The equilibrium field near the tip of a crack for finite plane strain of incompressible elastic materials

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Abstract

This investigation is concerned with the deformations and stresses in a slab of all-around infinite extent containing a traction-free plane crack, under conditions of plane strain. The analysis is carried out within the framework of the fully nonlinear equilibrium theory of homogeneous and isotropic incompressible elastic solids. For a fairly wide class of such materials and general loading conditions at infinity, asymptotic estimates appropriate to the various field quantities near the crack-tips are deduced. For a subclass of the materials considered, these results — in contrast to the analogous predictions of the linearized theory — lead to the conclusion that the crack opens up in a neighborhood of its tips even if the applied loading is antisymmetric about the plane of the crack, (e.g., Mode II loading). It is shown further that the nonlinear global crack problem corresponding to such a loading in general cannot admit an antisymmetric solution.

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Introduction

Early investigations of crack problems beyond the scope of the classical theory of elasticity typically retain the kinematic assumption of infinitesimal deformations while relinquishing the linear stress-strain law in favor of nonlinear constitutive relations of one kind or another.[1] Apparently the first investigation of a crack problem within the fully nonlinear equilibrium theory of elastic materials is due to Wong and Shield [2]. In [2] an approximative global solution is obtained to the problem of a finite crack in an all-around infinite thin incompressible sheet of a neo-Hookean material, subjected to biaxial tension at infinity, on the assumption that the deformations are large throughout the sheet.

Knowles and Sternberg [3], [4] deal with the problem of an infinite slab of a compressible elastic material containing a plane traction-free crack, within the equilibrium theory of finite plane strain; they restrict their attention to a loading of uniaxial tension at right angles to the plane of the crack (Mode I loading) and analyze the structure of the ensuing elastostatic field near the crack-tips. The Mode III crack problem, in which the loading at infinity is one of longitudinal shear parallel to the edges of the crack, is explored asymptotically in the finite theory for a class of incompressible elastic materials by Knowles [5]. Additional work on the nonlinear Mode III problem may be found in [6], [7], [8]. The investigations referred to above, as well as some related studies in finite elastostatics, are reviewed in several survey papers [9], [10], [11], [12].

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[1] See, for example, a comprehensive survey by Rice [1].

The present study, which is also set within finite elastostatics, retains the body geometry underlying [3], [4], as well as the assumption of plane strain, but supposes the material to be incompressible. The particular constitutive assumptions introduced here allow for both hardening and softening behavior in simple shear, but preclude a loss of ellipticity of the governing displacement equations of equilibrium.[1] In addition, more general loading conditions at infinity are considered, so as to encompass in particular the Mode II loading of simple shear parallel to the crack faces.

Section 1 contains a review of some prerequisites from the finite theory of plane strain for homogeneous, isotropic incompressible elastic solids and introduces a particular class of materials underlying the subsequent analysis. In Section 2 we formulate the global crack problem with which we are concerned and recall certain properties of the solution to the analogous crack problem in the linearized theory. We then show that, at least for the Mooney-Rivlin material, the global nonlinear Mode II problem, in contrast to its counterpart in the linear theory, does not admit a solution antisymmetric about the plane of the crack. Section 3 and Section 4 are devoted to an asymptotic analysis of the elastostatic field near the crack-tips. Finally Section 5 contains a discussion of the main conclusions reached concerning the ensuing local deformation and stresses.

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[1] A potential loss of ellipticity in the context of the Mode III problem is provided for in [7], [8].

1. Preliminaries from the finite theory of plane strain for incompressible elastic solids

Throughout this paper we shall be concerned with plane finite elastostatics of homogeneous and isotropic incompressible materials, in the absence of body forces. Consider a body which in an undeformed configuration occupies a cylindrical region of space. Denote by Π the open projection of this region onto a plane that is perpendicular to the generators of the lateral boundary. Introduce a two-dimensional rectangular cartesian coordinate system in this plane and thus let (x_1, x_2) be coordinates associated with the position vector[1] $\underset{\sim}{x}$.

A plane deformation of the body parallel to the (x_1, x_2)-plane is characterized by the mapping

$$\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) \quad \text{on} \quad \Pi \quad . \tag{1.1}$$

Thus $\underset{\sim}{y}$ is the position vector of points in the deformation image $\Pi^* = \hat{\underset{\sim}{y}}(\Pi)$, while $\underset{\sim}{u}$ is the displacement field. We suppose the function $\hat{\underset{\sim}{y}}$ to be twice continuously differentiable on $\bar{\Pi}$, and the mapping (1.1) to be invertible.

Let $\underset{\sim}{F}$ be the associated deformation-gradient field, so that

$$\underset{\sim}{F} = \nabla\hat{\underset{\sim}{y}} \quad \text{on} \quad \Pi \quad . \tag{1.2}$$

Since the body is incompressible, the deformation (1.1) must be locally

[1] Here and in what follows, letters underlined by a tilde designate vectors and second-order tensors in two dimensions.

volume-preserving and hence

$$J = \det \underset{\sim}{F} = 1 \quad \text{on} \quad \Pi \ . \tag{1.3}$$

Further, let

$$I = \operatorname{tr} \underset{\sim}{F}^T \underset{\sim}{F} \quad \text{on} \quad \Pi \ . \tag{1.4}$$[1]

With the aid of (1.3), the scalar deformation invariant I is readily found to obey

$$I \geq 2 \quad \text{on} \quad \Pi \ , \tag{1.5}$$

and $I = 2$ only in an undeformed configuration.

Next, let $\underset{\sim}{\tau}$ stand for the two-dimensional Cauchy (true) stress-tensor field, regarded as a function of position on Π^*. For an equilibrium deformation, in the absence of body forces,

$$\operatorname{div} \underset{\sim}{\tau} = \underset{\sim}{0}, \quad \underset{\sim}{\tau} = \underset{\sim}{\tau}^T \quad \text{on} \quad \Pi^* \ . \tag{1.6}$$

If $\underset{\sim}{\sigma}$ designates the associated Piola (nominal) stress field on Π, one has in view of (1.3),

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = \underset{\sim}{\tau}(\hat{\underset{\sim}{y}}(\underset{\sim}{x}))\underset{\sim}{F}^{-T}(\underset{\sim}{x}) \quad \text{for all} \quad \underset{\sim}{x} \quad \text{in} \quad \Pi \ . \tag{1.7}$$[2]

In terms of the nominal stresses, the equilibrium equations (1.6) become

$$\operatorname{div} \underset{\sim}{\sigma} = \underset{\sim}{0}, \quad \underset{\sim}{\sigma}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{\sigma}^T \quad \text{on} \quad \Pi \ . \tag{1.8}$$

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[1]The superscript T indicates transposition.

[2]Here $\underset{\sim}{F}^{-T}$ denotes the inverse of $\underset{\sim}{F}^T$.

For future reference we also recall the following result. Suppose Γ is a regular arc in Π, and $\Gamma^* = \hat{\underset{\sim}{y}}(\Gamma)$ its image in Π^*. Next let $\underset{\sim}{n}$ and $\underset{\sim}{n}^*$ be the oriented unit normal vectors of Γ and Γ^*, respectively, and denote by $\underset{\sim}{s}$ and $\underset{\sim}{t}$ the Piola and Cauchy traction vectors defined by

$$\underset{\sim}{s} = \underset{\sim}{\sigma}\underset{\sim}{n} \quad \text{on } \Gamma \; , \; \underset{\sim}{t} = \underset{\sim}{\tau}\underset{\sim}{n}^* \quad \text{on } \Gamma^* \; . \tag{1.9}$$

Then,

$$\underset{\sim}{s} = \underset{\sim}{0} \quad \text{on } \Gamma \quad \text{if and only if} \quad \underset{\sim}{t} = \underset{\sim}{0} \quad \text{on } \Gamma^* \; . \tag{1.10}$$

Moreover, if the deformation and the nominal stress field are suitably regular in the <u>closure</u> of Π, (1.10) continues to hold true for an arc Γ on the boundary of Π. The foregoing result thus enables one to impose boundary conditions at a traction-free edge without involving the unknown deformation image of such a part of the boundary.

Suppose now the elastic solid under consideration possesses an elastic potential, and let W stand for the strain-energy density per unit undeformed volume, regarded as a function of position on Π. For plane deformations of a homogeneous and isotropic incompressible elastic solid, W depends on the material position vector exclusively through the deformation invariant I. Hence,

$$W(\underset{\sim}{x}) = \overset{\circ}{W}(I(\underset{\sim}{x})) \quad \text{for all } \underset{\sim}{x} \text{ in } \Pi \; , \tag{1.11}$$

where $\overset{\circ}{W}(I)$ is the plane-strain elastic potential. The appropriate constitutive law, as far as the in-plane stress response is concerned, now becomes

$$\underset{\sim}{\sigma} = 2\overset{\circ}{W}'(I)\underset{\sim}{F} - p\underset{\sim}{F}^{-T} \quad \text{on} \quad \Pi \quad , \tag{1.12}$$

in which $\overset{\circ}{W}'$ is the derivative of $\overset{\circ}{W}$, while p is an arbitrary hydrostatic pressure field arising from the constraint of incompressibility. In view of (1.12), the second of the equilibrium equations (1.8) is satisfied a priori.

We now turn to certain restrictions to which the elastic potential $\overset{\circ}{W}(I)$ is subject. First, we shall take for granted that

$$\overset{\circ}{W}(I) \geq 0 \quad \text{for} \quad I > 2, \; \overset{\circ}{W}(2) = 0 \quad , \tag{1.13}$$

so that in particular the strain-energy density vanishes in the undeformed state. Next, according to the relevant Baker-Ericksen inequality,

$$\overset{\circ}{W}'(I) > 0 \quad \text{for} \quad I > 2 \quad . \tag{1.14}$$

This inequality admits a simple physical interpretation in the context of a plane homogeneous deformation corresponding to simple shear. In this case, the components of the deformation-gradient tensor are given by

$$\left[F_{\alpha\beta}\right] = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad , \qquad (1.15)^{1}$$

in which k is the amount of shear. It follows from (1.12), (1.7) that for such a deformation the true shearing-stress component τ_{12} obeys

$$\tau_{12} \equiv \tau(k) = 2k\overset{\circ}{W}'(2+k^2) \quad , \tag{1.16}$$

[1] Greek subscripts have the range (1,2) and summation over repeated subscripts is implied.

and we henceforth refer to the graph of $\tau(k)$ as the response curve for simple shear. The inequality (1.14) thus holds provided $\tau(k)$ and k have the same sign, as demanded by obvious physical considerations.

Finally, we recall from Abeyaratne [13] that the pertinent displacement equations of equilibrium — obtained by elimination of $\underset{\sim}{\sigma}$ between (1.8), (1.12) and subject to the constraint (1.3) — are elliptic at a solution $\underset{\sim}{u}$ and at a material point $\underset{\sim}{x}$ if and only if

$$\mathring{W}'(I) \neq 0 \ , \ \frac{2\mathring{W}''(I)}{\mathring{W}'(I)}(I-2)+1>0 \ , \tag{1.17}$$

where $I = I(\underset{\sim}{x})$ is the associated value of the deformation invariant defined in (1.4). The first of these inequalities is assured by (1.14), while the second is readily seen to be equivalent to the requirement that the slope of the response curve for simple shear be non-negative at $k = \sqrt{I-2}$.

At this point we introduce a particular class of incompressible elastic materials, hereafter referred to as power-law materials, which will play an important part in the analysis to follow. This class of materials is characterized by

$$\mathring{W}(I) = AI^{n} + BI^{n-1} + o(I^{n-1}) \quad \text{as} \quad I \to \infty \ , \tag{1.18}$$

in which A, B and n are constants with A and n positive. We shall assume further that the asymptotic identities obtained by two successive formal differentiations of (1.18) are also valid. It should be emphasized that (1.18) merely stipulates the asymptotic behavior of the elastic potential at <u>large</u> values of I, and does not otherwise restrict its specific form. A special case of an incompressible material conforming to (1.18) is supplied by the Mooney-Rivlin material, with the complete plane-strain elastic potential

$$\overset{\circ}{W}(I) = \frac{\mu}{2}(I - 2) \quad . \tag{1.19}$$

This particular potential evidently satisfies (1.18) with $n = 1$, $A = \mu/2$, $B = -\mu$.

