Arithmetic and Geometry on Triangular Shimura Curves

Thesis by

Shujuan Ji

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Abstract

By a triangular Shimura curve, we mean the canonical model X_Γ of $\Gamma\backslash\mathcal{H}$, the quotient of the upper half plane $\mathcal{H}$ by a cocompact arithmetic subgroup Γ of $Sl_2(\mathbf{R})$ with a triangular fundamental domain. To be concise, let F be a totally real algebraic number field of degree d, and B a quaternion algebra over F, with $B \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R}) \bigoplus \mathbf{H}^{d-1}$, where $\mathbf{H}$ is the Hamilton quaternion algebra. Let O be an order of B, and $\Gamma(O) = \{\gamma \in O : \gamma O = O, N_{B/F}(\gamma) \text{ is totally positive}\}$. A Fuchsian group Γ of the first kind is called arithmetic if it is commensurable with $\Gamma(O)$ for some B and O. Here we are only interested in the arithmetic triangular groups, i.e., those generated by three elliptic elements. If the three generators γ_1, γ_2, γ_3 are of order e_1, e_2, e_3, then we call (e_1, e_2, e_3) its signature.

Our main results are the follows:
We first exhibit, for each arithmetic triangle group Γ, positive integers k such that the space $\mathcal{S}_k(\Gamma)$ of modular forms for Γ of weight k is 1-dimensional (cf. Theorem A, Chapter 2). Then we establish a class of modular functions on a family of coverings of triangular Shimura curve X_Γ, satisfying some arithmetic properties analogous to those of the classical functions $\Delta(Nz)/\Delta(z)$ (cf. Theorem B, Chapter 4). Finally, we provide two explicit examples and illustrate the properties proved.

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Chapter 1 Introduction

Discrete subgroups Γ of $SL_2(\mathbf{R})$ operate on $\mathcal{H} = \{z \in \mathbf{C} | (Im)(z) > 0\}$ by fractional linear transformations $(\gamma, z) \mapsto \frac{az+b}{cz+d}$, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The class of Γ admitting triangular fundamental domains (of finite area) has a long history, and was first studied by H. A. Schwarz.

A well-known example is given by $\Gamma(2) := \{\gamma \in SL_2(\mathbf{Z}) | \gamma \equiv 1 \pmod 2\}$, whose fundamental domain Φ has all of its vertices at infinity ("cusp"), namely at 0, 1, ∞. Even the modular group $SL_2(\mathbf{Z})$ belongs to this class, as its fundamental domain has vertices i, $\rho = e^{2\pi i/3}$ and ∞. The function fields of $\Gamma(2)\backslash\mathcal{H}$ and $SL_2(\mathbf{Z})\backslash\mathcal{H}$ are generated by the classical elliptic modular functions $\lambda(z)$ and $j(z)$, respectively. Moreover, there is a distinguished modular form $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}$, $q = e^{2\pi i z}$ for $SL_2(\mathbf{Z})$, which spans the space of cusp forms of weight 12 for $SL_2(\mathbf{Z})$. By a well-known theorem, one knows that, for any $N \geq 1$, the modular function $\Delta(Nz)/\Delta(z)$ is, when suitably normalized, integral over $\mathbf{Q}[j]$ (see [K-L]). This fact leads to many interesting results in Number Theory and Geometry.

The goal of this thesis is to find analogs of $\Delta(z)$ and prove such an integrality result for "triangular " groups Γ, which are cocompact and arithmetic.

Being arithmetic means Γ is commensurable with the group of units of norm 1 of a maximal order O in a quaternion algebra B over a totally real number field F such that $B \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R}) \oplus \mathbf{H}^{[F:\mathbf{Q}]-1}$ where $\mathbf{H}$ is the algebra of Hamilton's quaternions over $\mathbf{R}$. Γ is cocompact unless $F = \mathbf{Q}$ and $B = M_2(F)$, in which case Γ is commensurable with $SL_2(\mathbf{Z})$.

A complete (finite) list of cocompact arithmetic triangle groups Γ, given by congruence conditions, is available ([Ta], [Sh 1]). Furthermore, one knows by Shimura that the algebraic curve $\Gamma\backslash\mathcal{H}$ and the three vertices are defined over an explicit extension M of F. For each such Γ, we first find weights k such that $\mathcal{S}_k(\Gamma)$ is one-dimensional, generated by an $M-$ rational modular function with a unique zero at one of the vertices of Φ; we call this function $\Delta_{B,\Gamma}$ (or Δ_B for short). There is also an analog j_B of j, given by Shimura's theory, which we normalize to have a simple zero at P_1, simple pole at P_2 and to be integral at P_3. For $\alpha \in B^+$, we also has an automorphy factor $\zeta(\alpha, z)$ (see [Chapter 5]). Our main result is the following:

Main Theorem Let (Γ, B, k) be as above. Then $\forall \alpha \in B^+$, $\zeta(\alpha, z)^k \Delta_B(\alpha z)/\Delta_B(z)$ is integral over $M[j_B]$.

By Shimura's theory of canonical model, we know that any arithmetically defined modular function relative to a congruence subgroup Γ takes values at any CM point z, in a class field of a totally imaginary quadratic extension K_z of F. This in particular applies to our functions $\zeta(\alpha, z)^k \Delta_B(\alpha z)/\Delta_B(z)$. One may view our result as a refinement in a very special case. Since for $F = \mathbf{Q}$, it gives abelian extensions of complex quadratic fields, we are only interested in those $F \neq \mathbf{Q}$.

In the last two chapters, I give two explicit examples Γ^*, Γ of arithmetic triangular groups. It turns out that Γ is a subgroup of Γ^* of index 2, and they are associated to the same B. Analogous to the classical result $j(\lambda) = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$, we also express j_B explicitly in terms of λ_B, where j_B and λ_B are the j analog of Γ^* and Γ respectively.

Chapter 2 Analogs of $\Delta(z)$

In this chapter, we find those Γ and k such that the space of modular forms for Γ of weight k is 1-dimensional.

Notations

F: totally real number field with $[F:\mathbf{Q}]=d$

B: quaternion algebra over F with $B\otimes_{\mathbf{Q}}\mathbf{R}\cong M_2(\mathbf{R})\oplus\mathbf{H}^{d-1}$

ξ: the composite map

$$\alpha\in B\xrightarrow{i} B\bigotimes_{\mathbf{Q}}\mathbf{R}\cong M_2(\mathbf{R})\bigoplus\mathbf{H}\bigoplus\cdots\bigoplus\mathbf{H}\xrightarrow{Pr_1} M_2(\mathbf{R})\ni\xi(\alpha)$$

$B^+=\{b\in B: N_{B/F}(b) \text{ is totally positive}\}$

O: a maximal order of B

τ: a two-sided integral O ideal of B

$\Gamma=\Gamma(O,\tau)=\{\gamma\in B^+:\gamma \text{ is a unit of } O \text{ and } \gamma-1\in\tau\}$

We also use Γ to denote the image of Γ under ξ.

F_Γ is the ray class field of F corresponding to $(\tau\cap O_F)\varpi_0$ where ϖ_0 is the product of all archimedean primes of F.

Fix an arithmetic triangular group Γ with (X_Γ,ϕ) the Shimura canonical model defined over F_Γ. Let $\mathcal{M}(\Gamma)$ (resp. $\mathcal{M}(\Gamma)_0$) be the space of meromorphic modular functions for Γ (resp. rational over F_Γ) and $\mathcal{S}_k(\Gamma)$ (resp. $\mathcal{S}_k(\Gamma)_0$) the space of holomorphic cusp forms of weight k for Γ (resp. rational over F_Γ).

Theorem A For the following Γ and k, $\mathcal{S}_k(\Gamma)$ is one-dimensional, generated by an F_Γ-rational modular form $\Delta_B = \Delta_B(\Gamma, k)$, which is an eigenform of Hecke operators. Moreover Δ_B is non-zero everywhere except at a unique elliptic point.

signature of Γ	k	order of the elliptic point where $\Delta_B = 0$
(2, 3, 8)	12, 16, 32	8, 3, 3
(2, 4, 5)	8, 16, 24, 32	5, 5, 5, 5
(2, 3, 10)	12, 20	10, 3
(2, 5, 6)	12	5
(2, 3, 7)	12, 24, 28, 36, 42, 48, 56, 60, 72	7, 7, 3, 7, 2, 7, 3, 7, 7
(2, 3, 9)	18	2
(2, 3, 11)	12, 24	11, 11

Proof.

