COMPUTING THE DEVELOPMENT OF DISTURBANCES IN THE BLASIUS BOUNDARY LAYER DUE TO LOCALIZED UNSTEADY TWO-DIMENSIONAL WALL INJECTION

Thesis by

Phuc The Doan

In Partial Fulfillment of the Requirements

For the Degree of

Aeronautical Engineer

California Institute of Technology

Pasadena, California

1981

Submitted: May 4, 1981

DEDICATION

To my parents

ACKNOWLEDGEMENTS

I would like to express sincere gratitude to Professor Toshi Kubota for his cooperation as my research adviser. Many thanks are also due to the programming consultants at the Booth Computing Center for their cheerful service, to Karen Valente and Susan Yewell for an efficient job of typing the manuscript, and to Betty Wood for her elegant draftsmanship.

The generous financial assistance of the California Institute of Technology in the form of an Earle C. Anthony Fellowship and subsequent teaching and research assistantships is gratefully acknowledged. This work was sponsored by the Office of Naval Research of the United States Department of the Navy under Contract No. N00014-78C-0457.

ABSTRACT

The localized unsteady two-dimensional wall injection of fluid into the Blasius boundary layer was mathematically modeled as a mixed initial-value boundary-value problem. Here the magnitude of the initial disturbance was sufficiently small so that the problem was linear. For the case of parallel mean flow, detailed formulation was discussed, and then, as an illustration, the problem of instantaneous blowing through a narrow slit in the wall was specifically analyzed and computed at a Reynolds number based on displacement thickness of 750 with the result showing a wave packet traveling at .44 times the speed of the freestream, amplifying like $\frac{e^{.0047x}}{\sqrt{8.4x}}$, spreading like 0.1x, where x is the distance from the slit, and eventually containing only frequencies whose imaginary parts lie between 0.009 and -0.010. To account for the effect of the boundary layer growth on the evolution of the disturbance, a perturbation method based on the idea of multiple scales was presented.

TABLE OF CONTENTS

I. INTRODUCTION

Classical studies of hydrodynamics stability dealt mainly with the steady-state problem of determining whether a given Tollmien-Schlichting wave will amplify or decay. How such a wave comes into existence in the first place, i.e., the process of creation, was largely unexplored until fairly recently. Among the pioneers was Gaster who analyzed the vibrating-ribbon problem in 1965 (Ref. 1) and computed the disturbance in the laminar boundary layer of a flat plate due to an acoustic pulse at a point on the wall in 1975 (Ref. 2a). The purpose of Gaster's second work was to provide a theoretical model for comparison with the result from an experiment which he had carried out earlier (Ref. 2b) where a wave packet was artificially generated by a short duration acoustic pulse injected into the boundary layer flow through a small hole in the plate. In his latest work (Ref. 3), Gaster numerically evaluated the evolution of a linear wave packet in the Blasius boundary layer produced by a two-dimensional localized impulse.

Because of its relevance to the present investigation, Gaster's third work will now be reviewed in some detail. In the case of a parallel two-dimensional mean flow, the linearized perturbation equations admit elementary solutions of the form

$$\psi(y;\alpha,\omega)\, e^{i(\alpha x-\omega t)}$$

where the wavenumber α and the frequency ω satisfy a dispersion relation

$$\mathcal{D}(\alpha,\omega) = 0$$

The disturbance resulting from an impulsive excitation can be represented by a sum of these elementary solutions as follows:

$$\int \psi(y;\alpha,\omega)\, e^{i(\alpha x-\omega t)}\, d\alpha \qquad (*)$$

if the algebraic weighting function associated with the x-distribution of the excitation is neglected. Further, the function defining the vertical structure of the eigensolutions is also ignored so far as evaluating the wave packet goes. The approximations seem reasonable for the integral (*) since this will be dominated by exponentially large elements. The subject of Gaster's computation is then:

$$I = \int e^{i(\alpha x-\omega t)}\, d\alpha$$

This was first evaluated by direct summation. The result was considered "exact" and thus would serve as a standard against which other solutions would be judged. Since the summation process typically involves an excessive amount of computing, it is desirable to have alternative methods of estimating I. Indeed, for large values of x and t, we can obtain an asymptotic representation for this integral via the method of steepest descent. Gaster computed the first term of the expansion and showed that the result agreed well with the summation solution (over the appropriate ranges of x and t). He also discussed further simplifications to the asymptotic formula; however, these are not of any interest here.

In our work, we consider the instantaneous blowing of fluid into the Blasius boundary layer through a narrow slit in the wall as a special case of the general problem of arbitrary localized unsteady two-dimensional wall injection. We use an eigenfunction expansion procedure to arrive at a completely theoretical prediction, i.e., one which directly relates the amplitude of the disturbance to that of the excitation. The method of steepest descent is used to study the asymptotic behavior of the wave packet. We shall discuss a number of characteristics of the packet not mentioned in Gaster's paper. Finally, regarding the effect of the boundary layer growth on the development of the disturbance, Gaster in his second work argued on the basis of the local balance of kinetic energy of the disturbance that the amplitude given by the parallel-flow model should be weighted by an algebraic term like $x^{-1/4}$. In the present work, we propose a rigorous scheme based on the idea of multiple scales as a solution to this difficulty.

II. THE PARALLEL-FLOW PROBLEM

(1) Formulation

In this section we present the solution of the linearized perturbation equations.

For a low-Mach-number, isothermal, zero-body force, two-dimensional flow, the governing equations are:

-Continuity:

$$U_x + V_y = 0 \tag{1}$$

-Momentum:

$$U_t + UU_x + VU_y = -\frac{1}{\rho} P_x + \nu(U_{xx} + U_{yy}) \tag{2a}$$

$$V_t + UV_x + VV_y = -\frac{1}{\rho} P_y + \nu(V_{xx} + V_{yy}) \tag{2b}$$

where the subscripts denote differentiations, e.g. $U_x \equiv \frac{\partial U}{\partial x}$.

We assume that the above flow can be written as the sum of a steady mean part and a small perturbation, namely,

$$U = \overline{U} + u \qquad |u| << |U|$$

$$V = \overline{V} + v \qquad |v| << |V|$$

$$P = \overline{P} + p \qquad |p| << |P|$$

$(\overline{U},\overline{V},\overline{P})$ satisfies

$$\overline{U}_x + \overline{V}_y = 0 \tag{3}$$

$$\overline{U}\overline{U}_x + \overline{V}\overline{U}_y = -\frac{1}{\rho}\overline{P}_x + \nu(\overline{U}_{xx} + \overline{U}_{yy}) \tag{4a}$$

$$\overline{U}\overline{V}_x + \overline{V}\overline{V}_y = -\frac{1}{\rho}\overline{P}_y + \nu(\overline{V}_{xx} + \overline{V}_{yy}) \tag{4b}$$

then it can be readily shown that (u,v,p) satisfies the following set of equations:

$$u_x + v_y = 0 \tag{5a}$$

$$u_t + \overline{U}u_x + u\overline{U}_x + \overline{V}u_y + v\overline{U}_y = -\frac{1}{\rho} p_x + \nu(u_{xx} + u_{yy}) \qquad (5b)$$

$$v_t + \overline{U}v_x + u\overline{V}_x + \overline{V}v_y + v\overline{V}_y = -\frac{1}{\rho} p_y + \nu(v_{xx} + v_{yy}) \qquad (5c)$$

when all terms nonlinear in perturbations have been neglected.

Equations (5a-c) are the linearized Navier-Stokes equations. In the present general form, these partial differential equations are inseparable in x and y.

When the mean flow is parallel, i.e.

$$\overline{V} = 0$$

$$\overline{U} = \overline{U}(y)$$

equations (5a-c) become, respectively,

$$u_x + v_y = 0 \qquad (6a)$$

$$u_t + \overline{U}u_x + v\overline{U}_y = -\frac{1}{\rho} p_x + \nu(u_{xx} + u_{yy}) \qquad (6b)$$

$$v_t + \overline{U}v_x = -\frac{1}{\rho} p_y + \nu(v_{xx} + v_{yy}) \qquad (6c)$$

It is convenient to work with dimensionless quantities. To this end, we introduce the following reference values:

U_∞ (freestream velocity)

δ_* (displacement thickness) $= \int_0^\infty (1 - \overline{U}/U_\infty)\, dy$

The dimensionless counterpart of (6) is

$$u_x + v_y = 0 \qquad (7a)$$

$$u_t + \overline{U}u_x + v\overline{U}_y = - p_x + \frac{1}{R}(u_{xx} + u_{yy}) \qquad (7b)$$

$$v_t + \overline{U}v_x = - p_y + \frac{1}{R}(v_{xx} + v_{yy}) \qquad (7c)$$

where $R \equiv \dfrac{U_\infty \delta_*}{\nu}$.

For the problem under consideration, the appropriate boundary conditions are

$$u = 0 \quad \text{at} \quad y = 0 \quad \text{(no slip)} \tag{8a}$$

$$v = f(x,t) \quad \text{at} \quad y = 0 \quad \text{(injection)} \tag{8b}$$

$$u,v,p \to 0 \quad \text{as} \quad y \to \infty \quad \text{(no freestream turbulence)} \tag{8c}$$

Equations (7) are completely separable. In particular, they admit elementary solutions of the form $Y(y)e^{i(\alpha x-\omega t)}$. Therefore, we can use the Fourier transform method to solve the inhomogeneous problem described by equations (7) and (8).

Introduce the transform pair

$$v(x,y,t) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \tilde{v}(y,\alpha,\omega)\; e^{i(\alpha x-\omega t)}\; d\alpha d\omega \tag{9a}$$

$$\tilde{v}(y,\alpha,\omega) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} v(x,y,t)\; e^{-i(\alpha x-\omega t)}\; dx dt \tag{9b}$$

and similar pairs for u, p.

In terms of $\tilde{u},\tilde{v},\tilde{p}$ the problem is restated as:

$$i\alpha\tilde{u} + D\tilde{v} = 0 \tag{10a}$$

$$-i\omega\tilde{u} + \overline{U}i\alpha\tilde{u} + (D\overline{U})\tilde{v} = -\, i\alpha\tilde{p} + \frac{1}{R}\,(-\,\alpha^2\tilde{u} + D^2\tilde{u}) \tag{10b}$$

$$-i\omega\tilde{v} + \overline{U}i\alpha\tilde{v} = -\, D\tilde{p} + \frac{1}{R}\,(-\,\alpha^2\tilde{v} + D^2\tilde{v}) \tag{10c}$$

$$\tilde{u}(0,\alpha,\omega) = 0 \tag{11a}$$

$$\tilde{v}(0,\alpha,\omega) = \tilde{f}(\alpha,\omega) \tag{11b}$$

$$\tilde{u},\tilde{v},\tilde{p} \to 0 \quad \text{as} \quad y \to \infty \tag{11c}$$

where

$$D \equiv d/dy \tag{12}$$

$$\tilde{f}(\alpha,\omega) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} f(x,t)\ e^{-i(\alpha x-\omega t)}\ dxdt \tag{13}$$

Eliminating $\tilde{u}$ and $\tilde{p}$ from equations (10a-c), we obtain

$$\{(D^2 - \alpha^2)^2 - iR[\alpha\overline{U} - \omega)(D^2 - \alpha^2) - \alpha D^2\overline{U}]\}\ \tilde{v} = 0 \tag{14}$$

which is the Orr-Somerfeld equation, a fourth-order linear ordinary differential equation with variable coefficients.

From (11a-c), the boundary conditions accompanying (14) are

$$\tilde{v}(0,\alpha,\omega) = \tilde{f}(\alpha,\omega) \tag{15a}$$

$$D\tilde{v}(0,\alpha,\omega) = 0 \tag{15b}$$

$$\tilde{v}(\infty,\alpha,\omega) = 0 \tag{15c}$$

$$D\tilde{v}(\infty,\alpha,\omega) = 0 \tag{15d}$$

The solution of problem [(14), (15)] can be expressed in terms of the eigenfunctions of the Orr-Sommerfeld equation as follows[1]:

$$\tilde{v}(y,\alpha,\omega) = \sum_{n=1}^{\infty} \frac{c_n(\alpha,\omega)\ \phi_n(y,\omega)}{\alpha - \alpha_n(\omega)} \tag{16}$$

where $\alpha_n(\omega)$ is the nth eigenvalue and $\phi_n(y,\omega)$ the corresponding eigenfunction. ϕ_n is a solution of equation (14), with boundary conditions $\phi_n(0) = D\phi_n(0) = \phi_n(\infty) = D\phi_n(\infty) = 0$. The coefficient c_n is given by

$$c_n(\alpha,\omega) = -\frac{\tilde{f}(\alpha,\omega)\chi_n^*(0,\omega)}{q_n(\omega)}$$

where $\chi_n^*(y,\omega)$ is used to denote a linear combination of the adjoint eigenfunction $\phi_n^*(y,\omega)$ and its y-derivatives:

$$\chi_n^* = \{(D-\alpha)\ [-(D^2-\alpha^2) + iR(\alpha\overline{U}-\omega)]\}\phi_n^*$$

and $q_n(\omega)$ is a normalization factor.

---

1 Details are given in Appendix A.

Substituting (16) into the inversion formula (9a) yields

$$v(x,y,t) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \iint_{-\infty}^{\infty} \frac{c_n(\alpha,\omega)\phi_n(y,\omega)}{\alpha - \alpha_n(\omega)} e^{i(\alpha x-\omega t)} d\alpha d\omega \tag{17}$$

Equation (17) represents a formal solution of the original problem in the physical space.

Our next task is to interpret the physical consequences of this expression.

First, we carry out the integration in the α-plane:

$$v(x,y,t) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \phi_n(y,\omega) \; e^{-i\omega t} \int_{-\infty}^{\infty} \frac{c_n(\alpha,\omega)}{\alpha-\alpha_n(\omega)} e^{i\alpha x} \, d\alpha d\omega \tag{18}$$

For $x > 0$, we close the integration contour in the upper half-plane and apply the residue theorem to obtain

$$v(x,y,t) = i \sum_{n} \int_{-\infty}^{\infty} \phi_n(y,\omega) \; c_n(\omega) \; e^{i[\alpha_n(\omega)x-\omega t]} \, d\omega \tag{19}$$

where only those $\alpha_n(\omega)$ that are in the upper half of the α-plane are taken.

Defining

$$H_n(y,\omega) = c_n(\omega) \; \phi_n(y,\omega) \tag{20}$$

and

$$I_n(x,y,t) = \int_{-\infty}^{\infty} H_n(y,\omega) \; e^{i[\alpha_n(\omega)x-\omega t]} \, d\omega \tag{21}$$

one rewrites (19) as

$$v(x,y,t) = i \sum_{n} I_n \tag{22}$$

It remains to study integrals of the form

$$I(x,y,t) = \int_{-\infty}^{\infty} H(y,\omega) \; e^{i[\alpha(\omega)x-\omega t]} \, d\omega$$

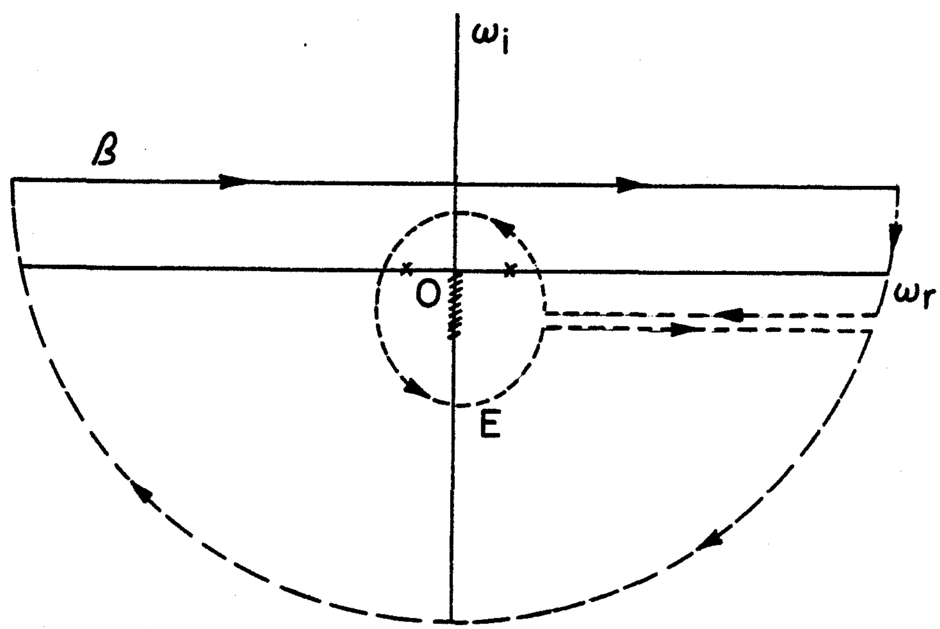
The argument that follows was inspired by an example given on pp. 277-282 in the book by Carrier, Krook, and Pearson (Ref. 4).

Suppose the forcing is turned on at time zero. At $x = 0$,

$$I = \int_{-\infty}^{\infty} H(y,\omega)\, e^{-i\omega t}\, d\omega$$

Since $v = 0$ for $t < 0$, it follows that $\mathcal{B}$, the integration path, must lie above all singularities.

The asymptotic behavior of $I(x,y,t)$ for large values of x and t can be examined by the method of steepest descents. To illustrate the procedure, we assume for now that the path of steepest descent E is a closed curve, e.g., an ellipse, and that the singularities of $H(y,\omega)$ consist of two poles on the real axis and a branch cut.



By the residue theorem,

$$I = \int_E + \text{contributions from the poles if they are located outside } E$$

In Appendix B, we prove that

$$\int_E \sim \pm i\sqrt{\frac{2\pi}{i}}\ H(y,\omega_s)\ \frac{e^{i[\alpha(\omega_s)x - \omega_s t]}}{\sqrt{\alpha''(\omega_s)x}} \tag{23}$$

for large values of x and t. Here s denotes the "saddle point" which is defined by

$$\alpha'(\omega_s) = \frac{t}{x} \quad .$$

where the prime indicates $\frac{d}{d\omega}$. In the asymptotic limit, most contributions to $\int_E$ come from the vicinity of this point.

For the case of fluid injection through a slit of zero width where the blowing process is an impulse in time, it can be shown that[1]

$$H_n(y,\omega) = -\frac{a}{2\pi}\ \frac{\chi_n^*(0,\omega)\phi_n(y,\omega)}{q_n(\omega)} \tag{24}$$

where a is the amplitude of the excitation.

