THE WING-BODY PROBLEM FOR LINEARIZED SUPERSONIC FLOW

Thesis by

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ABSTRACT

This work is concerned with one of the important problems remaining in the theory of linearized supersonic flow: the study of non-planar systems dealing with configurations which cannot be completely solved with the existing theory - in particular, the study of interaction or interference between fuselage and lifting or control surfaces in supersonic flow.

In Sections 1.2 and 1.3 the non-planar problems are classified and the problem considered to be the fundamental wing-body problem for linearized supersonic flow is presented. In Part II, this and related problems are formulated in a manner suitable for Laplace transform methods and subsequently the transformed solutions are presented in a general form by the Green's function method.

Due to the inherent difficulties arising in non-planar problems, related planar problems are solved in Part III. In Part IV, the fundamental wing-body problem is discussed in detail and in the light of the results of Part III an approximate solution (in terms of the pressure) in the region of greatest interest is presented; and a quantitative estimate of the increase in lift due to the interaction between wing and body is indicated.

CONTENTS

I. INTRODUCTION

1.1 Preliminary Remarks

The theory of linearized supersonic flow has received a considerable amount of attention during the last decade or so and an adequate framework of knowledge has been developed to handle many problems of supersonic flight. Many of the important concepts in supersonic flow and methods used in solving these problems have been presented in the most recent of von Karman's reviews given at the Tenth Wright Brothers Lecture (Ref. 1). A systematic and fairly comprehensive presentation of the theory of linearized supersonic flow has been given by W. D. Hayes (Ref. 2), "with special emphasis placed upon the study of planar systems". Both of these papers include extensive bibliographies. The present work is concerned with one of the important problems remaining in the theory of linearized supersonic flow: the study of non-planar systems dealing with configurations which cannot be completely solved with the existing theory. The stationary (steady-state) problem only will be given here.

Due to the complexity of general non-planar problems very few works on the subject have been presented with the exception of problems dealing with configurations having axial symmetry, or conical properties (Ref. 3). The development of the study of planar systems and of some of these special non-planar configurations is given in Hayes' paper. Specific reference to some of these and other works will be made whenever previous concepts and results are needed in the development of the present study. Before mentioning the investigations of the few workers on more general non-planar systems, a brief survey of existing problems and a general classification of them will be made.

1.2 Non-planar Problems

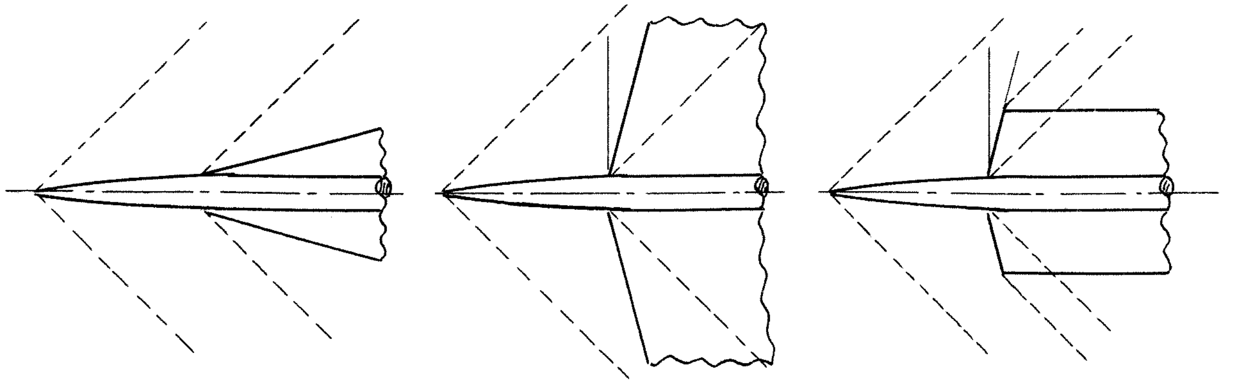
It is emphasized that only those problems which satisfy the conditions imposed by the linearization of the flow equations are considered here. Fortunately, it appears at the present time that a great many of these problems may be handled (at least formulated) under these rather stringent restrictions. However, other seemingly unavoidable difficulties arise, the main one being that there is a great unlikelihood that even ordinary non-planar problems, in the practical sense, can be solved with the generality that planar systems can be handled. Yet, some interesting and useful qualitative and a few quantitative results of wide generality have been found. Reference to these will be made later.

Many immediately important non-planar problems fall in the category of interaction or interference between fuselage and lifting or control surfaces, i.e. wing, tail, control fin, etc. The simplest configuration of this type will be a combination of an axially-symmetric fuselage or body and near-planar appendages located in one plane passing (within the linearized theory assumptions) through the axis of the body. Clearly a formulation of the complete problem for arbitrary planforms situated at arbitrary positions along the body is difficult. Consider then a single wing plus body combination. The study of this "simplified" non-planar system may be classified as follows, depending essentially upon the Mach number and planform of the wing

Class A: Body plus slender, pointed wing (Fig. 1.1a)

Class B: Body plus wide wing (Fig. 1.1b)

Class C: Body plus narrow wing (Fig. 1.1c)



a) Class A b) Class B c) Class C

Note: 1) Free-stream direction from left to right

2) Dotted lines represent Mach cone elements

Figure 1.1 Classification of Wing-Body Problems

This classification is given in the order of increasing difficulty of analysis. Many problems in Class A may be handled by the approximate method of Munk, assuming a slow variation of the variables in the flow direction. Spreiter (Ref. 4) has applied this method to slender wing-body configurations. This is essentially a subsonic analysis which is valid for slender, pointed configurations in both subsonic and supersonic flow. The present study is mainly concerned with problems in Class B. The first attempt to study such problems appears to have been made by Ferrari (Ref. 5). This method is essentially an iteration procedure: The body solution and wing solution are superimposed on each other. First, a correction body potential is introduced along the body axis to cancel the wing potential (uncorrected for body interaction) sufficiently to satisfy the body boundary conditions. Second, a correction wing potential is introduced in the plane of the wing to cancel the body potential (uncorrected for wing interaction) sufficiently to satisfy the wing boundary conditions. This completes the first iteration. Theoretically, this procedure may be continued using the corrected potentials after each iteration and if the process is convergent, a solution results. However, there appear to be several inherent difficulties: 1) the boundary value problem for the correction wing potential cannot be uniquely formulated,* 2) convergence of the iteration procedure, and 3) the rapidity of convergence. Hence, any results based on the first few iterations are questionable. The difficult study of Class C problems rests upon the accumulated efforts of the previous analysis.

A further classification of each of the above classes may be made roughly as follows:

1) Supersonic leading edge problems, i.e. wing leading edge-body junction, body nose → wing**,
2) Afterbody problems, i.e. wing trailing edge-body junction, wing tip → body, and wing → body → afterfin.

This is necessitated by the fact that additional physical conditions, e.g. Kutta condition, are usually needed to formulate the latter problems. Other considerations which must be made are, for example, the wing position relative to the body nose and fin position relative to other fins. The present indications are that in many cases only a qualitative analysis is feasible. Recent discussions with Lagerstrom and Van Dyke (Ref. 6) apparently verify this.

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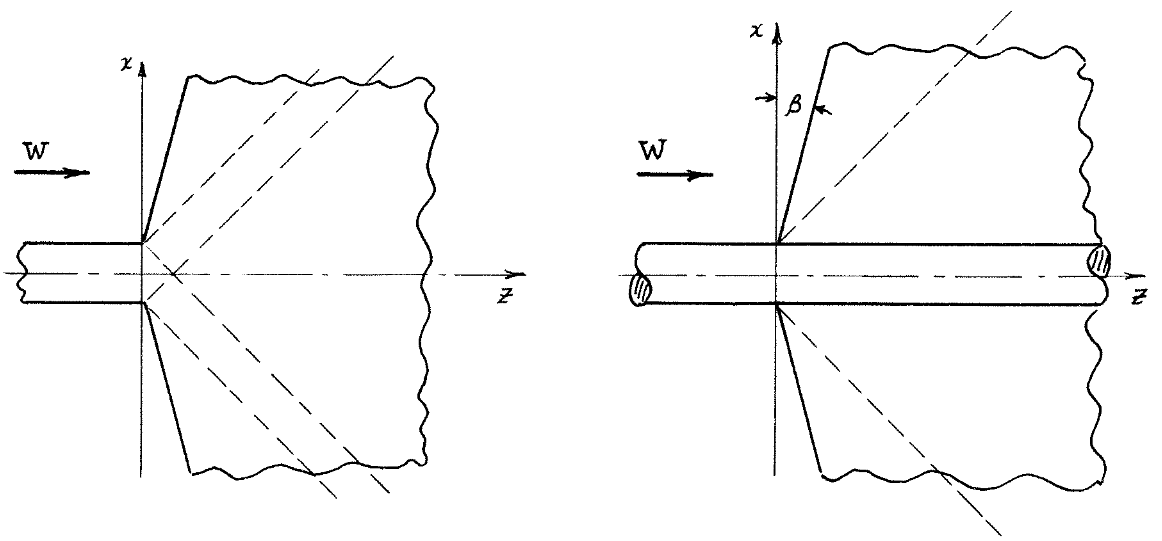
*This was pointed out by De Prima
** The arrow indicates the direction of influence

1.3 Remarks on the Present Study

As mentioned in the previous section, 1.2, the slender configuration problems of Class A can be handled by subsonic analysis, that is, conditions in planes perpendicular to the flow direction are assumed not to influence each other and the analysis reduces to the classical potential (Laplace) problem of hydrodynamics in two-dimensions.

The study of problems of Class B affords no such simplification although in certain regions of the flow some of the methods and results of the study of planar systems are useful. As a general rule, it will be found advisable to investigate completely planar systems initially, that is, configurations in which the axially-symmetric body is replaced by a planar one of the same planform. The principal leading edge problems are shown in Figures 1.2 and 1.3.

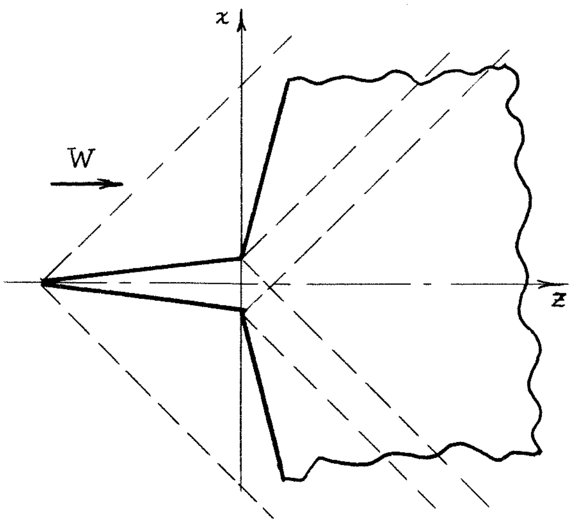


a) Case Ia, Planar b) Case IIa

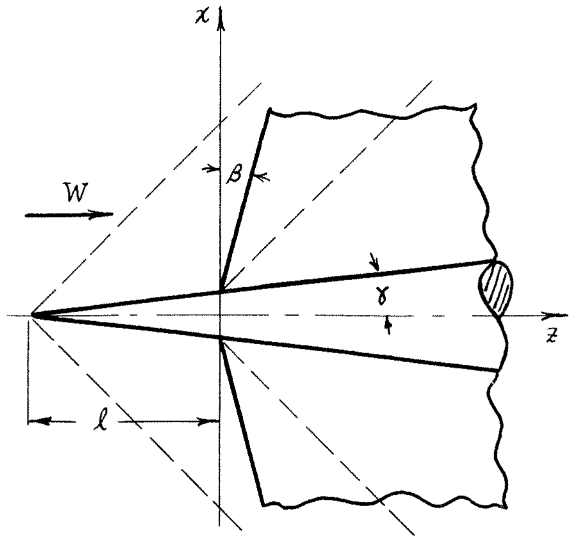
Note: W is the free-stream velocity

Figure 1.2 Leading Edge Problems:

Wing Leading Edge-Body Junction



a) Case Ib, planar



b) Case IIb

Figure 1.3 Leading Edge Problems:

Body Nose → Wing

Cases Ia and Ib are the planar systems corresponding to Cases IIa and IIb, respectively. Case IIa is a combination of an infinitely long circular cylinder and semi-infinite wing. This represents the physical problem in which the influence of the body nose may be neglected, i.e. the wing is sufficiently far downstream from the nose. In Case IIb the circular cylinder is replaced by a cone (or a more general pointed nose). Case IIa is clearly a limiting case of Case IIb as the distance, ℓ, from the leading edge to the nose becomes large and the half-angle, γ, of the cone becomes small. The study of <u>Case IIa</u> is considered <u>the fundamental wing-body problem for linearized supersonic flow</u>. Cases Ib and IIb will be studied here only in a qualitative way and some of the analytical difficulties pointed out.

II. FORMULATION OF THE PROBLEM

2.1 Linearized Supersonic Flow Theory

The basic assumptions of stationary linearized supersonic flow are well known and lead to the differential equation for the perturbation potential, φ,

$$\varphi_{xx} + \varphi_{yy} - (M^2-1)\,\varphi_{zz} = 0 \tag{2.1}$$

where M is the Mach number of the free stream velocity, W, nominally taken to be in the positive z direction. The perturbation velocity is given by

$$(u,\ v,\ w) = \text{grad}\ \varphi \tag{2.2a}$$

where u, v, and w are the velocity components in the x, y and z directions, respectively; the condition for isentropic, irrotational flow is identically satisfied and is written

$$\text{curl}\ (u,\ v,\ w) = 0 \tag{2.2b}$$

The perturbation pressure is

$$p = -\rho\, Ww \tag{2.3}$$

where ρ is the fluid density in the free stream.

A simplification of the analytical work may be made by the well-known similarity transformation as given by Hayes (Ref. 2)

$$\varphi = \frac{1}{\sqrt{M^2-1}}\ \bar{\varphi} \tag{2.4a}$$

$$(x,\ y,\ z) = (\bar{x},\ \bar{y},\ \sqrt{M^2-1}\ \ \bar{z}) \tag{2.4b}$$

which changes equation (2.1) into the equation

$$\bar{\varphi}_{\bar{x}\bar{x}} + \bar{\varphi}_{\bar{y}\bar{y}} - \bar{\varphi}_{\bar{z}\bar{z}} = 0 \tag{2.5a}$$

with the velocity given by

$$(u, v, w) = \left(\frac{\bar{u}}{\sqrt{M^2-1}}, \frac{\bar{v}}{\sqrt{M^2-1}}, \frac{\bar{w}}{(M^2-1)}\right) \tag{2.6}$$

where the angle of attack (or incidence) of the corresponding bodies is taken to be

$$\alpha = \frac{\bar{\alpha}}{\sqrt{M^2-1}} \tag{2.4c}$$

for the type of problems considered in this work (another transformation scheme is given in Ref. 22)*. Equation (2.5) corresponds to considering a supersonic flow at $M = \sqrt{2}$. There is no loss of generality here and supersonic flow at any other Mach number past a given body may be translated in terms of the transformed system. For future work, then, the bar notation will be dropped and equation (2.5a) will be written in the usual notation

$$\Delta\varphi - \varphi_{zz} \equiv \varphi_{xx} + \varphi_{yy} - \varphi_{zz} = 0 \quad , \quad M = \sqrt{2} \tag{2.5b}$$

where $\Delta(\ )$ is the two-dimensional Laplacian operator. Generally for planar problems, Cartesian coordinates will be used; for the non-planar problems of the present study, cylindrical polar coordinates fit naturally.

*This is for planar systems.

2.2 Remarks on Boundary Conditions

The physical boundary conditions imposed on all problems studied here are the usual conditions for inviscid flow theory:

1) The normal flow at a solid boundary is zero.

2) Uniform flow exists far upstream and away from the body. Since linearized supersonic flow is governed by the wave equation (2.5), the condition 2) is analogous to initial conditions for a problem where z is replaced by t. Specifically, condition 1) will be stated in terms of the normal perturbation velocity.

> 1') The normal velocity is given everywhere outside of the body in the plane of the wing (nominally, the x-z plane), and on the body surface.

No mixed type conditions such as those which enter in the planar tip problem are considered. Condition 2) will be replaced, after a suitable choice of the perturbation potential and the coordinate system relative to the wing-body configuration, by initial conditions in the plane, $z = 0$. Thus, the problem with the proper initial and boundary conditions is formulated in a manner suitable for Laplace transform methods, operating on the axial variable z. This is carried out in detail for the leading edge problem, Case IIa with zero sweepback, $\beta = 0$ (Figure 1.2).

2.3 Boundary Conditions for the Leading Edge Problem, Case IIa

Consider the total velocity potential, Φ, written as

$$\Phi = W_1 z + W_2 r \sin\theta + \varphi' + \varphi \qquad (2.7)$$

where $\quad W_1 = W\cos\alpha \doteq W \quad , \quad W_2 = W\sin\alpha \doteq W\alpha$

The first and second terms on the right hand side of equation (2.7) represent the potential of a uniform flow of magnitude W at an angle α in the y-z plane. $(\varphi' + \varphi)$ is the usual perturbation potential. Note that equation (2.2a) should now read

$$(u,\ v,\ w) = \text{grad}\ (\varphi' + \varphi) \qquad (2.2b)$$

for the perturbation velocity. Now let φ' be the solution which satisfies the boundary conditions for $z < 0$ where $z = 0$ passes through the wing leading edge. Then, determine φ to satisfy the boundary conditions for $z > 0$. Note, $\varphi \equiv 0$ for $z \leq 0$. This is expressed as the "initial" conditions

$$\varphi = \varphi_z \equiv 0 \quad {}^*, \text{for } z = 0 \qquad (2.8)$$

Three cases are considered. The body radius is normalized with no loss of generality.

