ON INEQUALITIES OF WEAK TYPE

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Stanley Arthur Sawyer

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ABSTRACT

Let $\{T_n\}$ be a sequence of continuous linear transformations on $L^p(X)$ for finite measure space X and $1 \leq p \leq 2$. Assume further that $\lim T_n f(x)$ exists a.e. for all f(x) in $L^p(X)$. Then, under the added assumptions that X is a compact group or homogeneous space and that each operator T_n commutes with translations on X, E.M.Stein was able to prove the existence of a constant Ω such that

$$(1) \qquad m\left[\left\{x : \sup_{1 \leq n < \infty} |T_n f(x)| \geq A\right\}\right] \leq \frac{\Omega}{A^p} \int_X |f(x)|^p dx$$

for all f(x) in $L^p(X)$ and $A > 0$. The first result of this paper is to prove (1) from convergence under the weaker assumption that the sequence $\{T_n\}$ commutes with each member of a family of measure-preserving transformations on X, a family which is large enough to have only trivial fixed sets. This result contains Stein's theorem, concludes maximal ergodic theorems from individual ergodic theorems, and applies in situations arising in probability theory.

The conditions above are then weakened so that the domain of $\{T_n\}$ becomes an F-space of functions satisfying a certain concordance condition on its topology, and the operators $\{T_n\}$ become continuous in measure with range in the space of measurable functions on X. Then, under

the assumption that $\{T_n\}$ commutes with enough measure-preserving transformations as above, a slightly weaker version of (1) is concluded.

Now, assume that $\{T_n\}$ is a sequence of continuous-in-measure linear transformations of an abstract F-space E into measurable functions on finite measure space X, and that $T^*f(x) = \sup_n |T_n f(x)| < \infty$ a.e. for every f in a dense subset of E. A decomposition of the measure space X is then obtained, such that $T^*f(x) < \infty$ a.e. on one of the sets for all f in E, and such that for all f in the complement of a set of the first category in E, $T^*f(x) = \infty$ a.e. on the other set of the decomposition. A theorem of Banach then applies on the first set to give a result which can be viewed as similar to (1). The decomposition is then applied to the preceeding results to prove new theorems.

<u>1. Introduction</u>: The purpose of this exposition will be to examine the relationship between almost everywhere convergence and inequalities of a certain sort, called "of weak type". To be more precise, assume we have a sequence $\{T_n\}$ of continuous linear transformations of a space $L^p(X)$ into itself, where X is some finite measure space, and that we have been able to prove the almost everywhere existence of

$$(1) \qquad \lim_{n\to\infty} T_n f(x)$$

for all $f(x)$ in $L^p(X)$. Questions of convergence or summability a.e. of orthogonal series fall into this class, but in general the limit function need not belong to $L^p(X)$. For example, assume

$$\sum a_k \cos kx + b_k \sin kx$$

is the Fourier series of $f(x)$ in $L^1(0,2\pi)$, and define

$$(2) \qquad T_n f(x) = \sum_0^\infty r^k (a_k \sin kx - b_k \cos kx)$$

for $r = 1 - 1/n$. Then, the limit (1) always exists a.e. by a theorem of Privaloff [13,p252,I], but the limit function, called the <u>conjugate function</u> $(\tilde{f}(x))$ of $f(x)$, is in general not integrable. However, by a result of Kolmogorov [7], there does exist a universal constant K such that for all $f(x)$ in $L^1(0,2\pi)$ and positive numbers A,

$$(3)\qquad m\left[\{y : |\tilde{f}(y)| \geq A\}\right] \leq \frac{K}{A}\int_0^{2\pi} |f(y)|\,dy.$$

Since this would be a corollary of

$$\int_0^{2\pi} |\tilde{f}(y)|\,dy \leq K\int_0^{2\pi} |f(y)|\,dy$$

if it were true, inequality (3) is called an inequality of weak type. In general, if S is any mapping of $L^p(X)$ into measurable functions on X, it is said to be of <u>weak type (p,p)</u> if there exists a constant Ω, depending only on S, such that

$$(4)\qquad m\left[\{y : |Sf(y)| \geq A\}\right] \leq \frac{\Omega}{A^p}\int_X |f(y)|^p\,dy, \quad 0 < A < \infty,$$

for any f(x) in $L^p(X)$. This inequality would also be a corollary of a related integral inequality; hence the name "weak type". For example, the Riesz inequalities [13,p253,I]

$$(5)\qquad \int_0^{2\pi} |\tilde{f}(y)|^p\,dy \leq A_p\int_0^{2\pi} |f(y)|^p\,dy, \quad 1 < p < \infty,$$

would imply that the operator $Sf = \tilde{f}$ is of weak type (p,p) for all p in the range $1<p<\infty$.

While an inequality of weak type may be strictly weaker than the corresponding integral inequality, one can still draw important implications. For example, if $g(x) = Sf(x)$ satisfies (4), then g(x) belongs to $L^r(X)$ for every $r < p$. In particular, Kolmogorov was able to conclude from (3) the inequality

$$\int_0^{2\pi} |\tilde{f}(y)|^{1-\varepsilon}\, dy \leq 2\pi + K_\varepsilon \int_0^{2\pi} |f(y)|\, dy$$

for all f(x) in $L^1(0,2\pi)$ and ε in the range $0 < \varepsilon < 1$, and furthermore that

$$\lim_{N\to\infty} \int_0^{2\pi} |\tilde{f}(y) - s_N'(y,f)|^{1-\varepsilon}\, dy = 0,$$

where $s_N'(x,f) = \sum_1^N (a_k \sin kx - b_k \cos kx)$. Alternately, one can often convert weak type inequalities into integral inequalities by the use of interpolation theorems. For example, assume the operator S is of weak types (a,a) and (b,b) for $1 \leq a < b$, and satisfies the inequality

$$|S(f+g)(x)| \leq |Sf(x)| + |Sg(x)| \quad \text{a.e.}$$

for all f,g in $L^a(X)$. Then, the inequalities

$$(6) \qquad \int_X |Sf(x)|^p\, dx \leq \Lambda_p \int_X |f(x)|^p\, dx, \quad a < p < b,$$

follow from the Marcinkiewicz Interpolation Theorem [13,p111,II]. If a=1, we can also get an inequality for p=1=a;

$$(7) \qquad \int_X |Sf(x)|\, dx \leq \Lambda_1 \int_X |f| \log^+ |f|\, dx + \Lambda_b.$$

In particular, the Riesz inequalities (5) are a consequence of (3) plus the trivial inequality $\|\tilde{f}\|_2 \leq \|f\|_2$.