Since H_n does not have any singularities, except a number of branch points associated with the eigenfunction ϕ_n and its adjoint ϕ_n^* we obtain

$$I^{(n)} = \int_E^{(n)}$$

where $q_n(\omega)$ has been assumed to be positive definite.

Moreover, far downstream of the disturbance source, contributions from other modes are diminishingly small compared to that from the first mode, and hence

---

[1] Details are provided in Appendix A.

$$v(x,y,t) \sim \pm \frac{a}{\sqrt{2\pi i}} \frac{\chi_1^*(0,\omega_s)\phi_1(y,\omega_s)}{q_1(\omega_s)} \frac{e^{i[\alpha_1(\omega_s)x-\omega_s t]}}{\sqrt{\alpha_1''(\omega_s)x}} \tag{25}$$

for large values of x and t. Henceforth, for convenience, the subscript indicating the mode number will be omitted.

In the next section, we present an algorithm for computing the expression on the right-hand side of equation (25).

(2) Computing Algorithm

Our objective is to compute the quantity appearing on the right-hand side of equation (25). It involves three separate computations; it is necessary to

(a) Locate the saddle point ω_s and calculate the values of $\alpha(\omega)$ and $\alpha''(\omega)$ at that point,

(b) Compute the eigenfunctions $\underset{\sim}{z}$ of the Orr-Sommerfeld equation and its adjoint $\underset{\sim}{x}$ for $\omega = \omega_s$, and

(c) Evaluate the normalization factor $q(\omega_s)$.

(a) The Saddle-Point Computation

Recall that the saddle point is defined by the condition:

$$\alpha'(\omega_s) = \frac{t}{x}$$

If we let $\alpha = \alpha_r + i\alpha_i$, then in the region where $\alpha(\omega)$ is analytic,

$$\alpha' = \frac{\partial \alpha_r}{\partial \omega_r} - i\,\frac{\partial \alpha_r}{\partial \omega_i}$$

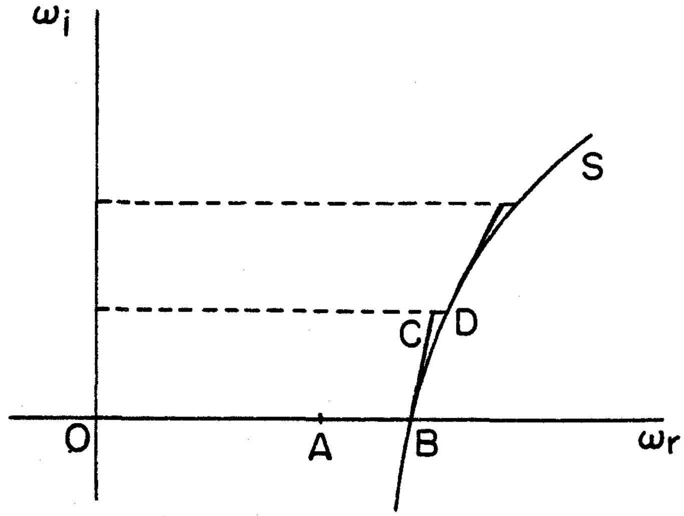
$$= \frac{\partial \alpha_i}{\partial \omega_i} + i\,\frac{\partial \alpha_i}{\partial \omega_r}$$

Hence,

$$\begin{cases} \dfrac{\partial \alpha_r}{\partial \omega_r} = \dfrac{\partial \alpha_i}{\partial \omega_i} = \dfrac{t}{x} \\[2ex] \dfrac{\partial \alpha_r}{\partial \omega_i} = \dfrac{\partial \alpha_i}{\partial \omega_r} = 0 \end{cases}$$

at $\omega = \omega_s$.

Let S denote the saddle-point locus. The following scheme is used to trace out S:



Starting at A, we use Newton's method to solve $\frac{\partial \alpha_i}{\partial \omega_r}(\omega_r) = 0$ to arrive at B on S. There is no difficulty in selecting the point A so that it is sufficiently close to S, because details of the stability diagram for spatial amplification are well known. Next, we follow the tangent to S at B to reach C, and then D again by Newton's method. By repeating the process, the locus S can be traced out.

To carry out the above scheme, we need to know α as a function of ω. For this purpose, direct solution of the Orr-Sommerfeld equation (to be presented in part (b)) is very time-consuming and, therefore, avoided. Instead, we use a series representation of the dispersion relation for the Blasius boundary layer developed by Gaster (Ref. 5). Here the frequency ω is expressed as a double series in terms of the Reynolds number R and the wavenumber α. When summing the series, a nonlinear transformation due to Shanks is applied to the sequences of partial sums to speed up the convergence. The inversion from $\omega(\alpha)$ to $\alpha(\omega)$ is accomplished by Newton's method. All derivatives required in the computation are approximated by finite differences.

(b) Numerical Solutions of the Orr-Sommerfeld Equation and its Adjoint

Consider equation (14) with homogeneous boundary conditions. Suppose R and ω are given.

The eigenvalue-eigenfunction problem is to determine the particular values of α for which nontrivial solutions exist. Those values of α are called eigenvalues and the corresponding solutions eigenfunctions.

Here we adopt the following solution procedure. By regarding α as another dependent variable, i.e. by adding $D\alpha = 0$ to the Orr-Sommerfeld equation, we obtain a system of nonlinear differential equations. The new problem is solved using PASVA3, an adaptive finite-difference FORTRAN code for nonlinear ordinary boundary-value problems developed by Pereyra and Lentini (Ref. 9), with additional boundary condition that fixes the magnitude of the eigenfunction. PASVA3 is also used to solve for the eigenfunction of the adjoint equation. The problem is linear because we have determined α.

For more details of the computation of eigenvalues and eigenfunctions, the reader is referred to Appendix C.

(c) Evaluating the Normalization Factor

From Appendix A,

$$q(\omega) = \int_0^\infty \underset{\sim}{x}(y,\omega)A(y)\underset{\sim}{z}(y,\omega)dy$$

where $\underset{\sim}{z}$ is a 4-column-vector whose components are linear combinations of the eigenfunction and its y-derivatives, $\underset{\sim}{x}$ a 4-row-vector whose components are linear combinations of the adjoint eigenfunction and its y-derivatives, and A a 4x4 matrix containing $\overline{U}$, $D^2\overline{U}$.

The above integral is broken down into two parts:

$$\int_0^\infty \underset{\sim}{x} A \underset{\sim}{z}\, dy = \underbrace{\int_0^{y_e} \underset{\sim}{x} A \underset{\sim}{z}\, dy}_{(I)} + \underbrace{\int_{y_e}^\infty \underset{\sim}{x} A \underset{\sim}{z}\, dy}_{(II)}$$

where $y_e = 4.931$ (the reference length being the displacement thickness). At this value of $y, \overline{U} = 0.999999$ and $D^2\overline{U} = -\ 0.000029$.

Part (I) is computed by the trapezoidal rule using numerically integrated $\underset{\sim}{z}$ and $\underset{\sim}{x}$.

Part (II) is computed using the following asymptotic solutions.

For $y \geqq y_e$,

$$\underset{\sim}{z} = c_2 e^{\lambda_2 y} \underset{\sim}{e}^{(2)} + c_4 e^{\lambda_4 y} \underset{\sim}{e}^{(4)} \tag{31}$$

$$\underset{\sim}{x} = d_2 e^{\lambda_2 y} \underset{\sim}{f}^{(2)} + d_4 e^{\lambda_4 y} \underset{\sim}{f}^{(4)} \tag{32}$$

(the notation used here is explained in Appendix C) where the constants c_2, c_4, d_2, d_4 are determined by matching equations (31), (32) with the computed values at $y = y_e$. The matrix A also takes on a particularly simple form, namely

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & iR & -1 \end{pmatrix}$$

and so the integral of $\underset{\sim}{x} A \underset{\sim}{z}$ over y from y_e to infinity can be analytically evaluated.

(3) Results and Discussion

Figure 1 shows the saddle-point locus computed for R=750. The directions of the paths of steepest descent are indicated by the line segments. Values of ω_r, ω_i, α_r, α_i, $\frac{\partial \alpha_r}{\partial \omega_r}$, $\frac{\partial^2 \alpha_r}{\partial \omega_r^2}$, and $\frac{\partial^2 \alpha_i}{\partial \omega_r^2}$ at the saddle points are given in Table 1. A few remarks about the accuracy of these values are in order. First, in applying Newton's method to solve the equation

$$\frac{\partial \alpha_i}{\partial \omega_r}(\omega_r) = 0 \quad ,$$

we set the tolerance ε at a value of 10^{-4}, i.e., when

$$\left| \frac{\omega_r^{(n)} - \omega_r^{(n-1)}}{\omega_r^{(n-1)}} \right|$$

decreases monotonically to 10^{-4} the iteration will stop and $\omega_r^{(n)}$ will then be taken as the desired root. The values of $\frac{\partial \alpha_i}{\partial \omega_r}$ at these roots are of the order 10^{-5}. The corresponding values of $\frac{\partial \alpha_r}{\partial \omega_r}$ are of the order unity, and hence the above convergence criterion is considered satisfactory. Second, the step size for finite-difference approximations of the derivatives (of the Gaster function $\omega(\alpha)$ with respect to α) was chosen by trial and error to be .002. It has to be small enough to bring the truncation error down to an acceptable level and at the same time sufficiently large to avoid incurring damaging loss of significant digits. As a check for the above choice of step size, we computed the derivatives in two directions: horizontally, i.e., $\Delta\alpha = .002$, and vertically, i.e., $\Delta\alpha = .002i$. In the region where the function is analytic, the results must be identical. For all values of α involved, there was excellent agreement (within one part in 10^{-6} of each other) between the approximations of the first derivative. Slight disagreements, up to 7%, occurred at a number of points in the case of the second derivative. Therefore, the computed values thereat are subject to possible errors of the same magnitude. These errors directly affect the values of $\alpha''(\omega_s)$ and manifest themselves through kinks in the graphs which we shall soon discuss. To smooth out these sharp corners, we need to adjust the size of $\Delta\alpha$ locally, i.e., to build in some step-size control mechanism.

If we define

$$P(x,y,t) = \frac{\chi^*(o,\omega_s)\phi(y,\omega_s)}{q(\omega_s)}$$

and

$$Q(x,t) = \frac{e^{i[\alpha(\omega_s)x-\omega_s t]}}{\sqrt{\alpha''(\omega_s)x}}$$

equation (25) becomes

$$v(x,y,t) \sim \pm \frac{a}{\sqrt{2\pi i}} PQ$$

The physical velocity perturbation is given by the real part of v.

Let

$$P = |P| \, e^{i \arg P}$$

and

$$Q = |Q| \, e^{i \arg Q}$$

Then

$$\mathrm{Re}\{v\} \sim \pm \frac{a}{\sqrt{2\pi}} |P| \, |Q| \cos(\arg P + \arg Q - \frac{\pi}{4})$$

Since the rate of change associated with Q is exponential whereas that due to P is only algebraic, the shape of the asymptotic wave packet depends largely on Q. The computed data will give an additional reason to support this conclusion. Thus to evaluate v, we can approximate P by its value at, say, the middle of the packet.

Figures 2-6 are graphs of $|Q| \cos(\arg Q)$ versus time for different values of x. For convenience, the coordinates have been scaled by the factors:

$$\text{Horizontal coordinate} = \frac{t - t_{min}}{t_{max} - t_{min}}$$

$$\text{Vertical coordinate} = \frac{Q}{|Q|_{max}}$$

First, we observe that as the magnitude of α'' remains relatively constant for the values of ω_s considered, the shape of the envelope is determined primarily by the factor $e^{-\alpha_i x + \omega_i t}$ (where for convenience the subscript s has been dropped). Differentiating $(-\alpha_i x + \omega_i t)$ with respect to t yields

$$-\left(-\frac{x}{t^2}\alpha_i'\right)x + \left(-\frac{x}{t^2}\omega_i'\right)t \quad \text{where } ' \text{ denotes } \frac{d}{d\frac{x}{t}}$$

$$= \left(\frac{x}{t}\right)^2 \alpha_i' - \frac{x}{t}\omega_i'$$

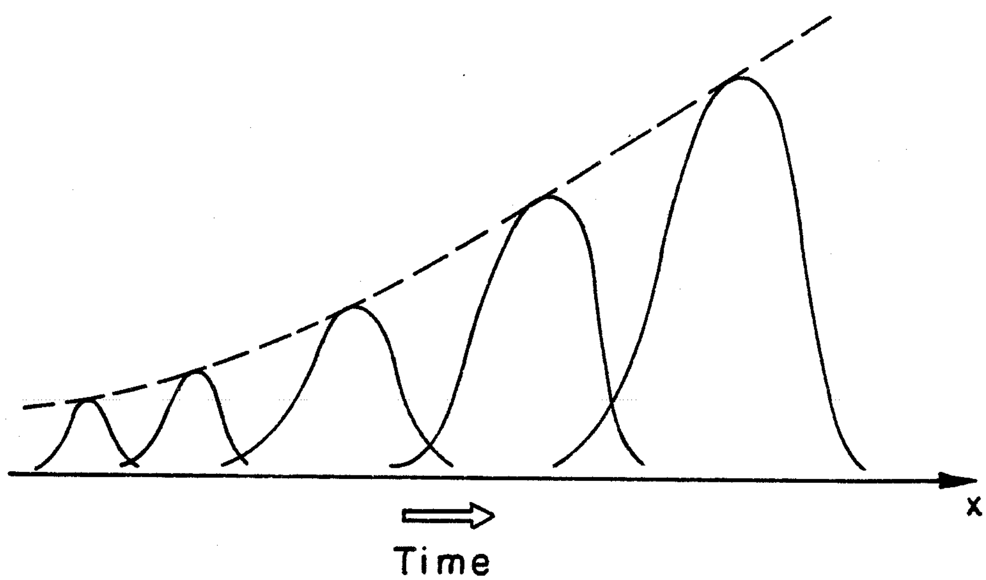
which vanishes when

$$\frac{\alpha_i'}{\omega_i'} = \frac{t}{x} \quad .$$

At this value of $\frac{t}{x}$, the envelope reaches a maximum. This maximum separates the initial temporal amplification from the subsequent decay in time as seen in the figures. We also notice from Table I that α_i and ω_i change sign at almost the same values of $\frac{t}{x}$. This explains the fact that the maximum occurs in the vicinity of $\omega_i = 0$. Specifically, $|Q|_{max}$ occurs at $\frac{t}{x} = 2.27$, for which $\omega_i = 0$, $\alpha_i = -.0047$, and $|\alpha''| = 8.4$. Therefore,

$$|Q|_{max} = \frac{e^{.0047x}}{\sqrt{8.4x}}$$

i.e., the envelope of the wave packets grows practically exponentially:



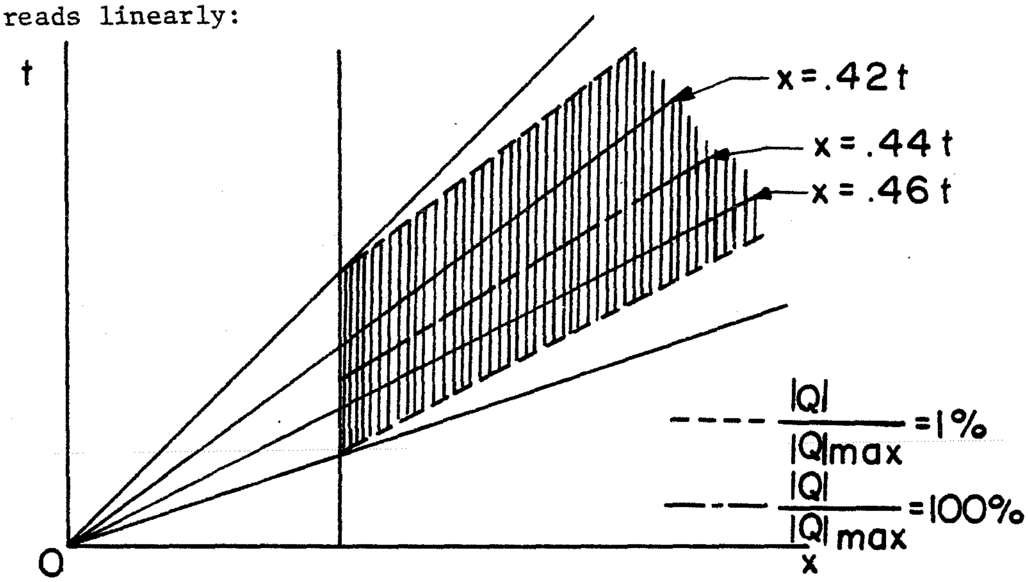
Second, the horizontal coordinate can be directly associated with the frequency ω_s because ω_s depends on $\frac{x}{t}$ (specifically, zero corresponds to (.1109, .024) and unity to (.1030,-.026)), and hence Figs. 2 - 6 clearly show that the frequency domain involved becomes smaller as the wave packet moves downstream. This gives further justification to the claim made earlier that the factor P does not significantly alter the shape of the packet.

If we define the leading and trailing edges by the condition $\frac{|Q|}{|Q|_{max}} = 1\%$, then the packet spreads linearly as it propagates downstream (Fig. 7):

$$\frac{d}{dx}\ \text{width}\ (\equiv t \text{ at trailing edge} - t \text{ at leading edge}) = 0.1$$

The crest travels at a constant speed equal to 0.44 that of the free-stream while the leading edge gradually decelerates to a final speed of .46 U_∞ and the trailing edge gradually accelerates to a final speed of .42 U_∞ .

The following sketch dramatizes the above two observations that the asymptotic wave packet contains progressively fewer frequencies and spreads linearly:



Note that the packet will eventually contain only those frequencies whose imaginary parts lie between -.010 and .009.

The computation was repeated for R = 1000. We observe the same behavior as described above for R = 750. However, here the disturbance travels at a speed somewhat lower than in the previous case.

III. A THEORY FOR SLIGHTLY NONPARALLEL FLOW

In this section, we present an idea to account for the effect of the boundary layer growth on the solution of the mixed initial-value boundary-value problem under investigation. Here the author benefited from the works by Nayfeh et al (Refs. 6 and 7) on the eigenvalue problem. Carrier's discussion of gravity waves on water of variable depth (Ref. 8) is also of some relevance.

The linear partial differential equations and boundary conditions governing the flow perturbation can be written symbolically as follows:

$$\mathcal{L}\,\psi(x,y,t) = f(x,y,t) \tag{33}$$

$\mathcal{L}\psi = 0$ admits solutions of the form $\tilde{\psi}(x,y,\omega)e^{-i\omega t}$.