1) Thickness (or symmetric) Problem (Figure 2.1):
Body at zero incidence with respect to W; Wing at α with respect to W.

Choose $\varphi' \equiv 0$; then the boundary conditions on φ for $z < 0$ are

i) $\frac{1}{r}\,\Phi_\theta(r,0,z) = 0 \quad \xRightarrow{(implies)} \quad W_2 + \frac{1}{r}\,\varphi_\theta(r,0,z) = 0 \qquad (2.9a)$

(Camber, f(r,z) say, may be considered by introducing it on the righthand side in place of zero)

*This is not precisely true, Cf. remarks following equation (2.11b)

ii) $\Phi_r(1,\theta,z) = W_2 \sin\theta \implies \varphi_r(1,\theta,z) = 0$ (2.9a)

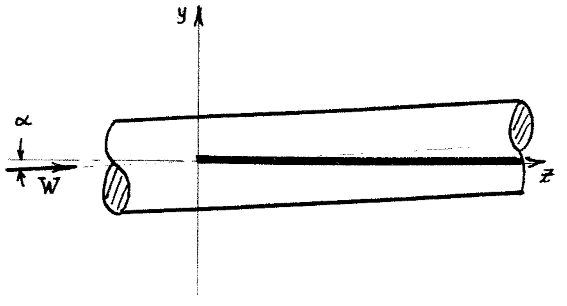


Figure 2.1 The Thickness Problem

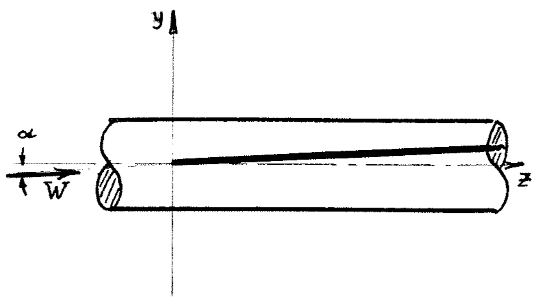


Figure 2.2 The Body Incidence Problem

2) Body Incidence (or lifting) Problem (Fig. 2.2):
Body at α w.r.t.W ; Wing at zero w.r.t.W.

Choose $\varphi' = \frac{W_2}{r} \sin\theta$, which is the perturbation potential of flow past an infinitely long circular cylinder (Ref. 7); then the boundary conditions on φ for $z > 0$ are

i) $\frac{1}{r}\Phi_\theta(r,0,z) = W_2,\ r \geq 1 \implies \frac{W_2}{r} + \frac{1}{r}\varphi_\theta(r,0,z) = 0,\ r \geq 1$ (2.9b)

ii) $\Phi_r(1,\theta,z) = 0 \implies \varphi_r(1,\theta,z) = 0$

It is remarked that the Figures 2.1 and 2.2 are given only to emphasize the orientation of the body and wing with respect to the free stream velocity, W; actually, within the linearized theory, all boundary conditions are prescribed on the body and wing as oriented in Fig. 2.3.

3) The Incidence Problem (Fig. 2.3):
Body at α w.r.t.W.
Wing at α w.r.t.W.

Choose $\varphi' = \frac{W_2}{r} \sin\theta$; then the boundary conditions on φ for $z > 0$ are, for zero sweepback angle, $\beta = 0$

i) $\frac{1}{r}\Phi_\theta(r,0,z) = 0,\ r \geq 1 \implies W_2\left(1 + \frac{1}{r^2}\right) + \frac{1}{r}\varphi_\theta(r,0,z) = 0$ (2.9c)

ii) $\Phi_r(1,\theta,z) = 0 \implies \varphi_r(1,\theta,z) = 0$

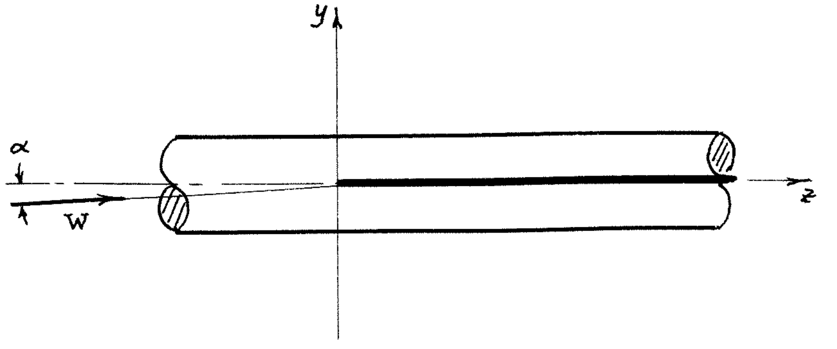


Figure 2.3 The Incidence Problem

In addition, if the wing is taken at a sweepback angle, $\beta \neq 0$, the boundary conditions (2.9c) become

$$\text{i)} \quad \frac{1}{r}\varphi_\theta(r,0,z) = -\frac{W_2}{r^2} \quad , \quad 0 < z < (r-1)\tan\beta$$

$$= -W_2\left(1+\frac{1}{r^2}\right) \quad , \quad z > (r-1)\tan\beta \qquad (2.9d)$$

$$\text{ii)} \quad \varphi_r(1,\theta,z) = 0$$

Two essential simplifications have been made in this section:

1) The anti-symmetric (with respect to the x-z plane) part of the solution, φ', has been separated out. This means that the problem need only be considered in the upper half space, $y \geq 0$.

2) Then, by appropriate choice of coordinates relative to the configuration, problems amenable to Laplace transform methods with zero initial conditions have resulted.

By the principle of superposition, the general problem of a body and cambered wing at incidence, may be given. Usually, an additional condition of symmetry of the flow with respect to the y-z plane is also made

$$\text{iii)} \quad \frac{1}{r}\Phi_\theta(r,\tfrac{\pi}{2},z) = 0 \ , \ r \geq 1 \implies \frac{1}{r}\varphi_\theta(r,\tfrac{\pi}{2},z) = 0 \qquad (2.10)$$

2.4 The Laplace Transformation and Green's Function Method

The previous section 2.3 has shown that the fundamental wing-body problem, Case IIa, may be formulated as an initial and boundary value problem for the wave equation (2.5b). This is a natural problem for the elegant and powerful methods of the Laplace transformation (one of many in the field of linear integral operators) which has been so widely studied by both theoretical physicists and mathematicians (Refs. 8 to 11). Several British workers, including Lighthill, Gunn and Ward (Ref. 12 to 14) have applied these methods recently to planar and axially-symmetric systems. Briefly, the method may be described as follows:

Original Problem	Laplace →	Transformed Problem
Wave equation plus initial and boundary conditions	Transformation on z	Modified Helmholtz equation plus boundary condition ↓
Solution of original problem	← Inverse Laplace Transformation	Solution of transformed problem

The solution of the transformed problem is most conveniently obtained by the Green's function method (Refs. 13, 15 and 16).

The method is presented for a problem with arbitrary boundary conditions. The original problem is then stated for the wave equation (2.5) with the conditions

Initial conditions: $\varphi(r,\theta,0+) = \varphi_z(r,\theta,0+) = 0$ (2.11a)

Boundary condition: φ_n, the normal perturbation velocity, is given on the body and on the x-z plane off the body (2.11b)

Actually, $\varphi_z(r,0,0+) \neq 0$ or $\varphi_z(x,0,0+) \neq 0$; that is to say $\varphi_z \neq 0$ if z = 0 is approached in the x-z plane. This causes no difficulty here*.

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*Stewart has pointed out a necessary condition for uniqueness: the condition of outgoing waves.

Assuming that the solution is twice differentiable and the second derivatives have Laplace transforms, the original problem with the conditions (2.11) becomes the <u>transformed problem</u>.

$$\psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} - s^2\psi = -s\,\varphi(r,\theta,0+) - \varphi_z(r,\theta,0+) \tag{2.12a}$$

or by the initial conditions (2.11a)

$$\Delta\psi - s^2\psi = 0 \tag{2.12b}$$

with the boundary condition: ψ_n is given (in the x-y plane) on the body and on the x-axis off the body (2.13)

where

$$\psi(r,\theta;s) = \int_0^\infty e^{-sz}\varphi(r,\theta,z)\,dz \tag{2.14a}$$

the Laplace transform of φ on the axial variable, z. The inverse Laplace transform is given by

$$\varphi(r,\theta,z) = \frac{1}{2\pi i}\int_\Gamma e^{zs}\,\psi(r,\theta;s)\,ds \tag{2.15a}$$

where the contour Γ runs from $-i\infty$ to $i\infty$ in the right-half of the complex s-plane. The required Laplace transform operations are given in Table I. For brevity, (2.14a) is sometimes written

$$\psi = \mathcal{L}\{\varphi;\, s\} \tag{2.14b}$$

and (2.15a) is sometimes written

$$\varphi = \mathcal{L}^{-1}\{\psi;\, z\} \tag{2.15b}$$

The differential equation (2.12b) is the two-dimensional modified Helmholtz equation with the parameter, s ($\mathcal{R}\ell$ s > 0) and the transformed problem is an elliptic boundary value problem in the x-y plane. This exterior problem (the boundary may include infinity) was first formulated by Sommerfeld (Refs. 16 and 17). For strong conditions on ψ within and on the boundary of the region, i.e., ψ is twice continuously differentiable in (x,y) or (r,θ), the solution within the region is expressible as an integral in terms of ψ and the normal derivative, ψ_n , on the boundary by application of Green's theorems; but ψ and ψ_n are related on the boundary and the solution takes the form

$$\psi(r,\theta;s) = -\frac{1}{2\pi}\int_C \left(\psi\frac{\partial G}{\partial n} - G\frac{\partial \psi}{\partial n}\right)d\ell \qquad (2.16)$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative on the boundary C (Fig. 2.4) and G is the sum of a singular and a regular function in the region (Cf. equation (2.20a)).

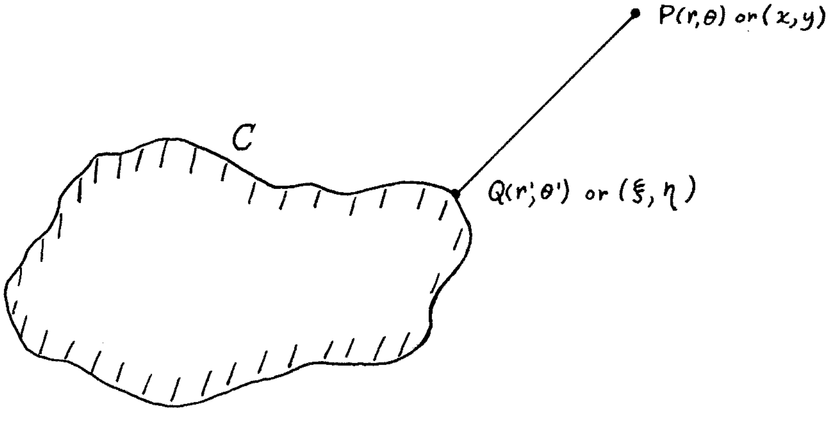


Figure 2.4 The Exterior Problem for $\Delta\psi - s^2\psi = 0$

If G is chosen such that G = 0 on C,

$$\psi(r,\theta;s) = -\frac{1}{2\pi}\int_C \psi\,\frac{\partial G_0}{\partial n}\,d\ell \qquad (2.17)$$

where G_0 is called the first Green's function. If G is chosen such that $\frac{\partial G}{\partial n} = 0$ on C,

$$\Psi(r, \theta; s) = \frac{1}{2\pi} \int_C G \frac{\partial \psi}{\partial n} \, d\ell \tag{2.18}$$

where G is called the second Green's function. The problems of the present study are of this second kind. For the solution (2.18) the Green's function is characterized then by

1) $G(r, \theta; r', \theta') \equiv G(P; Q)$ satisfies the equation $\Delta\psi - s^2\psi = 0$ for $P(r, \theta) \neq Q(r', \theta')$

2) $G(r, \theta; r', \theta')$ has the proper singularity at $P(r, \theta) = Q(r', \theta')$ (Cf. section 2.5) (2.19)

3) $\frac{\partial G}{\partial n} = 0$ on C

A consequence of conditions 1) and 2) is that $G(r, \theta; r', \theta') = G(r', \theta'; r, \theta)$, symmetric in (P;Q). Weaker conditions, say, $\psi_n \equiv \frac{\partial \psi}{\partial n}$ may be discontinuous but integrable* on the boundary, are permissible for equation (2.18) as in potential theory (Ref. 18).

*This statement is not precise, but will not be elaborated upon here.

2.5 Determination of Green's Functions for Various Boundaries

The fundamental solution of the modified Helmholtz equation (2.12b) that is, the solution independent of θ and singular at the origin, is $K_0(sr)$, which has a logarithmic-type singularity at $r = 0$. $K_0(sr)$ is the modified Bessel function of the second kind of zero order (Refs. 19 and 20). For a fixed point, $P(r,\theta)$, with the variable point $Q(r',\theta')$, the fundamental solution may be written immediately as $K_0(s\rho)$, since the differential equation is invariant under translation; $\rho = \sqrt{r^2 + r'^2 - 2rr'\cos(\theta-\theta')}$ is the distance between P and Q. This solution may be interpreted as the Green's function for the entire x-y plane since it satisfies the required properties (2.19). The determination of Green's functions for other boundaries, in general, is not a simple task although there is no difficulty in obtaining them for the boundaries of the present study. However, a few remarks which will be helpful later are made.

In determining the Green's function for a region with a given boundary, the function sought will always have the form

$$G(P;Q) = K_0(s\rho) + h(P;Q) \tag{2.20a}$$

where $h(P;Q)$ is regular in the region. Since $K_0(s\rho)$ is symmetric in $(P;Q)$ then $h(P;Q)$ must be also. The invariance properties of the differential equation will be helpful in determining $h(P;Q)$ and give some intuitive meaning to it. It is easily verified that the Helmholtz equation (2.12b) is invariant under the following transformations:

1) Translation, $(\bar{x}, \bar{y}) = (x + a, y + b)$ where a, b are constants
2) Reflection on the axes, e.g. on the y-axis, $(\bar{x}, \bar{y}) = (-x, y)$ (or reflection on any straight line) (2.21)
3) Rotation, $(\bar{r}, \bar{\theta}) = (r, \theta + c)$ where c is a constant

There appears to be no simple invariant transformation with respect to inversion on the unit circle as there is for Laplace's potential equation. Such a transformation would be helpful in obtaining an intuitive notion on constructing Green's functions for the circle.

The Green's functions for the half-plane, the circle, and the boundary of Case IIa are determined for later use.

1. Half-plane, $y \geq 0$ (Fig. 2.5)

The first Green's function is

$$G_0(P;Q) = K_0(s\rho) - K_0(s\rho_1) \tag{2.22}$$

where, $\rho = \sqrt{r^2 + r'^2 - 2rr'\cos(\theta-\theta')}$, the Euclidean distance from P to Q

$= \sqrt{(x-\xi)^2 + (y-\eta)^2}$

and $\rho_1 = \sqrt{r^2 + r'^2 - 2rr'\cos(\theta+\theta')}$, the distance from P to the reflection of Q on the x-axis

$= \sqrt{(x-\xi)^2 + (y+\eta)^2}$

The second Green's function is

$$G(P;Q) = K_0(s\rho) + K_0(s\rho_1) \tag{2.23}$$

The Green's functions for a quadrant or a sector (included angle, $\theta = \frac{\pi}{n}$; n, positive integer) may be obtained as simply.

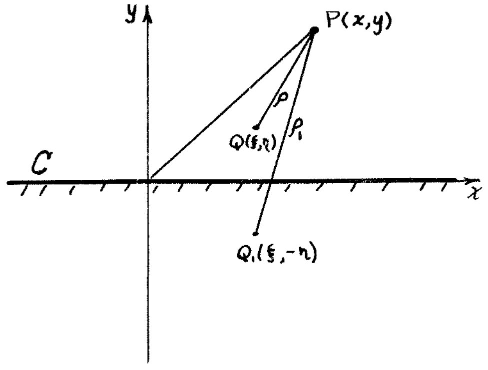


Figure 2.5. Half Plane, $y \geq 0$

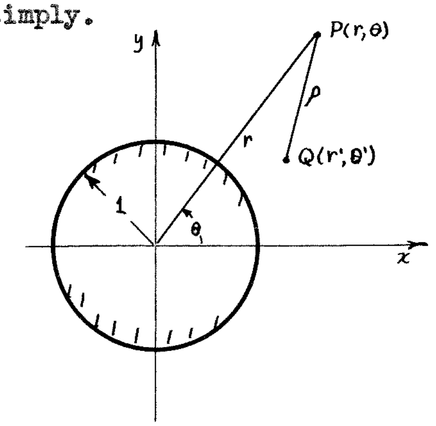


Figure 2.6. Exterior to the Unit Circle

2. Exterior to the Unit Circle (Fig. 2.6)

No invariance properties of the Helmholtz equation (2.12b) with respect to the unit circle has been found so other means are used to determine the Green's function. Here the addition theorem (Ref. 19) for $K_0(s\rho)$ is used

$$K_0(s\rho) = I_0(sr)K_0(sr') + 2\sum_{n=1}^{\infty} I_n(sr)K_n(sr')\cos n(\theta-\theta'), \quad r<r'$$

$$= I_0(sr')K_0(sr) + 2\sum_{n=1}^{\infty} I_n(sr')K_n(sr)\cos n(\theta-\theta'), \quad r>r' \qquad (2.24)$$

where I_n and K_n are modified Bessel functions of the first and second kinds, respectively, of n^{th} order. For $r < r'$, consider $h(P;Q)$ (Cf. equation (2.20a)) of the form

$$h(P;Q) = A_0 K_0(sr) + \sum_{n=1}^{\infty} A_n K_n(sr)\cos n(\theta-\theta') \qquad (2.20b)$$

where A_0 and A_n are determined to satisfy the boundary condition

$$G_0(P;Q)\Big|_{r=1} = K_0(s\rho)\Big|_{r=1} + h(P;Q)\Big|_{r=1} = 0 \qquad (2.25)$$

Hence

$$A_0 = -\frac{I_0(s)}{K_0(s)}K_0(sr'), \quad A_n = -2\frac{I_n(s)}{K_n(s)}K_n(sr') \qquad (2.26)$$

and the first Green's function is

$$G_0(P;Q) = K_0(sr')\left\{I_0(sr) - \frac{I_0(s)}{K_0(s)}K_0(sr)\right\} + \\ + 2\sum_{n=1}^{\infty} K_n(sr')\left\{I_n(sr) - \frac{I_n(s)}{K_n(s)}K_n(sr)\right\}\cos n(\theta-\theta'), \quad r<r' \qquad (2.27)$$

with r and r' interchanged for $r > r'$.