Inequalities of weak type often arise as in the following example. Let $w(x)$ be an ergodic transformation on finite measure space X, and define for all $f(x)$ in $L^1(X)$

$$(8) \qquad T_n f(x) = \frac{f(x) + f(w(x)) + \cdots + f(w^{n-1}(x))}{n}.$$

Almost everywhere convergence then follows from the Birkhoff Ergodic Theorem [8,p410], and the Maximal Ergodic Theorem predicts the existence of a constant Ω such that

$$(9) \qquad m\left[\left\{y : \sup_{1\le n<\infty} |T_n f(y)| \geq A\right\}\right] \leq \Omega/A \int_X |f(x)|dx$$

for all $f(x)$ in $L^1(X)$ and $A>0$. Alternately, the proof of (3) by Kolmogorov actually yields (9) as well, where $\{T_n\}$ is as before. In many instances, this inequality is more basic than convergence, and is actually used to prove convergence. Indeed, the step from (9) to convergence, or from convergence to a much-weakened form of (9), can be carried out in a very general context. For example, if $T^*f(x) = \sup_n |T_n f(x)|$, then either convergence or (9) implies that $T^*f(x) < \infty$ a.e.; and each operator T_n is <u>continuous in measure</u>--i.e. a converging sequence is mapped by it into a sequence converging in measure. Then, we have by a theorem of Banach [1; 5,p332]:

Theorem 1 (Banach)[1,2] Let $\{T_n\}$ be a sequence of linear transformations of an abstract F-space E into measurable functions on X. Assume that each operator T_n is continuous in measure, and that $T^*f(x) < \infty$ a.e. for each f in E. Then, there exists a continuous function $\Phi(a)$, decreasing to zero as $a \to 0$, such that

$$(10) \qquad m[\{x : T^*f(x) \geq A\}] \leq \Phi(|\tfrac{1}{A}f|)$$

for all f in E and $A > 0$, where $|f|$ is the norm of E.

The condition that $T^*f(x) < \infty$ a.e. for every f in E can be weakened somewhat; see Theorem 5. In any event, from inequality (10) one can conclude

Corollary: Assume in addition in Theorem 1 that $\lim T_nf(x)$ exists a.e. for every f in a dense subset of E. Then, the limit exists a.e. for all f in E.

Convergence on a dense subset is usually easy to come by. In the first example (2), the dense subset could be the trigonometric polynomials; in the second example (8) it

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1--Any Banach space is an F-space; in general an F-space [2, Chapter III] is a complete metric linear topological space. See Section 4 for further discussion.

2--Note that the existence of such a function $\Phi(a)$ is equivalent to T^* itself being continuous in measure.

follows from Hilbert space methods applied to $L^2(X)$[3]. Going from convergence to (9), however, is another question; inequality (10) is a long way from an inequality of weak type. Mathematics, however, abounds with situations in which convergence is accompanied with a weak-type inequality on T^*. Fourier analysis, in particular, is a haven of such examples; for instance the Lebesgue theorem in combination with the Hardy-Littlewood inequalities [13,p29,I]. Many of these examples have in common that they involve translation-invariant operators on the unit circle or unit torus, and are thus covered by a recent theorem of E.M.Stein[10]. In the following, G is a compact group with its natural Haar measure, although a homogeneous space of a compact group (such as the unit sphere in E_n under rotation) would also be allowed.

<u>Theorem 2</u> (Stein) Assume $\{T_n\}$ is a sequence of translation-invariant and continuous linear transformations of $L^p(G)$ into $L^p(G)$, where $1 \leq p \leq 2$. Then, if

$$\lim_{n\to\infty} T_n f(x)$$

exists a.e. for every $f(x)$ in $L^p(G)$, there exists a

3--This procedure has recently been used to prove an interesting generalization of the Birkhoff theorem, involving weighted means. See [6] and [3], and Example 1 in Section 6.

constant Ω such that

(11) $$m\left[\{y : \sup_{1\le n<\infty} |T_n f(y)| \geq A\}\right] \leq \frac{\Omega}{A^p} \int_G |f(x)|^p dx$$

for all f(x) in $L^p(G)$ and $A > 0$.

Since all that Stein requires in his proof is that $T^* f(x)$ be finite a.e. for all f(x) in $L^p(G)$, this theorem is actually a companion to Theorem 1.

Convergence, however, is often paired with inequality (11) in situations far removed from group theoretical considerations. The Birkhoff theorem is one example; others are found in probability. For example, let $\{x_n\}$ be a sequence of identically distributed independent random variables on a probability space $(X, \mathcal{L}, P)$. Assume for convenience they have been normalized to have mean zero and variance one, and are also bounded in absolute value by one. Then, by an old theorem of Kolmogorov, the series

$$\sum a_n x_n$$

converges a.e. iff the sequence $\{a_n\}$ is square-summable. If we define

(12) $$T_n(f) = \sum_1^n x_k \, E(f x_k),$$

we then have convergence a.e. for all f in $L^2(X)$; inequality (11) for p=2 is also due to Kolmogorov [8,p236].

It is instructive to look at this example in greater detail. By the Consistency Principle [8,p93], we can assume that X is the infinite product $\prod_1^\infty [-1,1]$, $\mathcal{L}$ is the Borel field of X, and that $\{x_n\}$ are the coordinate functions of X. Let $w(a)$ be the unilateral shift on X. Then, for any $f(a)$ in $L(X)$ and $g(a) = f(w(a))$, we have

$$\begin{aligned} T_n f(w(a)) &= \sum_1^n x_k(w(a))\, E(fx_k) \\ &= \sum_1^n x_{k+1}(a) \int_X f(w(a))x_k(w(a))\,dP \\ &= \quad T_{n+1} g(a), \end{aligned}$$

since $E(gx_1) = E(g)E(x_1) = 0$. Thus, both this example and the example (8) have the property that the sequences involved commute, or nearly commute, with certain ergodic transformations on X. Translation-invariant operators on the unit circle also fall into this category, since translation by an irrational forms an ergodic transformation. But many compact groups, for example any non-Abelian one, admit no ergodic transformations of this form. However, a certain homogeneousness condition is involved in both the theorem of Stein and the last two examples. In both cases we have operators commuting with enough measure-preserving transformations to "provide communication" inside the underlying measure space, in a sense shortly to be made precise.

