THE EGOROFF PROPERTY AND ITS RELATION TO THE ORDER TOPOLOGY IN THE THEORY OF RIESZ SPACES

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ABSTRACT

A sequence $(f_n : n = 1, 2, \ldots)$ in a Riesz space L is order convergent to an element $f \in L$ whenever there exists a sequence $u_n \downarrow 0$ in L such that $|f_n - f| \leq u_n$ holds for all n. Sequential order convergence defines the order topology on L. The closure of a subset S in this topology is denoted by cl(S). The pseudo order closure S' of a subset S is the set of all $f \in L$ such that there exists a sequence in S which is order convergent to f. If S' = cl(S) for every convex subset S, then S' = cl(S) for every subset S. L has the Egoroff property if and only if S' = cl(S) for every order bounded subset S of L. A necessary and sufficient condition for L to have the property that S' = cl(S) for every subset S of L is that L has the strong Egoroff property.

A sequence $(f_n : n = 1, 2, \ldots)$ in a Riesz space L is ru-convergent to an element $f \in L$ whenever there exists a real sequence $\epsilon_n \downarrow 0$ and an element $w \in L^+$ such that $|f_n - f| \leq \epsilon_n w$ holds for all n. Sequential ru-convergence defines the ru-topology on L. The closure of a subset S in this topology is denoted by $\overline{S}$. The pseudo ru-closure S'_{ru} of a subset S is the set of all $f \in L$ such that there exists a sequence in S which is ru-convergent to f. If L is Archimedean, then $S'_{ru} = \overline{S}$ for every convex subset S implies that $S'_{ru} = \overline{S}$ for every subset S. A characterization of those Archimedean Riesz spaces L with the property that $S'_{ru} = \overline{S}$ for every subset S of L is obtained.

If ρ is a monotone seminorm on a Riesz space L, then a necessary and sufficient condition for ρ to be σ-Fatou (i.e.,

$0 \leq u_n \uparrow u$ in L implies $\rho(u_n) \uparrow \rho(u)$) is that the set $S_\rho = \{f \in L: \rho(f) \leq 1\}$ is order closed. For every monotone seminorm ρ on L, the largest σ-Fatou monotone seminorm bounded by ρ is the Minkowski functional of the order closure of S_ρ.

A monotone seminorm ρ on a Riesz space L is called strong Fatou whenever $0 \leq u_\tau \uparrow u$ in L implies $\sup \rho(u_\tau) = \rho(u)$. A characterization of those Riesz spaces L which have the following property is given: "For every monotone seminorm ρ on L, the largest strong Fatou monotone seminorm bounded by ρ is $\rho_{\mathfrak{L}}(f) = \inf\{\sup \rho(u_\tau) : 0 \leq u_\tau \uparrow |f|\}$." A similar characterization for Boolean algebras is also obtained.

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INTRODUCTION

A well-known result from the theory of Banach function spaces is the following: "for any given monotone function seminorm ρ, the largest σ-Fatou monotone seminorm bounded by ρ is $\rho_L(f) = \inf\{\lim \rho(u_n) : 0 \le u_n \uparrow |f|\}$." J. A. R. Holbrook extended the result by determining those Riesz spaces L for which the family of monotone seminorms have the above property; he showed that they are precisely those Riesz spaces which have the almost Egoroff property. In this thesis, a characterization of those Riesz spaces L which have the following property is obtained: "For any given monotone seminorm ρ on L, the largest strong Fatou seminorm bounded by ρ is $\rho_{\mathfrak{L}}(f) = \inf\{\sup \rho(u_\tau) : 0 \le u_\tau \uparrow |f|\}$."

To obtain the above characterization, we introduce first in Section I the various equivalent forms of the Egoroff property of Riesz spaces. As a consequence, a new property called the strong Egoroff property arises naturally.

In Section II, we introduce the order topology and the relative uniform topology on a Riesz space. The result of Section I is then used to show that a Riesz space L has the strong Egoroff property if and only if, for every subset S of L, the pseudo order closure of S is order closed. We determine also those Riesz spaces in which the relative uniform topology has the property that, for every subset S of L, the pseudo relative uniform closure of S is relatively uniformly closed.

Section III starts with the various equivalent forms of the almost Egoroff property (of Riesz spaces) which are similar to the equivalent forms of the Egoroff property proved in Section I. The proof of J. A. R. Holbrook's result is then modified. This leads easily to a generalization to directed systems.

In Section IV we focus our attention on a particular monotone seminorm ρ of a given Riesz space L and determine necessary and sufficient conditions for ρ to be σ-Fatou in terms of the order topology of L.

Section V deals with Boolean algebras. In this section we show which form our results take on in the theory of Boolean algebras.

I. THE EGOROFF PROPERTY AND STRONG EGOROFF PROPERTY OF RIESZ SPACES

In this section we shall discuss the Egoroff property of Riesz spaces and a property which is stronger than the Egoroff property called the strong Egoroff property of Riesz spaces. Various equivalent forms of the two properties are given. We also introduce the d-property of Riesz spaces. A Riesz space has the strong Egoroff property if and only if it has the Egoroff property and the d-property.

Definition. An element f of a Riesz space L is said to have the Egoroff property if, given any double sequence of elements $(u_{nk} : n, k = 1, 2, \ldots)$ in L such that $0 \le u_{nk} \uparrow_k |f|$ for $n = 1, 2, \ldots$, there exists a sequence $0 \le v_m \uparrow |f|$ in L and for every m a sequence $k(m, n)$ of indices such that $v_m \le u_{nk(m,n)}$ for all m and n.

A Riesz space L is said to have the Egoroff property if every element of L has the Egoroff property.

It follows that a Riesz space L has the Egoroff property if and only if every element in the positive cone L^+ has the Egoroff property. Hence, we may restrict our discussion to positive elements.

For a subset S of a Riesz space L, we shall denote by $\langle S \rangle$ the convex hull of S in L. Hence, $f \in \langle S \rangle$ if and only if $f = \sum_n \lambda_n f_n$ where the λ_n are real numbers satisfying $\lambda_n \ge 0$, $\lambda_n = 0$ except for finitely many n, $\sum_n \lambda_n = 1$ and each $f_n \in S$. We shall next study various equivalent conditions for an element $u \in L^+$ to have the Egoroff property.

Theorem 1. Let L be a Riesz space and $u \in L^+$. Then the following statements are equivalent.

(1) u has the Egoroff property.

(2) If $0 \le u_{nk} \uparrow_k u$ in L, then there is a sequence $0 \le v_n \uparrow u$ in L and for every n an appropriate k(n) such that $v_n \le u_{nk(n)}$ for all n.

(3) If $0 \le u_{nk} \uparrow_k u_n \uparrow u$ in L, then there is a sequence $0 \le v_n \uparrow u$ in L and for every n an appropriate k(n) such that $v_n \le u_{nk(n)}$ for all n.

(4) If $0 \le u_{nk} \uparrow_k u_n \uparrow u$ in L, then there is a sequence $0 \le v_m \uparrow u$ in L such that, for every m, $v_m \le w_m$ for some element w_m in $\langle \{u_{nk}\} \rangle$.

We shall prove theorem 1 by means of the following theorem.

Theorem 2. Let L be a Riesz space and $u \in L^+$. Then the following statements are equivalent.

(1') If $u \ge u_{nk} \downarrow_k 0$ in L, then there is a sequence $v_m \downarrow 0$ in L and for every m a sequence k(m, n) of indices such that $v_m \ge u_{nk(m,n)}$ for all m and n.

(2') If $u \ge u_{nk} \downarrow_k 0$ in L, then there is a sequence $v_n \downarrow 0$ in L and for every n an appropriate k(n) such that $v_n \ge u_{nk(n)}$ for all n.

(3') If $u \ge u_{nk} \downarrow_k u_n \downarrow 0$ in L, then there is a sequence $v_n \downarrow 0$ in L and for every n an appropriate k(n) such that $v_n \ge u_{nk(n)}$ for all n.

(4') If $u \ge u_{nk} \downarrow_k u_n \downarrow 0$ in L, then there is a sequence

$v_m \downarrow 0$ in L such that, for every m, $v_m \geq w_m$ for some element w_m in $\langle\{u_{nk}\}\rangle$.

Proof. It is clear that (1') ⇒ (2') and (3') ⇒ (4').

(2') ⇒ (3'): Let $u \geq u_{nk} \downarrow_k u_n \downarrow 0$ in L. Set $w_{nk} = u_{nk} - u_n$. Then $u \geq w_{nk} \downarrow_k 0$ in L and so by (2') there is a sequence $v_n^* \downarrow 0$ in L such that, for every n, $v_n^* \geq w_{nk(n)}$ for some $k = k(n)$. Set $v_n = v_n^* + u_n$, then $v_n \downarrow 0$ and $v_n \geq w_{nk(n)} + u_n = u_{nk(n)}$.

(4') ⇒ (1'): Let $u \geq u_{nk} \downarrow_k 0$ in L. We may assume that $u_{nk} \uparrow_n$ for every fixed k (since we can always replace u_{nk} by $u_{1k} \vee \ldots \vee u_{nk}$). Furthermore, we may assume that there exists a sequence $(u_n : n = 1, 2, \ldots)$ in L^+ such that $u \geq u_n \downarrow 0$ and $u_n \neq 0$ for all n. Now $u \geq u_{nk} \vee u_n \downarrow_k u_n \downarrow 0$. By (4'), there is a sequence $v_m^* \downarrow 0$ such that, for every m, $v_m^* \geq w_m$ for some element $w_m \in \langle\{ u_{nk} \vee u_n\}\rangle$.

For every m, let λ_{nk}^m, $n, k = 1, 2, \ldots$ be the real numbers such that $\lambda_{nk}^m \geq 0$ for all n, k, $\lambda_{nk}^m = 0$ except for finitely many n, k, $\sum_{n,k} \lambda_{nk}^m = 1$ and $w_m = \sum_{n,k} \lambda_{nk}^m (u_{nk} \vee u_n)$. For every fixed m, n, set $k(m, n) = \max\{k : \lambda_{nk}^m \neq 0\}$ and $\lambda_n^m = \sum_k \lambda_{nk}^m$; then, for each fixed m, $\lambda_n^m \geq 0$ for all n, $\lambda_n^m = 0$ except for finitely many n, $\sum_n \lambda_n^m = 1$ and $v_m^* \geq w_m \geq \sum_n \lambda_n^m (u_{nk(m,n)} \vee u_n)$.

Set $v_m = 2v_m^*$. Clearly, $v_m \downarrow 0$ in L. It remains to be shown that for a particular M, N of natural numbers, there exists some $k = k(M, N)$ such that $v_M \geq u_{Nk(M,N)}$.

Let M, N be given. Set $\gamma = \sup\{ \sum_{n \geq N} \lambda_n^m : m \geq M\}$. We shall first show that $\gamma = 1$. It is clear that $0 \leq \gamma \leq 1$. Moreover, we have, for every $m \geq M$,

$$v_m^* \geq \sum_n \lambda_n^m (u_{nk(m,n)} \vee u_n) \geq \sum_{n<N} \lambda_n^m u_n$$

$$\geq (\sum_{n<N} \lambda_n^m) u_N = (\sum_n \lambda_n^m - \sum_{n\geq N} \lambda_n^m) u_N \geq (1-\gamma) u_N ;$$

but $\inf\{v_m^* : m \geq M\} = 0$ and $u_N \neq 0$ as we have chosen it so; therefore, we must have $\gamma = 1$.

By the above argument there exists a natural number $P \geq M$ such that $\sum_{n\geq N} \lambda_n^P \geq \frac{1}{2}$. Consider

$$2v_M^* \geq 2v_P^* \geq 2\sum_n \lambda_n^P (u_{nk(P,n)} \vee u_n) \geq 2\sum_{n\geq N} \lambda_n^P u_{nk(P,n)} ;$$

if we let $k(M,N) = \max\{k(P,n) : \lambda_n^P \neq 0\}$ and recall that $u_{nk} \uparrow_n$ for every k, we thus obtain

$$2v_M^* \geq 2(\sum_{n\geq N} \lambda_n^P) u_{Nk(M,N)} \geq u_{Nk(M,N)} ,$$

and so $v_M = 2v_M^* \geq u_{Nk(M,N)}$. This completes the proof of the theorem.

Theorem 1 now follows immediately from theorem 2 since (1) ⇔ (1'), (2) ⇔ (2'), (3) ⇔ (3'), (4) ⇔ (4'). An element $u \in L^+$ has the Egoroff property if and only if one of the statements (1) - (4) of theorem 1, (1') - (4') of theorem 2 holds. Moreover, a Riesz space L has the Egoroff property if and only if, for every $u \in L^+$, one of the statements (1) - (4) of theorem 1, (1') - (4') of theorem 2 holds.

In view of theorem 2, it is natural to introduce the following strong Egoroff property of Riesz spaces.

Definition. A Riesz space L is said to have the strong Egoroff property if, given any double sequence of elements $(u_{nk}: n, k = 1,2,\ldots)$ in L^+ such that $u_{nk} \downarrow_k 0$ for every n, there exists in L^+ a sequence $v_m \downarrow 0$ and for every m a sequence $k(m, n)$ of indices such that $v_m \geq u_{nk(m,n)}$ for all m and n.

As for the Egoroff property, we have a similar result concerning the equivalent forms for the strong Egoroff property of Riesz spaces.

Theorem 3. Let L be a Riesz space. Then the following statements are equivalent.

(1) L has the strong Egoroff property.

(2) If $u_{nk} \downarrow_k 0$ in L, then there is a sequence $v_n \downarrow 0$ in L and for every n an appropriate $k(n)$ such that $v_n \geq u_{nk(n)}$ for all n.

(3) If $u_{nk} \downarrow_k u_n \downarrow 0$ in L, then there is a sequence $v_n \downarrow 0$ in L and for every n an appropriate $k(n)$ such that $v_n \geq u_{nk(n)}$ for all n.

(4) If $u_{nk} \downarrow_k u_n \downarrow 0$ in L, then there is a sequence $v_m \downarrow 0$ in L such that, for every m, $v_m \geq w_m$ for some element w_m in $\langle \{u_{nk}\} \rangle$.

Proof. Exactly the same as the proof of theorem 2, omitting "$u \geq$" everywhere.

From theorem 2 and the above theorem 3, we have that a Riesz space L having the strong Egoroff property has also the Egoroff property. The converse of this statement does not hold. The Riesz space of all real bounded sequences with the pointwise ordering has the Egoroff property but not the strong Egoroff property (see [9]).

However, if a Riesz space L has the Egoroff property as well as the property that every decreasing (to zero) double sequence in L is eventually dominated by some element of L^+, then L has the strong Egoroff property. To be precise, we have the following definition.

Definition. A Riesz space L is said to have the d-property whenever, given any double sequence $(u_{nk} : n, k = 1, 2, \ldots)$ in L^+ with $u_{nk} \downarrow_k 0$ for every n, there is an element $w \in L^+$ such that for every n there is an appropriate $k = k(n)$ such that $u_{nk(n)} \leq w$.

The d-property in a stronger form was first introduced by L. V. Kantorovitch which he called regularity (see [4] and [11]). Later on, H. Nakano introduced the notion of complete regularity, which is the same as the d-property (see [10]).

Theorem 4. Let L be a Riesz space. Then the following two statements are equivalent.

(1) L has the strong Egoroff property.

(2) L has the Egoroff property and the d-property.

Proof. It is clear that if L has the strong Egoroff property, then L has the Egoroff property and the d-property. Conversely, let L have the Egoroff property and the d-property. Let $u_{nk} \downarrow_k 0$ in L for every n. By the d-property of L there is an element $w \in L^+$ such that, for every n, $u_{nk(n)} \leq w$ for some $k = k(n)$. Set $w_{ni} = u_{n,k(n)+i}$, $n, i = 1, 2, \ldots$, then $w \geq w_{ni} \downarrow_i 0$. There is, by the Egoroff property of L, a sequence $v_m \downarrow 0$ in L such that, for every m, n, $v_m \geq w_{ni(m,n)}$ for some $i = i(m, n)$. Now, for every m, n, $v_m \geq w_{ni(m,n)} = u_{n,k(n)+i(m,n)}$, and so L has the strong Egoroff property. This completes the proof of the theorem.