According to (1.16), the response of a power-law material in simple shear obeys

$$\tau(k) = 2nAk^{2n-1} + o(k^{2n-1}) \quad \text{as} \quad k \to \infty \quad . \tag{1.20}$$

The corresponding response curve, for a range of values of the parameter n, is indicated in Figure 1. For $n > 1$ the material hardens under simple shear, for $n < 1$ it softens, while for $n = 1$ the response function $\tau(k)$ is asymptotically linear. For this reason we shall from here on refer to n as the "hardening parameter." When $n = 1/2$, $\tau(k) \to A$ as $k \to \infty$; when $n < 1/2$, $\tau(k)$ is bound to reach a maximum beyond which it monotonically declines to zero. Such a "collapse under shear" is, as previously remarked, associated with a possible loss of ellipticity of the governing field equations. In contrast, if $n > 1/2$, the ellipticity conditions are necessarily met at all sufficiently large values of I.

2. Formulation of the global crack problem. Global considerations.

We proceed here to the global formulation of the problem with which we are concerned. Let $\mathcal{L}$ be the straight-line segment

$$\mathcal{L} = \{\underset{\sim}{x} \mid x_2 = 0, \ -b \le x_1 \le b\} \quad , \tag{2.1}$$

and suppose Π to be the region exterior to $\mathcal{L}$, as indicated in Figure 2. Accordingly, the undeformed body is an all-around infinite slab containing a plane crack of length $2b$. We shall assume that the faces of the crack are traction-free and that a known homogeneous state of plane deformation prevails at infinity.

The problem to be considered may be stated as follows. Given an elastic potential $\overset{\circ}{W}(I)$, we seek a suitably regular plane deformation $\underset{\sim}{\hat{y}}$ on Π, subject to (1.3), as well as a nominal stress field $\underset{\sim}{\sigma}$ and a pressure field p such that (1.12) holds, while $\underset{\sim}{\sigma}$ satisfies the equilibrium equations (1.8); in addition $\underset{\sim}{\hat{y}}$ is to be consistent with the pre-assigned deformation at infinity, whereas $\underset{\sim}{\sigma}$ should conform to the boundary condition for the crack faces. The latter, in view of (1.10), are satisfied provided

$$\sigma_{\alpha 2}(x_1, 0+) = 0, \ \sigma_{\alpha 2}(x_1, 0-) = 0 \quad (-b < x_1 < b) \quad . \tag{2.2}$$

Finally, we stipulate that

$$\underset{\sim}{\hat{y}}(\underset{\sim}{x}) = \underset{\sim}{\overset{\infty}{F}}\underset{\sim}{x} + O(1) \quad \text{as} \quad x_1^2 + x_2^2 \to \infty \ , \ \det \underset{\sim}{\overset{\infty}{F}} = 1 \quad , \tag{2.3}$$

where $\overset{\infty}{\underset{\sim}{F}}$ is a constant tensor.

The function $\hat{\underset{\sim}{y}}$ should be twice continuously differentiable on Π, as required before, and continuous up to $\mathcal{L}$, while $\underset{\sim}{F}$ is to be continuous up to the interior of $\mathcal{L}$. Further, we assume that $\overset{\circ}{W}$ is twice continuously differentiable on $[2,\infty)$ and throughout its interval of definition meets the ellipticity conditions (1.17). Thus, taking for granted the existence of a solution to the global problem at hand, ellipticity is bound to prevail on Π, and $\hat{\underset{\sim}{y}}$ is assured to have continuous partial derivatives of all orders on Π.

Let $\mathcal{S}$ be the class of all $\{\hat{\underset{\sim}{y}}, \underset{\sim}{\sigma}, p\}$ that satisfy the field equations referred to above, as well as the boundary conditions (2.2). Clearly, a solution to the complete problem formulated above belongs to $\mathcal{S}$; conversely any member of $\mathcal{S}$ that meets condition (2.3) is a solution to this problem. For future purposes it is essential to note that if $\{\hat{\underset{\sim}{y}}, \underset{\sim}{\sigma}, p\}$ is in $\mathcal{S}$, the same is true of $\{\underset{\sim}{Q}\hat{\underset{\sim}{y}}, \underset{\sim}{Q}\underset{\sim}{\sigma}, p\}$ for every proper orthogonal tensor $\underset{\sim}{Q}$. This claim is readily confirmed because of the objectivity of the constitutive law (1.12) and by virtue of the form of the boundary conditions (2.2).

At this stage we recall certain properties of the familiar solutions to the analogous crack problem within the linearized equilibrium theory of plane strain, which encompass both compressible and incompressible elastic solids.[1] In view of the principle of superposition it is sufficient in this connection to consider separately the following two loading modes at infinity.

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[1] See for example Rice [1].

$$\text{Mode I:}\quad \overset{\circ}{\sigma}_{1\alpha} = o(1),\ \overset{\circ}{\sigma}_{22} = \sigma + o(1) \quad \text{as} \quad x_1^2 + x_2^2 \to \infty \quad , \qquad (2.4)$$ [1]

$$\text{Mode II:}\quad \overset{\circ}{\sigma}_{11} = \overset{\circ}{\sigma}_{22} = o(1),\ \overset{\circ}{\sigma}_{12} = \sigma + o(1) \quad \text{as} \quad x_1^2 + x_2^2 \to \infty \quad . \qquad (2.5)$$

Thus Mode I corresponds to uniform uniaxial tension perpendicular to the crack faces, while Mode II corresponds to simple shear parallel to the crack faces.

For the Mode I loading the resulting displacement field is symmetric about the x_1-axis, so that

$$\overset{\circ}{u}_1(x_1,x_2) = \overset{\circ}{u}_1(x_1,-x_2), \quad \overset{\circ}{u}_2(x_1,x_2) = -u_2(x_1,-x_2) \quad . \qquad (2.6)$$

On the other hand, the displacement field associated with Mode II is anti-symmetric, whence

$$\overset{\circ}{u}_1(x_1,x_2) = -\overset{\circ}{u}_1(x_1,-x_2), \quad \overset{\circ}{u}_2(x_1,x_2) = u_2(x_1,-x_2) \quad . \qquad (2.7)$$

Further, the crack faces in Mode II undergo an in-plane gliding deformation and do not separate.

We now cite the asymptotic behavior of various field quantities near the crack-tips. To this end let (r,θ) be the local polar coordinates (see Figure 2) defined by

$$x_1 - b = r\cos\theta, \quad x_2 = r\sin\theta, \quad (0 \le r < \infty\ ,\ -\pi \le \theta \le \pi) \quad . \qquad (2.8)$$

Accordingly, the right-hand crack-tip corresponds to $r = 0$, while the crack faces are located at $\theta = -\pi$ and $\theta = \pi$, respectively.

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[1] $\overset{\circ}{u}_\alpha$ and $\overset{\circ}{\sigma}_{\alpha\beta}$ are the components of the displacement and stress fields in the linearized theory.

For the Mode I loading, as $r \to 0$:

$$\left.\begin{aligned} \overset{\circ}{u}_1 &= \frac{K_I}{\mu} r^{1/2} \cos\frac{\theta}{2}\,(1 - 2\nu + \sin^2\frac{\theta}{2}) + o(r^{1/2}) \quad , \\ \overset{\circ}{u}_2 &= \frac{K_I}{\mu} r^{1/2} \sin\frac{\theta}{2}\left[2(1-\nu) - \cos^2\frac{\theta}{2}\right] + o(r^{1/2}) \quad ; \end{aligned}\right\} \quad (2.7)$$

$$\left.\begin{aligned} \overset{\circ}{\sigma}_{11} &= K_I\, r^{-1/2} \cos\frac{\theta}{2}\,(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}) + o(r^{-1/2}) \quad , \\ \overset{\circ}{\sigma}_{12} &= K_I\, r^{-1/2} \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\sin\frac{\theta}{2} + o(r^{-1/2}) \quad , \\ \overset{\circ}{\sigma}_{22} &= K_I\, r^{-1/2} \cos\frac{\theta}{2}\,(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}) + o(r^{-1/2}) \quad . \end{aligned}\right\} \quad (2.10)$$

Here μ is the shear modulus for infinitesimal deformations, ν is Poisson's ratio, while K_I is the Mode I stress-intensity factor, given by

$$K_I = \sigma(b/2)^{1/2} \quad . \qquad (2.11)$$

For the Mode II loading, as $r \to 0$:

$$\left.\begin{aligned} \overset{\circ}{u}_1 &= \frac{K_{II}}{\mu} r^{1/2} \sin\frac{\theta}{2}\left[2(1-\nu) + \cos^2\frac{\theta}{2}\right] + o(r^{1/2}) \quad , \\ \overset{\circ}{u}_2 &= \frac{K_{II}}{\mu} r^{1/2} \cos\frac{\theta}{2}\left[-(1-2\nu) + \sin^2\frac{\theta}{2}\right] + o(r^{1/2}) \quad ; \end{aligned}\right\} \quad (2.12)$$

$$\left.\begin{aligned}\overset{\circ}{\sigma}_{11} &= K_{II}\, r^{-1/2} \sin\frac{\theta}{2}\left(-2-\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) + o(r^{-1/2}) \quad , \\ \overset{\circ}{\sigma}_{12} &= K_{II}\, r^{-1/2} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + o(r^{-1/2}) \quad , \\ \overset{\circ}{\sigma}_{22} &= K_{II}\, r^{-1/2} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + o(r^{-1/2}) \quad ,\end{aligned}\right\} \tag{2.13}$$

with

$$K_{II} = \sigma(b/2)^{1/2} \quad . \tag{2.14}$$

The displacements appropriate to the special case of an <u>incompressible</u> linearly-elastic body are obtained by setting $\nu = 1/2$ in the formulas cited above.

Although superposition no longer holds in the nonlinear theory, it is still useful to consider separately the kinematic counterparts of the Mode I and Mode II loadings. Corresponding to the <u>symmetric</u> loading case we now require the deformation at infinity to be one of uniaxial stretching, characterized by

$$\left[\overset{\infty}{F}_{\alpha\beta}\right] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} , \quad \lambda > 0 \quad , \tag{2.15}$$

in which λ and λ^{-1} are the principal stretch ratios at infinity. The analogue of the <u>antisymmetric</u> loading case is a state of simple shear governed by

$$\left[\overset{\infty}{F}_{\alpha\beta}\right] = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} , \tag{2.16}$$

and k is the amount of shear at infinity.

Since the governing field equations in the finite theory do not remain invariant if $u_1(x_1,x_2)$, $u_2(x_1,x_2)$ are replaced by $-u_1(x_1,-x_2)$, $u_2(x_1,-x_2)$, one is led to wonder whether or not the nonlinear Mode II problem admits a solution that is antisymmetric about the plane of the crack. We shall now show that, at least for the Mooney-Rivlin material, a solution with this parity cannot possibly exist. To this end suppose now that $\overset{\circ}{W}$ is given by (1.19). Eliminating $\underset{\sim}{\sigma}$ between (1.12) and (1.8), and invoking (1.2), (1.3), (1.4), (1.19), we find that

$$\frac{1}{\mu}\, p_{,\alpha} = \hat{y}_{\lambda,\alpha}\hat{y}_{\lambda,\beta\beta} \quad \text{on} \quad \Pi \ . \tag{2.17}$$[1]

Equation (1.3), in view of (1.1), becomes

$$1 + u_{1,1} + u_{2,2} + u_{1,1}u_{2,2} - u_{1,2}u_{2,1} = 1 \quad \text{on} \quad \Pi \ . \tag{2.18}$$

Now assume $\underset{\sim}{u}$ has the parity of $\overset{\circ}{\underset{\sim}{u}}$ in (2.7). Then (2.18) implies

$$1 - u_{1,1} - u_{2,2} + u_{1,1}u_{2,2} - u_{1,2}u_{2,1} = 1 \quad \text{on} \quad \Pi \ . \tag{2.19}$$

Therefore

$$u_{1,1} + u_{2,2} = 0 \quad \text{on} \quad \Pi \ , \tag{2.20}$$

$$u_{1,1}u_{2,2} - u_{1,2}u_{2,1} = 0 \quad \text{on} \quad \Pi \ . \tag{2.21}$$

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[1] Here and in the sequel subscripts preceded by a comma indicate partial differentiation with respect to the appropriate cartesian coordinate.