To formalize the above, let $(X, \mathcal{L}, m)$ be a fixed unit measure space, and let $\{T_n\}$ be a sequence of continuous linear transformations of $L^p(X)$ into $L^p(X)$. Set $T^*f(x) = \sup_n |T_nf(x)| = T^*(x,f)$ as before, and let $w(x)$ be a measure-preserving transformation on X. Then, the sequence $\{T_n\}$ shall here be said to commute with $w(x)$ if for every $f(x)$ in $L^p(X)$ and $g(x) = f(w(x))$, we have $T^*(w(x),f) \leq T^*(x,g)$ a.e. (Notice that this is actually a condition on T^*. Indeed, in (12) none of the operators themselves commute with the unilateral shift.) Now, let $\mathcal{F}$ be a collection of measure-preserving transformations on X. Then, $\mathcal{F}$ will be called an ergodic family on X if for any two sets A,B in $\mathcal{L}$ with $m(A)>0$, $m(B)>0$, there exists $w(x)$ in $\mathcal{F}$ such that $m(A \cap w^{-1}(B)) > 0$. If $\mathcal{F}$ is closed under composition, an equivalent formulation (by Lemma 1, Section 2) is that if any set F in $\mathcal{L}$ is fixed (i.e. $w^{-1}(F) = F$ essentially) by every w in $\mathcal{F}$, then either $m(F)=0$ or $m(F)=1$. Finally, if the sequence $\{T_n\}$ commutes with every member of some ergodic family on X, it will be said to be distributive (on X or on $L^p(X)$).

Examples of distributive sequences would be any sequence of operators commuting with a given ergodic transformation, or a translation-invariant sequence on a compact group or homogeneous space.[4] Thus, the next theorem includes

4--See the second remark after Lemma 2, Section 2.

the theorem of Stein, as well as the last two examples (8) and (12).

<u>Theorem 3</u> Assume $\{T_n\}$ is a distributive sequence of linear operators on $L^p(X)$, where $1 \leq p \leq 2$, and that $T^*f(x) < \infty$ a.e. for all $f(x)$ in $L^p(X)$. Then, the operator T^* is of weak type (p,p); i.e. inequality (11) holds.

The proof of Theorem 3 will be deferred until Section 3. The same techniques can also be used to obtain an interesting extension of Theorem 1. If E is an F-space[5] of measurable functions on X, it will be called a <u>basic space</u> (on X) if every sequence of functions in E, converging in the topology of E to a given function, contains a subsequence which converges a.e. to that function. In particular, if convergence in E to a function always implies convergence in measure to the same function, then E would be a basic space. Indeed, it turns out in general that convergence in the topology of a basic space implies convergence in measure. Now, if $w(x)$ is a measure-preserving transformation on X, the basic space E will be called <u>invariant</u> under $w(x)$ if $f(x)$ in E implies $g(x) = f(w(x))$ also belongs to E, and has a smaller or equal norm. Assume $\{T_n\}$ is a sequence of linear transformations of

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5--See footnote 1.

basic space E into measurable functions on X, where the transformations are continuous in measure. Then, if $\mathcal{F}$ is an ergodic family on X, under all of whose members E is invariant, we can define commutativity of $\{T_n\}$ with elements of $\mathcal{F}$ as before. If the sequence $\{T_n\}$ commutes with every member of some ergodic family on X, all of whose members leave E invariant, it will be called distributive, again as before. We can now state the following theorem, which will be proved in Section 4.

Theorem 4 Let $\{T_n\}$ be as in Theorem 1, except that we assume E to be a basic space on X and $\{T_n\}$ to be distributive. Then, we can assume that the function $\Phi(a)$ is linear; i.e. that there exists a constant Ω such that

$$(13) \qquad m[\{x : T^*f(x) \geq A\}] \leq \Omega \; |\tfrac{1}{A}f|$$

for all f in E and $A > 0$, where $|f|$ is the norm of E.

For example, Theorem 4 contains Theorem 3 for p=1, and gives a weaker version for $1 < p \leq 2$, with the right-hand side of (11) being replaced by its pth root. In the range $0 < p < 1$ (see the remark in Section 3) the two theorems agree. Stein [10,p159] also has a theorem along these lines, which applies to Banach spaces contained in L^1.

So far we have assumed that the sequence $\{T_n\}$ is reasonably well-behaved; i.e. that $T^*f(x) < \infty$ a.e. for all f.

Remarkably enough, a certain regularity remains if this condition is removed, at least under some supplementary conditions; enough so that we can prove theorems to the effect that if $\{T_n\}$ is not reasonably well-behaved, then it must be uniformly pathological. The appropriate setting for such a theorem seems to be as a generalization of Theorem 1, and the theorem itself is proven in Section 5.

<u>Theorem 5</u> Let $\{T_n\}$ be as in Theorem 1, except that we only require that $T^*f(x) < \infty$ a.e. on a dense subset of E. Then, there exists a measurable subset $X_0 \subseteq X$ such that

(a) For any f in E, $T^*f(x) < \infty$ a.e. on the complement of X_0, and thus Theorem 1 applies on the complement of X_0,

(b) $T^*f(x) = \infty$ a.e. on X_0 for every f in E, with the exception of a set of the first category in E,

(c) If E is a basic space on X, invariant under a measure-preserving transformation w(x), and $\{T_n\}$ commutes with w(x), then w(x) fixes X_0. Thus, if $\{T_n\}$ is distributive, $m(X_0) = 0$ or 1.

<u>Corollary</u> (the "Alternative") Let $\{T_n\}$ be as in Theorem 3, except that we only require that $T^*f(x) < \infty$ a.e. on a dense subset of $L^p(X)$. Then, either T^* is of weak type (p,p), or $T^*f(x) = \infty$ a.e. for every f(x) in $L^p(X)$, with the exception of a set of the first category in $L^p(X)$.