Let $\Gamma =< \gamma_1, \gamma_2, \gamma_3 >$ with $\gamma_1^{e_1} = \gamma_2^{e_2} = \gamma_3^{e_3} = 1$ as an automorphism of $\mathcal{H}$. Assume P_1, P_2, P_3 are fixed points of $\gamma_1, \gamma_2, \gamma_3$ respectively. For any $P \in \mathcal{H}$, denote by $\overline{P}$ the image of P under the projection $\mathcal{H} \to \Gamma \backslash \mathcal{H}$. Let G_P be the isotropy group of P and $e(P)$ the order of G_P; then $e(P_i) = e_i$ for $i = 1, 2, 3$ and $e(P) = 1$ for other P. Choose a local parameter z_P such that G_P operates on z_P by multiplication by e-th roots of unity; then $t = (z_P)^e$ is a local parameter of $\overline{P}$ in $\Gamma \backslash \mathcal{H}$.

Let $O_P(f)$ be the order of f at P and $O_{\overline{P}}(\omega)$ the order of ω at $\overline{P}$ in $\Gamma \backslash \mathcal{H}$. First, we will use the Riemann-Roch theorem to prove the following formula:

If the signature of Γ is (e_1, e_2, e_3), then

$$\dim \mathcal{S}_2(\Gamma) = 0; \tag{2.1}$$

for even $k > 2$, we have

$$\dim \mathcal{S}_k(\Gamma) = \left[\frac{(e_1 - 1)k}{2e_1}\right] + \left[\frac{(e_2 - 1)k}{2e_2}\right] + \left[\frac{(e_3 - 1)k}{2e_3}\right] - k + 1, \tag{2.2}$$

and for any $f \in \mathcal{M}(\Gamma)$,

$$\text{and} \sum_{P\neq P_1,P_2,P_3} O_P(f) + \frac{O_{P_1}(f)}{e_1} + \frac{O_{P_2}(f)}{e_2} + \frac{O_{P_3}(f)}{e_3} = \left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right)\frac{k}{2}. \quad (2.3)$$

Let $f \in \mathcal{S}_k(\Gamma)$, then $\omega = f(dz)^{k/2}$ is invariant under Γ, hence represents a holomorphic differential form of $\Gamma\backslash\mathcal{H}$. We have

$$\begin{aligned} \omega &= f(t)(dt)^{k/2} \\ &= ut^{(O_{\overline{P}}(\omega))}(dt)^{k/2} \\ &= u(z_P)^{e(O_{\overline{P}}(\omega))}(e(z_P)^{e-1}dz_P)^{k/2} \\ &= ue^{k/2}(z_P)^{e(O_{\overline{P}}(\omega))+k(e-1)/2}(dz_P)^{k/2}, \end{aligned}$$

where u is locally holomorphic and nonzero around P.

Thus $O_P(f) = eO_{\overline{P}}(\omega) + k(e-1)/2$.

As we know, on an algebraic curve of genus g, the sum of the orders of a differential form of degree 1 is equal to $2g-2$. Here $g = 0$. Hence:

$$\sum_{P\neq P_1,P_2,P_3} O_{\overline{P}}(\omega) + O_{\overline{P_1}}(\omega) + O_{\overline{P_2}}(\omega) + O_{\overline{P_3}}(\omega) = (-2)\frac{k}{2} = -k,$$

i.e.

$$\sum_{P\neq P_1,P_2,P_3} O_P(f) + \frac{O_{P_1}(f)}{e_1} - \left(1 - \frac{1}{e_1}\right)\frac{k}{2} + \frac{O_{P_2}(f)}{e_2} - \left(1 - \frac{1}{e_2}\right)\frac{k}{2} + \frac{O_{P_3}(f)}{e_3} - \left(1 - \frac{1}{e_3}\right)\frac{k}{2} = -k,$$

$$\sum_{P\neq P_1,P_2,P_3} O_P(f) + \frac{O_{P_1}(f)}{e_1} + \frac{O_{P_2}(f)}{e_2} + \frac{O_{P_3}(f)}{e_3} = \left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right)\frac{k}{2},$$

which is (2.3).

The map $f \mapsto f(dz)^{k/2}$ gives an isomorphism between $\mathcal{S}_k(\Gamma)$ and Ω_k, the space of differential forms ω on $\Gamma\backslash\mathcal{H}$ of degree $k/2$ such that

$$O_{P_i}(f) = e_i O_{\overline{P_i}}(\omega) + k(e_i - 1)/2 \geq 0.$$

As the $O_{P_i}(f)$ are always integers, for $k = 1$, this map is an the isomorphism between $\mathcal{S}_k(\Gamma)$ and the space of all holomorphic differential forms on $\Gamma\backslash\mathcal{H}$. Hence $\dim\mathcal{S}_2(\Gamma) = g = 0$, and (2.1) is proved.

If K denotes the canonical class of $\Gamma\backslash\mathcal{H}$, and

$$D_k = (k/2)K + \sum_{i=1}^{3}[k(e_i - 1)/2e_i]P_i,$$

then Ω_k is isomorphic to
$L(D_k) = \{f \in$ meromorphic functions on $\Gamma\backslash\mathcal{H} : f = 0$ or $\mathrm{div}(f) \geq -D_k\}$. Hence $\dim\mathcal{S}_k(\Gamma) = l(D_k)$.

By the Riemann-Roch Theorem,

$$l(D_k) = \deg(D_k) - g + 1 + l(\mathrm{div}(\omega) - D_k)$$

where ω is a non-zero differential form on $\Gamma\backslash\mathcal{H}$.
Here

$$g = 0,$$

and

$$\begin{aligned} \deg(D_k) &= (k/2)(2g-2) + \sum_{i=1}^{3}\left[\frac{k(e_i - 1)}{2e_i}\right] \\ &= -k + \sum_{i=1}^{3}\left[\frac{k(e_i - 1)}{2e_i}\right]. \end{aligned}$$

Now, we need the following

Lemma 2.1 *For any pair of positive integers m, n,*

$$\left[\left(1-\frac{1}{m}\right)n\right] \geq \left(1-\frac{1}{m}\right)(n-1).$$

Proof of the Lemma.

If $m|n$, then $[(1-\frac{1}{m})n] = (1-\frac{1}{m})n \geq (1-\frac{1}{m})(n-1)$.

If m does not divide n, then $[\frac{n}{m}] \leq \frac{n-1}{m}$.

Hence

$$\left[\left(1-\frac{1}{m}\right)n\right] = n - \left[\frac{n}{m}\right] - 1 \geq n - \frac{n-1}{m} - 1 = \left(1-\frac{1}{m}\right)(n-1).$$

Applying the Lemma with $m = e_i$ and $n = \frac{k}{2}$, we have

$$\left[\frac{k(e_i-1)}{2e_i}\right] = \left[\left(1-\frac{1}{e_i}\right)\frac{k}{2}\right] \geq \left(1-\frac{1}{e_i}\right)\left(\frac{k}{2}-1\right)$$

for $i = 1, 2, 3$. Since Φ is a hyperbolic triangle, by Theorem 11 [Ford, page 247], we also know that $\sum_{i=1}^{3}\frac{1}{e_i} < 1$. Hence for $k > 2$,

$$\deg(D_k) \geq -k + \left(3 - \sum_{i=1}^{3}\frac{1}{e_i}\right)\left(\frac{k}{2}-1\right) > -k + 2\left(\frac{k}{2}-1\right) = -2.$$

So $\deg(D_k) \geq -1$, and $\deg(\mathrm{div}(\omega) - D_k) = \deg(K) - \deg(D_k) < 0$.

Therefore $l(\mathrm{div}(\omega) - D_k) = 0$.