To cope with the inhomogeneity presented by f, we superpose these elementary solutions to obtain the following general solution:

$$\psi(x,y,t) = \int_{-\infty}^{\infty} \tilde{\psi}(x,y,\omega)\, e^{-i\omega t}\, d\omega \tag{34a}$$

Inverting equation (34a) gives

$$\tilde{\psi}(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x,y,t)\, e^{i\omega t}\, dt \tag{34b}$$

Applying the Laplace transform defined by equations (34) to (33) yields

$$\tilde{\mathcal{L}}\,\tilde{\psi}(x,y,\omega) = \tilde{f}(x,y,\omega) \tag{35}$$

Equation (35) is inseparable.

However, as the x-dependence of the coefficients is rather weak, it is possible to introduce a perturbation scheme to break equation (35) down into a sequence of separable equations.

Specifically, the streamfunction Ψ of the Blasius boundary layer flow depends on y and εx where ε is a small parameter. Let

$$x_1 = \varepsilon x$$

and

$$x_0 = g(x)$$

$$g'(x) = G(\varepsilon x)$$

(Throughout this section, all primes indicate differentiations with respect to the variables shown in brackets.)

If we assume that

$$\tilde{\psi}(x,y,\omega) = \tilde{\psi}_0(x_0,y,\omega,x_1) + \varepsilon\tilde{\psi}_1(x_0,y,\omega,x_1) + \ldots \tag{36}$$

Then

$$\tilde{\psi}_{ix} = G\tilde{\psi}_{ix_0} + \varepsilon(\ )$$

$$\tilde{\psi}_{ixx} = G^2\tilde{\psi}_{ix_0x_0} + \varepsilon(\ ) + \varepsilon^2(\ )$$

$$\tilde{\psi}_{ixxx} = G^3\tilde{\psi}_{ix_0x_0x_0} + \varepsilon(\ ) + \varepsilon^2(\ ) + \varepsilon^3(\ )$$

$$\tilde{\psi}_{ixxxx} = G^4\tilde{\psi}_{ix_0x_0x_0x_0} + \varepsilon(\ ) + \varepsilon^2(\ ) + \varepsilon^3(\ ) + \varepsilon^4(\ )$$

$\tilde{\mathcal{L}}$ contains Ψ_x, Ψ_{xx}, and Ψ_{xxx} which are of the orders ε, ε^2, and ε^3, respectively. But they appear in a way that makes the highest-order terms in $\tilde{\mathcal{L}}\tilde{\psi}_i$ proportional to ε^4.

Since the forcing depends strongly on x, we can write

$$\tilde{f}(x,y,\omega) = \tilde{F}(x_0,y,\omega)$$

where higher-order terms have been neglected. (For details the reader is referred to Appendix D.)

The sequence of equations corresponding to equation (35) is

$$\varepsilon^0: \ \underline{\tilde{\mathcal{L}}}\ \tilde{\psi}_0 = \tilde{F} \tag{37a}$$

$$\varepsilon^1: \; \tilde{\underline{\mathcal{L}}}\,\tilde{\psi}_1 = -\mathcal{R}_1\tilde{\psi}_0 \tag{37b}$$

$$\varepsilon^2: \; \tilde{\underline{\mathcal{L}}}\,\tilde{\psi}_2 = -\mathcal{R}_1\tilde{\psi}_1 - \mathcal{R}_2\tilde{\psi}_0 \tag{37c}$$

$$\varepsilon^3: \; \tilde{\underline{\mathcal{L}}}\,\psi_3 = -\mathcal{R}_1\tilde{\psi}_2 - \mathcal{R}_2\tilde{\psi}_1 - \mathcal{R}_3\tilde{\psi}_0 \tag{37d}$$

$$\varepsilon^4: \; \tilde{\underline{\mathcal{L}}}\,\tilde{\psi}_4 = -\mathcal{R}_1\tilde{\psi}_3 - \mathcal{R}_2\tilde{\psi}_2 - \mathcal{R}_3\tilde{\psi}_1 - \mathcal{R}_4\tilde{\psi}_0 \tag{37e}$$

$$\varepsilon^5: \; \tilde{\underline{\mathcal{L}}}\,\tilde{\psi}_5 = -\mathcal{R}_1\tilde{\psi}_4 - \mathcal{R}_2\tilde{\psi}_3 - \mathcal{R}_3\tilde{\psi}_2 - \mathcal{R}_4\tilde{\psi}_1 \tag{37f}$$

etc.

In solving equations (37), the variables x_0 and x_1 representing the fast and slow variables respectively are treated as though they were completely independent of each other. Their interdependence will be restored as soon as solutions are obtained.

$\tilde{\underline{\mathcal{L}}}\,\tilde{\psi}_i = 0$ admits solutions of the form $\tilde{\tilde{\psi}}_i(\beta,y,\omega,x_1)\, e^{i\beta x_0}$, where β is in general a function of x_1.

To solve the inhomogeneous equations (37), we invoke the superposition principle to write down the general solution as a Fourier transform:

$$\tilde{\psi}_i(x_0,y,\omega,x_1) = \int_{-\infty}^{\infty} \tilde{\tilde{\psi}}_i(\beta,y,\omega,x_1)\, e^{i\beta x_0}\, d\beta \tag{38a}$$

Equation (38a) inverted:

$$\tilde{\tilde{\psi}}_i(\beta,y,\omega,x_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\psi}_i(x_0,y,\omega,x_1)\, e^{-i\beta x_0}\, dx_0 \tag{38b}$$

Equations (37) transform to:

$$\varepsilon^0: \quad L\tilde{\tilde{\psi}}_0 = \tilde{\tilde{F}} \tag{39a}$$

$$\varepsilon^1: \quad L\tilde{\tilde{\psi}}_1 = -R_1\tilde{\tilde{\psi}}_0 \tag{39b}$$

etc. where $L\tilde{\tilde{\psi}}_i = 0$ is the Orr-Sommerfeld equation with $\alpha = \beta G$.

Let α_n denote the nth root of the dispersion relation $\mathcal{D}(\alpha,\omega,x_1)$. ϕ_n and ϕ_n^* respectively stand for the corresponding eigenfunction of the Orr-Sommerfeld equation and its adjoint.

As shown in Appendix A, the solution of equation (39a) can be expressed in terms of the ϕ_n's as follows:

$$\tilde{\tilde{\psi}}_0 = \sum_{n=1}^{\infty} \frac{c_n \phi_n}{\alpha - \alpha_n} \tag{40}$$

Substituting (40) into (38a), we obtain

$$\tilde{\psi}_0 = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{c_n \phi_n}{\alpha - \alpha_n} e^{i\beta x_0} d\beta$$

$$= \frac{1}{G} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{c_n \phi_n}{\alpha - \alpha_n} e^{i\alpha \frac{x_0}{G}} d\alpha \tag{41}$$

For all practical purposes, the disturbance far downstream of the source can be adequately represented by contribution from the first mode, i.e.

$$\tilde{\psi}_0(x_0,y,\omega,x_1) = \frac{1}{G} 2\pi i\ c_1\phi_1\ e^{i\alpha_1 \frac{x_0}{G}} \tag{42}$$

Henceforth, for convenience, c_1, ϕ_1, and α_1 will be written as c, ϕ, and α.

To avoid secularity, we require that $\frac{\alpha}{G}$ be a constant. Without loss of generality, we take $G = \alpha$. Then

$$x_0 = g = \int_0^x \alpha\ (\varepsilon\bar{x})\ d\bar{x}$$

and

$$\tilde{\psi}_0 = 2\pi i \frac{c\ \phi}{\alpha}\ e^{i\int_0^x \alpha\ d\bar{x}} \tag{43}$$

Now, if the expansion (36) is to have an extended domain of validity, $\tilde{\psi}_1$, $\tilde{\psi}_2$, etc., must also be of the form (function of x_1) e^{ix_0} . We are left with no freedom to ensure that.

To remedy this situation, we propose a new expansion:

$$\tilde{\psi}(x,y,\omega) = h_0(\omega,x_1)\ \tilde{\psi}_0(x_0,y,\omega,x_1) + \varepsilon\tilde{\chi}_1(x_0,y,\omega,x_1) + \ldots \tag{44}$$

where $\tilde{\psi}_0$ is given by (42) and h_0 provides the desired freedom

Since

$$(h_0\ \tilde{\psi}_0)_x = h_0\alpha\ \tilde{\psi}_{0_{x_0}} + \varepsilon(\ )$$

$$(h_0\ \tilde{\psi}_0)_{xx} = h_0\alpha^2\tilde{\psi}_{0_{x_0x_0}} + \varepsilon(\ ) + \varepsilon^2(\ )$$

$$(h_0\ \tilde{\psi}_0)_{xxx} = h_0\alpha^3\tilde{\psi}_{0_{x_0x_0x_0}} + \varepsilon(\ ) + \varepsilon^2(\ ) + \varepsilon^3(\ )$$

$$(h_0\ \tilde{\psi}_0)_{xxxx} = h_0\alpha^4\tilde{\psi}_{0_{x_0x_0x_0x_0}} + \varepsilon(\ ) + \varepsilon^2(\ ) + \varepsilon^3(\ ) + \varepsilon^4(\ )$$

(35) is broken down into:

$$\varepsilon^0:\quad h_0\ \underline{\tilde{\mathcal{L}}}\ \tilde{\psi}_0 = \tilde{F} \tag{45a}$$

$$\varepsilon^1:\quad \underline{\tilde{\mathcal{L}}}\ \tilde{\psi}_1 = -\ \mathcal{R}_1 h_0\tilde{\psi}_0 \tag{45b}$$

etc. where $\mathcal{R}_1 h_0\tilde{\psi}_0$ has the form

$$[\mathcal{P}(y,\omega,x_1)h_0 + \mathcal{Q}(y,\omega,x_1)h_0']\ e^{ix_0}$$

(Explicit expressions for $\mathcal{P}$ and $\mathcal{Q}$ are given in Appendix D.)

We seek a solution of the form:

$$\tilde{\psi}_1(x_0,y,\omega,x_1) = \tilde{\tilde{\psi}}_1(y,\omega,x_1)\ e^{ix_0}$$

From (45b),

$$L\tilde{\tilde{\psi}}_1 = -\ [\mathcal{P}h_0 + \mathcal{Q}\,h_0'] \tag{46}$$

Equation (46) has a unique solution if and only if

$$\int_0^\infty \phi^*[\mathcal{P}h_0 + \mathcal{Q}h_0']\ dy = 0 \tag{47}$$

$$\therefore\ [\int_0^\infty \phi^*\mathcal{Q}\,dy]h_0' = -\ [\int_0^\infty \phi^*\mathcal{P}\ dy]\ h_0$$

$$\therefore\ h_0(\omega,x_1) = h_0(\omega,0)\ e^{-\int_0^{x_1}\gamma(\omega,\bar{x}_1)\ d\bar{x}_1} \tag{48}$$

where

$$\gamma(\omega,x_1) \equiv \frac{\int_0^\infty \phi^* \mathcal{P}\, dy}{\int_0^\infty \phi^* \mathcal{Q}\, dy} \tag{49}$$

Having determined $\tilde{\psi}_1$, we replace (44) by a revised expansion:

$$\tilde{\psi}(x,y,\omega) = h_0\tilde{\psi}_0 + \varepsilon h_1(\omega,x_1)\tilde{\psi}_1 + \varepsilon^2\tilde{\psi}_2 + \ldots \tag{50}$$

and choose h_1 to guarantee the existence and uniqueness of

$$\tilde{\psi}_2 = \tilde{\tilde{\psi}}_2(y,\omega,x_1)\, e^{ix_0} .$$

The final expansion obtained by repeated applications of this "term-wise improvement" scheme is

$$\tilde{\psi}(x,y,\omega) = [h_0(\omega,x_1)\, \tilde{\tilde{\psi}}_0(y,\omega,x_1) + \varepsilon h_1\tilde{\tilde{\psi}}_1 + \ldots]\, e^{ix_0}$$

Approximating $\tilde{\psi}$ by the leading term only:

$$\tilde{\psi}(x,y,\omega) = h_0(\omega,0)\, e^{-\int_0^{x_1} \gamma(\omega,\bar{x}_1)d\bar{x}_1}\, 2\pi i\left(\frac{c\,\phi}{\alpha}\right)\, e^{ix_0} \tag{51}$$

To determine $h_0(\omega,0)$, we observe that

$$e^{-\int_0^{x_1} \gamma(\omega,\bar{x}_1)\, d\bar{x}_1} = e^{-\varepsilon\int_0^{x} \gamma(\omega,\varepsilon\bar{x})\, d\bar{x}}$$

and hence, if we are sufficiently near the source so that x_0 can be approximated by $\alpha\, x$ and yet far enough from it so that the first mode really dominates the solution, then

$$\tilde{\psi}(x,y,\omega) = h_0(\omega,0)[2\pi i\, \left(\frac{c\,\phi}{\alpha}\right)_{x_1=0}\, e^{i\alpha\,(\omega,0)x}] \tag{52}$$

The expression in the square brackets in (52) is just what we would get if we solved the local parallel-flow problem around the slit. Therefore, $h_0(\omega,0) = 1$. Substituting (51) into (34a) yields

$$\psi(x,y,t) = 2\pi i \int_{-\infty}^{\infty} e^{-\varepsilon \int_0^x \gamma d\bar{x}} \frac{c\,\phi}{\alpha} e^{i\left(\int_0^x \alpha\, d\bar{x} - \omega t\right)} d\omega \quad . \tag{53}$$

IV. CONCLUSION

This work dealt with the problem of following the evolution of a disturbance in the laminar boundary layer on a flat plate created by localized unsteady two-dimensional blowing at the wall. For the parallel-flow model, a formulation was presented which directly relates the amplitude of the disturbance to that of the excitation. The asymptotic behavior of the disturbance far downstream of the source was studied using the method of steepest descent. As an illustration, the problem of impulsive injection through a narrow slit in the wall was specifically analyzed and computed at a Reynolds number based on displacement thickness of 750 with the result showing a wave packet traveling at .44 times the speed of the freestream, amplifying like $\frac{e^{.0047x}}{\sqrt{8.4x}}$, spreading like 0.1x, where x is the distance from the slit, and eventually containing only frequencies whose imaginary parts lie between 0.009 and -0.010. To account for the effect of the boundary layer growth on the development of the wave packet, we propose a solution based on the method of multiple scales.

The result of our effort in this investigation is a solution procedure which makes it possible to put forth a completely theoretical prediction of the disturbance due to a given excitation of the type mentioned earlier. Once verified by comparison with experimental results, the procedure can be useful in designing a feedback control system to cancel natural disturbances occurring randomly in the laminar boundary layer, and thereby to delay the transition to turbulence.

APPENDIX A

Eigenfunction Expansion

We seek to solve the following inhomogeneous problem:

$$\{(D^2-\alpha^2)^2 - iR[(\alpha\overline{U}-\omega)(D^2-\alpha^2) - \alpha D^2\overline{U}]\}\ \tilde{v} = 0 \tag{A.0}$$

$$\tilde{v}(0,\alpha,\omega) = \tilde{\tilde{f}}(\alpha,\omega) \tag{A.1.a}$$

$$D\tilde{v}(0,\alpha,\omega) = 0 \tag{A.1.b}$$

$$\tilde{v}(\infty,\alpha,\omega) = 0 \tag{A.1.c}$$

$$D\tilde{v}(\infty,\alpha,\omega) = 0 \tag{A.1.d}$$

where $D \equiv d/dy$.

(For origin, the reader is referred to page 7 of the main text.)

Define:

$$z_1 = \tilde{v}$$

$$z_2 = (D-\alpha)z_1$$

$$z_3 = (D+\alpha)z_2 = (D^2-\alpha^2)\tilde{v}$$

$$z_4 = (D-\alpha)z_3$$

$$\Rightarrow (D+\alpha)z_4 = (D^2-\alpha^2)^2\ \tilde{v}$$

Then, equation (A.0) can be written as:

$$\begin{cases} Dz_1 = \alpha z_1 + z_2 & \text{(A.2.a)} \\ Dz_2 = -\alpha z_2 + z_3 & \text{(A.2.b)} \\ Dz_3 = \alpha z_3 + z_4 & \text{(A.2.c)} \\ Dz_4 = -iR\alpha(D^2\overline{U})z_1 + iR(\alpha\overline{U}-\omega)z_3 - \alpha z_4 & \text{(A.2.d)} \end{cases}$$

In matrix notation, equation (A.2) becomes

$$D\underset{\sim}{z} = [\alpha A(y) + B(\omega)]\underset{\sim}{z} \tag{A.3}$$

where

$$\underset{\sim}{z} = (z_1, z_2, z_3, z_4)^T$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -iRD^2\overline{U} & 0 & iR\overline{U} & -1 \end{pmatrix} \qquad \text{(A.4.a)}$$

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -iR\omega & 0 \end{pmatrix} \qquad \text{(A.4.b)}$$

The boundary conditions (A.1.a-d) now appear as follows:

$$z_1(0,\alpha,\omega) = f(\alpha,\omega) \qquad \text{(A.5.a)}$$

$$z_2(0,\alpha,\omega) = 0 \qquad \text{(A.5.b)}$$

$$\underset{\sim}{z}(\infty,\alpha,\omega) = 0 \qquad \text{(A.5.c)}$$

At this point, we must <u>assume</u> that for the values of ω being investigated the eigenvalues of the Orr-Sommerfeld equation form a denumerably infinite spectrum, and hence enable us to express the solution $\underset{\sim}{z}$ of problem [(A.3), (A.5)] in the following manner:

$$\underset{\sim}{z} = \sum_{n=1}^{\infty} a_n \underset{\sim}{z}^{(n)} \qquad \text{(A.6)}$$

where $\underset{\sim}{z}^{(n)}(y,\omega)$ is the eigenfunction corresponding to the eigenvalue α_n, i.e.

$$D\underset{\sim}{z}^{(n)} = (\alpha_n A + B)\underset{\sim}{z}^{(n)} \qquad \text{(A.7)}$$

$$z_1^{(n)}(0,\alpha,\omega) = 0 \qquad \text{(A.8.a)}$$

$$z_2^{(n)}\ (0,\alpha,\omega) = 0 \tag{A.8.b}$$

$$\underset{\sim}{z}^{(n)}\ (\infty,\alpha,\omega) = 0 \tag{A.8.c}$$

To determine the coefficients a_n, consider the adjoint equation

$$D\underset{\sim}{x}^{(m)} = -\ \underset{\sim}{x}^{(m)}\ (\alpha_m A + B) \tag{A.9}$$

where $\underset{\sim}{x}^{(m)}$ is a four-row-vector.