A Riesz space L is called Archimedean if, for every $u \in L^+$, the sequence $(n^{-1}u : n = 1, 2, \ldots)$ satisfies $n^{-1}u \downarrow 0$. It follows that a Riesz space L is Archimedean if and only if, for every $u \in L^+$, $\varepsilon_n u \downarrow 0$ for every sequence $(\varepsilon_n : n = 1, 2, \ldots)$ of positive real numbers satisfying $\varepsilon_n \downarrow 0$.

The following theorem is due to W. A. J. Luxemburg.

Theorem 5. If L is an Archimedean Riesz space, then L has the strong Egoroff property if and only if L has the d-property.

Proof. One implication of the theorem holds always. On the other hand, assume that L is Archimedean and has the d-property. Let $u_{nk} \downarrow_k 0$. Then $nu_{nk} \downarrow_k 0$, and so by the d-property there is an element $u \in L^+$ such that $nu_{nk(n)} \leq u$ for all n. Let $v_n = n^{-1}u$. Then $v_n \downarrow 0$ (since L is Archimedean), and $u_{nk(n)} \leq v_n$ for all n. Hence L has the strong Egoroff property.

II. ORDER CONVERGENCE AND RELATIVE UNIFORM CONVERGENCE IN A RIESZ SPACE

The Egoroff and the strong Egoroff properties of a Riesz space discussed in the previous section are analogous to properties of monotonely convergent sequences of real numbers. In this section we shall consider in a Riesz space two kinds of convergence of sequences which are not necessarily monotone. They are the order convergence of sequences which induces on the Riesz space the order topology, and the relative uniform convergence of sequences which induces on the Riesz space the relative uniform topology.

Definition. A sequence $(f_n : n = 1, 2, \ldots)$ in a Riesz space L is order convergent to an element $f \in L$ whenever there exists a sequence $u_n \downarrow 0$ in L such that $|f_n - f| \leq u_n$ holds for all n. This will be denoted by $f_n \to_n f$ or simply $f_n \to f$.

It can be easily shown that the limit of an order convergent sequence is unique, i.e., if $f_n \to f$ and $f_n \to g$, then $f = g$. Monotone convergence is a special case of order convergence, i.e., if $f_n \uparrow f$ or $f_n \downarrow f$, then $f_n \to f$; moreover, if $f_n \to f$, $g_n \to g$ and λ, μ are real numbers, then $\lambda f_n + \mu g_n \to \lambda f + \mu g$.

It is not true that, in every Riesz space L, if $f_n \to f$ in L and $\lambda_n \to \lambda$ in the real number space, then $\lambda_n f_n \to \lambda f$. However, if L is Archimedean, then $f_n \to f$ and $\lambda_n \to \lambda$ imply that $\lambda_n f_n \to \lambda f$ (see [9]).

A subset S of a Riesz space L is called order closed if for every order convergent sequence in S the order limit of the sequence is also a member of S. It follows immediately from the definition that the empty set and the space L itself are order closed and that arbitrary intersections and finite unions of order closed sets are order closed. Hence, the order closed sets are exactly the closed sets of a certain topology in L, the order topology (see [9]). It is evident that the order topology in L satisfies the T_1-separation axiom, i.e., every subset of L consisting of one point is order closed. A subset V of L is open in the order topology if and only if $f_n \to f$ in L and $f \in V$ implies that $f_n \in V$ for all but a finite number of the f_n. It follows that if $f_n \to f$, then f_n converges to f in the order topology.

For any subset S of L, the pseudo order closure S' of S is the set of all $f \in L$ such that there exists a sequence in S converging in order to f. The order closure of S, i.e., the closure of S in the order topology, will be denoted by cl(S). Evidently we have $S \subseteq S' \subseteq$ cl(S). Taking closures, we obtain $cl(S) \subseteq cl(S') \subseteq cl(S)$, so $cl(S') =$ cl(S). Hence, replacing S by S' in the first formula, we obtain also that

$$S' \subseteq (S')' \subseteq cl(S') = cl(S),$$

and by induction it follows that

$$S \subseteq S' \subseteq (S')' \subseteq S''' \subseteq \dots \subseteq cl(S).$$

Furthermore, we have the following theorem.

Theorem 6. (1) A subset S of L is order closed if and only if $S = S'$, i.e., $S = S'$ already implies that $S = cl(S')$.

(2) A subset S of L satisfies $S' = cl(S)$ if and only if $S' = (S')'$.

Proof. (1) If S is order closed, then the order limit of every order convergent sequence in S is in S, hence $S = S'$. Conversely, assume that $S = S'$. Let $f_n \in S$ and $f_n \to f$. Then $f \in S'$ by the definition of S', and so $f \in S$ by the assumption that $S = S'$. This shows that S' is order closed.

(2) It follows from the formula $S' \subseteq (S')' \subseteq \mathrm{cl}(S)$ that $S' = \mathrm{cl}(S)$ implies $S' = (S')'$. On the other hand, assume that $S' = (S')'$. By (1), S' is order closed and so $S' = \mathrm{cl}(S')$. But $\mathrm{cl}(S') = \mathrm{cl}(S)$ holds always, hence $S' = \mathrm{cl}(S)$.

We are interested in finding a condition on a Riesz space L which ensures that $S' = \mathrm{cl}(S)$ for every subset S of L. From the last theorem it is sufficient to find a condition under which $S' = (S')'$ for every subset S of L. In order to do this, first we prove the following theorem.

Theorem 7. Let L be a Riesz space. Then the following statements are equivalent.

(1) L has the strong Egoroff property.

(2) If $f_{nk} \to_k f$ in L, then, for every n, there is $k = k(n)$ such that $f_{nk(n)} \to f$.

(3) If $f_{nk} \to_k f_n \to f$ in L, then, for every n, there is $k = k(n)$ such that $f_{nk(n)} \to f$.

(4) If $f_{nk} \to_k f_n \to f$ in L, then there is a sequence $(g_m : m = 1, 2, \dots)$ in $\langle \{f_{nk}\} \rangle$ such that $g_m \to f$.

Proof. (1) $\Rightarrow$ (2): Let $f_{nk} \to_k f$ in L. By the definition of order convergence there exists for every n a sequence $u_{nk} \downarrow_k 0$ in

L such that $|f_{nk}-f| \le u_{nk}$ for all n, k. By the assumption that L has the strong Egoroff property and by theorem 3 of the last section, there is a sequence $v_n \downarrow 0$ such that, for every n, $u_{nk(n)} \le v_n$ for some k = k(n). Now $|f_{nk(n)}-f| \le u_{nk(n)} \le v_n$ for every n; hence, $f_{nk(n)} \to f$.

(2) ⇒ (3): Let $f_{nk} \to_k f_n \to f$ in L. There exists, for every n, a sequence $u_{nk} \downarrow_k 0$ and a sequence $u_n \downarrow 0$ in L^+ such that $|f_{nk}-f_n| \le u_{nk}$ for all n, k and $|f_n-f| \le u_n$ for all n. Then, for every fixed n, $u_{nk} \to_k 0$ and so by (2), for every n, there is an appropriate k(n) such that $u_{nk(n)} \to 0$. Let the sequence (v_n : v = 1, 2, ...) in L^+ be such that $v_n \downarrow 0$ and $u_{nk(n)} \le v_n$ for all n. Then $w_n = u_n+v_n$ satisfies $w_n \downarrow 0$ and $|f_{nk(n)}-f| \le u_{nk(n)}+u_n \le w_n$ for all n; hence, $f_{nk(n)} \to f$.

It is clear that (3) ⇒ (4).

(4) ⇒ (1): It is sufficient by theorem 3 of the last section to show that if $u_{nk} \downarrow_k u_n \downarrow 0$ in L, then there is a sequence $v_m \downarrow 0$ in L such that for every m, $v_m \ge w_m$ for some element w_m in $\langle\{u_{nk}\}\rangle$. Let $u_{nk} \downarrow_k u_n \downarrow 0$ in L. Then $u_{nk} \to_k u_n \to 0$. By (4) there is a sequence (w_m : m = 1, 2, ...) in $\langle\{u_{nk}\}\rangle$ such that $w_m \to 0$. Thus, there is a sequence $v_m \downarrow 0$ in L such that $v_m \ge w_m$ for all m. This completes the proof of the theorem.

A Riesz space L is said to have the <u>diagonal property (for order convergence)</u>, whenever $f_{nk} \to_k f_n \to f$ in L, there is for every n an appropriate k(n) such that $f_{nk(n)} \to_n f$. A Riesz space L is said to have the <u>diagonal gap property (for order convergence)</u>, whenever

$f_{nk} \to_k f_n \to f$ in L, there is for every m an appropriate $k(m), n(m)$ with $n(1) < n(2) < \ldots$ such that $f_{n(m)k(m)} \to_m f$. The diagonal property and the diagonal gap property have been introduced by W. A. J. Luxemburg and A. C. Zaanen (see [9]). Condition (2) in theorem 7 is the diagonal property, and it is clear that (2) implies the diagonal gap property, while the diagonal gap property implies condition (3) of theorem 7. It follows then that the diagonal property, the diagonal gap property, and the strong Egoroff property are equivalent.

Now we are ready to answer the question: under what conditions does a Riesz space L have the property that, for every subset S of L, the pseudo order closure and the order closure of S coincide? A surprising result is that, if, for every convex subset S of L, the pseudo order closure and the order closure of S coincide, then, for every subset S of L, the pseudo order closure and the order closure of S coincide.

Theorem 8. *Let L be a Riesz space. Then the following conditions are mutually equivalent.*

(1) L *has the strong Egoroff property.*

(2) $S' = (S')'$ *for every subset* S *of* L.

(3) $S' = (S')'$ *for every convex subset* S *of* L.

Proof. (1) ⇒ (2): Let S be any subset of L. It is clear that $S' \subseteq (S')'$. For the reverse inclusion, let $f \in (S')'$. By the definition of pseudo order closure, there is a sequence $(f_n : n = 1, 2, \ldots)$ in S' and, for every n, a sequence $(f_{nk}: k = 1, 2, \ldots)$ in S such that $f_{nk} \to_k f_n \to f$. By the assumption that L has the strong Egoroff property and theorem 7, we have $f_{nk(n)} \to_n f$ for some $k = k(n)$; hence, $f \in S'$. Thus, $(S')' \subseteq S'$.

It is clear that (2) ⇒ (3).

(3) ⇒ (1): By theorem 7, it is sufficient to show that, if $f_{nk} \to_k f_n \to f$ in L, then there is a sequence $(g_m: m = 1, 2, \ldots)$ in $\langle\{f_{nk}\}\rangle$ such that $g_m \to f$. Let $f_{nk} \to_k f_n \to f$ in L. We write $S = \langle\{f_{nk}\}\rangle$. Then $f \in (S')'$, and so by our assumption $f \in S'$. Hence, there is a sequence $(g_m: m = 1, 2, \ldots)$ in $\langle\{f_{nk}\}\rangle$ such that $g_m \to f$. This completes the proof of theorem 8.

We have, in Riesz spaces with the strong Egoroff property, the order topology has the property that the closure of any set consists of the set itself plus all its order sequential limit points.

A subset S of a Riesz space L is called order bounded if there exists an element u in L^+ such that $|f| \leq u$ for all f in S. The Egoroff property, in a way, is the strong Egoroff property restricted to order bounded subsets. This last statement is clarified by the following theorems 9 and 10.

Theorem 9. *Let L be a Riesz space. Then the following statements are all equivalent.*

(1) *L has the Egoroff property.*

(2) *If* $f_{nk} \to_k f$ *in* L *and the set* $S = \{f_{nk}: n, k = 1, 2, \ldots\}$ *is order bounded, then, for every* n, *there is* $k = k(n)$ *such that* $f_{nk(n)} \to f$.

(3) *If* $f_{nk} \to_k f_n \to f$ *in* L *and the set* $S = \{f_{nk}: n, k = 1, 2, \ldots\}$ *is order bounded, then, for every* n, *there is* $k = k(n)$ *such that*

$f_{nk(n)} \to f$.

(4) If $f_{nk} \to_k f_n \to f$ in L and the set $S = \{f_{nk}: n, k = 1, 2, \ldots\}$ is order bounded, then there is a sequence $(g_m: m = 1, 2, \ldots)$ in $\langle S \rangle$ such that $g_m \to f$.

Proof. Similar to the proof of theorem 7.

Theorem 10. Let L be a Riesz space. Then the following statements are all equivalent.

(1) L has the Egoroff property.

(2) $S' = (S')'$ for every order bounded subset S of L.

(3) $S' = (S')'$ for every order bounded convex subset S of L.

Proof. Similar to the proof of theorem 8.

Besides order convergence, we introduce the relative uniform convergence of sequences in a Riesz space.

Definition. Given an element $u \geq 0$ in a Riesz space L, we shall say that the sequence $(f_n: n = 1, 2, \ldots)$ in L converges u-uniformly to the element $f \in L$ whenever, for every $\varepsilon > 0$, there exists a natural number N_ε such that $|f_n - f| \leq \varepsilon u$ holds for all $n \geq N_\varepsilon$. It is said that the sequence $(f_n: n = 1, 2, \ldots)$ in L converges relatively uniformly to f whenever f_n converges u-uniformly to f for some $u \in L^+$. Relative uniform convergence of f_n to f will be denoted by $f_n \to_n f$ (ru) or simply $f_n \to f$ (ru).

It follows immediately that if $f_n \to f$(ru), $f_n \to g$(ru) and λ, μ are real numbers, then $\lambda f_n + \mu g_n \to \lambda f + \mu g$ (ru). Furthermore, a sequence $(f_n: n = 1, 2, \ldots)$ converges relatively uniformly to f if and only if there is $u \in L^+$ and a real sequence $(\varepsilon_n: n = 1, 2, \ldots)$ with

$\varepsilon_n \downarrow 0$ such that $|f_n - f| \le \varepsilon_n u$ for all n. Thus, if the Riesz space L is Archimedean, then f_n converges relatively uniformly to f implies that f_n converges in order to f; and so if L is Archimedean, then the limit of relative uniform convergence is unique, i.e., $f_n \to f\,(ru)$ and $f_n \to g\,(ru)$ imply that $f = g$. In a non-Archimedean Riesz space, the limit of a relatively uniformly convergent sequence is not necessarily unique. By way of example, let R^2 be the lexicographically ordered plane (i.e., R^2 is the linear space of all ordered pairs $f = (f_1, f_2)$ of real numbers, with the ordering defined as: $f \le g$ for $f = (f_1, f_2)$, $g = (g_1, g_2)$ whenever $f_1 < g_1$, or $f_1 = g_1$ and $f_2 \le g_2$). If $f = (1, 1)$ and $f_n = n^{-1} f$ for $n = 1, 2, \ldots$, then $f_n \to 0\,(ru)$ in R^2. On the other hand, if $g = (0, 1)$, then $0 \le f_n - g < n^{-1} f$ for all n and so also $f_n \to g\,(ru)$ in R^2.

If a Riesz space L has the d-property, then order convergent sequences are relatively uniformly convergent. To see this, let $f_n \to 0$. Then $|f_n| \le u_n$ and $u_n \downarrow 0$. From the d-property, $nu_k \downarrow_k 0$ implies the existence of a sequence $k(n) \uparrow$ of indices and an element $u \in L^+$ such that $u_{k(n)} \le n^{-1} u$. For $k(n) \le m < k(n+1)$ we set $\lambda_m = n$. Then $|f_m| \le \lambda_m^{-1} u$. Hence, $f_n \to 0\,(ru)$.