Moreover, the Riemann-Roch Theorem

$l(D_k) = \deg(D_k) - g + 1 + l(\mathrm{div}(\omega) - D_k)$ gives $l(D_k) = -k + 1 + \sum_{i=1}^{3}[\frac{k(e_i-1)}{2e_i}]$, i.e.

$$\dim\mathcal{S}_k(\Gamma) = \left[\frac{(e_1-1)k}{2e_1}\right] + \left[\frac{(2e_2-1)k}{e_2}\right] + \left[\frac{(2e_3-1)k}{e_3}\right] - k + 1$$

for $k > 2$.

Hence (2.2) is proved.

For the proof of (2.1) & (2.2), one can also use [Sh 2, Section 2.6] which is valid for a general Fuchsian group of the first kind.

Next we will use the formula (2.1) to (2.3) to find those arithmetic triangular groups Γ and integers k such that $\dim\mathcal{S}_k(\Gamma) = 1$. We will use the table in [Sh 1, page 82]. For those groups in the table, we calculate $\dim\mathcal{S}_k(\Gamma)$ and the possible divisors for even $k \geq 2$ such that $\left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right)\frac{k}{2} < 1$. See the last Chapter for a list of the tables.

The table in the Proposition is a complete list of Γ's in Shimura's table and k's for which one knows explicitly from the above formular that $\Delta_B(\Gamma, k)$ takes zero at only one elliptic point .

Since X_Γ is defined over F_Γ, and $\mathcal{S}_k(\Gamma) \cong H^0(X_{\Gamma/\mathbf{C}}, \underline{\omega}_k)$ where $\underline{\omega}_k$ is the sheaf of modular forms of weight k which is also rational over F_Γ, this cohomology group evidently admits an F_Γ structure $\mathcal{S}_k(\Gamma)_0$. We choose Δ_B to come from $\mathcal{S}_k(\Gamma)_0$. It is obviously a Hecke eigenform as $\dim\mathcal{S}_k(\Gamma) = 1$.

Chapter 3 Anologs of $j(z)$

Without loss of generality, we may assume the Δ_B in the previous chapter has zeroes only at P_2. In this chapter, we modify the Shimura Canonical model to get a new parametrization j_B with a simple zero at P_1 and a simple pole at P_2, and such that it is integral at P_3.

For any CM point z, let K_z be the associated totally imaginary quadratic extension of F which can be F linearly embedded into B. By Shimura's Main Theorem 1 [Sh 1, page 73], $F_\Gamma(\phi(z)) = M_z$ is a finite abelian unramified extension of K_z. If the class number of K_z is 1, then $M_z = K_z$. Since P_i (for $i = 1$ to 3) is the fixed point of γ_i, P_i is a CM point (see [Sh 1, page 66]). Let $M_\Gamma = M_{P_1} M_{P_2} M_{P_3}$.

Proposition 3.1 *There exists a modular function $j_B = j_B(\Gamma, k)$ rational over M_Γ, such that $\mathcal{M}(\Gamma)_0 \otimes_{F_\Gamma} M_\Gamma = M_\Gamma(j_B)$, $div(j_B) = (P_1) - (P_2)$, and $j_B(P_3)$ is integral (in M_Γ).*

Proof. As (X_Γ, ϕ) is the Shimura canonical model, ϕ gives a birational isomorphism of $\Gamma\backslash\mathcal{H}$ to $X_\Gamma(\mathbf{C})$. Therefore ϕ has a simple zero X and a simple pole Y which are both $F_\Gamma-$ rational. From the above argument, our j_B can be obtained, up to a non-zero scalar in M_Γ, from ϕ via an automorphism of $\mathbf{P}^1$ over M_Γ which sends X, Y to P_1, P_2 respectively. Consequently, j_B is rational over M_Γ. For any CM point z, $j_B(z)$ will take values in $M_z M_\Gamma$. In particular, $j_B(P_3) \in M_\Gamma$. Now, we normalize j_B such that $j_B(P_3)$ is integral.

Q. E. D.

Remarks:

1. This property of j_B is an anolog of the classical property of the j–function, namely: $j(\infty) = \infty$, $j(i) = 0$ and $j(\rho) = 1728 \in \mathbf{Z}$.

2. Some explicit examples have been developed in Chapters 5 & 6, where the class number of the relevant CM fields are always 1, so $M_\Gamma = K_{P_1}K_{P_2}K_{P_3}$, the compositum of the fields attached to P_1, P_2, P_3.

Chapter 4 Main Theorem

Theorem B Fixing any Γ and k in the table of Theorem A, for $\alpha \in B^+$, set

$$\phi_\alpha(z) = \left(\frac{(det(\xi(\alpha))}{j(\xi(\alpha), z)^2}\right)^k \frac{\Delta_B(\alpha z)}{\Delta_B(z)}.$$

Then ϕ_α is a modular function for $\Gamma_\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$. Moreover, ϕ_α is integral over $M_\Gamma[j_B]$.

Now we will first give an immediate Corollary and a remark, leaving the proofs of the Theorem and the Corollary to the end of the Chapter.

Corollary For each α as above, there exists a non-zero $\beta_\alpha \in O_{M_\Gamma}$ such that for any CM point z, $\beta_\alpha\phi_\alpha(z)$ is an algebraic integer whenever $j_B(z)$ is. In particular, $\beta_\alpha\phi_\alpha(P_3)$ is integral in $M_\Gamma K(P_3)^{ab}$.

Remark: The theorem above is the analog of the classical result that $\Delta(Nz)/\Delta(z)$ is integral over $\mathbf{Q}[j]$, $\forall N \geq 1$. But in that case, one can further conclude, by using $q-$ expansions, that $\Delta(Nz)/\Delta(z)$ is in fact integral over $\mathbf{Z}[j]$. We are not able to obtain this refinement, owing to the lack of cusps (and Fourier expansions) in our case. For us, the role of ∞ is played by the elliptic CM point P_2. Since P_2 is associated to an (anisotropic) form T, one may however expand modular functions at P_2 using the characters of $T(\mathbf{A})/T(F)$. Such things have been investigated by A. Mori ([Mo]) at arbitary CM points. One of our future goals is to use this to give a finer version of Theorem B.

Now we begin the proof of Theorem B. We first need the following

Lemma 4.1 $\Delta_B|_\alpha = \left(\frac{det(\xi(\alpha))}{j(\xi(\alpha),z)^2}\right)^k \Delta_B(\alpha z)$ *is a modular form for* $\alpha^{-1}\Gamma\alpha$. *Hence* $\phi_\alpha = \frac{\Delta_B|_\alpha}{\Delta_B}$ *is a modular function for* $\Gamma_\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$. *Moreover, for any* $\gamma \in \Gamma$, *we have*

$$\phi_{\gamma\alpha}(z) = \phi_\alpha(z) \tag{4.1}$$

and

$$\phi_{\alpha\gamma}(z) = \phi_\alpha(\gamma z) = \phi_\alpha|_\gamma(z). \tag{4.2}$$

Proof.

$$\begin{aligned}
(\Delta_B|_\alpha)|_{\alpha^{-1}\gamma\alpha}(z) &= \frac{det(\xi(\alpha^{-1}\gamma\alpha))^k \Delta_B|_\alpha(\alpha^{-1}\gamma\alpha z)}{j(\xi(\alpha^{-1}\gamma\alpha), z)^{2k}} \\
&= \frac{det(\xi(\alpha^{-1}\gamma\alpha))^k det(\xi(\alpha))^k \Delta_B(\gamma\alpha z)}{j(\xi(\alpha^{-1}\gamma\alpha), z)^{2k} j(\xi(\alpha), \alpha^{-1}\gamma\alpha z)^{2k}} \\
&= \frac{det(\xi(\gamma\alpha))^k \Delta_B(\gamma\alpha z)}{j(\xi(\gamma\alpha), z)^{2k}} \\
&= \frac{det(\xi(\gamma))^k det(\xi(\alpha))^k \Delta_B(\alpha z) j(\xi(\gamma), \alpha z)^{2k}}{j(\xi(\gamma), \alpha z)^{2k} j(\xi(\alpha), z)^{2k} det(\xi(\gamma))^k} \\
&= \Delta_B|_\alpha(z).
\end{aligned}$$

Hence $\phi(\alpha)$ is invariant under Γ_α.