Pre-multiplying (A.7) by $\underset{\sim}{x}^{(m)}$, aft-multiplying (A.9) by $\underset{\sim}{z}^{(n)}$, and adding the resultant equations, we obtain

$$D\underset{\sim}{x}^{(m)}\ \underset{\sim}{z}^{(n)} = (\alpha_n - \alpha_m)\ \underset{\sim}{x}^{(m)}\ A\underset{\sim}{z}^{(n)} \tag{A.10}$$

Integrating equation (A.10) with respect to y from zero to infinity yields

$$[\underset{\sim}{x}^{(m)}\ \underset{\sim}{z}^{(n)}]_0^\infty = (\alpha_n - \alpha_m) \int_0^\infty \underset{\sim}{x}^{(m)}\ A\underset{\sim}{z}^{(n)}\ dy \tag{A.11}$$

The left-hand side of (A.11) vanishes if

$$x_3^{(m)}\ (0,\alpha,\omega) = 0 \tag{A.12.a}$$

$$x_4^{(m)}\ (0,\alpha,\omega) = 0 \tag{A.12.b}$$

$$\underset{\sim}{x}^{(m)}\ (\infty,\alpha,\omega) = 0 \tag{A.12.c}$$

Then,

$$\int_0^\infty \underset{\sim}{x}^{(m)}\ A\underset{\sim}{z}^{(n)}\ dy = 0 \quad \text{if} \quad m \neq n \tag{A.13}$$

Equation (A.13) is called the orthogonality condition.

Now, from $\underset{\sim}{z} = \sum_{n=1}^{\infty} a_n\ \underset{\sim}{z}^{(n)}$,

$$\underset{\sim}{x}^{(m)}\ A\underset{\sim}{z} = \sum_{n=1}^{\infty} a_n\ \underset{\sim}{x}^{(m)}\ A\underset{\sim}{z}^{(n)}$$

Hence,

$$\int_0^\infty \underset{\sim}{x}^{(m)} A\underset{\sim}{z}\, dy = \sum_{n=1}^{\infty} a_n \int_0^\infty \underset{\sim}{x}^{(m)} A\underset{\sim}{z}^{(n)}\, dy$$
$$= a_m \int_0^\infty \underset{\sim}{x}^{(m)} A\underset{\sim}{z}^{(m)}\, dy \qquad (A.14)$$

Next, we pre-multiply (A.3) by $\underset{\sim}{x}^{(m)}$, aft-multiply (A.9) by $\underset{\sim}{z}$, and add the results to arrive at

$$D\underset{\sim}{x}^{(m)} \underset{\sim}{z} = (\alpha - \alpha_m)\, \underset{\sim}{x}^{(m)} A\underset{\sim}{z} \qquad (A.15)$$

Integrating equation (A.15) with respect to y from zero to infinity gives

$$[\underset{\sim}{x}^{(m)} \underset{\sim}{z}]_0^\infty = (\alpha - \alpha_m) \int_0^\infty \underset{\sim}{x}^{(m)} A\underset{\sim}{z}\, dy \qquad (A.16)$$

Applying the boundary conditions on $\underset{\sim}{x}^{(m)}$ and $\underset{\sim}{z}$, and using (A.14) we obtain

$$- x_1^{(m)} (0,\omega)\, \tilde{f}(\alpha,\omega) = (\alpha - \alpha_m)\, a_m \int_0^\infty \underset{\sim}{x}^{(m)} A\underset{\sim}{z}^{(m)}\, dy$$

Therefore,

$$a_m = - \frac{x_1^{(m)} (0,\omega)\, \tilde{f}(\alpha,\omega)}{(\alpha - \alpha_m) \int_0^\infty \underset{\sim}{x}^{(m)} A\underset{\sim}{z}^{(m)}\, dy} \qquad (A.17)$$

Finally, we substitute (A.17) into (A.6) to get the desired solution:

$$\underset{\sim}{z} = - \sum_{n=1}^{\infty} \frac{x_1^{(n)} (0,\omega)\, \tilde{f}(\alpha,\omega)}{(\alpha - \alpha_n) \int_0^\infty \underset{\sim}{x}^{(n)} A\underset{\sim}{z}^{(n)}\, dy}\, \underset{\sim}{z}^{(n)} \qquad (A.18)$$

Since $\underset{\sim}{z}^{(n)}$ and $\underset{\sim}{x}^{(n)}$ appear in both the numerator and the denominator of the right-hand side of (A.18), $\underset{\sim}{z}$ is unique. (A.18) is rewritten as:

$$\underset{\sim}{z} = \sum_{n=1}^{\infty} \frac{c_n\, \underset{\sim}{z}^{(n)}}{\alpha - \alpha_n} \qquad (A.19)$$

where

$$c_n \equiv \frac{\tilde{f}(\alpha,\omega)\ x_1^{(n)}\ (0,\omega)}{\int_0^\infty \underset{\sim}{x}^{(n)}\ A\underset{\sim}{z}^{(n)}\ dy} \tag{A.20}$$

For the case of impulse injection of fluid through a slit of zero width located at $x = 0$ we have

$$f(x,t) = a\ \delta(x)\ \delta(t) \tag{A.21}$$

where a is a positive constant. The Fourier transform is

$$\begin{aligned} \tilde{f}(\alpha,\omega) &= \frac{1}{2\pi} \iint_{-\infty}^{\infty} a\ \delta(x)\ \delta(t)\ e^{-\ i(\alpha x-\omega t)}\ dx\ dt \\ &= \frac{a}{2\pi} \end{aligned} \tag{A.22}$$

Consequently,

$$c_n = -\frac{a}{2\pi} \frac{x_1^{(n)}\ (0,\omega)}{\int_0^\infty \underset{\sim}{x}^{(n)}\ A\underset{\sim}{z}^{(n)}\ dy} \tag{A.23}$$

Substituting (A.23) into equation (20) yields

$$H_n\ (y,\omega) = -\frac{a}{2\pi} \frac{x_1^{(n)}\ (0,\omega)\ z_1^{(n)}\ (y,\omega)}{\int_0^\infty \underset{\sim}{x}^{(n)}\ A\underset{\sim}{z}^{(n)}\ dy} \tag{A.24}$$

Note: In the main text, $x_1^{(n)}$ and $z_1^{(n)}$ are respectively denoted by χ_n^* and ϕ_n. $\int_0^\infty \underset{\sim}{x}^{(n)} A\underset{\sim}{z}^{(n)} dy$ is referred to only as a normalization factor.

APPENDIX B

The Steepest-Descent Argument

We seek to determine the asymptotic behavior of the following integral:

$$\int_E H(y,\omega)\, e^{i[\alpha(\omega)x-\omega t]}\, d\omega$$

for large values of x and t.

Define

$$K(\omega) = i[\alpha(\omega) - \frac{\omega}{c}]$$

where $c \equiv \frac{x}{t}$.

Then

$$\int_E H(y,\omega)\, e^{i[\alpha(\omega)x-\omega t]}\, d\omega = \int_E H(y,\omega)\, e^{xK(\omega)}\, d\omega$$

c is kept fixed.

Introduce the saddle point ω_s via

$$K'(\omega_s) = 0$$

The contour E is chosen to have the following properties:

(1) K_i, the imaginary part of K, is constant on E

(2) E passes through ω_s.

Consequently,

$$\int_E = e^{ix(K_i)_s} \int_{-c}^{d} H(y,\ell)\, e^{xK_r} \frac{d\omega}{d\ell}\, d\ell$$

where ℓ is the distance from the saddle point measured along E, and $c \geq 0$, $d > 0$.

Expanding $K_r(\ell)$ in a Taylor series about ω_s, we obtain

$$K_r(\ell) = (K_r)_s + \frac{1}{2}\, \ell^2 (K_r'')_s + \ldots$$

Hence, assuming $(K_r'')_s \neq 0$, near ω_s,

$$e^{xK_r} \approx e^{x(K_r)_s} e^{\frac{1}{2} x(K_r'')_s \ell^2}$$

The two properties listed earlier do not uniquely determine E. In this case $[(K_r'')_s \neq 0]$, there are two curves both satisfying those properties and intersecting orthogonally at the saddle point. $(K_r'')_s$ is positive on one of these curves and negative on the other.

(The proof is as follows:

Let $\omega - \omega_s = re^{i\theta}$ and $(K'')_s = Re^{i\Theta}$. Near ω_s,

$$K \approx (K)_s + \frac{1}{2} (\omega - \omega_s)^2 (K'')_s$$

$$\therefore K_i \approx (K_i)_s + \frac{1}{2} Rr^2 \sin(2\theta + \Theta)$$

To satisfy the constant-phase requirement (1) we must choose θ to be either $-\frac{\Theta}{2}$

or $-\frac{\Theta}{2} + \frac{\pi}{2}$

The first direction corresponds to $(K_r'')_s$ being positive and the second $(K_r'')_s$ negative.)

Consider the function

$$f(\ell) = e^{xK_r(\ell)}$$

$$\Rightarrow f(\ell) > 0$$

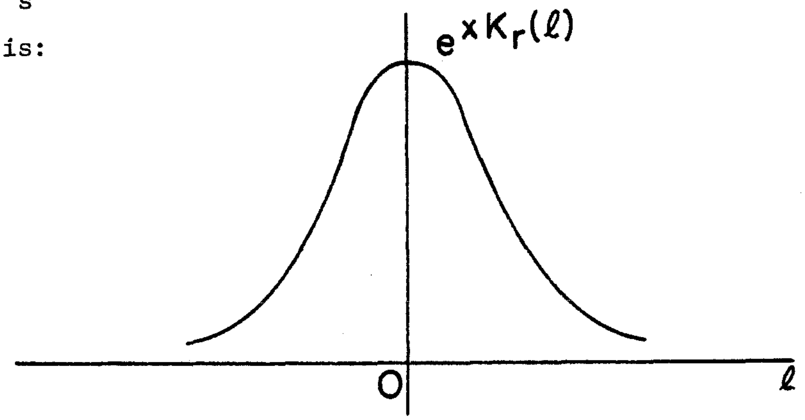
$$f'(\ell) = xK_r' e^{xK_r}$$

$$\Rightarrow f'(0) = 0$$

$$f''(\ell) = x(K_r'' + xK_r'^2)\, e^{xK_r}$$

$$\Rightarrow \quad f''(0) = x(K_r'')_s\, e^{x(K_r)_s}$$

If $(K_r'')_s$ is negative, then so is $f''(0)$, and hence $f(\ell)$ must look like this:



(We are assuming that there is one and only one saddle point in the interval $-c \le \ell \le d$.)

Thus, for large values of x, the graphs of e^{xK_r} and

$$e^{x(K_r)_s}\; e^{\frac{1}{2}x(K_r'')_s\,\ell^2}$$

will almost coincide. Consequently,

$$\int_E \sim e^{ix(K_i)_s}\,(H)_s\, e^{x(K_r)_s}\left(\frac{d\omega}{d\ell}\right)_s \int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}x|(K_r'')_s|\ell^2}\, d\ell$$

$$\sim (H)_s\; e^{x(K)_s}\left(\frac{d\omega}{d\ell}\right)_s \int_{-\infty}^{\infty} e^{-\frac{1}{2}x|(K_r'')_s|\ell^2}\, d\ell$$

Define

$$\ell' = \sqrt{\frac{1}{2}x|(K_r'')_s|}\;\ell$$

Then

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2} x |(K''_r)_s| \ell^2} \, d\ell = \frac{1}{\sqrt{\frac{1}{2} x |(K''_r)_s|}} \int_{-\infty}^{\infty} e^{-\ell'^2} \, d\ell'$$

$$= \sqrt{\frac{2\pi}{x |(K''_r)_s|}}$$

And so

$$\int_E \sim \sqrt{2\pi} \; H(y,\omega_s) \frac{e^{xK(\omega_s)}}{\sqrt{x}} \frac{\left(\frac{d\omega}{d\ell}\right)_s}{\sqrt{-(K''_r)_s}}$$

Now,

$$K = K(\ell(\omega))$$

$$\frac{dK}{d\omega} = \frac{d\ell}{d\omega} \frac{dK}{d\ell}$$

$$\frac{d^2K}{d\omega^2} = \left(\frac{d\ell}{d\omega}\right)^2 \frac{d^2K}{d\ell^2}$$

or

$$\frac{d^2K}{d\ell^2} = \left(\frac{d\omega}{d\ell}\right)^2 \frac{d^2K}{d\omega^2}$$

But, on E, K_i = constant.

Hence,

$$\frac{d^2K}{d\ell^2} = \frac{d^2K_r}{d\ell^2}$$

Therefore,

$$K''_{r,s} = \left(\frac{d\omega}{d\ell}\right)_s^2 K''(\omega_s)$$

Thus,

$$\int_E \sim \pm i \sqrt{2\pi} \; H(y,\omega_s) \frac{e^{xK(\omega_s)}}{\sqrt{K''(\omega_s) x}}$$

where, as usual, the square root of a complex number is chosen to have positive real part.

In terms of $\alpha(\omega)$,

$$\mathscr{E} \sim \pm i\sqrt{\frac{2\pi}{i}}\; H(y,\omega_s)\; \frac{e^{i[\alpha(\omega_s)x - \omega_s t]}}{\sqrt{\alpha''(\omega_s)x}} \quad .$$

APPENDIX C

Numerical Solutions of the Orr-Sommerfeld Equation and Its Adjoint

In Appendix A, we wrote the Orr-Sommerfeld equation as a system of four first-order equations, namely, equations (A.2.a-d):

$$\begin{cases} Dz_1 = \alpha z_1 + z_2 & \text{(C.1.a)} \\ Dz_2 = -\alpha z_2 + z_3 & \text{(C.1.b)} \\ Dz_3 = \alpha z_3 + z_4 & \text{(C.1.c)} \\ Dz_4 = -iR\alpha(D^2\overline{U})z_1 + iR(\alpha\overline{U}-\omega)z_3 - \alpha z_4 & \text{(C.1.d)} \end{cases}$$

The boundary conditions are

$$z_1(0) = 0 \qquad \text{(C.2.a)}$$

$$z_2(0) = 0 \qquad \text{(C.2.b)}$$

$$\underset{\sim}{z}\,(\infty) = 0 \qquad \text{(C.2.c)}$$

By adding

$$D\alpha = 0 \qquad \text{(C.3)}$$

to equations (C.1.a-d), we obtain a system of five nonlinear first-order equations. The additional boundary condition is arbitrary, e.g.

$$z_3(0) = 1 \qquad \text{(C.4)}$$

and amounts to a normalization of the eigenfunction. Thus, the eigenvalue-eigenfunction problem is reduced to solving the inhomogeneous system defined by equations (C.1.a-d) and (C.3) with boundary conditions (C.2.a-c) and (C.4), henceforth to be known as System I.

Similarly, the adjoint equation and associated boundary conditions were represented as follows:

$$\begin{cases} Dx_1 = -\alpha x_1 + iR(D^2\overline{U})\alpha x_4 & \text{(C.5.a)} \\ Dx_2 = -x_1 + \alpha x_2 & \text{(C.5.b)} \\ Dx_3 = -x_2 - \alpha x_3 - iR(\overline{U}\alpha-\omega)x_4 & \text{(C.5c)} \\ Dx_4 = -x_3 + \alpha x_4 & \text{(C.5.d)} \end{cases}$$

$$x_3(0) = 0 \tag{C.6.a}$$

$$x_4(0) = 0 \tag{C.6.b}$$

$$\underset{\sim}{x}(\infty) = 0 \tag{C.6.c}$$

Since the eigenvalue α has been determined above, the problem is linear. To normalize the adjoint eigenfunction, we replace one of the boundary conditions (C.6.a-c) by some arbitrary condition, e.g. impose

$$x_1(0) = 1 \tag{C.7}$$

instead of (C.6.b).

The inhomogeneous problem defined by equations (C.5.a-d) with boundary conditions (C.6.a), (C.7), (C.6.c) will be referred to as System II.

Before using PASVA3 to solve Systems I and II, we must approximate the boundary conditions at infinity, namely, (C.2.c) and (C.6.c)

As y approaches infinity, $\overline{U}$ and $D^2\overline{U}$ approach 1. and 0., respectively. Hence, the asymptotic limit of the Orr-Sommerfeld equation is

$$D\underset{\sim}{z} = \mathcal{A}\underset{\sim}{z} \tag{C.8}$$

where

$$\mathcal{A} = \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & -\alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & \sigma & -\alpha \end{pmatrix}$$

$$\sigma \equiv iR(\alpha - \omega)$$

The eigenvalues of $\mathcal{A}$ can easily be shown to be

$$\lambda_1 = \alpha$$
$$\lambda_2 = -\alpha$$
$$\lambda_3 = (\alpha^2 + \sigma)^{1/2}$$
$$\lambda_4 = -(\alpha^2 + \sigma)^{1/2}$$

α and $(\alpha^2 + \sigma)^{1/2}$ are taken to have positive real parts.

The corresponding eigenvectors are respectively

$$\underset{\sim}{e}^{(1)} = (1,0,0,0)^T$$
$$\underset{\sim}{e}^{(2)} = (-1,2\alpha,0,0)^T$$
$$\underset{\sim}{e}^{(3)} = (-1,(\alpha - \lambda_3),-\sigma,(\alpha - \lambda_3)\sigma)^T$$
$$\underset{\sim}{e}^{(4)} = (-1,(\alpha - \lambda_4),-\sigma,(\alpha - \lambda_4)\sigma)^T$$

The general solution of (C.1) is

$$\underset{\sim}{z} = \sum_{i=1}^{4} c_i e^{\lambda_i y} \underset{\sim}{e}^{(i)} \qquad \text{(C.9)}$$

or

$$\underset{\sim}{z} = \sum_{i=1}^{4} c_i \underset{\sim}{\phi}^{(i)} \qquad \text{(C.10)}$$

(Each $\underset{\sim}{\phi}^{(i)} \equiv e^{\lambda_i y} \underset{\sim}{e}^{(i)}$ is called a fundamental solution and the matrix which has $\underset{\sim}{\phi}^{(i)}$'s as columns the fundamental matrix.)