Relative uniform convergence is <u>stable</u>, i.e., it has the property that for any sequence $f_n \to 0\,(ru)$ there exists a sequence of real numbers $(\lambda_n : n = 1, 2, \ldots)$ such that $0 \le \lambda_n \uparrow \infty$ and $\lambda_n f_n \to 0\,(ru)$. Indeed, given that $f_n \to 0\,(ru)$, there exists an element $u \in L^+$ and a sequence of real numbers $(\varepsilon_n : n = 1, 2, \ldots)$ with $\varepsilon_n \downarrow 0$ and $|f_n| \le \varepsilon_n u$ for all n, and so $\lambda_n = \varepsilon_n^{-\frac{1}{2}}$ satisfies the conditions mentioned above.

Order convergence is not necessarily stable. By way of example, let f_n, $n = 1, 2, \ldots$, be the elements of the space $\ell\infty$ with the first n coordinates zero and all other coordinates equal to 1. Then $f_n \downarrow 0$, but for any sequence of real numbers λ_n satisfying $0 \le \lambda_n \uparrow \infty$ it is impossible that $\lambda_n f_n \to 0$, simply because $\lambda_n f_n$ is not bounded from above. We have, however, the following theorem (see [9]).

Theorem 11. In an Archimedean Reisz space, order convergence is stable if and only if order convergence and relative uniform convergence are equivalent.

Proof. In an Archimedean Riesz space, relative uniform convergence implies order convergence. Assuming stability of order convergence, it will be sufficient, therefore, to prove that order convergence implies relative uniform convergence. To this end, let $f_n \to 0$. Since order convergence is stable by hypothesis, there exists a sequence $0 < \lambda_n \uparrow \infty$ such that $\lambda_n f_n \to 0$. It follows that the sequence $(\lambda_n |f_n| : n = 1, 2, \ldots)$ is bounded, i. e., there exists an element $u \in L^+$ such that $\lambda_n |f_n| \le u$ holds for all n. Then $|f_n| \le \lambda_n^{-1} u$ holds for all n, and so $f_n \to 0\,(ru)$.

Conversely, if order convergence and relative uniform convergence are equivalent, then order convergence is stable because relative uniform convergence is so.

If L is a Riesz space, then the property that order convergence in L is stable is weaker than the property that L has the strong Egoroff property.

Theorem 12. If a Riesz space L has the strong Egoroff property, then order convergence in L is stable. (The strong Egoroff property of an Archimedean Riesz space L is, therefore, a sufficient condition for order convergence and relative uniform convergence to be equivalent in L.)

Proof. Let L be a Riesz space having the strong Egoroff property. Let $f_n \to 0$ in L. There exists, by definition, a sequence $u_n \downarrow 0$ in L and $|f_n| \leq u_n$ for all n. Then the double sequence $u_{nk} = n u_k$, n, k = 1, 2, ... satisfies $u_{nk} \downarrow_k 0$ for every n ; hence, by our assumption that L has the strong Egoroff property and theorem 3 of the last section, there is a sequence $w_n \downarrow 0$ in L and for every n, there is $k = k(n)$ such that $w_n \geq u_{nk(n)}$. We may assume that k(n) is strictly increasing in n. For every m = 1, 2, ... there is some n such that $k(n) \leq m < k(n+1)$, we set $\lambda_m = n$ and $v_m = w_n$. We then have $\lambda_m \uparrow \infty$ and $v_m \downarrow 0$. Furthermore, for every m, if n is such that $k(n) \leq m < k(n+1)$, then

$$\lambda_m u_m = n u_m \leq n u_{k(n)} = u_{nk(n)} \leq w_n = v_m .$$

Thus, for every m, $\lambda_m |f_m| \leq \lambda_m u_m \leq v_m$; hence, $\lambda_m f_m \to 0$. This shows that order convergence in L is stable.

Stability of order convergence in a Riesz space is not a sufficient condition for the space to have the strong Egoroff property. The following example shows the existence of a Riesz space in which order convergence is stable but the space does not have the strong Egoroff property.

Example. Let L be the Riesz space of all real sequences

$f = \{f(1), f(2), \ldots\}$ with only finitely many non-zero terms; the ordering is coordinatewise. We shall first show that order convergence in L is stable. For this, it is sufficient to show that, for $u_n \downarrow 0$ in L, there is a sequence $0 \le \lambda_n \uparrow \infty$ such that $\lambda_n u_n \to 0$. Let $u_n \downarrow 0$ in L. There exists a finite set F of natural numbers such that u_n vanishes outside of F for all n. For every n, set $\mu_n = \max\{u_n(i) : i \in F\}$, then $\mu_n \downarrow 0$. We may assume that $\mu_n \neq 0$ for all n. Let (v_n) be the sequence in L such that $v_n(i) = \mu_n^{\frac{1}{2}}$ if $i \in F$ and $v_n(i) = 0$ otherwise; let $\lambda_n = \mu_n^{-\frac{1}{2}}$. Clearly, $v_n \downarrow 0$ in L and $\lambda_n \uparrow \infty$. For every natural number i, if $i \in F$, then $\lambda_n u_n(i) \le (\mu_n)^{-\frac{1}{2}} \mu_n = \mu_n^{\frac{1}{2}} = v_n(i)$; if $i \notin F$, then $\lambda_n u_n(i) = 0 = v_n(i)$. It follows that $\lambda_n u_n \le v_n$ for all n and so $\lambda_n u_n \to 0$. This completes the proof that order convergence in L is stable. To see that L does not have the strong Egoroff property, we take the sequence $u_n \in L^+$ such that, for every n, u_n has its first n coordinates equal to 1 and all other coordinates zero, and we let $u_{nk} = k^{-1} u_n$ for $n, k = 1, 2, \ldots$. Then $u_{nk} \downarrow_k 0$ for every n. There is, however, no $v \in L$ satisfying $v \ge u_{nk(n)}$ for all n because it would imply that $v(i) \ge u_{ik(i)}(i) = k(i)^{-1} > 0$ for all $i = 1, 2, \ldots$.

If L is an Archimedean Riesz space, then we can show that L has the strong Egoroff property if and only if order convergence in L is stable, provided L has the following property which we will call the σ-property.

Definition. A Riesz space L is said to have the σ-property if, for any sequence $(u_n : n = 1, 2, \ldots)$ in L^+, there exists an element $u \in L^+$ and a sequence $(\lambda_n : n = 1, 2, \ldots)$ of real numbers such that

$u_n \leq \lambda_n u$ for all n.

Theorem 13. Let L be an Archimedean Riesz space. Then the following conditions are equivalent.

(1) L has the strong Egoroff property.

(2) L has the d-property.

(3) L has the σ-property and order convergence in L is stable.

Proof. Let L be an Archimedean space.

(1) $\Leftrightarrow$ (2): Theorem 5 of Section I.

(1) $\Rightarrow$ (3): Since L has the strong Egoroff property, order convergence is stable in L; it will be sufficient to show that the strong Egoroff property of L implies the σ-property of L. Let $(u_n : n = 1, 2, \ldots)$ be a sequence in L^+. Then $k^{-1}u_n \downarrow_k 0$ for every n. Hence, there exists, by the strong Egoroff property of L, an element $u \in L^+$ and for every n an appropriate $k(n)$ such that $u \geq k(n)^{-1}u_n$ for all n. Set $\lambda_n = k(n)$; then $u_n \leq \lambda_n u$ for all n.

(3) $\Rightarrow$ (1): Assume that order convergence in L is stable and L has the σ-property. Let $u_{nk} \downarrow_k 0$ in L. Under our hypothesis that order convergence is stable and L is Archimedean, order convergence and relative uniform convergence are then equivalent; hence, $u_{nk} \to_k 0$ (ru). There exists a double sequence of real numbers $\varepsilon_{nk} \downarrow_k 0$ and $w_n \in L^+$ such that $u_{nk} \leq \varepsilon_{nk} w_n$ for all n, k. By the σ-property of L, there is an element $w \in L^+$ and for every n a real sequence $(\delta_{nk} : k = 1, 2, \ldots)$ such that $u_{nk} \leq \delta_{nk} w$ for all n, k and $\delta_{nk} \downarrow_k 0$ for every n. Since $\delta_{nk} \downarrow_k 0$ for every n, by the strong

Egoroff property of the space of real numbers there is a sequence (μ_m) of real numbers such that $\mu_m \downarrow 0$ and, for every m, n, $\mu_m \geq \delta_{nk(m,n)}$ for some $k = k(m,n)$. Now the sequence $v_m = \mu_m w$ satisfies $v_m \downarrow 0$ and, for every m, n,

$$u_{nk(m,n)} \leq \delta_{nk(m,n)} w < v_m .$$

Hence, L has the strong Egoroff property. This completes the proof of the theorem.

It is an interesting fact that almost all the results we have proved for order convergence hold for relative uniform convergence.

A subset S of a Riesz space L is called <u>(relatively) uniformly closed</u> whenever, for every relatively uniformly convergent sequence in S, all relative uniform limits of the sequence are also members of S. The empty set and L itself are uniformly closed, and arbitrary intersections and finite unions of uniformly closed sets are uniformly closed. Hence, the uniformly closed sets are exactly the closed sets of a certain topology in L, the <u>relative uniform topology</u> (see [9]).

If L is Archimedean, then the relative uniform topology satisfies the T_1-separation axiom, i.e., every set consisting of one point is closed. Conversely, if every set consisting of one point is relatively uniformly closed, then L is Archimedean. Indeed, if not, there exist strictly positive elements u and v in L such that $v \leq n^{-1}u$ holds for $n = 1, 2, \ldots$. It follows that the sequence $(f_n : n = 1, 2, \ldots)$, with $f_n = 0$ for all n, satisfies $|f_n - v| \leq n^{-1}u$ for all n, so $f_n \to v$ (ru) as well as (trivially) $f_n \to 0$ (ru). This contradicts the hypothesis that the set $\{0\}$ is relatively uniformly closed. To summarize, we have that L is Archimedean if and only if the set $\{0\}$ is relatively uniformly closed.

A subset V of a Riesz space L is open in the relative uniform topology if and only if, for every sequence $(f_n : n = 1, 2, \ldots)$ in L which converges relatively uniformly to a point $f \in V$, we have $f_n \in V$ for all but a finite number of the f_n. It follows that if $f_n \to f(ru)$, then f_n converges to f in the relative uniform topology.

For any subset S of L, the pseudo uniform closure S'_{ru} of S is the set of all $f \in L$ such that there exists a sequence in S converging relatively uniformly to f. The closure of S in the relative uniform topology will be denoted by $\overline{S}$. Evidently we have $S \subseteq S'_{ru} \subseteq (S'_{ru})'_{ru} \subseteq \ldots \subseteq \overline{S}$. We have also the following theorem.

Theorem 14. Let L be a Riesz space. Then

(1) A subset S of L is relatively uniformly closed if and only if $S = S'_{ru}$, i.e., $S = S'_{ru}$ already implies that $S = \overline{S}$.

(2) A subset S of L satisfies $S'_{ru} = \overline{S}$ if and only if $S'_{ru} = (S'_{ru})'_{ru}$.

Proof. Similar to the proof of theorem 6.

We shall discuss under what conditions the pseudo uniform closure and the relative uniform closure of S coincide for every subset S of a Riesz space L, or, $S'_{ru} = (S'_{ru})'_{ru}$ for every subset S of L. We prove first the following theorem.

Theorem 15. Let L be an Archimedean Riesz space. Then the following statements are equivalent.

(1) L has the σ-property.

(2) If $f_{nk} \to_k f(ru)$ in L, then, for every n, there is $k = k(n)$ such that $f_{nk(n)} \to f(ru)$.

(3) If $f_{nk} \to_k f_n$ (ru) and $f_n \to f(ru)$ in L, then, for every n, there is $k = k(n)$ such that $f_{nk(n)} \to f(ru)$.

(4) If $f_{nk} \to_k f_n(ru)$ and $f_n \to f(ru)$ in L, then there is a sequence $(g_m : m = 1, 2, \ldots)$ in $\langle \{f_{nk}\} \rangle$ such that $g_m \to f(ru)$.

Proof. (1) $\Rightarrow$ (2): Let $f_{nk} \to_k f(ru)$ in L. There exists a sequence $(u_n : n = 1, 2, \ldots)$ in L^+ and a double sequence $(\varepsilon_{nk} : n, k = 1, 2, \ldots)$ of real numbers such that $\varepsilon_{nk} \downarrow_k 0$ for all n and $|f_{nk} - f| \leq \varepsilon_{nk} u_n$ for all n, k. L has the σ-property implies the existence of an element $u \in L^+$ and a real double sequence $(\delta_{nk} : n, k = 1, 2, \ldots)$ such that $|f_{nk} - f_n| \leq \delta_{nk} u$ for all n, k and $\delta_{nk} \downarrow_k 0$ for every n. Since the space of real numbers has the strong Egoroff property, there is a real sequence $(\gamma_n : n = 1, 2, \ldots)$ such that $\gamma_n \downarrow_n 0$ and, for every n, $\gamma_n \geq \delta_{nk(n)}$ for some $k = k(n)$. Now $|f_{nk(n)} - f| \leq \delta_{nk(n)} u \leq \gamma_n u$ for all n and so $f_{nk(n)} \to f(ru)$.

(2) $\Rightarrow$ (3): Let $f_{nk} \to_k f_n$ (ru) and $f_n \to f(ru)$ in L. There exists a sequence $(u_n : n = 1, 2, \ldots)$ in L^+, an element u in L^+, a real double sequence $(\varepsilon_{nk} : n, k = 1, 2, \ldots)$ and a real sequence $(\varepsilon_n : n = 1, 2, \ldots)$ such that $\varepsilon_{nk} \downarrow_k 0$ for every n, $\varepsilon_n \downarrow 0$, $|f_{nk} - f_n| \leq \varepsilon_{nk} u_n$ for all n, k and $|f_n - f| \leq \varepsilon_n u$ for all n. For each fixed n, $\varepsilon_{nk} u_n \to 0(ru)$; hence, by (2), for every n, there is $k = k(n)$ such that $\varepsilon_{nk(n)} u_n \to 0$ (ru), and so there is an element $w \in L^+$ and a real sequence $(\delta_n : n = 1, 2, \ldots)$ such that $\delta_n \downarrow 0$ and $\varepsilon_{nk(n)} u_n \leq \delta_n w$ for

all n. Now we have

$$|f_{nk(n)}-f| \le \varepsilon_{nk(n)}u_n + \varepsilon_n u \le \delta_n w + \varepsilon_n u .$$

If we write $\gamma_n = \delta_n+\varepsilon_n$ and $v = w+u$, then $\gamma_n \downarrow 0$ and $|f_{nk(n)}-f| \le \gamma_n v$ for every n; hence, $f_{nk(n)} \to f\,(ru)$.

It is clear that (3) ⇒ (4).

(4) ⇒ (1): Let $(u_n : n = 1, 2, \dots)$ be any sequence of elements of L^+. There is no loss in generality to assume that $u_n \uparrow$. We shall show that there is an element $w \in L^+$ and a real sequence $(\lambda_n : n = 1, 2, \dots)$ such that $u_n \le \lambda_n w$ for all n.

Let u be an element of L^+ such that $u > 0$. Set $w_{nk} = k^{-1}u_n + n^{-1}u$ for $n, k = 1, 2, \dots$ and $w_n = n^{-1}u$ for $n = 1, 2, \dots$. Clearly, $w_{nk} \to_k w_n\,(ru)$ for every n, and $w_n \to 0\,(ru)$. By (4), there is a sequence $(v_m : m = 1, 2, \dots)$ in $\langle\{w_{nk}\}\rangle$ such that $v_m \to 0\,(ru)$.

For each fixed m, since v_m is in $\langle\{w_{nk}\}\rangle$, there is a sequence of real numbers $(\lambda_n^m : n = 1, 2, \dots)$ and for every m an appropriate index k(m, n) such that $\lambda_n^m \ge 0$ for all n, $\lambda_n^m = 0$ except for finitely many n, $\sum_n \lambda_n^m = 1$ and $v_m \ge \sum_n \lambda_n^m w_{nk(m,n)}$.