$$\begin{aligned}
\phi_{\gamma\alpha}(z) &= \frac{det(\xi(\gamma\alpha))^k \Delta_B(\gamma\alpha z)}{j(\xi(\gamma\alpha), z)^{2k} \Delta_B(z)} \\
&= \frac{det(\xi(\gamma))^k det(\xi(\alpha))^k \Delta_B(\alpha z) j(\xi(\gamma), \alpha z)^{2k}}{j(\xi(\gamma), \alpha z)^{2k} j(\xi(\alpha), z)^{2k} det(\xi(\gamma))^k \Delta_B(z)} \\
&= \phi_\alpha(z).
\end{aligned}$$

$$\phi_\alpha|_\gamma(z) = \phi_\alpha(\gamma z)$$

$$
\begin{aligned}
&= \frac{det(\xi(\alpha))^k \Delta_B(\alpha\gamma z)}{j(\xi(\alpha),\gamma z)^{2k}\Delta_B(\gamma z)} \\
&= \frac{det(\xi(\alpha))^k \Delta_B(\alpha\gamma z) det(\xi(\gamma))^k}{j(\xi(\alpha),\gamma z)^{2k} j(\xi(\gamma),z)^{2k}\Delta_B(z)} \\
&= \frac{det(\xi(\alpha\gamma))^k \Delta_B(\alpha\gamma z)}{j(\xi(\alpha\gamma),z)^{2k}\Delta_B(z)} \\
&= \phi_{\alpha\gamma}(z).
\end{aligned}
$$

Hence the lemma is proved.

Proof of Theorem B (contd.)

Now, let $\Gamma\alpha\Gamma = \bigcup_{i=1}^r \Gamma\alpha_i$ be disjoint union of right cosets, ψ be any elementary symmetric function of $\{\phi_{\alpha_i}, i = 1 \cdots r\}$. Then from the above lemma, ϕ_{α_i} depends only on the right coset where α_i lies and $\{\phi_{\alpha_i}|_\gamma, i = 1 \cdots r\}$ is just a permutation of $\{\phi_\alpha, i = 1 \cdots r\}$ for any $\gamma \in \Gamma$. So $\psi|_\gamma = \psi$. Consequently, $\psi \in \mathbf{C}(j_B)$.

Assume $\psi = \frac{f(j_B)}{g(j_B)}$ where f, g are relatively prime polynomials, then ψ has a pole at any point z such that $j_B(z)$ is a root of g. Since ϕ_{α_i} (hence ψ) has poles only at points Γ-equivalent to P_2 and $j_B(P_2) = \infty$, g must be a constant, i.e. $\psi \in \mathbf{C}[j_B]$.

Since Δ_B is F_Γ rational, the map $\tilde{\alpha} : \mathcal{S}_k(\Gamma)_0 \to \mathcal{S}_k(\alpha^{-1}\Gamma\alpha)_0$ with $\Delta_B \mapsto \Delta_B|_\alpha$ is defined over F_Γ from the theory of canonical models. Therefore $\psi \in M_\Gamma[j_B]$. Hence ϕ_α is a root of the monic polynomial $\prod_{i=1}^r (x - \phi_{\alpha_i}) \in M_\Gamma[j_B][x]$.

Q. E. D.

Proof of the Corollary.

Assume ϕ_α is a root of the polynomial

$$\beta_\alpha x^n + a_{n-1}(j_B)x^{n-1} + \cdots + a_i(j_B)x^i + \cdots + a_0(j_B)$$

with $0 \neq \beta_\alpha \in O_{M_\Gamma}, a_i(j_B) \in O_{M_\Gamma}[j_B]$ for $i = 0 \cdots n-1$, then one can check that $\beta_\alpha\phi_\alpha$ is a root of the polynomial

$$x^n + a_{n-1}(j_B)x^{n-1} + \cdots + \beta_\alpha^{n-1-i}a_i(j_B)x^i + \cdots + \beta_\alpha^{n-1}a_0(j_B).$$

Evaluate the polynomial at CM point z. If $j_B(z)$ is an algebraic integer, then it is a mononic polynomial with integral coefficients. Therefore $\beta_\alpha\phi_\alpha(z)$ is an algebraic integer.

Q. E. D.

Chapter 5 Two explicit examples

In this chapter, we will give two examples of arithmetic triangular groups.

Proposition 5.1 *Let $F = \mathbf{Q}(\sqrt{2})$, $B = F + Fi + Fj + Fk$ where $i^2 = -3$, $j^2 = \sqrt{2}$ and $k = ij = -ji$. $x = \frac{1+i}{2}$, $y = \frac{\sqrt{2}-1}{2} + \frac{(\sqrt{2}-1)i}{6} + \frac{j}{2} + \frac{k}{2}$, $z = \frac{j}{2} + \frac{k}{2}$.*
$O = Z[\sqrt{2}][1, x, y, z]$. Then O is a maximal order of B.

Proof

Step 1 Let's check that x, y, z are integers in B.

	x	y	z
reduced trace $Tr_{B/F}$	1	$\sqrt{2}-1$	0
reduced norm $N_{B/F}$	1	$1-\sqrt{2}$	$-\sqrt{2}$

Step2 We prove that every element in O is an integer.

Let B/F be any quaternion algebra. For $u_1, u_2, \cdots u_{n+1} \in B$, define

$$D(u_1, u_2, \cdots u_{n+1}) = N_{B/F}(u_1+u_2+\cdots+u_{n+1}) - N_{B/F}(u_1+u_2+\cdots+u_n) - N_{B/F}(u_{n+1}).$$

Lemma 5.1

$$D(su_1, tu_2) = stD(u_1, u_2),$$

for any $s, t \in F$. And

$$D(u_1, u_2, \cdots u_{n+1}) = \sum_{r=1}^{n} D(u_r, u_{n+1}).$$

Proof

Assume $B = F[1, i, j, k]$ where $i^2 = a$, $j^2 = b$ and $k = ij = -ji$.

Let $u_r = a_r + b_r i + c_r j + d_r k$ with $a_r, b_r, c_r, d_r \in F$ for $r = 1, \cdots n+1$, then

$$D(u_1, u_2) = N_{B/F}(u_1 + u_2) - N_{B/F}(u_1) - N_{B/F}(u_2)$$

$$
\begin{aligned}
&= (a_1+a_2)^2 - a(b_1+b_2)^2 - b(c_1+c_2)^2 + ab(d_1+d_2)^2 \\
&\quad - (a_1^2 - ab_1^2 - bc_1^2 + abd_1^2) \\
&\quad - (a_2^2 - ab_2^2 - bc_2^2 + abd_2^2) \\
&= 2(a_1a_2 - ab_1b_2 - bc_1c_2 + abd_1d_2).
\end{aligned}
$$

Hence

$$D(su_1, tu_2) = stD(u_1, u_2).$$

$$
\begin{aligned}
D(u_1, u_2, \cdots u_{n+1}) &= D(\sum_{r=1}^{n} u_r, u_{n+1}) \\
&= 2((\sum_{r=1}^{n} a_r)a_{n+1} - a(\sum_{r=1}^{n} b_r)b_{n+1} - b(\sum_{r=1}^{n} c_r)c_{n+1} + ab(\sum_{r=1}^{n} d_r)d_{n+1}) \\
&= \sum_{r=1}^{n}(2(a_ra_{n+1} - ab_rb_{n+1} - bc_rc_{n+1} + abd_rd_{n+1})) \\
&= \sum_{r=1}^{n} D(u_r, u_{n+1}).
\end{aligned}
$$

Therefore we have the following

Lemma 5.2 *Let O_F be the integer ring of F. If for all $1 \leq p, q \leq n$, $D(u_p, u_q) \in O_F$ and u_p are integers of B, then $\sum_{p=1}^{n} a_pu_p$ are integers of B for any $a_p \in O_F$.*

And also, we have the following

$D-$**table**:

	1	x	y	z
1	2	1	$\sqrt{2}-1$	0
x	1	2	$\sqrt{2}-1$	0
y	$\sqrt{2}-1$	$\sqrt{2}-1$	$2(1-\sqrt{2})$	$-\sqrt{2}$
z	0	0	$-\sqrt{2}$	$-2\sqrt{2}$

From the above Lemma and the $D-$table, it is easy to see that every element in O is an integer of B.

Step 3 We show that O ia a ring.

Obviously, O is closed under addition. So we only need to check its multiplication table.

