is necessary to substitute for ξ in terms of the physical coordinates in the expressions 2.23 to 2.25. Thus, in terms of r, θ and t, the stresses are:

$$
\begin{aligned}
\frac{v_2^4\pi}{2\rho}\sigma_y = & -\frac{H(t-v_1 r)}{\sqrt{t^2-v_1^2 r^2}}\frac{t^3}{r^3}\cos\theta\Big\{3\sin^2\theta-\cos^2\theta-3v_1^2\frac{r^2}{t^2}\sin^2\theta \\
& + v_2^2\frac{r^2}{t^2}-\frac{v_2^4}{4}\frac{r^4}{t^4-v_1^2r^2t^2\sin^2\theta}\Big\} \\
& + \frac{v_2^4\pi}{4v_1}H(-\cos\theta)\,\delta(t-v_1 y) \\
& + \frac{H(t-v_2 r)}{\sqrt{t^2-v_2^2 r^2}}\frac{t^3\cos\theta}{r^3}\Big\{\frac{v_2^2r^2}{t^2}\cos^2\theta-2\sin^2\theta\frac{v_2^2r^2}{t^2} \\
& + 3\sin^2\theta-\cos^2\theta\Big\}
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
\frac{v_2^4\pi}{2\rho}\sigma_x = & \frac{H(t-v_1 r)}{\sqrt{t^2-v_1^2 r^2}}\frac{t^3}{r^3}\cos\theta\Big\{3\sin^2\theta-\cos^2\theta \\
& -3\frac{v_2^2r^2}{t^2}\sin^2\theta+\left(v_1^2-\frac{v_2^2}{2}\right)\frac{r^2}{t^2} \\
& -\left(v_1^2-\frac{v_2^2}{2}\right)\frac{v_2^2}{2}\frac{r^4}{t^4-v_1^2r^2t^2\sin^2\theta}\Big\} \\
& -\frac{(2v_1^2-v_2^2)v_2^2\pi}{4v_1}H(-\cos\theta)\,\delta(t-v_1 y)
\end{aligned}
$$

(cont'd)

$$- \frac{H(t-v_2 r)}{\sqrt{t^2 - v_2^2 r^2}} \frac{t^3 \cos\theta}{r^3} \left\{ \frac{v_2^2 r^2}{t^2} \cos^2\theta - 2 \sin^2\theta \, v_2^2 \frac{r^2}{t^2} + 3 \sin^2\theta - \cos^2\theta \right\} \tag{2.27}$$

and

$$\frac{v_2^4 \pi \tau_{xy}}{2\rho} = \frac{H(t-v_1 r)}{\sqrt{t^2 - v_1^2 r^2}} \frac{t^3 \sin\theta}{r^3} \left\{ \frac{v_2^2 r^2}{2t^2} + \left(1 - \frac{v_1^2 r^2}{t^2}\right) \sin^2\theta - 3\cos^2\theta + 2\cos^2\theta \frac{v_1^2 r^2}{t^2} \right\}$$
$$- \frac{H(t-v_2 r)}{\sqrt{t^2 - v_2^2 r^2}} \frac{t^3 \sin\theta}{r^3} \left\{ \frac{v_2^2 r^2}{2t^2} + \left(1 - \frac{v_2^2 r^2}{t^2}\right) \sin^2\theta - 3\cos^2\theta + 2\cos^2\theta \frac{v_2^2 r^2}{t^2} \right\} \tag{2.28}$$

From these expressions, it is seen that as $r \to 0$, the dilatational stresses are of the order $\frac{1}{r}$. This corresponds to a concentrated force at the edge. Furthermore, it is remarked that for $t > 0$, in the plane of the dislocation ($\theta = \pi$), the dilatational stresses are everywhere zero except behind the fronts of the cylindrical waves. This means that if the edge were infinitely far away, the corresponding picture would be a one-dimensional one, as this should be the case.

As mentioned in section V above, there exists a skew dislocation* traveling along the plane $x = 0$, away from the edge. It would be natural to expect, therefore, that at a given time and a given radial

---

* By skew dislocation is meant a discontinuity of the displacement component tangential to the locus of the discontinuities.

position, the shear is a maximum at $x = 0$. This suspicion is further strengthened by the fact that the plane $x = 0$ is under pure shear deformation -- c.f. the stress expressions 2.26 to 2.28. It turns out, however, that this is not necessarily the case. As shown in Figure 5*, at $t = 10^{-5}$ sec, $r = 4$ cm, the absolute value of the shear, although a relative maximum in the plane $x = 0$, attains its absolute maximum at some other positions, in fact in the neighborhoods of $\theta = 150^\circ$ and $\theta = 30^\circ$. A glance at the graphs of Figure 5 shows, on the other hand, that the shear is not the item of major concern, since the normal stresses are much higher in magnitude in the neighborhoods of $\theta = 50^\circ$ and $\theta = 130^\circ$.

---

* Figure 5 represents the stress ratios σ_x/σ_o, σ_y/σ_o and τ_{xy}/σ_o, the quantity σ_o being some constant of proportionality. It is the plot of equations 2.26--2.28 for $v_1^{-1} = 5 \times 10^5$ cm/sec and $v_2^{-1} = 3 \times 10^5$ cm/sec.

CHAPTER 3

UNLOADING WAVES GENERATED BY A SUDDEN CUT

Having solved the problem of a cut that is suddenly contact-welded, it seems natural to investigate the inverse problem, that of a cut suddenly generated in a stressed medium. In fact, the latter has been solved by Maue using conical coordinates (Reference 11). Our excuse for investigating this same problem is that we would like to relate it to the more general one of a line load moving along a half plane crack*, which problem is readily amenable to the present Wiener-Hopf techniques.

Consider an infinite elastic medium subjected to a uniform tension T at infinity as shown in Figure 6. At time $t = 0$, the medium is suddenly cut along the half-plane $y = 0 \quad x < 0$. It is proposed to investigate the wave patterns generated.

---

* These ideas will be cleared up in the next chapter.

I. THE INITIAL AND BOUNDARY VALUE PROBLEM

The problem is to solve the two wave equations subjected to the following conditions:

$$\sigma_y = T \qquad |y| = \infty, \qquad \text{all } t < \infty \tag{3.1a}$$

$$= T \qquad \text{all } x, y \qquad t > 0 \tag{3.1b}$$

$$= 0 \qquad y = 0, \qquad x < 0, \qquad t > 0 \tag{3.1c}$$

$$\tau_{xy} = 0 \qquad |y| = \infty, \qquad t < \infty \tag{3.2a}$$

$$= 0 \qquad y = 0 \qquad \text{all } x, \qquad \text{all } t \tag{3.2b}$$

The stress patterns will be split into two parts: an initial static part and a radiated part, i.e.,

$$\sigma_y = \sigma_y^{(r)} + T \tag{3.3a}$$

$$\sigma_x = \sigma_x^{(r)} + \nu' T \quad (*) \tag{3.3b}$$

$$\tau_{xy} = \tau_{xy}^{(r)} \tag{3.3c}$$

where the superscript (r) indicates the radiated part.

The representations 3.3 automatically satisfy conditions 3.1a, 3.1b and 3.2a. The radiated stresses are derivable from the wave functions ϕ and Ψ as defined previously. The method followed here is much the same as in Chapter 2. We first take the Laplace transforms with respect to t, of all the quantities involved, i.e., stresses, strains, and the equations of motion. The latter then take the forms 2.3 and 2.4.

---

* For $t < 0$, $E\epsilon_x = \sigma_x - \nu(\sigma_y + \sigma_z) = 0$ and $E\epsilon_z = \sigma_z - \nu(\sigma_x + \sigma_y) = 0$. Hence $\sigma_x = \nu'\sigma_y$ where $\nu' = \frac{\nu}{1-\nu}$, ν being Poisson's ratio, and E Young's modulus.

As in Chapter 2, let

$$\overline{\Phi} = \int_{\Gamma} P(\xi) e^{-p\left\{\pm\sqrt{v_1^2+\xi^2}\,y - i\xi x\right\}} d\xi \tag{3.4}$$

$$\overline{\Psi} = \mp \int_{\Gamma} Q(\xi) e^{-p\left\{\pm\sqrt{v_2^2+\xi^2}\,y - i\xi x\right\}} d\xi \tag{3.5}$$

be the wave functions (in the Laplace plane). Here again, the upper and lower signs refer to $y > 0$ and $y < 0$ respectively. In view of equation 2.14, the conditions 3.1c and 3.2b give, in terms of the representations 3.4 and 3.5:

$$\int_{\Gamma} \left\{ \left(\xi^2 + \frac{v_2^2}{2}\right) P(\xi) + i\xi\sqrt{v_2^2+\xi^2}\, Q(\xi) \right\} e^{ip\xi x} d\xi = \frac{\delta_o}{p^3} \tag{3.6}$$

$$x < 0$$

and

$$\int_{\Gamma} \left\{ i\xi\sqrt{v_1^2+\xi^2}\, P(\xi) - \left(\frac{v_2^2}{2} + \xi^2\right) Q(\xi) \right\} e^{ip\xi x} d\xi = 0 \tag{3.7}$$

$$\text{all} \quad x$$

where

$$\delta_o = v_2^2 T/2\rho$$

Condition 3.7 is satisfied by:

$$P(\xi) = \left(\xi^2 + \frac{v_2^2}{2}\right) R(\xi) \tag{3.8a}$$

$$Q(\xi) = i\xi\sqrt{v_1^2 + \xi^2}\, R(\xi) \tag{3.8b}$$

where $R(\xi)$ is a newly introduced unknown.

In terms of relations 3.8, condition 3.6 can be written:

$$\int_{\Gamma} \left\{ \left(\xi^2 + \frac{v_2^2}{2}\right)^2 - \xi^2 \sqrt{(v_1^2+\xi^2)(v_2^2+\xi^2)} \right\} R(\xi) e^{ip\xi x} d\xi = \frac{\delta_o}{p^3} \qquad x < 0 \tag{3.9}$$

On the other hand, the conditions of continuity at $y = 0$, $x \quad 0$, are

$$\frac{\partial \bar{\phi}}{\partial y} = \bar{\Psi} = 0 \qquad y = 0 \qquad x > 0 \tag{3.10}$$

which is satisfied by

$$\int_{\Gamma} \sqrt{v_1^2 + \xi^2}\, R(\xi) e^{ip\xi x} d\xi = 0 \qquad x > 0 \tag{3.11}$$

Equations 3.9 and 3.11 are the integral equations of the problem.

II. SOLUTION OF THE INTEGRAL EQUATIONS

Let us first define the path of integration Γ as in Figure 7. Since p is real and positive, equation 3.9, upon applying the theorem of residues, is satisfied by:

$$\left\{\left(\xi^2+\frac{v_2^2}{2}\right)^2 - \xi^2\sqrt{(v_1^2+\xi^2)(v_2^2+\xi^2)}\right\}R(\xi) = -\frac{1}{2\pi i}\frac{\sigma_o}{p^3}\frac{L(\xi)}{L(0)}\frac{1}{\xi} \qquad (3.12)$$

where $L(\xi)$ is any function without zeros or singularities in the lower ξ half-plane, except at infinity where it is only required to be of algebraic behavior. Similarly equation 3.11 is satisfied by

$$\sqrt{v_1^2+\xi^2}\,R(\xi) = U(\xi) \qquad (3.13)$$

where $U(\xi)$ is a function with corresponding properties in the upper half-plane. Eliminating $R(\xi)$ from relations 3.12 and 3.13 gives:

$$\frac{U(\xi)}{\sqrt{v_1^2+\xi^2}} = -\frac{1}{2\pi i}\frac{\sigma_o}{p^3}\frac{1}{v_2^2-v_1^2}\frac{L(\xi)}{L(0)}\frac{1}{\xi}\frac{2}{\sqrt{(v_1^2+\xi^2)(v_2^2+\xi^2)}}\frac{1}{F(\xi)} \qquad (3.14)$$

where

$$F(\xi) = \frac{2}{v_2^2-v_1^2}\left\{\frac{\left(\xi^2+\frac{v_2^2}{2}\right)^2}{\sqrt{(v_1^2+\xi^2)(v_2^2+\xi^2)}} - \xi^2\right\} \qquad (3.15)$$

Suppose that the function $F(\xi)$ can be split into two factors, one being a U-function, the other an L-function, i.e.,

$$F(\xi) = F_U(\xi)\,F_L(\xi) \qquad (3.16)$$

where the subscripts U and L have obvious meanings.