Since $v_m \to 0\,(ru)$, there is an element w in L^+ and a real sequence $\varepsilon_m \downarrow 0$ such that $v_m \le \varepsilon_m w$ for all m. Then $\sum_n \lambda_n^m w_{nk(m,n)} \le \varepsilon_m w$ for all m.

Denote by A the set of all natural numbers n such that $\lambda_n^m \ne 0$ for some m. Clearly, A is non-empty. We claim that A has infinitely many elements. Suppose A is finite. Let M be the largest number in A. Then, for every m, we have

$$\varepsilon_m w \geq \sum_n \lambda_n^m w_{nk(m,n)} \geq \sum_n \lambda_n^m n^{-1} u \geq M^{-1} u ;$$

but by the assumption that L is Archimedean, $\varepsilon_m w \downarrow 0$ and so $M^{-1}u \leq 0$, which contradicts that $u > 0$.

It remains to be shown that for a given natural number N, there is some positive real number λ_N such that $u_N \leq \lambda_N w$. Let N be given. By the above argument that A is an infinite set, there is a natural number n(N) in A such that $n(N) \geq N$; so $\lambda_{n(N)}^{m(N)} \neq 0$ for some $m = m(N)$. We then have

$$\varepsilon_{m(N)} w \geq \sum_n \lambda_n^{m(N)} w_{nk(m(N),n)} \geq \sum_n \lambda_n^{m(N)} \cdot k(m(N),n)^{-1} u_n$$

$$\geq \lambda_{n(N)}^{m(N)} \cdot k(m(N), n(N))^{-1} u_{n(N)} \quad .$$

Set $\lambda_N = \varepsilon_{m(N)} \cdot (\lambda_{n(N)}^{m(N)})^{-1} \cdot k(m(N), n(N))$. Recall that $u_n \uparrow_n$; we thus have $\lambda_N w \geq u_N$. This completes the proof of the theorem.

As a direct consequence of the last theorem we have:

Theorem 16. *Let L be an Archimedean Riesz space. Then the following statements are equivalent.*

(1) L *has the* σ-*property.*

(2) $S'_{ru} = (S'_{ru})'_{ru}$ *for every subset* S *of* L.

(3) $S'_{ru} = (S'_{ru})'_{ru}$ *for every convex subset* S *of* L.

If L is an Archimedean Riesz space having the σ-property, then the relative uniform topology in L has the property that the closure of any set consists of the set itself plus all its relatively uniform sequential limit points.

From theorem 13 and theorem 16 we have:

Theorem 17. *If* L *is an Archimedean Riesz space and* L *has the* d-*property, then* $S'_{ru} = (S'_{ru})'_{ru}$ *for every subset* S *of* L.

We conclude this section by summarizing several important results.

If L is an Archimedean Riesz space, then relative uniform convergence of a sequence f_n to f implies order convergence of f_n to f, which implies immediately that every order closed set in L is relatively uniformly closed. Hence, in an Archimedean Riesz space, the relative uniform topology is stronger than the order topology.

If a Riesz space L has the d-property, then order convergence of a sequence f_n to f implies relative uniform convergence of f_n to f; hence, in this case the order topology is stronger than the relative uniform topology.

If, in an Archimedean Riesz space L, order convergence and relative uniform convergence are equivalent (or, equivalently, if L is Archimedean and order convergence in L is stable), then the pseudo order closure S' and the pseudo uniform closure S'_{ru} of any set S are identical, and the same holds for the topological closures cl(S) and $\overline{S}$. Thus, in an Archimedean Riesz space where order convergence and relative uniform convergence are equivalent, the order topology and the relative uniform topology are identical.

If L is Archimedean and L has the strong Egoroff property, then order convergence and relative uniform convergence are equivalent in L, the order topology and the relative uniform topology are identical, and this topology has the property that the closure of any set consists of the set itself plus all its sequential limit points.

III. THE ALMOST EGOROFF PROPERTY OF RIESZ SPACES

In Section I we discussed the Egoroff and strong Egoroff properties of Riesz spaces. A related property which has been introduced by J. A. R. Holbrook is the almost Egoroff property. In a Riesz space with the almost Egoroff property, the largest σ-Fatou monotone seminorm ρ_M majorized by a given monotone seminorm ρ can be constructed explicitly. We show that, in a Riesz space with the generalized almost Egoroff property, the largest strong Fatou monotone seminorm $\rho_{\mathfrak{M}}$ majorized by a given monotone seminorm ρ can also be constructed explicitly.

Definition. An element f of a Riesz space L is said to have the almost Egoroff property if, given any real number ε with $0 < \varepsilon < 1$ and any double sequence of elements $(u_{nk} : n, k = 1, 2, \dots)$ in L such that $0 \le u_{nk} \uparrow_k |f|$ for $n = 1, 2, \dots$, there exists a sequence $0 \le v_m^{\varepsilon} \uparrow |f|$ in L and for every m a sequence $k(m, n)$ of indices such that $(1-\varepsilon)v_m^{\varepsilon} \le u_{nk(m,n)}$ for all m and n.

A Riesz space L is said to have the almost Egoroff property if every element of L has the almost Egoroff property.

We can easily see that the Egoroff property of a Riesz space implies the almost Egoroff property of the same space. The converse is not necessarily true. We shall see later that for Archimedean Riesz spaces the two properties are equivalent. Let us first discuss the various equivalent forms of the almost Egoroff property.

Theorem 18. Let L be a Riesz space and $u \in L^+$. Then the following statements are equivalent.

(1) u has the almost Egoroff property.

(2) If $0 \le u_{nk} \uparrow_k u$ in L and $0 < \varepsilon < 1$, then there is a sequence $0 \le v_n^\varepsilon \uparrow u$ in L and for every n an appropriate k(n) such that $(1-\varepsilon)v_n^\varepsilon \le u_{nk(n)}$ for all n.

(3) If $0 \le u_{nk} \uparrow_k u_n \uparrow u$ in L and $0 < \varepsilon < 1$, then there is a sequence $0 \le v_n^\varepsilon \uparrow u$ in L and for every n an appropriate k(n) such that $(1-\varepsilon)v_n^\varepsilon \le u_{nk(n)}$ for all n.

(4) If $0 \le u_{nk} \uparrow_k u_n \uparrow u$ in L and $0 < \varepsilon < 1$, then there is a sequence $0 \le v_m^\varepsilon \uparrow u$ in L such that, for every m, $(1-\varepsilon)v_m^\varepsilon \le z_m$ for some element z_m in $\langle\{u_{nk}\}\rangle$.

In order to prove theorem 18, we first introduce the following:

Theorem 19. Let L be a Riesz space and $u \in L^+$. Then the following statements are all equivalent.

(0') u has the almost Egoroff property.

(1') If $u \ge u_{nk} \downarrow_k 0$ in L and $0 < \varepsilon < 1$, then there is a sequence $u \ge v_m^\varepsilon \downarrow 0$ in L and for every m a sequence k(m, n) of indices such that $v_m^\varepsilon + \varepsilon u \ge u_{nk(m,n)}$ for all m and n.

(2') If $u \ge u_{nk} \downarrow_k 0$ in L and $0 < \varepsilon < 1$, then there is a sequence $u \ge v_n^\varepsilon \downarrow 0$ in L and for every n an appropriate k(n) such that $v_n^\varepsilon + \varepsilon u \ge u_{nk(n)}$ for all n.

(3') If $u \ge u_{nk} \downarrow_k u_n \downarrow 0$ in L and $0 < \varepsilon < 1$, then there is a sequence $u \ge v_n^\varepsilon \downarrow 0$ in L and for every n an appropriate k(n) such that $v_n^\varepsilon + \varepsilon u \ge u_{nk(n)}$ for all n.

(4') If $u \ge u_{nk} \downarrow_k u_n \downarrow 0$ in L and $0 < \varepsilon < 1$, then there is a sequence $u \ge v_m^\varepsilon \downarrow 0$ in L such that, for every m, $v_m^\varepsilon + \varepsilon u \ge z_m$ for some element z_m in $\langle\{u_{nk}\}\rangle$.

Proof. We shall show that $(1') \Rightarrow (0') \Rightarrow (1') \Rightarrow (2') \Rightarrow (3') \Rightarrow (4') \Rightarrow (1')$.

$(1') \Rightarrow (0')$: Let $0 \le u_{nk} \uparrow_k u$ in L and $0 < \varepsilon < 1$. Set $w_{nk} = u - u_{nk}$. Then $u \ge w_{nk} \downarrow_k 0$ and so by $(1')$ there is a sequence $u \ge w_m^\varepsilon \downarrow 0$ such that, for every m, n, $w_m^\varepsilon + \varepsilon u \ge w_{nk(m,n)}$ for some $k = k(m,n)$. Let $v_m^\varepsilon = (1-\varepsilon)^{-1}[(1-\varepsilon)u - w_m^\varepsilon]^+$. We then have $0 \le v_m^\varepsilon \uparrow u$ and $(1-\varepsilon)v_m^\varepsilon \le u_{nk(m,n)}$.

$(0') \Rightarrow (1')$: Let $u \ge u_{nk} \downarrow_k 0$ in L and $0 < \varepsilon < 1$. Set $w_{nk} = u - u_{nk}$. Then $0 \le w_{nk} \uparrow_k u$ and so by $(0')$ there is a sequence $0 \le w_m^\varepsilon \uparrow u$ such that, for every m, n, $(1-\varepsilon)w_m^\varepsilon \le w_{nk(m,n)}$ for some $k = k(m,n)$. Now $v_m^\varepsilon = u - w_m^\varepsilon$ satisfies $u \ge v_m^\varepsilon \downarrow 0$ and $v_m^\varepsilon + \varepsilon u \ge u_{nk(m,n)}$.

It is clear that $(1') \Rightarrow (2')$.

$(2') \Rightarrow (3')$: Let $u \ge u_{nk} \downarrow u_n \downarrow 0$ and $0 < \varepsilon < 1$. Then $u \ge (u_{nk} - u_n) \downarrow_k 0$ for every n; there is by $(2')$ a sequence $u \ge w_n^\varepsilon \downarrow 0$ such that, for every n, $w_n^\varepsilon + \varepsilon u \ge u_{nk(n)} - u_n$ for some $k = k(n)$. Set $v_n^\varepsilon = u - (u - w_n^\varepsilon - u_n)^+$. Clearly, $u \ge v_n^\varepsilon \downarrow 0$. It remains to be shown that $v_n^\varepsilon + \varepsilon u \ge u_{nk(n)}$. To this end, we observe that $u - w_n^\varepsilon - u_n \le (1+\varepsilon)u - u_{nk(n)}$ and so $u - (u - w_n^\varepsilon - u_n)^+ \ge u_{nk(n)} - \varepsilon u$, i.e., $v_n^\varepsilon + \varepsilon u \ge u_{nk(n)}$.

It is also clear that $(3') \Rightarrow (4')$.

$(4') \Rightarrow (1')$: Let $u \ge u_{nk} \downarrow_k 0$ in L and $0 < \varepsilon < 1$. We may assume that $u_{nk} \uparrow_n$ for every k (since we can always replace u_{nk} by $u_{1k} \vee \dots \vee u_{nk}$). Furthermore, we may assume that there exists a sequence (y_n) in L^+ such that:

(i) $u \geq y_n \downarrow 0$,

(ii) $y_n \nleq \varepsilon u$ for all n.

For if such a sequence (y_n) does not exist, then $u \geq u_{nk} \downarrow_k 0$ implies that for every n there is some index k(n) such that $u_{nk(n)} \leq \varepsilon u$, and so the sequence (v_m^ε) with $v_m^\varepsilon = 0$ for all m satisfies the required conditions.

We shall construct from the sequence (y_n) a sequence $u \geq u_n \downarrow 0$ such that $\inf\{\alpha \geq 0 : u_n \leq \alpha u \text{ for some } n\} = \varepsilon$. To this end, consider the number $\beta = \inf\{\alpha \geq 0 : y_n \leq \alpha u \text{ for some } n\}$. Since $y_n \nleq \varepsilon u$ for all n, we have $\varepsilon \leq \beta$. Set $u_n = \varepsilon\beta^{-1} y_n$; then the sequence (u_n) has the above required conditions.

Now we have $u \geq (u_{nk} \vee u_n) \downarrow_k u_n \downarrow 0$. Set $\delta = \varepsilon/4$. By (4'), there is a sequence $u \geq w_m^\delta \downarrow 0$ such that, for every m, $w_m^\delta + \delta u \geq z_m$ for some z_m in $\langle\{u_{nk} \vee u_n\}\rangle$. Then, for every fixed m, there are real numbers λ_n^m, n = 1, 2, ... satisfying $\lambda_n^m \geq 0$ for all n, $\lambda_n^m = 0$ except for finitely many n, $\sum_n \lambda_n^m = 1$ and $w_m^\delta + \delta u \geq \sum_n \lambda_n^m (u_{nk(m,n)} \vee u_n)$ for some $k = k(m, n)$.

Set $v_m^\varepsilon = u - (u - 2w_m^\delta)^+$. Clearly, $u \geq v_m^\varepsilon \downarrow 0$. It remains to be shown that for a particular pair M, N of positive integers, there exists some $k = k(M, N)$ such that $v_M^\varepsilon + \varepsilon u \geq u_{Nk(M,N)}$.

Let M, N be given. Set $\gamma = \sup\{\sum_{n \geq N} \lambda_n^m : m \geq M\}$. We shall first prove that $\gamma \geq 3/4$. It is clear that $0 \leq \gamma \leq 1$. For every $m \geq M$ we have

$$w_m^\delta + \delta u \geq \sum_n \lambda_n^m (u_{nk(m,n)} \vee u_n)$$
$$\geq \sum_{n<N} \lambda_n^m u_n \geq (\sum_{n<N} \lambda_n^m) u_N$$
$$= (\sum_n \lambda_n^m - \sum_{n \geq N} \lambda_n^m) u_N \geq (1-\gamma) u_N ;$$

since $\inf\{w_m^\delta : m \geq M\} = 0$, we then have $\delta u \geq (1-\gamma)u_N$. Recall that $\varepsilon = \inf\{\alpha \geq 0 : u_n \leq \alpha u$ for some $n\}$, so either $1-\gamma = 0$ or $\varepsilon \leq (1-\gamma)^{-1}\delta$. Substituting $\delta = \varepsilon/4$, we obtain $\gamma \geq 3/4$.

By the above argument, there exists an integer $P \geq M$ such that $\sum_{n \geq N} \lambda_n^P \geq 1/2$. Consider

$$2w_P^\delta \geq 2 \sum_n \lambda_n^P u_{nk(P,n)} - 2\delta u ;$$

if we let $k(M,N) = \max\{k(P,n) : \lambda_n^P \neq 0\}$ and recall that $u_{nk} \uparrow_n$, we obtain

$$2w_P^\delta \geq 2 (\sum_{n \geq N} \lambda_n^P) u_{Nk(M,N)} - 2\delta u \geq u_{Nk(M,N)} - 2\delta u .$$

Hence

$$v_M^\varepsilon \geq v_P^\varepsilon = u - (u - 2w_P^\delta)^+ \geq u_{Nk(M,N)} - 2\delta u ,$$

and so $v_M^\varepsilon + \varepsilon u \geq u_{Nk(M,N)}$ as required. This completes the proof of the theorem.

Proof of theorem 18. It is clear that (2) ⇔ (2'), (3) ⇔ (3'), (4) ⇔ (4') in a similar way as the above proof of (0') ⇔ (1'). Hence, theorem 18 follows immediately from theorem 19.

In a Riesz space L, an element f has the Egoroff property implies that f has the almost Egoroff property. The converse implication does not always hold. For counter-example, see [3], p. 72. However, the following theorem due to J. A. R. Holbrook (see [3]) is true.