Multiplication table:

	1	x	y	z
1	1	x	y	z
x	x	$x-1$	$(\sqrt{2}-1)x-y+z$	$(\sqrt{2}-1)+(\sqrt{2}-1)x-3y+2z$
y	y	$-(\sqrt{2}-1)+2y-z$	$(\sqrt{2}-1)y+(\sqrt{2}-1)$	$1-(\sqrt{2}-1)x-(\sqrt{2}-1)y+(\sqrt{2}-1)z$
z	z	$-(\sqrt{2}-1)-(\sqrt{2}-1)x+3y-z$	$(\sqrt{2}-1)+(\sqrt{2}-1)x+(\sqrt{2}-1)y$	2

Hence, we showed that O is an order of B.

Step 4 Let $u_1=1, u_2=x, u_3=y, u_4=z$, then

$$(Tr_{B/F}(u_iu_j)) = \begin{pmatrix} 2 & 1 & \sqrt{2}-1 & 0 \\ 1 & -1 & 0 & 0 \\ \sqrt{2}-1 & 0 & 1 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 2\sqrt{2} \end{pmatrix}.$$

Hence the reduced discriminant of O

$$\mathrm{disc}(O) = |\det((Tr_{B/F}(u_iu_j))|^{1/2} = \sqrt{2}.$$

Since the only finite prime of F which ramifies in B is $\sqrt{2}$, O is a maximal order of B by Corollary 5.3 [Vi, Page 94].

Q. E. D.

Let $\Gamma^* = \{\gamma \in B^+ : \gamma O = O\gamma\}$ and $\Gamma = \{\gamma \in B^+ : N_{B/F}(\gamma) = 1\}$.

Let $K = \mathbf{Q}(\sqrt[4]{2})$, a real quadratic extension of F. Fix an embedding

$$B \hookrightarrow M_2(K) \hookrightarrow M_2(\mathbf{R})$$

with

$$i \mapsto \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$$

$$j \mapsto \begin{pmatrix} \sqrt[4]{2} & 0 \\ 0 & -\sqrt[4]{2} \end{pmatrix}$$

$$k \mapsto \begin{pmatrix} 0 & -\sqrt[4]{2} \\ -3\sqrt[4]{2} & 0 \end{pmatrix}.$$

Then

$$x \mapsto \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

$$y \mapsto \begin{pmatrix} \frac{\sqrt{2}-1+\sqrt[4]{2}}{2} & \frac{\sqrt{2}-1-\sqrt[4]{2}}{6} \\ -\frac{\sqrt{2}-1+\sqrt[4]{2}}{2} & \frac{\sqrt{2}-1-\sqrt[4]{2}}{2} \end{pmatrix}.$$

$$z \mapsto \begin{pmatrix} \frac{\sqrt[4]{2}}{2} & -\frac{\sqrt[4]{2}}{2} \\ -\frac{3\sqrt[4]{2}}{2} & -\frac{\sqrt[4]{2}}{2} \end{pmatrix}$$

Identifying B with its image in $M_2(K)$, let

$$\eta_1 = x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

$$\eta_2 = 1 + x + y = \begin{pmatrix} \frac{2+\sqrt{2}+\sqrt[4]{2}}{2} & \frac{2+\sqrt{2}-\sqrt[4]{2}}{6} \\ -\frac{2+\sqrt{2}+\sqrt[4]{2}}{2} & \frac{2+\sqrt{2}-\sqrt[4]{2}}{2} \end{pmatrix}$$

$$\eta_3 = \eta_1\eta_2 = \begin{pmatrix} 0 & \frac{2+\sqrt{2}-\sqrt[4]{2}}{3} \\ -(2+\sqrt{2}+\sqrt[4]{2}) & 0 \end{pmatrix}$$

$$\gamma_1 = \eta_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

$$\gamma_2 = \frac{1}{2+\sqrt{2}}{\eta_2}^2 = \begin{pmatrix} \frac{\sqrt{2}+\sqrt[4]{2}}{2} & \frac{2+\sqrt{2}-\sqrt[4]{2}}{6} \\ -\frac{2+\sqrt{2}+\sqrt[4]{2}}{2} & \frac{\sqrt{2}-\sqrt[4]{2}}{2} \end{pmatrix}$$

$$\gamma_3 = \gamma_1\gamma_2 = \frac{1}{2+\sqrt{2}}\eta_3\eta_2 = \begin{pmatrix} -\frac{1}{2} & \frac{2\sqrt{2}+1-2\sqrt[4]{2}}{6} \\ -\frac{2\sqrt{2}+1+2\sqrt[4]{2}}{2} & -\frac{1}{2} \end{pmatrix}.$$

Then as an element of the group Aut$\mathcal{H}$ of all analytic automorphisms on $\mathcal{H}$, ${\eta_1}^3 = {\eta_2}^8 = {\eta_3}^2 = 1$ and $\gamma^3 = \gamma^4 = \gamma^3 = 1$. It is easy to check that $\Gamma^* = \langle \eta_1, \eta_2, \eta_3 \rangle$, $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$, and $[\Gamma^* : \Gamma] = 2$ with $\Gamma^* = \Gamma \cup \Gamma\eta_2 = \Gamma \cup \Gamma\eta_3$.

Let $Q_1, Q_2, Q_3; P_1, P_2, P_3$ be the fixed point of $\eta_1, \eta_2, \eta_3, \gamma_1, \gamma_2, \gamma_3$ respectively. Now we look for the fields K_{Γ^*} and K_Γ.

The characteristic polynomials for η_1, η_2, η_3 are:

$$P_{\eta_1}(x) = x^2 - x + 1$$

$$P_{\eta_2}(x) = x^2 - (2+\sqrt{2})x + (2+\sqrt{2})$$

$$P_{\eta_3}(x) = x^2 + (2+\sqrt{2}).$$

Hence the CM fields corresponding to Q_1, Q_2, Q_3 respectively are:

$$K_{Q_1} = F(\sqrt{3}i) = \mathbf{Q}(\sqrt{2}, \sqrt{3}i)$$

$$K_{Q_2} = F(\sqrt{2}i) = \mathbf{Q}(\sqrt{2}, i)$$

$$K_{Q_3} = F(\sqrt{2+\sqrt{2}}i).$$

Using the software tool "Pari", one knows the class numbers of $K_{Q_1}, K_{Q_2}, K_{Q_3}$ are all 1. Hence $M_{Q_i} = K_{Q_i}$ for $i = 1, 2, 3$. We have
$M_{\Gamma^*} = K_{Q_1} K_{Q_2} K_{Q_3} = \mathbf{Q}(\sqrt{2+\sqrt{2}}, \sqrt{3}, i)$.

The characteristic polynomials for $\gamma_1, \gamma_2, \gamma_3$ are:

$$P_{\gamma_1}(x) = x^2 - x + 1$$

$$P_{\gamma_2}(x) = x^2 - \sqrt{2}x + 1$$

$$P_{\gamma_3}(x) = x^2 = x + 1.$$

So

$$K_{P_1} = F(\sqrt{3}i) = \mathbf{Q}(\sqrt{2}, \sqrt{3}i)$$

$$K_{P_2} = F(\sqrt{2}i) = \mathbf{Q}(\sqrt{2}, i)$$

$$K_{P_3} = K_{P_1}.$$

Note that here P_1 and P_3 come from the same CM field. But they correspond to different embedding, hence representing different branches (see [Sh 1, page 72]).

Again $\phi(P_i) \in M_{P_i}$. Hence

$$M_\Gamma = K_{P_1} K_{P_2} K_{P_3} = F(\sqrt{3}, i) = \mathbf{Q}(\sqrt{2}, \sqrt{3}, i).$$

Chapter 6 An explicit relation

In classical case, for the canonical level 2 modular function

$$\lambda : \Gamma(2)\backslash\mathcal{H}^* \xrightarrow{\lambda} \mathbf{P}^1(\mathbf{C}),$$

the map from the λ-line to the j-line is given by

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

We also have this kind of result for certain triangular groups.