Next, $p = p' + p''$ on Π, provided

$$\left.\begin{aligned} p' &= \tfrac{1}{2}[p(x_1,x_2) - p(x_1,-x_2)] \quad , \\ p'' &= \tfrac{1}{2}[p(x_1,x_2) + p(x_1,-x_2)] \quad . \end{aligned}\right\} \tag{2.22}$$

By virtue of the assumed parity of $\underset{\sim}{u}$, (2.17) and (1.1) give

$$\frac{1}{\mu} p'_{,\alpha} = u_{\alpha,\beta\beta} \quad \text{on} \quad \Pi \quad . \tag{2.23}$$

From (2.23), (2.20) it follows that p' is harmonic and u_α is biharmonic on Π:

$$\nabla^2 p' = 0 \,, \quad \nabla^4 u_\alpha = 0 \quad \text{on} \quad \Pi \quad . \tag{2.24}$$

For the Mode II problem one draws from (2.16), (2.3), (1.1) that

$$u_1 = kx_2 + O(1) \,, \quad u_2 = O(1) \quad \text{as} \quad \rho = \sqrt{x_1^2 + x_2^2} \to \infty \quad . \tag{2.25}$$

We shall now prove that (2.20), (2.21), (2.24), (2.25) imply

$$u_1 = kx_2 \,, \quad u_2 = c \quad \text{on} \quad \Pi, \qquad c = \text{constant} \quad , \tag{2.26}$$

so that the entire deformation field is one of homogeneous simple shear, which contradicts the boundary condition (2.2) for traction-free faces of the crack.

With a view to establishing this claim, we observe first that $u_1 - kx_2$ and u_2 are both biharmonic on Π, and hence in the neighborhood of infinity defined by $\mathcal{N} = \{\underset{\sim}{x} \,|\, \rho > b\}$, 2b being the crack length;

further both of these functions are bounded as $\rho \to \infty$. As shown in the Appendix, u_α then admits an expansion of the form

$$u_\alpha(x_1,x_2) = k\delta_{\alpha 1}x_2 + \sum_{j=0}^{\infty}[a_j^{(\alpha)}\cos j\varphi + b_j^{(\alpha)}\sin j\varphi + c_j^{(\alpha)}\cos(j-2)\varphi$$

$$+ d_j^{(\alpha)}\sin(j-2)\varphi]\rho^{-j} \quad \text{on} \quad \mathcal{N}, \qquad (2.27)$$

where φ is the polar angle defined by $x_1 = \rho\cos\varphi$, $x_2 = \rho\sin\varphi$. The series in (2.27) is absolutely and uniformly convergent on every closed subset of $\mathcal{N}$. Substitution from (2.27) into (2.20), (2.21), in view of the parity of u_α, confirms that these equations cannot hold unless all coefficients apart from $a_0^{(2)}$ in the above series vanish. Continuation of the biharmonic functions u_α from $\mathcal{N}$ onto Π now yields the desired conclusion (2.26).

In view of the above result it is a plausible conjecture that the Mode II global crack problem under consideration fails to admit an anti-symmetric solution for <u>every</u> legitimate choice of the plane-strain elastic potential $\mathring{W}$.

3. Asymptotic analysis of the elastostatic field near the tips of the crack.

In this section we aim at the local structure near one of the crack-tips of the solution to the global problem stated in Section 2. For this purpose it is sufficient to consider the right-hand crack-tip, situated at $x_1 = b$, $x_2 = 0$. We suppose that the material has an elastic potential of the power-law type (1.18). Further, if (r,θ) are the local polar coordinates introduced earlier (see Figure 2), we assume that the deformation field of the solution admits an asymptotic representation of the form

$$\hat{y}_\alpha(x_1,x_2) \sim r^{m_\alpha} U_\alpha(\theta) \quad \text{as} \quad r \to 0 \text{ (no sum on } \alpha) \quad . \qquad (3.1)^1$$

Here m_1 and m_2 are originally unknown constants obeying

$$0 < m = \min\{m_1, m_2\} < 1 \quad , \qquad (3.2)$$

whereas U_α are as yet undetermined functions that fail to vanish identically on $[-\pi,\pi]$. In addition we take for granted the validity of the asymptotic equalities obtained by two successive formal differentiations of (3.1). Note that (3.2) implies that the deformation is continuous at the crack-tip, and that at least one of the deformation gradient components $F_{\alpha\beta}$ becomes unbounded there. Observe also that according to (3.1), (3.2) the deformation image of the crack-tip has for convenience been placed

[1] The asymptotic equality symbol "$\sim$" is used in its standard connotation. Thus, $f(r) \sim g(r)$ abbreviates $f(r) = g(r) + o(g(r))$ as $r \to 0$.

at the origin of the cartesian coordinates, which can always be arranged by means of a suitable translation. Finally, we suppose that the pressure field associated with the global solution satisfies

$$p(x_1,x_2) \sim r^{\ell} P(\theta) \quad \text{as} \quad r \to 0 \quad , \tag{3.3}$$

ℓ being another constant exponent, and assume that this asymptotic identity may be formally differentiated at least once.

As pointed out in Section 2, the unknown global solution $\{\hat{\underset{\sim}{y}},\underset{\sim}{\sigma},p\}$ necessarily belongs to the class $\mathcal{S}$ introduced there, regardless of the particular prescription (2.3) at infinity. It follows that the determination of $\hat{y}_\alpha$ and p in accordance with (3.1) and (3.3) for an <u>arbitrary</u> member of $\mathcal{S}$ is bound to encompass the asymptotic behavior of the solution to the complete crack problem at hand.

We now show that for the present purpose (3.1) may — without loss of generality — be replaced by

$$\hat{y}_\alpha(x_1,x_2) \sim r^m U_\alpha(\theta) \quad \text{as} \quad r \to 0 \quad . \tag{3.4}$$

With this aim in mind we first note on the basis of the discussion in Section 2 that whenever $\{\hat{\underset{\sim}{y}},\underset{\sim}{\sigma},p\}$ is in $\mathcal{S}$, the same is true of $\{\hat{\underset{\sim}{y}}^*,\underset{\sim}{\sigma}^*,p^*\}$, where

$$\hat{\underset{\sim}{y}}^* = \underset{\sim}{Q}\,\hat{\underset{\sim}{y}}, \quad \underset{\sim}{\sigma}^* = \underset{\sim}{Q}\,\underset{\sim}{\sigma}, \quad p^* = p \quad , \tag{3.5}$$

and $\underset{\sim}{Q}$ is an <u>arbitrary</u> proper orthogonal tensor. The components of every such tensor in the underlying coordinate frame admit the representation

$$\left[Q_{\alpha\beta}\right] = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} , \tag{3.6}$$

for some φ in the interval $[0,2\pi)$. From (3.6), (3.5), (3.1) one finds

$$\left.\begin{aligned} \hat{y}_1^* &= r^{m_1}U_1(\theta)\cos\varphi - r^{m_2}U_2(\theta)\sin\varphi + o\left(r^{m_1}\right) + o\left(r^{m_2}\right) , \\ \hat{y}_2^* &= r^{m_1}U_1(\theta)\sin\varphi + r^{m_2}U_2(\theta)\cos\varphi + o\left(r^{m_1}\right) + o\left(r^{m_2}\right) , \end{aligned}\right\} \tag{3.7}$$

whence there are functions U_α^* such that

$$\hat{y}_\alpha^*(x_1,x_2) = r^m U_\alpha^*(\theta) + o(r^m) \quad . \tag{3.8}$$

Further, since U_α is not permitted to vanish identically, it is always possible to choose φ so as to assure that U_α^* fails to vanish identically. Now, dropping the asterisk in (3.8) we arrive at (3.4).

Note that the transformation (3.5) represents a rigid rotation of the deformed body about the origin, determined by the rotation tensor $\underset{\sim}{Q}$; further, the same rotation is applied to the nominal stress field $\underset{\sim}{\sigma}$ associated with the original deformation. Consequently the asymptotic analysis to follow can determine the local behavior of the global solution at most to within an unknown rigid rotation, which would ultimately depend on the particular prescription at infinity. In this connection it should be emphasized, however, that most of the desired local field quantities are in fact invariant under such a rotation. For the special case of the Mode I loading the local solution is bound to be symmetric about the x_1-axis and this additional restriction supplies essential information relevant to the determination of the rotation tensor $\underset{\sim}{Q}$. As is clear from the discussion at the end of Section 2, analogous parity restrictions are not available for the Mode II loading.

We now seek to determine the smallest exponent m in (0,1) and the functions U_α appearing in (3.4) consistent with the governing field equations and boundary conditions. This objective is most easily reached by recourse to the polar coordinates (r,θ). Writing $y_\alpha(r,\theta)$ in place of $\hat{y}_\alpha(x_1(r,\theta),\ x_2(r,\theta))$ one finds that the incompressibility condition (1.3) becomes

$$J = \frac{1}{r}\left(\frac{\partial y_1}{\partial r}\frac{\partial y_2}{\partial \theta} - \frac{\partial y_2}{\partial r}\frac{\partial y_1}{\partial \theta}\right) = 1 \quad . \tag{3.9}$$

Further (1.3), (1.8), (1.12) lead to

$$\left.\begin{aligned} \frac{\partial p}{\partial r} &= 2\mathring{W}'(I)H_r + 2\mathring{W}''(I)\left[g_{rr}\frac{\partial I}{\partial r} + \frac{1}{r^2}\,g_{r\theta}\frac{\partial I}{\partial \theta}\right] \ , \\ \frac{\partial p}{\partial \theta} &= 2\mathring{W}'(I)H_\theta + 2\mathring{W}''(I)\left[g_{r\theta}\frac{\partial I}{\partial r} + \frac{1}{r^2}\,g_{\theta\theta}\frac{\partial I}{\partial \theta}\right] \ , \end{aligned}\right\} \tag{3.10}$$

and (1.4) furnishes

$$I = g_{rr} + \frac{1}{r^2}\,g_{\theta\theta} \quad , \tag{3.11}$$

with

$$\left.\begin{aligned} H_r &= \frac{\partial y_\alpha}{\partial r}\nabla^2 y_\alpha \ , \quad H_\theta = \frac{\partial y_\alpha}{\partial \theta}\nabla^2 y_\alpha \ , \\ g_{rr} &= \frac{\partial y_\alpha}{\partial r}\frac{\partial y_\alpha}{\partial r} \ , \quad g_{r\theta} = \frac{\partial y_\alpha}{\partial r}\frac{\partial y_\alpha}{\partial \theta} \ , \quad g_{\theta\theta} = \frac{\partial y_\alpha}{\partial \theta}\frac{\partial y_\alpha}{\partial \theta} \quad . \end{aligned}\right\} \tag{3.12}$$

The boundary conditions (2.2) take the form

$$\left.\begin{aligned} 2\mathring{W}'(I)\frac{\partial y_1}{\partial \theta} + rp\frac{\partial y_2}{\partial r} &= 0 \quad , \\ 2\mathring{W}'(I)\frac{\partial y_2}{\partial \theta} - rp\frac{\partial y_1}{\partial r} &= 0 \quad \text{on } \theta = \pm\pi \quad . \end{aligned}\right\} \tag{3.13}$$

With the aid of (3.12) and (3.9), equations (3.13) are readily found to be equivalent to

$$p = 2\mathring{W}'(I)/g_{rr} = 2\mathring{W}'(I)g_{\theta\theta}/r^2 \ , \ g_{r\theta} = 0 \quad \text{for} \quad \theta = \pm\pi \quad . \tag{3.14}$$

We thus need to determine m and U_α in (3.4) consistent with (3.9), (3.10) and (3.14).

Equation (3.11), together with (3.4), gives

$$I \sim r^{2(m-1)}\{\dot{U}_1^2(\theta) + \dot{U}_2^2(\theta) + m^2[U_1^2(\theta) + U_2^2(\theta)]\} \quad , \tag{3.15}$$

where the dot denotes differentiation with respect to θ. We shall henceforth take for granted that U_1 and U_2 do not have a common multiple zero on $[-\pi,\pi]$ so that the coefficient of $r^{2(m-1)}$ in (3.15) does not vanish on $[-\pi,\pi]$.

Next (3.4), (3.9) lead to

$$J = mr^{2(m-1)}[U_1(\theta)\dot{U}_2(\theta) - U_2(\theta)\dot{U}_1(\theta)] + o(r^{2(m-1)}) = 1 \quad . \tag{3.16}$$

Dividing this identity by $r^{2(m-1)}$, proceeding to the limit as $r \to 0$, and bearing in mind that $m < 1$, we obtain

$$U_1\dot{U}_2 - U_2\dot{U}_1 = 0 \quad \text{on} \quad [-\pi,\pi] \quad . \tag{3.17}$$

Thus,

$$U_\alpha = a_\alpha U \quad \text{on} \quad [-\pi,\pi] \ , \ a_\alpha \neq 0 \ , \tag{3.18}$$

in which a_1 and a_2 are constants and U an unknown function.