The last theorem is of a slightly different character, and in most examples gives an extension of Theorem 3. (See Section 5 for the proof.)

<u>Theorem 6</u> Let $\{T_n\}$ be a sequence of linear transformations of $L^p(X)$ into $L^p(X)$, where

(i) T^* is of weak type (p,p),

(ii) For all $f(x)$ in a dense subset of $L^p(X)$,[6]

$$m[\{x: T^*f(x) \geq A\}] = o(1/A^p) \text{ as } A \to \infty. \tag{14}$$

Then, (14) holds for all $f(x)$ in $L^p(X)$.

For example, we could apply Theorem 6 to our first example (2), and obtain the result of Titchmarsh [12] that

$$m[\{y: |\tilde{f}(y)| \geq A\}] = o(1/A)$$

for all $f(x)$ in $L^1(0,2\pi)$.

Further examples of the theorems presented here will be found in Section 6. Section 2 is devoted to a discussion of ergodic families which is drawn upon in later sections, and proofs of Theorems 3-6 take up Sections 3-5.

Remark: D.L.Burkholder [4] has extended Stein's results

6--In particular, if $\{T_n\}$ is distributive and $p<2$, and if each operator T_n when restricted to $L^2(X)$ forms a continuous linear transformation of $L^2(X)$ into $L^2(X)$, then by Theorem 3 we could take $L^2(X)$ as the dense subset.

in another direction, to prove weak-type inequalities from convergence for certain classes of sequences of functions. His approach is probabilistic in nature, and is strong enough to apply to the martingale theorem and certain of the operator ergodic theorems.

2. Ergodic Families In the following, let $\mathcal{F}$ be a collection of measure-preserving transformations of a unit measure space $(X, \mathcal{L}, m)$, which we assume is closed under composition. Now, set $A^w = w^{-1}(A) = \{x;\ w(x) \in A\}$ for any A in $\mathcal{L}$ and w in $\mathcal{F}$. We recall that $\mathcal{F}$ is ergodic iff A, B in $\mathcal{L}$ of positive measure implies there exists w in $\mathcal{F}$ such that $m(B \cap A^w) > 0$.

Lemma 1 $\mathcal{F}$ is an ergodic family iff A in $\mathcal{L}$, $A = A^w$ essentially for all w in $\mathcal{F}$ implies $m(A)=0$ or $m(A)=1$.

$\Longrightarrow$ Clear, since for such an A, $m(A' \cap A^w)=0$ for all w, where $A'=X-A$.

$\Longleftarrow$ It is sufficient to prove that for A in $\mathcal{L}$, $m(A)>0$, there exists a sequence $\{w_n\} \subseteq \mathcal{F}$ such that $m(\bigcup A^{w_n}) = 1$. Given some A in $\mathcal{L}$ of positive measure, set

$$q = \sup m(\bigcup A^{w_n}), \qquad \{w_n\} \subseteq \mathcal{F}.$$

Since a countable union of countable unions is still a countable union, the supremum must be attained. Now, if a set $\bigcup A^{w_n}$ whose measure is q is fixed by all w in $\mathcal{F}$, then $q=1$ by hypothesis; but if $\bigcup A^{w_n}$ is not fixed by some $w(x)$ in $\mathcal{F}$, then $m(\bigcup A^{w_n} \cup \bigcup A^{w_n w}) > q$. Hence $\mathcal{F}$ is an ergodic family.

Lemma 2 Assume $\mathcal{F}$ is an ergodic family. Then, for any two sets A,B in $\mathcal{L}$ and real number $\theta>1$, one can choose $w(x)$ in $\mathcal{F}$ such that

$$(15) \qquad m(B \cap A^{w}) \leq \theta m(A)m(B).$$

Given A,B and θ, assume no such $w(x)$ exists; in particular A and B have positive measure. If $f(x) = \chi_A(w(x))$ for some $w(x)$ in $\mathcal{F}$, where $\chi_A(x)$ is the characteristic function of the set A, we then have

$$\begin{aligned} &(a) \quad \int_X f(x)dx = m(A) \\ &(b) \quad \int_B f(x)dx \geq \theta m(A)m(B). \end{aligned}$$

Now, define Q as the closed convex hull in $L^2(X)$ of the set of all functions of the form $\chi_A(w(x))$ for $w(x)$ in $\mathcal{F}$. Every function $f(x)$ is Q also satisfies (a) and (b), and if $g(x) = f(w(x))$ for w in $\mathcal{F}$, then $g(x)$ also belongs to Q, since $\mathcal{F}$ is closed under composition. Now, Q is a closed convex subset of Hilbert space $L^2(X)$; it thus contains a unique element $q(x)$ of smallest norm [11,p243]. But if $r(x) = q(w(x))$ for some w in $\mathcal{F}$, $r(x)$ also falls into Q, and has the same norm as $q(x)$. Thus, $q(x) = r(x)$ a.e., and for any real number y and $w(x)$ in $\mathcal{F}$ we conclude

$$E_y = \{x: q(x)>y\} = \{x: q(w(x))>y\} = E_y^{w}$$

essentially. Hence $m(E_y) = 0$ or 1 for all y, and $q(x)$ is essentially constant. By (a), $q(x)$ is essentially the constant $m(A)$, and (b) provides a contradiction.

<u>Remarks</u>: (1) If $\mathcal{F}$ is the set of iterates of a single ergodic transformation $w(x)$, the above can be simplified. Indeed, given any function $f(x)$ in $L^2(X)$, the sequence

$$\frac{f(x) + f(w(x)) + \cdots + f(w^{n-1}(x))}{n}$$

converges a.e. and in $L^2(X)$ to the constant function $\int_X f(x)dx$; Lemma 2 then follows from the Mean Ergodic Theorem. In fact, Lemma 2 can be viewed as an extension of the Mean Ergodic Theorem, for we have shown

<u>Corollary</u> Let $\mathcal{F}$ be an ergodic family on X, and $f(x)$ an arbitrary function in $L^2(X)$. Then, the closed convex hull in $L^2(X)$ of $\{f(w(x)): w \text{ in } \mathcal{F}\}$ contains a unique constant function.