Theorem 20. If an element u in a Riesz space L is such that $n^{-1}u \downarrow 0$ (in particular, if L is Archimedean), then u has the Egoroff property if and only if u has the almost Egoroff property.

Proof. One implication of the theorem holds always. On the other hand, assume the element u in L has the almost Egoroff property and $n^{-1}u \downarrow 0$. Let $0 \le u_{nk} \uparrow_k u$ in L. For every $p = 1, 2, \ldots$ there exists, by the almost Egoroff property of u, a sequence $(v_m^p : m = 1, 2, \ldots)$ such that $0 \le v_m^p \uparrow_m (1-p^{-1})u$ and, for every m, n, $v_m^p \le u_{nk(m,n,p)}$ for some $k = k(m,n,p)$. Let $v_m = v_m^1 \vee \ldots \vee v_m^m$. Clearly, $v_m \uparrow_m$ and $v_m \le u$ for all m. Moreover, if w is such that $v_m \le w$ for all m, then $v_m^p \le w$ for all m, p, so that $(1-p^{-1})u \le w$ for all p or $u-w \le p^{-1}u$ for all p; but $p^{-1}u \downarrow 0$, therefore, $u \le w$. Hence, $v_m \uparrow u$. For every m, n, p, since $v_m^p \le u_{nk(m,n,p)}$ for some $k = k(m,n,p)$, we have, for every m, n, $v_m \le u_{nk(m,n)}$ where $k(m,n) = \max\{k(m,n,p) : 1 \le p \le m\}$. Therefore, u has the Egoroff property and the proof of the theorem is complete.

We shall next study a characterization of those Riesz spaces which have the almost Egoroff property.

An extended real valued function ρ on a Riesz space L is called a monotone seminorm on L if:

(i) $0 \le \rho(f) \le \infty$, $\rho(f+g) \le \rho(f) + \rho(g)$, and $\rho(\lambda f) = \lambda\rho(f)$ for all f, g in L and real $\lambda \ge 0$,

(ii) ρ is monotone, i.e., $|f| \le |g|$ implies $\rho(f) \le \rho(g)$.

It follows immediately from (ii) that $\rho(f) = \rho(|f|)$ for all f in L.

A monotone seminorm ρ on a Riesz space L is <u>σ-Fatou</u> if $0 \leq u_n \uparrow u$ in L implies that $\rho(u_n) \uparrow \rho(u)$.

A given monotone seminorm ρ may not itself be σ-Fatou, but the existence of the largest element ρ_M among those monotone σ-Fatou seminorms majorized by ρ is not difficult to show. In fact, we can easily show that:

$$\rho_M(f) = \{\sup \rho'(f) : \rho' \text{ is } \sigma\text{-Fatou monotone seminorm and } \rho' \leq \rho\}.$$

In general, it is not known how to construct ρ_M explicitly in terms of the given monotone seminorm ρ. However, there are three cases of interest in which the seminorm ρ_M may be constructed explicitly.

To facilitate the discussion, we introduce the Lorentz seminorm ρ_L associated with a given monotone seminorm ρ. If ρ is a monotone seminorm on a Riesz space L, then, for every f in L, $\rho_L(f)$ is defined by:

$$\rho_L(f) = \inf\{\lim_n \rho(u_n) : 0 \leq u_n \uparrow |f|\}.$$

We can easily see that, for any given monotone seminorm ρ on L, ρ_L is again a monotone seminorm on L. Moreover, $\rho \geq \rho_L \geq \rho_M$, $\rho_L = \rho$ if and only if ρ is σ-Fatou, $\rho_1 \leq \rho_2$ implies that $\rho_{1L} \leq \rho_{2L}$.

We now go back to the discussion of special cases when ρ_M can be constructed explicitly in terms of ρ.

If the Riesz space L is a real Banach function space, then $\rho_M = \rho_L$ for every monotone seminorm ρ on L (see [6]).

If a monotone seminorm ρ on a Riesz space L is of the form

$\rho(f) = \phi(|f|)$ for every f in L, where ϕ is a positive linear functional on L (i.e., ϕ is a real linear functional on L such that $\phi(u) \geq 0$ for all u in L^+), then $\rho_M = \rho_L$ (see [6]).

The third result is due to J. A. R. Holbrook. This result gives also a characterization of those Riesz spaces having the almost Egoroff property. Hence, we shall study it in detail.

Before we state and prove the result, notice first that $\rho \geq \rho_M$; hence, we have $\rho_L \geq \rho_{ML} = \rho_M$, so that $\rho_L = \rho_M$ if and only if ρ_L is σ-Fatou, i.e., $\rho_{LL} = \rho_L$.

Theorem 21. Let L be a Riesz space and $u \in L^+$.

(1) If u has the almost Egoroff property, then $\rho_L(u) = \rho_{LL}(u)$ for every monotone seminorm ρ on L.

(2) If $\rho_L(u) = \rho_{LL}(u)$ for every monotone seminorm ρ on L such that $\rho(u) < \infty$, then u has the almost Egoroff property.

Remark. The following proof is a modified version of J. A. R. Holbrook's proof (see [3]). We will see later that this modified proof leads to the desired generalization we have in mind.

Proof of theorem 21. (1) It is clear that $\rho_{LL}(u) \leq \rho_L(u)$. On the other hand, suppose $\rho_{LL}(u) < \alpha$; in this case, there must exist $0 \leq u_n \uparrow u$ and, for every n, $0 \leq u_{nk} \uparrow_k u_n$ such that $\rho(u_{nk}) < \alpha$ for all n, k. By the assumption that u has the almost Egoroff property and by theorem 18, there exists, for every $0 < \varepsilon < 1$, a sequence $0 \leq v_m^{\varepsilon} \uparrow u$ such that, for every m, $(1-\varepsilon)v_m^{\varepsilon} \leq z_m$ for some z_m in $\langle\{u_{nk}\}\rangle$. We have then $(1-\varepsilon)\rho(v_m^{\varepsilon}) \leq \rho(z_m) < \alpha$ for all m; hence, $\rho_L(u) \leq (1-\varepsilon)^{-1}\alpha$. Since this is true for every $0 < \varepsilon < 1$, we obtain

$\rho_L(u) \le \alpha$. Therefore, $\rho_L(u) \le \rho_{LL}(u)$.

(2) It is sufficient to show that if $0 \le u_{nk} \uparrow u_n \uparrow u$ in L and $0 < \varepsilon < 1$, then there is a sequence $0 \le v_m^\varepsilon \uparrow u$ such that, for every m, $(1-\varepsilon)v_m^\varepsilon \le z_m$ for some z_m in $\langle\{u_{nk}\}\rangle$.

Let $0 \le u_{nk} \uparrow_k u_n \uparrow u$ and $0 < \varepsilon < 1$. Let $\varepsilon_1 = \varepsilon/2$ and define ρ on L as follows: for f in L,

$$\rho(f) = \inf\{\sum_{n,k}\alpha_{nk}: \alpha_{nk} \ge 0 \text{ for all } n,k,\ \alpha_{nk} = 0 \text{ except for finitely many } n,k,\ \sum_{n,k}\alpha_{nk}(u_{nk} \vee \varepsilon_1 u) \ge |f|\},$$

$$= \infty \text{ if there is no such finite sum covering } |f|.$$

It can be easily verified that ρ is a monotone seminorm on L. Moreover, $\rho(u) < \infty$; in fact, $\rho(u) \le \varepsilon_1^{-1}$.

Now $\rho(u_{nk}) \le 1$ for all n, k; then $\rho_L(u_n) \le 1$ for all n and so $\rho_{LL}(u) \le 1$. By the assumption that $\rho_L(u) = \rho_{LL}(u)$ for all monotone seminorms ρ with $\rho(u) < \infty$, we have $\rho_L(u) \le 1$. Set $\varepsilon_2 = (2-\varepsilon)^{-1}\varepsilon$. Then $\rho_L(u) < 1+\varepsilon_2$ implies that there exists a sequence $0 \le w_m \uparrow u$ and, for each m, $\rho(w_m) < 1+\varepsilon_2$. If we let $v_m^\varepsilon = (1-\varepsilon)^{-1}(w_m - \varepsilon u)^+$, we have $0 \le v_m^\varepsilon \uparrow u$. It remains to be shown that, for every m, $(1-\varepsilon)v_m^\varepsilon \le z_m$ for some element z_m in $\langle\{u_{nk}\}\rangle$.

For each fixed m, since $\rho(w_m) < 1+\varepsilon_2$, there exist, by the definition of ρ, real numbers α_{nk}^m, $n, k = 1, 2, \ldots$, such that $\alpha_{nk}^m \ge 0$ for all n, k, $\alpha_{nk}^m = 0$ except for finitely many n, k, $0 < \sum_{n,k}\alpha_{nk}^m < 1+\varepsilon_2$ and $\sum_{n,k}\alpha_{nk}^m(u_{nk} \vee \varepsilon_1 u) \ge w_m$. We then have

$$(1+\varepsilon_2)^{-1}w_m \le \sum_{n,k}(1+\varepsilon_2)^{-1}\alpha_{nk}^m(u_{nk} \vee \varepsilon_1 u)$$

$$\le \sum_{n,k}(1+\varepsilon_2)^{-1}\alpha_{nk}^m u_{nk} + \varepsilon_1 u .$$

Set $\lambda_{nk}^{m} = (\Sigma_{n,k}\alpha_{nk}^{m})^{-1} \cdot \alpha_{nk}^{m}$ and $z_m = \Sigma_{n,k}\lambda_{nk}^{m} u_{nk}$; then, for every m, z_m is an element in $\langle\{u_{nk}\}\rangle$ and $(1+\varepsilon_2)^{-1} w_m \le z_m + \varepsilon_1 u$. Note that $(w_m - \varepsilon u)^+ \le [w_m - \frac{\varepsilon}{2}(w_m+u)]^+ = [(1+\varepsilon_2)^{-1} w_m - \varepsilon_1 u]^+$; hence, $(1-\varepsilon)v_m^{\varepsilon} = (w_m - \varepsilon u)^+ \le z_m$ as required. This completes the proof of (2).

The following results can be obtained directly from theorems 20 and 21.

Theorem 22. Let L be a Riesz space. Then $\rho_M = \rho_L$ for every monotone seminorm ρ on L if and only if L has the almost Egoroff property.

Theorem 23. Let L be an Archimedean Riesz space. Then $\rho_M = \rho_L$ for every monotone seminorm on L if and only if L has the Egoroff property.

Theorems 21 and 22 can be generalized in a natural way to directed systems instead of sequences.

Let ρ be a monotone seminorm on a Riesz space L. ρ is said to have the strong Fatou property whenever, for every $u \in L^+$ and for every indexed subset (u_τ) of L^+ such that $0 \le u_\tau \uparrow u$ in L, $\sup \rho(u_\tau) = \rho(u)$.

Every monotone seminorm ρ dominates a largest monotone seminorm $\rho_{\mathfrak{m}}$ having the strong Fatou property, namely,

$$\rho_{\mathfrak{m}}(f) = \sup\{\rho'(f) : \rho' \le \rho \text{ and } \rho' \text{ is a monotone seminorm having the strong Fatou property}\}.$$

If, for instance, ϕ is a positive linear functional on L and

ρ_ϕ is defined by $\rho_\phi(f) = \phi(|f|)$ for every $f \in L$, then $\rho_{\phi\mathfrak{m}}$ can be obtained explicitly as follows: for every $f \in L$,

$$\rho_{\phi\mathfrak{m}}(f) = \inf\{\sup \rho(u_\tau) : 0 \le u_\tau \uparrow |f|\}$$

(see [7]).

We can have the same explicit construction for $\rho_{\mathfrak{m}}$ in terms of ρ, for every monotone seminorm ρ on a Riesz space L, provided L has the generalized almost Egoroff property.

Definition. An element f of a Riesz space L is said to have the generalized almost Egoroff property whenever $0 \le u_{\varkappa_\tau} \uparrow_{\varkappa_\tau} u_\tau \uparrow |f|$ and $0 < \varepsilon < 1$, there is an indexed subset (v_μ^ε) of L^+ such that $0 \le v_\mu^\varepsilon \uparrow |f|$ and, for every μ, $(1-\varepsilon)v_\mu^\varepsilon \le z_\mu$ for some element z_μ in $\langle\{u_{\varkappa_\tau}\}\rangle$.

A Riesz space L is said to have the generalized almost Egoroff property if every one of its elements has the generalized almost Egoroff property.

For every monotone seminorm ρ on a Riesz space L, we define $\rho_{\mathfrak{L}}$ by: for every $f \in L$,

$$\rho_{\mathfrak{L}}(f) = \inf\{\sup_\tau \rho(u_\tau) : 0 \le u_\tau \uparrow |f|\} .$$

Clearly, $\rho_{\mathfrak{L}}$ is again a monotone seminorm on L, $\rho \ge \rho_{\mathfrak{L}} \ge \rho_{\mathfrak{m}}$, $\rho_{\mathfrak{L}} = \rho$ if and only if ρ has the strong Egoroff property, $\rho_1 \le \rho_2$ implies that $\rho_{1\mathfrak{L}} \le \rho_{2\mathfrak{L}}$. Since $\rho \ge \rho_{\mathfrak{m}}$, we have $\rho_{\mathfrak{L}} > \rho_{\mathfrak{m}\mathfrak{L}} = \rho_{\mathfrak{m}}$, so $\rho_{\mathfrak{m}} = \rho_{\mathfrak{L}}$ if and only if $\rho_{\mathfrak{L}} = \rho_{\mathfrak{L}\mathfrak{L}}$.

Theorem 24. Let L be a Riesz space and $u \in L^+$.

(1) If u has the generalized almost Egoroff property, then

$\rho_{\mathfrak{L}}(u) = \rho_{\mathfrak{L}\mathfrak{L}}(u)$ for every monotone seminorm ρ on L.

(2) If $\rho_{\mathfrak{L}}(u) = \rho_{\mathfrak{L}\mathfrak{L}}(u)$ for every monotone seminorm ρ on L such that $\rho(u) < \infty$, then u has the generalized almost Egoroff property.

Proof. (The following proof is an exact analogue of the proof of theorem 21 except that sequences are replaced by directed systems.)

(1) It is clear that $\rho_{\mathfrak{L}\mathfrak{L}}(u) \le \rho_{\mathfrak{L}}(u)$. Conversely, suppose that $\lambda > \rho_{\mathfrak{L}\mathfrak{L}}(u)$; in this case, there must exist $0 \le u_\tau \uparrow u$ and, for every τ, there is $0 \le u_{\varkappa_\tau} \uparrow_{\varkappa_\tau} u_\tau$ such that $\rho(u_{\varkappa_\tau}) < \lambda$ for all $\varkappa_\tau$. Since u has the generalized almost Egoroff property, there exists for each ε with $0 < \varepsilon < 1$ a directed set $0 \le v_\mu^\varepsilon \uparrow u$ such that, for every μ, $(1-\varepsilon)v_\mu^\varepsilon \le z_\mu$ for some element z_μ in $\langle\{u_{\varkappa_\tau}\}\rangle$. We then have $(1-\varepsilon)\rho(v_\mu^\varepsilon) \le \rho(z_\mu) < \lambda$ for all μ; hence, $(1-\varepsilon)\rho_{\mathfrak{L}}(u) \le \lambda$ for every $0 < \varepsilon < 1$, thus $\rho_{\mathfrak{L}}(u) \le \lambda$. Therefore, $\rho_{\mathfrak{L}}(u) \le \rho_{\mathfrak{L}\mathfrak{L}}(u)$.