Theorem C Let Γ^*, Γ be as in the previous chapter. Let j_B be the modular function of Γ^* such that $\mathrm{div}(j_B) = (Q_1) - (Q_2)$ and $j_B(Q_3) = 1$, and λ_B be the modular function of Γ with $\mathrm{div}(\lambda_B) = (P_1) - (P_3)$ and $\lambda_B(P_2) = 1$. Then

$$j_B = -\frac{4\lambda_B}{(1 - \lambda_B)^2}.$$

Proof. (See the end of the Chapter for fundamental domains of Γ^* and Γ)

It is easy to see

$$Q_1 = P_1 = \eta_3 P_3 = \frac{\sqrt{3}}{3} i$$

$$Q_2 = P_2 = \frac{-\sqrt[4]{2} + \sqrt{2} i}{2 + \sqrt{2} + \sqrt[4]{2}}$$

$$Q_3 = \frac{\sqrt{2 + \sqrt{2}} i}{2 + \sqrt{2} + \sqrt[4]{2}}$$

$$P_3 = \eta_3 P_1 = \frac{\sqrt{3} i}{2\sqrt{2} + 1 + 2\sqrt[4]{2}}.$$

Denote by $[A]^*$ ($[A]$) the Γ^*-equivalent (Γ-equivalent) class represented by A.

Let P_r be the natural projection:

$$\Gamma\backslash\mathcal{H} \stackrel{P_r}{\to} \Gamma^*\backslash\mathcal{H}.$$

One sees $P_r^{-1}\{[Q_1]^*\} = \{[P_1],[P_3]\}$, $P_r^{-1}\{[Q_2]^*\} = \{[Q_2]\}$, $P_r^{-1}\{[Q_3]^*\} = \{[Q_3]\}$.

Noticing that $\lambda_B|_{\eta_3} \in \mathcal{M}(\Gamma)$ and

$$\begin{aligned}
\lambda_B|_{\eta_3}(P_1) &= \lambda_B(\eta_3 P_1) = \lambda_B(P_3) = \infty \\
\lambda_B|_{\eta_3}(P_3) &= \lambda_B(\eta_3 P_3) = \lambda_B(P_1) = 0 \\
\lambda_B|_{\eta_3}(P_2) &= \lambda_B|_{\eta_2}(P_2) = \lambda_B(\eta_2 P_2) = \lambda_B(P_2) = 1,
\end{aligned}$$

we have $\lambda_B|_{\eta_3} = \frac{1}{\lambda_B}$.

Now look at $\frac{1}{1-\lambda_B} \in \mathcal{M}(\Gamma)$. We have

$$\begin{aligned}
&\tfrac{1}{1-\lambda_B}(P_1) = 1 \\
&\tfrac{1}{1-\lambda_B}(P_3) = 0 \\
&\tfrac{1}{1-\lambda_B}(P_2) = \infty \\
&(\tfrac{1}{1-\lambda_B})|_{\eta_3}(P_1) = \tfrac{1}{1-\lambda_B}(\eta_3 P_1) = \tfrac{1}{1-\lambda_B}(P_3) = 0 \\
&(\tfrac{1}{1-\lambda_B})|\eta_3(P_2) = (\tfrac{1}{1-\lambda_B})|_{\eta_2}(P_2) = \tfrac{1}{1-\lambda_B}(\eta_2 P_2) = \tfrac{1}{1-\lambda_B}(P_2) = \infty.
\end{aligned}$$

Hence as a modular function of Γ

$$\operatorname{div}\left(\frac{1}{1-\lambda_B}(\frac{1}{1-\lambda_B})|_{\eta_3}\right) = ([P_1]) + ([P_3]) - 2([P_2]).$$

View it as a modular function of Γ^*,

$$\operatorname{div}\left(\frac{1}{1-\lambda_B}(\frac{1}{1-\lambda_B})|_{\eta_3}\right) = ([Q_1]^*) - ([Q_2]^*) = \operatorname{div}(j_B),$$

so they are the same modular function of Γ^* up to a scalar multiplication.

As

$$(\frac{1}{1-\lambda_B})|_{\eta_3} = \frac{1}{1-\lambda_B|_{\eta_3}} = \frac{1}{1-\frac{1}{\lambda_B}} = \frac{\lambda_B}{\lambda_B - 1},$$

we have

$$-\frac{\lambda_B}{(1-\lambda_B)^2} = Cj_B$$

for some nonzero constant C.

Observe that

$$\lambda_B{}^2(Q_3) = \lambda_B(Q_3)\lambda_B(\eta_3 Q_3) = \lambda_B(Q_3)\lambda_B|_{\eta_3}(Q_3) = 1,$$

so $\lambda_B(Q_3) = \pm 1$.

Since $\lambda_B(P_2) = 1$, P_2 and Q_3 are not Γ-equivalent, $\lambda_B(Q_3)$= -1. Combining this with the fact that $j_B(Q_3) = 1$, we conclude $C = \frac{1}{4}$. So

$$-\frac{\lambda_B}{(1-\lambda_B)^2} = \frac{j_B}{4}$$

i.e.