Similarly, the second Green's function $G(P;Q)$ must satisfy the boundary condition

$$\left.\frac{\partial G(P;Q)}{\partial r}\right|_{r=1} = 0 \tag{2.19'}$$

Therefore

$$\begin{aligned} G(P;Q) = {} & K_o(sr')\left\{I_o(sr) - \frac{I_o'(s)}{K_o'(s)} K_o(sr)\right\} + \\ & + 2\sum_{n=1}^{\infty} K_n(sr')\left\{I_n(sr) - \frac{I_n'(s)}{K_n'(s)} K_n(sr)\right\}\cos n(\theta-\theta'), \quad r<r' \end{aligned} \tag{2.28}$$

with r and r' interchanged for r > r', and where the primes on the Bessel functions mean differentiation with respect to r(or r').

3. Case IIa (Fig. 2.7)

The Green's functions are obtained simply by using the reflection property on the x-axis with the results for the circle

$$\begin{aligned} G_o(P;Q) = {} & 2K_o(sr')\left\{I_o(sr) - \frac{I_o(s)}{K_o(s)} K_o(sr)\right\} + \\ & + 4\sum_{n=1}^{\infty} K_n(sr')\left\{I_n(sr) - \frac{I_n(s)}{K_n(s)} K_n(sr)\right\}\cos n\theta \cos n\theta', \quad r<r' \end{aligned} \tag{2.29}$$

$$\begin{aligned} G(P;Q) = {} & 2K_o(sr')\left\{I_o(sr) - \frac{I_o'(s)}{K_o'(s)} K_o(sr)\right\} + \\ & + 4\sum_{n=1}^{\infty} K_n(sr')\left\{I_n(sr) - \frac{I_n'(s)}{K_n'(s)} K_n(sr)\right\}\cos n\theta \cos n\theta', \quad r<r' \end{aligned} \tag{2.30a}$$

with r and r' interchanged for r > r'.

Sometimes it is convenient to write

$$\begin{aligned} G(P;Q) = {} & K_o(s\rho) + K_o(s\rho_1) - \\ & - 2\left\{\frac{I_o'(s)}{K_o'(s)} K_o(sr)K_o(sr') + 2\sum_{n=1}^{\infty}\frac{I_n'(s)}{K_n'(s)} K_n(sr)K_n(sr')\cos n\theta \cos n\theta'\right\} \end{aligned} \tag{2.30b}$$

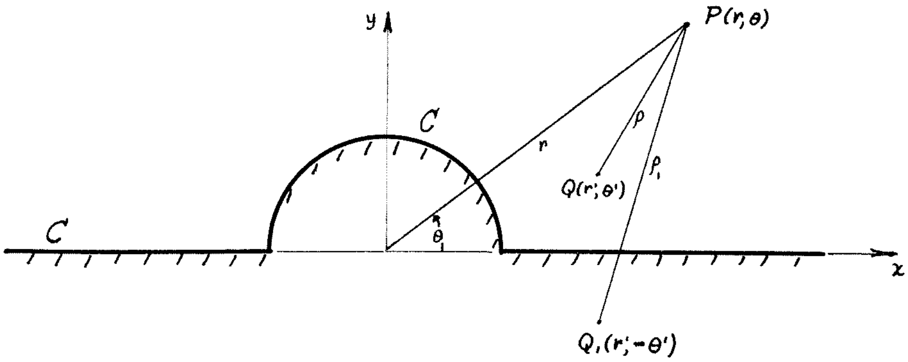


Figure 2.7 Boundary for Case IIa

With the determination of Green's functions for appropriate boundaries the transformed problem is formally solved. There follows, for each solution of specific problems, the interpretation by the inverse Laplace transformation into the solution of the original physical problem. Sometimes, this is a formidable task.

It is remarked here that some intuitive meaning may be given to the Green's functions for the transformed problem. For example, the inverse Laplace transform of the fundamental solution $K_0(s\rho)$ is given by Table II or by equation (3.6); $L^{-1}\left\{ K_0(s\rho)\right\}$ represents a supersonic source singularity in the physical space.

III. PLANAR PROBLEMS

3.1 Preliminary Remarks

Some of the implications of possible difficulties arising in non-planar problems are seen in section 2.5 by comparing the Green's functions for a planar system and a non-planar one, e.g. equation (2.23) and (2.30b), respectively. This is verified in Part IV. Thus, the study of planar systems, related to non-planar systems especially in planform, is not only advisable but, as will be shown, important. By a propitious choice of planar systems, some qualitative results applicable to non-planar systems are obtained, e.g. general regions of pressure concentration and relative distribution of pressure over the configuration. Also some important physical insight helpful in handling the more complex non-planar problems is obtained.

The main problem which will be considered here will be the supersonic leading edge problem, Case Ia. Case Ib may be formulated but the boundary conditions are complicated; the incidence problem may be handled more easily by conical methods. Also, the trailing edge problem will be discussed.

3.2 The Leading Edge Problem, Case Ia

The incidence problem, corresponding to Case IIa which resulted in boundary conditions (2.9c) or (2.9d), is considered where the "body" is now a narrow flat plate. The wide wing is taken at a sweepback angle, $\beta < \frac{\pi}{4}$, i.e., less than the complement of the Mach angle, as shown in Figure 3.1.

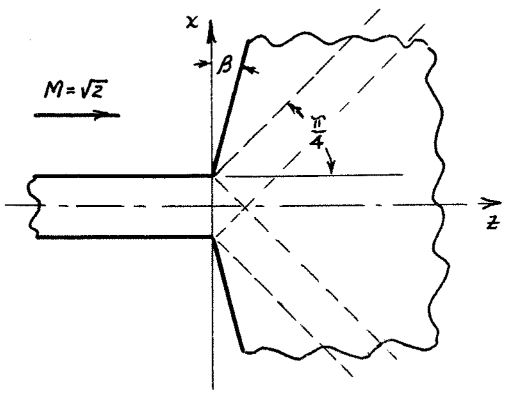


Figure 3.1 Case Ia: Incidence Problem

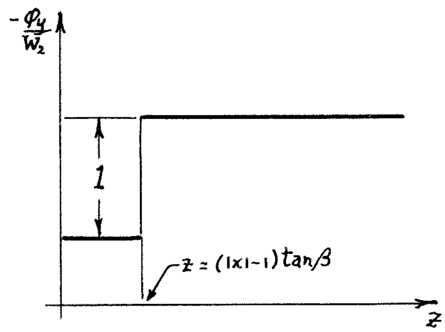


Figure 3.2 Case Ia: Distribution in z-direction, $|x| \geq 1$

Choose $\varphi' = W_2 \,\mathcal{I}m\,(\xi^2-1)^{1/2}$, the perturbation potential for flow past an infinitely long flat plate, where $\xi = x+iy$ and $\mathcal{I}m$ means "imaginary part of". Then, the boundary conditions in Cartesian coordinates for $z > 0$ are

i) $$\varphi_y(x,0,z) = -W_2\left(\frac{|x|}{\sqrt{x^2-1}} - 1\right), \quad 0 < z < (|x|-1)\tan\beta, \quad |x| \geq 1$$

$$= -W_2 \frac{|x|}{\sqrt{x^2-1}}, \quad z > (|x|-1)\tan\beta, \quad |x| \geq 1$$

The distribution in the z-direction is shown in Fig. 3.2.

(3.1)

ii) $$\varphi_y(x,0,z) = 0, \quad 0 \leq |x| < 1$$

iii) $$\varphi_x(0,y,z) = 0$$

The singularity at $x = 1$ corresponds to a subsonic leading edge singularity. The symmetry condition iii) implies that the transformed problem may be confined to the first quadrant. If β is near $\frac{\pi}{2}$ the assumptions of Class A problems are satisfied.

The boundary conditions for the transformed problem, operating on both sides of conditions (3.1) are

$$\text{i)} \quad \psi_y(x,0;s) = -\frac{W_2}{s}\left\{\left(\frac{|x|}{\sqrt{x^2-1}} - 1\right) + e^{-s(|x|-1)\tan\beta}\right\}, \quad |x| \geq 1$$

$$\text{ii)} \quad \psi_y(x,0;s) = 0 \quad , \quad 0 \leq |x| < 1 \tag{3.2}$$

$$\text{iii)} \quad \psi_x(0,y;s) = 0$$

For zero sweepback, condition i) becomes simply

$$\text{i')} \quad \psi_y(x,0;s) = -\frac{W_2}{s}\frac{|x|}{\sqrt{x^2-1}} \quad , \quad |x| \geq 1 \tag{3.2'}$$

For simplicity, pressure distribution calculations are carried out only for this latter case. By equation (2.18) in Cartesian coordinates the transformed solution may be written immediately in the form

$$\psi(x,y;s) = \frac{1}{2\pi}\int_C \psi_n(\xi,0)\, G(x,y;\xi,0)\, d\xi \tag{3.3a}$$

where

$$\psi_n = -\psi_y \tag{3.3b}$$

the contour C, running from $-\infty$ to ∞ on the x-axis; and for the half-plane, from equation (2.23), the Green's function is

$$G(x,y;\xi,0) = 2K_0\left(s\sqrt{(x-\xi)^2+y^2}\right) \tag{3.3c}$$

The integral along the negative x-axis can be expressed with positive limits and

$$\psi(x,y;s) = \frac{W_2}{\pi s}\left\{\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}} K_0(s\sqrt{(x-\xi)^2+y^2})\,d\xi + \int_{-\infty}^{-1} \frac{|\xi|}{\sqrt{\xi^2-1}} K_0(s\sqrt{(x-\xi)^2+y^2})\,d\xi\right\} \qquad (3.3d)$$

or, $$\psi(x,y;s) = \frac{W_2}{\pi s}\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}}\left\{K_0(s\sqrt{(x-\xi)^2+y^2}) + K_0(s\sqrt{(x+\xi)^2+y^2})\right\} d\xi$$

This is the transformed solution for the potential of the incidence problem for Case Ia. The second part of the solution may be regarded as the reflected part off of a solid boundary (corresponding to the condition (2.10)) in the y-z plane. If the pressure is desired, the transformed solution is written in terms of $s\cdot\psi(x,y;s)$ since, for zero initial conditions (Table I)

$$w = \varphi_z = \mathcal{L}^{-1}\left\{s\psi;\, z\right\} \qquad (3.4)$$

In supersonic leading edge problems, this is usually the component of interest. The pressure is given immediately by equation (2.3). Hence, the solution for the pressure component may be written

$$\varphi_z(x,y,z) = \frac{W_2}{\pi}\left[\frac{1}{2\pi i}\int_\Gamma e^{sz}ds\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}}\left\{K_0(s\sqrt{(x-\xi)^2+y^2})+K_0(s\sqrt{(x+\xi)^2+y^2})\right\}d\xi\right] \qquad (3.5)$$

The most convenient way to carry out this integration is to formally interchange the order of integration and carry out the inverse Laplace transformation. The integral over ξ is essentially a finite integral since the integrand is zero over part of the range. By appropriately restricting the variables (x, y, z) convergence requirements are satisfied. By Table II,

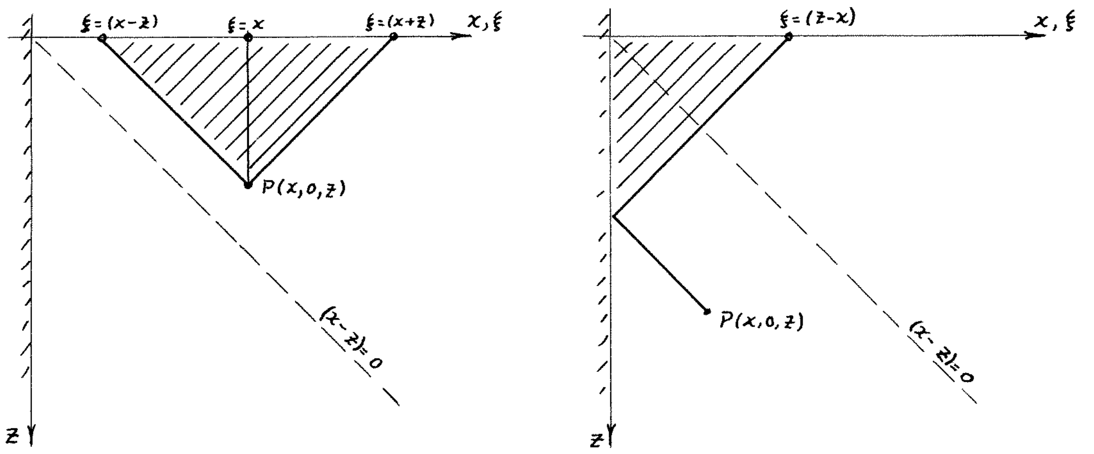
$$\mathcal{L}^{-1}\left\{K_0\left(s\sqrt{(x-\xi)^2+y^2}\right); z\right\} = 0 \qquad , \xi < (x-\sqrt{z^2-y^2}),\ (x+\sqrt{z^2-y^2}) < \xi$$

$$= \frac{1}{\sqrt{z^2-y^2-(x-\xi)^2}} \quad , (x-\sqrt{z^2-y^2}) < \xi < (x+\sqrt{z^2-y^2}) \tag{3.6}$$

$$\mathcal{L}^{-1}\left\{K_0\left(s\sqrt{(x+\xi)^2+y^2}\right); z\right\} = 0 \qquad , \xi > (\sqrt{z^2-y^2}-x)$$

$$= \frac{1}{\sqrt{z^2-y^2-(x+\xi)^2}} \quad , 0 < \xi < (\sqrt{z^2-y^2}-x) \tag{3.7}$$

Thus, two regions of integration are defined. These are most easily seen for points in the x-z plane as shown by the cross-hatched areas in Figure 3.3.



a) First integral b) Second integral

Figure 3.3. Regions of Integration for Equation (3.5)

Since the lower limit of the integral over ξ is one, actually three regions are defined and have a physical interpretation (Figure 3.4). Note that the condition of flow symmetry with respect to the y-z plane has been replaced by a solid boundary.

Region I:	$(x-z) > 1$,	Region of influence of the Mach cones from the wing leading edge
Region II:	$-1 < (x-z) < 1$,	Region of influence of the Mach cone from the wing leading edge-body junction plus Region I
Region III:	$(x-z) < -1$,	Region of influence of the Mach cone from the opposite wing leading edge-body junction plus Region I and Region II.

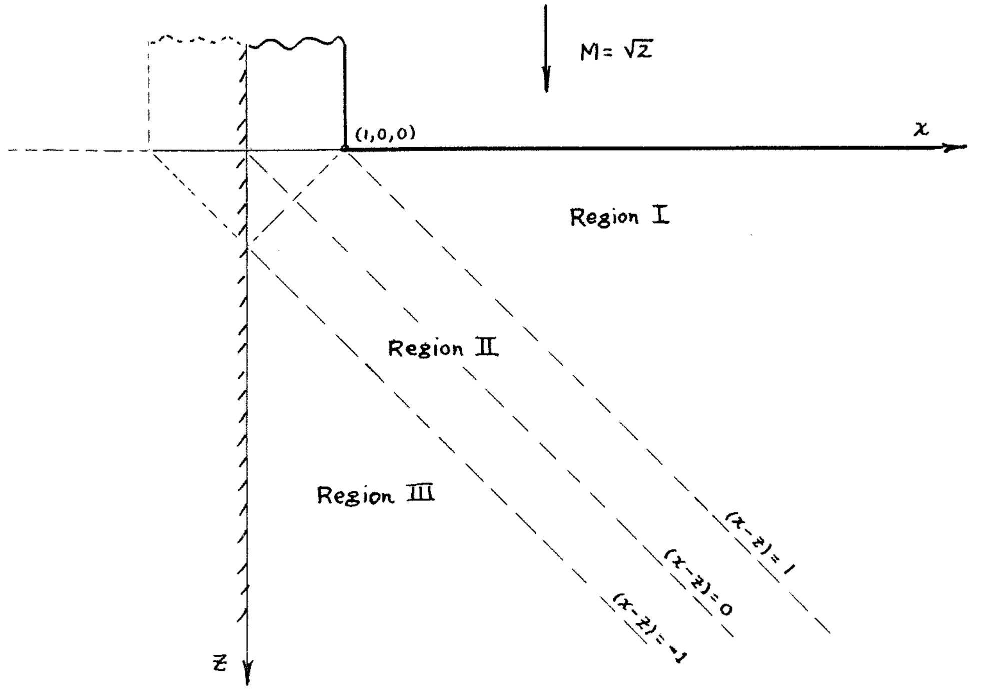


Figure 3.4. Regions of Influence in the x-z plane.