(2) Let $0 \le u_{\varkappa_\tau} \uparrow_{\varkappa_\tau} u_\tau \uparrow u$ and $0 < \varepsilon < 1$. Set $\varepsilon_1 = \varepsilon/2$ and define, for every $f \in L$,

$$\rho(f) = \inf\{\sum_{\varkappa_\tau} \alpha_{\varkappa_\tau} : \alpha_{\varkappa_\tau} \ge 0 \text{ for all } \tau \text{ and } \varkappa_\tau,\ \alpha_{\varkappa_\tau} = 0 \text{ except for finitely many } \tau \text{ and } \varkappa_\tau,\ \sum_{\varkappa_\tau} \alpha_{\varkappa_\tau}(u_{\varkappa_\tau} \vee \varepsilon_1 u) \ge |f|\},$$

$$= \infty \text{ if there is no such finite sum covering } |f|.$$

It can be easily verified that ρ is a monotone seminorm on L and $\rho(u) \le \varepsilon_1^{-1}$.

Now $\rho(u_{\varkappa_\tau}) \le 1$ for all τ and $\varkappa_\tau$; then $\rho_{\mathfrak{L}}(u_\tau) \le 1$ for all τ and so $\rho_{\mathfrak{L}\mathfrak{L}}(u) \le 1$. Hence, by assumption, $\rho_{\mathfrak{L}}(u) \le 1$. Set $\varepsilon_2 = (2-\varepsilon)^{-1}\varepsilon$. Then $\rho_{\mathfrak{L}}(u) < 1+\varepsilon_2$ implies that there exists a directed set

$0 \leq w_\mu \uparrow u$ and $\rho(w_\mu) < 1+\varepsilon_2$ for all μ.

If we let $v_\mu^\varepsilon = (1-\varepsilon)^{-1}(w_\mu - \varepsilon u)^+$, we have $0 \leq v_\mu^\varepsilon \uparrow_\mu u$. It remains to find, for every μ, an element z_μ in $\langle\{u_{\varkappa_\tau}\}\rangle$ such that $(1-\varepsilon)v_\mu^\varepsilon \leq z_\mu$.

For every fixed μ, since $\rho(w_\mu) < 1+\varepsilon_2$, there exist real numbers $\alpha_{\varkappa_\tau}^\mu$ such that $\alpha_{\varkappa_\tau}^\mu \geq 0$ for all τ and $\varkappa_\tau$, $\alpha_{\varkappa_\tau}^\mu = 0$ except for finitely many τ and $\varkappa_\tau$, $0 < \sum_{\varkappa_\tau} \alpha_{\varkappa_\tau}^\mu < 1+\varepsilon_2$ and $\sum_{\varkappa_\tau} \alpha_{\varkappa_\tau}^\mu (u_{\varkappa_\tau} \vee \varepsilon_1 u) \geq w_\mu$. We then have

$$(1+\varepsilon_2)^{-1} w_\mu \leq \sum_{\varkappa_\tau} (1+\varepsilon_2)^{-1} \alpha_{\varkappa_\tau}^\mu u_{\varkappa_\tau} + \varepsilon_1 u .$$

Set $\lambda_{\varkappa_\tau}^\mu = (\sum_{\varkappa_\tau} \alpha_{\varkappa_\tau}^\mu)^{-1} \cdot \alpha_{\varkappa_\tau}^\mu$ and $z_\mu = \sum_{\varkappa_\tau} \lambda_{\varkappa_\tau}^\mu u_{\varkappa_\tau}$; then, for every μ, z_μ is an element in $\langle\{u_{\varkappa_\tau}\}\rangle$ and $(1+\varepsilon_2)^{-1} w_\mu \leq z_\mu + \varepsilon_1 u$. Note that

$$(w_\mu - \varepsilon u)^+ \leq [w_\mu - \frac{\varepsilon}{2}(w_\mu + u)]^+ = [(1+\varepsilon_2)^{-1} w_\mu - \varepsilon_1 u]^+ ;$$

hence, we have $(1-\varepsilon)v_\mu^\varepsilon = (w_\mu - \varepsilon u)^+ \leq z_\mu$ as required. This completes the proof of (2).

Theorem 25. If L is a Riesz space, then $\rho_{\mathfrak{M}} = \rho_{\mathfrak{L}}$ for every monotone seminorm ρ on L if and only if L has the generalized almost Egoroff property.

IV. σ-FATOU PROPERTY OF A MONOTONE SEMINORM ON A RIESZ SPACE

In the last section, we obtained a characterization of those Riesz spaces in which ρ_L is σ-Fatou for every monotone seminorm ρ. In this section, we focus our attention on a fixed monotone seminorm ρ in an arbitrary Riesz space L, and we obtain necessary and sufficient conditions for ρ as well as ρ_L to be σ-Fatou, in terms of the order topology on L.

A monotone seminorm ρ on a Riesz space L is called a Riesz seminorm if $\rho(f) < \infty$ for all $f \in L$. A linear subspace S of L is called an ideal in L whenever S is solid (i.e., whenever it follows from $f \in S$, $g \in L$ and $|g| \le |f|$ that $g \in S$). If ρ is a monotone seminorm on a Riesz space L, then $L^\rho = \{f \in L : \rho(f) < \infty\}$ is an ideal in L and the restriction of ρ on L^ρ is a Riesz seminorm.

Theorem 26. Let ρ be a Riesz seminorm on a Riesz space L. Then $f_n \to f\,(ru)$ in L implies that $\rho(f_n - f) \to 0$, and so in particular $\rho(f_n) \to \rho(f)$.

Proof. Let $f_n \to f\,(ru)$ in L; then there is $u \in L^+$ and a real sequence $\varepsilon_n \downarrow 0$ such that $|f_n - f| \le \varepsilon_n u$ for all n. It follows that $\rho(f_n - f) \to 0$.

Since $\rho(f) - \rho(f_n - f) \le \rho(f_n) \le \rho(f) + \rho(f_n - f)$, so $\rho(f_n - f) \to 0$ implies $\rho(f_n) \to \rho(f)$.

Theorem 27. Let ρ be a Riesz seminorm on a Riesz space L. Then $\rho(u_n) \downarrow 0$ whenever $u_n \downarrow 0$ in L, if and only if $\rho(f_n - f) \to 0$ whenever $f_n \to f$ in L.

Proof. Assume that $\rho(u_n) \downarrow 0$ whenever $u_n \downarrow 0$ in L. Let $f_n \to f$. Then there is $u_n \downarrow 0$ such that $|f_n - f| \le u_n$ for all n and hence $\rho(f_n - f) \to 0$.

Conversely, assume that $\rho(f_n - f) \to 0$ whenever $f_n \to f$ in L. Let $u_n \downarrow 0$. Then $u_n \to 0$, and so by assumption $\rho(u_n) \downarrow 0$.

Theorem 28. A monotone seminorm ρ on a Riesz space L is σ-Fatou if and only if ρ is lower semi-continuous with respect to order convergence. (ρ is lower semi-continuous with respect to order convergence means that $\rho(f) \le \liminf \rho(f_n)$ whenever $f_n \to f$ in L.)

Proof. Assume that ρ is σ-Fatou. Let $f_n \to f$ in L. Then there exists a sequence $0 \le u_n \uparrow |f|$ such that $u_n \le |f_n|$ for all n. Hence, by the σ-Fatou property of ρ, $\rho(f) \le \liminf \rho(f_n)$.

Conversely, assume that $\rho(f) \le \liminf \rho(f_n)$ whenever $f_n \to f$ in L. Let $0 \le u_n \uparrow u$ in L. Then $\rho(u) \le \liminf \rho(u_n) = \lim \rho(u_n)$ implies that $\rho(u_n) \uparrow \rho(u)$. Hence, ρ is σ-Fatou.

For every monotone seminorm ρ on a Riesz space L, we shall denote S_ρ by $S_\rho = \{f \in L: \rho(f) \le 1\}$. Then S_ρ is a convex solid subset of L. Moreover, ρ is a Riesz seminorm on L if and only if S_ρ is absorbent (i.e., for each $f \in L$ there is some real number $\lambda > 0$ such that $f \in \mu S_\rho$ for all $\mu \ge \lambda$).

Theorem 29. A monotone seminorm ρ on a Riesz space L is σ-Fatou if and only if S_ρ is order closed.

Proof. Assume that ρ is σ-Fatou. Let $f_n \to f$ and $f_n \in S_\rho$ for all n. There is $u_n \downarrow 0$ such that $|f_n - f| \le u_n$ for all n. Then

$(|f| - u_n)^+ \leq |f_n| \in S_\rho$ and $(|f| - u_n)^+ \uparrow |f|$. Hence, $f \in S_\rho$.

Conversely, assume that S_ρ is order closed. Let $0 \leq u_n \uparrow u$. Clearly, $\rho(u_n) \uparrow \leq \rho(u)$. Let λ be such that $\rho(u_n) < \lambda$ for all n. Then $\lambda^{-1}u_n \in S_\rho$ for all n and so $\lambda^{-1}u \in S_\rho$, i.e., $\rho(u) \leq \lambda$. It follows then $\rho(u_n) \uparrow \rho(u)$ and hence ρ is σ-Fatou.

To each convex solid subset S of a Riesz space L corresponds a monotone seminorm ψ_S in L defined by:

$$\psi_S(f) = \inf\{\lambda > 0 : f \in \lambda S\} \text{ for every } f \in L.$$

ψ_S is called <u>the Minkowski functional of</u> S.

<u>Theorem 30.</u> <u>If</u> ρ <u>is a monotone seminorm on a Riesz space</u> L, <u>then</u> ρ_L <u>is the Minkowski functional of the pseudo order closure of</u> S_ρ.

<u>Proof.</u> We shall first show that $\psi_{S'_\rho} \leq \rho_L$. To this end, let $u \in L^+$ and $\rho_L(u) < \alpha < \infty$. There is $0 \leq u_n \uparrow u$ such that $\rho(u_n) < \alpha$ for all n. Then $\alpha^{-1}u_n \in S_\rho$ for all n and so $\alpha^{-1}u \in S'_\rho$. Hence, $\psi_{S'_\rho}(u) \leq \alpha$.

We shall next show that $\rho_L \leq \psi_{S'_\rho}$. Let $u \in L^+$ and $\psi_{S'_\rho} < \beta < \infty$. Then $u \in \beta S'_\rho$. There is $0 \leq u_n \uparrow u$ such that $u_n \in \beta S_\rho$ for all n. It follows that $\rho_L(u) \leq \beta$.

<u>Theorem 31.</u> <u>Let</u> ρ <u>be a monotone seminorm on a Riesz space</u> L. <u>Then the largest monotone</u> σ-<u>Fatou seminorm</u> ρ_M <u>dominated by</u> ρ <u>is the Minkowski functional of the order closure of</u> S_ρ.

<u>Proof.</u> Let ψ be the Minkowski functional of the order closure of S_ρ. If $0 \leq u_n \uparrow u$ and $\psi(u_n) < \alpha < \infty$ for every n, then $\alpha^{-1}u_n \in \text{cl}(S_\rho)$ for all n, and so $\alpha^{-1}u \in \text{cl}(S_\rho)$, i.e., $\psi(u) \leq \alpha$. This

shows that ψ is σ-Fatou.

Let η be a monotone σ-Fatou seminorm such that $\eta \leq \rho$. Then $S_\rho \leq S_\eta$ and so $cl(S_\rho) \subseteq cl(S_\eta) = S_\eta$. If $\psi(f) < \alpha < \infty$, then $\alpha^{-1} f \in cl(S_\rho) \subseteq S_\eta$, and hence $\eta(f) \leq \alpha$; so $\eta \leq \psi$. It follows then that $\psi = \rho_M$ and the proof of the theorem is complete.

Theorem 32. Let ρ be a monotone seminorm on a Riesz space L. Then $S'_\rho \subseteq S_{\rho_L} \subseteq (S'_\rho)'_{ru}$.

Proof. Let $f \in S'_\rho$. Then there exists $0 \leq u_n \in S_\rho$ such that $u_n \uparrow |f|$. Hence, $\rho_L(f) \leq 1$. This shows that $S'_\rho \subseteq S_{\rho_L}$.

Let $f \in S_{\rho_L}$. If $g_n = (1-n^{-1})f$, then $\rho_L(g_n) < 1$ for all n and $g_n \to f(ru)$. For every n, since $\rho_L(g_n) < 1$, there is $0 \leq u_{nk} \uparrow_k |g_n|$ such that $\rho(u_{nk}) < 1$ for all k. It follows that $g_n \in S'_\rho$ for all n and so $f \in (S'_\rho)'_{ru}$. This shows that $S_{\rho_L} \subseteq (S'_\rho)'_{ru}$.

Let ρ be a monotone seminorm on a Riesz space L. If S'_ρ is order closed, then by theorems 30 and 31, ρ_L is σ-Fatou. Whether the converse implication, i.e., ρ_L is σ-Fatou implies that S'_ρ is order closed, is true appears to be an open question. However, the following theorem holds.

Theorem 33. Let ρ be a monotone seminorm on a Riesz space L. Then ρ_L is σ-Fatou if and only if the ρ_L-closure of S'_ρ is order closed.

Proof. Since ρ_L is σ-Fatou if and only if S_{ρ_L} is order closed, it is sufficient to show that the ρ_L-closure of S'_ρ is S_{ρ_L}. We denote the ρ_L-closure of S'_ρ by $\overline{S'_\rho}^{\rho_L}$.

Since S_{ρ_L} is ρ_L-closed and $S'_\rho \subseteq S_{\rho_L}$, so $\overline{S'_\rho}^{\rho_L} \subseteq S_{\rho_L}$. For the reverse inclusion, let $f \in S_{\rho_L}$. Then $\rho_L(f) \leq 1$ and so $g_n = (1-n^{-1})f \in S'_\rho$. Hence, $\rho_L(g_n - f) \to 0$, and thus $f \in \overline{S'_\rho}^{\rho_L}$. This completes the proof of the theorem.

Theorem 34. If L is Archimedean and ρ_L is σ-Fatou, then the ρ_L-closure of S'_ρ is S''_ρ.

Proof. From the proof of the above theorem, ρ_L-closure of S'_ρ is S_{ρ_L}. Under the assumption, it is sufficient to show that $S_{\rho_L} = S''_\rho$.

We have, by theorem 32 and that L is Archimedean, $S_{\rho_L} \subseteq (S'_\rho)'_{ru} \subseteq S''_\rho$. On the other hand, ρ_L is σ-Fatou implies that S_{ρ_L} is order closed, and so that $S'_\rho \subseteq S_{\rho_L}$ implies that $S''_\rho \subseteq S_{\rho_L}$. Therefore, $S_{\rho_L} = S''_\rho$.

Theorem 35. Let S be a convex solid subset of a Riesz space L. Then $(\psi_S)_L = \psi_{S'}$.

Proof. Similar to the proof of theorem 30.

Theorem 36. Let S, T be convex solid absorbent subsets of a Riesz space L. Then $\psi_S = \psi_T$ if and only if $S'_{ru} = T'_{ru}$.

Proof. Assume that $\psi_S = \psi_T$. It is sufficient to show that $S'_{ru} \subseteq T'_{ru}$. To this end, let $0 \leq u \in S'_{ru}$. There is a sequence $0 \leq u_n \in S$ such that $u_n \to u(ru)$. Since ψ_S is a Riesz seminorm, it follows from theorem 26 that $\psi_S(u_n) \to \psi_S(u)$ and so $\psi_S(u) \leq 1$. Then, by assumption, $\psi_T(u) \leq 1$, and hence $u \in T'_{ru}$.

Assume that $S'_{ru} = T'_{ru}$. It is sufficient to show that $\psi_S \leq \psi_T$. Let $u \in L^+$. If α is such that $\psi_T(u) < \alpha$, then $\alpha^{-1}u \in T$ and so $\alpha^{-1}u \in S'_{ru}$. There is a sequence $0 \leq u_n \in S$ such that $u_n \to \alpha^{-1}u(ru)$.

Then, by theorem 26, $\psi_S(u_n) \to \psi_S(\alpha^{-1}u)$ and hence $\psi_S(u) \le \alpha$.

Theorem 37. Let ρ be a Riesz seminorm on a Riesz space L. Then ρ_L is σ-Fatou if and only if $(S'_\rho)'_{ru} = (S''_\rho)'_{ru}$.