(2) In general, the condition $\theta>1$ cannot be weakened to $\theta=1$. However, if $\mathcal{F}$ is the set of right translations on a compact group or its homogeneous space, we can push θ to 1 by compactness. The resulting theorem is due to Calderon [13,p165,II], and is usually proven by integrating (15) over the group concerned and applying Fubini's Theorem. (Actually, Calderon's result seems the simplest way of proving that a family of this kind is ergodic. Alternately, it follows from the theory of representations of a compact group that any measurable set $A \subseteq X$ which is fixed by all right translations must have measure either zero or one. Thus, Lemma 1 and Lemma 2 provide an

alternate proof of the result of Calderon.)

Lemma 3 Assume $\mathcal{F}$ is an ergodic family. Then, if $\{A_n\}$ is a sequence of measurable sets such that $\sum m(A_n) = \infty$, there exists a sequence of transformations $\{w_n\} \subseteq \mathcal{F}$ such that

$$(16) \qquad m\Big(\bigcup_{N}^{\infty} A_n^{w_n}\Big) = 1, \quad \text{all } N.$$

That is, such that $w_n(x) \in A_n$ infinitely often for almost every x in X.

Consider any sequence of sets $\{A_n\} \subseteq \mathcal{L}$. Then, by Lemma 2 and induction, we can choose transformations $\{w_n\} \subseteq \mathcal{F}$ such that for $N = 1,2,3, \ldots$,

$$m(A_1^{w_1} \cap A_2^{w_2} \cap A_3^{w_3} \cap \cdots \cap A_N^{w_N}) \leq \theta_1 \theta_2 \cdots \theta_N \prod_1^N m(A_k),$$

where $\{\theta_n\}$ is any sequence of real numbers with $\theta_n > 1$. In particular, if $\prod m(A_k) = 0$ and $\prod \theta_n$ converges, then $\bigcap A_n^{w_n}$ must be a null set. By complementation, given any sequence $\{A_n\}$ with $\sum m(A_k) = \infty$, we can choose $\{w_n\}$ such that $\bigcup A_n^{w_n}$ has full measure. But now we are through; for by induction we can choose $\{w_n\} \subseteq \mathcal{F}$ and integers $\{N_k\}$ such that

$$m\Big(\bigcup_{N_k+1}^{N_{k+1}} A_n^{w_n}\Big) \geq 1 - 1/k^2, \quad k = 1,2,3, \ldots$$

from which (16) follows.

3. The purpose of this section is to prove Theorem 3. First, we remark that the first step in Stein's proof of Theorem 2 is to prove Lemma 3 for this situation [10,p146], and that no other use is made of the group theoretical structure involved. In other words, once we have Lemma 3, the proof of Stein is also sufficient for Theorem 3. Thus, at this point we could refer the reader to Stein's paper, and be done. However, for completeness we will give the proof of Stein here.

We will need some information about the Rademacher functions, which are defined as follows. Let the binary expansion of a number in the unit interval be

$$\theta = .\varepsilon_1 \varepsilon_2 \varepsilon_3 \cdots$$

and set $r_n(\theta) = 2\varepsilon_n - 1$. These functions are then defined a.e. on the unit interval for $n \geq 1$, and form an orthonormal system. The following two facts are standard results about Rademacher functions [13,p212,213,I] which we will have occasion to use.

1. If $\{a_n\}$ is a sequence of real numbers, $\sum a_n r_n(\theta)$ converges a.e. iff $\sum a_n^2 < \infty$. (This is sometimes stated as $\sum \pm a_n$ converges for almost every choice of signs iff $\{a_n\}$ is square-summable.)

2. Let A be a subset of the unit interval of positive

measure. Then, there exists an integer N, depending only on the set A, such that the inequality

$$(17)\quad \left(\sum a_n^2\right)^{1/2} \leq \sqrt{2} \operatorname*{ess.\,sup}_{\theta \in A} \left|\sum_1^{\infty} a_n r_n(\theta)\right| + 3 \sum_1^{N} |a_n|$$

holds for every square-summable sequence $\{a_n\}$.

The proof of Stein depends on the following lemma, stated in a form due essentially to Burkholder [4].

<u>Lemma 4</u> Let $\{a_{mn}\}$ be a double sequence of real numbers such that

(i) For any n, $a_n^* = \sup_k |a_{kn}| < \infty$,

(ii) For any m, $\sum a_{mk}^2 < \infty$,

(iii) $\sup_m \left|\sum a_{mn} r_n(\theta)\right| < \infty$ almost everywhere.

Then, $\sup_{m,n} |a_{mn}| < \infty$.

By (iii) and inequality (17), we can find a set $A \subseteq [0,1]$, a constant Ω, and an integer N such that for all m,

$$\begin{aligned}\left(\sum a_{mk}^2\right)^{1/2} &\leq \sqrt{2} \sup_{\theta \in A} \left|\sum a_{mk} r_k(\theta)\right| + 3 \sum_1^N |a_{mk}| \\ &\leq \sqrt{2}\,\Omega + 3 \sum_1^N a_k^*\end{aligned}$$

and thus $\sup_{m,n} |a_{mn}| \leq \sqrt{2}\,\Omega + 3\sum_1^N a_k^* < \infty$.

The following simple lemma will also be needed.