If we write

$$z_1 = \psi_1 + \psi_2 + \psi_3 + \psi_4 \qquad \text{(C.11.a)}$$

where $\psi_1 \equiv c_1 e^{\lambda_1 y} e_1^{(1)}$, $\psi_2 \equiv c_2 e^{\lambda_2 y} e_1^{(2)}$, $\psi_3 \equiv c_3 e^{\lambda_3 y} e_1^{(3)}$, $\psi_4 \equiv c_4 e^{\lambda_4 y} e_1^{(4)}$.

Then

$$Dz_1 = \lambda_1\psi_1 + \lambda_2\psi_2 + \lambda_3\psi_3 + \lambda_4\psi_4 \qquad \text{(C.11.b)}$$

$$D^2z_1 = \lambda_1^2\psi_1 + \lambda_2^2\psi_2 + \lambda_3^2\psi_3 + \lambda_4^2\psi_4 \qquad \text{(C.11.c)}$$

$$D^3z_1 = \lambda_1^3\psi_1 + \lambda_2^3\psi_2 + \lambda_3^3\psi_3 + \lambda_4^3\psi_4 \qquad \text{(C.11.d)}$$

Solving (C.11) for $\psi_1, \psi_2, \psi_3, \psi_4$, we obtain

$$\psi_1 = (D + \lambda_3)(D^2 - \alpha^2)\ z_1$$

$$\psi_2 = (D^2 - \lambda_3^2)(D + \alpha)\ z_1$$

But $z_1 \to 0$ as $y \to \infty$. Thus, ψ_1 and ψ_3 must vanish, i.e.

$$(D + \lambda_3)(D^2 - \alpha^2)\ z_1(y_e) = 0 \qquad \text{(C.12.a)}$$

$$(D^2 - \lambda_3^2)(D + \alpha)\ z_1(y_e) = 0 \qquad \text{(C.12.b)}$$

are the correct approximation of (C.2.c). Here y_e is chosen to be 4.931 where $\overline{U} = 0.999999$ and $D^2\overline{U} = -\ 0.000029$.

Similarly, the asymptotic limit of the adjoint equation is

$$D\underset{\sim}{x} = \mathcal{A}\underset{\sim}{x} \qquad \text{(C.13)}$$

where

$$\mathcal{A} = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ -1 & \alpha & 0 & 0 \\ 0 & -1 & -\alpha & -\sigma \\ 0 & 0 & -1 & \alpha \end{pmatrix}$$

which naturally shares the same eigenvalues with the matrix defined earlier in connection with equation (C.8).

Here the eigenvectors are

$$\underset{\sim}{f}^{(1)} = (0,\ ,0,-\ 1)^T$$
$$\underset{\sim}{f}^{(2)} = (2\ ,1,-\frac{2\alpha}{\sigma},-\frac{1}{\sigma})^T$$
$$\underset{\sim}{f}^{(3)} = (0,0,\alpha-\lambda_3,1)^T$$
$$\underset{\sim}{f}^{(4)} = (0,0,\alpha-\lambda_4,1)^T$$

The general solution of (C.13) is

$$\underset{\sim}{x} = \sum_{i=1}^{4} d_i \, e^{\lambda_i y} \, \underset{\sim}{f}^{(i)} \tag{C.7}$$

or

$$\underset{\sim}{x} = \sum_{i=1}^{4} d_i \, \underset{\sim}{\chi}^{(i)} \tag{C.8}$$

It can be shown that the correct approximation of (C.6.c) is

$$(D + \alpha)(D^2 - \lambda_3^2) \; x_4(y_e) = 0 \tag{C.9.a}$$

$$(D + \lambda_3)(D^2 - \alpha^2) \; x_4(y_e) = 0 \tag{C.9.b}$$

Note: Listings of FORTRAN programs which implement the above algorithm are given at the end of the thesis.

APPENDIX D

Some Detailed Expressions

(1) Definition of $F(x_0,y,\omega)$ (introduced on page 22 of the main text.)

Expanding $x_0 = g(x)$ in a Taylor series about $x = 0$, we obtain

$$x_0 = g(0) + x\, G(0) + \frac{1}{2}\, \varepsilon\, x^2\, G'(0) + \ldots \tag{D.1}$$

We must require that $g(0) = 0$ so that, for small ε, x_0 behaves like x. For the same reason, we assume

$$x = a_1 x_0 + \varepsilon a_2 x_0^2 + \ldots \tag{D.2}$$

Substituting (D.2) into (D.1) yields

$$\begin{aligned} x_0 &= G(0)[a_1 x_0 + \varepsilon\, a_2 x_0^2 + \ldots] \\ &\quad + \frac{1}{2}\, G'(0)\ \varepsilon[a_1 x_0 + \varepsilon a_2 x_0^2 + \ldots]^2 + \ldots \\ &= [G(0)a_1]\ x_0 + \varepsilon[G(0)a_2 + \frac{1}{2}\, G'(0)a_1^2]\ x_0^2 + \ldots \end{aligned}$$

Hence,

$$a_1 = \frac{1}{G(0)}$$

$$a_2 = = \frac{1}{2}\,\frac{G'(0)}{G(0)}\quad a_1^2 = -\,\frac{1}{2}\,\frac{G'(0)}{[G(0)]^3}$$

etc.

Expanding $\tilde{f}(x,y,\omega)$ about $(a_1 x_0,y,\omega)$ yields

$$\tilde{f}(x,y,\omega) = \tilde{f}(a_1 x_0,y,\omega) + (\varepsilon a_2 x_0^2 + \ldots)\ \tilde{f}'(a_1 x_0,y,\omega) + \ldots$$

Define

$$\tilde{F}(x_0,y,\omega) = \tilde{f}(a_1 x_0,y,\omega)\quad .$$

(2) Explicit expressions for P and Q (introduced on page 25 of the main text.)

It can be shown that the linearized Navier-Stokes equation for the case of a nonparallel steady mean flow is

$$\mathscr{L}\psi = 0$$

where

$$\mathscr{L} \equiv \frac{\partial}{\partial t}\nabla^2 + \Psi_y \frac{\partial}{\partial x}\nabla^2 + (\nabla^2\Psi)_x \frac{\partial}{\partial y} - \Psi_x \frac{\partial}{\partial y}\nabla^2 - (\nabla^2\Psi)_y \frac{\partial}{\partial x} - \frac{1}{R}\nabla^4$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

ψ and Ψ stand for the perturbation and mean-flow streamfunctions, respectively.

Now,

$$(h_0\tilde{\psi}_0)_x = h_0 G\tilde{\psi}_{0_{x_0}} + \varepsilon(h_0'\tilde{\psi}_0 + h_0\tilde{\psi}_{0_{x_1}}) \qquad \text{(D.4.a)}$$

$$(h_0\tilde{\psi}_0)_{xx} = h_0 G^2\tilde{\psi}_{0_{x_0x_0}} + \varepsilon[(2h_0'G + h_0G')\tilde{\psi}_{0_{x_0}} + 2h_0G\tilde{\psi}_{0_{x_0x_1}}] + \varepsilon^2[\ ] \qquad \text{(D.4.b)}$$

$$(h_0\tilde{\psi}_0)_{xxx} = h_0 G^3\tilde{\psi}_{0_{x_0x_0x_0}} + \varepsilon[3(h_0'G + h_0G')G\tilde{\psi}_{0_{x_0x_0}} + 3h_0G^2\tilde{\psi}_{0_{x_0x_0x_1}}] + \varepsilon^2[\ ] + \varepsilon^3[\ ] \qquad \text{(D.4.c)}$$

$$(h_0\tilde{\psi}_0)_{xxxx} = h_0 G^4\tilde{\psi}_{0_{x_0x_0x_0x_0}} + \varepsilon[(4h_0'G + 6h_0G')G^2\tilde{\psi}_{0_{x_0x_0x_0}} + 4h_0G^3\tilde{\psi}_{0_{x_0x_0x_0x_1}}] + \varepsilon^2[\ ] + \varepsilon^3[\ ] + \varepsilon^4[\ ] \qquad \text{(D.4.d)}$$

But,

$$G = \alpha \quad ,$$

$$G' = \alpha' \quad ,$$

$$\tilde{\psi}_0 = \tilde{\tilde{\psi}}_0(y,\omega,x_1)\; e^{ix_0} \quad .$$

Therefore,

$$\begin{aligned}
\mathcal{R}_1 h_0 \tilde{\psi}_0 = \{ & -i\omega[(2\alpha\; h_0' + \alpha' h_0)\; i\tilde{\tilde{\psi}}_0 + 2\alpha\;\; i\tilde{\tilde{\psi}}_{0_{x_1}}\; h_0 \\
& + \Psi_y[3(\alpha\; h_0' + \alpha' h_0)\;\alpha\;\; i^2\;\; \tilde{\tilde{\psi}}_0 + 3\alpha^2\; i^2\; \tilde{\tilde{\psi}}_{0_{x_1}} h_0 \\
& \quad + (D^2\tilde{\tilde{\psi}}_0)\; h_0' + (D^2\tilde{\tilde{\psi}}_{0_{x_1}})h_0] \\
& + \Psi_{yyx_1}(D\tilde{\tilde{\psi}}_0)\; h_0 \\
& - \Psi_{x_1}[\alpha^2 i^2(D\tilde{\tilde{\psi}}_0)h_0 + (D^3\tilde{\tilde{\psi}}_0)h_0] \\
& - \Psi_{yyy}(\tilde{\tilde{\psi}}_0 h_0' + \tilde{\tilde{\psi}}_{0_{x_1}} h_0) \\
& - \frac{1}{R}\;[(4\alpha\; h_0' + 6\alpha' h_0)\alpha^2 i^3\; \tilde{\tilde{\psi}}_0 + 4\alpha^3 i^3 \tilde{\tilde{\psi}}_{0_{x_1}} h_0 \\
& + 2((2\alpha\; h_0' + \alpha' h_0) i D^2\tilde{\tilde{\psi}}_0 \;\; + 2\alpha\; i(D^2\tilde{\tilde{\psi}}_0)_{x_1} h_0)]\}\; e^{ix_0}
\end{aligned} \tag{D.5}$$

$$\begin{aligned}
= \{ & [(2\alpha\;\omega - 3\Psi_y\alpha^2 - \Psi_{yyy} + \frac{i4\;\alpha^3}{R})\tilde{\tilde{\psi}}_{0_{x_1}} + (\Psi_y - \frac{i4\alpha}{R})D^2\tilde{\tilde{\psi}}_{0_{x_1}} \\
& + (\Psi_{yyx_1} + \alpha^2\Psi_{x_1})D\tilde{\tilde{\psi}}_0 \;\; - \Psi_{x_1} D^3\tilde{\tilde{\psi}}_0 \\
& + ((\omega - 3\Psi_y\alpha \;\; + \frac{i6\alpha^2}{R})\tilde{\tilde{\psi}}_0 - i\,\frac{2}{R}\; D^2\tilde{\tilde{\psi}}_0)\;\alpha']h_0 + \\
& [(2\alpha\;\omega - 3\Psi_y\alpha^2 - \Psi_{yyy} + \frac{i4\alpha^3}{R})\tilde{\tilde{\psi}}_0 + (\Psi_y - \frac{i4\alpha}{R})D^2\tilde{\tilde{\psi}}_0]h_0'\}e^{ix_0}
\end{aligned} \tag{D.6}$$

Define

$$a_1 = 2\alpha\,\omega - 3\Psi_y\alpha^2 - \Psi_{yyy} + i\,\frac{4\alpha^3}{R} \quad , \tag{D.7.a}$$

$$a_2 = \Psi_y - i\,\frac{4\alpha}{R} \quad , \tag{D.7.b}$$

$$a_3 = \Psi_{yyx_1} + \alpha^2\Psi_{x_1} \quad , \tag{D.7.c}$$

$$a_4 = \omega - 3\Psi_y\alpha + i\,\frac{6\alpha^2}{R} \quad . \tag{D.7.d}$$

Then

$$\mathcal{P} = [(a_3 - \Psi_{x_1}D^2)D + (a_4 - i\,\frac{2}{R}\,D^2)\alpha']\tilde{\tilde{\psi}}_0$$

$$+ [a_1 + a_2D^2]\tilde{\tilde{\psi}}_{0_{x_1}} \quad , \tag{D.8.a}$$

$$\mathcal{Q} = [a_1 + a_2D^2]\tilde{\tilde{\psi}}_0 \quad . \tag{D.8.b}$$

REFERENCES

1. Gaster, M., "On the Generation of Spatially Growing Waves in a Boundary Layer," Journal of Fluid Mechanics, Vol. 22, 1965, pp. 433-441.

2a. Gaster, M., "A Theoretical Model of a Wave Packet in the Boundary Layer on a Flat Plate," Proceedings of the Royal Society of London, A.347, 1975, pp. 271-289.

2b. Gaster, M. and Grant, I., "An Experimental Investigation of the Formation and Development of a Wave Packet in a Laminar Boundary Layer," Proceedings of the Royal Society of London, A.347, 1975, pp. 253-269.

3. Gaster, M., "Propagation of Linear Wave Packets in Laminar Boundary Layers," AIAA Journal, Vol. 19, No. 4, April 1981, pp. 419-423.

4. Carrier, G.F., Krook, M. and Pearson, C.E., "Functions of a Complex Variable-Theory and Technique," McGraw-Hill, 1966.

5. Gaster, M., "Series Representation of the Eigenvalues of the Orr-Sommerfeld Equation," Journal of Computational Physics, Vol. 29, No. 2, 1978, p. 243.

6. Nayfeh, A.H., Saric, W.S. and Mook, D.T., "Stability of Nonparallel Flows," Archives of Mechanics, 26.3, 1974, pp. 401-406.

7. Saric, W.S. and Nayfeh, A.H., "Nonparallel Stability of Boundary-Layer Flows," The Physics of Fluids, Vol. 18, No. 8, August 1975, pp. 945-950.

8. Carrier, G.F., "Gravity Waves on Water of Variable Depth," Journal of Fluid Mechanics, Vol. 24, Part 4, 1966, pp. 641-659.

9. Pereyra, V., "PASVA3: An adaptive finite-difference FORTRAN program for first-order nonlinear ordinary boundary problems," Proceedings of the "Working Conference on Codes for Boundary Value Problems in O.D.E.'s," Springer-Verlag, 1978.

Table 1

Saddle-point data for R = 750

$\frac{\partial\alpha_r}{\partial\omega_r}$	ω_r	ω_i	α_r	α_i	$\frac{\partial^2\alpha_r}{\partial\omega_r^2}$	$\frac{\partial^2\alpha_i}{\partial\omega_r^2}$
2.0640	0.1109	0.0240	0.2948	0.0473	-2.1	7.9
2.0725	0.1106	0.0230	0.2943	0.0452	-2.1	8.0
2.0811	0.1104	0.0220	0.2937	0.0431	-2.2	7.9
2.0897	0.1101	0.0210	0.2932	0.0410	-2.2	7.9
2.0982	0.1098	0.0200	0.2926	0.0389	-2.2	8.0
2.1070	0.1096	0.0190	0.2921	0.0368	-2.0	8.1
2.1156	0.1093	0.0180	0.2915	0.0347	-2.0	8.1
2.1243	0.1091	0.0170	0.2910	0.0326	-2.0	8.1
2.1330	0.1088	0.0160	0.2905	0.0305	-2.0	8.1
2.1417	0.1086	0.0150	0.2899	0.0283	-1.9	8.2
2.1504	0.1083	0.0140	0.2894	0.0262	-1.9	8.2
2.1590	0.1081	0.0130	0.2889	0.0240	-1.9	8.2
2.1677	0.1079	0.0120	0.2884	0.0219	-1.9	8.2
2.1765	0.1076	0.0110	0.2879	0.0197	-1.9	8.2
2.1852	0.1074	0.0100	0.2874	0.0175	-1.8	8.3
2.1940	0.1072	0.0090	0.2869	0.0153	-1.8	8.3
2.2027	0.1070	0.0080	0.2864	0.0131	-1.8	8.3
2.2115	0.1067	0.0070	0.2859	0.0109	-1.7	8.3
2.2201	0.1065	0.0060	0.2855	0.0087	-1.7	8.5
2.2288	0.1063	0.0050	0.2851	0.0065	-1.7	8.5
2.2376	0.1061	0.0040	0.2846	0.0043	-1.6	8.5
2.2463	0.1059	0.0030	0.2842	0.0020	-1.6	8.5
2.2551	0.1057	0.0020	0.2837	-0.0002	-1.5	8.2
2.2638	0.1056	0.0010	0.2833	-0.0024	-1.4	8.3
2.2726	0.1054	0.0000	0.2829	-0.0047	-1.4	8.3
2.2813	0.1052	-0.0010	0.2825	-0.0070	-1.3	8.3
2.2900	0.1050	-0.0020	0.2821	-0.0092	-1.4	8.4
2.2987	0.1049	-0.0030	0.2817	-0.0115	-1.3	8.5
2.3074	0.1047	-0.0040	0.2814	-0.0138	-1.3	8.5
2.3162	0.1046	-0.0050	0.2810	-0.0161	-1.2	8.5
2.3249	0.1044	-0.0060	0.2807	-0.0185	-1.2	8.5
2.3337	0.1043	-0.0070	0.2804	-0.0208	-1.1	8.5
2.3424	0.1041	-0.0080	0.2800	-0.0231	-1.1	8.5
2.3511	0.1040	-0.0090	0.2797	-0.0255	-1.0	8.6
2.3599	0.1039	-0.0100	0.2795	-0.0278	-0.99	8.6
2.3686	0.1038	-0.0110	0.2792	-0.0302	-0.94	8.6
2.3774	0.1037	-0.0120	0.2790	-0.0326	-0.88	8.6
2.3861	0.1036	-0.0130	0.2787	-0.0349	-0.82	8.6
2.3950	0.1035	-0.0140	0.2785	-0.0374	-0.76	8.7
2.4038	0.1034	-0.0150	0.2783	-0.0398	-0.70	8.7
2.4126	0.1033	-0.0160	0.2781	-0.0422	-0.64	8.7
2.4214	0.1033	-0.0170	0.2780	-0.0446	-0.58	8.7
2.4302	0.1032	-0.0180	0.2778	-0.0470	-0.63	8.6
2.4391	0.1031	-0.0190	0.2777	-0.0494	-0.55	8.6
2.4479	0.1031	-0.0200	0.2775	-0.0519	-0.47	8.6