Then we can rewrite 3.14 as:

$$\frac{U(\xi)}{L(\xi)} = -\frac{1}{\pi i}\frac{\delta_o}{v_2^2-v_1^2}\frac{1}{p^3}\frac{1}{L(0)}\left\{\frac{1}{\xi}\frac{1}{\sqrt{\xi+iv_2}\,F_U(\xi)}\right\}\left\{\frac{1}{\sqrt{\xi-iv_2}\,F_L(\xi)}\right\} \tag{3.17}$$

If we introduce the branch cuts from $\mp iv_2$ to $\mp i\infty$, then $\sqrt{\xi+iv_2}$ and $\sqrt{\xi-iv_2}$ are U- and L-functions respectively, and a solution of equation 3.17 is:

$$U(\xi) = -\frac{1}{\pi i}\frac{\delta_o}{v_2^2-v_1^2}\frac{1}{p^3}\frac{1}{F_L(0)\sqrt{-iv_2}}\frac{1}{\xi}\frac{1}{\sqrt{\xi+iv_2}\,F_U(\xi)} \tag{3.18a}$$

$$L(\xi) = \sqrt{\xi-iv_2}\,F_L(\xi) \tag{3.18b}$$

From relations 3.13 and 3.8

$$R(\xi) = -\frac{1}{\pi i}\frac{\delta_o}{v_2^2-v_1^2}\frac{1}{p^3}\frac{2}{F_L(0)\sqrt{-iv_2}}\frac{1}{\xi}\frac{1}{F_U(\xi)\sqrt{(\xi+iv_2)(v_1^2+\xi^2)}} \tag{3.19a}$$

$$P(\xi) = -\frac{1}{\pi i}\frac{\delta_o}{v_2^2-v_1^2}\frac{1}{p^3}\frac{1}{F_L(0)\sqrt{-iv_2}}\frac{\xi^2+\frac{v_2^2}{2}}{\xi\sqrt{(\xi+v_2)(v_1^2+\xi^2)}}\frac{1}{F_U(\xi)} \tag{3.19b}$$

$$Q(\xi) = -\frac{1}{\pi i}\frac{\delta_o}{v_2^2-v_1^2}\frac{1}{p^3}\frac{1}{F_L(0)\sqrt{-iv_2}}\frac{i}{F_U(\xi)\sqrt{\xi+iv_2}} \tag{3.19c}$$

III. FACTORIZATION OF THE KERNEL FUNCTION $F(\xi)$*

Consider

$$F(\xi) = \frac{2}{v_2^2 - v_1^2}\left\{ \frac{(\xi^2 + \frac{v_2^2}{2})^2}{\sqrt{(v_1^2+\xi^2)(v_2^2+\xi^2)}} - \xi^2 \right\}$$

This function is regular in the cut ξ-plane shown in Figure 7. At infinity

$$F(\xi) \underset{|\xi|\to\infty}{\sim} 1 + 0(\frac{1}{\xi^2}) \tag{3.20}$$

Furthermore, its zeros are at $\pm i\xi_R$ (simple zeros) where ξ_R is the reciprocal of the Rayleigh surface wave velocity:

$$\xi_R > v_{1,\,2} \tag{3.21}$$

Hence

$$\log F(\xi) = \log F_U(\xi) + \log F_L(\xi) \tag{3.22}$$

where

$$\log F_U(\xi) = \frac{1}{2\pi i}\int_{C_U} \frac{\log F(z)}{z - \xi}\, dz \tag{3.23}$$

$$\log F_L(\xi) = \frac{1}{2\pi i}\int_{C_L} \frac{\log F(z)}{z - \xi}\, dz \tag{3.24}$$

C_U and C_L being as represented in Figure 8 — the circle at infinity gives no contribution since $\log F(\xi) \underset{|\xi|\to\infty}{\to} 0$ from relation 3.20. Therefore

$$F(\xi) = F_U(\xi)\; F_L(\xi) \tag{3.25}$$

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*See e.g. Reference 2.

is the required factorization, $F_U(\xi)$ being regular and free from zeros in the upper half-plane, and $F_L(\xi)$ being a corresponding function in the lower half-plane.

IV. THE STRESS WAVE PATTERNS

Because of symmetry it is sufficient to consider only the half-plane of positive y. In the Laplace transformed plane, the stresses, in view of relations 2.14, 3.4, 3.5 and 3.19 are given by:

$$\bar{\sigma}_y = \frac{1}{2\pi i}\frac{K}{p}\int_\Gamma A(\xi)\frac{e^{-p\left\{\sqrt{v_1^2+\xi^2}\,y-i\xi x\right\}}}{\xi}d\xi - \frac{1}{2\pi i}\frac{K}{p}\int_\Gamma B(\xi)\,e^{-p\left\{\sqrt{v_2^2+\xi^2}\,y-i\xi x\right\}}d\xi + \frac{\sigma_o}{p} \qquad (3.26)$$

$$\bar{\sigma}_x = -\frac{1}{2\pi i}\frac{K}{p}\int_\Gamma C(\xi)\frac{e^{-p\left\{\sqrt{v_1^2+\xi^2}\,y-i\xi x\right\}}}{\xi}d\xi + \frac{1}{2\pi i}\frac{K}{p}\int_\Gamma B(\xi)e^{-p\left\{\sqrt{v_2^2+\xi^2}\,y-i\xi x\right\}}d\xi + \frac{\nu\sigma_o}{p} \qquad (3.27)$$

$$\bar{\tau}_{xy} = -\frac{1}{2\pi}\frac{K}{p}\int_\Gamma D(\xi)e^{-p\left\{\sqrt{v_1^2+\xi^2}\,y-i\xi x\right\}}d\xi + \frac{1}{2\pi}\frac{K}{p}\int_\Gamma D(\xi)e^{-p\left\{\sqrt{v_2^2+\xi^2}\,y-i\xi x\right\}}d\xi \qquad (3.28)$$

where

$$K = \frac{4\sigma_o}{v_2^2-v_1^2}\frac{\rho}{v_2}\frac{1}{F_L(0)\sqrt{-iv_2}} \qquad (3.29)$$

$$A(\xi) = \frac{\left(\xi^2+\frac{v_2^2}{2}\right)}{\sqrt{(\xi+iv_2)(v_1^2+\xi^2)}\;F_U(\xi)} \qquad (3.30)$$

$$B(\xi) = \frac{\xi\sqrt{\xi-iv_2}}{F_U(\xi)} \qquad (3.31)$$

$$C(\xi) = \frac{(\xi^2+v_1^2-\frac{v_2^2}{2})(\xi^2+\frac{v_2^2}{2})}{\sqrt{(\xi+iv_2)(v_1^2+\xi^2)}\, F_U(\xi)} \qquad (3.32)$$

$$D(\xi) = \frac{(\xi^2+\frac{v_2^2}{2})}{\sqrt{\xi+iv_2}\, F_U(\xi)} \qquad (3.33)$$

For the evaluation of the Laplace inversion integrals, we follow, as in Chapter 2, the Cagniard-De Hoop method. Consider the first integral of formula 3.26:

$$\int_{\Gamma} A(\xi)\frac{e^{-p\left\{\sqrt{v_1^2+\xi^2}\,y-i\xi x\right\}}}{\xi}\,d\xi = \overline{I}_1(x\ y) \qquad \text{(say)} \qquad (3.34)$$

Define a new variable:

$$t = \sqrt{v_1^2+\xi^2}\,y-i\xi x \qquad (3.35)$$

and choose a new path of integration such that on it, t is real and positive so that expression 3.34 would be recognizable as the Laplace transform of some function of the time t.

Solving for ξ from formula 3.35 gives:

$$\xi = \pm\sqrt{\frac{t^2}{r^2}-v_1^2}\,\sin\theta + i\frac{t}{r}\cos\theta \qquad (3.36)$$

where

$$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$0 \leq \theta \leq \pi$$

Then, for $\frac{t}{r} > v_1$, relation 3.36 represents a hyperbola (Figure 9). The integrand of $\overline{I}_1(x, y)$ is regular everywhere in the ξ-plane except along the branch cuts from $\pm iv_1$ to $\pm i\infty$, and at the pole $\xi = 0$.

Therefore for $0 \leq \theta < \frac{\pi}{2}$, there are no singularities between the original path Γ and the new path represented as Γ_1 in Figure 9. Since on the other hand, the two circular arcs at infinity give no contribution, we can write:

$$\bar{I}_1 = \int_{v_1 r}^{\infty} \left\{ \frac{A(\xi_+^{(1)})}{\xi_+^{(1)}} \frac{\partial \xi_+^{(1)}}{\partial t} - \frac{A(\xi_-^{(1)})}{\xi_-^{(1)}} \frac{\partial \xi_-^{(1)}}{\partial t} \right\} e^{-pt}\, dt \tag{3.37}$$

where

$$\xi_\pm^{(1)} = \pm\sqrt{\frac{t^2}{r^2} - v_1^{\,2}}\, \sin\theta + i\frac{t}{r}\cos\theta \tag{3.38}$$

By inspection, the inverse Laplace transform of $\bar{I}_1$ is

$$I_1(r, \theta, t) = H(t-v_1 r)\left\{ \frac{A(\xi_+^{(1)})}{\xi_+^{(1)}} \frac{\partial \xi_+^{(1)}}{\partial t} - \frac{A(\xi_-^{(1)})}{\xi_-^{(1)}} \frac{\partial \xi_-^{(1)}}{\partial t} \right\} \qquad 0 \leq \theta < \frac{\pi}{2} \tag{3.39}$$

with $A(\xi)$ given by formula 3.30 and $\xi_\pm^{(1)}$ by formula 3.38.

For $\frac{\pi}{2} \leq \theta \leq \pi$, the contribution of the pole $\xi = 0$ should be taken into account, i.e.,

$$I_1(r, \theta, t) = H(t-v_1 r)\left\{ \frac{A(\xi_+^{(1)})}{\xi_+^{(1)}} \frac{\partial \xi_+^{(1)}}{\partial t} - \frac{A(\xi_-^{(1)})}{\xi_-^{(1)}} \frac{\partial \xi_-^{(1)}}{\partial t} \right\} - 2\pi i\, A(0)\, \delta(t-v_1 y) \qquad \frac{\pi}{2} < \theta \leq \pi \tag{3.40}$$

where δ is the Dirac delta function.

Consider the second integral of formula 3.26:

$$\int_{\Gamma} B(\xi) e^{-p\left\{\sqrt{v_2^2+\xi^2}\, y - i\xi x\right\}} d\xi = \bar{I}_2(x\ y) \qquad \text{(say)} \qquad (3.41)$$

Define

$$t = \sqrt{v_2^2+\xi^2}\, y - i\xi x$$

Solving for ξ gives:

$$\xi = \pm\sqrt{\frac{t^2}{r^2} - v_2^2}\, \sin\theta + i\frac{t}{r}\cos\theta \qquad (3.42)$$

The integrand of $\bar{I}_2(x\ y)$ is regular everywhere except along the branch cuts from $\pm iv_2$ to $\pm i\infty$ and from $-iv_1$ to $-iv_2$. For $0 < \theta < \frac{\pi}{2}$, no singularities lie between Γ and the hyperbola represented by equation 3.42, and we can write:

$$\bar{I}_2(r\ \theta\ p) = \int_{v_2 r}^{\infty} \left\{ B(\xi_+^{(2)}) \frac{\partial \xi_+^{(2)}}{\partial t} - B(\xi_-^{(2)}) \frac{\partial \xi_-^{(2)}}{\partial t} \right\} e^{-pt}\, dt \qquad (3.43)$$

where

$$\xi_{\pm}^{(2)} = \pm\sqrt{\frac{t^2}{r^2} - v_2^2}\, \sin\theta + i\frac{t}{r}\cos\theta \qquad (3.44)$$

By inspection, $I_2(r,\ \theta,\ t)$, the inverse Laplace transform of $\bar{I}_2$, is:

$$I_2(r,\ \theta,\ t) = H(t-v_2 r)\left\{ B(\xi_+^{(2)} \frac{\partial \xi_+^{(2)}}{\partial t} - B(\xi_-^{(2)} \frac{\partial \xi_-^{(2)}}{\partial t} \right\} \qquad (3.45)$$

$$0 < \theta < \frac{\pi}{2}$$

where $B(\xi)$ is given by formula 3.31 and $\xi_{\pm}^{(2)}$ by formula 3.44.

For $\frac{\pi}{2} < \theta < \pi$, the situation is slightly more complex since the hyperbola may cross the lower branch cut between $-iv_1$ and $-iv_2$ if $\pi > \theta > \pi - \cos^{-1}\frac{v_1}{v_2}$. In the latter case, in order to have a new path of integration equivalent to Γ (the original path), we must add to the hyperbola Γ_2 (Figure 9) the two segments represented by

$$\xi = i\left\{-\sqrt{v_2^2 - \frac{t^2}{r^2}}\sin\theta + \frac{t}{r}\cos\theta\right\} \mp \delta \qquad (\delta \to 0)$$

$$\pi - \cos^{-1}\frac{v_1}{v_2} \leq \theta \leq \pi \tag{3.46}$$

and a circular arc centered at $-iv_1$ and of radius δ ($\delta \to 0$). In relation 3.46 the range of t is:

$$t_S \leq t \leq v_2 r \tag{3.47a}$$

where

$$t_S = -v_1 r\cos\theta + r\sqrt{v_2^2 - v_1^2}\sin\theta \tag{3.47b}$$

While the contribution of the δ-circular arc is nil, the path 3.46 together with the hyperbolic path Γ_2 (Figure 9) gives:

$$I_2(r, \theta, t) = H(t - v_2 r)\left\{B(\xi_+^{(2)})\frac{\partial\xi_+^{(2)}}{\partial t} - B(\xi_-^{(2)})\frac{\partial\xi_-^{(2)}}{\partial t}\right\}$$

$$f_\theta(\theta)\, f_S(t)\left\{B(\xi_+^{(S)}) - B(\xi_-)\right\}\frac{\partial\xi_+^{(S)}}{\partial t} \tag{3.48}$$

$$\frac{\pi}{2} < \theta < \pi$$

where $B(\xi)$ is given by formula 3.31, and:

$$\xi_\pm^{(S)} = \pm\delta + i\left\{-\sqrt{v_2^2 - \frac{t^2}{r^2}}\sin\theta + \frac{t}{r}\cos\theta\right\} \tag{3.49}$$

$$(\delta \to 0)$$

$$f_\theta(\theta) = 1 \qquad \pi - \cos^{-1}\frac{v_1}{v_2} \leq \theta \leq \pi \tag{3.50}$$

$$= 0 \qquad \text{outside}$$

$$f_S(t) = 1 \qquad t_S \leq t \leq v_2 r \tag{3.51}$$

$$= 0 \qquad \text{outside}$$

We have essentially found $\frac{\partial \sigma_y}{\partial t}$ in quadrants $0 < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \pi$. We now investigate the angular positions $\theta = \frac{\pi}{2}$, $\theta = 0$ and $\theta = \pi$. For $\theta = \frac{\pi}{2}$, the hyperbola degenerates into the axis of real ξ. The integral $\overline{I}_1$ therefore receives half the contribution of the pole $\xi = 0$ whereas $\overline{I}_2$ is not affected by the latter. At $\theta = 0$,

$$\overline{I}_1 + \overline{I}_2 = \int_\Gamma \frac{A(\xi) - \xi B(\xi)}{\xi} e^{ip\xi x} d\xi \tag{3.52}$$

where $A(\xi)$ and $B(\xi)$ are given by relations 3.30 and 3.31. By an asymptotic expansion of $A(\xi)$ and $B(\xi)$, it is readily shown that the integrand of expression 3.52 is of the order $\frac{e^{ip\xi x}}{\sqrt{\xi}}$ as $|\xi| \to \infty$. The integral in equation 3.52 therefore converges. The combination $\overline{I}_1 + \overline{I}_2$ is thus continuous at $\theta = 0$, and its value is obtained by taking the limits of expressions 3.39 and 3.45 as $\theta \to 0$. At $\theta = \pi$, the integrand of $\overline{I}_1 + \overline{I}_2$ is of the order $\frac{e^{ip\xi x}}{\sqrt{\xi}}$, $|\xi| \to \infty$, and $(\overline{I}_1 + \overline{I}_2)$ again converges (to zero).