<u>Lemma 5</u> Let $\{a_n\}$ be a sequence of real numbers, and choose integers $N \leq M$. Then for all p, $1 \leq p \leq 2$,

(18) $$\int_0^1 \Big|\sum_N^M a_n r_n(\theta)\Big|^p d\theta \le \sum_N^M |a_n|^p$$

Proof: $$\int_0^1 \Big|\sum_N^M a_n r_n(\theta)\Big|^p d\theta \le \Big(\int_0^1 \Big|\sum_N^M a_n r_n(\theta)\Big|^2 d\theta\Big)^{p/2}$$
$$= \Big(\sum_N^M a_n^2\Big)^{p/2} \le \sum_N^M |a_n|^p.$$

Proof of Theorem 3 Assume T^* is not of weak type (p,p); i.e. that inequality (11) holds for no choice of constant Ω. Then, we can choose a sequence of functions $\{f_n\}$ in $L^p(X)$ and positive constants $\{A_n\}$ such that for all n,

(19) $$m\left[\{x:\ T^*f_n(x) \ge A_n\}\right] \ge \frac{n^3}{A_n^p}\int_X |f_n(x)|^p dx.$$

Since f_n, A_n enter in (19) as the combination $\frac{1}{A_n}f_n$, one can redefine the functions $\{f_n\}$ so that (19) holds with, say, $A_n = n^{2/p}$. Then, $A_n \to \infty$, $\frac{n^3}{A_n^p} \to \infty$, and since $m(X) < \infty$, $\int_X |f_n(x)|^p dx \to 0$. After a process of eliminating some members of $\{f_n\}$ and repeating others, we arrive at a new sequence $\{f_n\}$ which satisfies

$$\sum m\left[\{x: T^*f_n(x) \ge R_n\}\right] = \infty, \qquad \sum \int_X |f_n(x)|^p dx < \infty$$

where $R_n \to \infty$. By hypothesis, $\{T_n\}$ commutes with the members of some ergodic family $\mathcal{F}$ on X. Thus, by Lemma 3, there exists a sequence of measure-preserving transformations $\{w_n\} \subseteq \mathcal{F}$ such that $T^*f_n(w_n(x)) \geq R_n$ infinitely often for a.e. x in X. Since each $w_n(x)$ was from $\mathcal{F}$, $T^*g_n(x) \geq T^*f_n(w_n(x))$ a.e. for $g_n(x) = f_n(w_n(x))$, and

$$\text{(20)} \qquad \sup_n T^* g_n(x) = \infty \ \text{a.e.}, \qquad\qquad \int_X \sum |g_n(x)|^p\,dx < \infty.$$

We have assumed $p \leq 2$; thus the series

$$F(x,\theta) = \sum r_n(\theta) g_n(x), \qquad F_m(x,\theta) = \sum r_n(\theta) T_m g_n(x)$$

converge almost everywhere in the square $X \times [0,1]$. Now, for an arbitrary increasing sequence of integers $\{N_k\}$ define

$$q_k(\theta) = \left(\int_X \left| \sum_{N_k+1}^{N_{k+1}} r_n(\theta) g_n(x) \right|^p dx \right)^{1/p}.$$

By Holder's inequality and (18),

$$\int_0^1 q_k(\theta)\,d\theta \leq \left(\int_X \int_0^1 \left| \sum_{N_k+1}^{N_{k+1}} r_n(\theta) g_n(x) \right|^p d\theta\,dx \right)^{1/p}$$
$$\leq \left(\int_X \sum_{N_k+1}^{N_{k+1}} |g_n(x)|^p\,dx \right)^{1/p}.$$

In particular, it follows from (20) that the $\{N_k\}$ could be chosen so that $\sum q_k(\theta)$ is integrable, and thus convergent almost everywhere. In other words, some sequence of partial sums of $F(x,\theta)$ converges in $L^p(X)$ for a.e. θ. For these θ, then, F and F_m belong to $L^p(X)$ and satisfy $F_m = T_m F$; thus $\sup_m |F_m(x,\theta)| < \infty$ a.e. in $X \times [0,1]$. Putting all of this together, we see that for almost every x_0 in X, $T^* g_n(x_0) < \infty$ for all n, $\sum T_m g_n(x_0)^2 < \infty$ for all m, and

$$\sup_m \left| \sum r_n(\theta) T_m g_n(x_0) \right| < \infty \text{ for a.e. } \theta.$$

Now we can apply Lemma 4 with $a_{mn} = T_m g_n(x_0)$, and conclude $\sup_n T^* g_n(x) < \infty$ a.e. Inequality (20) now provides a contradiction, and T' must have originally been of weak type (p,p).

<u>Remark</u>: The condition $p \leq 2$ in the above is essential; in fact Stein [10,p157] gives a counterexample for $p > 2$. However, the condition $p \geq 1$ is not essential, and was never used. Thus, Theorem 3 also holds in the $L^p(X)$ spaces for $0 < p < 1$. Indeed, this is a special case of Theorem 4.

4. A linear topological space [5,p49] is a vector space E enjoying a topology with respect to which addition and scalar multiplication are bicontinuous operations. If the topology is a complete metric topology, given by a metric ρ which satisfies

$$\rho(x,y) = \rho(x-y,0) \tag{21}$$

for all x,y in E, then E is called an <u>F-space</u> [2,p35; 5, p51][7]. For example, any Banach space is an F-space. The <u>norm</u> of E is the function $|x| = \rho(x,0)$; as in the case of Banach spaces, this function completely determines the topology and metric structure of E.

For example, let $L^{\alpha}(X)$, for $0<\alpha<1$, be the space of all measurable functions on X for which the integral

$$\int_X |f(x)|^{\alpha} dx$$

is finite. With this integral as the norm, $L^{\alpha}(X)$ forms an F-space, and even a basic space as defined in the introduction. Another example would be the space of all finite a.e. measurable functions on X, with the norm

$$\int_X \frac{|f(x)|}{1+|f(x)|}\, dx.$$

7--Since a Hausdorff first countable linear topological space is always metrizable with a metric satisfying (21), this is not as exacting a condition as it might appear.

This last space we will call $L^0(X)$; it has the interesting property that convergence in its topology is equivalent to convergence in measure. It is also a basic space, as, of course, are all the spaces $L^p(X)$ for $1 \leq p \leq \infty$.

Let E be a basic space on X, and f an element of E. Then, $af \to 0$ in the topology of E as $a \to 0$, since E is a linear topological space, and thus f(x) must be finite a.e. by the basic property of basic spaces. In particular, E is contained in $L^0(X)$. Moreover, since both spaces are basic spaces, the embedding operation is a closed operation, in the sense of the Closed Graph Theorem [2,p41; 5,p57][8]. Thus, it is a continuous operation as well. In other words, convergence in E implies convergence in $L^0(X)$, and convergence in any basic space implies convergence in measure of the functions involved. Conversely, if E is any F-space of measurable functions on X, with the property that convergence in the topology of E implies convergence in measure, then of course E must be a basic space. Thus, we have an alternate characterization of basic spaces.