Since the solution for the pressure component, equation (3.5) becomes unwieldy, results are presented only in regions of greatest interest, i.e. on the surface of the configuration.

In <u>Region I.</u> $(x-z) > 1$

$$\frac{\varphi_z(x,o,z)}{W_z} = \frac{1}{\pi}\left\{\int_{(x-z)}^{x} \frac{\xi}{\sqrt{\xi^2-1}} \frac{d\xi}{\sqrt{z^2-(x-\xi)^2}} + \int_{x}^{(x+z)} \frac{\xi}{\sqrt{\xi^2-1}} \frac{d\xi}{\sqrt{z^2-(x-\xi)^2}}\right\}$$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{(x-z\tau)\;d\tau}{\sqrt{[(x-z\tau)^2-1](1-\tau^2)}} \tag{3.8a}$$

This is an elliptic type integral which may be expressed in terms of the standard complete elliptic integrals of the first and third kinds by known methods of reduction (Ref. 21). For $z = 1$, for example,

$$\frac{\varphi_z(x,o,1)}{W_z} = \frac{z}{\pi}\left\{\left(\frac{x}{g}-1\right)K(k) + \frac{(g^2-1)}{g^2}\Pi(k,\lambda,1)\right\} \tag{3.8b}$$

where

$$f = \frac{x-\sqrt{x^2-4}}{2} \quad , \quad g = \frac{x+\sqrt{x^2-4}}{2}$$

$$k^2 = \frac{f^2}{g^2} \quad , \quad \lambda^2 = f^2$$

$$K(k) = \int_0^1 \frac{d\sigma}{\sqrt{(1-\sigma^2)(1-k^2\sigma^2)}}$$, complete elliptic integral of the first kind

$$\Pi(k,\lambda,1) = \int_0^1 \frac{d\sigma}{(1-\lambda^2\sigma^2)\sqrt{(1-\sigma^2)(1-k^2\sigma^2)}}$$, complete elliptic integral of the third kind

The integral (3.8a) for $z = 1$ may also be obtained by superposition of conical flow solutions (Ref. 22)

In Region II, $-1 < (x-z) < 1$

$$\frac{\varphi_z(x,o,z)}{W_z} = \frac{1}{\pi}\int_{1}^{(z+x)} \frac{\xi}{\sqrt{\xi^2-1}} \frac{d\xi}{\sqrt{z^2-(\xi-x)^2}} \tag{3.9a}$$

For x = 1,

$$\frac{\varphi_z(1,0,z)}{W_2} = \frac{1}{\pi}\int_1^{(z+1)} \frac{\xi}{\sqrt{\xi^2-1}}\,\frac{d\xi}{\sqrt{z^2-(\xi-1)^2}}$$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{[(\frac{2}{z}+1)+\tau]\;d\tau}{\sqrt{[(\frac{4}{z}+1)+\tau](3+\tau)(1-\tau^2)}} \qquad (3.9b)$$

which is the most convenient form for reduction.

For z = 1,

$$\frac{\varphi_z(x,0,1)}{W_2} = \frac{1}{\pi}\int_1^{(x+1)} \frac{\xi}{\sqrt{\xi^2-1}}\,\frac{d\xi}{\sqrt{1-(x-\xi)^2}} \qquad (3.9c)$$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{[(\frac{2}{x}+1)+\tau]\;d\tau}{\sqrt{[(\frac{4}{x}+1)+\tau][(\frac{4}{x}-1)+\tau](1-\tau^2)}}$$

In Region III, $(x-z) < -1$

$$\frac{\varphi_z(x,0,z)}{W_2} = \frac{1}{\pi}\left\{\int_1^{(z+x)} \frac{\xi}{\sqrt{\xi^2-1}}\,\frac{d\xi}{\sqrt{z^2-(\xi-x)^2}} + \int_1^{(z-x)} \frac{\xi}{\sqrt{\xi^2-1}}\,\frac{d\xi}{\sqrt{z^2-(\xi+x)^2}}\right\} \qquad (3.10a)$$

The first integral is the same form as in Region II; the second integral is the "reflected" solution.

For x = 0,

$$\frac{\varphi_z(0,y,z)}{W_2} = \frac{2}{\pi}\int_1^{+\sqrt{z^2-y^2}} \frac{\xi}{\sqrt{\xi^2-1}}\,\frac{d\xi}{\sqrt{(z^2-y^2)-\xi^2}} \qquad (3.10b)$$

$$= \frac{1}{\pi}\int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)}}$$

hence,

$$\frac{\varphi_z(0,y,z)}{W_2} = 1 \quad , \; (z^2-y^2) \geq 1$$

$$= 0 \quad , \; (z^2-y^2) < 1$$

This shows an interesting and unexpected result; the flow from the two sides of the body combines to give the same pressure as that of a two-dimensional flat plate at incidence.

For $x = 1$,

$$\frac{\varphi_z(1,0,z)}{W_z} = \frac{1}{\pi}\left\{\int_1^{(z+1)} \frac{\xi}{\sqrt{\xi^2-1}}\,\frac{d\xi}{\sqrt{z^2-(\xi-1)^2}} + \int_1^{(z-1)} \frac{\xi}{\sqrt{\xi^2-1}}\,\frac{d\xi}{\sqrt{z^2-(\xi+1)^2}}\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-1}^{1} \frac{\left[\left(\frac{2}{z}+1\right)+\tau\right]\,d\tau}{\sqrt{\left[\left(\frac{4}{z}+1\right)+\tau\right](3+\tau)(1-\tau^2)}} + \int_{-1}^{1} \frac{\left[\left(\frac{z}{z-2}\right)+\tau\right]\,d\tau}{\sqrt{\left[\left(\frac{z+2}{z-2}\right)+\tau\right]\left[\left(\frac{3z+2}{z-2}\right)+\tau\right](1-\tau^2)}}\right\} \tag{3.10c}$$

Calculations show that the sum of these two integrals also has the value one, a result which certainly is not obviously obtained by simple transformation in an attempt to combine the two integrals.*

The spanwise (parallel to the x-axis) and chordwise (parallel to the z-axis) pressure distributions on the top surface ($y = 0+$) along significant lines are shown in Figures 3.6 and 3.7** respectively. For the incidence problem, the pressure on the bottom surface ($y = 0-$) takes the negative value. The significant features of these calculations are

1) The reciprocal square root type pressure discontinuity arising at the wing leading edge body junction is propagated downstream along the Mach cone on the wing but falls off rapidly as a finite discontinuity along the Mach cone on the body.

2) The pressure reaches its asymptotic value, i.e. its value as $(x^2 + z^2)$ becomes large, very rapidly in the downstream direction on the body and in the vicinity of the body on the wing. In fact, the pressure is the asymptotic value throughout region III.

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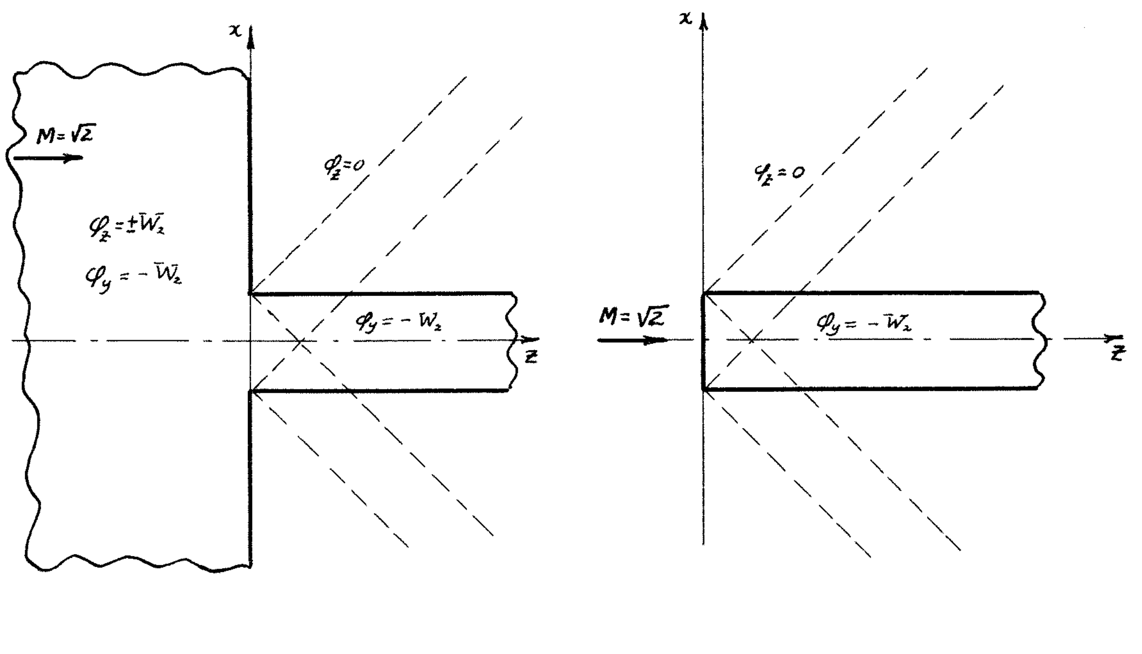
*All calculations are referred to a report to be published by the Jet Propulsion Laboratory, California Institute of Technology.

**These figures follow Table II at the end of the work.

3.3 A Trailing Edge Problem

In the study of the afterbody problem in the vicinity of the trailing edge, it is instructive and simple to consider the limiting incidence case where the pressure is constant on the wing and body upstream of the trailing edge, taking initially the case of the unswept trailing edge, as shown in Figure 3.5a. This problem results by cutting off the wing in Case IIa (section 3.2) far downstream of the leading edge. However, as the pressure distribution has shown for that case, the asymptotic value is already reached in Region III (Figure 3.4). The boundary conditions for this problem are shown in Figure 3.5a.



a) Trailing Edge Problem* b) Analogous Problem for $z > 0$.

Figure 3.5. Trailing Edge Problem Analogy.

*Note that the boundary conditions on the wing are precisely those of a two-dimensional flat plate at incidence.

Clearly the analogous problem is given by the problem of supersonic flow past a low aspect ratio rectangular wing (Figure 3.5b). This problem has been studied by Lagerstrom (Ref. 23) and Coleman (Ref. 24). The pressure distribution is given in the latter paper. Now if the trailing edge is swept back, an additional side flow is introduced over the body; this flow will tend to increase the lift, i.e. ϕ_z is increased on the top surface of the body.

IV. NON-PLANAR PROBLEMS

4.1 Preliminary Remarks

The principal difficulty encountered in non-planar problems is mathematical complexity. Needless to say, most physical problems are often fraught with such difficulties, but in many cases, guided by physical intuition and experimental work, researchers have made reasonable assumptions yielding satisfactory solutions and, in the exceptional case, opening up new fields of research. In the present study, the mathematical complexity has been overcome only partially. Two references are made to other works along these lines. The flow field of an axial sinusoidal distribution of source or lift elements near a circular cylinder has been set up by Hayes (Cf. Ref. 2). Application to problems in this section would lead to the closely related Fourier transform methods. An approach to these problems using similar methods has been initiated independently by Lagerstrom and Van Dyke (Cf. Ref. 6).

The main problem considered in this part is the leading edge problem, Case IIa. Due to the analytical complications of the problem, the presentation is made in several sections. Most of the analysis is given for zero sweepback and the sweepback case is discussed qualitatively. The trailing edge problem corresponding to the planar problem in section 3.3. is discussed.

4.2 <u>The Leading Edge Problem, Case IIa: The Transformed Problem.</u>

The boundary conditions for this problem are given in section 2.3. For convenience, since all the essential features are included, the incidence problem with <u>zero sweepback</u>, given by conditions (2.9c) for $z > 0$, will be studied. Repeating the conditions,

Initial Conditions: $\varphi = \varphi_z = 0$ in the plane $z = 0$.

Boundary Conditions: i) $\frac{1}{r}\varphi_\theta(r,0,z) = -W_2(1+\frac{1}{r^2})$ (2.9c)

ii) $\varphi_r(1,\theta,z) = 0$

The boundary conditions for the <u>transformed problem</u> are then

$$\text{i)}\quad \frac{1}{r}\psi_\theta(r,0;s) = -\frac{W_2}{s}(1+\frac{1}{r^2})$$

$$\text{ii)}\quad \psi_r(1,\theta;s) = 0 \tag{4.1a}$$

With sweepback, $\beta < \frac{\pi}{4}$, the conditions (corresponding to conditions (2.9d)) are

$$\text{i)}\quad \frac{1}{r}\psi_\theta(r,0;s) = -\frac{W_2}{s}(e^{-s(r-1)\tan\beta}+\frac{1}{r^2})$$

$$\text{ii)}\quad \psi_r(1,\theta;s) = 0 \tag{4.1b}$$

By equation (2.18), the transformed solution may be written

$$\psi(r,\theta;s) = \frac{1}{2\pi}\int_C \psi_n(r',\theta')\,G(r,\theta;r',\theta')\,d\ell \tag{2.18}$$

where $G(r,\theta;r',\theta')$ is given by equation (2.30). Since, on the circle, $\psi_n = -\psi_r = 0$, then by conditions (4.1a),

$$\psi(r,\theta;s) = \frac{W_2}{2\pi s}\int_1^\infty (1+\frac{1}{r'^2})\left\{G(r,\theta;r',\pi)+G(r,\theta;r',0)\right\}dr' \tag{4.2a}$$

Using the form of the Green's function given by equation (2.30a)

$$G(r,\theta;r',\pi) = 2K_o(sr')\left\{I_o(sr) - \frac{I_o'(s)}{K_o'(s)}K_o(sr)\right\} +$$
$$+ 4\sum_{n=1}^{\infty} K_n(sr')\left\{I_n(sr) - \frac{I_n'(s)}{K_n'(s)}K_n(sr)\right\}(-1)^n \cos n\theta \ , \quad r<r' \qquad (4.3a)$$

with r and r' interchanged for r > r', and

$$G(r,\theta;r',o) = 2K_o(sr')\left\{I_o(sr) - \frac{I_o'(s)}{K_o'(s)}K_o(sr)\right\} +$$
$$+ 4\sum_{n=1}^{\infty} K_n(sr')\left\{I_n(sr) - \frac{I_n'(s)}{K_n'(s)}K_n(sr)\right\}\cos n\theta \ , \quad r<r' \qquad (4.3b)$$

with r and r' interchanged for r > r'.

Hence, the sum $\left\{G(r,\theta;r',\pi)+G(r,\theta;r',o)\right\}$ will yield the even terms only under the summation; and

$$\Psi(r,\theta;s) = \frac{2W_2}{\pi s}\int_1^r \left(1+\frac{1}{r'^2}\right)\left[K_o(sr)\left\{I_o(sr') - \frac{I_o'(s)}{K_o'(s)}K_o(sr')\right\} + \right.$$
$$\left. + 2\sum_{n=1}^{\infty} K_{2n}(sr)\left\{I_{2n}(sr') - \frac{I_{2n}'(s)}{K_{2n}'(s)}K_{2n}(sr')\right\}\cos 2n\theta\right] dr' +$$
$$+ \frac{2W_2}{\pi s}\int_r^{\infty}\left(1+\frac{1}{r'^2}\right)\left[K_o(sr')\left\{I_o(sr) - \frac{I_o'(s)}{K_o'(s)}K_o(sr)\right\} + \right. \qquad (4.2b)$$
$$\left. + 2\sum_{n=1}^{\infty} K_{2n}(sr')\left\{I_{2n}(sr) - \frac{I_{2n}'(s)}{K_{2n}'(s)}K_{2n}(sr)\right\}\cos 2n\theta\right] dr'$$

Using the second form of the Green's function, equation (2.30b) the solution may be written in simpler form

$$\Psi(r,\theta;s) = \frac{W_2}{\pi s}\int_1^{\infty}\left(1+\frac{1}{r'^2}\right)\left\{K_o\left(s\sqrt{r^2+r'^2-2rr'\cos\theta}\right) + K_o\left(s\sqrt{r^2+r'^2+2rr'\cos\theta}\right)\right\} dr' -$$
$$- \frac{2W_2}{\pi s}\int_1^{\infty}\left(1+\frac{1}{r'^2}\right)\left\{\frac{I_o'(s)}{K_o'(s)}K_o(sr)K_o(sr') + \right. \qquad (4.2c)$$
$$\left. + 2\sum_{n=1}^{\infty}\frac{I_{2n}'(s)}{K_{2n}'(s)}K_{2n}(sr)K_{2n}(sr')\cos 2n\theta\right\} dr'$$

This transformed solution for Case IIa is again (as in Case Ia) confined to the first quadrant; and the solutions in the other quadrants are obtained by symmetry arguments.

The solution (4.2c) is written as two integrals which may be given an important physical interpretation; the first integral solution will be called $\psi^{(1)}(r,\theta;s)$ and the second, $\psi^{(2)}(r,\theta;s)$. Comparing $\psi^{(1)}$ with the planar solution, (3.3d), it is clear that $\psi^{(1)}$ may be regarded as the transformed solution of the problem in which the flat plate for z < 0 in Case Ia is replaced by a circular cylinder in a certain sense (Figure 4.1). Thus

$$\psi^{(1)}(r,\theta;s) = \frac{W_2}{\pi s}\int_1^{\infty}\left(1+\frac{1}{r'^2}\right)\left\{K_0\left(s\sqrt{r^2+r'^2-2rr'\cos\theta}\right)+K_0\left(s\sqrt{r^2+r'^2+2rr'\cos\theta}\right)\right\}dr' \qquad (4.4a)$$

is called the <u>"Flat Plate" solution for Case IIa.</u> This solution will be discussed in detail in Section 4.3.