In view of relations 3.26, 3.34, 3.39, 3.40, 3.43, 3.45 and 3.48, we finally have:

$$\sigma_y = \frac{K}{2\pi i} H(t-v_1 r) \int_{v_1 r}^{t} \left\{ \frac{A(\xi_+^{(1)})}{\xi_+^{(1)}} \frac{\partial \xi_+^{(1)}}{\partial t} - \frac{A(\xi_-^{(1)})}{\xi_-^{(1)}} \frac{\partial \xi_-^{(1)}}{\partial t} \right\} dt$$

$$-KA(0)\, H(-\cos\theta)\, H(t-v_1 \bar{y})$$

$$- \frac{K}{2\pi i} H(t-v_2 r) \int_{v_2 r}^{t} \left\{ B(\xi_+^{(2)}) \frac{\partial \xi_+^{(2)}}{\partial t} - B(\xi_-^{(2)}) \frac{\partial \xi_-^{(2)}}{\partial t} \right\} dt$$

(3.53)

$$- \frac{K}{2\pi i} f_\theta(\theta)\, f_S(t) \int_{t}^{v_2 r} \left\{ B(\xi_+) - B(\xi_-) \right\} \frac{\partial \xi_+^{(S)}}{\partial t}\, dt$$

$$+ \sigma_o$$

$$0 \leq \theta \leq \pi$$

where $A(\xi)$ and $B(\xi)$ are given by formulae 3.30 and 3.31, $\xi_{\pm}^{(1)}$ by 3.38, $\xi_{\pm}^{(2)}$ by 3.44, t_S by 3.47, $\xi_{\pm}^{S}$ by 3.49, $f_\theta(\theta)$ by 3.50 and $f_S(t)$ by 3.51.

The evaluation of σ_x is carried out in exactly the same manner in the angular interval $0 \leq \theta \leq \frac{\pi}{2}$. The situation changes when we consider the interval $\frac{\pi}{2} \leq 0 \leq \pi$, because of the contribution of the pole $\xi = -i\xi_R$ (*), in the neighborhood of $\theta = \pi$. Consider $\overline{\delta_x}$ in equation 3.27. We again define a new variable t in each of the integrals of this equation so that in the first integral

$$\xi = \pm\sqrt{\frac{t^2}{r^2} - v_1^2}\,\sin\theta + i\frac{t}{r}\cos\theta \tag{3.53a}$$

and, in the second integral,

$$\xi = \pm\sqrt{\frac{t^2}{r^2} - v_2^2}\,\sin\theta + i\frac{t}{r}\cos\theta \tag{3.53b}$$

If we let $\theta \to \pi$ in formula 3.52, the hyperbolas 3.53a and 3.53b tend to the negative imaginary ξ-axis (see Γ_3 of Figure 9). If $\frac{t}{r} > \xi_R$, the pole $\xi = -i\xi_R$ lies between Γ_3 and Γ. We can now write the complete expression for δ_x:

$$\delta_x = -\frac{K}{2\pi i} H(t-v_1 r)\int_{v_1 r}^{t}\left\{\frac{C(\xi_+^{(1)})}{\xi_+^{(1)}}\frac{\partial\xi_+^{(1)}}{\partial t} - \frac{C(\xi_-^{(1)})}{\xi_-^{(1)}}\frac{\partial\xi_-^{(1)}}{\partial t}\right\}dt$$

$$+ KC(0)\,H(-\cos\theta)\,H(t - v_1 y)$$

$$+ \frac{K}{2\pi i} H(t-v_2 r)\int_{v_2 r}^{t}\left\{B(\xi_+^{(2)})\frac{\partial\xi_+^{(2)}}{\partial t} - B(\xi_-^{(2)})\frac{\partial\xi_-^{(2)}}{\partial t}\right\}dt$$

$$+ \frac{K}{2\pi i} f_\theta(\theta)\, f_S(t)\int_{t_S}^{v_2 r}\left\{B(\xi_+) - B(\xi_-)\right\}\frac{\partial\xi_+^{S}}{\partial t}\,dt$$

(cont'd)

---

* It is recalled that $-i\xi_R$ is a simple zero of $F_U(\xi)$.

$$+ K \lim_{\xi \to -i\xi_R} \left\{ (C(\xi) - \xi B(\xi))(\xi + i\xi_R) \right\} \delta(\theta - \pi) H(t - \xi_R r) + \nu \delta_0 \tag{3.54}$$

$$0 \leq \theta \leq \pi$$

where $C(\xi)$ and $B(\xi)$ are given by formulae 3.32 and 3.31, $\xi_{\pm}^{(1)}$ by 3.38, $\xi_{\pm}^{(2)}$ by 3.44, t_S by 3.47, $\xi_{\pm}^{S}$ by 3.49, $f_\theta(\theta)$ and $f_S(t)$ by 3.50 and 3.51.

The expression for τ_{xy} is obtained in a similar fashion, except that there are no difficulties associated with surface waves:

$$\tau_{xy} = -\frac{K}{2\pi} H(t - v_1 r) \int_{v_1 r}^{t} \left\{ D(\xi_+^{(1)}) \frac{\partial \xi_+^{(1)}}{\partial t} - D(\xi_-^{(1)}) \frac{\partial \xi_-^{(1)}}{\partial t} \right\} dt$$

$$+ \frac{K}{2\pi} H(t - v_2 r) \int_{v_2 r}^{t} \left\{ D(\xi_+^{(2)}) \frac{\partial \xi_+^{(2)}}{\partial t} - D(\xi_-^{(2)}) \frac{\partial \xi_-^{(2)}}{\partial t} \right\} dt$$

$$+ \frac{K}{2\pi} f_\theta(\theta) f_S(t) \int_{t_S}^{v_2 r} \left\{ D(\xi_+^S) - D(\xi_-^S) \right\} \frac{\partial(\xi_+^S)}{\partial t} dt$$

$$0 \leq \theta \leq \pi \tag{3.55}$$

where $D(\xi)$ is given by formula 3.33, $\xi_{\pm}^{(1)}$ by 3.38, $\xi_{\pm}^{(2)}$ by 3.44, $\xi_{\pm}^{S}$ by 3.49, $f_\theta(\theta)$ by 3.50 and $f_S(t)$ by 3.51.

The wave patterns are now clear (Figure 10). First there is a "plane wave" emanating from the surface of the cut. Next there are the two cylindrical waves radiating from the edge with the velocities of the shear and dilatation waves respectively. In contrast to the previous chapter, there now appear two new waves: a surface wave propagating with the velocity of the Rayleigh wave, and a head wave propagating with the velocity of a shear wave. The head wave can be considered as the envelope of the shear waves resulting from the interaction of the dilatation wave with the free boundary. It is noted that this is not the only type of boundary condition that gives rise to this type of wave. It was found in (Reference 12) that a head wave could be generated by interaction of a dilatation wave with a boundary constrained to a given displacement and a given shear distribution.

V. THE STRESS SINGULARITIES

To obtain the stress field at an arbitrary position, considerable numerical work would be necessary. However, if one is interested only in the immediate neighborhood of the root of the crack, then some approximation can be made. In effect, as $r \to 0$, $|\xi| \to \infty$ and $F_U(\xi) \to 1$ in view of equation 3.23. Therefore, for $r \to 0$, the stress integrals of the preceding section -- which are in fact double integrals -- can be approximated by single integrals by replacing the integral $F_U(\xi)$ by unity. While it is not our present purpose to go into numerical work, it is of interest to point out that the stresses are everywhere finite in the interior of the whole medium, even on the wave fronts. This is seen by substituting for ξ in the stress integrals of the preceding section (equations 3.53 to 3.55) in terms of the physical coordinates r, θ, t. Such a substitution shows that as $t \to v_{1,2}r$, the integrands of the stress integrals are of the order $\frac{1}{\sqrt{t^2 - v_{1,2}^2 r^2}}$, which is an integrable singularity. This is to be contrasted with the situation prevailing in the preceding chapter where the stresses were infinite on the wave fronts. As to the stress singularity in the neighborhood of the root of the crack, it is of the order $\frac{1}{\sqrt{r}}$ for $t > 0$. Take the stress σ_y for instance (equation 3.53):

$$\sigma_y \underset{r \to 0}{\sim} \frac{1}{\sqrt{r}} \int_{v_2 r}^{t} \frac{dt}{\sqrt{t^2 - v_2^2 r^2}}^{(*)} = \frac{1}{\sqrt{r}} \log \frac{t + \sqrt{t^2 - v_2^2 r^2}}{v_2 r}$$

$$\sim \frac{1}{\sqrt{r}}$$

This result is in agreement with Maue's (Reference 11). It is of interest to note, furthermore, that the same order of stress singularity has been obtained for the static case (Reference 13).

* This expression was obtained by an asymptotic expansion of the integrand in equation 3.53 for $|\xi| \to \infty$, *i.e. for* $r \to 0$, *noting that* $\lim_{|\xi| \to \infty} F_U(\xi) = 1$.

CHAPTER 4

CONCENTRATED FORCE MOVING ALONG A CRACK

In this chapter, we deal with a problem which is closely related to the preceding one. The latter was concerned with the sudden initiation of a semi-infinite crack in an infinite elastic medium initially subjected to a uniform σ_y stress. The physical process involved therein can be considered as due to a force traveling along the x-axis, in the negative direction (Figure 11), and of such a magnitude as to cancel the initial surface stresses in the half-plane $y = 0 \quad x < 0$. The present chapter treats of the motion of a concentrated force* along a crack, at various velocities. It will be instructive to study how the wave patterns change with the velocity of the moving force, and, in particular, when the latter velocity tends to infinity.

Consider an infinite elastic medium with a semi-infinite crack as shown in Figure 11. The medium being initially free from stresses, a concentrated force whose normal component (in the y-direction) is unity and whose tangential component is ϵ**, starts moving at $t = 0^+$, at a constant velocity c_o^{-1} in the negative x-direction, from its initial position at $y = 0 \quad x = 0$ (Figure 11). It is proposed to study the wave patterns generated.

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* The motion of a force distributed over an area can be treated by a proper superposition of motions of concentrated forces.

** ϵ can be considered as a coefficient of friction.

I. THE INITIAL AND BOUNDARY VALUE PROBLEM

The initial and boundary conditions are, in the Laplace plane:

$$\bar{\sigma}_y = -e^{pc_o x} \qquad x < 0, \; y = 0 \tag{4.1}$$

$$\tau_{xy} = -\epsilon e^{pc_o x} \qquad x < 0, \; y = 0 \tag{4.2}$$

where p is the parameter of the Laplace transformation, and $e^{pc_o x}$ was obtained as the Laplace transform of $\delta(t + c_o x)$, δ being a Dirac delta function.

Let

$$\bar{\phi} = \int_{\Gamma} \left\{ P_1(\xi) \pm P_2(\xi) \right\} e^{\mp p \left\{ \sqrt{v_1^2 + \xi^2}\, y - i\xi x \right\}} d\xi \tag{4.3}$$

$$\bar{\Psi} = \int_{\Gamma} \left\{ \pm Q_1(\xi) + Q_2(\xi) \right\} e^{\mp p \left\{ \sqrt{v_2^2 + \xi^2}\, y - i\xi x \right\}} d\xi \tag{4.4}$$

where the upper and lower signs refer to y > 0 and y < 0 respectively, be the solutions of the Laplace transformed equations 2.3 and 2.4. In view of equations 2.14a, 2.14c, 4.3 and 4.4, the conditions 4.1 and 4.2 can be written:

$$\int_{\Gamma} \left\{ (P_1 \pm P_2)(-\xi^2 - \frac{v_2^2}{2}) \pm (\pm Q_1 + Q_2) i\xi \sqrt{v_2^2 + \xi^2} \right\} e^{ipx\xi} d\xi = -D \frac{e^{pc_o x}}{p^2}$$

$$x < 0 \tag{4.5}$$

$$\int_{\Gamma} \left\{ \pm(P_1 \pm P_2) i\xi \sqrt{v_1^2 + \xi^2} + (\pm Q_1 + Q_2)(\frac{v_2^2}{2} + \xi^2) \right\} e^{ipx\xi} d\xi = -\epsilon \frac{De^{pc_o x}}{p^2}$$

$$x < 0 \tag{4.6}$$

where the upper and lower signs refer to the values of the stresses obtained by approaching the negative x-axis from above and from below respectively and $D = 2\rho/v_2^2$.