Proof. It follows from theorem 30 that $\rho_L = \psi_{S'_\rho}$. Hence, $\rho_L = \rho_{LL}$ if and only if $\psi_{S'_\rho} = (\psi_{S'_\rho})_L$, or $\psi_{S'_\rho} = \psi_{S''_\rho}$. Then by theorem 36, ρ_L is σ-Fatou if and only if $(S'_\rho)'_{ru} = (S''_\rho)'_{ru}$.

For every subset S of a Riesz space L, we define S^n inductively by

$$S^1 = S',$$
$$S^n = (S^{n-1})'.$$

For every monotone seminorm ρ on L, we define ρ_{L^n} inductively by:

$$\rho_{L^1} = \rho_L$$
$$\rho_{L^n} = (\rho_{L^{n-1}})_L.$$

The following theorem is a generalization of the result of theorem 37.

Theorem 38. Let ρ be a Riesz seminorm on a Riesz space L. Then, for every $n = 1, 2, \ldots$, ρ_{L^n} is σ-Fatou if and only if $(S^n_\rho)'_{ru} = (S^{n+1}_\rho)'_{ru}$.

Furthermore, if L is Archimedean, then ρ_{L^n} is σ-Fatou implies that S^{n+1}_ρ is relatively uniformly closed.

Proof. Since $\rho_L = \psi_{S'_\rho}$, and $(\psi_S)_L = \psi_{S'}$ for every convex solid subset S, so $\rho_{L^n} = \psi_{S^n_\rho}$ for every $n = 1, 2, \ldots$. It follows then from theorem 36 that ρ_{L^n} is σ-Fatou if and only if $(S^n_\rho)'_{ru} = (S^{n+1}_\rho)'_{ru}$.

If L is Archimedean, then $(S^n_\rho)'_{ru} \subseteq S^{n+1}_\rho \subseteq (S^{n+1}_\rho)'_{ru}$; and so the σ-Fatou property of ρ_{L^n} implies that $S_\rho^{n+1} = (S_\rho^{n+1})'_{ru}$, i.e., S_ρ^{n+1} is relatively uniformly closed.

In conclusion, we observe that theorem 33 may be used to state the result of J. A. R. Holbrook (theorem 21) in the following form: a Riesz space L has the almost Egoroff property if and only *if, for every solid convex subset S of L, the* $\psi_{S'}$*-closure of* S' *is* order closed.

V. THE EGOROFF PROPERTY IN BOOLEAN ALGEBRAS

In this section we shall discuss a characterization (which is similar to the one for Riesz spaces proved in Section III) of those Boolean algebras having the Egoroff property. This result is also generalized to deal with directed systems instead of sequences. We shall also introduce order topology on a Boolean algebra.

Definition. An element a of a Boolean algebra B is said to have the Egoroff property, whenever $a_{nk} \uparrow_k a$ for every n in B, there exists a sequence $b_m \uparrow a$ in B and for every m a sequence $k(m, n)$ of indices such that $b_m \leq a_{nk(m,n)}$ for all m and n.

A Boolean algebra is said to have the Egoroff property if every one of its elements has the Egoroff property (see [8]).

If an element a of a Boolean algebra B has the Egoroff property, then every element b satisfying $b \leq a$ has the Egoroff property. Hence, a Boolean algebra B has the Egoroff property if and only if its unit element 1 has the Egoroff property.

The complement of an element a of a Boolean algebra B will be denoted by a'.

Theorem 39. Let B be a Boolean algebra and $a \in B$. Then the following statements are equivalent.

(1) a has the Egoroff property.

(2) If $a_{nk} \uparrow_k a$ in B, then there is a sequence $b_n \uparrow a$ in B and for every n an appropriate $k(n)$ such that $b_n \leq a_{nk(n)}$ for all n.

(3) If $a_{nk} \uparrow_k a_n \uparrow a$ in B, then there is a sequence $b_n \uparrow a$

in B *and for every* n *an appropriate* k(n) *such that* $b_n \le a_{nk(n)}$ *for all* n.

(4) *If* $a_{nk} \uparrow_k a_n \uparrow a$ *in* B, *then there is a sequence* $b_m \uparrow a$ *in* B *and for every* m *an appropriate* n(m), k(m) *such that* $b_m \le a_{n(m)k(m)}$ *for all* m.

Proof. It is clear that (1) ⇒ (2).

(2) ⇒ (3): Let $a_{nk} \uparrow_k a_n \uparrow a$ in B. Then, for every n, $a_{nk} \vee (a \wedge a_n') \uparrow_k a$, and hence by (2) there is a sequence $c_n \uparrow a$ such that, for every n, $c_n \le a_{nk(n)} \vee (a \wedge a_n')$ for some $k = k(n)$. Set $b_n = c_n \wedge a_n$, then $b_n \uparrow a$ and $b_n \le a_{nk(n)}$.

It is also clear that (3) ⇒ (4).

(4) ⇒ (1): Let $a_{nk} \uparrow_k a$ for every n. We may assume that $a_{nk} \downarrow_n$ for every k (since we can always replace a_{nk} by $a_{1k} \wedge \ldots \wedge a_{nk}$). We may also assume that there exists a sequence $a_n \uparrow a$ in B such that $a_n \neq a$ for all n. Now $a_{nk} \wedge a_n \uparrow_k a_n \uparrow a$. By (4), there is a sequence $b_n \uparrow a$ such that, for every m, $b_m \le a_{n(m)k(m)} \wedge a_{n(m)}$ for some $n = n(m)$, $k = k(m)$. We shall show that for a given pair of natural numbers M, N, there is some $k = k(M, N)$ such that $b_M \le A_{Nk(M,N)}$.

First, we claim that $n(m) \to \infty$ as $m \to \infty$. To see this, observe that if, for some p, $n(m) = p$ for infinitely many m, then for each m, there exists $m' \ge m$ such that $n(m') = p$; but then, for each b_m,

$$b_m \le b_{m'} \le a_{n(m')k(m')} \wedge a_{n(m')} \le a_p$$

(recall that a_p is strictly less than a), so that $b_m \not\uparrow a$, a contradiction.

Now let M, N be given. From the above argument there exists $m'' \geq M$ such that $n(m'') \geq N$. From this, together with the fact that $a_{nk} \downarrow_n$, it follows that

$$b_M \leq b_{m''} \leq a_{n(m'')k(m'')} \wedge a_{n(m'')} \leq a_{n(m'')k(m'')}$$
$$\leq a_{Nk(M,N)} \quad \text{where } k(M,N) = k(m'') .$$

This completes the proof of the theorem.

From the above theorem and duality, we have the following theorems 40, 41.

Theorem 40. *Let* B *be a Boolean algebra and* $a \subset B$. *Then the following statements are equivalent.*

(0) a *has the Egoroff property.*

(1) *If* $a_{nk} \downarrow_k a'$ *in* B, *then there is a sequence* $b_m \downarrow a'$ *in* B *and for every* m *a sequence* $k(m,n)$ *of indices such that* $b_m \geq a_{nk(m,n)}$ *for all* m *and* n.

(2) *If* $a_{nk} \downarrow_k a'$ *in* B, *then there is a sequence* $b_n \downarrow a'$ *in* B *and for every* n *an appropriate* $k(n)$ *such that* $b_n \geq a_{nk(n)}$ *for all* n.

(3) *If* $a_{nk} \downarrow_k a_n \downarrow a'$ *in* B, *then there is a sequence* $b_n \downarrow a'$ *in* B *and for every* n *an appropriate* $k(n)$ *such that* $b_n \geq a_{nk(n)}$ *for all* n.

(4) *If* $a_{nk} \downarrow_k a_n \downarrow a'$ *in* B, *then there is a sequence* $b_m \downarrow a'$ *in* B *and for every* m *an appropriate* $n(m)$, $k(m)$ *such that* $b_m \geq a_{n(m)k(m)}$ *for all* m.

Theorem 41. *Let* B *be a Boolean algebra. Then the following statements are all equivalent.*

(0) B _has the Egoroff property._

(1) _If_ $a_{nk} \uparrow_k 1$ _in_ B, _then there is a sequence_ $b_m \uparrow 1$ _in_ B _and for every_ m _a sequence_ k(m, n) _of indices such that_ $b_m \leq a_{nk(m,n)}$ _for all_ m _and_ n.

(1') _If_ $a_{nk} \downarrow_k 0$ _in_ B, _then there is a sequence_ $b_m \downarrow 0$ _in_ B _and for every_ m _a sequence_ k(m, n) _of indices such that_ $b_m \geq a_{nk(m,n)}$ _for all_ m _and_ n.

(2) _If_ $a_{nk} \uparrow_k 1$ _in_ B, _then there is a sequence_ $b_n \uparrow 1$ _in_ B _and for every_ n _an appropriate_ k(n) _such that_ $b_n \leq a_{nk(n)}$ _for all_ n.

(2') _If_ $a_{nk} \downarrow_k 0$ _in_ B, _then there is a sequence_ $b_n \downarrow 0$ _in_ B _and for every_ n _an appropriate_ k(n) _such that_ $b_n \geq a_{nk(n)}$ _for all_ n.

(3) _If_ $a_{nk} \uparrow_k a_n \uparrow 1$ _in_ B, _then there is a sequence_ $b_n \uparrow 1$ _in_ B _and for every_ n _an appropriate_ k(n) _such that_ $b_n \leq a_{nk(n)}$ _for all_ n.

(3') _If_ $a_{nk} \downarrow_k a_n \downarrow 0$ _in_ B, _then there is a sequence_ $b_n \downarrow 0$ _in_ B _and for every_ n _an appropriate_ k(n) _such that_ $b_n \geq a_{nk(n)}$ for _all_ n.

(4) _If_ $a_{nk} \uparrow_k a_n \uparrow 1$ _in_ B, _then there is a sequence_ $b_m \uparrow 1$ _in_ B _and for every_ m _an appropriate_ n(m), k(m) _such that_ $b_m \leq a_{n(m)k(m)}$ _for all_ m.

(4') _If_ $a_{nk} \downarrow_k a_n \downarrow 0$ _in_ B, _then there is a sequence_ $b_m \downarrow 0$ _in_ B _and for every_ m _an appropriate_ n(m), k(m) _such that_ $b_m \geq a_{n(m)k(m)}$ _for all_ m.

A real function ϕ on a Boolean algebra B is called a _finitely additive measure_ if ϕ satisfies the following conditions: (i) $0 \leq \phi(a) < \infty$ for all $a \in B$; (ii) $\phi(a \vee b) = \phi(a) + \phi(b)$, whenever

$a \wedge b = 0$; (iii) $\phi(1) \neq 0$. If ϕ is a finitely additive measure on B, then ϕ is monotone, i.e., $a \leq b$ implies $\phi(a) \leq \phi(b)$; $\phi(0) = 0$; $\phi(a) + \phi(b) = \phi(a \vee b) + \phi(a \wedge b)$ for all $a, b \in B$; $\phi(a_1 \vee \ldots \vee a_n) = \sum_1^n \phi(a_i)$, whenever the set $\{a_1, \ldots, a_n\}$ is disjointed in B.

A finitely additive measure ϕ on a Boolean algebra B is called a <u>countably additive measure</u> if, for every countable disjointed subset $\{a_1, a_2, \ldots\}$ of B such that $\bigvee_1^\infty a_n$ exists, $\phi(\bigvee_1^\infty a_n) = \sum_1^\infty \phi(a_n)$. It follows immediately that for every finitely additive measure ϕ the following conditions are mutually equivalent: (i) ϕ is countably additive; (ii) if $a_n \uparrow a$, then $\phi(a_n) \uparrow \phi(a)$; (iii) if $a_n \downarrow 0$, then $\phi(a_n) \downarrow 0$.

We say that a finitely additive measure ϕ is <u>purely finitely additive</u> if every countably additive measure ϕ' such that $0 \leq \phi' \leq \phi$ is identically zero. K. Yosida and E. Hewitt (see [13]) proved the important result that every finitely additive measure ϕ on a Boolean algebra can be uniquely written as the sum of a countably additive measure ϕ_c and a purely finitely additive measure ϕ_p. We shall call ϕ_c the countably additive part of ϕ and ϕ_p the purely finitely additive part of ϕ.

For every finitely additive measure ϕ on a Boolean algebra B we define a related function ϕ_L on B in the following way: for each $a \in B$, set

$$\phi_L(a) = \inf\{\lim_n \phi(a_n) : a_n \uparrow a\}.$$

It follows that ϕ_L is finitely additive; $\phi_L \leq \phi$; $\phi_L = \phi$ if and only if ϕ is countably additive; if $\psi \leq \phi$, then $\psi_L \leq \phi_L$.

A result due to M. A. Woodbury [12] and H. Bauer [1] states that, for all finitely additive measures ϕ on a Boolean algebra B, its countably additive part ϕ_c is equal to ϕ_L.

If ϕ is a given finitely additive measure on a Boolean algebra B and ψ is a countably additive measure dominated by ϕ (i. e., $\psi \leq \phi$), then $\phi_L \geq \psi_L = \psi$; so that $\phi_c = \phi_L$ is simply the statement that ϕ_L is countably additive. The fundamental result concerning finitely additive measures, then, may be expressed by saying that, for every finitely additive measure ϕ on a Boolean algebra B, $\phi_{LL} = \phi_L$.

We are interested in the condition under which $\phi_L = \phi_{LL}$ remains true for every ϕ in a larger class of monotone functions on the Boolean algebra B. A function $\rho : B \to [0, \infty]$ is called an <u>outer measure</u> on the Boolean algebra B if ρ is monotone (i. e., $a \leq b$ implies $\rho(a) \leq \rho(b)$), $\rho(0) = 0$ and ρ is countably subadditive, i. e., $a \leq \bigvee_1^\infty a_n$ implies $\rho(a) \leq \sum_1^\infty \rho(a_n)$. Given any function $\rho : B \to [0, \infty]$ which is monotone, we define ρ_L by:

$$\rho_L(a) = \inf \{ \lim_n \rho(a_n) : a_n \uparrow a \} .$$

The following theorem due to J. A. R. Holbrook represents a characterization of those Boolean algebras which are Egoroff; it is also an analogous to the result concerning seminorm on Riesz spaces discussed in Section III.

<u>Theorem 49.</u> <u>A Boolean algebra</u> B <u>has the Egoroff property if and only if, for every outer measure</u> ρ <u>on</u> B, $\rho_L = \rho_{LL}$. <u>In fact,</u>

(1) <u>if</u> B <u>has the Egoroff property, then</u> $\rho_L = \rho_{LL}$ <u>for every</u> monotone function $\rho : B \to [0, \infty]$,

(2) <u>if</u> $\rho_L = \rho_{LL}$ <u>for every finitely-valued outer measure</u> ρ <u>on</u> B, <u>then</u> B <u>has the Egoroff property</u>.

(The following proof is a modified version of J. A. R. Holbrook's proof (see [2]).)

<u>Proof.</u> (1) Clearly $\rho_{LL} \leq \rho_L$. On the other hand, suppose $\rho_{LL}(a) < \lambda$; in this case, there must exist $a_n \uparrow a$ and, for each n, $a_{nk} \uparrow_k a_n$ such that, for all n, k, $\rho(a_{nk}) < \lambda$. Since $a_{nk} \uparrow_k a_n \uparrow a$ in B and B has the Egoroff property, by theorem 26, there exists a sequence $b_m \uparrow a$ such that, for every m, $b_m \leq a_{n(m)k(m)}$ for some $n = n(m)$, $k = k(m)$. Thus, $\rho(b_m) \leq \rho(a_{n(m)k(m)}) < \lambda$ for all m, so that $\rho_L(a) \leq \lambda$. Since this holds for all $\lambda > \rho_{LL}(a)$, we therefore have $\rho_L(a) \leq \rho_{LL}(a)$.