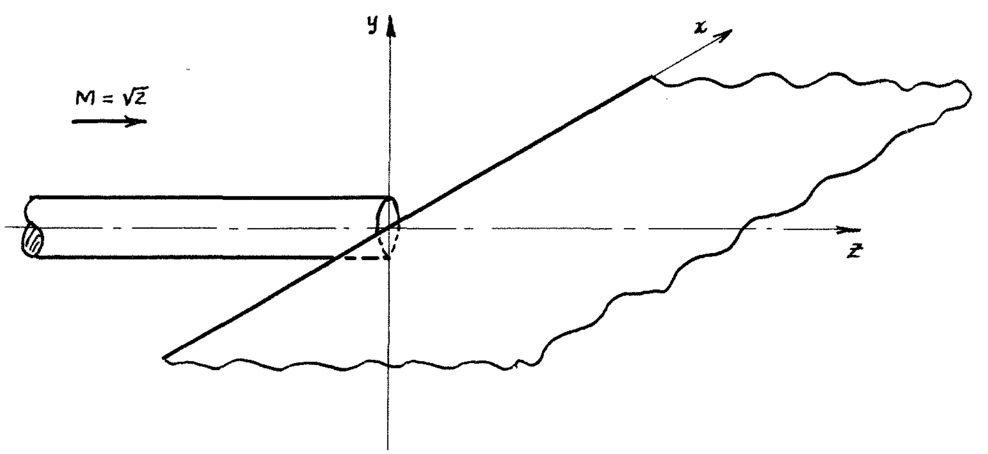


Figure 4.1. Configuration* for "Flat Plate" Solution, $\psi^{(1)}$.

The second solution, $\psi^{(2)}$, may be interpreted then as the solution needed to satisfy the boundary condition of zero flow through the circular

*Actually, a true configuration with a cylindrical body for the solution, $\psi^{(1)}$, is not possible. The circular cylinder here is to be interpreted as producing the correct upwash distribution only.

cylinder, i.e. the circular cylinder is a stream tube. Thus,

$$\psi^{(2)}(r,\theta;s) = -\frac{2\bar{W}_2}{\pi s}\int_1^{\infty}\left(1+\frac{1}{r'^2}\right)\left\{\frac{I_0'(s)}{K_0'(s)}K_0(sr)K_0(sr') + 2\sum_{n=1}^{\infty}\frac{I_{2n}'(s)}{K_{2n}'(s)}K_{2n}(sr)K_{2n}(sr')\cos 2n\theta\right\}dr' \tag{4.5a}$$

is called the "<u>Body" solution for Case IIa</u>. The principal mathematical difficulties arise in attempting to perform the inverse Laplace transform of $\psi^{(2)}$. This will be discussed in greater detail in Section 4.4.

The important feature of this splitting of the solution is that the relatively simpler planar part of the solution has been separated out. This implies that the second solution, $\varphi^{(2)}$, is effective only within the Mach cones from the wing leading edge-body junctions and is at most a constant outside of this region.

4.3 Case IIa: The "Flat Plate" Solution, $\varphi^{(1)}$.

The "Flat Plate" solution, $\varphi^{(1)}$, or the corresponding pressure component, $\varphi_z^{(1)}$, is obtained in the same manner as in the planar leading edge problem, Case Ia; and the results are relatively simpler, being expressible in closed form. In order to obtain the pressure distribution on the surface, $s\psi^{(1)}$ may conveniently be handled in Cartesian coordinates.

$$s\cdot\psi^{(1)}(x,y;s) = \frac{W_z}{\pi s}\int_1^\infty \left(1+\frac{1}{\xi^2}\right)\left\{K_0\left(s\sqrt{(x-\xi)^2+y^2}\right)+K_0\left(s\sqrt{(x+\xi)^2+y^2}\right)\right\}d\xi \qquad (4.4b)$$

Regions of influence similar to those shown in Figure 3.4 in Part III are defined by the inverse Laplace transform of the Green's function within the curly bracket of Equation (4.4b)*. The pressure distribution on the surface of the "Flat Plate" configuration follows.

In Region I, $(x - z) > 1$

$$\frac{\varphi_z^{(1)}(x,0,z)}{W_z} = \frac{1}{\pi}\left\{\int_{(x-z)}^{x}\left(1+\frac{1}{\xi^2}\right)\frac{d\xi}{\sqrt{z^2-(x-\xi)^2}} + \int_x^{(x+z)}\left(1+\frac{1}{\xi^2}\right)\frac{d\xi}{\sqrt{z^2-(x-\xi)^2}}\right\} \qquad (4.6a)$$

$$= 1 + \frac{1}{\pi}\int_0^\infty\left\{\frac{1}{(x-z\tau)^2}+\frac{1}{(x+z\tau)^2}\right\}\frac{d\tau}{\sqrt{1-\tau^2}}$$

which is an elementary integral.

For $z = 1$,

$$\frac{\varphi_z^{(1)}(x,0,1)}{W_z} = 1 + \frac{x}{(x^2-1)\sqrt{x^2-1}} \qquad (4.6b)$$

which may also be obtained by conical methods.

In Region II, $-1 < (x - z) < 1$

---

*The solution, $\varphi^{(1)}$ or $\varphi_z^{(1)}$ may also be obtained using the addition formula and the inverse Laplace transforms given in Table II; the details are believed to have more than academic interest but will not be presented here.

$$\frac{\varphi_z^{(1)}(x,0,z)}{W_2} = \frac{1}{\pi}\int_1^{(x+z)}\left(1+\frac{1}{\xi^2}\right)\frac{d\xi}{\sqrt{z^2-(x-\xi)^2}} \tag{4.7a}$$

For example, if this integral is evaluated,

$$\frac{\varphi_z^{(1)}(x,0,z)}{W_2} = \frac{1}{\pi}\left\{\frac{\pi}{2} + \frac{\sqrt{z^2-(x-1)^2}}{(z^2-x^2)} + \sin^{-1}\left(\frac{x-1}{z}\right) + \right.$$
$$\left. + \frac{x}{(z^2-x^2)\sqrt{z^2-x^2}}\ln\left(\frac{z^2-x(x-1)-\sqrt{(z^2-x^2)[z^2-(x-1)^2]}}{z}\right)\right\} x<z$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{2} + \frac{(x^2+1)}{3x^2}\sqrt{z^2-(x-1)^2} + \sin^{-1}\left(\frac{x-1}{z}\right)\right\},\quad x=z$$

(4.7b)

$$= \frac{1}{\pi}\left\{\frac{\pi}{2} - \frac{\sqrt{z^2-(x-1)^2}}{(x^2-z^2)} + \sin^{-1}\left(\frac{x-1}{z}\right) + \right.$$
$$\left. + \frac{x}{(x^2-z^2)\sqrt{x^2-z^2}}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{z^2-x(x-1)}{z}\right)\right]\right\},\quad x>z$$

In <u>Region III</u>, $(x-z) < -1$

$$\frac{\varphi_z^{(1)}(x,0,z)}{W_2} = \frac{1}{\pi}\left\{\int_1^{(x+z)}\left(1+\frac{1}{\xi^2}\right)\frac{d\xi}{\sqrt{z^2-(x-\xi)^2}} + \int_1^{(z-x)}\left(1+\frac{1}{\xi^2}\right)\frac{d\xi}{\sqrt{z^2-(x+\xi)^2}}\right\} \tag{4.8c}$$

The first integral on the right-hand side is evaluated above in equation (4.7b) for $x < z$; the second elementary integral is the "reflected" solution. In regions II and III, the solution is desired on the "body", i.e. on a circular cylinder of radius $r = 1$. The ranges of integration are easily found from the inverse Laplace transforms (3.6) and (3.7), obtained from Table II and most conveniently written in cylindrical polar coordinates.

In Region II,

$$\frac{\varphi_z^{(1)}(r,\theta,z)}{W_z} = \frac{1}{\pi}\int_1^{r\cos\theta+\sqrt{z^2-r^2\sin^2\theta}} \left(1+\frac{1}{r'^2}\right)\frac{dr'}{\sqrt{z^2-(r^2+r'^2-2rr'\cos\theta)}} \tag{4.7c}$$

In Region III,

$$\frac{\varphi_z^{(1)}(r,\theta,z)}{W_z} = \frac{1}{\pi}\left\{\int_1^{r\cos\theta+\sqrt{z^2-r^2\sin^2\theta}} \left(1+\frac{1}{r'^2}\right)\frac{dr'}{\sqrt{z^2-(r^2+r'^2-2rr'\cos\theta)}} + \right.$$
$$\left. + \int_1^{-r\cos\theta+\sqrt{z^2-r^2\sin^2\theta}} \left(1+\frac{1}{r'^2}\right)\frac{dr'}{\sqrt{z^2-(r^2+r'^2+2rr'\cos\theta)}}\right\} \tag{4.8b}$$

For example, at r = 1,

$$\frac{\varphi_z^{(1)}(1,\frac{\pi}{2},z)}{W_z} = \frac{2}{\pi}\int_1^{\sqrt{z^2-1}}\left(1+\frac{1}{r'^2}\right)\frac{dr'}{\sqrt{(z^2-1)-r'^2}}$$
$$= 1+\frac{2}{\pi}\left\{\frac{\sqrt{z^2-1}}{(z^2-1)} - \sin^{-1}\left(\frac{1}{\sqrt{z^2-1}}\right)\right\}, \quad z \geq \sqrt{2} \tag{4.8c}$$

The pressure distribution given by the above relations are shown in Figures 4.7 and 4.8*. Significantly, the pressure distribution obtained from this "Flat Plate" part of the complete solution approaches the asymptotic value rapidly in the downstream direction on the "body" and in the vicinity of the "body" on the wing (Cf. Discussion for Case Ia and corresponding Figure (3.7))*.

An additional solution, which may be obtained immediately, will be important for later discussions. The "Flat Plate" configuration (Figure 4.1) is modified now by inserting a semi-infinite plane barrier at x = 1, parallel to the y-z plane downstream of the leading edge-body junction (Figure 4.2).

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*These figures are at the end of this work following Table II.

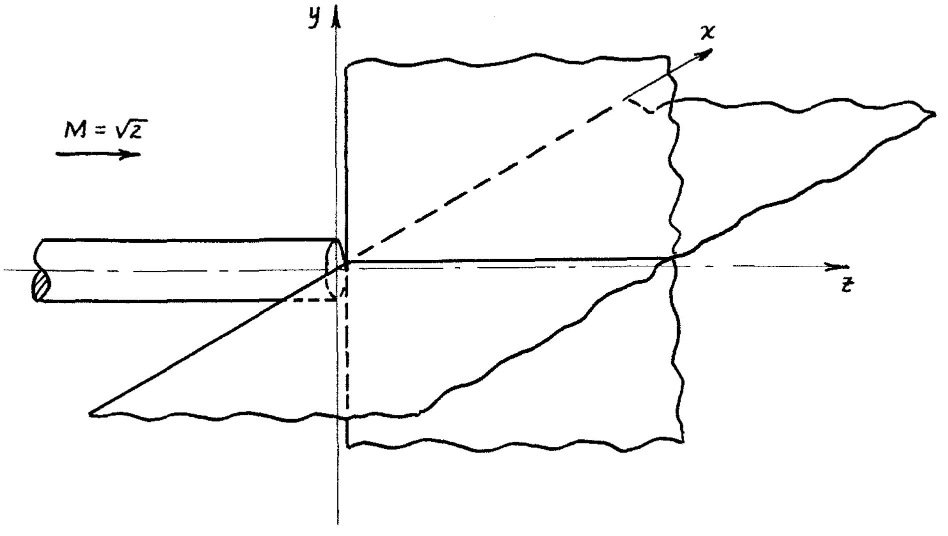


Figure 4.2. Modified "Flat Plate" Configuration: Solution φ^* for $x \geq 1$.

This does not affect the solution, $\varphi^{(1)}$, outside of the Mach cone from the leading edge-body junction; but inside of this region of influence for $x \geq 1$, the inserted plane simulates the limiting case of a body with infinite radius. Thus, in this region, the solution (which will be called φ^*) obtained, say in terms of the pressure, may be interpreted as an upperbound (at least in Region II) to the complete solution of Case IIa. The pressure is given by twice the integral of equation (4.7a) or twice the first form ($x < z$) of equation (4.7b). The pressure distribution is shown in Figures 4.7 and 4.8.

4.4 Case IIa: Discussion of the "Body" Solution, $\varphi^{(2)}$

The "body" solution may be written formally by operating on equation (4.5a) with the inverse Laplace transform (2.15),

$$\varphi^{(2)}(r,\theta,z) = -\frac{2\bar{W}_2}{\pi}\int_1^\infty \left(1+\frac{1}{r'^2}\right)dr'\left[\frac{1}{2\pi i}\int_\Gamma\left\{\frac{I_0'(s)}{K_0'(s)}K_0(sr)K_0(sr') + \right.\right. \tag{4.9}$$
$$\left.\left. + 2\sum_{n=1}^\infty \frac{I_{2n}'(s)}{K_{2n}'(s)}K_{2n}(sr)K_{2n}(sr')\cos 2n\theta\right\}\frac{ds}{s}\right]$$

and the solution is obtained by quadratures. The integrations, however, appear formidable. Yet, in order to indicate a method of solution the procedure will be outlined for the complete solution <u>on the body</u>, which solution will be "simpler" but still retains the difficulties. Consider the transformed solution in the form of equation (4.2b) for r = 1,

$$\Psi(1,\theta;s) = \frac{2\bar{W}_2}{\pi s}\int_1^\infty\left(1+\frac{1}{r'^2}\right)\left[K_0(sr')\left\{I_0(s) - \frac{I_0'(s)}{K_0'(s)}\right\} + \right. \tag{4.2d}$$
$$\left. + 2\sum_{n=1}^\infty K_{2n}(sr')\left\{I_{2n}(s) - \frac{I_{2n}'(s)}{K_{2n}'(s)}K_{2n}(s)\right\}\cos 2n\theta\right]dr'$$

Since the Wronskian (Ref. 19)

$$I_m(s)K_m'(s) - K_m(s)I_m'(s) = -\frac{1}{s} \tag{4.10}$$

$$\Psi(1,\theta;s) = -\frac{2W_2}{\pi s^2}\int_1^\infty\left(1+\frac{1}{r'^2}\right)\left\{\frac{K_0(sr')}{K_0'(s)} + 2\sum_{n=1}^\infty\frac{K_{2n}(sr')}{K_{2n}'(s)}\cos 2n\theta\right]dr' \tag{4.2e}$$

Then, assuming the uniform convergence of the series, the inverse Laplace transform required for the pressure is

$$\mathcal{L}^{-1}\left\{\frac{K_{2n}(sr')}{s\,K_{2n}'(s)}\,;\,z\right\} \equiv \frac{1}{2\pi i}\int_\Gamma e^{sz}\left\{\frac{K_{2n}(sr')}{s\,K_{2n}'(s)}\right\}ds\ ,\quad r'\geq 1 \tag{4.11}$$

A possible method (Ref. 11) of determining this is to rewrite the integral over Γ in the complex s-plane (with a cut on the negative real axis to obtain single-valuedness) as an integral on the negative real axis plus additional terms by Cauchy's Theorem of Residues. Then,

$$\mathcal{L}^{-1}\left\{\frac{K_{2n}(sr')}{sK_{2n}'(s)}\,;\,z\right\} = -\int_0^\infty e^{-z\sigma}\left\{\frac{K_{2n}(\sigma r')I_{2n}'(\sigma)-K_{2n}'(\sigma)I_{2n}(\sigma r')}{[K_{2n}'(\sigma)]^2+\pi^2\,[I_{2n}'(\sigma)]^2}\right\}\frac{d\sigma}{\sigma} + \tag{4.12}$$

$$+\sum \operatorname{Res}\left\{\frac{K_{2n}(sr')}{sK_{2n}'(s)}\right\}$$

There are two complications here: 1) the integral probably can be evaluated only by numerical methods, and 2) the residues of the function (although the singularities, which are in the left half plane, are simple poles) can be evaluated only after determining the location and number, which are of the order of 2n, of poles. This is a formidable calculation procedure. A similar procedure must be carried out to determine explicitly the "body" solution, (4.9). Thus, the necessity for approximate methods of solution is clear.