Conditions 4.5 and 4.6 can be satisfied only if:

$$P_1(\xi) = -(\frac{v_2^2}{2} + \xi^2) R_1(\xi) \qquad (4.7a)$$

$$Q_1(\xi) = i\xi \sqrt{v_1^2+\xi^2} R_1(\xi) \qquad (4.7b)$$

$$P_2(\xi) = i\xi \sqrt{v_2^2+\xi^2} R_2(\xi) \qquad (4.7c)$$

$$Q_2(\xi) = (\xi^2 + \frac{v_2^2}{2}) R_2(\xi) \qquad (4.7d)$$

where $R_1(\xi)$ and $R_2(\xi)$ are two newly introduced unknown functions.

In terms of relations 4.7, the conditions 4.5 and 4.6 become respectively:

$$\int_\Gamma \left\{(\xi^2 + \frac{v_2^2}{2})^2 - \xi^2 \sqrt{(v_1^2 + \xi^2)(v_2^2 + \xi^2)}\right\} R_1(\xi)\, e^{ip\xi x}\, d\xi$$

$$= \frac{-D}{p^2} e^{pc_o x} \qquad x < 0 \qquad (4.8)$$

$$\int_\Gamma \left\{(\xi^2 + \frac{v_2^2}{2})^2 - \xi^2 \sqrt{(v_1^2 + \xi^2)(v_2^2 + \xi^2)}\right\} R_2(\xi)\, e^{ip\xi x}\, d\xi$$

$$= -\frac{\epsilon D}{p^2} e^{pc_o x} \qquad x < 0 \qquad (4.9)$$

The conditions of continuity are:

$$\frac{\partial\bar{\phi}_1}{\partial y} = \frac{\partial\bar{\Psi}_2}{\partial y} = \bar{\phi}_2 = \bar{\Psi}_1 = 0$$

$$y = 0, \quad x > 0 \qquad (4.10)$$

where

$$\overline{\phi}_{1,2} = \int_{\Gamma} P_{1,2}(\xi) e^{\mp p\left\{\sqrt{v_1^2+\xi^2}\, y - i\xi x\right\}} d\xi$$

$$\overline{\Psi}_{1,2} = \int_{\Gamma} Q_{1,2}(\xi) e^{\mp p\left\{\sqrt{v_2^2+\xi^2}\, y - i\xi x\right\}} d\xi$$

In view of relations 4.7, the conditions 4.10 are satisfied by

$$\int_{\Gamma} \sqrt{v_1^2 + \xi^2}\, R_1(\xi)\, e^{ipx\xi}\, d\xi = 0 \qquad x > 0 \qquad (4.11)$$

$$\int_{\Gamma} \sqrt{v_2^2 + \xi^2}\, R_2(\xi)\, e^{ipx\xi}\, d\xi = 0 \qquad x > 0 \qquad (4.12)$$

The equations 4.8 and 4.11, 4.9 and 4.12 are the integral equations of the problem.

II. SOLUTION OF THE INTEGRAL EQUATIONS

Consider the dual integral equations 4.8 and 4.11. They are of the same type as considered in previous chapters, and are amenable to the same techniques, the Wiener-Hopf techniques. Define the path of integration Γ as being the ξ real axis, and introduce the branch cuts of $\sqrt{v_1^2 + \xi^2}$ and $\sqrt{v_2^2 + \xi^2}$ as running from $\pm iv_1$ to $\pm i\infty$ and $\pm iv_2$ to $\pm i\infty$. Then, a solution of the system 4.8 and 4.11 is:

$$R_1(\xi) = \frac{D}{\pi i} \frac{1}{v_2^2 - v_1^2} \frac{2}{p^2} \frac{2}{F_L(-ic_o)\sqrt{-iv_2}} \frac{1}{\xi + ic_o} \frac{1}{F_U(\xi)\sqrt{(\xi + iv_2)(v_1^2 + \xi^2)}} \tag{4.13}$$

where $F_U(\xi)$ and $F_L(\xi)$ are given by equations 3.23 and 3.24.

Solution 4.13 is to be compared with the solution 3.19a of the dual integral equations 3.11 and 3.12. Similarly, a solution of the equations 4.9 and 4.12 is

$$R_2(\xi) = \frac{D}{\pi i} \frac{\epsilon}{v_2^2 - v_1^2} \frac{1}{p^2} \frac{2}{F_L(-ic_o)\sqrt{-iv_2}} \frac{1}{\xi + ic_o} \frac{1}{F_U(\xi)\sqrt{(\xi + iv_2)(\xi^2 + v_1^2)}} \tag{4.14}$$

III. THE STRESS WAVE PATTERNS

For the sake of simplicity we consider the case where the moving force has only a normal component. Accordingly, $\epsilon = 0$ and $R_2(\xi) = 0$. Then, from relation 4.7, for $y > 0$, we get:

$$\bar{\phi} = - \int_{\Gamma} \left(\frac{v_2^2}{2} + \xi^2\right) R_1(\xi) e^{-p\left\{\sqrt{v_1^2+\xi^2}\, y - i\xi x\right\}} d\xi \tag{4.15}$$

$$\bar{\Psi} = i \int_{\Gamma} \xi \sqrt{v_1^2+\xi^2}\, R_1(\xi) e^{-p\left\{\sqrt{v_2^2+\xi^2}\, y - i\xi x\right\}} d\xi \tag{4.16}$$

We can limit ourselves to considering the half space $y > 0$, because of the symmetry of the problem.

In view of relation 2.14, the stresses, in the Laplace plane, are:

$$\bar{\sigma}_y = \int_{\Gamma} \frac{E(\xi)}{\xi + ic_o} e^{-p\left\{\sqrt{v_1^2+\xi^2}\, y - i\xi x\right\}} d\xi + \int_{\Gamma} \frac{G(\xi)}{\xi + ic_o} e^{-p\left\{\sqrt{v_2^2+\xi^2}\, y - i\xi x\right\}} d\xi \tag{4.17}$$

$$\bar{\sigma}_x = \int_{\Gamma} \frac{L(\xi)}{\xi + ic_o} e^{-p\left\{\sqrt{v_1^2+\xi^2}\, y - i\xi x\right\}} d\xi - \int_{\Gamma} \frac{G(\xi)}{\xi + ic_o} e^{-p\left\{\sqrt{v_2^2+\xi^2}\, y - i\xi x\right\}} d\xi \tag{4.18}$$

$$\bar{\tau}_{xy} = \int_{\Gamma} \frac{M(\xi)}{\xi + ic_o} e^{-p\left\{\sqrt{v_1^2+\xi^2}\, y - i\xi x\right\}} d\xi - \int_{\Gamma} \frac{M(\xi)}{\xi + ic_o} e^{-p\left\{\sqrt{v_2^2+\xi^2}\, y - i\xi x\right\}} d\xi \tag{4.19}$$

where

$$E(\xi) = \frac{2\rho p^2}{v_2^2}(\xi^2 + \frac{v_2^2}{2})^2 R_1(\xi)(\xi + ic_o) \tag{4.20}$$

$$G(\xi) = -\frac{2\rho p^2}{v_2^2}\xi^2\sqrt{(v_1^2+\xi^2)(v_2^2+\xi^2)}\, R_1(\xi)(\xi + ic_o) \tag{4.21}$$

$$L(\xi) = -\frac{2\rho p^2}{v_2^2}(v_1^2 - \frac{v_2^2}{2} + \xi^2)(\frac{v_2^2}{2} + \xi^2) R_1(\xi)(\xi + ic_o) \tag{4.22}$$

$$M(\xi) = -\frac{2\rho p^2}{v_2^2}\, i\xi\sqrt{v_1^2+\xi^2}\,(\frac{v_2^2}{2} + \xi^2) R_1(\xi)(\xi + ic_o) \tag{4.23}$$

The location of the pole $\xi = -ic_o$ depends on the velocity of propagation of the force (equal to $\frac{1}{c_o}$). We consider the three règimes: (A) $\frac{1}{c_o} < \frac{1}{v_2} < \frac{1}{v_1}$ corresponding to a force moving more slowly than both shear waves and dilatation waves; this règime will be referred to as a "subsonic" one; (B) $\frac{1}{v_2} < \frac{1}{c_o} < \frac{1}{v_1}$ which will be called a "transonic" règime; and (C) $\frac{1}{v_2} < \frac{1}{v_1} < \frac{1}{c_o}$ which will be called a "supersonic" règime.

A. "Subsonic" Règime

If the force moves more slowly than both shear waves and dilatation waves in the solid, the singularities of the integrands of the stress integrals are as represented in Figure 12. The evaluation of the stresses in the (x, y, t) space is carried out in exactly the same manner as in previous chapters. We simply record the results:

$$\sigma_y = H(t-v_1r)\left\{\frac{E(\xi_+^{(1)})}{\xi_+^{(1)}+ic_o}\frac{\partial\xi_+^{(1)}}{\partial t} - \frac{E(\xi_-^{(1)})}{\xi_-^{(1)}+ic_o}\frac{\partial\xi_-^{(1)}}{\partial t}\right\}$$

$$+ H(t-v_2r)\left\{\frac{G(\xi_+^{(2)})}{\xi_+^{(2)}+ic_o}\frac{\partial\xi_+^{(2)}}{\partial t} - \frac{G(\xi_-^{(2)})}{\xi_-^{(2)}+ic_o}\frac{\partial\xi_-^{(2)}}{\partial t}\right\}$$

$$+ f_S(t)\,f_\theta(\theta)\left\{\frac{G(\xi_+^S)}{\xi_+^S+ic_o} - \frac{G(\xi_-^S)}{\xi_-^S+ic_o}\right\}\frac{\partial\xi_+^S}{\partial t}$$

$$+ \delta(\theta-\pi)\,\delta(t+c_ox) \tag{4.24}$$

$$\sigma_x = H(t-v_1r)\left\{\frac{L(\xi_+^{(1)})}{\xi_+^{(1)}+ic_o}\frac{\partial\xi_+^{(1)}}{\partial t} - \frac{L(\xi_-^{(1)})}{\xi_-^{(1)}+ic_o}\frac{\partial\xi_-^{(1)}}{\partial t}\right\}$$

$$- H(t-v_2r)\left\{\frac{G(\xi_+^{(2)})}{\xi_+^{(2)}+ic_o}\frac{\partial\xi_+^{(2)}}{\partial t} - \frac{G(\xi_-^{(2)})}{\xi_-^{(2)}+ic_o}\frac{\partial\xi_-^{(2)}}{\partial t}\right\}$$

$$- f_S(t)\,f_\theta(\theta)\left\{\frac{G(\xi_+^S)}{\xi_+^S+ic_o} - \frac{G(\xi_-^S)}{\xi_-^S+ic_o}\right\}\frac{\partial\xi_+^S}{\partial t}$$

$$- 2\pi i\left\{L(-ic_o) - G(-ic_o)\right\}\delta(\theta)-\pi)\,\delta(t+c_ox)$$

$$- 2\pi i\lim_{\xi\to -i\xi_R}\left\{\frac{L(\xi)-G(\xi)}{\xi+ic_o}(\xi+i\xi_R)\right\}\delta(\theta-\pi)\,\delta(t_+\xi_Rx) \tag{4.25}$$

$$\tau_{xy} = H(t-v_1 r)\left\{\frac{M(\xi_+^{(1)})}{\xi_+^{(1)}+ic_o}\frac{\partial\xi_+^{(1)}}{\partial t} - \frac{M(\xi_-^{(1)})}{\xi_-^{(1)}+ic_o}\frac{\partial\xi_-^{(1)}}{\partial t}\right\}$$

$$- H(t-v_2 r)\left\{\frac{M(\xi_+^{(2)})}{\xi_+^{(2)}+ic_o}\frac{\partial\xi_+^{(2)}}{\partial t} - \frac{M(\xi_-^{(2)})}{\xi_-^{(2)}+ic_o}\frac{\partial\xi_-^{(2)}}{\partial t}\right\}$$

$$- f_S(t)\, f_\theta(\theta)\left\{\frac{M(\xi_+^S)}{\xi_+^S+ic_o} - \frac{M(\xi_-^S)}{\xi_-^S+ic_o}\right\}\frac{\partial\xi_+^S}{\partial t} \qquad (4.26)$$

where E(ξ), G(ξ), L(ξ) and M(ξ) are given by relations 4.20 to 4.23, and

$$\xi_\pm^{(1)} = \pm\sqrt{\frac{t^2}{r^2} - v_1^2}\ \sin\theta + i\frac{t}{r}\cos\theta \qquad (4.27)$$

$$\xi_\pm^{(2)} = \pm\sqrt{\frac{t^2}{r^2} - v_2^2}\ \sin\theta + i\frac{t}{r}\cos\theta \qquad (4.28)$$

$$\xi_\pm^S = i\left\{-\sqrt{v_2^2 - \frac{t^2}{r^2}}\ \sin\theta + \frac{t}{r}\cos\theta\right\} \pm \delta \qquad (4.29)$$

$$(\delta \rightarrow 0^+)$$

$$f_S(t) = 1 \qquad (-v_1 r\cos\theta + r\sqrt{v_2^2 - v_1^2}\ \sin\theta) \leq t \leq v_2 r$$

$$= 0 \qquad \text{otherwise} \qquad (4.30)$$

$$f_\theta(\theta) = 1 \qquad \pi - \cos^{-1}\frac{v_1}{v_2} \leq \theta \leq \pi \qquad (4.31)$$

$$= 0 \qquad \text{otherwise}$$

The wave patterns are represented in Figure 15. They differ from those exhibited in the preceding chapter in that there no longer exists the "plane" dilatation wave emanating from the crack.