Equations (3.18), (3.15), (3.12), (3.10), (1.18) eventually lead to

$$\left.\begin{aligned} \frac{\partial p}{\partial r} &\sim 2Aa^{2n} r^{2(m-1)n-1} mU(\theta)Z(\theta) \ , \\ \frac{\partial p}{\partial \theta} &\sim 2Aa^{2n} r^{2(m-1)n} \dot{U}(\theta)Z(\theta) \ , \end{aligned}\right\} \tag{3.19}$$

provided

$$\left.\begin{aligned} &Z = G^{n-2}\{G(\ddot{U} + m^2 U) + (n-1)[\dot{G}\dot{U} + 2m(m-1)GU]\} \quad \text{on} \quad [-\pi,\pi] \ , \\ &G = \dot{U}^2 + m^2 U^2 \quad \text{on} \quad [-\pi,\pi], \quad a = \sqrt{a_1^2 + a_2^2} \ . \end{aligned}\right\} \tag{3.20}$$

On the other hand, the boundary conditions (3.14) yield

$$U\dot{U} = 0 \quad \text{at} \quad \theta = \pm\pi \ , \tag{3.21}$$

$$p \sim 2Ana^{2n} r^{2(m-1)n} G^{n-1} \dot{U}^2 \quad \text{at} \quad \theta = \pm\pi \ . \tag{3.22}$$

We now show on the basis of (3.22), (3.21), (3.19) that

$$Z = 0 \quad \text{on} \quad [-\pi,\pi], \ \dot{U}(-\pi) = \dot{U}(\pi) = 0 \ . \tag{3.23}$$

For this purpose we integrate the first of (3.19) to obtain

$$p \sim 2Aa^{2n} r^{2(m-1)n} \frac{mU(\theta)Z(\theta)}{2(m-1)n} \ . \tag{3.24}$$

According to (3.21) either $U(\pi)$ or $\dot{U}(\pi)$ must vanish. By assumption U has no multiple zeros on $[-\pi,\pi]$ so that $U(\pi)$ and $\dot{U}(\pi)$ cannot vanish together. Suppose $U(\pi)=0$, $\dot{U}(\pi)\neq 0$. Then (3.24) gives

$$p(r,\pi) = o\left(r^{2(m-1)}\right) \quad \text{as} \quad r\to 0 \quad , \tag{3.25}$$

contradicting (3.22). Thus $U(\pi)\neq 0$, $\dot{U}(\pi)=0$ and similarly $U(-\pi)\neq 0$, $\dot{U}(-\pi)=0$. On comparing (3.24) and (3.22) we infer that

$$Z(\pi) = Z(-\pi) = 0 \quad . \tag{3.26}$$

Next, we differentiate (3.24) with respect to θ and use the second of (3.19) to confirm

$$U\dot{Z} + \left[1+\frac{2(1-m)n}{m}\right]Z\dot{U} = 0 \quad \text{on} \quad [-\pi,\pi] \quad . \tag{3.27}$$

Equations (3.27) and (3.26) necessitate

$$Z = 0 \quad \text{on} \quad [-\pi,\pi] \quad , \tag{3.28}$$

which proves (3.23). Thus

$$G(\ddot{U}+m^2U) + (n-1)[\ddot{G}U + 2m(m-1)GU] = 0, \ \dot{U}(-\pi) = \dot{U}(\pi) = 0 \ . \tag{3.29}$$

Equations (3.29) constitute a nonlinear eigenvalue problem that arose also in connection with asymptotic analyses of other crack problems in finite elasticity [3], [4], [5] and was solved in [3]. The unique eigenvalue in the range $(0,1)$ is given by

$$m = 1 - 1/2n, \quad 1/2 < n < \infty \quad , \tag{3.30}$$

while the associated eigenfunction U is supplied by

$$U(\theta) = \sin(\theta/2)\left[1 - \frac{2\kappa^2\cos^2(\theta/2)}{1+\omega(\theta,n)}\right]^{1/2} [\omega(\theta,n) + \kappa\cos\theta]^{\kappa/2} \quad , \qquad (3.31)$$

where

$$\omega(\theta,n) = [1 - \kappa^2\sin^2\theta]^{1/2} \quad , \quad \kappa = (n-1)/n \quad . \qquad (3.32)$$

The representation of the local deformation field furnished by (3.4), (3.18), (3.31) yields only the weak estimates

$$J = o\left(r^{2(m-1)}\right), \; p = o\left(r^{-1}\right) \quad \text{as} \quad r \to 0 \quad , \qquad (3.33)$$

and is therefore inadequate. Furthermore, the Jacobian determinant of the dominant terms in (3.4) is found to vanish, which reflects the degenerate character of the asymptotic approximation established so far. With a view towards refining this approximation we now replace (3.4) by

$$y_\alpha(r,\theta) = a_\alpha r^m U(\theta) + r^{m'} V_\alpha(\theta) + o\left(r^{m'}\right) \; , \quad m' > m \quad , \qquad (3.34)$$

with m' another unknown exponent and $V_\alpha(\theta)$ unknown functions that are not permitted to vanish identically, while m and U are now given by (3.30), (3.31). We suppose that (3.34) may be formally differentiated at least twice. Again, by virtue of the discussion that led to the adoption of (3.4) in place of (3.1), no generality is lost in assuming equal exponents in the second term of (3.34).

From (3.34), (3.9) one now draws

$$J = r^{m+m'-2}(mU\dot{\Psi} - m'\dot{U}\Psi) + o\left(r^{m+m'-2}\right) = 1 \quad , \tag{3.35}$$

where

$$\Psi = a_1 V_2 - a_2 V_1 \quad \text{on} \quad [-\pi,\pi] \quad . \tag{3.36}$$

Consequently

$$m + m' - 2 \le 0 \quad , \tag{3.37}$$

and further

$$mU\dot{\Psi} - m'\dot{U}\Psi = 0 \quad \text{on} \quad [-\pi,\pi] \quad \text{if} \quad m < m' < 2 - m \quad , \tag{3.38}$$

$$mU\dot{\Psi} - m'\dot{U}\Psi = 1 \quad \text{on} \quad [-\pi,\pi] \quad \text{if} \quad m' = 2 - m \quad . \tag{3.39}$$

Next, combining (3.34), (3.10), (3.11), bearing in mind (1.18), and recalling that U satisfies (3.29), one confirms with the aid of arguments similar to those used in deriving (3.28) that

$$Y = 0 \quad \text{on} \quad [-\pi,\pi] \quad \text{if} \quad m < m' < 2 - m \quad , \tag{3.40}$$

$$Y = -\frac{B}{nA}(\ddot{U} + m^2 U) \quad \text{on} \quad [-\pi,\pi] \quad \text{if} \quad m' = 2 - m \quad , \tag{3.41}$$

where Y is an auxiliary function to be defined presently. In fact,

$$Y = G(\ddot{\chi} + m'^2\chi) + 2K(\ddot{U} + m^2 U) + (n-1)\dot{\chi}\dot{G}$$
$$+ 2(n-1)[m'(m-1)\chi G + m(m'+m-2)UK + \dot{U}\dot{K}] \quad , \tag{3.42}$$

in which

$$\chi = a_1 V_1 + a_2 V_2 , \quad K = \dot{\chi}\dot{U} + mm'\chi U \quad \text{on} \quad [-\pi,\pi] \quad . \tag{3.43}$$

Finally, the boundary conditions (3.14) lead to

$$\dot{\chi}(\pm\pi) = 0 \quad \text{if} \quad m < m' \leq 2 - m \quad , \tag{3.44}$$

$$\dot{\Psi}(\pm\pi) = 0 \quad \text{if} \quad m < m' < 2 - m \quad , \tag{3.45}$$

$$\dot{\Psi}(\pi) = 1/mU(\pi), \; \dot{\Psi}(-\pi) = 1/mU(-\pi) \quad \text{if} \quad m' = 2 - m \quad . \tag{3.46}$$

In view of (3.42), equation (3.40) together with (3.44) constitute an eigenvalue problem for χ with m' as eigenvalue parameter. On the other hand, (3.41) is an inhomogeneous differential equation for χ, to be solved subject to the boundary conditions (3.44). Consider next the characterization of Ψ through the differential equations (3.38), (3.39) together with (3.45), (3.46). Since these boundary conditions are readily seen to be implied by (3.38), (3.39), additional information is needed to determine the appropriate solutions. Such information is supplied by the assumption that the solution Ψ to (3.38) or (3.39) possesses continuous derivatives of <u>all</u> orders on $[-\pi,\pi]$. Equation (3.38) now poses an unconventional eigenvalue problem: one is to determine the values of m' on the interval $(m,2-m)$ for which the solution to (3.38) is infinitely many times continuously differentiable on $[-\pi,\pi]$.

We turn at present to the determination of the relevant eigenvalues of (3.40), (3.44) or (3.38). If at least one of the foregoing problems has eigenvalues in the appropriate range, we choose for m' the smallest such eigenvalue. The following three possibilities now arise. In case m' so determined is an eigenvalue for (3.40), (3.44) but not for (3.38), χ must be the eigenfunction associated with m', while $\Psi = 0$ on $[-\pi,\pi]$.

Next, if m' is an eigenvalue for (3.38) but not for (3.40), (3.44), Ψ has to be the eigenfunction associated with m', while $\chi = 0$ on $[-\pi,\pi]$. Further, if m' is an eigenvalue common to both problems, Ψ and χ are the eigenfunctions associated with this value of m'.

Finally, suppose neither of the eigenvalue problems considered above has an eigenvalue in $(m,2-m)$. Then evidently $m' = 2 - m$ and the inhomogeneous equations (3.39) and (3.41), (3.44) are to be solved for χ and Ψ, which in turn determine V_1 and V_2.

At this stage we investigate the solutions to (3.38) and (3.39). Consider first (3.38). In view of (3.31), we find

$$\Psi = a_3 U^{m'/m} \quad \text{on} \quad (0,\pi], \quad \Psi = a_4 (-U)^{m'/m} \quad \text{on} \quad [-\pi,0) \quad , \tag{3.47}$$

in which a_3 and a_4 are constants. If Ψ is to possess unlimited smoothness on $[-\pi,\pi]$, m'/m must be a positive integer. Thus the smallest eigenvalue $m' > m$ is given by

$$m' = 2m = 2 - 1/n \quad , \tag{3.48}$$

and the associated eigenfunction by

$$\Psi = b_0 U^2 \quad , \tag{3.49}$$

where $b_0 \neq 0$ is a constant. In addition m' defined by (3.48) conforms to (3.37), provided

$$n < 3/2 \quad . \tag{3.50}$$

In the event that $n \geq 3/2$, (3.38) does not possess an eigenvalue in (m,2-m). With a view to the determination of Ψ when $n \geq 3/2$ we consider (3.39) merely on the interval $[0,\pi]$, which will presently be seen to be sufficient. Evidently

$$\Psi = \Psi' + \Psi'' \quad \text{on} \quad [-\pi,\pi] \quad , \tag{3.51}$$

if one sets

$$\Psi'(\theta) = \frac{1}{2}[\Psi(\theta) - \Psi(-\theta)], \ \Psi''(\theta) = \frac{1}{2}[\Psi(\theta) + \Psi(-\theta)], \quad -\pi \leq \theta \leq \pi \quad . \tag{3.52}$$

Moreover, Ψ' and Ψ'' are found to satisfy

$$mU\dot{\Psi}' - m'\dot{U}\Psi' = 0 \quad \text{on} \quad [0,\pi] \quad , \tag{3.53}$$

$$mU\dot{\Psi}'' - m'\dot{U}\Psi'' = 1 \quad \text{on} \quad [0,\pi] \quad . \tag{3.54}$$