2.4569	0.1031	-0.0210	0.2774	-0.0543	-0.39	8.6
2.4658	0.1030	-0.0220	0.2773	-0.0568	-0.31	8.6
2.4748	0.1030	-0.0230	0.2773	-0.0593	-0.22	8.7
2.4837	0.1030	-0.0240	0.2772	-0.0617	-0.13	8.8
2.4927	0.1030	-0.0250	0.2772	-0.0642	-0.12	9.0
2.5017	0.1030	-0.0260	0.2772	-0.0667	-0.08	9.0

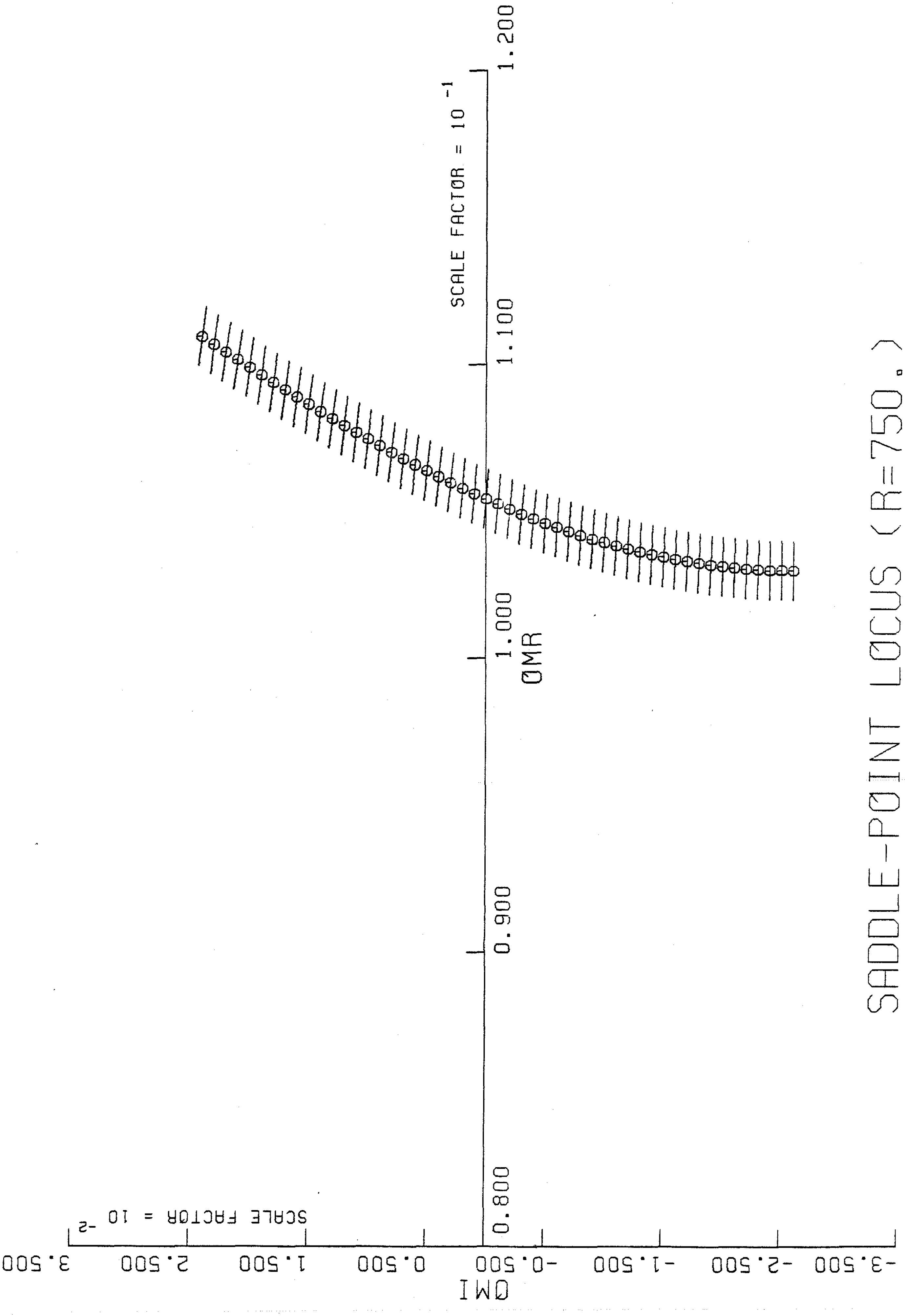
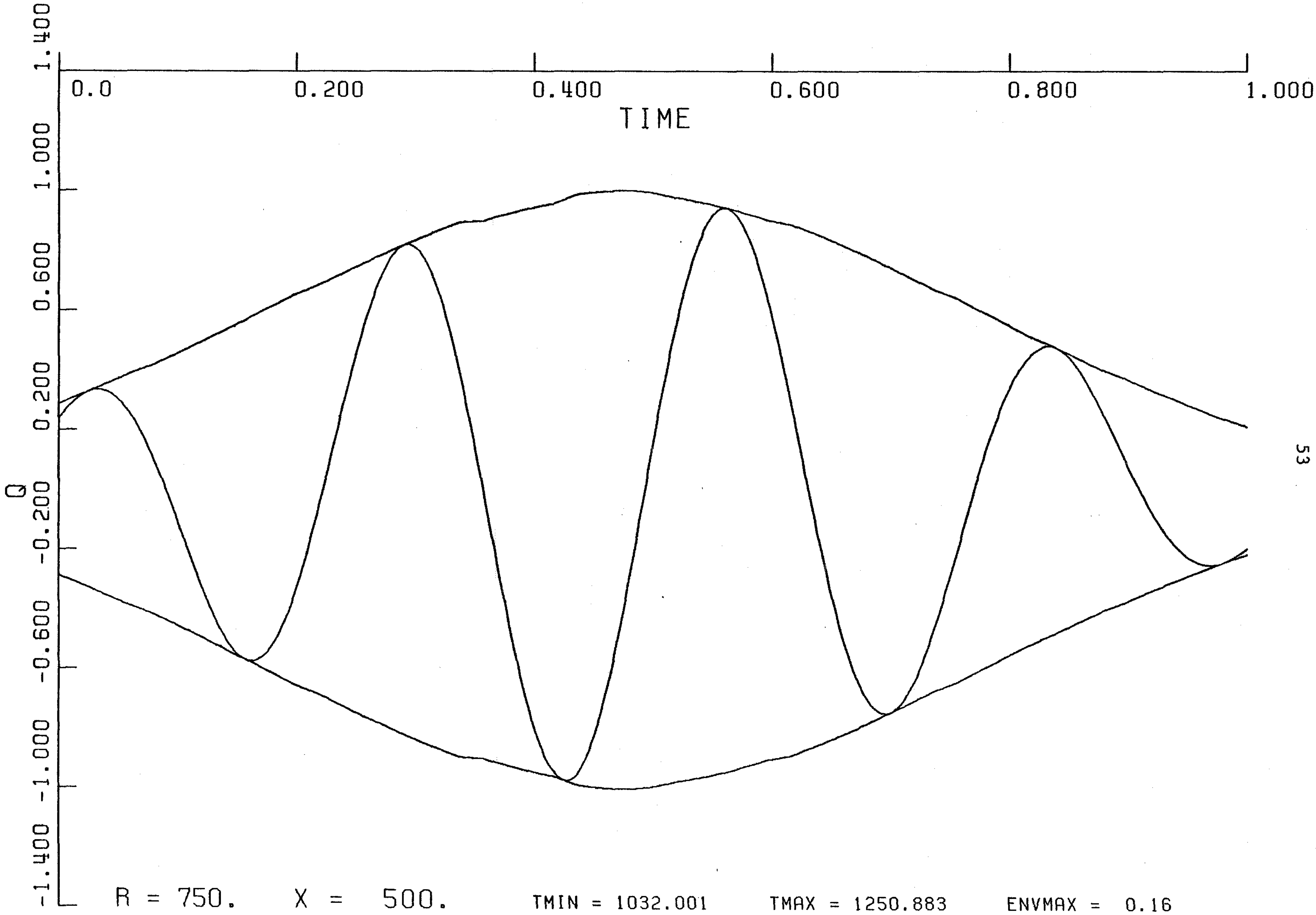

SADDLE-POINT LOCUS (R=750.)
SCALE FACTOR = 10 -1
1.200
1.100
1.000
0.900
0.800
ØMR
SCALE FACTOR = 10 -2
3.500
2.500
1.500
0.500
-0.500
-1.500
-2.500
-3.500
ØMI

Fig. 1



1.400
1.000
0.600
0.200
-0.200
-0.600
-1.000
-1.400
Q
0.0
0.200
0.400
0.600
0.800
1.000
TIME
R = 750. X = 500.
TMIN = 1032.001 TMAX = 1250.883 ENVMAX = 0.16

Fig. 2

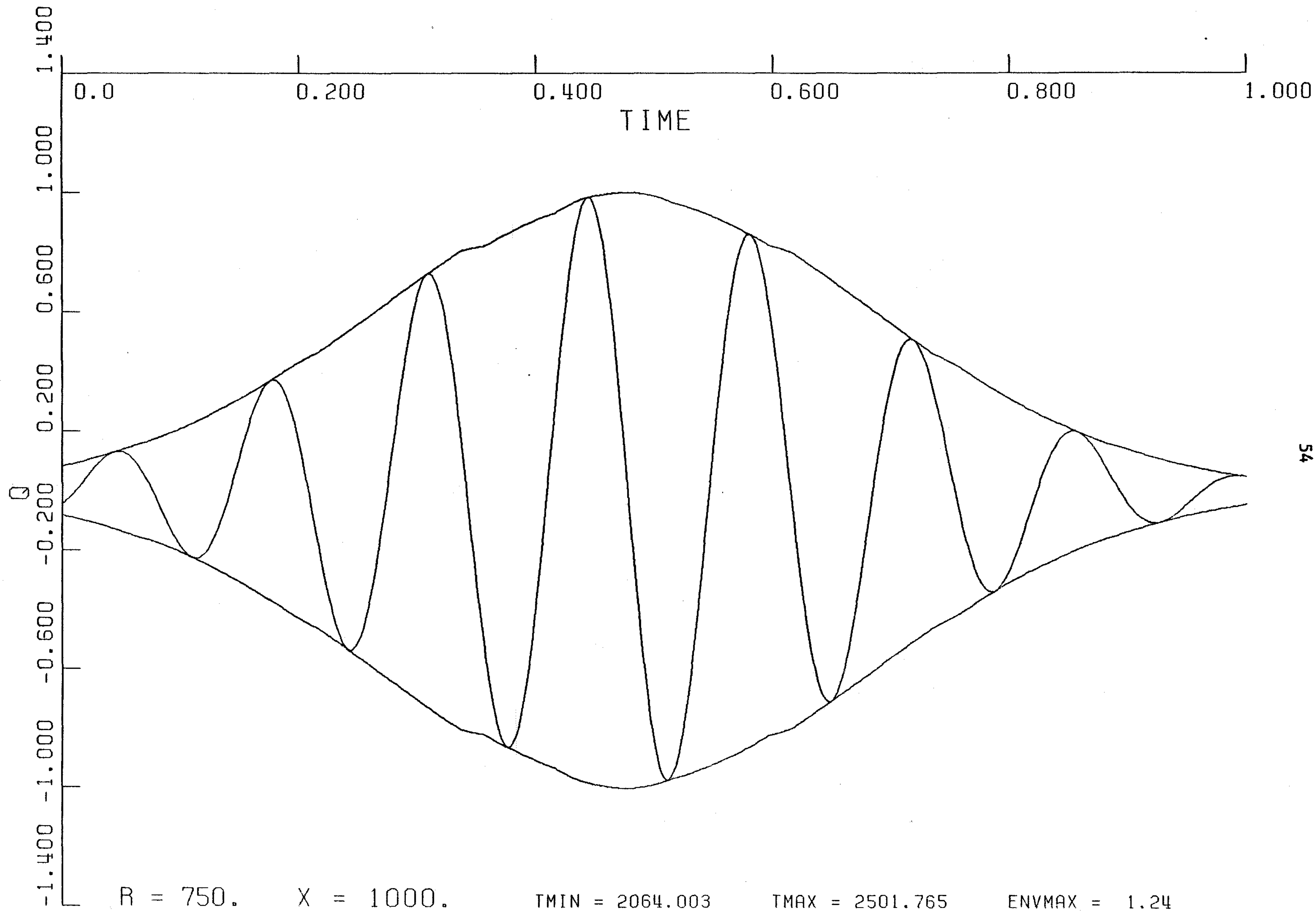
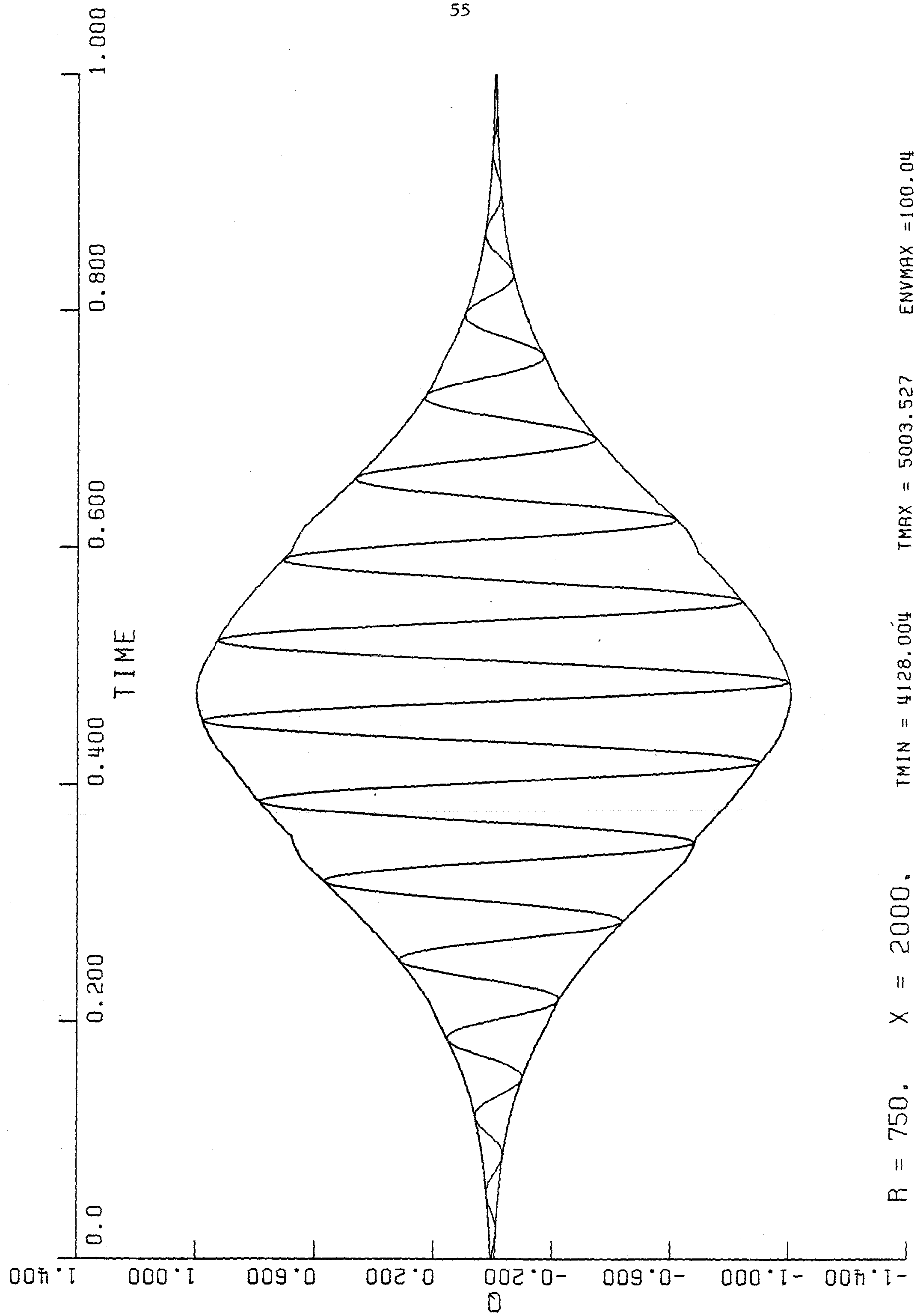

1.400
1.000
0.600
0.200
-0.200
-0.600
-1.000
-1.400
Q
0.0
0.200
0.400
0.600
0.800
1.000
TIME
R = 750.
X = 1000.
TMIN = 2064.003
TMAX = 2501.765
ENVMAX = 1.24

Fig. 3



TIME
Q
0.0
0.200
0.400
0.600
0.800
1.000
1.400
1.000
0.600
0.200
-0.200
-0.600
-1.000
-1.400
R = 750.   X = 2000.
TMIN = 4128.004   TMAX = 5003.527   ENVMAX =100.04