B "Transonic" Règime.

If the velocity of propagation of the force along the crack is between the velocities of shear waves and of dilatation waves, the singularities of the integrands of the stress integrals are represented in Figure 13. The stresses, in this case, are given by:

$$
\begin{aligned}
\sigma_y = {} & H(t-v_1 r)\left\{\frac{E(\xi_+^{(1)})}{\xi_+^{(1)}+ic_o}\,\frac{\partial\xi_+^{(1)}}{\partial t} - \frac{E(\xi_-^{(1)})}{\xi_-^{(1)}+ic_o}\,\frac{\partial\xi_-^{(1)}}{\partial t}\right\} \\
& + H(t-v_2 r)\left\{\frac{G(\xi_+^{(2)})}{\xi_+^{(2)}+ic_o}\,\frac{\partial\xi_+^{(2)}}{\partial t} - \frac{G(\xi_-^{(2)})}{\xi_-^{(2)}+ic_o}\,\frac{\partial\xi_-^{(2)}}{\partial t}\right\} \\
& + f_S(t)\, f_\theta(\theta)\left\{\frac{G(\xi_+^S)}{\xi_+^S+ic_o} - \frac{G(\xi_-^S)}{\xi_-^S+ic_o}\right\}\frac{\partial\xi_+^S}{\partial t} \\
& - g_\theta^S(\theta)G(-ic_o)\,\delta\!\left(c_o x+\sqrt{v_2^{\,2}-c_o^{\,2}}\,y+t\right)2\pi i \\
& - 2\pi i\,\delta(\theta-\pi)\,E(-ic_o)\,\delta(t+c_o x)
\end{aligned}
\qquad (4.32)
$$

$$
\begin{aligned}
\sigma_x = {} & H(t-v_1 r)\left\{\frac{L(\xi_+^{(1)})}{\xi_+^{(1)}+ic_o}\,\frac{\partial\xi_+^{(1)}}{\partial t} - \frac{L(\xi_-^{(1)})}{\xi_-^{(1)}+ic_o}\,\frac{\partial\xi_-^{(1)}}{\partial t}\right\} \\
& - H(t-v_2 r)\left\{\frac{G(\xi_+^{(2)})}{\xi_+^{(2)}+ic_o}\,\frac{\partial\xi_+^{(2)}}{\partial t} - \frac{G(\xi_-^{(2)})}{\xi_-^{(2)}+ic_o}\,\frac{\partial\xi_-^{(2)}}{\partial t}\right\} \\
& - f_S(t)\, f_\theta(t)\left\{\frac{G(\xi_+^S)}{\xi_+^S+ic_o} - \frac{G(\xi_-^S)}{\xi_-^S+ic_o}\right\}\frac{\partial\xi_+^S}{\partial t}
\end{aligned}
$$

(cont'd)

$$+ 2\pi i\, G(-ic_o)\, g_\theta^S(\theta)\, \delta(c_o x + \sqrt{v_2^2 - c_o^2}\, y + t)$$

$$- 2\pi i\, L(-ic_o)\, \delta(\theta - \pi)\, \delta(t + c_o x)$$

$$- 2\pi i \lim_{\xi \to -i\xi_R} \left\{ \frac{L(\xi) - G(\xi)}{\xi + ic_o} (\xi + i\xi_R) \right\} \delta(\theta - \pi)\, \delta(t + \xi_R x) \qquad (4.33)$$

$$\tau'_{xy} = H(t - v_1 r) \left\{ \frac{M(\xi_+^{(1)})}{\xi_+^{(1)} + ic_o} \frac{\partial \xi_+^{(1)}}{\partial t} - \frac{M(\xi_-^{(1)})}{\xi_-^{(1)} + ic_o} \frac{\partial \xi_-^{(1)}}{\partial t} \right\}$$

$$- H(t - v_2 r) \left\{ \frac{M(\xi_+^{(2)})}{\xi_+^{(2)} + ic_o} \frac{\partial \xi_+^{(2)}}{\partial t} - \frac{M(\xi_-^{(2)})}{\xi_-^{(2)} + ic_o} \frac{\partial \xi_-^{(2)}}{\partial t} \right\}$$

$$- f_S(t)\, f_\theta(\theta) \left\{ \frac{M(\xi_+^S)}{\xi_+^S + ic_o} - \frac{M(\xi_-^S)}{\xi_-^S + ic_o} \right\} \frac{\partial \xi_+^S}{\partial t}$$

$$+ 2\pi i\, g_\theta^S(\theta)\, \delta(c_o x + \sqrt{v_2^2 - c_o^2}\, y + t)\, M(-ic_o) \qquad (4.34)$$

where $\xi_\pm^{(1)}$, $\xi_\pm^{(2)}$, $\xi_\pm^S$ are given by formulae 4.27 to 4.29, $f_S(t)$, $f_\theta(\theta)$ by 4.30 to 4.31 and

$$g_\theta^S(\theta) = 1 \qquad \pi - \cos^{-1}\frac{c_o}{v_2} \lesssim \theta \lesssim \pi$$

$$= 0 \qquad \text{otherwise} \qquad (4.35)$$

It is seen that in addition to the two cylindrical waves plus a head wave and a Rayleigh wave, there now appears a fifth wave whose front may be considered as the envelope of all cylindrical shear waves generated by the moving force interacting with the boundary. The whole picture is represented in Figure 16.

C The "Supersonic" Règime

If the velocity of propagation of the force is greater than both the velocities of sound, the singularities of the integrands of the stress integrals are as represented in Figure 14. The stresses are then given by:

$$
\begin{aligned}
\sigma_y = {} & H(t-v_1 r)\left\{\frac{E(\xi_+^{(1)})}{\xi_+^{(1)}+ic_o}\frac{\partial\xi_+^{(1)}}{\partial t} - \frac{E(\xi_-^{(1)})}{\xi_-^{(1)}+ic_o}\frac{\partial\xi_-^{(1)}}{\partial t}\right\} \\
& - g_\theta^c(\theta)\, 2\pi i\, \delta\left(c_o x + \sqrt{v_1^{\,2}-c_o^{\,2}}\, y + t\right) E(-ic_o) \\
& + H(t-v_2 r)\left\{\frac{G(\xi_+^{(2)})}{\xi_+^{(2)}+ic_o}\frac{\partial\xi_+^{(2)}}{\partial t} - \frac{G(\xi_-^{(2)})}{\xi_-^{(2)}+ic_o}\frac{\partial\xi_-^{(2)}}{\partial t}\right\} \\
& - g_\theta^S(\theta)\, 2\pi i\, G(-ic_o)\, \delta\left(c_o x + \sqrt{v_2^{\,2}-c_o^{\,2}}\, y + t\right) \\
& + f_S(t)\, f_\theta(\theta)\left\{\frac{G(\xi_+^S)}{\xi_+^S+ic_o} - \frac{G(\xi_-^S)}{\xi_-^S+ic_o}\right\}\frac{\partial\xi_+^S}{\partial t}
\end{aligned}
\qquad (4.36)
$$

$$
\begin{aligned}
\sigma_x = {} & H(t-v_1 r)\left\{\frac{L(\xi_+^{(1)})}{\xi_+^{(1)}+ic_o}\frac{\partial\xi_+^{(1)}}{\partial t} - \frac{L(\xi_-^{(1)})}{\xi_-^{(1)}+ic_o}\frac{\partial\xi_-^{(1)}}{\partial t}\right\} \\
& - g_\theta^c(\theta)\, 2\pi i\, L(-ic_o)\, \delta\left(c_o x + \sqrt{v_1^{\,2}-c_o^{\,2}}\, y + t\right) \\
& - H(t-v_2 r)\left\{\frac{G(\xi_+^{(2)})}{\xi_+^{(2)}+ic_o}\frac{\partial\xi_+^{(2)}}{\partial t} - \frac{G(\xi_-^{(2)})}{\xi_-^{(2)}+ic_o}\frac{\partial\xi_-^{(2)}}{\partial t}\right\} \\
& + g_\theta^S(\theta)\, 2\pi i\, G(-ic_o)\, \delta\left(c_o x + \sqrt{v_2^{\,2}-c_o^{\,2}}\, y + t\right) \\
& - 2\pi i \lim_{\xi\to -i\xi_R}\left\{\frac{L(\xi)-G(\xi)}{\xi+ic_o}(\xi+i\xi_R)\right\}\delta(\theta-\pi)\,\delta(t+\xi_R x)
\end{aligned}
\qquad (4.37)
$$

$$\tau_{xy} = H(t-v_1 r)\left\{\frac{M(\xi_+^{(1)})}{\xi_+^{(1)}+ic_o}\frac{\partial\xi_+^{(1)}}{\partial t} - \frac{M(\xi_-^{(1)})}{\xi_-^{(1)}+ic_o}\frac{\partial\xi_-^{(1)}}{\partial t}\right\}$$

$$- g_\theta^{(c)}(\theta)\, 2\pi i\, M(-ic_o)\,\delta(c_o x + \sqrt{v_1^2 - c_o^2}\, y + t)$$

$$+ H(t-v_2 r)\left\{\frac{M(\xi_+^{(2)})}{\xi_+^{(2)}+ic_o}\frac{\partial\xi_+^{(2)}}{\partial t} - \frac{M(\xi_-^{(2)})}{\xi_-^{(2)}+ic_o}\frac{\partial\xi_-^{(2)}}{\partial t}\right\}$$

$$+ g_\theta^{S}(\theta)\, 2\pi i\, M(-ic_o)\,\delta(c_o x + \sqrt{v_2^2 - c_o^2}\, y + t)$$

$$- f_S(t)\, f_\theta(\theta)\left\{\frac{M(\xi_+^S)}{\xi_+^S+ic_o} - \frac{M(\xi_-^S)}{\xi_-^S+ic_o}\right\}\frac{\partial\xi_+^S}{\partial t} \qquad (4.38)$$

where the notations are the same as in the foregoing subsections A and B, except for

$$g_\theta^{(c)}(\theta) = 1 \qquad \pi - \cos^{-1}\frac{c_o}{v_1} \leq \theta \leq \pi$$

$$= 0 \qquad \text{otherwise} \qquad (4.39)$$

The wave patterns are represented in Figure 17. The only difference with those of the preceding subsection is the present existence of an additional shock wave propagating with the velocity of a dilatation wave. As the velocity of the moving force tends to infinity, it would be natural to expect that both shear and dilatation shock waves (Figure 17) tend to horizontal positions so as to become "plane" waves. It turns out that this is not the case. As the velocity of the moving force approaches infinity, the dilatation shock wave becomes a "plane wave", but the shear shock wave actually disappears. To see this fact, let us look at any of the equations 4.17 to 4.19. For example, in equation 4.17, the contribution of the shear wave is represented by the second integral. If we let $c_o = 0$, c_o^{-1} being the velocity of the force, then, the residue of the second integral's integrand at $\xi = 0$ is zero. The foregoing assertion is therefore deduced.

IV. THE STRESS SINGULARITIES

It is shown in the same way as in the preceding chapter that for $t > 0$, the stresses in the neighborhood of the root of the crack vary as $\frac{1}{\sqrt{r}}$. What is more, the stress distribution in the immediate neighborhood of the crack's root at any $t > 0$, can be closely approximated by replacing $F_U(\xi)$ by unity in the stress expressions of the previous section. However, we shall not further indulge in this matter, since a detailed discussion of the stresses would be outside the scope of the present investigation. Suffice it to mention that the stresses here are infinite on the wave fronts, being of the order $\frac{1}{\sqrt{R}}$ where R is the distance from the front.

CHAPTER 5

CONCLUDING REMARKS

To conclude, a word must be said about the uniqueness of the solutions. With regards to the problems of dislocation (Chapter 2) and of crack initiation (Chapter 3), it is observed that new solutions could be generated by differentiating the present solutions with respect to the x-coordinate any number of times. The reason for this is simple. First, the x-derivatives of the solutions of the wave equations are also solutions of the latter. Second, since the boundary conditions in these problems are either homogeneous — i.e. zero surface stresses along the half-plane crack $y = 0$, $x < 0$ — or else constant — i.e. zero shear and constant discontinuity in the v-displacement along the slip half-plane $y = 0$, $x < 0$ — it is clear that the x-derivatives of the solutions satisfy homogeneous boundary conditions along the half-plane $y = 0$, $x < 0$. Thus we would obtain an infinite number of solutions by adding to the present solutions their x-derivatives of any orders.* However, we have determined the stress singularities to be of the order $\frac{1}{r}$, $r \to 0$, for the dislocation, and of the order $\frac{1}{\sqrt{r}}$, $r \to 0$, for the crack. If we differentiate the present solutions with respect to x, the stresses in the neighborhood of the edges would vary as $\frac{1}{r^{(1+n)}}$ and $\frac{1}{r^{(\frac{1}{2}+n)}}$ $(n = 1, 2, \ldots)$ respectively, which would violate the criteria developed in Chapter 1.