$$j_B = -\frac{4\lambda_B}{(1-\lambda_B)^2}.$$

Q. E. D.

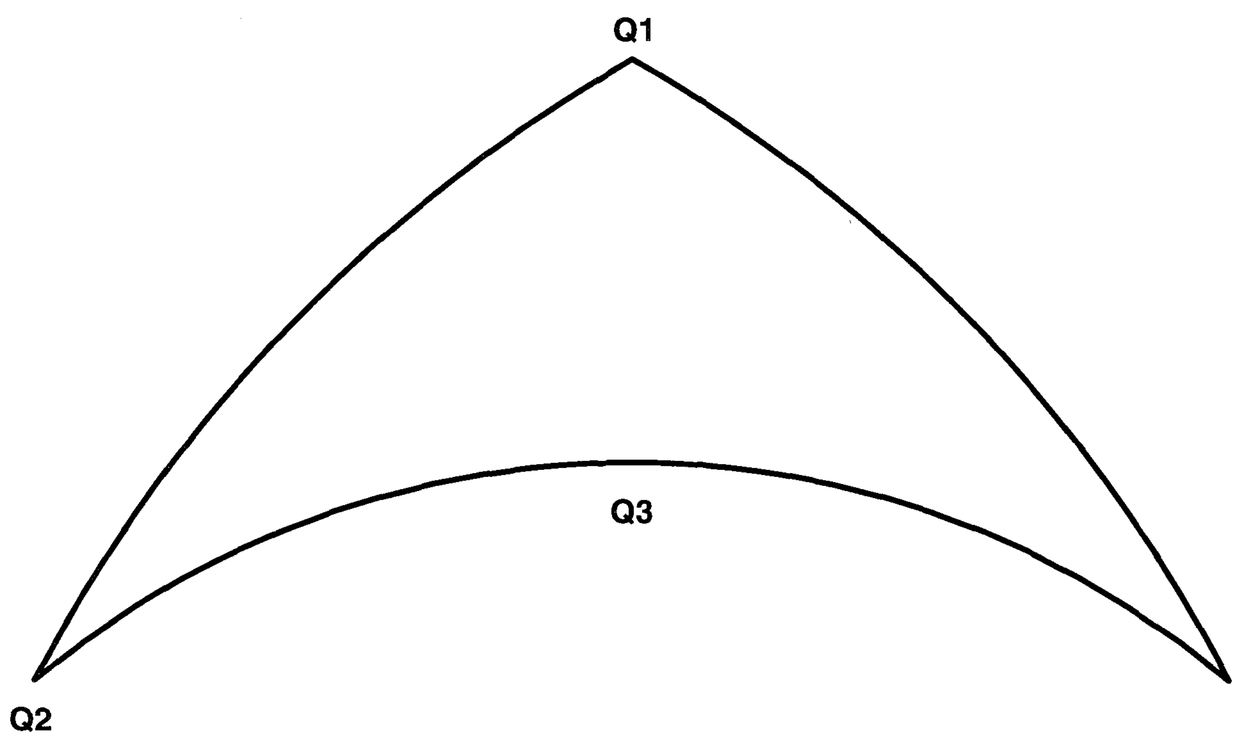

Figure 6.1: Fundamental Domain of Γ^*

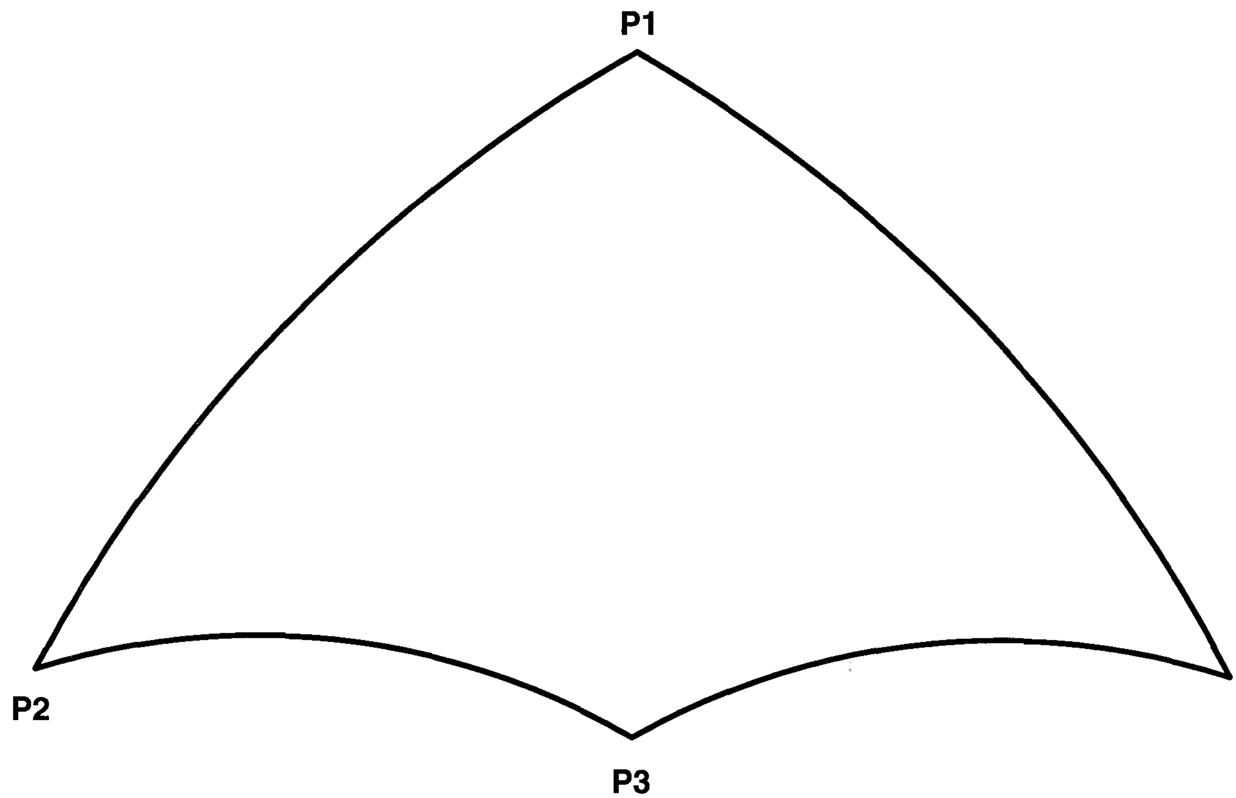

Figure 6.2: Fundamental Domain of Γ

Chapter 7 List of the Tables

In the following tables, we use P_1, P_2, P_3 to denote the elliptic points of order e_1, e_2, e_3 respectively if the signature of Γ is (e_1, e_2, e_3). And let $D(B/F)$ be the product of all prime ideals of F which are ramified in B

The signature of Γ is $(2,3,8)$		
$F = \mathbf{Q}(\sqrt{2})$ and $N_{F/\mathbf{Q}}(D(B/F)) = 2$		
$\dim\mathcal{S}_k(\Gamma) = [\frac{k}{4}] + [\frac{k}{3}] + [\frac{7k}{16}] - k + 1$		
$\sum_{P \neq P_1, P_2, P_3} O_P(f) + \frac{O_{P_1}(f)}{2} + \frac{O_{P_2}(f)}{3} + \frac{O_{P_3}(f)}{8} = \frac{1}{48}k$		
k	$\dim\mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	0	
10	0	
12	1	$2(P_3)$
14	0	
16	1	(P_2)
18	0	
20	0	
22	0	
24	1	$4(P_3)$
26	0	
28	1	$(P_2) + 2(P_3)$
30	1	$(P_1) + (P_3)$ or $5(P_3)$
32	1	$2(P_2)$
34	0	
36	1	$6(P_3)$
38	0	
40	1	$(P_2) + 4(P_3)$
42	1	$(P_1) + 3(P_3)$ or $7(P_3)$
44	1	$2(P_2) + 2(P_3)$
46	1	$(P_1) + (P_2) + (P_3)$ or $(P_2) + 5(P_3)$

The signature of Γ is $(2,3,12)$		
$F=\mathbf{Q}(\sqrt{3})$ and $N_{F/\mathbf{Q}}(D(B/F))=2$		
$\dim\mathcal{S}_k(\Gamma)=[\frac{k}{4}]+[\frac{k}{3}]+[\frac{11k}{24}]-k+1$		
$\sum_{P\neq P_1,P_2,P_3} O_P(f)+\frac{O_{P_1}(f)}{2}+\frac{O_{P_2}(f)}{3}+\frac{O_{P_3}(f)}{12}=\frac{1}{24}k$		
k	$\dim\mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	0	
10	0	
12	1	(P_1) or $6(P_3)$ or $(P_2)+2(P_3)$
14	0	
16	1	$2(P_2)$ or $8(P_3)$ or $(P_1)+2(P_3)$ or $(P_2)+4(P_3)$
18	1	$(P_1)+3(P_3)$ or $9(P_3)$ or $2(P_2)+(P_3)$ or $(P_2)+5(P_3)$
20	1	$(P_1)+(P_2)$ or $(P_2)+6(P_3)$ or $(P_1)+4(P_3)$ or $10(P_3)$ or $2(P_2)+2(P_3)$
22	1	$(P_1)+(P_2)+(P_3)$ or $(P_2)+7(P_3)$ or $(P_1)+5(P_3)$ or $11(P_3)$

The signature of Γ is $(2,4,12)$		
$F = \mathbf{Q}(\sqrt{3})$ and $N_{F/\mathbf{Q}}(D(B/F)) = 3$		
$\dim \mathcal{S}_k(\Gamma) = [\frac{k}{4}] + [\frac{3k}{8}] + [\frac{11k}{24}] - k + 1$		
$\sum_{P \neq P_1, P_2, P_3} O_P(f) + \frac{O_{P_1}(f)}{2} + \frac{O_{P_2}(f)}{4} + \frac{O_{P_3}(f)}{12} = \frac{1}{12}k$		
k	$\dim \mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	1	$(P_1) + 2(P_3)$ or $2(P_2) + 2(P_3)$ or $(P_2) + 5(P_3)$ or $8(P_3)$
10	0	

The signature of Γ is $(2,4,5)$		
$F = \mathbf{Q}(\sqrt{5})$ and $N_{F/\mathbf{Q}}(D(B/F)) = 4$		
$\dim \mathcal{S}_k(\Gamma) = [\frac{k}{4}] + [\frac{3k}{8}] + [\frac{2k}{5}] - k + 1$		
$\sum_{P \neq P_1, P_2, P_3} O_P(f) + \frac{O_{P_1}(f)}{2} + \frac{O_{P_2}(f)}{4} + \frac{O_{P_3}(f)}{5} = \frac{1}{40}k$		
k	$\dim \mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	1	(P_3)
10	0	
12	0	
14	0	
16	1	$2(P_3)$
18	0	
20	1	(P_1) or $2(P_2)$
22	0	
24	1	$3(P_3)$
26	0	
28	1	$(P_1) + (P_3)$ or $2(P_2) + (P_3)$
30	1	$(P_1) + (P_2)$ or $3(P_2)$
32	1	$4(P_3)$
34	0	
36	1	$(P_1) + 2(P_3)$ or $2(P_2) + 2(P_3)$
38	1	$(P_1) + (P_2) + (P_3)$ or $3(P_2) + (P_3)$