4.5 Case IIa: Approximate "Body" Solution, $\varphi^{(2)}$.

The difficulties in obtaining an exact "body" solution are now apparent. Also, the study of the "Flat Plate" solution, $\varphi^{(1)}$, implies that the main contribution to the pressure of the "body" solution will be found near the leading edge-body junction. Keeping these ideas in mind an approximation to the second Green's function (2.28) for the region exterior to the unit circle is constructed. Consider a Green's function of the form

$$G(r,\theta;r',\theta') = K_0\left(s\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\right) + A\,K_0\left(s\sqrt{r^2+\frac{1}{r'^2}-\frac{2r}{r'}\cos(\theta-\theta')}\right) \qquad (4.13a)$$

where A is to be determined. It is clear from section 2.5 that the exact Green's function (2.28) cannot be put in this form since the second function on the right hand side of equation (4.13a) is the fundamental solution (with respect to P(r,Θ)) placed at the inverse point of Q(r',Θ') with respect to the unit circle. But, by the addition formula (Cf. equation (2.24))

$$\begin{aligned} K_0\left(s\sqrt{r^2+\frac{1}{r'^2}-\frac{2r}{r'}\cos(\theta-\theta')}\right) &= K_0(sr)\,I_0\left(\frac{s}{r'}\right) + \\ &+ 2\sum_{n=1}^{\infty} K_n(sr)\,I_n\left(\frac{s}{r'}\right)\cos n(\theta-\theta'), \quad r > \frac{1}{r'} \end{aligned} \qquad (4.14)$$

Replacing this form in equation (4.13a) and comparing with the exact Green's function (2.28) such that the first term is exactly matched, the arbitrary function A becomes

$$A = -\frac{K_0(sr')}{I_0\left(\frac{s}{r'}\right)}\,\frac{I_0'(s)}{K_0'(s)} \qquad (4.15)$$

Then, the approximate Green's function for the region exterior to the unit circle is

$$G(r,\theta;r',\theta') = K_o\left(s\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\right) -$$

$$- \frac{K_o(sr')}{I_o(\frac{s}{r'})}\frac{I_o'(s)}{K_o'(s)} K_o\left(s\sqrt{r^2+\frac{1}{r'^2}-\frac{2r}{r'}\cos(\theta-\theta')}\right) \qquad (4.16)$$

This approximate Green's function satisfies the differential equation (2.12b) with respect to the point P(r,Θ) but is not symmetric in (r,Θ; r',Θ') and does not satisfy the boundary conditions. However, comparing term by term, the difference in the Green's function occurs only under the summation sign where

$$\frac{I_n'(s)}{K_n'(s)} K_n(sr)K_n(sr')\cos n(\theta-\theta') \qquad (4.17)$$

for the exact Green's function has been replaced

$$\frac{I_n(\frac{s}{r'})}{I_o(\frac{s}{r'})}\frac{I_o'(s)}{K_o'(s)} K_n(sr)K_o(sr')\cos n(\theta-\theta') \qquad (4.18)$$

for the approximate Green's function. Then, for fixed r' and <u>large</u> s (for $\Re\ell s>0$), terms (4.18) approach terms (4.17), and the approximation is best there*. This may be shown by using the asymptotic expansions for the modified Bessel functions (Ref. 19). More will be said about this approximation in section 4.6.

The approximate Green's function for the region and boundary of Case IIa (Fig. 2.7) may be written immediately

$$G(r,\theta;r',\theta') = \left\{K_o\left(s\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\right)+K_o\left(s\sqrt{r^2+r'^2-2rr'\cos(\theta+\theta')}\right)\right\} - \qquad (4.13b)$$

$$- \frac{K_o(sr')}{I_o(\frac{s}{r'})}\frac{I_o'(s)}{K_o'(s)}\left\{K_o\left(s\sqrt{r^2+\frac{1}{r'^2}-\frac{2r}{r'}\cos(\theta-\theta')}\right)+K_o\left(s\sqrt{r^2+\frac{1}{r'^2}-\frac{2r}{r'}\cos(\theta+\theta')}\right)\right\}$$

---

*Other approximate Green's functions may be set up in a similar manner, e.g. a distribution of singularities along a line or in a region within the circle in the neighborhood of the inverse point to Q(r',Θ').

The approximate Green's function (4.13b) is, thus, a good representation for large s (and r' bounded); this implies by a known theorem in Laplace transform theory (Ref. 11) that the solution obtained using this Green's function is a good approximation near the leading edge-body junction. This solution will now be obtained.

The transformed approximate "body" solution corresponding to equation (4.5a) is

$$\psi^{(2)}(r,\theta;s) = -\frac{W_2}{\pi s}\int_1^\infty \left(1+\frac{1}{r'^2}\right)\frac{K_0(sr')}{I_0(\frac{s}{r'})}\frac{I_0'(s)}{K_0'(s)}\left\{K_0\left(s\sqrt{r^2+\frac{1}{r'^2}-\frac{2r}{r'}\cos\theta}\right) + \right.$$
$$\left. + K_0\left(s\sqrt{r^2+\frac{1}{r'^2}+\frac{2r}{r'}\cos\theta}\right)\right\}dr' \tag{4.19}$$

The second term in the curly bracket of equation (4.19) corresponds to the "reflected" solution. Now, the solution for the pressure component, $\varphi_z^{(2)}$, may be obtained in Region II (Cf. Figure 3.4, Part III) where the approximation is best (the "reflected" solution does not enter here)*. Then, by the asymptotic expansions for the modified Bessel functions

$$-\frac{K_0(sr')}{I_0(\frac{s}{r'})}\frac{I_0'(s)}{K_0'(s)} \sim \frac{1}{r'}e^{-s\frac{(r'-1)^2}{r'}}\left\{1+O\left(\frac{1}{s}\right)\right\} \tag{4.20}$$

Neglecting the higher order terms,

$$-\frac{K_0(sr')}{I_0(\frac{s}{r'})}\frac{I_0'(s)}{K_0'(s)}K_0\left(s\sqrt{r^2+\frac{1}{r'^2}-\frac{2r}{r'}\cos\theta}\right) \sim \frac{1}{r'}e^{-\frac{s(r'-1)^2}{r'}}K_0\left(s\sqrt{r^2+\frac{1}{r'^2}-\frac{2r}{r'}\cos\theta}\right) \tag{4.21}$$

The solution will be carried out for $\theta = 0$ (on the x-z plane). Applying the inverse Laplace transform to the right side of equation (4.21),

---

*Cf. discussion in paragraph following equation (4.23b)

$$\mathcal{L}^{-1}\left\{\frac{1}{r'}e^{-\frac{s(r'-1)^2}{r'}}K_o(s|r-\tfrac{1}{r'}|)\,;\bar{z}\right\} = 0 \qquad , \; r' > (\bar{z}-r+z)$$

$$= \frac{1}{\sqrt{[\bar{z}r'-(r'-1)^2]^2-(rr'-1)^2}} \; , \; 1 < r' < (\bar{z}-r+z) \qquad (4.22)$$

Then, in Region II (in the plane of the wing)

$$\frac{\varphi_{\bar{z}}^{(2)}(r,0,\bar{z})}{W_z} = \frac{1}{\pi}\int_1^{(\bar{z}-r+z)}\left(1+\frac{1}{r'^2}\right)\frac{dr'}{\sqrt{[\bar{z}r'-(r'-1)^2]^2-(rr'-1)^2}} \qquad (4.23a)$$

As to be expected the integral vanishes for $(r-z) = 1$, $z \neq 0$. This is an elliptic type integral which may be expressed in terms of the standard complete and <u>incomplete</u> elliptic integrals of the first and third kinds by known methods of reduction (Cf. Ref. following equation (3.8a)). For $r = 1$, equation (4.23a) reduces to (See Appendix)

$$\frac{\varphi_{\bar{z}}^{(2)}(1,0,\bar{z})}{W_z} = \frac{1}{\pi}\int_1^{(\bar{z}+1)}\left(1+\frac{1}{r'^2}\right)\frac{dr'}{\sqrt{[\bar{z}r'-(r'-1)^2]^2-(r'-1)^2}}$$

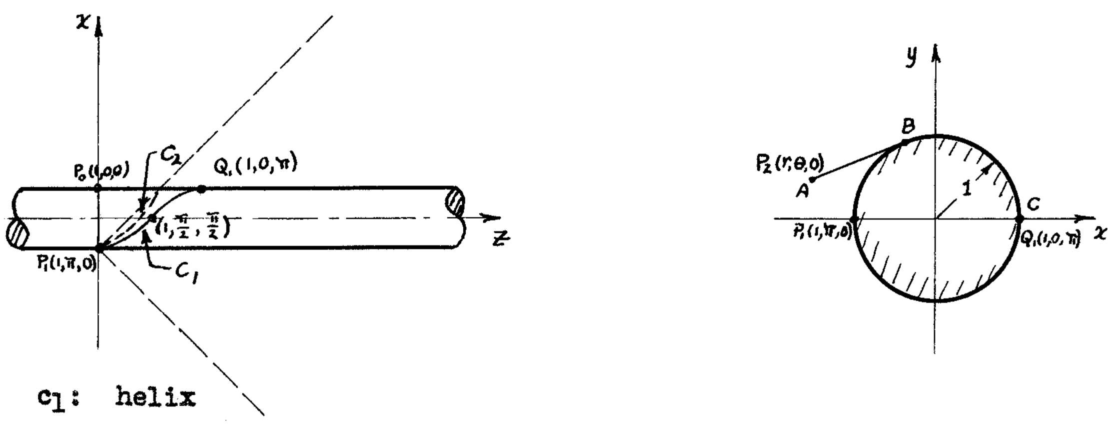
$$= \frac{1}{\pi}\int_0^1\left(1+\frac{1}{(1+\bar{z}\tau)^2}\right)\frac{d\tau}{\sqrt{[1+(1+\bar{z})\tau-\bar{z}\tau^2](1+\bar{z}\tau)(1-\tau)}} \qquad (4.23b)$$

The distribution is shown in Figure 4.8* for $0 \leq z \leq 2$. The solution approaches zero as $z \to \infty$ (as it should) but quantitatively the result is meaningless for large z. Remarkably enough, the asymptotic value $\left(\frac{\varphi_{\bar{z}}}{W_z} = 1\right)$ is essentially reached by the complete solution at $z = 2$ (Figure 4.5)*; $\varphi_{\bar{z}} = \varphi_{\bar{z}}^{(1)} + \varphi_{\bar{z}}^{(2)}$.

---

*This figure follows Table II at the end of this work. Also Fig. 4.7 gives the spanwise distribution.

Due to the amount of labor involved, further calculations on the body have not been made.* However, it is clear that the complete solution, $\varphi_z = \varphi_z^{(1)} + \varphi_z^{(2)}$, approaches the asymptotic value (Cf. discussion following equation (3.10c), Part III) very rapidly in the downstream direction on the body and in the vicinity of the body on the wing. Some remarks should be made here concerning the "reflected" solution which has not been obtained quantitatively. It can be indicated in at least two ways (also from the exact solution, equation (4.2c) if the solution were known) that the region of influence (on the body) of a disturbance initiating at the leading edge-body junction, say at $P(r,\theta,z) = P_1(1, \pi, 0)$, lies downstream of a helix of lead $= 2\pi$, i.e. the disturbance is first felt on the diametrically opposite element of the circular cylinder at $Q(r,\theta,z) = Q_1(1, 0, \pi)$, a distance π in the z-direction from the initial point** (Fig. 4.3a). This implies that



c_1: helix

c_2: intersection of Mach cone from $(1, \pi, 0)$ and circular cylinder

a) Propagation of a disturbance on a circular cylinder

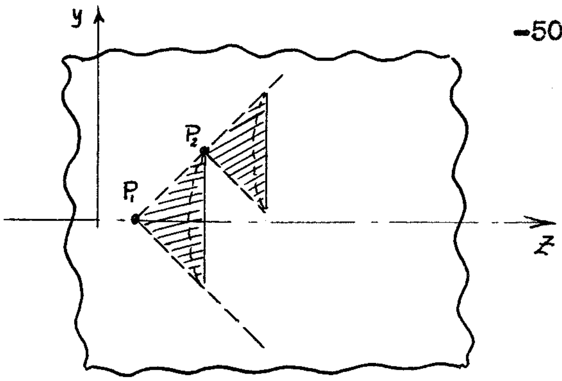
b) Two-dimensional analogy: $\varphi_{xx} + \varphi_{yy} - \varphi_{tt} = 0$

Figure 4.3 Region of Influence of "Reflected" Solution

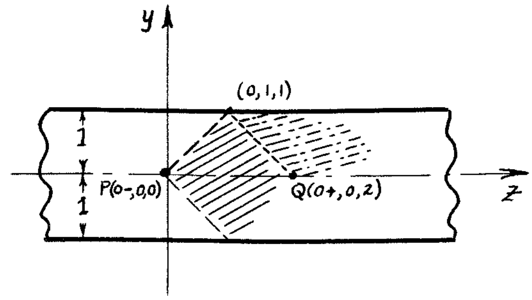
*Cf. footnote following equation (3.10c)

**This was pointed out by Stewart.

the solution ϕ on the body is zero upstream of the two helices initiating from $P_0(1,0,0)$ and $P_1(1,\pi,0)$. This region of influence on the cylinder may be argued by considering initially the <u>local</u> region of influence of a disturbance at a point P on the circular cylinder and proceeding step by step around the cylinder. The argument may be initiated: Consider a disturbance, say a point source, at $P(x,y,z) = P(0,y,z)$ on an infinite plane barrier (say, the y-z plane). The plane may be replaced by an image source and the region of influence is the same as that of a point source (twice the strength) in an infinite region, i.e. it is a cone of the same included angle, $\frac{\pi}{2}$. Then for a point source at $P_1(1,\pi,0)$ on a circular cylinder (Fig. 4.3a) the region of influence on the circular cylinder <u>in the neighborhood of P_1</u> is identical to that of an infinite plane barrier. Then consider a point P_2 (Fig. 3.4c) in the neighborhood of and on the cone from P_1 and hence consider the region of influence of a disturbance at P_2. Continuing this process around the cylinder yields the helical region of influence of a disturbance at P_1 on the circular cylinder barrier. Another example is shown in Fig. 4.3d for a finite width flat plate barrier of zero thickness and width = 2. The region of influence of the point P on <u>both sides</u> of the plate is indicated by the cross-hatching. The above reasoning, however, does not yield the strength at Q of the disturbance at P. These results also follow by direct analogy with the two-dimensional diffraction problem (Fig. 4.3b) with a circular barrier. Then, a consequence of Fermat's Variational Principle of Least Time (Ref. 15) is that the time required for a light signal to travel from a point P to another point Q is a



c) Local region of influence of P_1 on circular cylinder



d) Example: finite width plane barrier

Figure 4.3 Region of Influence of "Reflected" Solution

minimum, e.g. between P_1 and Q_1, the time is $\frac{\pi}{c}$ (c = velocity of light = 1 in this analogy), and between P_2 and Q_1, $\frac{\overline{ABC}}{c}$.

An estimate of the complete solution on the body (r = 1) for large z is readily made, referring to the transformed solution for the pressure component, $s\cdot\psi$, from equation (4.2d) and considering the behavior of the Green's function for small s (Ref. 11). Then

$$K_0'(s) = -\frac{1}{s} + O(s\,\ln s)$$

$$K_{2n}(sr') = \frac{(2n-1)!}{(sr')^{2n}} + O\left(\frac{1}{s^{2n-2}}\right) \tag{4.24}$$

$$K_{2n}'(s) = -\frac{(2n)!}{s^{2n+1}} + O\left(\frac{1}{s^{2n-1}}\right)$$

and

$$\left\{\frac{K_0(sr')}{K_0'(s)} + 2\sum_{n=1}^{\infty}\frac{K_{2n}(sr')}{K_{2n}'(s)}\cos 2n\theta\right\} \doteq -s\left\{K_0(sr') + 2\sum_{n=1}^{\infty}\frac{\cos 2n\theta}{2n\,(r')^{2n}}\right\} \tag{4.25}$$

Since $K_0(sr')$ is the dominant term for small s ($R\ell\, s > 0$), equation (4.22) becomes

$$\frac{s\cdot\psi(1,\theta;s)}{W_z} = \frac{2}{\pi}\int_1^{\infty}\left(1+\frac{1}{r'^2}\right)K_0(sr')\,dr' \tag{4.2e}$$

Taking the inverse Laplace transform of $K_0(sr')$, the pressure component is

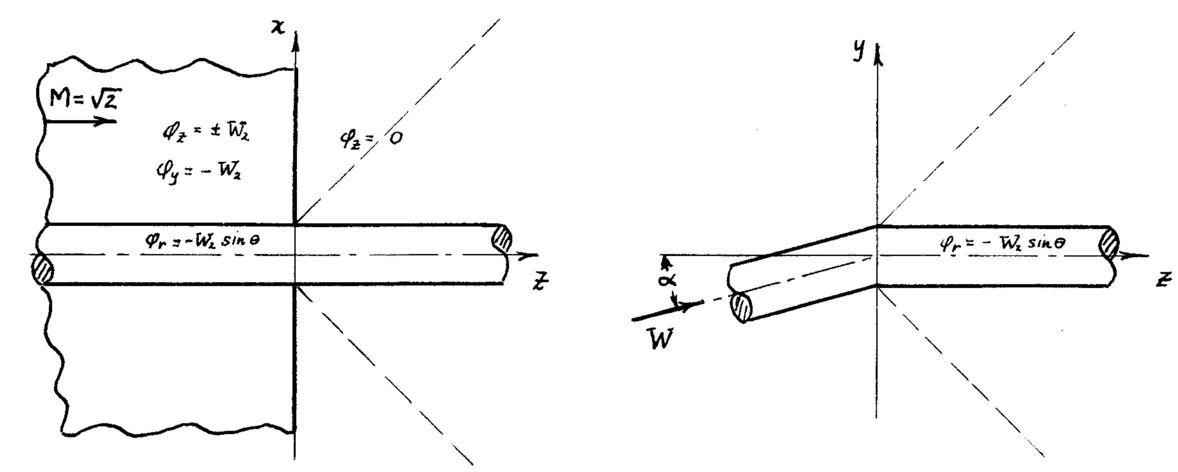
$$\frac{\varphi_z(1,\theta,z)}{W_z} = \frac{2}{\pi}\int_1^{z}\left(1+\frac{1}{r'^2}\right)\frac{dr'}{\sqrt{z^2-r'^2}} = 1+\frac{2}{\pi}\,\frac{\sqrt{z^2-1}}{z^2} \tag{4.26}$$

and

$$\lim_{z\to\infty}\frac{\varphi_z(1,\theta,z)}{W_z} = 1 \tag{4.27}$$

4.6 A Trailing Edge Problem

Following, again, the suggestions obtained in the study of planar systems, in Part III, the trailing edge problem considered here corresponds to the problem discussed in Section 3.3. The configuration and boundary conditions are shown in Figure 4.4. (Cf. Figure 3.5).



a) Boundary Conditions

b) An Approximate Analogy

Figure 4.4. A Trailing Edge Problem

In this case, however, an exactly analogous problem, for the body alone, such as the one found in the planar case, cannot be formulated. However, an approximate analogy for low supersonic Mach numbers is given by an infinite circular cylinder with a discontinuity in slope at one section (Figure 4.4b). This is the same problem as that of the external flow past an open-ended tube at incidence, which problem has been solved by G. N. Ward (Cf. Ref. 14). More complicated problems such as the wing tip effect on the body and wing $\rightarrow$ body $\rightarrow$ afterfin problems will require additional study of the flow variables, i.e. u(sidewash) and v(upwash), followed by proper, ingenious approximations.