* With regards to the problem of a force moving at an arbitrary velocity along a half-plane crack (Chapter 4), differentiations do not give new solutions, since the boundary conditions are not constant along $y = 0$, $x < 0$.

Thus, we have at least ruled out the possibility of obtaining an infinite number of solutions by differentiations. Although this fact certainly does not constitute a uniqueness proof, the inference tends to support our intuition based upon physical grounds. It is of interest to note that similar situations arose in the case of a crack in an elastostatic medium (Reference 13) and in the electromagnetic diffraction by a conducting half-plane (Reference 3), where, out of the infinite number of solutions obtained, only one was selected as compatible with physical requirements.

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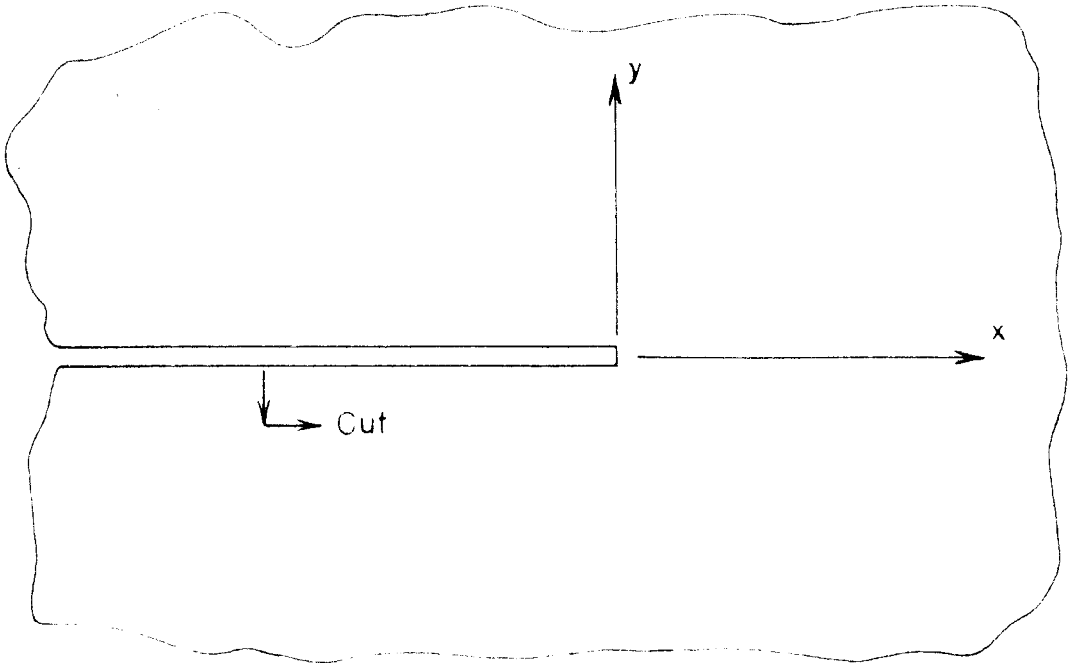


FIG. 1 - INITIALLY CUT STRESS FREE MEDIUM

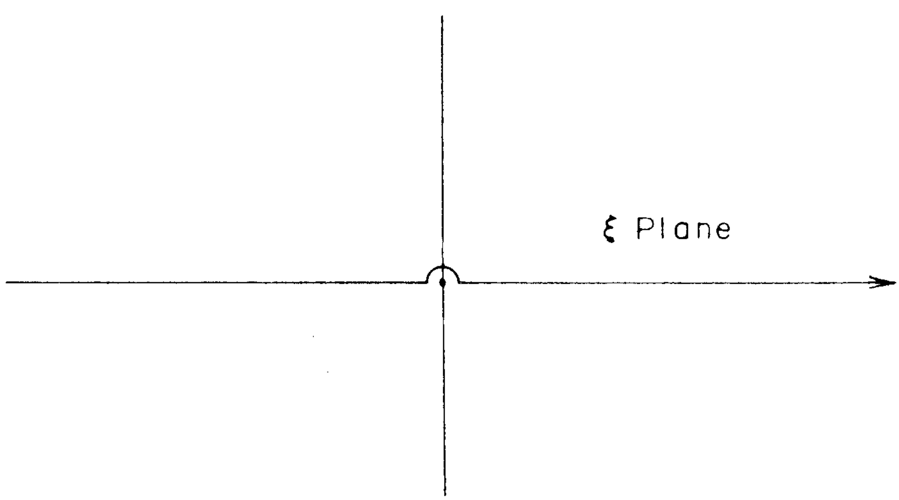


FIG. 2 - PATH OF INTEGRATION FOR DISLOCATION PROBLEM

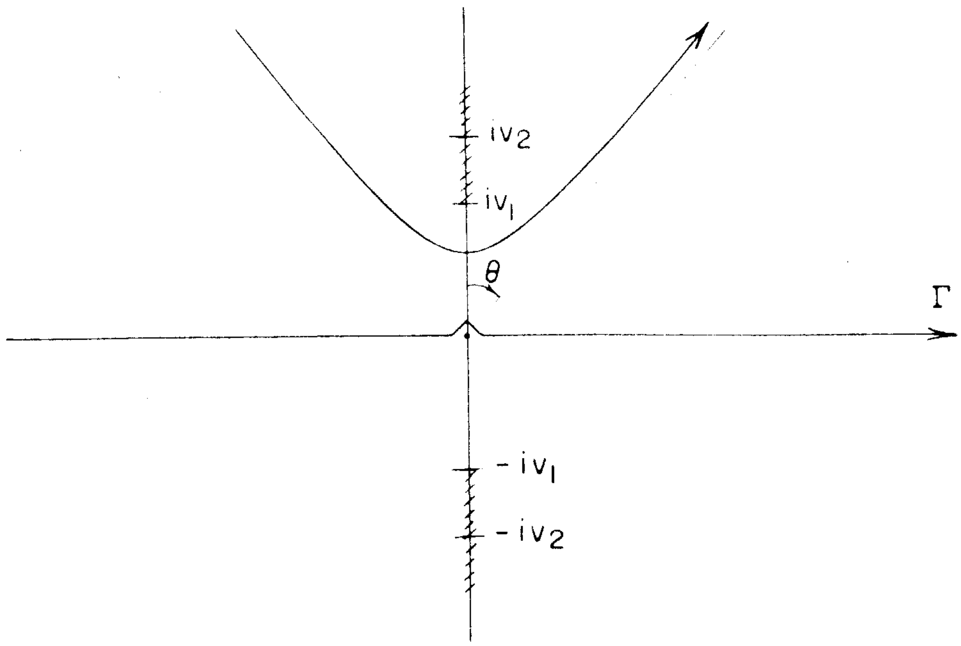
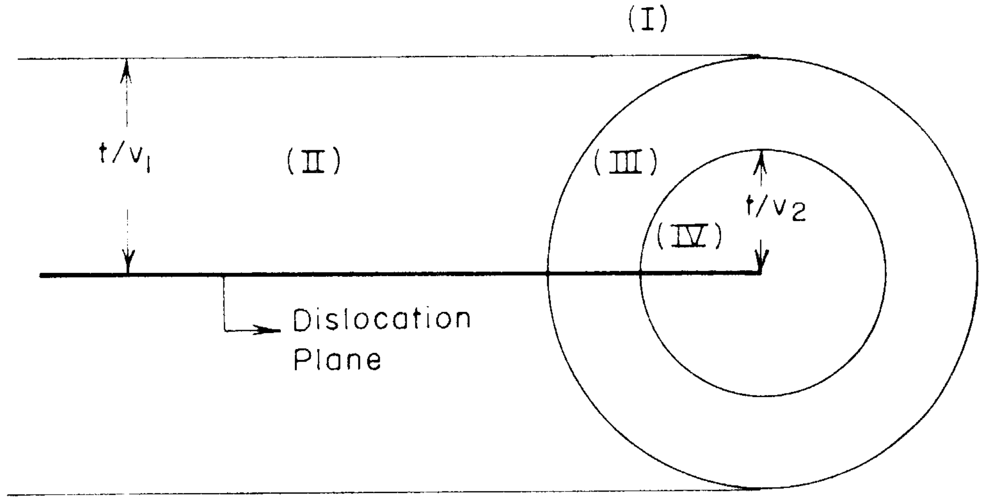


FIG. 3 – DEFORMATION OF PATH OF INTEGRATION IN DISLOCATION PROBLEM



I Undisturbed Zone
II Plane Dilatation Wave
III Cylindrical Dilatation Wave
IV Cylindrical Shear Wave

FIG. 4 – WAVES RADIATING FROM A DISLOCATION

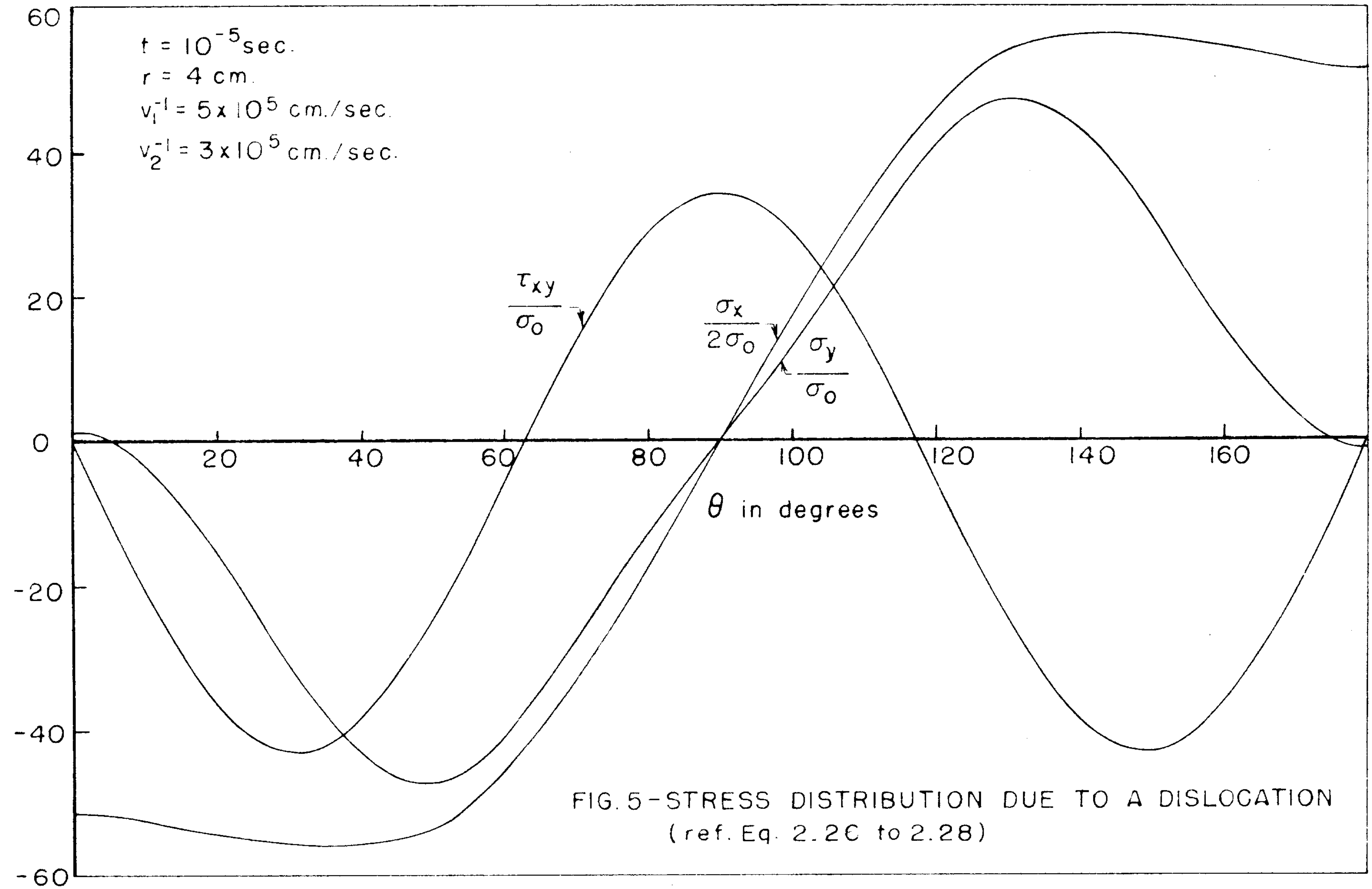


FIG. 5 - STRESS DISTRIBUTION DUE TO A DISLOCATION
(ref. Eq. 2.20 to 2.28)