If $n \geq 3/2$, the only solution to (3.53) having an infinitely smooth odd extension to $[-\pi,\pi]$ is

$$\Psi' = 0 \quad \text{on} \quad [0,\pi] \quad . \tag{3.55}$$

Further, (3.54) gives

$$\Psi''(\theta) = U^{m'/m} \left\{ b_1 - \frac{1}{m} \int_\theta^\pi \frac{d\varphi}{[U(\varphi)]^{2/m}} \right\}, \ m' = 2 - m, \quad (0 \leq \theta \leq \pi), \tag{3.56}$$

with b_1 an arbitrary constant. We now change the variable of integration in (3.56) from φ to ξ by means of the one-to-one mapping

$$\cos\xi = \frac{1}{n\sqrt{2}} \frac{[1 + \kappa\sin^2\varphi - \omega(\varphi,n)\cos\varphi]^{1/2}}{\omega(\varphi,n) + \kappa\cos\varphi} \qquad (0 \leq \xi \leq \frac{\pi}{2}) \quad , \tag{3.57}$$

ω and κ being given by (3.32). This change of variable carries (3.56) into

$$\Psi''(\theta) = U^{m'/m} \left\{ b_1 - \frac{n^{1-1/m}}{2m^3} \int_0^{\xi_0} \frac{1 + 2(n-1)\cos^2\xi}{[\cos\xi]^{2/m}} \, d\xi \right\}, \tag{3.58}$$

where

$$\cos\xi_0 = \frac{1}{n\sqrt{2}} \frac{[1 + \kappa\sin^2\theta - \omega(\theta,n)\cos\theta]^{1/2}}{\omega(\theta,n) + \kappa\cos\theta} \qquad (0 \le \theta \le \pi) \quad . \tag{3.59}$$

The integral in (3.58) may be evaluated in terms of hypergeometric functions, provided $n > 3/2$. Following an analysis, the details of which will be omitted here, one is led to the representation

$$\Psi'' = b_2 U^{m'/m} + H \quad \text{on} \quad [0,\pi], \quad m' = 2 - m \quad . \tag{3.60}$$

Here

$$b_2 = b_1 - \frac{\sqrt{\pi}\, n^{3-1/m}}{2m^2} \frac{\Gamma(1/2 - 1/m)}{\Gamma(1 - 1/m)} \quad , \tag{3.61}$$

Γ denoting the Gamma function, while

$$H(\theta) = -\frac{n^{5/2}}{m^2} [\omega(\theta,n) + \kappa\cos\theta]^{m'}$$

$$\times \left[\frac{m}{m'} F(1/2 - 1/m, 1/2; 3/2 - 1/m; \cos^2\xi_0) - \kappa\sin\xi_0\right] \quad . \tag{3.62}$$

in which F stands for the hypergeometric function.[1] Now if Ψ'' is to possess continuous derivatives of all orders on $[0,\pi]$ then b_2 must vanish.[2] From (3.51), (3.55), (3.60) one infers

$$\Psi(\theta) = H(\theta) \qquad (-\pi \leq \theta \leq \pi) \quad . \tag{3.63}$$

One draws from (3.62), (3.63) and familiar properties of the hypergeometric function that

$$\left.\begin{aligned} \Psi(0) &= -2n^{1/2}(2m)^{-m}/m' \quad , \\ \Psi(\pm\pi) &= -\frac{\sqrt{n\pi}\, n^m}{mm'}\,\frac{\Gamma(3/2 - 1/m)}{\Gamma(1 - 1/m)} \quad , \end{aligned}\right\} \tag{3.64}$$

the first of which is also a direct consequence of the differential equation (3.39). The second of (3.64) will be used later in discussing the shape of the deformed crack.

In case $n = 3/2$, (3.39) fails to admit a solution of the requisite smoothness. We postpone consideration of this special case, which requires separate attention, until Section 4 and turn to the determination of χ. Consider first (3.40) and set

$$\chi = \chi' + \chi'' \quad \text{on} \quad [-\pi,\pi] \quad , \tag{3.65}$$

where

$$\chi'(\theta) = \frac{1}{2}[\chi(\theta) - \chi(-\theta)], \;\; \chi''(\theta) = \frac{1}{2}[\chi(\theta) + \chi(-\theta)] \quad (-\pi \leq \theta \leq \pi) \quad . \tag{3.66}$$

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[1] See for example [15], p.556.

[2] Note that $H(\theta)$ is an even function of θ with continuous derivatives of all orders on $[-\pi,\pi]$ and recall that $U(\theta)$ has a simple zero at $\theta = 0$.

From (3.65), (3.66), (3.40), (3.42) we have, bearing in mind the parity of U,

$$G(\ddot{\chi}'+m'^2\chi') + 2K'(\ddot{U}+m^2U) + (n-1)\dot{\chi}'\dot{G} + 2(n-1)[m'(m-1)\chi' G$$

$$+ m(m'+m-2)UK' + \dot{U}\dot{K}'] = 0 \quad \text{on} \quad [0,\pi] \quad , \qquad (3.67)$$

$$G(\ddot{\chi}''+m'^2\chi'') + 2K''(\ddot{U}+m^2U) + (n-1)\dot{\chi}''\dot{G} + 2(n-1)[m'(m-1)\chi'' G$$

$$+ m(m'+m-2)UK'' + \dot{U}\dot{K}''] = 0 \quad \text{on} \quad [0,\pi] \quad , \qquad (3.68)$$

where

$$K' = \dot{\chi}'U + mm'\chi'U, \qquad K'' = \dot{\chi}''\dot{U} + mm'\chi''U \quad . \qquad (3.69)$$

Next, (3.66), (3.44) furnish

$$\dot{\chi}'(\pi) = 0 \quad , \quad \chi'(0) = 0 \quad , \qquad (3.70)$$

$$\dot{\chi}''(\pi) = 0 \quad , \quad \dot{\chi}''(0) = 0 \quad . \qquad (3.71)$$

The differential equation (3.67) and the boundary conditions (3.70) are also encountered in [4]. As indicated there, the transformations

$$\cos\xi = \frac{1}{n\sqrt{2}} \frac{[1+\kappa\sin^2\theta - \omega(\theta,n)\cos\theta]^{1/2}}{\omega(\theta,n)+\kappa\cos\theta} \qquad (0 \le \theta \le \pi) \quad , \qquad (3.72)$$

$$W_1(\xi) = [\omega(\theta,n)+\kappa\cos\theta]^{-m'}\chi'(\theta) \qquad (0 \le \theta \le \pi) \quad , \qquad (3.73)$$

reduce (3.67), (3.70) to

$$\ddot{W}_1(\xi) + \lambda^2 W_1(\xi) = 0 \quad (0 \le \xi \le \pi/2), \; \dot{W}_1(0) = W_1(\pi/2) = 0 \quad , \qquad (3.74)$$

where

$$\lambda^2 = 4nm'(nm' - n + 1)/(2n - 1) \quad . \tag{3.75}$$

The transformed eigenvalue problem (3.74) has the solutions

$$W_1(\xi) = c_1 \cos \lambda \xi, \ \lambda = j \quad (j = 1,3,5,\ldots) \tag{3.76}$$

in which c_1 is a constant. Under the transformation

$$W_2(\xi) = [\omega(\theta,n) + \kappa\cos\theta]^{-m'} \chi''(\theta) \qquad (0 \le \theta \le \pi) \quad , \tag{3.77}$$

together with (3.72), equations (3.68), (3.71) become

$$\ddot{W}_2(\xi) + \lambda^2 W_2(\xi) = 0 \quad (0 \le \xi \le \pi/2), \ \dot{W}_2(0) = \dot{W}_2(\pi/2) = 0 \quad , \tag{3.78}$$

and the solution to (3.78) is given by

$$W_2(\xi) = c_2 \cos \lambda \xi, \ \lambda = j \quad (j = 2,4,6,\ldots) \ . \tag{3.79}$$

Thus the eigenvalues of the original problem (3.40), (3.44) are the solutions for m' of

$$4nm'(nm' - n + 1)/(2n - 1) = j^2 \quad (j = 1,2,3,\ldots) \quad . \tag{3.80}$$

For j an <u>odd</u> positive integer, $\chi'' = 0$ on $[-\pi,\pi]$ and $\chi'(\theta)$ is furnished by (3.76), (3.73), (3.72), while if j is <u>even</u>, $\chi'(\theta) = 0$ on $[-\pi,\pi]$ and $\chi''(\theta)$ is furnished by (3.79), (3.77), (3.72).

On discarding negative roots m' of (3.80) one arrives at

$$m' = \frac{\kappa}{2} + \sqrt{\frac{\kappa}{4}^2 + \frac{m}{2n} j^2} \quad (j = 1,2,3,\ldots) \quad . \tag{3.81}$$

Further,

$$m' = 1 - 1/2n = m \quad \text{if} \quad j = 1 \quad , \tag{3.82}$$

so that $j \geq 2$ since $m' > m$. The minimum admissible eigenvalue m' occurs at $j = 2$ and is given by

$$m' = \frac{1}{2}\left[\kappa + \sqrt{\kappa^2 + 8m/n}\,\right], \quad \kappa = (n-1)/n \quad ; \tag{3.83}$$

the corresponding eigenfunction is readily shown to be

$$\chi(\theta) = c_2[\omega(\theta,n) + \kappa\cos\theta]^{m'} \left\{ \frac{1 + \kappa\sin^2\theta - \omega(\theta,n)\cos\theta}{[\omega(\theta,n) + \kappa\cos\theta]^2 n^2} - 1 \right\} \quad (-\pi \leq \theta \leq \pi). \tag{3.84}$$

The eigenvalue (3.83) conforms to (3.37) as long as

$$n < 7/2 \quad . \tag{3.85}$$

Finally suppose that $n \geq 7/2$, in which case (3.40), (3.44) do not possess an eigenvalue on $(m, 2-m)$. Equations (3.41), (3.44), (3.65), (3.66) in conjunction with the transformations (3.72), (3.73), (3.77) lead to

$$\left.\begin{aligned} &\ddot{W}_1(\xi) + \lambda^2 W_1(\xi) = -\frac{B\kappa\sqrt{n}}{m^3 A}\cos\xi, \quad \dot{W}_1(0) = W_1(\pi/2) = 0 \quad , \\ &\ddot{W}_2(\xi) + \lambda^2 W_2(\xi) = 0, \quad \dot{W}_2(0) = \dot{W}_2(\pi/2) = 0 \quad , \end{aligned}\right\} \tag{3.86}$$

where

$$\lambda^2 = 3(2n+1)/(2n-1), \quad \kappa = (n-1)/n \quad . \tag{3.87}$$

Thus,

$$W_1(\xi) = \frac{-B\kappa n\sqrt{n}}{4Am^2(1+n)} \cos \xi \quad \text{if} \quad n \geq 7/2 \quad , \tag{3.88}$$

$$\left.\begin{aligned} &W_2(\xi) = c_2 \cos 2\xi \quad \text{if} \quad n = 7/2 \qquad (c_2 = \text{constant}) \quad , \\ &W_2(\xi) = 0 \quad \text{if} \quad n > 7/2 \quad . \end{aligned}\right\} \tag{3.89}$$

Equations (3.88), (3.89), together with (3.77), (3.73), (3.72), in turn furnish

$$\chi(\theta) = -\frac{B\kappa\sqrt{2n}}{4Am^2(1+n)} [\omega(\theta,n) + \kappa\cos\theta]^{m'-1} \sqrt{1 + \kappa\sin^2\theta - \omega(\theta,n)\cos\theta}$$

$$+ c_2[\omega(\theta,n) + \kappa\cos\theta]^{m'} \left\{ \frac{1+\kappa\sin^2\theta - \omega(\theta,n)\cos\theta}{[\omega(\theta,n)+\kappa\cos\theta]^2 n^2} - 1 \right\} \quad (-\pi \leq \theta \leq \pi),$$

$$m' = 2 - m \quad , \tag{3.90}$$

for $n \geq 7/2$, with $c_2 = 0$ when $n > 7/2$.