Although the analogy problem given here is not exact indicating only an order of magnitude for the pressure distribution, it is instructive in another sense -- that of giving an estimate of the error for the solution using the approximate Green's function (4.13b). The problem is formulated for the wave equation (2.5b):

Initial conditions: $\varphi = \varphi_z = 0$ in the plane $z = 0$

Boundary conditions: i) $\varphi_r(1, \theta, z) = -W_2 \sin\theta$

ii) $\frac{1}{r}\varphi_\theta(r, \frac{\pi}{2}, z) = 0$

Then, the transformed conditions are

i) $\psi_r(1, \theta; s) = -\frac{W_2}{s}\sin\theta$

ii) $\frac{1}{r}\psi_\theta(r, \frac{\pi}{2}; s) = 0$

and the solution by the Green's function method is given by equation (2.18) where the contour C is the unit circle and the Green's function by equation (2.28) for the point Q $(1, \theta')$ on the circle is

$$G(r, \theta; 1, \theta') = -\frac{1}{s}\left\{\frac{K_0(sr)}{K_0'(s)} + 2\sum_{n=1}^{\infty}\frac{K_n(sr)}{K_n'(s)}\cos n(\theta - \theta')\right\} \tag{4.28}$$

and by equation (2.18) the solution may be written

$$\begin{aligned}\psi(r, \theta; s) &= -\frac{W_2}{2\pi s}\int_0^{2\pi} G(r, \theta; 1, \theta')\sin\theta' d\theta' \\ &= -\frac{W_2\, K_1(sr)}{s^2 K_1'(s)}\sin\theta\end{aligned} \tag{4.29}$$

which is the same transformed solution as given by Ward* (Cf. Ref. 14)

In precisely the same manner, if the approximate Green's function (4.13b) (expanded by the addition formula) is used the approximate transformed solution corresponding to equation (4.29) becomes

*There is a difference in defining θ.

$$\Psi(r,\theta;s) = -\frac{W_2\, K_1(sr)\, I_1(s)}{s^2\, K_0'(s)\, I_0(s)} \sin\theta \tag{4.30}$$

An exact comparison of the two solutions would necessitate carrying out the approximate solution, completely, in terms of the pressure, say. Actually, a comparison of the first few terms of the asymptotic forms of the two solutions should be sufficient. Then, simply comparing the denominator, $K_1'(s)$, with the corresponding denominator of the approximate solution (4.30),

For the <u>exact solution</u>,

$$K_1'(s) \sim \left(\frac{\pi}{2s}\right)^{1/2} e^{-s}\left\{1 + \frac{1}{2}\left[\frac{7}{4s} - \frac{7}{2!(4s)^2} + \cdots\right]\right\} \tag{4.31}$$

For the <u>approximate solution</u>,

$$\frac{K_0'(s)\, I_0(s)}{I_1(s)} \sim \left(\frac{\pi}{2s}\right)^{1/2} e^{-s}\left\{1 + \frac{1}{2}\left[\frac{7}{4s} - \frac{3}{2!(4s)^2} + \cdots\right]\right\} \tag{4.32}$$

The difference occurs beginning with the third term of the expansion.

4.7 Discussion

Clearly, the Laplace transform method of solution is most suitable for configurations whose boundary conditions may be expressed independent of the variable operated upon, i.e. the method is essentially a form of the well known method of separation of variables. Thus, configurations such as those which come up in the consideration of the body nose, e.g. Case IIb, are not natural problems for this method. For Case IIb, even conical coordinates do not make the problem susceptible to transform methods; the source or doublet distribution method probably should be applied here.

An example of a configuration which may readily be handled by transform methods is shown in Figure 4.5. The body is a cylindrical tube (Cf. Ward, Ref. 14).

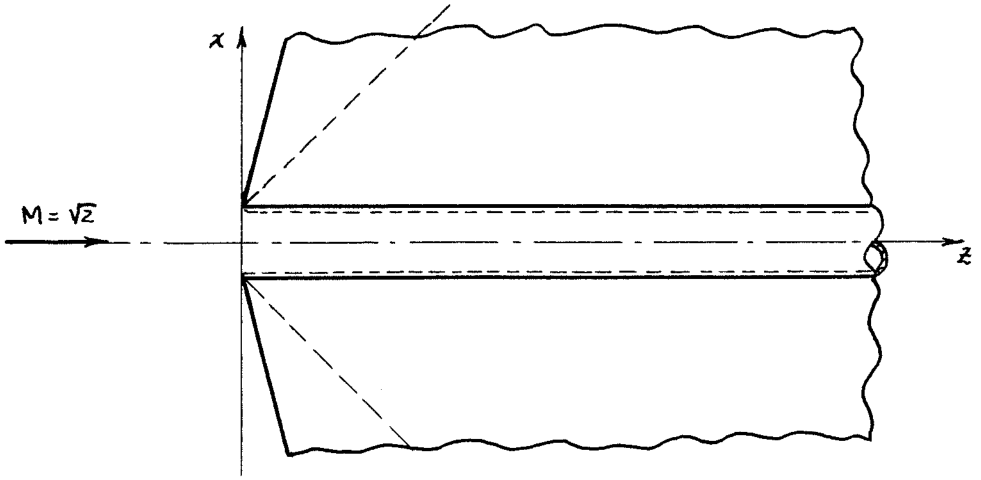


Figure 4.5 A Leading Edge Problem

Another example of a configuration which possibly may be handled by transform methods is shown in Figure 4.6 (Cf. Figure 1.3). Here, the probable necessity for expressing the boundary conditions by careful approximations, due to the nature of the body, will lead to some labor.

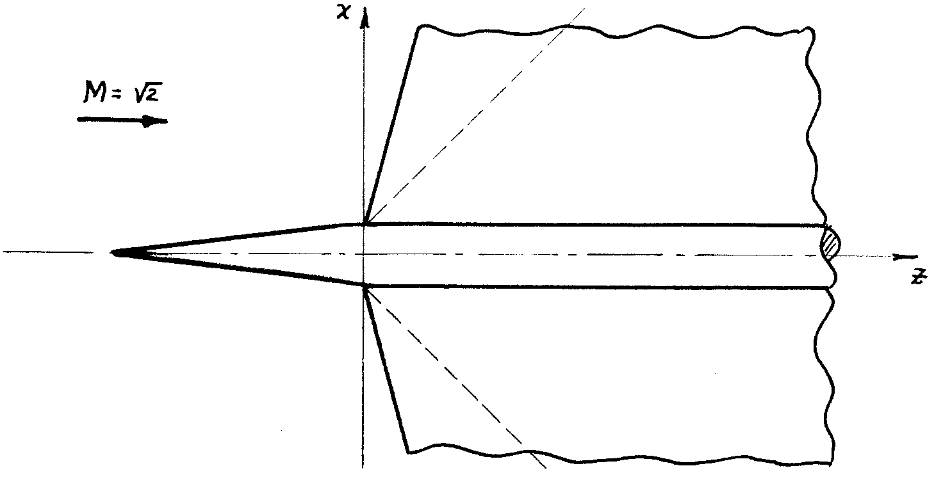


Figure 4.6 A Body Nose Problem

V. GENERAL CONSIDERATIONS

5.1 Remarks on Application

In the application of some of the results of the present study integrated values such as total lift, drag, moment, etc. for a particular configuration are of interest. Generally, in this respect further numerical work is necessary. However, some useful observations may be made.

In his work on the lift of a wide delta wing (supersonic leading edges) in supersonic flow, Puckett (Ref. 25) obtains an interesting result that the total lift coefficient, C_L (total lift divided by the free stream dynamic pressure), is the same value as that for the two dimensional flat plate at the same incidence, α. For $M = \sqrt{2}$, this states that

$$C_L = 4\alpha \tag{5.1}$$

This result has been generalized by Lagerstrom (Ref. 6) for any wing having an arbitrary incidence distribution with supersonic edges whose leading edge is perpendicular to the flow direction and states that

$$C_L = 4\alpha_{av} = \frac{4}{S}\iint_{wing} \alpha(\xi,\zeta)\,d\xi\,d\zeta \tag{5.2}$$

where α_{av} is the average incidence distribution, S is the area of the wing and the integral is taken over the wing nominally in the x-z plane. Clearly, this result applies to the planar problem, Case Ia, discussed in Part III -- either for the semi-infinite flat plate or the flat plate cut-off at a finite distance from the leading edge such that the trailing edge is normal to the flow direction. This may be checked for a chord length of one (1), say, with the incidence distribution (constant in the flow direction) shown in Figure 5.1.

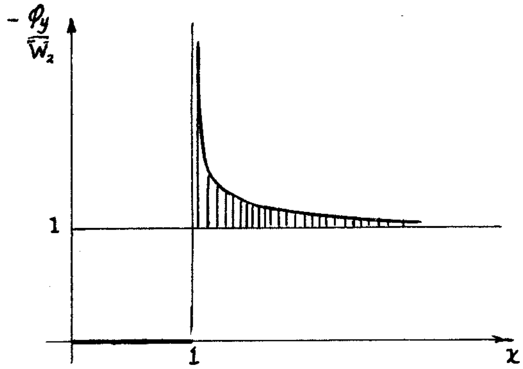


Figure 5.1 Incidence Distribution for Case Ia

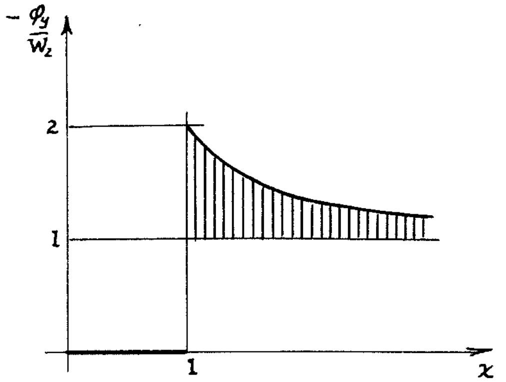


Figure 5.2 Incidence Distribution for Case IIa: the "Flat Plate" solution

Now, the integral of the cross-hatched area,

$$\int_1^{\infty}\left(\frac{\xi}{\sqrt{\xi^2-1}}-1\right)d\xi = \lim_{\xi\to\infty}\left[\sqrt{\xi^2-1}-\xi\right]\Big|_1^{\xi} = 1 \tag{5.3}$$

which implies that $\alpha_{av.} = 1$ for the incidence problem. Similarly, considering the "Flat Plate" part of Case IIa whose incidence distribution is given by Figure 5.2,

$$\int_1^{\infty}\left(\frac{1}{\xi^2}\right)d\xi = 1 \tag{5.4}$$

which implies that $\alpha_{av.} = 1$ for the incidence problem.

Hence, the following observation which has been applied loosely to non-planar systems, is made:

1) For planar systems which satisfy Lagerstrom's conditions, the total lift on the wing-body combination is equal to the sum of the lift on the body alone plus the lift on the "wing" alone

(the portion covered by the body is assumed to be at the incidence of the body).

2) For non-planar systems with cylindrical bodies downstream of the wing leading edge-body junction, satisfying Lagerstrom's conditions for the wing, the total lift is given by 1) for the "Flat Plate" portion (Cf. Figure 4.1) of the system. Since the "Body" portion of the solution (Cf. Figure 4.7) for Case IIa yields positive lift, the total lift on the wing-body combination is greater than*the sum of the lift on body alone plus the lift on the "wing" alone. Additional calculations and numerical integration are necessary to obtain a quantitative estimate.

Generally, it is somewhat dangerous to speak in terms of the lift on the body alone and lift on the "wing" alone since the "wing" is defined rather arbitrarily for the leading edge problem (Cf. definition in parenthesis in 1), above). The actual interaction between wing and body is more complex than these statements alone indicate, particularly within the Mach cones from the leading edge-body junction, e.g. one might speak of the lift on the body ($z > 0$) due to the influence of the body ($z < 0$) on the wing.

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*Cf. Spreiter (Ref. 4) who has shown that this reads "less than" for lifting cases of Class A problems.

5.2 Further Studies

Due to the seemingly unavoidable mathematical complexities encountered in the study of non-planar systems, "local" experimental studies (in contrast to integrated information, such as total lift and drag) are sorely needed before a full scaled computational attack of many of these problems are made. Undoubtedly, viscous effects will necessitate a modification of the boundary conditions. For example, the boundary layer growth (or even separation) on a long axially-symmetric, nearly-cylindrical body, where the results of the present study might be applied, will modify the upwash field upstream of the attached wing or fin.

Certainly, the analytical work has just begun on non-planar problems.

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APPENDIX

The integral solution (4.23b) in Part IV takes the formidable form

$$\frac{\varphi_Z^{(2)}(1,0,z)}{W_2} = \frac{1}{\pi}\left[\frac{B}{g z} M_1 - \frac{1}{3(z+1)}\left\{z + \frac{B}{g}\left[\left(d_1 + \frac{d_2}{a_2}\right)M_1 + \frac{d_2(g-f)}{a_2}\left(a_1 M_2 - g a_2 M_3\right)\right]\right\}\right]$$

where

$$M_1 = \int_0^f \frac{d\sigma}{\sqrt{(1-\sigma^2)(1-k^2\sigma^2)}} - \int_0^1 \frac{d\sigma}{\sqrt{(1-\sigma^2)(1-k^2\sigma^2)}}$$

$$\equiv F(k,f) - K(k)$$

(Cf. equation (3.8b) in Part III)

$$M_2 = \int_0^f \frac{d\sigma}{(1-\lambda^2\sigma^2)\sqrt{(1-\sigma^2)(1-k^2\sigma^2)}} - \int_0^1 \frac{d\sigma}{(1-\lambda^2\sigma^2)\sqrt{(1-\sigma^2)(1-k^2\sigma^2)}}$$

$$\equiv \Pi(k,\lambda,f) - \Pi(k,\lambda,1)$$

$$M_3 = \int_{-f}^{\frac{g(f-1)}{1-g}} \frac{\xi\, d\xi}{(a_1^2 - a_2^2\xi^2)\sqrt{(1+m\xi^2)(1+n\xi^2)}}$$

an elementary integral,

and,

$$B = \frac{g-f}{\delta} \quad ; \quad f+g = \frac{\tau_1}{z\tau_2} + 1 \quad , \quad fg = -\frac{1}{z}$$

$$\tau_1 = \frac{1}{2z}\left(1+z+\sqrt{z^2+6z+1}\right) \quad , \quad \tau_2 = \frac{1}{2z}\left(1+z-\sqrt{z^2+6z+1}\right)$$

$$\delta^2 = (f^2+pf+q)(f^2+rf+s)$$

$$p = \left(\frac{1}{z} - \tau_1\right) , \quad q = -\frac{\tau_1}{z} \quad , \quad r = -(1+\tau_2) \quad , \quad s = \tau_2$$

$$m = \frac{(g^2+pg+q)}{(f^2+pf+q)} \quad , \quad n = \frac{(g^2+rg+s)}{(f^2+rf+s)}$$

$$k^2 = \frac{-m}{-n} \quad , \quad \lambda^2 = \frac{a_2^2}{g^2 a_1^2}$$

$$a_1 = f + \frac{1}{z} \quad , \quad a_2 = g + \frac{1}{z}$$

$$d_1 = 1 + \frac{2}{z} \quad , \quad d_2 = -\left(1 + \frac{4}{z} + \frac{5}{z^2}\right)$$

TABLE I

Laplace Transforms

If $\varphi(x,y,z),\ z>0$ has a Laplace transform with respect to the variable z, then define

$$\psi(x,y;s) = \int_0^\infty e^{-sz}\varphi(x,y,z)\,dz$$

	Given Function of z z > 0	Laplace Transform
1.	b (constant)	$\frac{b}{s}$
2.	φ_{xx}	ψ_{xx}
3.	φ_{yy}	ψ_{yy}
4.	φ_z	$s\cdot\psi - \varphi(x,y,0+)$
5.	φ_{zz}	$s^2\psi - s\varphi(x,y,0+) - \varphi_z(x,y,0+)$

TABLE II

Inverse Laplace Transforms

If $\psi(x,y;s)$ is a Laplace transform (variable s), then the inverse Laplace transform is

$$\varphi(x,y,z) = \frac{1}{2\pi i}\int_{\Gamma} e^{zs}\psi(x,y;s)\,ds$$

where Γ is the contour described in Part II, Section 2.4.