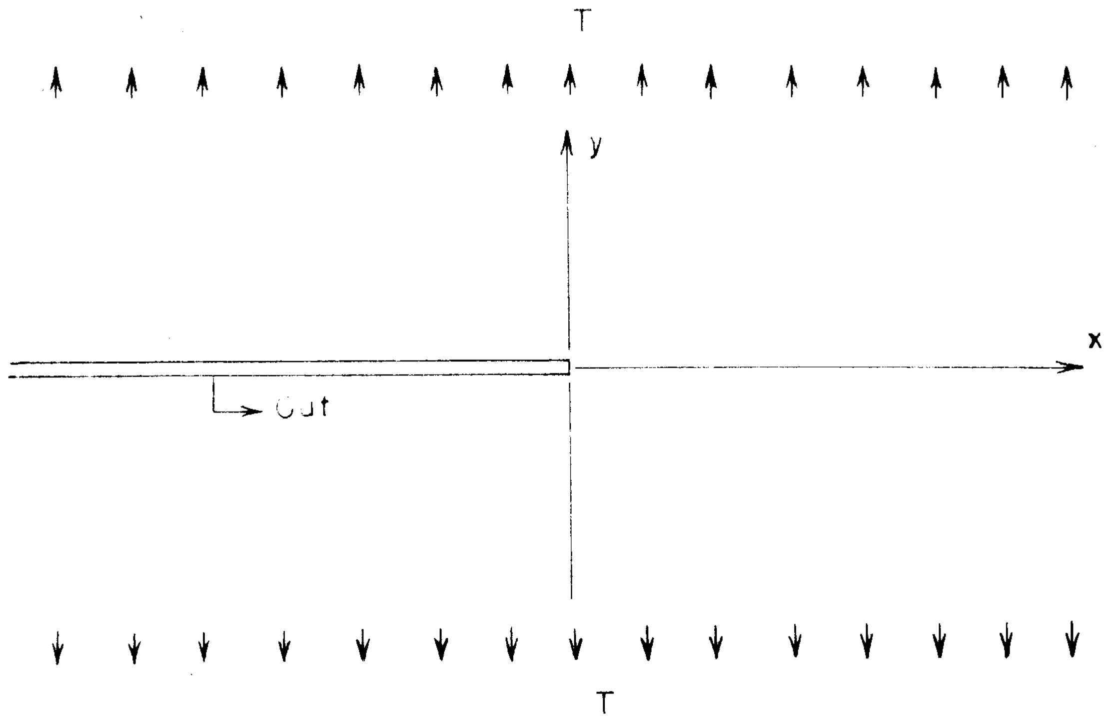


FIG. 6 — SUDDENLY CUT ELASTIC MEDIUM

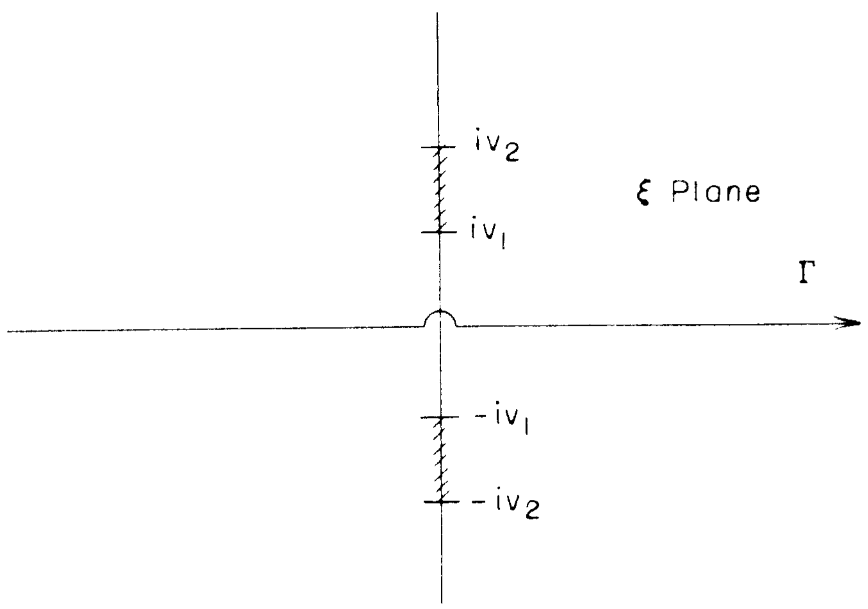


FIG. 7 — PATH OF INTEGRATION Γ AND BRANCH CUTS FOR CRACK INITIATION PROBLEM

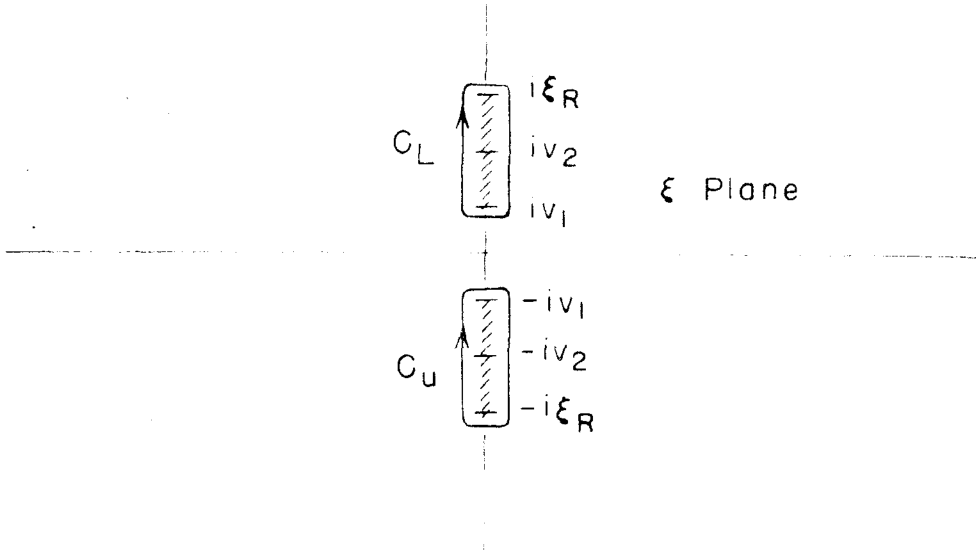


FIG. 8 – PATHS OF INTEGRATION FOR THE FACTORIZATION OF THE KERNEL FUNCTION $F(\xi)$

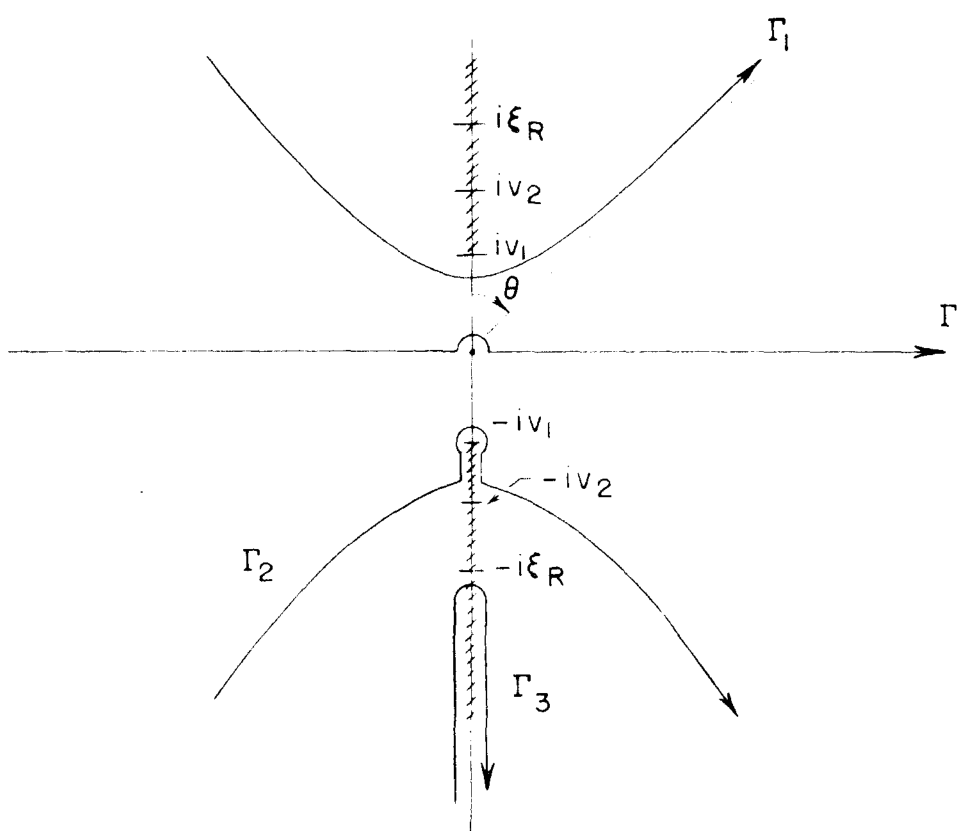
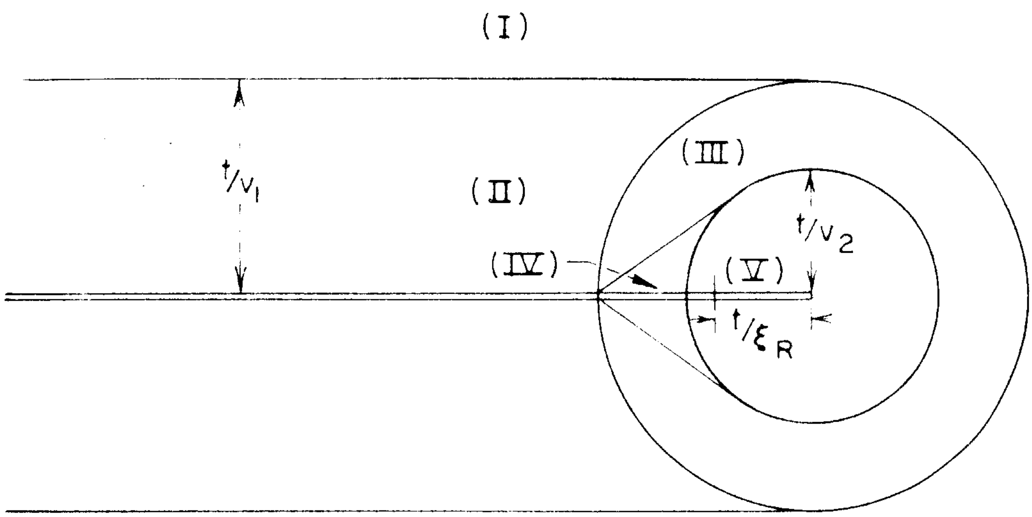


FIG. 9 – DEFORMATION OF PATHS OF INTEGRATION IN CRACK PROBLEM



I Undisturbed Zone

II Plane Wave

III Cylindrical Dilatation Wave

IV Head Wave

V Cylindrical Shear Wave

FIG. 10 — STRESS WAVE PROPAGATING FROM A SUDDEN CUT

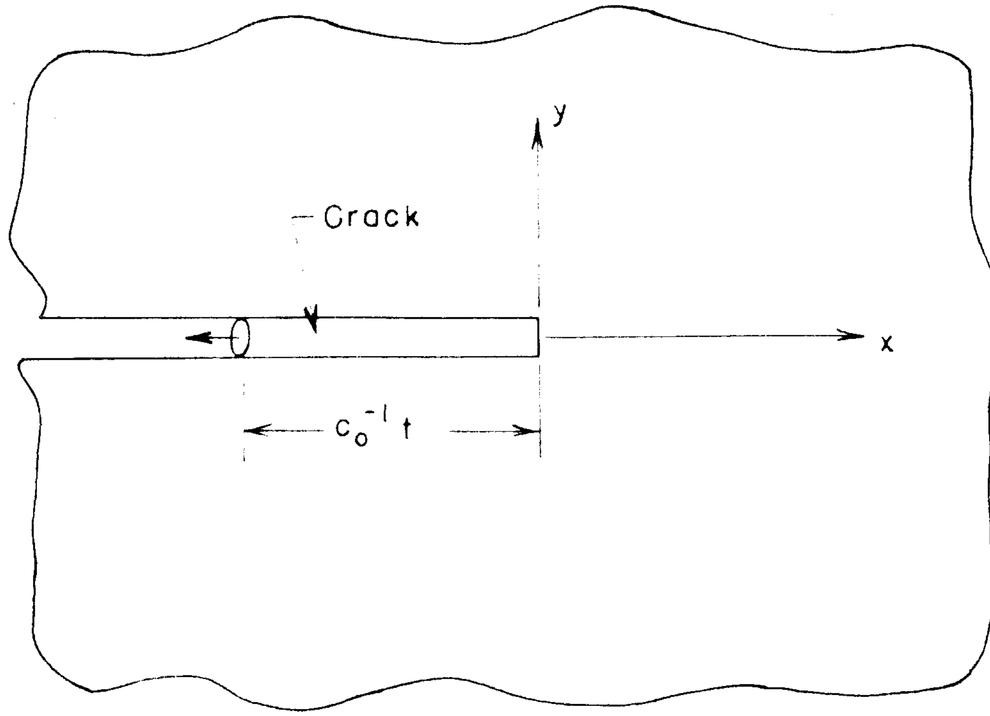


FIG. 11 — CONCENTRATED PRESSURE MOVING ALONG A CRACK

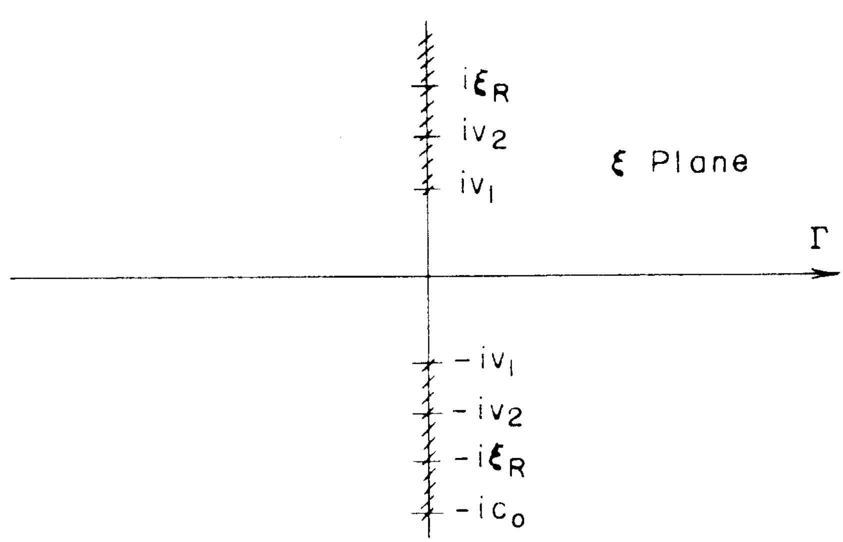


FIG. 12 — SINGULARITIES OF THE INTEGRANDS OF THE STRESS INTEGRALS ("SUBSONIC" REGIME)