The proof of Theorem 4 depends on the following lemma, due

8--A linear operator U with domain E_1 and range E_2 is said to be <u>closed</u> if $x_n \to a$ in E_1, $Ux_n \to b$ in E_2 always implies $Ua=b$. The <u>Closed Graph Theorem</u> is the assertion that any closed linear mapping of one F-space into another is necessarily continuous.

to Orlicz [9].

<u>Lemma 6</u> Let $\{f_n\}$ be a sequence of finite a.e. measurable functions on X, and suppose that $\sum f_n$, together with every subseries, converges in measure. Then, $\sum f_n(x)^2 < \infty$ a.e.

Let $\{r_n(\theta)\}$ be the Rademacher functions, as in Section 3. Then, the series $\sum r_n(\theta) f_n(x)$ converges in measure on X for every irrational value of θ. By bounded convergence, then, it must converge in measure in the square $M = X \times [0,1]$. Thus, some sequence of partial sums of this series converges a.e. in M or, by Fubini's theorem, almost everywhere in $[0,1]$ for almost every value of x. It then follows from inequality (17) that for these values of x, $\sum f_n(x)^2 < \infty$.

<u>Proof of Theorem 4</u> Assume inequality (13) holds for no choice of constant Ω. Then, we can choose a sequence of functions $\{f_n\} \subseteq E$ and positive constants $\{A_n\}$ such that for $n = 1,2,3, \ldots$,

(22) $$m\left[\{x : T^* f_n(x) \geq A_n\}\right] \geq n^3 \left| \tfrac{1}{A_n} f_n \right|.$$

Since elements of E enter (22) only in the form $\frac{1}{A_n} f_n$, we can assume $A_n = n$. By the triangle inequality $|f_n| \leq n \left|\frac{1}{n} f_n\right|$ and thus for all n,

$$m\left[\{x : T^* f_n(x) \geq n\}\right] \geq n^2 |f_n|.$$

In particular, $\sum|f_n| < \infty$; at the possible cost of repeating some of the sequence $\{f_n\}$ we can assume

$$\sum m[\{x: T^* f_n(x) \geq R_n\}] = \infty, \quad R_n \to \infty,$$

as well. Thus by distributiveness and Lemma 3 there exists a sequence of measure-preserving transformations $\{w_n\}$ such that $g_n(x) = f_n(w_n(x))$ falls into E and satisfies $T^* g_n(x) \geq T^* f_n(w_n(x))$ a.e. for all n, and

$$(23) \qquad \sup_n T^* g_n(x) = \infty \text{ a.e.,} \qquad \sum |g_n| < \infty.$$

In particular, $\sum g_n$, together with every subseries, converges in E and thus by continuity $\sum T_m g_n$, with all of its subseries, converges in measure for each m. Hence by Lemma 6

$$\sum g_n(x)^2 < \infty \text{ a.e.,} \qquad \sum T_m g_n(x)^2 < \infty \text{ a.e.}$$

for all m, and thus

$$F(x,\theta) = \sum r_n(\theta) g_n(x), \qquad F_m(x,\theta) = \sum r_n(\theta) T_m g_n(x)$$

converge almost everywhere in the square $M = X \times [0,1]$. Since $F(x,\theta)$ converges in E and in measure for all θ, $F_m = T_m F$ for a.e. θ and m, and $\sup_m |F_m(x,\theta)| < \infty$ a.e. in M. Putting all of this together, we see that for almost every x_0 in X, $T^* g_n(x_0) < \infty$ for all n, $\sum T_m g_n(x_0)^2 < \infty$ for all m, and

$$\sup_m \left|\sum r_n(\theta) T_m g_n(x_o)\right| < \infty \text{ for a.e. } \theta.$$

Now we can apply Lemma 4 with $a_{mn} = T_m g_n(x_o)$, and conclude $\sup_n T^* g_n(x) < \infty$ a.e. Inequality (23) now provides a contradiction, and T^* must have originally satisfied inequality (13) for some constant $\mathcal{R}$.

5. Proof of Theorem 5 Let $\overline{\mathcal{L}}$ be the set of all measurable sets A in $\mathcal{L}$ such that for some f in E, $T^*f(x) = \infty$ a.e. on A, and define

$$(24) \qquad q = \sup m(A), \qquad A \text{ in } \overline{\mathcal{L}}.$$

The key assertion in this proof is that if a set A belongs to $\overline{\mathcal{L}}$, then $T^*f(x) = \infty$ a.e. on A for all f in E, except for a set of the first category in E. Given a set of positive measure A in $\overline{\mathcal{L}}$, define a sequence of subsets of E by

$$K_N = \left\{ f : \int_A \frac{T^*f(x)}{1+T^*f(x)}\,dx \le (1-1/N)\,m(A) \right\},$$

where we interpret $\frac{\infty}{1+\infty} = 1$. Now, I claim that each set K_N is closed in E. For, assume we had a sequence $\{f_n\} \subseteq K_N$ such that $f_n \to f$ in E, and choose an integer M. Then, we can select of subsequence of $\{f_n\}$, which we shall also call $\{f_n\}$, such that $T_kf_n(x) \to T_kf(x)$ a.e. for $1 \le k \le M$. If we define

$$T_M^*g(x) = \max_{1\le k\le M} |T_kg(x)|, \qquad g \text{ in } E,$$

we then have $T_M^*f_n(x) \to T_M^*f(x)$ a.e. in X. Hence by Fatou's lemma,

$$\int_A \frac{T_M^*f(x)\,dx}{1+T_M^*f(x)} \le \liminf_{n\to\infty} \int_A \frac{T_M^*f_n(x)\,dx}{1+T_M^*f_n(x)} \le (1-1/N)\,m(A).$$

But this holds for arbitrary M; thus f must fall into K_N, and K_N is closed. Now, the union of the sets $\{K_N\}$ is precisely the set of all f such that $T^*f(x)$ is finite on a set of positive measure in A. Thus, it only remains to show that each set K_N is actually nowhere dense.