Laplace transform on z	Inverse Laplace transform
1. $K_0(s\rho)$, $\rho>0$	$\begin{cases} 0 & , 0<z<\rho \\ (z^2-\rho^2)^{-1/2} & , z>\rho \end{cases}$
2. $\dfrac{K_0(s\rho)}{s}$, $\rho>0$	$\begin{cases} 0 & , 0<z<\rho \\ \cosh^{-1}\left(\frac{z}{\rho}\right) & , z>\rho \end{cases}$
3. $e^{-sb}K_0(s\rho)$, $b>0$, $\rho>0$	$\begin{cases} 0 & , 0<z<\rho+b \\ \left[(z-b)^2-\rho^2\right]^{-1/2} & , z>\rho+b \end{cases}$
4. $\left.\begin{array}{l} I_n(sr')K_n(sr),\ r>r' \\ I_n(sr)K_n(sr'),\ r<r' \end{array}\right\}$	$\begin{cases} 0 & , z<\lvert r-r'\rvert \\ \dfrac{1}{\pi}\displaystyle\int_0^{\cos^{-1}\left(\frac{r^2+r'^2-z^2}{2rr'}\right)} \frac{\cos(n\omega)\,d\omega}{\sqrt{z^2-\bar\rho^2}} & , \lvert r-r'\rvert<z<(r+r') \\ \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \frac{\cos(n\omega)\,d\omega}{\sqrt{z^2-\bar\rho^2}} & , z>(r+r') \end{cases}$ $\bar\rho^2 = r^2+r'^2-2rr'\cos\omega$

Laplace transform on z	Inverse Laplace transform
5. $\left.\begin{array}{ll} \dfrac{I_n(sr')K_n(sr)}{s}, & r>r' \\ \dfrac{I_n(sr)K_n(sr')}{s}, & r<r' \end{array}\right\}$	$\begin{cases} 0 & , z<\lvert r-r'\rvert \\ \dfrac{1}{\pi}\displaystyle\int_0^{\cos^{-1}\left(\frac{r^2+r'^2-z^2}{2rr'}\right)} \cos(n\omega)\cosh^{-1}\left(\frac{z}{\bar\rho}\right)d\omega & , \lvert r-r'\rvert<z<(r+r') \\ \left(\text{or } \dfrac{zrr'}{n\pi}\displaystyle\int_0^{\cos^{-1}\left(\frac{r^2+r'^2-z^2}{2rr'}\right)} \frac{\sin(n\omega)\sin\omega\, d\omega}{\bar\rho^2\sqrt{z^2-\bar\rho^2}}\right) & \\ \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(n\omega)\cosh^{-1}\left(\frac{z}{\bar\rho}\right)d\omega & , z>(r+r') \\ \left(\text{or } \dfrac{zrr'}{n\pi}\displaystyle\int_0^{\pi} \frac{\sin(n\omega)\sin\omega\, d\omega}{\bar\rho^2\sqrt{z^2-\bar\rho^2}}\right) & \end{cases}$

Figure 3.6

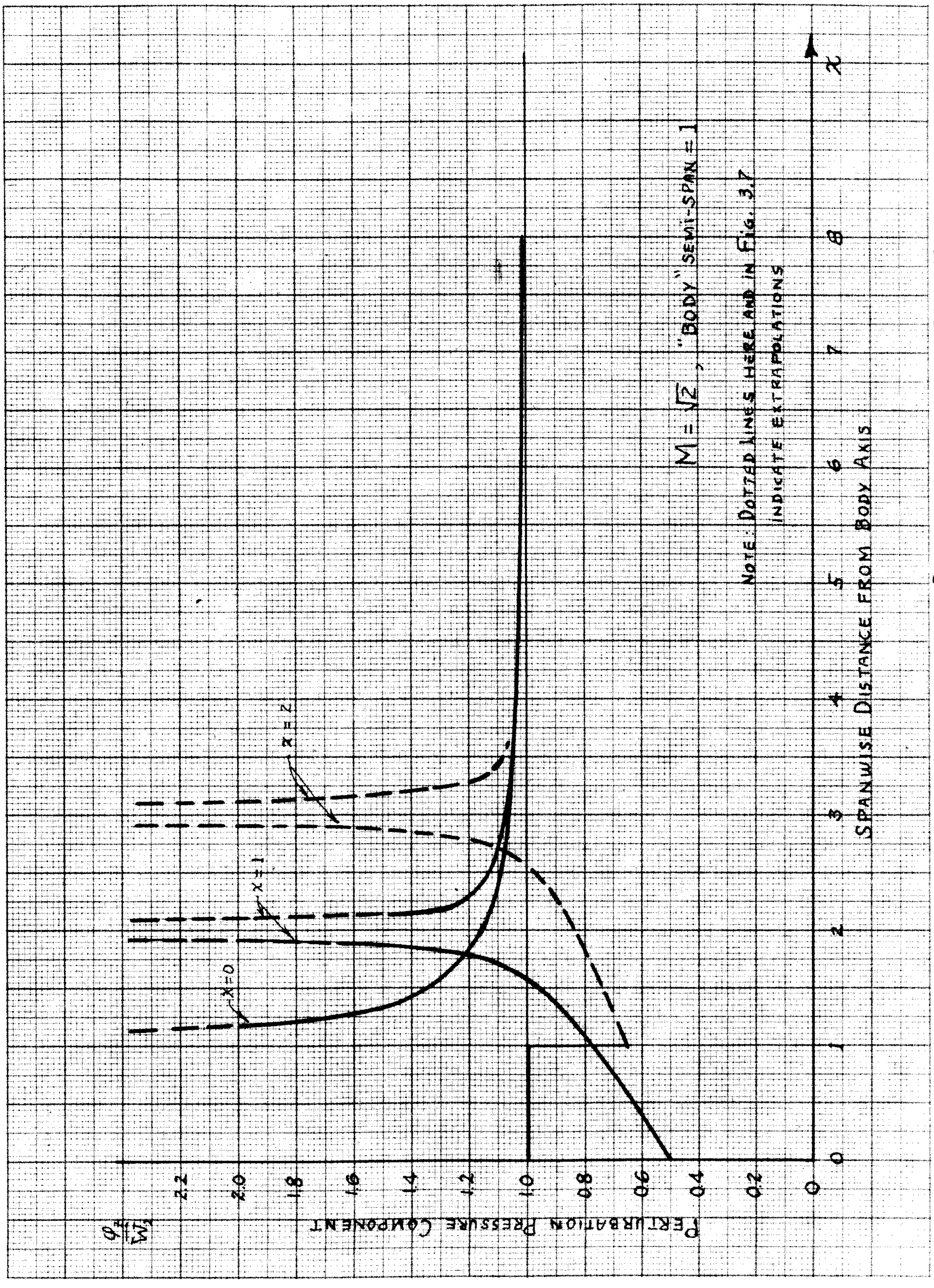

Figure 3.6 Planar Case Ia: Spanwise Pressure Distribution

Figure 3.7

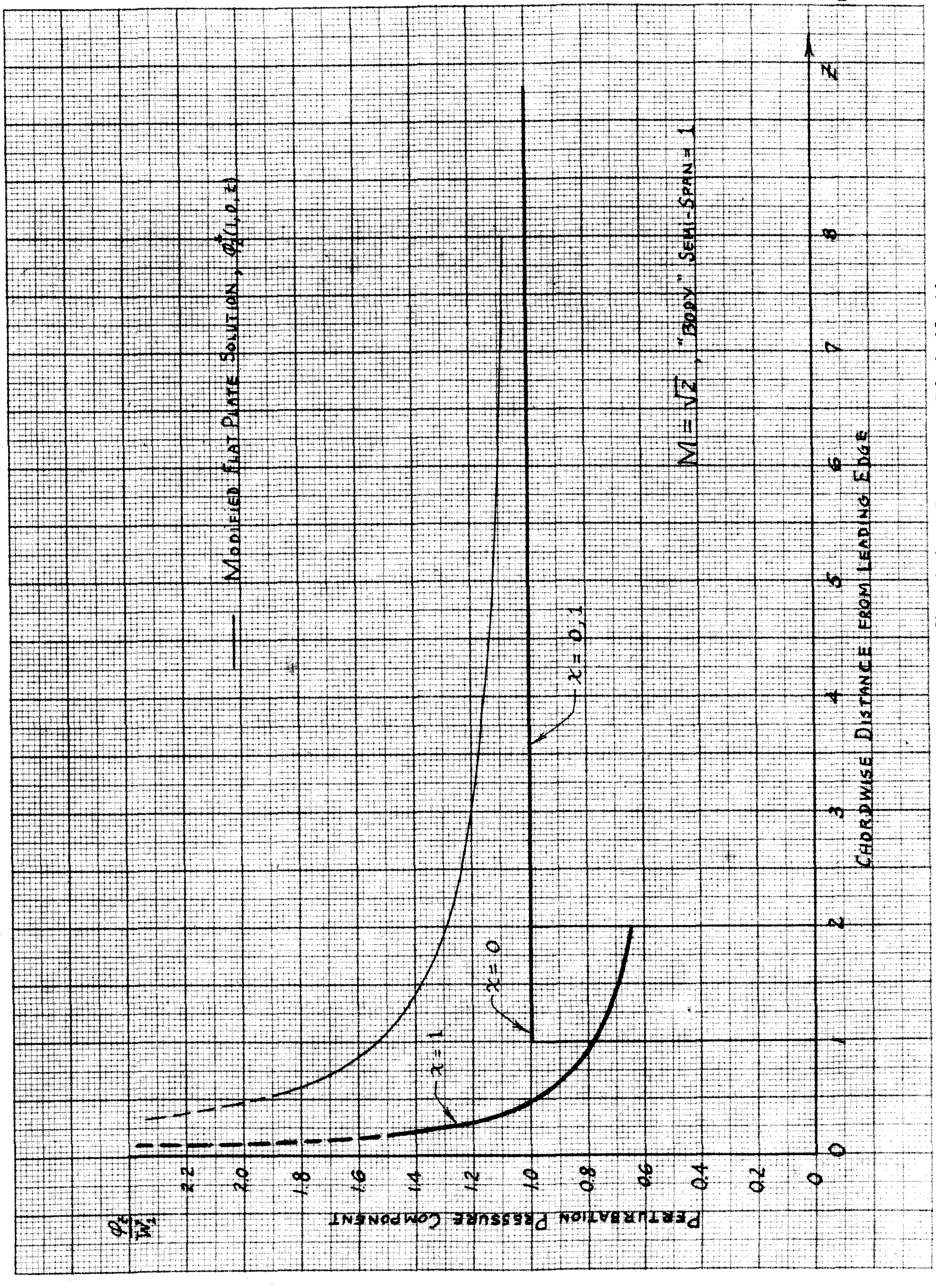

Figure 3.7 Planar Case Ia: Chordwise Pressure Distribution

Figure 4.7

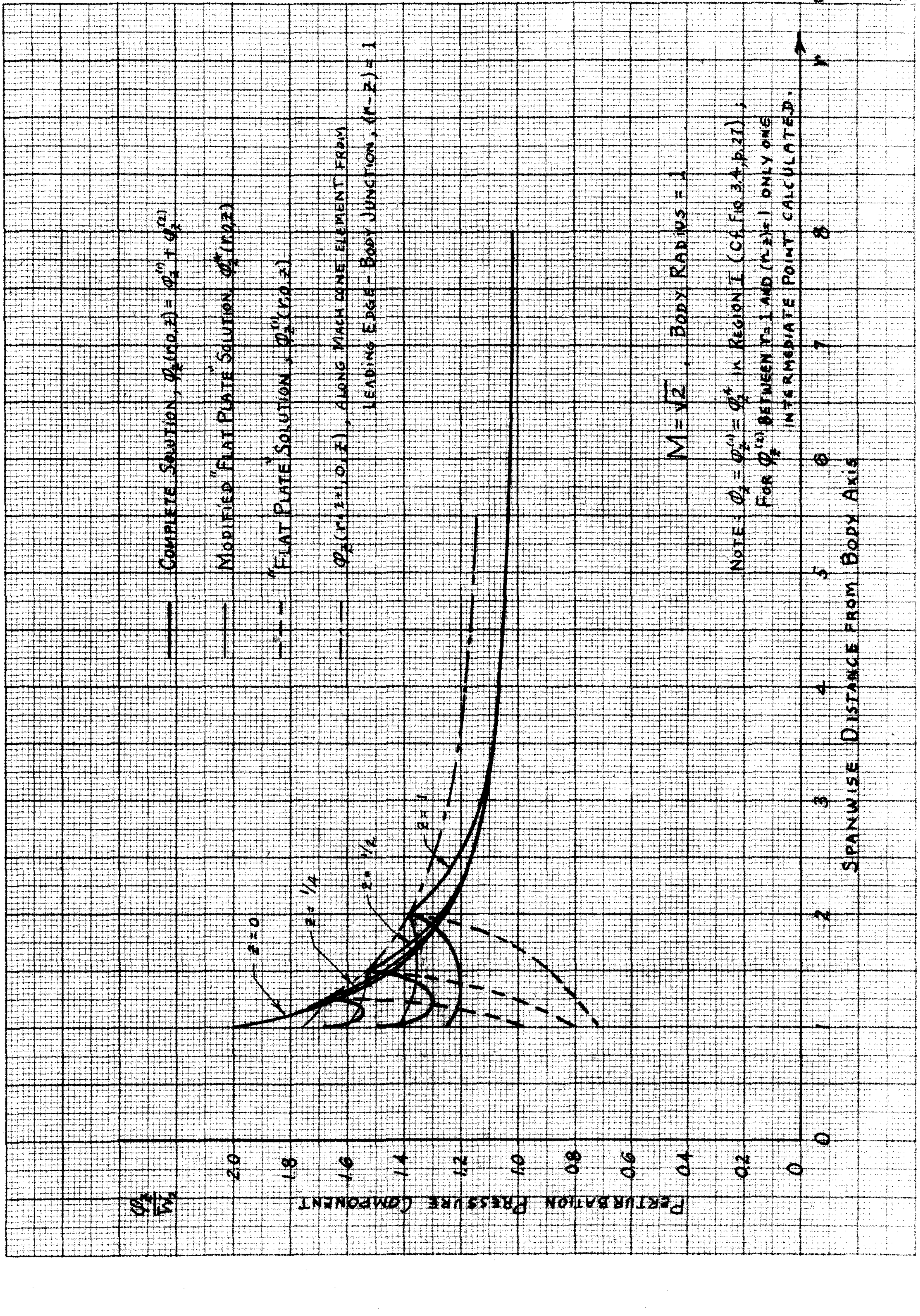

Figure 4.7 Case IIa : Spanwise Pressure Distribution

Figure 4.8

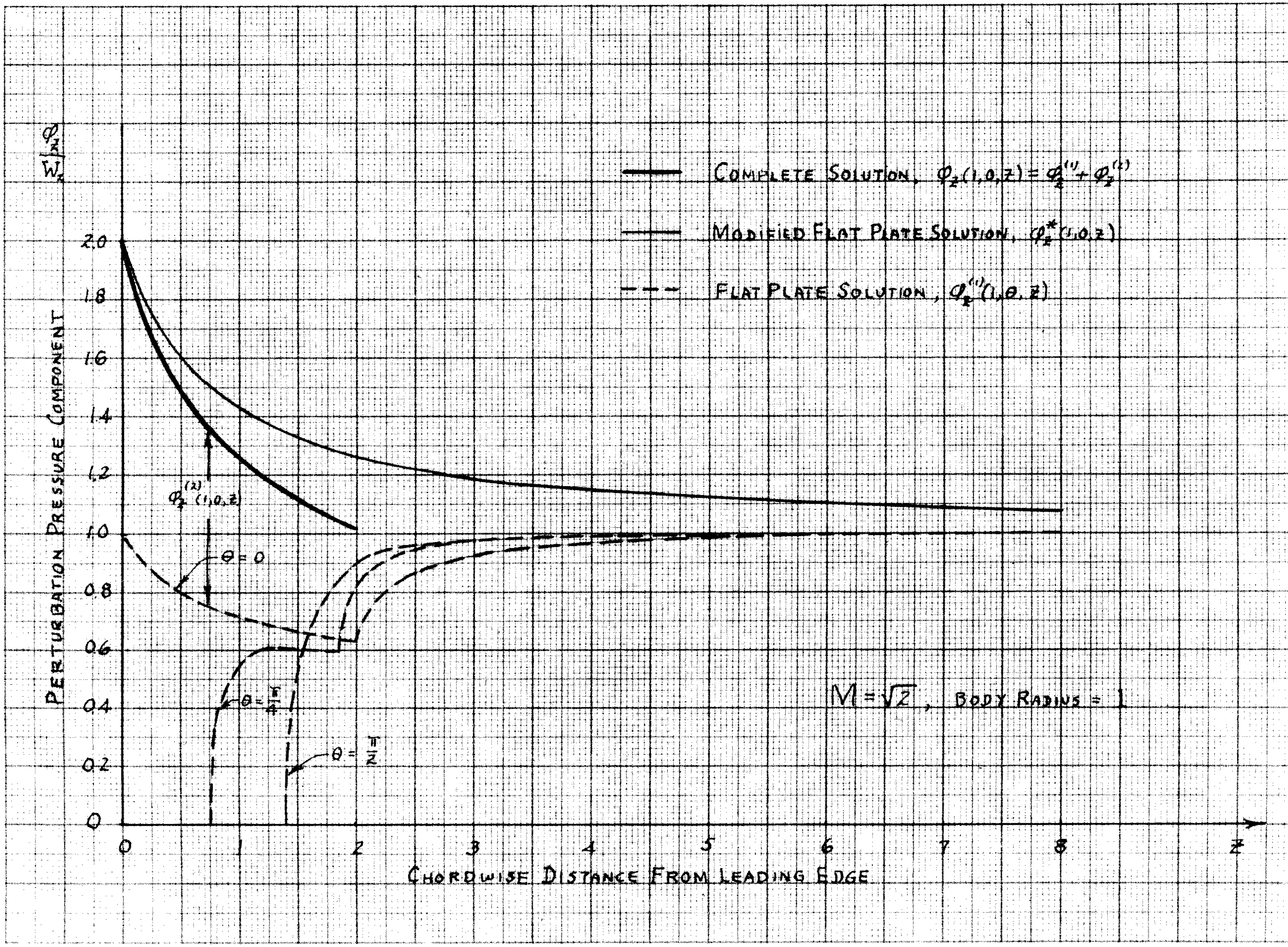

Figure 4.8 Case IIa: Chordwise Pressure Distribution