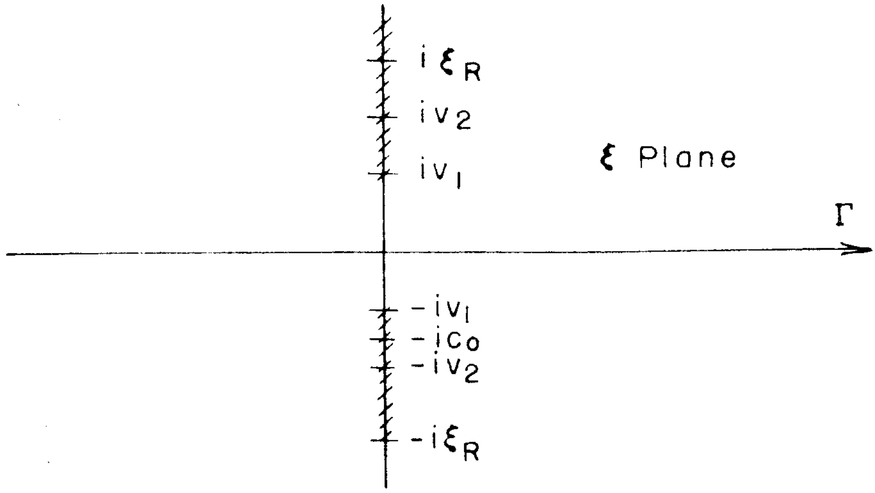


FIG. 13 – SINGULARITIES OF THE INTEGRANDS OF THE STRESS INTEGRALS ("TRANSONIC" REGIME)

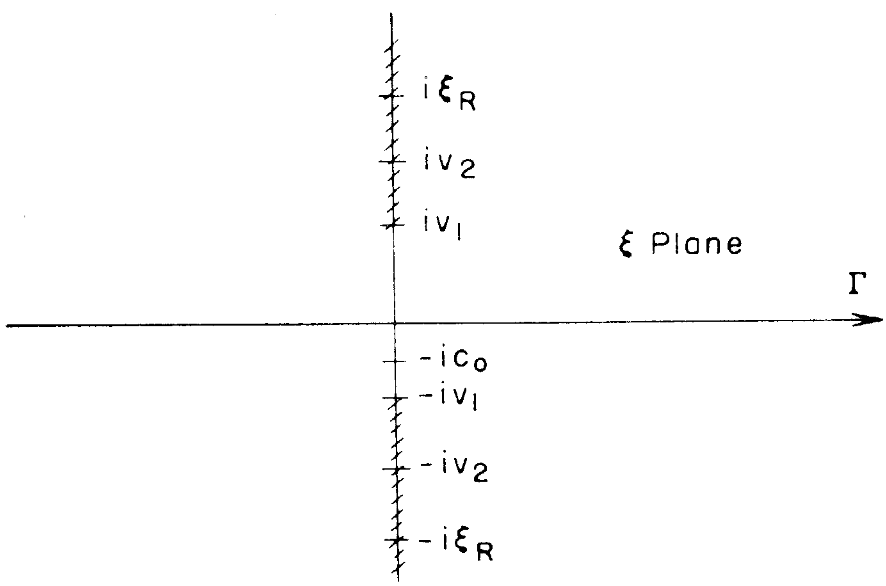
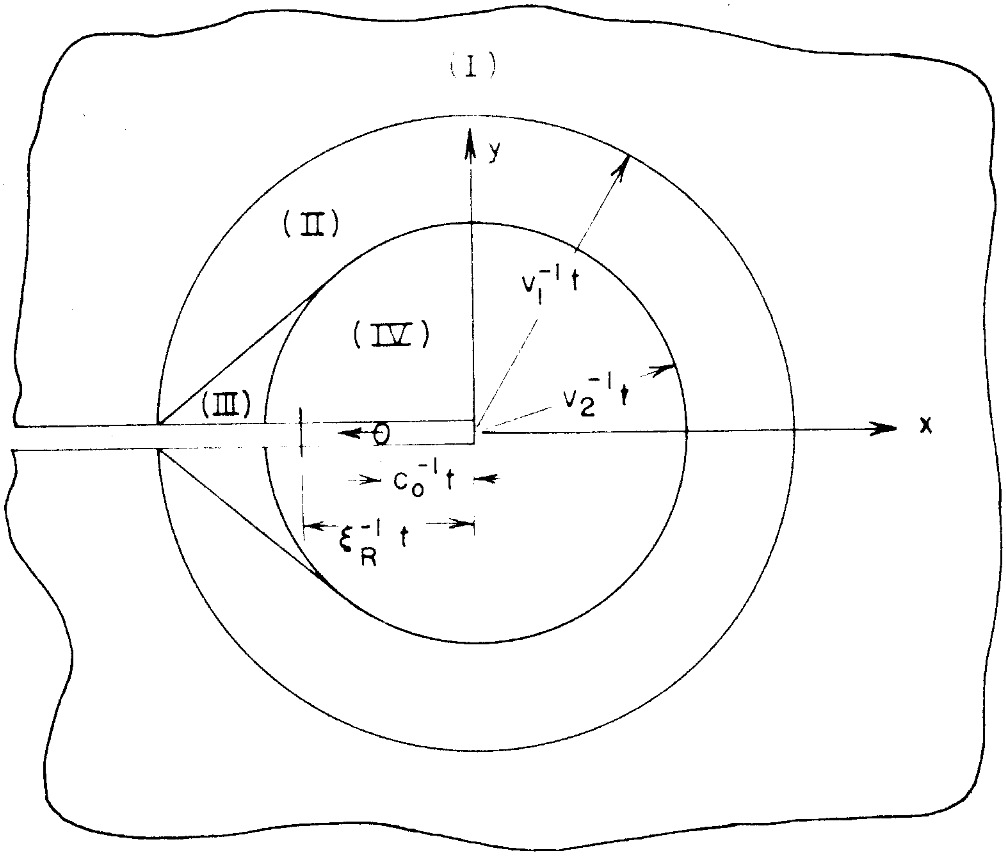
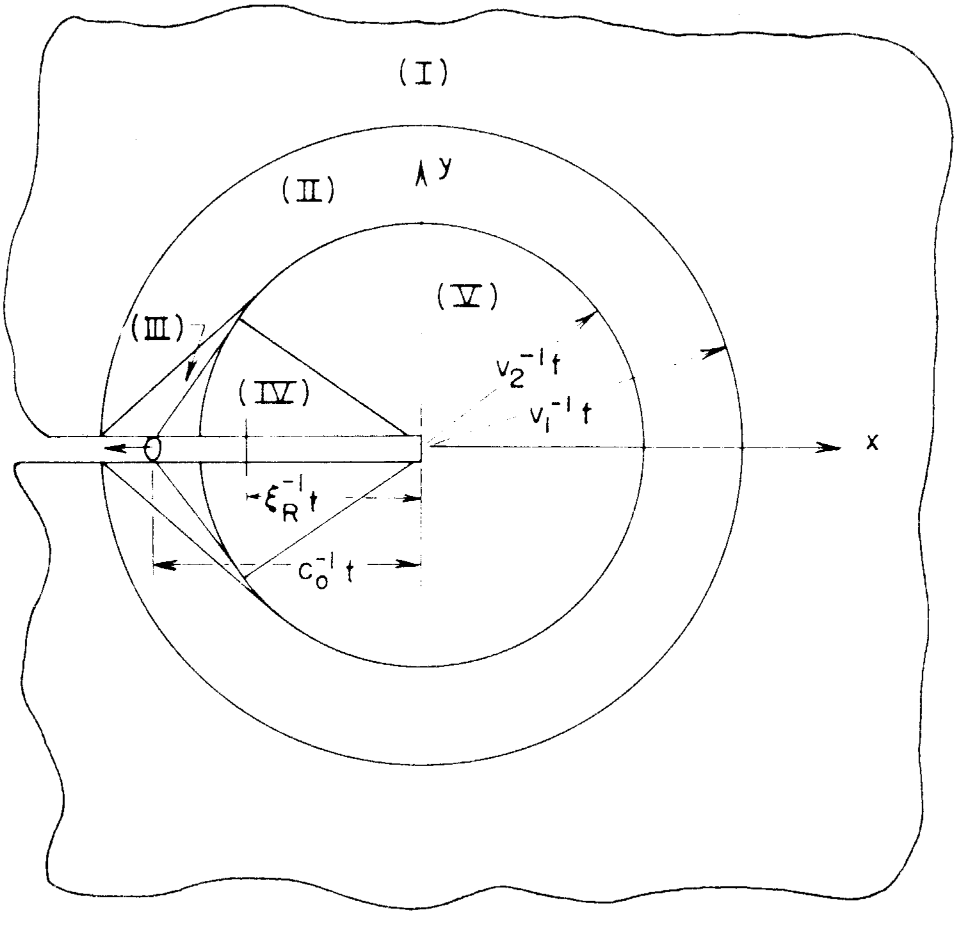


FIG. 14 – SINGULARITIES OF THE INTEGRANDS OF THE STRESS INTEGRALS ("SUPERSONIC" REGIME)



I Undisturbed Zone
II Cylindrical Dilatation Wave
III Head Wave
IV Cylindrical Shear Wave

FIG. 15 — WAVE PATTERNS ("SUBSONIC" REGIME)



I Undisturbed Zone

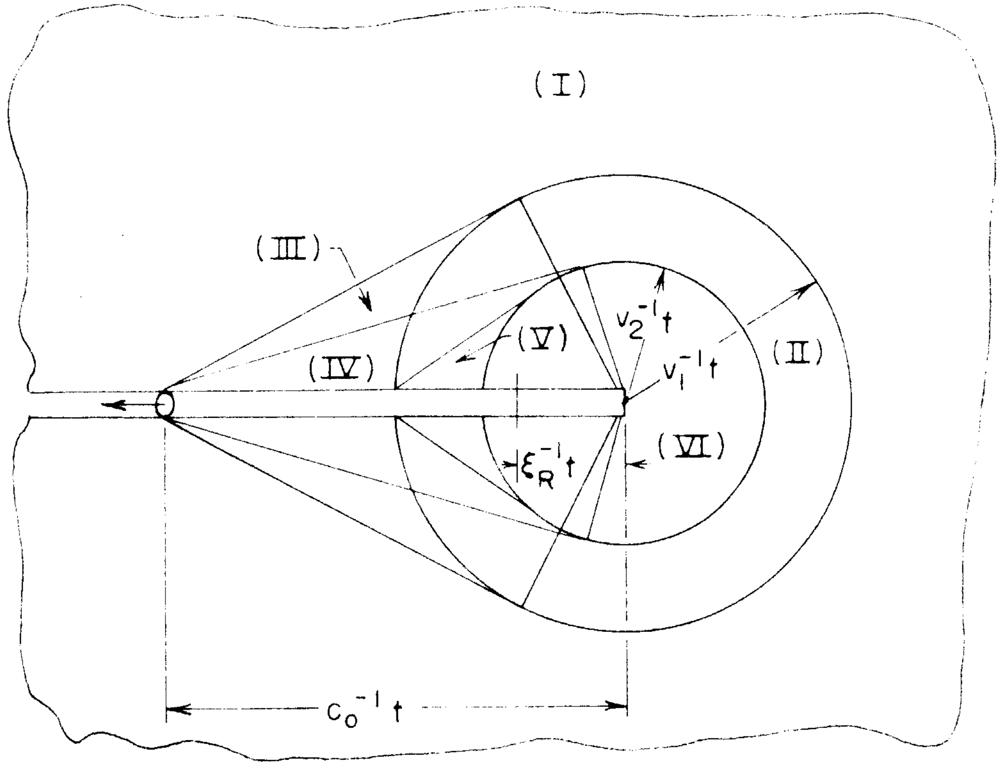
II Cylindrical Dilatation Wave

III Head Wave

IV Shear "Supersonic" Shock Wave

V Cylindrical Shear Wave

FIG. 16 — WAVE PATTERNS ("TRANSONIC" REGIME)



I	Undisturbed Zone
II	Cylindrical Dilatation Wave
III	Dilatation "Supersonic" Shock Wave
IV	Shear "Supersonic" Shock Wave
V	Head Wave
VI	Cylindrical Shear Wave

FIG. 17 — WAVE PATTERNS ("SUPERSONIC" REGIME)