At this point we summarize the various possibilities that arise in the determination of the second term in (3.34). First, if $n \geq 7/2$, there are no eigenvalues of (3.38) or (3.40), (3.44) on the interval $(m, 2-m)$. Thus $m' = 2 - m$ in this instance and χ, Ψ are given by (3.90), (3.63). On solving (3.36) and the first of (3.43) for V_α, one is led to

$$V_1 = (a_1\chi - a_2\Psi)/a^2, \quad V_2 = (a_1\Psi + a_2\chi)/a^2 \quad \text{on} \quad [-\pi, \pi] \quad . \tag{3.91}$$

Next, suppose $3/2 < n < 7/2$. In this case only the χ-problem admits an eigenvalue on $(m,2-m)$, which is given by (3.83), whereas χ is supplied by (3.84). V_α is now found by setting $\Psi = 0$ in (3.91), which gives

$$V_1 = a_1\chi/a^2, \quad V_2 = a_2\chi/a^2 \quad \text{on} \quad [-\pi,\pi] \quad . \tag{3.92}$$

Finally, for the range $1/2 < n < 3/2$, in which case both the χ-problem and the Ψ-problem admit eigenvalues on the appropriate range, we seek the smallest such eigenvalue. This criterion, along with (3.48), (3.83), leads to

$$\left.\begin{aligned} m' &= \frac{1}{2}\left[\kappa + \sqrt{\kappa^2 + 8m/n}\right] \quad \text{if} \quad 1 < n < 3/2 \quad , \\ m' &= 1 \quad \text{if} \quad n = 1 \quad , \\ m' &= 2 - 1/n \quad \text{if} \quad 1/2 < n < 1 \quad . \end{aligned}\right\} \tag{3.93}$$

Further,

$$\Psi = 0 \ \text{on} \ [-\pi,\pi] \ \text{for} \ 1 < n < 3/2, \ \chi = 0 \ \text{on} \ [-\pi,\pi] \ \text{for} \ 1/2 < n < 1 \ . \tag{3.94}$$

For $1/2 < n < 1$, V_α is supplied by

$$V_1 = -a_2\Psi/a^2, \quad V_2 = a_1\Psi/a^2 \quad \text{on} \quad [-\pi,\pi] \quad , \tag{3.95}$$

where Ψ is given by (3.49). When $n = 1$, both problems have the common eigenvalue $m' = 1$. In the following section we shall explore in greater detail the Mooney-Rivlin material, which is a power-law material with $n = 1$.

It is convenient, for reasons that will be made clear presently, to apply an additional rigid rotation to the deformation field characterized by (3.34). In particular, consider the rotation tensor whose component matrix in the underlying frame is given by

$$\left[Q_{\alpha\beta}\right] = \begin{bmatrix} a_2/a & -a_1/a \\ a_1/a & a_2/a \end{bmatrix}, \quad a = \sqrt{a_1^2 + a_2^2} \quad . \tag{3.96}$$

Calculating $\underset{\sim}{y}^*$ in the first of (3.5) from (3.34), (3.96), bearing in mind (3.36), (3.43) and dropping the asterisk from the resulting estimates for the components y^*_α, one arrives at

$$\left.\begin{aligned} y_1(r,\theta) &= -\frac{1}{a} r^{m'} \Psi(\theta) + o(r^{m'}) \quad , \\ y_2(r,\theta) &= a r^m U(\theta) + \frac{1}{a} r^{m'} \chi(\theta) + o(r^{m'}) \quad . \end{aligned}\right\} \tag{3.97}$$

The significance of (3.97) is as follows. Suppose, in particular, that $n > 7/2$ and consider the deformation-image of the extended crack axis $\theta = 0$. From (3.97), (3.90), (3.64), (3.31) follows

$$y_1(r,0) = \frac{2n^{1/2}(2m)^{-m}}{m' a} r^{m'} + o(r^{m'}) \,, \quad y_2(r,0) = o(r^{m'}) \quad , \tag{3.98}$$

and thus the ray $\theta = 0$ after deformation is tangent to the x_1-axis at the origin.

In case $1 < n < 7/2$ $(n \neq 3/2)$, $\Psi = 0$ on $[-\pi,\pi]$ and the first of (3.97) yields merely the weak estimate $y_1 = o(r^{m'})$. There is, however, no difficulty in establishing a non-degenerate estimate for y_1. Indeed, let

$$y_1(r,\theta) = -\frac{1}{a}\, r^{m''} X(\theta) + o(r^{m''}), \quad m'' > m \quad , \tag{3.99}$$

and suppose that y_2 is given by the second of (3.97). Entering (3.9) with this Ansatz, one has

$$\left.\begin{aligned} m\dot{X}U - m''X\dot{U} &= 0 \quad \text{if} \quad m < m'' < 2-m \quad , \\ m\dot{X}U - m''X\dot{U} &= 1 \quad \text{if} \quad m'' = 2-m \quad . \end{aligned}\right\} \tag{3.100}$$

These equations are simply (3.38), (3.39) with m', Ψ replaced by m'', X, and hence

$$\left.\begin{aligned} m'' &= 2-\frac{1}{n}, \; X = b_0U^2 \quad \text{on } [-\pi,\pi] \quad \text{for} \quad 1 < n < 3/2 \quad , \\ m'' &= 1+\frac{1}{2n}, \; X = H \quad \text{on } [-\pi,\pi] \quad \text{for} \quad n > 3/2 \quad . \end{aligned}\right\} \tag{3.101}$$

Thus for $1/2 < n < 3/2$,

$$\left.\begin{aligned} y_1(r,\theta) &= b_0 r^{2m}[U(\theta)]^2 + o(r^{2m}) \quad , \\ y_2(r,\theta) &= ar^m U(\theta) + \frac{1}{a} r^{m'} \chi(\theta) + o(r^{m'}) \quad , \end{aligned}\right\} \tag{3.102}$$

while for $3/2 < n < 7/2$,

$$\left.\begin{aligned} y_1(r,\theta) &= -\frac{1}{a} r^{2-m} H(\theta) + o(r^{2-m}) \quad , \\ y_2(r,\theta) &= ar^m U(\theta) + \frac{1}{a} r^{m'} \chi(\theta) + o(r^{m'}) \quad , \end{aligned}\right\} \tag{3.103}$$

m', χ and H being given by (3.83), (3.84), (3.62). In case $n \geq 7/2$,

$$\left.\begin{aligned} y_1(r,\theta) &= -\frac{1}{a} r^{2-m} H(\theta) + o(r^{2-m}) \quad , \\ y_2(r,\theta) &= a r^m U(\theta) + \frac{1}{a} r^{2-m} \chi(\theta) + o(r^{2-m}) \quad , \end{aligned}\right\} \quad (3.104)$$

with χ now furnished by (3.90). U and m are supplied in all cases by (3.30), (3.31). As mentioned earlier, the special case $n = 3/2$ will be dealt with in the next section.

We now turn to the determination of the local structure of the pressure field, which has been assumed to admit the representation (3.3). From (3.9), (3.10), (3.12) follows

$$2\overset{\circ}{W}'(I)\nabla^2 y_1 + 2\overset{\circ}{W}''(I)\left(\frac{\partial y_1}{\partial r}\frac{\partial I}{\partial r} + \frac{1}{r^2}\frac{\partial y_1}{\partial \theta}\frac{\partial I}{\partial \theta}\right) = \frac{1}{r}\left(\frac{\partial p}{\partial r}\frac{\partial y_2}{\partial \theta} - \frac{\partial p}{\partial \theta}\frac{\partial y_2}{\partial r}\right) \quad . \quad (3.105)$$

Suppose first $1/2 < n < 3/2$, in which case y_α is given by (3.102). Substituting from (3.102) and (3.3) into (3.105), one is led to

$$4nAb_0 a^{2n-2} G^n r^{-1} + o(r^{-1}) = a r^{m+\ell-2}(\ell P\dot{U} - m\dot{P}U) + o(r^{m+\ell-2}) \quad , \quad (3.106)$$

where G is given by (3.20). By virtue of the boundary conditions (3.14), along with (3.102),

$$p(r,\pm\pi) \sim 2nA[maU(\pi)]^{2(n-2)} r^{(2-n)/n} \quad , \quad (3.107)$$

whereas from (3.3),

$$p(r,\pm\pi) \sim r^\ell P(\pm\pi) \quad . \quad (3.108)$$

Since $b_0 \neq 0$ one has

$$\ell \le 1 - m = 1/2n \quad . \qquad (3.109)$$

From (3.107), (3.108), (3.109) one now draws

$$P(\pm\pi) = 0 \quad . \qquad (3.110)$$

Suppose $\ell < 1/2n$. Then (3.106) gives

$$\ell P\dot{U} - m\dot{P}U = 0 \quad \text{on} \quad [-\pi,\pi] \quad , \qquad (3.111)$$

which together with (3.110) yields

$$P = 0 \quad \text{on} \quad [-\pi,\pi] \quad . \qquad (3.112)$$

Therefore

$$\ell = 1/2n \quad , \qquad (3.113)$$

$$\ell P\dot{U} - m\dot{P}U = 4nAb_0 a^{2n-3} G^n \quad \text{on} \quad [-\pi,\pi], \ P(\pm\pi) = 0 \quad . \qquad (3.114)$$

The solution to (3.114) is found to be given by

$$P(\theta) = 4b_0 A a^{2n-3} n^{5/2-n} m^{2n-2} [\omega(\theta,n) + \kappa\cos\theta]^{1/2n} \sqrt{1 - \cos^2\xi_0} \quad , \qquad (3.115)$$

with $\cos\xi_0$ and $\omega(\theta,n)$ given by (3.59), (3.32).

When $n > 3/2$, substitution from (3.103) or (3.104) into (3.105) eventually yields

$$\ell = \frac{2}{n} - 1 \quad , \qquad (3.116)$$

$$m\dot{P}U - \ell P\dot{U} = 2nAa^{2n-4} G^{n-2} S \quad \text{on} \quad [-\pi,\pi] \quad , \qquad (3.117)$$

where

$$S = G[\ddot{H} + (2-m)^2 H] + (n-1)[\dot{G}\dot{H} + 2(2-m)(m-1)GH] \quad . \tag{3.118}$$

The boundary conditions (3.107), (3.108) give

$$P(\pm\pi) = 2nA[amU(\pi)]^{2n-4} \quad . \tag{3.119}$$

The boundary-value problem consisting of (3.117), (3.119) fails to admit a solution with continuous derivatives of all orders at $\theta = 0$, and consequently the Ansatz (3.3) cannot possibly be consistent with the field equations when $n > 3/2$.

In summary, the behavior of the pressure near the crack-tip is governed by

$$\left.\begin{aligned} p(r,\theta) &= r^{1/2n} P(\theta) + o(r^{1/2n}) \quad \text{for} \quad 1/2 < n < 3/2 \quad , \\ p(r,\theta) &= o(r^{-1+1/n}) \quad \text{for} \quad n > 3/2 \quad , \end{aligned}\right\} \tag{3.120}$$

with P furnished by (3.15); the weak estimate for p appropriate to $n > 3/2$ follows from (3.10) together with (3.103) or (3.104). A non-degenerate pressure estimate valid for $n > 3/2$ would presumably necessitate a higher-order asymptotic analysis of the deformation field.

We note that if $n > 3/2$ the dominant terms in (3.103), (3.104) have a Jacobian determinant $J \sim 1$ and hence constitute a locally one-to-one mapping. In contrast, J computed from the dominant terms in (3.102), which apply to $1/2 < n < 3/2$, is merely $o(r^{1-3/2n})$. The discussion in the next section of the Mooney-Rivlin material $(n = 1)$ and of the special case $n = 3/2$ will, however, lead to an improved approximation to the deformation at least when $1 \leq n < 3/2$.

4. The transition case $n = 3/2$. Higher-order analysis for the Mooney-Rivlin material.

We first investigate the modifications necessary in the asymptotic structure of the deformation and pressure fields when $n = 3/2$. The following analysis parallels that in [4] for the derivation of the logarithmic term in the deformation field when the hardening parameter there takes the value $7/6$.

Recall from the first of (3.103) that

$$y_1(r,\theta) = -\frac{1}{a} r^{2-m} H(\theta,n) + o(r^{2-m}), \quad (n > 3/2) \quad , \qquad (4.1)^1$$

and note on the basis of (3.62) that

$$H(\theta,n) = \frac{c_0[U(\theta,3/2)]^2}{n-3/2} + O(1) \quad \text{as} \quad n \to \frac{3}{2}, \quad c_0 = \frac{27}{16}\sqrt{\frac{3}{2}} \quad . \qquad (4.2)$$

Thus $y_1(r,\theta) \to \infty$ as $n \to 3/2$ from above for all $\theta \neq 0$ and every sufficiently small $r > 0$. Consequently the preceding estimate for y_1 is inadequate near $n = 3/2$.