Since A fell in $\bar{\mathcal{L}}$ by assumption, there exists at least one element of E, call it f_0, such that $T^*f_0(x) = \infty$ a.e. on A. Also, for all g in a dense subset of E we have $T^*g(x) < \infty$ a.e. on X and thus $T^*(f_0+g)(x) = \infty$ a.e. on A. But a translation in E of a dense subset of E is still dense in E; thus no set K_N could possibly contain an open set. Hence $\cup K_N$ is of the first category in E.

Now, I claim that $\bar{\mathcal{L}}$ is closed under countable unions; this follows from the fact that a countable intersection of sets with complements of the first category in E is necessarily nonempty. The supremum in (24) can then be attained, and there exists some set X_0 in $\bar{\mathcal{L}}$ such that $m(X_0)=q$. But now, for any f in E we must have $T^*f(x) < \infty$ a.e. on the complement of X_0, and thus parts (a) and (b) of Theorem 5 follow. To prove part (c), let $Y_0 = X_0^w$ and choose f(x) in E such that $T^*f(x) = \infty$ a.e. on X_0. Then, $g(x) = f(w(x))$ belongs to E by hypothesis, and $T^*g(x) \geqslant T^*f(w(x))$ a.e. in X. Hence, there exist null sets N_1, N_2 such that $T^*g(x) = \infty$ as soon as $x \in X-N_1$ and $w(x) \in X_0-N_2$, and $T^*g(x) = \infty$ a.e. on $w^{-1}(X_0) = Y_0$. By part (a), $Y_0 \subseteq X_0$

essentially, and since $w(x)$ is measure-preserving, we have $X_0 = Y_0 = X_0^W$ essentially. Finally, if this takes place for all $w(x)$ in an ergodic family on X, by Lemma 1 $m(X_0)$ equals zero or one.

Proof of Theorem 6 Let $f(x)$ be an arbitrary element of $L^p(X)$, and choose $g(x)$ in the dense subset referred to in condition (ii). Then, $T^*f(x) \leq T^*(f-g)(x) + T^*g(x)$ for all x, and

$$m[\{x : T^*f(x) \geq 2A\}] \leq m[\{x : T^*(f-g)(x) \geq A\}] + m[\{x : T^*g(x) \geq A\}]$$

$$(2A)^p m[\{x : T^*f(x) \geq 2A\}] \leq 2^p \Omega \int_X |f(x)-g(x)|^p dx + o(1)$$

where Ω is a constant independent of $g(x)$. But now $\|f-g\|_p$ can be made arbitrarily small, and (14) follows.

<u>6. Further Examples</u> (1) Let $w(x)$ be an ergodic transformation[9] on X, and $\{u_{nk}\}$ some infinite matrix. Define

$$T_nf(x) = u_{no}f(x) + u_{n1}f(w(x)) + \cdots + u_{nn}f(w^n(x))$$

for all $f(x)$ in $L^1(X)$. The sequence $\{T_n\}$ is then distributive on X, and if we are given that $\lim\sup |T_nf(x)| < \infty$ a.e. for all $f(x)$ in a set of the second category in $L^1(X)$, we conclude that T^* is of weak type (p,p) for all p, $1 \leq p \leq 2$. Inequalities (6) and (7) for $S=T^*$ would then be corollaries. In particular, if $\lim T_nf(x)$ exists a.e. for all $f(x)$ in $L^1(X)$, the limit function would also satisfy these inequalities, and define a bounded linear transformation of $L^p(X)$ into $L^p(X)$ for $1 < p < 2$. Also, it follows from Theorem 6[10] that

$$m\left[\left\{x : \sup_{1 \leq n < \infty} |T_nf(x)| \geq A\right\}\right] = o(1/A^p)$$

for any individual $f(x)$ in $L^p(X)$, $1 \leq p < 2$. Indeed, this could imply that an inequality of maximal ergodic type is involved in all theorems of this sort.

(2) Using Theorem 5, we can prove the existence of a

9--Actually, it is sufficient to assume that $w(x)$ commutes under composition with every member of some ergodic family on X, as for example a power of an ergodic transformation.

10--See footnote 6.

function continuous on the unit interval but nondifferentiable on a set of full measure. More precisely, we can prove that the class of continuous functions which are differentiable on a set of positive measure form a set of the first category in the space of all (periodic) continuous functions on the unit interval.

Following an example of S. Banach, we define

$$(25)\qquad T_n f(x) = \frac{f(x+a_n) - f(x)}{a_n}, \quad a_n \to 0,$$

for all $f(x)$ in the Banach space E of continuous functions on the unit interval satisfying $f(0)=f(1)$, where addition is modulo one. We will now apply Theorem 5 to the sequence $\{T_n\}$. Since it commutes with translations, it is distributive; thus the set X_o of Theorem 5 has measure zero or one. If $m(X_o)=0$, then by Theorem 1 the operator T^* is continuous in measure on E; but $h_n(x) = \frac{1}{n}\sin \pi nx$ satisfies $h_n \to 0$ in E although $T^* h_n(x) \geq 1$ on a set of measure bigger than $\frac{1}{2}$ for all n. Thus, T^* is not continuous in measure, and $m(X_o)=1$, which implies what was to be proven.

(3) It is known [13,p310,I] that there exist integrable functions whose trigonometric Fourier series fail to converge at a single point. Thus, if $s^*(x,f) = \sup_n |s_n(x,f)|$, where $s_n(x,f)$ is the nth partial sum of the Fourier series of $f(x)$, by the corollary to Theorem 1 s^* is not of weak type (1,1). It then follows from the

"alternative" that the Fourier series of every integrable function diverges unboundedly a.e., with the exception of a set of the first category in $L^1(0,2\pi)$.

(4) A still unsettled conjecture of N. Luzin is that the Fourier series of any function in $L^2(0,2\pi)$ converges almost everywhere. By a result of A. C. Calderon [13,p165,II], Luzin's conjecture is equivalent to the assertion that s^* is of weak type (2,2), where s^* is as in Example 3. From Theorem 6, we have that Luzin's conjecture holds if and only if

$$m[\{x : s^*(x,f) \geq A\}] = o(1/A^2)$$

for all f(x) in $L^2(0,2\pi)$. Alternately, from the "alternative" it follows that the set of L^2 functions whose Fourier series converges a.e. is either the entire space or of the first category in $L^2(0,2\pi)$.

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