(2) It is sufficient to show that the unit element 1 of B has the Egoroff property. Let $a_{nk} \uparrow a_n \uparrow 1$ in B. We shall find a sequence $b_m \uparrow 1$ in B such that, for every m, $b_m \leq a_{n(m)k(m)}$ for some $n = n(m)$, $k = k(m)$. We may assume that $a_n \neq 1$ for all n (otherwise, if there exists $a_N = 1$, then the sequence $b_m = a_{Nm}$ satisfies the condition).

Define a function ρ on B as follows: for each $c \in B$,

$$\begin{aligned} \rho(c) &= 0 \text{ if } c = 0, \\ &= \tfrac{1}{2} \text{ if } 0 < c \leq a_{nk} \text{ for some } a_{nk}, \\ &= 1 \text{ otherwise.} \end{aligned}$$

It is evident that ρ is a finite-valued outer measure on B. Moreover, $a_{nk} \uparrow_k a_n \uparrow 1$ and $\rho(a_{nk}) = \frac{1}{2}$ for all n, k imply that $\rho_{LL}(1) = \frac{1}{2}$. Hence, by the assumption, $\rho_L(1) = \frac{1}{2}$. This means that there exists a sequence $b_m \uparrow 1$ such that, for every m, $\rho(b_m) = \frac{1}{2}$, and so, for every m, $b_m \le a_{n(m)k(m)}$ for some $n = n(m)$, $k = k(m)$ as required. This completes the proof of the theorem.

Replacing increasing sequences by systems which are directed upward, we can get a result which is similar to the result of the last theorem. First, we define the generalized Egoroff property in Boolean algebras.

Definition. *An element* a *of a Boolean algebra* B *is said to have the generalized Egoroff property whenever* $a_{\varkappa_\tau} \uparrow_{\varkappa_\tau} a_\tau \uparrow a$ *in* B, *there is an indexed subset* $\{b_\mu\}$ *of* B *such that* $b_\mu \uparrow a$ *and, for every* μ, *there exists an appropriate* τ *(depending on* μ*) and a* $\varkappa_\tau = \varkappa_\tau(\mu)$ *such that* $b_\mu \le a_{\varkappa_\tau(\mu)}$.

A Boolean algebra is said to have the generalized Egoroff property if every one of its elements has the generalized Egoroff property.

If an element a of a Boolean algebra B has the generalized Egoroff property, then every element $b \in B$ satisfying $b \le a$ has the generalized Egoroff property. Hence, a Boolean algebra has the generalized Egoroff property if and only if its unit element has the generalized Egoroff property.

For every monotone function $\rho : B \to [0, \infty]$, we define the function $\rho_{\mathfrak{L}}$ on B by: for each element a in B,

$$\rho_{\mathfrak{L}}(a) = \sup \{ \inf_\tau \rho(a_\tau) : a_\tau \uparrow a \} .$$

It follows immediately that $\rho_{\mathfrak{L}}$ is monotone; $\rho_{\mathfrak{L}} \le \rho$ for every monotone function ρ; $\rho_{\mathfrak{L}} = \rho$ if and only if, for every $a \in B$, $a_\tau \uparrow a$ implies $\sup_\tau \rho(a_\tau) = \rho(a)$.

Theorem 43. A Boolean algebra B has the generalized Egoroff property if and only if, for every outer measure ρ on B, $\rho_{\mathcal{L}} = \rho_{\mathcal{L}\mathcal{L}}$. In fact,

(1) if B has the generalized Egoroff property, then $\rho_{\mathcal{L}} = \rho_{\mathcal{L}\mathcal{L}}$ for every monotone function $\rho : B \to [0, \infty]$,

(2) if $\rho_{\mathcal{L}} = \rho_{\mathcal{L}\mathcal{L}}$ for every finite-valued outer measure ρ on B, then B has the generalized Egoroff property.

Proof. (1) Clearly $\rho_{\mathcal{L}\mathcal{L}} \le \rho_{\mathcal{L}}$. On the other hand, suppose $\rho_{\mathcal{L}\mathcal{L}}(a) < \lambda$; in this case, there must exist $a_\tau \uparrow a$ and, for each τ, there exists $a_{\varkappa_\tau} \uparrow_{\varkappa_\tau} a_\tau$ such that for all τ and $\varkappa_\tau$, $\rho(a_{\varkappa_\tau}) < \lambda$. Since $a_{\varkappa_\tau} \uparrow_{\varkappa_\tau} a_\tau \uparrow a$ and the Boolean algebra B has the generalized Egoroff property, there exists $b_\mu \uparrow a$ such that, for every μ, there is some τ and $\varkappa_\tau = \varkappa_\tau(\mu)$ satisfying $b_\mu \le a_{\varkappa_\tau(\mu)}$. Thus, $\rho(b_\mu) \le \rho(a_{\varkappa_\tau(\mu)}) < \lambda$ for all μ, so that $\rho_{\mathcal{L}}(a) \le \lambda$. Since this holds for any $\lambda > \rho_{\mathcal{L}\mathcal{L}}(a)$, we therefore have $\rho_{\mathcal{L}}(a) \le \rho_{\mathcal{L}\mathcal{L}}(a)$.

(2) It is sufficient to show that the unit element of B has the generalized Egoroff property. Let $a_{\varkappa_\tau} \uparrow_{\varkappa_\tau} a_\tau \uparrow 1$ in B. We may assume that $a_\tau \ne 1$ for all τ. Define a function ρ on B as follows: for each $c \in B$,

$$\begin{aligned} \rho(c) &= 0 \text{ if } c = 0, \\ &= \tfrac{1}{2} \text{ if } 0 < c \le a_{\varkappa_\tau} \text{ for some } \tau \text{ and } \varkappa_\tau, \\ &= 1 \text{ otherwise.} \end{aligned}$$

It is evident that ρ is a finite-valued outer measure on B. Moreover, since $a_{\varkappa_\tau} \uparrow_{\varkappa_\tau} a_\tau \uparrow 1$ and $\rho(a_{\varkappa_\tau}) = \frac{1}{2}$ for all τ and $\varkappa_\tau$, we have $\rho_{\mathcal{L}\mathcal{L}}(1) = \frac{1}{2}$. By the assumption, $\rho_{\mathcal{L}}(1) = \frac{1}{2}$. This means that there exists a directed system $b_\mu \uparrow 1$ such that for every μ, $\rho(b_\mu) = \frac{1}{2}$; thus, for every μ, there is some τ and $\varkappa_\tau = \varkappa_\tau(\mu)$ such that $b_\mu \le a_{\varkappa_\tau(\mu)}$. Therefore, the unit element of B has the generalized Egoroff property and the proof of the theorem is complete.

We shall next discuss order convergence of sequences and order topology in a Boolean algebra.

For every pair of elements a, b of a Boolean algebra B, we shall write $a \Delta b = (a' \wedge b) \vee (a \wedge b')$. Then the following inequality holds:

$$a \Delta b \le (a \Delta c) \vee (c \Delta b) . \qquad (A)$$

Furthermore, $a \Delta b = 0$ if an only if $a = b$.

Definition. A sequence $(a_n : n = 1, 2, \dots)$ in a Boolean algebra B is order convergent to an element $a \in B$ whenever there exists a sequence $b_n \downarrow 0$ in B such that $a_n \Delta a \le b_n$ for all n. This will be denoted by $a_n \to_n a$ or simply $a_n \to a$.

It follows from the inequality (A) that the limit of an order convergent sequence is unique. Moreover, it can be easily shown that $a_n \uparrow a$ or $a_n \downarrow a$ implies that $a_n \to a$.

A subset S of a Boolean algebra B is called order closed if, for every order convergent sequence in S, the order limit of the sequence is also a member of S. As for Riesz spaces, the order closed sets of a Boolean algebra B are exactly the closed sets of a certain topology in B, the order topology.

For any subset S of a Boolean algebra B, the pseudo order closure S' of S is the set of all $a \in B$ such that there exists a sequence in S converging in order to a. We shall denote the order closure of S, i.e., the closure of S in the order topology, by cl(S). Then we have $S \subseteq S' \subseteq (S')' \subseteq \ldots \subseteq cl(S)$; S is order closed if and only if $S = S'$; $S' = cl(S)$ if and only if $S' = S''$.

Theorem 44. Let B be a Boolean algebra. Then the following statements are equivalent.

(1) B has the Egoroff property.

(2) If $a_{nk} \to_k a$ in B, then for every n there is some k(n) such that $a_{nk(n)} \to a$.

(3) If $a_{nk} \to_k a_n \to a$ in B, then for every n there is some k(n) such that $a_{nk(n)} \to a$.

(4) If $a_{nk} \to_k a_n \to a$ in B, then for every m there is some k(m), n(m) such that $a_{n(m)k(m)} \to a$.

Proof. (1) ⇒ (2): Let $a_{nk} \to_k a$ in B. Then for every n there is $b_{nk} \downarrow_k 0$ such that $a_{nk} \Delta a \leq b_{nk}$ for all k. By the Egoroff property of B, there is $b_n \downarrow 0$ and for every n an appropriate k(n) such that $b_{nk(n)} \leq b_n$ for all n. Hence, $a_{nk(n)} \to a$.

(2) ⇒ (3): Let $a_{nk} \to_k a_n \to a$ in B. Then $a_{nk} \Delta a_n \to_k 0$ for all n. By (2), there is for every n an appropriate k(n) such that $a_{nk(n)} \Delta a_n \to 0$. It follows from $a_{nk(n)} \Delta a \leq (a_{nk(n)} \Delta a_n) \vee (a_n \Delta a)$ that $a_{nk(n)} \to a$.

It is clear that (3) ⇒ (4).

(4) ⇒ (1): Let $a_{nk} \downarrow_k a_n \downarrow 0$ in B. Then $a_{nk} \to_k a_n \to 0$. Hence, by (4) there is for every m an appropriate n(m), k(m) such that $a_{n(m)k(m)} \to 0$, and so there exists $b_m \downarrow 0$ such that $a_{n(m)k(m)} \leq b_m$ for all m. Therefore, B has the Egoroff property.

An immediate consequence of the last theorem is:

Theorem 45. Let B be a Boolean algebra. Then B has the Egoroff property if and only if $S' = (S')'$ for every subset S of B.

The following theorem is similar to theorem 29 of Section IV.

Theorem 46. Let ρ be an outer measure on a Boolean algebra B. Then $\rho = \rho_L$ if and only if, for every real $\alpha > 0$, $S_\alpha = \{a \in B : \rho(a) \leq \alpha\}$ is order closed.

Proof. Assume that $\rho = \rho_L$. Let $\alpha > 0$. If $a_n \in S_\alpha$ and $a_n \to a$, then there exists $b_n \downarrow 0$ such that $a_n \Delta a \leq b_n$ for all n. The sequence $c_n = a \wedge b_n'$ satisfies $c_n \uparrow a$ and $c_n \leq a_n$ for all n. Thus, by the assumption $\rho = \rho_L$, we have $\rho(c_n) \uparrow \rho(a)$ and so $a \in S_\alpha$. This proves that S_α is order closed.

Conversely, assume that for every $\alpha > 0$, S_α is order closed. Let $a_n \uparrow a$. If a real number α is such that $\rho(a_n) < \alpha$ for all n, then $a_n \in S_\alpha$ for all n and so by assumption $a \in S_\alpha$, i.e., $\rho(a) \leq \alpha$. This proves that $\rho = \rho_L$.

Theorem 47. Let ρ be an outer measure on a Boolean algebra B. For every $\alpha > 0$, define

$$S_\alpha^{\rho_L} = \{a \in B : \rho_L(a) \leq \alpha\}$$
$$S_\alpha = \{a \in B : \rho(a) \leq \alpha\}.$$

Then $S_\alpha^{\rho_L} = \bigcap_{\varepsilon > 0} S'_{\alpha+\varepsilon}$.

Proof. Let $a \in S_\alpha^{\rho_L}$. Then, for every $\varepsilon > 0$, $\rho_L(a) < \alpha+\varepsilon$ implies the existence of $a_n^\varepsilon \uparrow_n a$ such that $\rho(a_n^\varepsilon) < \alpha+\varepsilon$ for all n. Hence, $a \in S'_{\alpha+\varepsilon}$ for all $\varepsilon > 0$ and so $a \in \bigcap_{\varepsilon > 0} S'_{\alpha+\varepsilon}$.

For the other inclusion, let $a \in \bigcap_{\varepsilon > 0} S'_{\alpha+\varepsilon}$. For every $\varepsilon > 0$,

$a \in S'_{\alpha+\varepsilon}$ implies the existence of $a_n^\varepsilon \uparrow_n a$ such that $a_n^\varepsilon \in S_{\alpha+\varepsilon}$ for all n. Then $\rho_L(a) \leq \alpha+\varepsilon$ for all $\varepsilon > 0$ and so $\rho_L(a) \leq \alpha$, i.e., $a \in S_\alpha^{\rho_L}$.

From the last two theorems we have:

Theorem 48. Let ρ be an outer measure on a Boolean algebra B. Then $\rho_L = \rho_{LL}$ if and only if, for every $\alpha > 0$, the set $\bigcap_{\varepsilon>0} S'_{\alpha+\varepsilon}$ is order closed.

REFERENCES

[1] Bauer, H., Darstellung additiver Funktionen auf Booleschen Algebren als Mengenfunktionen, Archiv der Mathematik, v. 6 (1955), pp. 215-222.

[2] Holbrook, J. A. R., The Egoroff property and related properties in the theory of Riesz spaces, Thesis, California Institute of Technology (1965).

[3] Holbrook, J. A. R., Seminorms and the Egoroff property in Riesz spaces, Trans. Amer. Math. Soc., v. 132 (1968), pp. 67-77.

[4] Kantorovitch, L., Lineare halbgeordnete Räume, Trans. Moscow Math. Soc., v. 44 (1937), pp. 121-165.

[5] Luxemburg, W. A. J. and L. C. Moore, Jr., Archimedean quotient Riesz spaces, Duke Math. J., v. 34 (1967), pp. 725-739.

[6] Luxemburg, W. A. J. and A. C. Zaanen, Notes on Banach Function Spaces, Proc. Acad. Sci., Amsterdam; Note I, A66, pp. 135-147 (1963); Note II, A66, pp. 148-153 (1963); Note III, A66, pp. 239-250 (1963); Note IV, A66, pp. 251-263 (1963); Note V, A66, pp. 496-504 (1963); Note VI, A66, pp. 655-668 (1963); Note VII, A66, pp. 669-681 (1963); Note VIII, A67, pp. 104-119 (1964); Note IX, A67, pp. 360-376 (1964); Note X, A67, pp. 493-506 (1964); Note XI, A67, pp. 507-518 (1964); Note XII, A67, pp. 519-529 (1964); Note XIII, A67, pp. 530-543 (1964).

[7] Luxemburg, W. A. J., Notes on Banach Function Spaces, Proc. Acad. Sci., Amsterdam; Note XIV, A68, pp. 229-248 (1965); Note XV, A68, pp. 415-446 (1965); Note XVI, A68, pp. 646-667 (1965).

[8] Luxemburg, W. A. J., On Finitely Additive Measures in Boolean Algebras, J. Reine Angew. Math., v. 213 (1964), pp. 165-173.

[9] Luxemburg, W. A. J. and A. C. Zaanen, Riesz spaces (linear vector lattices), Part 1, preprint.

[10] Nakano, H., Über ein lineares Funktional auf dem teilweise geordneten Modul, Proc. of the Imperial Academy (Tokyo), v. XVIII, no. 9 (1942), pp. 548-552.

[11] Orihara, M., On the regular vector lattice, Proc. of the Imperial Academy (Tokyo), v. XVIII, no. 9 (1942), pp. 525-529.

[12] Woodbury, M. A., A decomposition theorem for finitely additive set functions (Preliminary Report), Bull. Amer. Math. Soc., v. 56 (1950), pp. 171-162.

[13] Yosida, K. and E. Hewitt, Finitely additive measures, Trans. Amer. Math. Soc., v. 72 (1952), pp. 46-66.