We shall now show that in a deleted neighborhood of $n = 3/2$,

$$y_1(r,\theta) = -\frac{1}{a} r^{1+1/2n} H(\theta,n) + b_0 U^2(\theta,n) r^{2-1/n} + o(r^{m_*}) \quad , \qquad (4.3)$$

with $m_* = \max(1 + 1/2n,\ 2 - 1/n)$ and $b_0 \neq 0$ a constant. Suppose first that $n > 3/2$ and replace (4.1) by

---

[1] We now write $U(\theta,n)$, $H(\theta,n)$ in place of $U(\theta)$, $H(\theta)$ in order to emphasize the dependence of these functions upon the hardening parameter.

$$y_1(r,\theta) = -\frac{1}{a} r^{2-m} H(\theta,n) + r^{m''} T(\theta,n) + o(r^{m''}), \quad m'' > 2-m \quad . \tag{4.4}$$

Substitution from (4.4) and the second of (3.103) into (3.9) leads to

$$m U \dot{T} - m'' T \dot{U} = 0 \quad \text{on} \quad [-\pi,\pi] \quad , \tag{4.5}$$

which is merely (3.38) with m', Ψ replaced by m'', T. Thus

$$m'' = 2m = 2 - 1/n, \quad T = b_0 U^2 \quad . \tag{4.6}$$

In case $1 < n < 3/2$, a similar procedure yields

$$y_1(r,\theta) = b_0 r^{2m} U^2(\theta,n) - \frac{1}{a} r^{2-m} H(\theta,n) + o(r^{2-m}) \quad , \tag{4.7}$$

which confirms (4.3). It should be noted that the <u>first</u> term of (4.3) dominates as $r \to 0$ for $n > 3/2$, while for $n < 3/2$ the <u>second</u> term dominates; in either case the respective dominant term agrees with our previous results. The two terms trade the dominant role as n passes through 3/2, the respective powers of r having the common exponent 4/3 at $n = 3/2$.

From (4.2), (4.3) now follows

$$y_1(r,\theta) = \left[-\frac{1}{a} \frac{c_0 r^{1+1/2n}}{n - 3/2} + b_0 r^{2-1/n} \right] \left[U(\theta,3/2) \right]^2 + r^{1+1/2n} \eta_1(\theta)$$

$$+ b_0 (n - 3/2) r^{2-1/n} \eta_2(\theta) + o(r^{m_*}) \quad \text{as} \quad n \to 3/2 \quad . \tag{4.8}$$

where $\eta_1(\theta)$ and $\eta_2(\theta)$ are functions that stay bounded as $n \to 3/2$. Thus if $y_1(r,\theta)$ is to remain bounded as $n \to 3/2$ one must have

$$b_0 = \frac{1}{a} \frac{c_0}{n - 3/2} + O(1) \quad \text{as} \quad n \to 3/2 \quad . \tag{4.9}$$

Proceeding to the limit in (4.8) as $n \to 3/2$ one arrives at

$$y_1(r,\theta) = \frac{3c_0}{2an^2} [U(\theta,3/2)]^2 r^{4/3} \log r + r^{4/3} \eta(\theta) + o(r^{4/3}) \quad , \tag{4.10}$$

in which $\eta(\theta)$ is bounded. This result motivates the following modification of (4.3) in case $n = 3/2$. We suppose

$$y_1(r,\theta) = r^{4/3} \log r \, v_1(\theta) + r^{4/3} v_2(\theta) + o(r^{4/3}) \quad , \tag{4.11}$$

and, recalling (3.102), (3.103), assume

$$y_2(r,\theta) = ar^{2/3} U(\theta,3/2) + O(r^{2/3+\delta}) \quad , \tag{4.12}$$ [1]

where $\delta > 0$. Substitution from (4.11), (4.12) into (3.9) yields

$$U\dot{v}_1 - 2\dot{U}v_1 = 0, \quad U\dot{v}_2 - 2\dot{U}v_2 = \frac{3}{2}\left(v_1\dot{U} - \frac{1}{a}\right) \quad \text{on} \quad [-\pi,\pi] \quad . \tag{4.13}$$

The first of (4.13) gives

$$v_1 = c_1 U^2 \quad \text{on} \quad [-\pi,\pi] \quad , \tag{4.14}$$

so that the second of (4.13) may now be written as

$$U\dot{v}_2 - 2\dot{U}v_2 = \frac{3}{2}\left(c_1\dot{U}U^2 - \frac{1}{a}\right) \quad \text{on} \quad [-\pi,\pi] \quad . \tag{4.15}$$

---

[1] Merely requiring the error to be $o(r^{2/3})$ is inadequate for the present purpose.

If (4.15) is to possess a solution v_2 of unrestricted smoothness on $[-\pi,\pi]$, then $c_1 = 3c_0/2an^2$ in agreement with (4.10).

Next, observe that in view of (4.9), (3.115), P becomes unbounded as $n \to 3/2$ for $-\pi < \theta < \pi$. The foregoing analysis suggests that we replace (3.3) by

$$p(r,\theta) \sim r^{1/3} \log r \, P(\theta) \quad \text{for} \quad n = 3/2 \quad . \tag{4.16}$$

Substitution from (4.11), (4.12), (4.16) into (3.105) gives

$$\frac{1}{3} P\dot{U} - \frac{2}{3} \dot{P} U = 6Ac_0 G^{3/2} \quad \text{on} \quad [-\pi,\pi] \quad . \tag{4.17}$$

Moreover, from the boundary conditions (3.14),

$$P(\pm\pi) = 0 \quad . \tag{4.18}$$

Consequently,

$$P(\theta) = 4c_0 A[\omega(\theta,3/2) + \frac{1}{3}\cos\theta]^{1/3} \sqrt{1-\cos^2\xi_0} \quad \text{for} \quad n = 3/2 \quad , \tag{4.19}$$

in which $\cos^2\xi_0$ is given by (3.59) with $n = 3/2$.

Equations (4.16), (4.19) supply a non-degenerate estimate for the pressure in case $n = 3/2$. In addition, (4.3) together with the second of (3.102) yield a locally one-to-one approximation of the deformation if $1 < n < 3/2$.

We turn now to the Mooney-Rivlin material $(n = 1)$, which has the complete elastic potential (1.19), and seek improved estimates for the corresponding deformation and pressure fields.

The equilibrium equations (3.10) in this instance reduce to

$$\frac{1}{\mu}\frac{\partial p}{\partial r} = H_r, \quad \frac{1}{\mu}\frac{\partial p}{\partial \theta} = H_\theta \quad , \tag{4.20}$$

with H_r and H_θ supplied by (3.12). The boundary conditions (3.14), in turn, at present become

$$\frac{1}{\mu}p = \frac{1}{g_{rr}} = \frac{g_{\theta\theta}}{r^2} \; , \quad g_{r\theta} = 0 \quad \text{at} \quad \theta = \pm\pi \quad . \tag{4.21}$$

Setting $n = 1$ in (3.30), (3.31) one finds that

$$m = 1/2, \; U(\theta) = \sin(\theta/2) \quad . \tag{4.22}$$

We recall also that the eigenvalue problems determining χ, Ψ — and hence V_α in (3.34) — admit the common eigenvalue $m' = 1$. Upon substituting $n = 1$ in the corresponding solutions (3.49), (3.84) one arrives at

$$\chi = b_1 \cos\theta, \quad \Psi = b_2 \sin^2\frac{\theta}{2} \quad , \tag{4.23}$$

where b_1 and b_2 are constants. Finally, (3.113), (3.115) supply a first estimate for the pressure:

$$\ell = 1/2, \; P(\theta) = -\frac{2\mu b_2}{a}\cos\frac{\theta}{2} \quad . \tag{4.24}$$

Equations (3.3), (3.34), (3.91) together with (4.22), (4.23), (4.24) now give

$$\left.\begin{aligned} y_1(r,\theta) &= a_1 r^{1/2}\sin\frac{\theta}{2} + \frac{1}{a^2}[a_1b_1\cos\theta - a_2b_2\sin^2\frac{\theta}{2}]r + o(r) \quad ,\\ y_2(r,\theta) &= a_2 r^{1/2}\sin\frac{\theta}{2} + \frac{1}{a^2}[a_1b_2\sin^2\frac{\theta}{2} + a_2b_1\cos\theta]r + o(r) \quad ,\\ p(r,\theta) &= -\frac{2\mu b_2}{a^2} r^{1/2}\cos\frac{\theta}{2} + o(r^{1/2}) \quad . \end{aligned}\right\} \quad (4.25)$$

With a view to refining these estimates we first replace (3.34) by

$$y_\alpha(r,\theta) = a_\alpha r^{1/2}U(\theta) + rV_\alpha(\theta) + r^{m_*}R_\alpha(\theta) + r^{m'_*}S_\alpha(\theta) + o(r^{m'_*}) \quad . \quad (4.26)$$

where $m'_* > m_* > 1$ and R_α, S_α are as yet undetermined, whereas the functions U, V_α are already known. Combining (4.26) with (3.9), (4.20) and invoking the boundary conditions (4.21) one finds after considerable computations that

$$m_* = 3/2, \; m'_* = 2 \quad , \quad (4.27)$$

$$R_1 = \frac{1}{a^2}[a_1\chi_1 - a_2\Psi_1], \; R_2 = \frac{1}{a^2}[a_2\chi_1 + a_1\Psi_1] \quad \text{on} \quad [-\pi,\pi] \quad , \quad (4.28)$$

$$S_1 = \frac{1}{a^2}[a_1\chi_2 - a_2\Psi_2], \; S_2 = \frac{1}{a^2}[a_2\chi_2 + a_1\Psi_2] \quad \text{on} \quad [-\pi,\pi] \quad , \quad (4.29)$$

with Ψ_1, χ_1, Ψ_2, χ_2 supplied by

$$\left.\begin{aligned} \Psi_1(\theta) &= c_2\sin^3\frac{\theta}{2} + \frac{2b_1b_2}{a^2}\sin\frac{\theta}{2} - 4\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{4}{3}\cos^3\frac{\theta}{2} \quad ,\\ \chi_1(\theta) &= c_1\sin\frac{3\theta}{2} - \frac{b_2^2}{2a^2}\sin\frac{\theta}{2} \quad , \end{aligned}\right\} \quad (4.30)$$

$$\left.\begin{aligned}\Psi_2(\theta) &= d_1 \sin^4\frac{\theta}{2} + \frac{1}{a^2}\left(3b_1c_2 - \frac{b_2^3}{a^2} + \frac{8b_1^2b_2}{a^2} + 6b_2c_1\right)\sin^2\frac{\theta}{2} \\ &+ \frac{b_1^2b_2}{a^4} - \frac{8b_1}{a^2}\cos\frac{\theta}{2}\sin^3\frac{\theta}{2} - \frac{2b_1}{a^2}\cos^3\frac{\theta}{2}\sin\frac{\theta}{2} \quad , \\ \chi_2(\theta) &= \frac{b_2}{a^2}[2\sin\theta - \sin 2\theta] + d_2\cos 2\theta \\ &+ \frac{b_2}{a^2}\left[\frac{2b_1b_2}{a^2} + \frac{3c_2}{4}\right]\cos\theta - \frac{b_2}{a^2}\left[\frac{7}{4}\,\frac{b_1b_2}{a^2} + \frac{9}{16}c_2\right] \quad .\end{aligned}\right\} \quad (4.31)$$

Here c_1, c_2, d_1, d_2 are constants.