Fig. 4

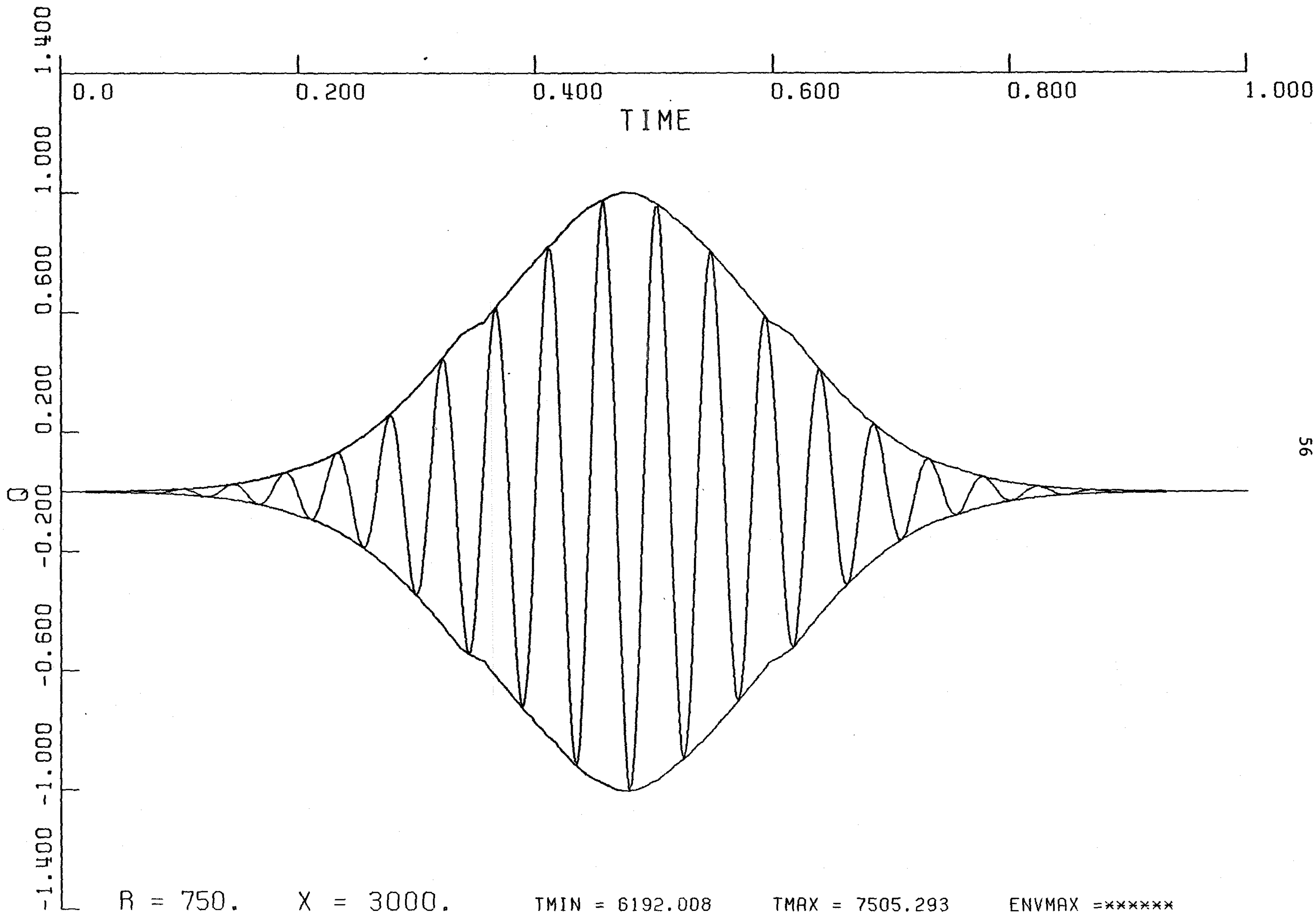

1.400
1.000
0.600
0.200
-0.200
-0.600
-1.000
-1.400
Q
0.0
0.200
0.400
0.600
0.800
1.000
TIME
R = 750.
X = 3000.
TMIN = 6192.008
TMAX = 7505.293
ENVMAX =******

Fig. 5

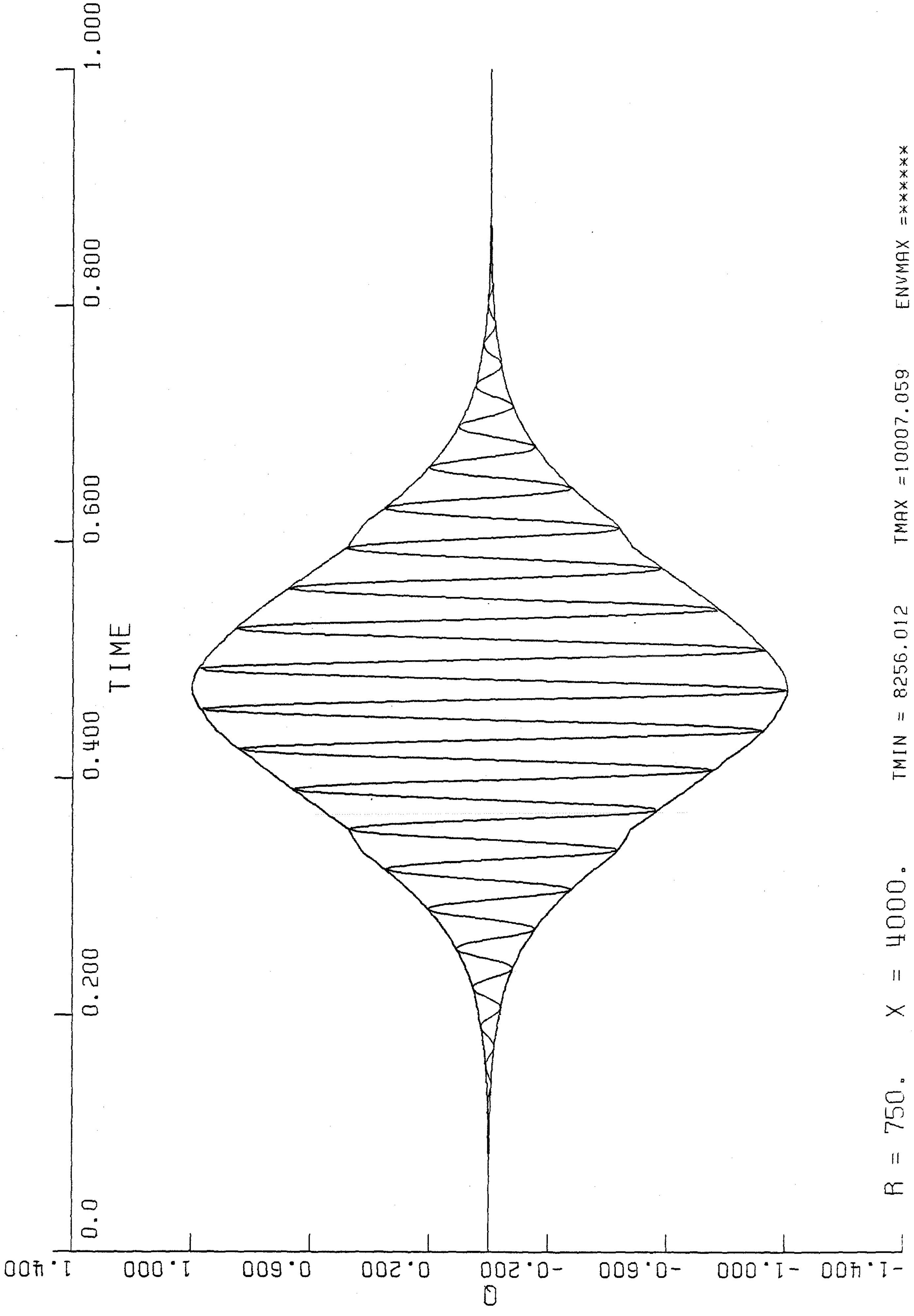
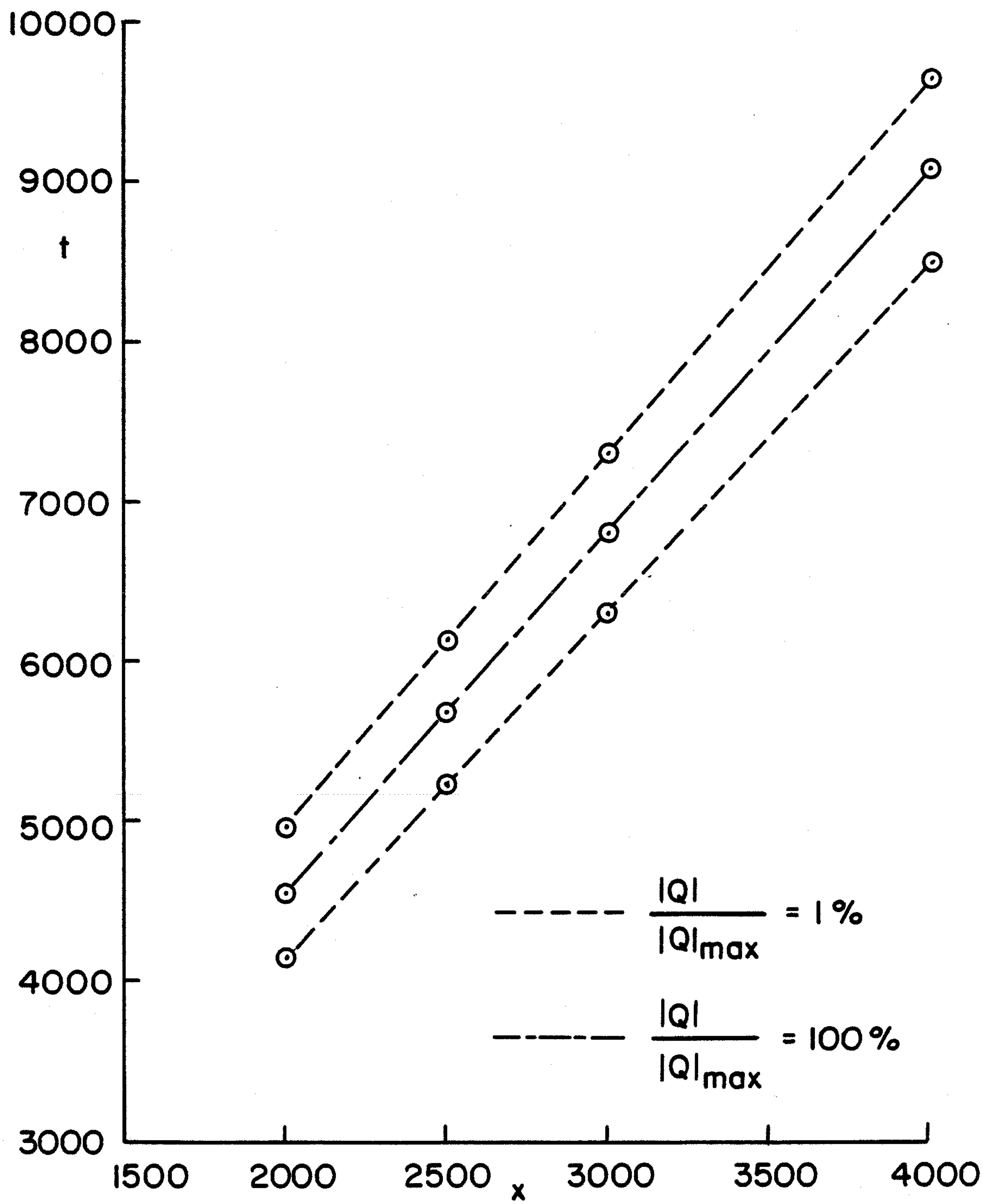

TIME
0.0
0.200
0.400
0.600
0.800
1.000
Q
1.400
1.000
0.600
0.200
-0.200
-0.600
-1.000
-1.400
R = 750. X = 4000. TMIN = 8256.012 TMAX =10007.059 ENVMAX =******