The signature of Γ is $(2,3,10)$		
$F=\mathbf{Q}(\sqrt{5})$ and $N_{F/\mathbf{Q}}(D(B/F))=5$		
$\dim\mathcal{S}_k(\Gamma)=[\frac{k}{4}]+[\frac{k}{3}]+[\frac{9k}{20}]-k+1$		
$\sum_{P\neq P_1,P_2,P_3} O_P(f)+\frac{O_{P_1}(f)}{2}+\frac{O_{P_2}(f)}{3}+\frac{O_{P_3}(f)}{10}=\frac{1}{30}k$		
k	$\dim\mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	0	
10	0	
12	1	$4(P_3)$
14	0	
16	1	$(P_2)+2(P_3)$
18	1	$6(P_3)$ or $(P_1)+(P_3)$
20	1	$2(P_2)$
22	0	
24	1	$8(P_3)$
26	0	
28	1	$(P_2)+6(P_3)$

The signature of Γ is $(2,5,6)$		
$F = \mathbf{Q}(\sqrt{5})$ and $N_{F/\mathbf{Q}}(D(B/F)) = 9$		
$\dim\mathcal{S}_k(\Gamma) = [\frac{k}{4}] + [\frac{2k}{5}] + [\frac{5k}{12}] - k + 1$		
$\sum_{P \neq P_1,P_2,P_3} O_P(f) + \frac{O_{P_1}(f)}{2} + \frac{O_{P_2}(f)}{5} + \frac{O_{P_3}(f)}{6} = \frac{1}{15}k$		
k	$\dim\mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	1	$(P_2) + 2(P_3)$
10	1	$(P_1) + (P_3)$ or $4(P_3)$
12	1	$4(P_2)$
14	0	
16	1	$2(P_2) + 4(P_3)$

The signature of Γ is $(2,3,7)$		
$F=\mathbf{Q}(e^{2\pi i/7}+e^{-2\pi i/7})$ and $N_{F/\mathbf{Q}}(D(B/F))=1$		
$\dim\mathcal{S}_k(\Gamma)=[\frac{k}{4}]+[\frac{k}{3}]+[\frac{3k}{7}]-k+1$		
$\sum_{P\neq P_1,P_2,P_3} O_P(f)+\frac{O_{P_1}(f)}{2}+\frac{O_{P_2}(f)}{3}+\frac{O_{P_3}(f)}{7}=\frac{1}{84}k$		
k	$\dim\mathcal{S}_k(\Gamma)$	possible divisor
12	1	(P_3)
24	1	$2(P_3)$
28	1	(P_2)
36	1	$3(P_3)$
40	1	$(P_2)+(P_3)$
42	1	(P_1)
48	1	$4(P_3)$
52	1	$(P_2)+2(P_3)$
54	1	$(P_1)+(P_3)$
56	1	$2(P_2)$
60	1	$5(P_3)$
64	1	$(P_2)+3(P_3)$
66	1	$(P_1)+2(P_3)$
68	1	$2(P_2)+(P_3)$
70	1	$(P_1)+(P_2)$
72	1	$6(P_3)$
76	1	$(P_2)+4(P_3)$
78	1	$(P_1)+3(P_3)$
80	1	$2(P_2)+2(P_3)$
82	1	$(P_1)+(P_2)+(P_3)$

The signature of Γ is $(2,3,9)$		
$F = \mathbf{Q}(e^{2\pi i/9} + e^{-2\pi i/9})$ and $N_{F/\mathbf{Q}}(D(B/F)) = 1$		
$\dim \mathcal{S}_k(\Gamma) = [\frac{k}{4}] + [\frac{k}{3}] + [\frac{4k}{9}] - k + 1$		
$\sum_{P \neq P_1, P_2, P_3} O_P(f) + \frac{O_{P_1}(f)}{2} + \frac{O_{P_2}(f)}{3} + \frac{O_{P_3}(f)}{9} = \frac{1}{36}k$		
k	$\dim \mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	0	
10	0	
12	1	(P_2) or $3(P_3)$
14	0	
16	1	$(P_2) + (P_3)$ or $4(P_3)$
18	1	(P_1)
20	0	
22	0	
24	1	$2(P_2)$ or $6(P_3)$
26	0	
28	1	$2(P_2) + (P_3)$ or $(P_2) + 4(P_3)$ or $7(P_3)$
30	1	$(P_1) + (P_2)$ or $(P_1) + 3(P_3)$
32	1	$2(P_2) + 2(P_3)$ or $(P_2) + 5(P_3)$ or $8(P_3)$

The signature of Γ is $(2,3,11)$		
$F=\mathbf{Q}(e^{2\pi i/11}+e^{-2\pi i/11})$ and $N_{F/\mathbf{Q}}(D(B/F))=1$		
$\dim\mathcal{S}_k(\Gamma)=[\frac{k}{4}]+[\frac{k}{3}]+[\frac{5k}{11}]-k+1$		
$\sum_{P\neq P_1,P_2,P_3} O_P(f)+\frac{O_{P_1}(f)}{2}+\frac{O_{P_2}(f)}{3}+\frac{O_{P_3}(f)}{11}=\frac{5}{132}k$		
k	$\dim\mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	0	
10	0	
12	1	$5(P_3)$
14	0	
16	1	$(P_2)+3(P_3)$
18	1	$(P_1)+2(P_3)$
20	1	$2(P_2)+(P_3)$
22	1	$(P_1)+(P_2)$
24	1	$10(P_3)$
26	0	
28	1	$(P_2)+8(P_3)$
30	1	$(P_1)+7(P_3)$
32	1	$2(P_2)+6(P_3)$
34	1	$(P_1)+(P_2)+5(P_3)$

The signature of Γ is $(2,3,16)$		
$F=\mathbf{Q}(e^{2\pi i/16}+e^{-2\pi i/16})$ and $N_{F/\mathbf{Q}}(D(B/F))=2$		
$\dim\mathcal{S}_k(\Gamma)=[\frac{k}{4}]+[\frac{k}{3}]+[\frac{15k}{32}]-k+1$		
$\sum_{P\neq P_1,P_2,P_3} O_P(f)+\frac{O_{P_1}(f)}{2}+\frac{O_{P_2}(f)}{3}+\frac{O_{P_3}(f)}{16}=\frac{5}{96}k$		
k	$\dim\mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	0	
10	0	
12	1	$(P_1)+2(P_3)$ or $10(P_3)$
14	0	
16	1	$(P_!)+(P_2)$ or $(P_2)+8(P_3)$
20	1	$2(P_2)+6(P_3)$
22	1	$(P_1)+(P_2)+5(P_3)$ or $(P_2)+13(P_3)$

The signature of Γ is $(2,3,24)$		
$F=\mathbf{Q}(e^{2\pi i/24}+e^{-2\pi i/24})$ and $N_{F/\mathbf{Q}}(D(B/F))=2$		
$\dim\mathcal{S}_k(\Gamma)=[\frac{k}{4}]+[\frac{k}{3}]+[\frac{23k}{48}]-k+1$		
$\sum_{P\neq P_1,P_2,P_3} O_P(f)+\frac{O_{P_1}(f)}{2}+\frac{O_{P_2}(f)}{3}+\frac{O_{P_3}(f)}{24}=\frac{1}{16}k$		
k	$\dim\mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	0	
10	0	
12	1	$(P_1)+6(P_3)$ or $(P_2)+10(P_3)$ or $2(P_2)+2(P_3)$ or $18(P_3)$
14	0	

The signature of Γ is $(2,3,30)$		
$F=\mathbf{Q}(e^{2\pi i/30}+e^{-2\pi i/30})$ and $N_{F/\mathbf{Q}}(D(B/F))=5$		
$\dim\mathcal{S}_k(\Gamma)=[\frac{k}{4}]+[\frac{k}{3}]+[\frac{29k}{60}]-k+1$		
$\sum_{P\neq P_1,P_2,P_3} O_P(f)+\frac{O_{P_1}(f)}{2}+\frac{O_{P_2}(f)}{3}+\frac{O_{P_3}(f)}{30}=\frac{1}{15}k$		
k	$\dim\mathcal{S}_k(\Gamma)$	possible divisor
2	0	
4	0	
6	0	
8	0	
10	0	
12	1	$(P_1)+9(P_3)$ or $(P_2)+14(P_3)$ or $2(P_2)+4(P_3)$ or $24(P_3)$
14	0	

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