Fig. 6



THE WAVE PACKET ON AN x – t DIAGRAM

Fig. 7

Listings of FORTRAN Programs

```
C  TESTE: This program tests subroutine EIGEN.  The reference length is t
C          displacement thickness.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y(101),Z(10,101),IWORK(2200),WORK(37000)
      COMMON/MNFLO/TU(1001),TUPP(1001),B
C
C
      OI     =      0.
      B      =      6.*DSQRT(2.D0)/1.7208
      NMAX   =    101
      TOL    =       .0001
C
C Read mean flow
C
      READ (20) (TU(J),TUPP(J),J=1,1001)
 1310 READ(20,END=1320)
      GO TO 1310
C
C Initial guess
C
 1320 READ (20) N,R,OR,OI,(Y(J),(Z(I,J),I=1,10),J=1,98)
 1330 READ(20,END=1340)
      GO TO 1330
 1340 R = 1000.
      OR = .10
      DO 33 J=1,98
   33 Y(J) = DSQRT(2.D0)/1.7208*Y(J)
C
C
      CALL EIGEN(R,OR,OI,IWORK,WORK,N,NMAX,TOL,Y,Z)
      WRITE (20) N,R,OR,OI,(Y(J),(Z(I,J),I=1,10),J=1,N)
C
C
      STOP
      END
      SUBROUTINE EIGEN(RR,OOR,OOI,IWORK,WORK,N,NMAX,TOL,Y,Z)
C
C  This subroutine computes the eigenvalues and eigenfunctions of
C  the Orr-Sommerfeld equation.
C  PASVA3 is used to solve nonlinear boundary-value problem.
C
C  Required input:
C   _Reynolds number R.
C   _Frequency (OR,OI).
C   _Initial guess.
C
C  Output:
C   _Eigenwavenumber (Z(9,I),Z(10,I)).
C   -Eigenfunction (Z(1,I),Z(2,I),...,Z(8,I)).
C
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER P,Q,IWORK(1)
      DIMENSION Y(1),Z(10,1),ABT(10),PAR(5),WORK(1)
      COMMON/MNFLO/TU(1001),TUPP(1001),B
      COMMON/RO/R,OR,OI
      EXTERNAL FF,JACOB
      R  = RR
      OR = OOR
      OI = OOI
C
C Write job heading
C
      WRITE(6,1100) R,OR,OI
 1100 FORMAT(1H1,' EIGENVALUE AND EIGENFUNCTION OF THE ORR-
     2              SOMMERFELD EQUATION'//
     3        '        R      =',F7.0/
     4        '        OR     =',F7.4/
     5        '        OI     =',F7.4//)
C
C Set PASVA3 parameters
C
C   Number of equations
      M = 10
C   Maximum number of equations
      MMAX = 10
C   Number of points in the initial mesh
C     N IS INPUT
C   Maximum number of mesh points allowed
C     NMAX IS INPUT
C   Maximum value of the product M*NMAX
      MTNMAX = M*NMAX
C   Number of initial conditions
      P = 6
C   Number of coupled conditions
      Q = 0
C   End points of integration
      A = 0.
C     B is input
C   No default option
      PAR(1) = 1.
C   No continuation
      PAR(2) = 0.
C   Print intermediary results
      PAR(3) = 1.
C   Initial Y and Z are provided by user
      PAR(4) = 1.
C   Nonlinear problem
      PAR(5) = 0.
C
C
C  Call PASVA3
C
C
```

```
      CALL PASVA3(MMAX,M,MTNMAX,NMAX,N,P,Q,A,B,TOL,Y,Z,ABT,FF,JACOB,PAR,
     2             WORK,IWORK,JERROR)
      IF(JERROR.GT.0) GO TO 101
C
C  Print output
C
      WRITE(6,2000)
 2000 FORMAT(1H1,'RESULTS'/)
      WRITE(6,2100)
 2100 FORMAT(1H0,'  J',T10,'Y(J)',T21,'Z(1,J)',T32,'Z(2,J)',T43,'Z(3,J)'
     2              ,T54,'Z(4,J)',T65,'Z(5,J)',T76,'Z(6,J)',T87,'Z(7,J)
     3               ',T98,'Z(8,J)',T109,'Z(9,J)',T120,'Z(10,J)'/)
      WRITE(6,2200) (J,Y(J),Z(1,J),Z(2,J),Z(3,J),Z(4,J),Z(5,J),Z(6,J),
     2               Z(7,J),Z(8,J),Z(9,J),Z(10,J),J=1,N)
 2200 FORMAT(I4,1P11D11.3)
C
C
C
 101  RETURN
      END
      FUNCTION U(Y)
C
C  This function computes the mean velocity at required y-location
C  by linear interpolation from the data supplied (1001 values at equal
C  intervals)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/MNFLO/TU(1001),TUPP(1001),B
      L1 = 1000.*Y/B
      L1 = L1+1
      L2 = L1+1
      IF((L1.EQ.1).OR.(L1.EQ.1001)) L2=L1
      U = (TU(L1)+TU(L2))/2.
      RETURN
      END
      FUNCTION UPP(Y)
C
C  This function computes the second derivative of the mean velocity
C  At required y-location by linear interpolation from the data supplied
C  (1001 values at equal intervals)
```

```
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/MNFLO/TU(1001),TUPP(1001),B
      L1 = 1000.*Y/B
      L1 = L1+1
      L2 = L1+1
      IF((L1.EQ.1).OR.(L1.EQ.1001)) L2=L1
      UPP = (TUPP(L1)+TUPP(L2))/2.
      RETURN
      END
      SUBROUTINE FF(Y,Z,N,F,ALPHA)
C
C   This subroutine supplies F, the right-hand side of the equation, and
C   ALPHA, the left-hand side of the boundary conditions.
C   Equation:
      DZ/DY = F(Y,Z)
C   Boundary conditions:
C     ALPHA = 0.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y(1),Z(10,1),F(10,1),ALPHA(10)
      COMMON/RO/R,OR,OI
      COMMON/GEN/NEF,NEJ
      NEF = NEF+N
C
C   Convenient parameters
      GR = Z(9,N)**2-Z(10,N)**2-R*(Z(10,N)-OI)
      GI = 2.*Z(9,N)*Z(10,N)+R*(Z(9,N)-OR)
      E1 = R*(Z(10,N)-OI)
      E2 = R*(OR-Z(9,N))
      E3 = Z(3,N)+2.*(Z(9,N)*Z(1,N)-Z(10,N)*Z(2,N))
      E4 = Z(4,N)+2.*(Z(9,N)*Z(2,N)+Z(10,N)*Z(1,N))
      BR = DSQRT(0.5*(GR+DSQRT(GR**2+GI**2)))
      BI = GI/(2.*BR)
      E5 = Z(9,N)+BR
      E6 = Z(10,N)+BI
      E7 = 2.*(Z(9,N)*Z(5,N)-Z(10,N)*Z(6,N))
      E8 = 2.*(Z(9,N)*Z(6,N)+Z(10,N)*Z(5,N))
      E9 = (2.*Z(9,N)*BR+(2.*Z(10,N)+R)*BI)/(2.*(BR**2+BI**2))
      E10 = (-2.*Z(9,N)*BI+(2.*Z(10,N)+R)*BR)/(2.*(BR**2+BI**2))
C
C
      ALPHA(1) = Z(1,1)
      ALPHA(2) = Z(2,1)
      ALPHA(3) = Z(3,1)



      ALPHA(4) = Z(4,1)
      ALPHA(5) = Z(5,1)-1.
      ALPHA(6) = Z(6,1)
      ALPHA(7) = Z(7,N)+E7+E1*E3-E2*E4
      ALPHA(8) = Z(8,N)+E8+E1*E4+E2*E3
      ALPHA(9) = Z(7,N)+E5*Z(5,N)-E6*Z(6,N)
      ALPHA(10) = Z(8,N)+E5*Z(6,N)+E6*Z(5,N)
```

```
C
C
      DO 10 I = 1,N
C   Convenient parameters
      CF1 = R*(Z(9,I)*Z(2,I)+Z(10,I)*Z(1,I))
      CF2 = R*(Z(9,I)*Z(6,I)+Z(10,I)*Z(5,I))
      CF3 = R*(OR*Z(6,I)+OI*Z(5,I))
      CF4 = Z(9,I)*Z(7,I)-Z(10,I)*Z(8,I)
      CF5 = R*(Z(9,I)*Z(1,I)-Z(10,I)*Z(2,I))
      CF6 = R*(Z(9,I)*Z(5,I)-Z(10,I)*Z(6,I))
      CF7 = R*(OR*Z(5,I)-OI*Z(6,I))
      CF8 = Z(9,I)*Z(8,I)+Z(10,I)*Z(7,I)
      CU = U(Y(I))
      CUPP = UPP(Y(I))
C
      F(1,I)  =  Z(9,I)*Z(1,I)-Z(10,I)*Z(2,I)+Z(3,I)
      F(2,I)  =  Z(9,I)*Z(2,I)+Z(10,I)*Z(1,I)+Z(4,I)
      F(3,I)  =  -(Z(9,I)*Z(3,I)-Z(10,I)*Z(4,I))+Z(5,I)
      F(4,I)  =  -(Z(9,I)*Z(4,I)+Z(10,I)*Z(3,I))+Z(6,I)
      F(5,I)  =  Z(9,I)*Z(5,I)-Z(10,I)*Z(6,I)+Z(7,I)
      F(6,I)  =  Z(9,I)*Z(6,I)+Z(10,I)*Z(5,I)+Z(8,I)
      F(7,I)  =  CF1*CUPP-CF2*CU+CF3-CF4
      F(8,I)  =  -CF5*CUPP+CF6*CU-CF7-CF8
      F(9,I)  =  0.
      F(10,I) =  0.
   10 CONTINUE
      RETURN
      END
      SUBROUTINE JACOB(Y,Z,I,C,N,A1,B1)
C
C   This subroutine supplies C, the Jacobian of F with respect to the
C   dependent variables Z; A1, the Jacobian of ALPHA with respect to
C   Z(Y=0.); and B1, the Jacobian of ALPHA with respect to Z(Y=6.)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y(1),Z(10,1),C(10,10),A1(10,10),B1(10,10)
      COMMON/RO/R,OR,OI
      COMMON/GEN/NEF,NEJ
      NEJ = NEJ+1
      IF(I.NE.1) GO TO 50
C
C   Convenient parameters
      GR = Z(9,N)**2-Z(10,N)**2-R*(Z(10,N)-OI)
      GI = 2.*Z(9,N)*Z(10,N)+R*(Z(9,N)-OR)
      E1 = R*(Z(10,N)-OI)
      E2 = R*(OR-Z(9,N))
      E3 = Z(3,N)+2.*(Z(9,N)*Z(1,N)-Z(10,N)*Z(2,N))
      E4 = Z(4,N)+2.*(Z(9,N)*Z(2,N)+Z(10,N)*Z(1,N))
      BR = DSQRT(0.5*(GR+DSQRT(GR**2+GI**2)))
      BI = GI/(2.*BR)
      E5 = Z(9,N)+BR
      E6 = Z(10,N)+BI
      E7 = 2.*(Z(9,N)*Z(5,N)-Z(10,N)*Z(6,N))
```

```
      E8  = 2.*(Z(9,N)*Z(6,N)+Z(10,N)*Z(5,N))
      E9  = (2.*Z(9,N)*BR+(2.*Z(10,N)+R)*BI)/(2.*(BR**2+BI**2))
      E10  = (-2.*Z(9,N)*BI+(2.*Z(10,N)+R)*BR)/(2.*(BR**2+BI**2))




C
C
C
      DO 10 I10 = 1,10
      DO 10 J = 1,10
      A1(I10,J) = 0.
      IF((I10.LE.6).AND.(J.EQ.I10)) A1(I10,J) = 1.
      B1(I10,J) = 0.
   10 C(I10,J) = 0.
C
      B1(7,1) = 2.*(E1*Z(9,N)-E2*Z(10,N))
      B1(7,2) = -2.*(E1*Z(10,N)+E2*Z(9,N))
      B1(7,3) = E1
      B1(7,4) = -E2
      B1(7,5) = 2.*Z(9,N)
      B1(7,6) = -2.*Z(10,N)
      B1(7,7) = 1.
      B1(7,9) = 2.*Z(5,N)+2.*E1*Z(1,N)+R*E4-2.*E2*Z(2,N)
      B1(7,10) = -2.*Z(6,N)+R*E3-2.*E1*Z(2,N)-2.*E2*Z(1,N)
      B1(8,1) = 2.*(E1*Z(10,N)+E2*Z(9,N))
      B1(8,2) = 2.*(E1*Z(9,N)-E2*Z(10,N))
      B1(8,3) = E2
      B1(8,4) = E1
      B1(8,5) = 2.*Z(10,N)
      B1(8,6) = 2.*Z(9,N)
      B1(8,8) = 1.
      B1(8,9) = 2.*(Z(6,N)+E1*Z(2,N)+E2*Z(1,N))-R*E3
      B1(8,10) = 2.*(Z(5,N)+E1*Z(1,N)-E2*Z(2,N))+R*E4
      B1(9,5) = E5
      B1(9,6) = -E6
      B1(9,7) = 1.
      B1(9,9) = Z(5,N)*(1.+E9)-Z(6,N)*E10
      B1(9,10) = -Z(5,N)*E10-Z(6,N)*(1.+E9)
      B1(10,5) = E6
      B1(10,6) = E5
      B1(10,8) = 1.
      B1(10,9) = Z(6,N)*(1.+E9)+Z(5,N)*E10
      B1(10,10) = -Z(6,N)*E10+Z(5,N)*(1.+E9)
C
   50 CONTINUE
C  Convenient parameters
      CU   = U(Y(I))
      CUPP = UPP(Y(I))
C
      C(1,1)  = Z(9,I)
      C(1,2)  = -Z(10,I)
      C(1,3)  = 1.
      C(1,9)  = Z(1,I)
      C(1,10) = -Z(2,I)
      C(2,1)  = Z(10,I)
      C(2,2)  = Z(9,I)
```

```
C(2,4)   = 1.
C(2,9)   = Z(2,I)
C(2,10)  = Z(1,I)
C(3,3)   = -Z(9,I)
C(3,4)   = Z(10,I)
C(3,5)   = 1.
C(3,9)   = -Z(3,I)
C(3,10)  = Z(4,I)
C(4,3)   = -Z(10,I)
C(4,4)   = -Z(9,I)
C(4,6)   = 1.
C(4,9)   = -Z(4,I)


C(4,10)  = -Z(3,I)
C(5,5)   = Z(9,I)
C(5,6)   = -Z(10,I)
C(5,7)   = 1.
C(5,9)   = Z(5,I)
C(5,10)  = -Z(6,I)
C(6,5)   = Z(10,I)
C(6,6)   = Z(9,I)
C(6,8)   = 1.
C(6,9)   = Z(6,I)
C(6,10)  = Z(5,I)
C(7,1)   = R*Z(10,I)*CUPP
C(7,2)   = R*Z(9,I)*CUPP
C(7,5)   = R*(OI-CU*Z(10,I))
C(7,6)   = R*(-CU*Z(9,I)+OR)
C(7,7)   = -Z(9,I)
C(7,8)   = Z(10,I)
C(7,9)   = R*(CUPP*Z(2,I)-CU*Z(6,I))-Z(7,I)
C(7,10)  = R*(CUPP*Z(1,I)-CU*Z(5,I))+Z(8,I)
C(8,1)   = -R*Z(9,I)*CUPP
C(8,2)   = R*Z(10,I)*CUPP
C(8,5)   = -C(7,6)
C(8,6)   = C(7,5)
C(8,7)   = -Z(10,I)
C(8,8)   = -Z(9,I)
C(8,9)   = R*(-CUPP*Z(1,I)+CU*Z(5,I))-Z(8,I)
C(8,10)  = R*(CUPP*Z(2,I)-CU*Z(6,I))-Z(7,I)
RETURN
END
```

```
C  TESTA: This program tests subroutine ADJOIN.  The reference length is
C         0.822 times the displacement thickness.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y(101),X(8,101),IWORK(2000),WORK(25000)
      COMMON/MNFLO/TU(1001),TUPP(1001)
      R  = 1000.
      OR =     .1000
      OI =    0.
      AR =     .2785
      AI =   -0.0007
      B  =    6.
      N  =   51
      NMAX= 101
      TOL =    .0001
      READ (20) (TU(J),TUPP(J),J=1,1001)
      CALL ADJOIN(R,OR,OI,AR,AI,IWORK,WORK,B,N,NMAX,TOL,Y,X)
      STOP
      END
      SUBROUTINE ADJOIN(RR,OOR,OOI,AAR,AAI,IWORK,WORK,B,N,NMAX,TOL,Y,X)
C
C This subroutine computes the eigenfunctions of the adjoint system.
C PASVA3 is used to solve linear boundary-value problem.
C
C   Required input:
C    -Reynolds number R.
C    -Frequency (OR,OI).
C    -Eigenvalue (AR,AI).
C   Output:
C    -Eigenfunction of the adjoint system (X(1,J),X(2,J),...,X(8,J)).
C
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER P,Q,IWORK(1)
      DIMENSION Y(1),X(8,1),ABT(8),PAR(5),WORK(1)
      COMMON/MNFLO/TU(1001),TUPP(1001)
      COMMON/ROA/R,OR,OI,AR,AI
      EXTERNAL FF1,JACOB1
      R  = RR
      OR = OOR
      OI = OOI
      AR = AAR
      AI = AAI
C
C Write job heading
      WRITE(6,1000) R,OR,OI,AR,AI
 1000 FORMAT(1H1,' EIGENFUNCTION OF THE ADJOINT SYSTEM'//
     2      '        R     =',F7.0/
     3      '        OR    =',F7.4/
     4      '        OI    =',F7.4/
     5      '        AR    =',F7.4/
     6      '        AI    =',F7.4//)
C
C Set PASVA3 parameters
```

```
C
C    Number of equations
       M = 8
C    Maximum number of equations
       MMAX = 8
C    Number of points in the initial mesh
C      N is input
C    Maximum number of mesh points allowed
C      NMAX is input
C    Maximum value of the product M*NMAX
       MTNMAX = M*NMAX
C    Number of initial conditions
       P = 4
C    Number of coupled conditions
       Q = 0
C    End points of integration
       A = 0.
C      B is input
C    No default option
       PAR(1) = 1.
C    No continuation
       PAR(2) = 0.
C    Print intermediary results
       PAR(3) = 1.
C    Initial guess is provided by PASVA3
       PAR(4) = 0.
C    Linear problem
       PAR(5) = 1.
C
C
C
      CALL PASVA3(MMAX,M,MTNMAX,NMAX,N,P,Q,A,B,TOL,Y,X,ABT,FF1,JACOB1,
     2            PAR,WORK,IWORK,JERROR)
      IF(JERROR.GT.0) GO TO 800
C
C
C  Print output
C
      WRITE(6,1500)
 1500 FORMAT(1H1,'RESULTS'/)
      WRITE(6,1600)
 1600 FORMAT(1H0,'  J',T10,'Y(J)',T21,'X(1,J)',T32,'X(2,J)',T43,'X(3,J)'
     2           ,T54,'X(4,J)',T65,'X(5,J)',T76,'X(6,J)',T87,'X(7,J)
     3             ',T98,'X(8,J)'/)
      WRITE(6,1700) (J,Y(J),X(1,J),X(2,J),X(3,J),X(4,J),X(5,J),
     2                 X(6,J),X(7,J),X(8,J),J=1,N)
 1700 FORMAT(I4,1P9D11.3)
C
C
C
  800 RETURN
      END
      SUBROUTINE FF1(Y,X,N,F,ALPHA)
```

```
C
C  This subroutine supplies F, the right-hand side of the equation, and
C  ALPHA, the left-hand side of the boundary conditions.
C  Equation:
C    DX/DY = F(Y,X)
C  Boundary conditions:
C    ALPHA = 0.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y(1),X(8,1),F(8,1),ALPHA(8)
      COMMON/ROA/R,OR,OI,AR,AI
      COMMON/GEN/NEF,NEJ
      NEF = NEF+N
      DO 10 I = 1,N
      CU = U(Y(I))
      CUPP = UPP(Y(I))
      F(1,I)  = -(AR*X(1,I)-AI*X(2,I))-R*CUPP*(X(7,I)*AI+X(8,I)*AR)
      F(2,I)  = -(AR*X(2,I)+AI*X(1,I))+R*CUPP*(X(7,I)*AR-X(8,I)*AI)
      F(3,I)  = -X(1,I)+(AR*X(3,I)-AI*X(4,I))
      F(4,I)  = -X(2,I)+(AR*X(4,I)+AI*X(3,I))
      F(5,I)  = -X(3,I)-(AR*X(5,I)-AI*X(6,I))+R*CU*(AR*X(8,I)+AI*X(7,I))
     2          -R*(OR*X(8,I)+OI*X(7,I))
      F(6,I)  = -X(4,I)-(AR*X(6,I)+AI*X(5,I))-R*CU*(AR*X(7,I)-AI*X(8,I))
     2          +R*(OR*X(7,I)-OI*X(8,I))
      F(7,I)  = -X(5,I)+(AR*X(7,I)-AI*X(8,I))
      F(8,I)  = -X(6,I)+(AR*X(8,I)+AI*X(7,I))
   10 CONTINUE
C  Convenient parameters
      E1  =  R*(OI-AI)
      E2  =  R*(AR-OR)
      GR  =  AR**2-AI**2+E1
      GI  =  2.*AR*AI+E2
      BR  =  DSQRT((GR+DSQRT(GR**2+GI**2))/2.)
      BI  =  GI/(2.*BR)
      E5  =  AR+BR
      E6  =  AI+BI
      E9  =  E1*E5-E2*E6
      E10 =  E1*E6+E2*E5
      ALPHA(1)  =  X(5,1)
      ALPHA(2)  =  X(6,1)
      ALPHA(3)  =  X(1,1)-1.
      ALPHA(4)  =  X(2,1)
      ALPHA(5) = X(1,N)-2.*(AR*X(3,N)-AI*X(4,N))
      ALPHA(6) = X(2,N)-2.*(AR*X(4,N)+AI*X(3,N))
      ALPHA(7) = X(1,N)-(E5*X(3,N)-E6*X(4,N))+(E1*X(5,N)-E2*X(6,N))
     2           -(E9*X(7,N)-E10*X(8,N))
      ALPHA(8) = X(2,N)-(E5*X(4,N)+E6*X(3,N))+(E1*X(6,N)+E2*X(5,N))
     2           -(E9*X(8,N)+E10*X(7,N))
      RETURN
      END
      SUBROUTINE JACOB1(Y,X,I,C,N,A1,B1)
C
C  This subroutine supplies C, the Jacobian of F with respect to the
C  dependent variables X; A1, the Jacobian of ALPHA with respect to
```

```
C  X(Y=0.); and B1, the Jacobian of ALPHA with respect to X(Y=6.).
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y(1),X(8,1),C(8,8),A1(8,8),B1(8,8)
      COMMON/ROA/R,OR,OI,AR,AI
      COMMON/GEN/NEF,NEJ
      NEJ = NEJ+1
      IF(I.NE.1) GO TO 50
      DO 10 L1=1,8
      DO 10 L2=1,8
      A1(L1,L2) = 0.
      B1(L1,L2) = 0.
   10 C(L1,L2) = 0.


C  Convenient parameters
      E1   =  R*(OI-AI)
      E2   =  R*(AR-OR)
      GR   =  AR**2-AI**2+E1
      GI   =  2.*AR*AI+E2
      BR   =  DSQRT((GR+DSQRT(GR**2+GI**2))/2.)
      BI   =  GI/(2.*BR)
      E5   =  AR+BR
      E6   =  AI+BI
      E9   =  E1*E5-E2*E6
      E10  =  E1*E6+E2*E5
      A1(1,5) = 1.
      A1(2,6) = 1.
      A1(3,1) = 1.
      A1(4,2) = 1.
      B1(5,1) = 1.
      B1(5,3) =-2.*AR
      B1(5,4) = 2.*AI
      B1(6,2) = 1.
      B1(6,3) =-2.*AI
      B1(6,4) =-2.*AR
      B1(7,1) = 1.
      B1(7,3) =-E5
      B1(7,4) = E6
      B1(7,5) = E1
      B1(7,6) =-E2
      B1(7,7) =-E9
      B1(7,8) = E10
      B1(8,2) = 1.
      B1(8,3) =-E6
      B1(8,4) =-E5
      B1(8,5) = E2
      B1(8,6) = E1
      B1(8,7) =-E10
      B1(8,8) =-E9
   50 CONTINUE
      CU = U(Y(I))
      CUPP = UPP(Y(I))
      C(1,1)     = -AR
      C(1,2)     =  AI
      C(1,7)     = -R*CUPP*AI
```

```
      C(1,8)     = -R*CUPP*AR
      C(2,1)     = -AI
      C(2,2)     = -AR
      C(2,7)     =  R*CUPP*AR
      C(2,8)     = -R*CUPP*AI
      C(3,1)     = -1.
      C(3,3)     =  AR
      C(3,4)     = -AI
      C(4,2)     = -1.
      C(4,3)     =  AI
      C(4,4)     =  AR
      C(5,3)     = -1.
      C(5,5)     = -AR
      C(5,6)     =  AI
      C(5,7)     =  R*(CU*AI-OI)
      C(5,8)     =  R*(CU*AR-OR)
      C(6,4)     = -1.
      C(6,5)     = -AI
      C(6,6)     = -AR
      C(6,7)     = -C(5,8)
      C(6,8)     =  C(5,7)
      C(7,5)     = -1.


      C(7,7)     =  AR
      C(7,8)     = -AI
      C(8,6)     = -1.
      C(8,7)     =  AI
      C(8,8)     =  AR
      RETURN
      END
      FUNCTION U(Y)
C
C   This function computes the mean velocity at required y-location by
C   linear interpolation from the data supplied (1001 values at equal
C   intervals).
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/MNFLO/TU(1001),TUPP(1001)
      L1 = 1000.*Y/6
      L1 = L1+1
      L2 = L1+1
      IF((L1.EQ.1).OR.(L1.EQ.1001)) L2=L1
      U = (TU(L1)+TU(L2))/2.
      RETURN
      END
      FUNCTION UPP(Y)
C
C   This function computes the second derivative of the mean velocity
C   at required y-location by linear interpolation from the data
C   supplied (1001 values at equal intervals).
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/MNFLO/TU(1001),TUPP(1001)
      L1 = 1000.*Y/6
      L1 = L1+1
```

```
L2 = L1+1
IF((L1.EQ.1).OR.(L1.EQ.1001)) L2=L1
UPP = (TUPP(L1)+TUPP(L2))/2.
RETURN
END
```