In order to calculate an additional term for the pressure field, we apply the rotation with the component matrix (3.96) to the deformation field (4.26).[1] This yields

$$\left.\begin{aligned} y_1(r,\theta) &= -\frac{1}{a}r\Psi(\theta) - \frac{1}{a}r^{3/2}\Psi_1(\theta) - \frac{1}{a}r^2\Psi_2(\theta) + o(r^2) \quad , \\ y_2(r,\theta) &= ar^{1/2}U(\theta) + \frac{1}{a}r\chi(\theta) + \frac{1}{a}r^{3/2}\chi_1(\theta) + \frac{1}{a}r^2\chi_2(\theta) + o(r^2) \ .\end{aligned}\right\} \quad (4.32)$$

Next, for the Mooney-Rivlin material, (3.105) reduces to

$$\mu\nabla^2 y_1 = \frac{1}{r}\left(\frac{\partial p}{\partial r}\frac{\partial y_2}{\partial \theta} - \frac{\partial p}{\partial \theta}\frac{\partial y_2}{\partial r}\right) \ . \qquad (4.33)$$

We now suppose that the pressure field conforms to

[1] Recall that the pressure field is not affected by such a rotation.

$$p(r,\theta) = r^{\ell}P(\theta) + r^{\ell'}P_1(\theta) + o(r^{\ell'}), \quad \ell' > \ell \quad , \tag{4.34}$$

with ℓ, P supplied by (4.24) and ℓ', P_1 unknown. Equations (4.32), (4.33), (4.34) and the boundary conditions lead to

$$\ell' = 1, \quad P_1(\theta) = \frac{2\mu}{a^2}\left[4 - 2\sin^2\frac{\theta}{2} - \left(\frac{3c_2}{2} + \frac{5b_1b_2}{a^2}\right)\sin\theta\right] \quad . \tag{4.35}$$

Finally, suppose that the Mode I loading (2.15) prevails at infinity, in which case the local deformation field (4.26) is to be symmetric about the x_1-axis. In this case,

$$a_1 = 0, \; a_2 = a, \; b_1 = 0, \; c_2 = 0, \; d_2 = 0 \quad , \tag{4.36}$$

and (4.26) reduces to

$$\left.\begin{aligned} y_1(r,\theta) &= -\frac{1}{a}rb_2\sin^2\frac{\theta}{2} + \frac{1}{a}r^{3/2}\left[4\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} + \frac{4}{3}\cos^3\frac{\theta}{2}\right] \\ &\quad -\frac{1}{a}r^2\left[d_1\sin^4\frac{\theta}{2} + \frac{2b_2}{a^2}\left(3c_1 - \frac{b_2^2}{2a^2}\right)\sin^2\frac{\theta}{2}\right] + o(r^2) \quad , \\ y_2(r,\theta) &= ar^{1/2}\sin\frac{\theta}{2} + \frac{1}{a}r^{3/2}\left[c_1\sin\frac{3\theta}{2} - \frac{b_2^2}{2a^2}\sin\frac{\theta}{2}\right] \\ &\quad + \frac{b_2}{a^3}r^2[2\sin\theta - \sin 2\theta] + o(r^2) \quad . \end{aligned}\right\} \tag{4.37}$$

Further, the associated pressure field conforms to

$$p(r,\theta) = -\frac{2\mu b_2}{a^2}\cos\frac{\theta}{2} + \frac{2\mu}{a^2} r\,(3-\cos\theta) + o(r) \quad . \tag{4.38}$$

The above results for the symmetric loading mode are identical with those arrived at in an unpublished study by Knowles and Sternberg, which is summarized in [9]. In contrast, it is not possible to specialize the constants in (4.26) so that the displacement field $u_\alpha = y_\alpha - x_\alpha$ has the parity of $\mathring{u}_\alpha$ in (2.7).

This fact is consistent with the conclusion reached at the end of Section 2, according to which the Mode II problem for the Mooney-Rivlin material fails to admit an antisymmetric global solution.

5. Discussion of the deformation and stresses near the crack-tip.

Our initial objective here is to explore the asymptotic results for the deformation field established in the preceeding two sections. Of particular concern is the deformation image of the crack faces $\theta = \pm\pi$. Suppose first that $n > 3/2$ and consider (3.103). Then, to leading order,

$$\left.\begin{aligned} y_1(r,\theta) &= -\frac{1}{a} r^{1+1/2n} H(\theta) + o(r^{1+1/2n}) \quad , \\ y_2(r,\theta) &= a r^{1-1/2n} U(\theta) + o(r^{1-1/2n}) \quad . \end{aligned}\right\} \quad (5.1)^{1}$$

In view of (3.31), (3.64), one has in particular

$$\left.\begin{aligned} y_1(r,\pm\pi) &= -\frac{1}{a} r^{1+1/2n} \frac{\sqrt{n\pi}}{2m^2 m' n^{1/2n}} \frac{\Gamma(3/2 - 1/m)}{\Gamma(2 - 1/m)} \\ &\qquad + o(r^{1+1/2n}), \quad m = 1 - \frac{1}{2n} \quad , \\ y_2(r,\pi) &= a r^{1-1/2n} n^{-\kappa/2} + o(r^{1-1/2n}) \quad , \\ y_2(r,-\pi) &= -a r^{1-1/2n} n^{-\kappa/2} + o(r^{1-1/2n}) \quad . \end{aligned}\right\} \quad (5.2)$$

Thus, to dominant order, the two crack faces are carried into the curve represented by

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[1] The shape of the deformation image of the crack faces is of course invariant under a rotation such as (3.96).

$$y_2 = \pm K(n)a^{2n}\left|\frac{y_1}{a^{2n}}\right|^{\frac{2n-1}{2n+1}}, \quad y_1 < 0, \ K(n) \neq 0 \quad . \qquad (5.3)^1$$

From (5.1) it follows that points near the crack-tip in the undeformed body lie to the right of this curve after deformation. We note on the basis of $(5.3)^2$ and (3.34) that the crack is <u>bound to open</u> when $n > 3/2$, regardless of the magnitude and nature of the particular loading at infinity. This conclusion, which is readily shown to hold also for $1 < n \leq 3/2$, is in marked contrast to the predictions of the linearized theory for a Mode II loading.

It should be emphasized, however, that the foregoing result is contingent upon the applicability of the Ansatz (3.4), in which the functions U_α are not permitted to vanish identically. For example, in the presence of a loading of uniform uniaxial tension <u>parallel</u> to the crack faces, the ensuing deformation field is homogeneous and the crack fails to open. In this instance the deformation field is non-singular at the crack-tips, so that (3.4) cannot possibly apply. We shall remark on the question of the <u>scale</u> of the crack opening later in this section.

Note that the local deformation field (3.34) may be regarded as the result of two successive mappings: a deformation of the undeformed body, in which the crack opens, at least when $n > 1$, followed by a rigid rotation, with $Q_{\alpha\beta}$ given by (3.96). In view of the principle of objectivity we shall therefore be entitled to base the computation of the local stress

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[1]The specific form of $K(n)$, which is expressible in terms of the Gamma function, is of no particular interest here.

[2]Recall from (3.18), (3.20) that the amplitude parameter "a" cannot vanish in this instance.

distribution on (3.102), (3.103), (3.104).

We cite next the symmetric deformation fields[1] appropriate to a Mode I loading at infinity. Since U in (3.34) is an odd function, one has $a_1 = 0$, $a_2 = a$. This specialization leads to the following results. For $1/2 < n < 3/2$,

$$\left.\begin{aligned} y_1(r,\theta) &= b_0[U(\theta)]^2 r^{2-1/n} + o(r^{2-1/n}) \quad , \\ y_2(r,\theta) &= aU(\theta) r^{1-1/2n} + o(r^{2-1/n}) \quad , \end{aligned}\right\} \quad (5.4)$$

while for $n > 3/2$,

$$\left.\begin{aligned} y_1(r,\theta) &= -\frac{1}{a} r^{1+1/2n} H(\theta) + o(r^{1+1/2n}) \quad , \\ y_2(r,\theta) &= ar^{1-1/2n} U(\theta) + \chi(\theta) r^{1+1/2n} + o(r^{1+1/2n}) \quad , \end{aligned}\right\} \quad (5.5)$$

in which χ is furnished by (3.90) with $c_2 = 0$. Finally, for $n = 3/2$,

$$\left.\begin{aligned} y_1(r,\theta) &= b_0 U^2(\theta,3/2) r^{4/3} \log r + O(r^{4/3}) \quad , \\ y_2(r,\theta) &= ar^{2/3} U(\theta,3/2) + o(r^{2/3}) \quad . \end{aligned}\right\} \quad (5.6)$$

It is not possible, however, to specialize the asymptotic results for the deformation so as to arrive at an antisymmetric local displacement field, at least when $n > 1$. To see this, recall that $u_\alpha = y_\alpha - x_\alpha$, and observe that for $n > 1$ one has $m' > 1$, whence (3.34) give

[1]Observe that the associated displacement field $u_\alpha = y_\alpha - x_\alpha$ is also bound to be symmetric by virtue of the parity of x_α.

$$\left.\begin{aligned} u_1(r,\theta) &= a_1 r^{1-1/2n} U(\theta) - r\cos\theta + o(r) \quad , \\ u_2(r,\theta) &= a_2 r^{1-1/2n} U(\theta) - r\sin\theta + o(r) \quad . \end{aligned}\right\} \quad (5.7)$$

The constants in (5.7) evidently cannot be chosen to yield a displacement field of the desired parity.

As we observed in the previous section, the local asymptotic solution obtained for the Mooney-Rivlin material $(n=1)$ also fails to admit an antisymmetric displacement field. The foregoing conclusions concerning $n \geq 1$ lend support to the conjecture that there does not exist an antisymmetric global solution for all values of the hardening parameter.

At this stage we calculate the associated local true-stress field, as well as the strain-energy density. On account of (1.7), (1.12) one has

$$\tau_{\alpha\beta} = 2\mathring{W}'(I) F_{\alpha\gamma} F_{\beta\gamma} - p\delta_{\alpha\beta} \quad . \qquad (5.8)$$

Suppose first that $1/2 < n < 3/2$. In this case, (3.102) and the first of (3.120), together with (5.8), yield

$$\left.\begin{aligned} \tau_{11}(r,\theta) &= r^{1-1/n} T_{11}(\theta,n) + o(r^{1-1/n}) \quad , \\ \tau_{12}(r,\theta) &= r^{-1/2n} T_{12}(\theta,n) + o(r^{-1/2n}) \quad , \\ \tau_{22}(r,\theta) &= r^{-1} T_{22}(\theta,n) + o(r^{-1}) \quad . \end{aligned}\right\} \quad (5.9)^{1}$$

If $n > 3/2$, the appropriate stresses obey

[1] Observe that the true stresses here are referred to the material polar coordinates (r,θ).

$$\left.\begin{aligned}
\tau_{11}(r,\theta) &= o(r^{-1+1/n}) \quad , \\
\tau_{12}(r,\theta) &= r^{-1+1/n}\,\hat{T}_{12}(\theta,n) + o(r^{-1+1/n}) \quad , \\
\tau_{22}(r,\theta) &= r^{-1}T_{22}(\theta,n) + o(r^{-1}) \quad .
\end{aligned}\right\} \tag{5.10}$$

The auxiliary functions $T_{\alpha\beta}$ appearing in (5.9), (5.10) are defined by

$$\left.\begin{aligned}
T_{11}(\theta,n) &= 8Ab_0^2a^{2n-2}m^{2n}n^{2-n}[\omega(\theta,n) + \kappa\cos\theta]^{1-1/n}\cos^2\xi_0 \quad , \\
T_{12}(\theta,n) &= 4Ab_0a^{2n-1}m^{2n}n^{3/2-n}[\omega(\theta,n) + \kappa\cos\theta]^{-1/2n}\cos\xi_0 \quad , \\
T_{22}(\theta,n) &= 2Aa^{2n}m^{2n}n^{1-n}[\omega(\theta,n) + \kappa\cos\theta]^{-1} \quad ,
\end{aligned}\right\} \tag{5.11}$$

with $\cos\xi_0$ given by (3.59), while $\hat{T}_{12}$ involves the hypergeometric function and will not be cited explicitly.

In the transition case $n = 3/2$, equations (4.11), (4.12), (4.16) may be used to deduce the corresponding stress field. We note that in this instance τ_{22} is given by the last of (5.9) or (5.10) with $n = 3/2$ and thus the representation

$$\tau_{22}(r,\theta) = r^{-1}T_{22}(\theta,n) + o(r^{-1}) \quad , \tag{5.12}$$

is valid for $1/2 < n < \infty$.

The components of stress in (5.9), (5.10) are those associated with the rotated deformation fields (3.102), (3.103). If $\tau^*_{\alpha\beta}$ are the true-stress components associated with the original field (3.34), one has

$$\tau^*_{\alpha\beta} = Q_{\mu\alpha} Q_{\nu\beta} \tau_{\mu\nu} \quad , \tag{5.13}$$

in which $Q_{\alpha\beta}$ are the components (3.96) of the rotation tensor $\underset{\sim}{Q}$.

Observe that a more satisfactory estimate for τ_{11} than that in the first of (5.10) is precluded by the weak estimate for the pressure in (3.120). It is interesting to recall in this connection that the asymptotic solution in [3], [4] of the Mode I crack problem for a class of compressible elastic solids also fails to yield a non-degenerate estimate for the stress τ_{11} at sufficiently large values of the corresponding hardening parameter. The most singular of the stress components in (5.9), (5.10) is τ_{22}, which becomes infinite at the crack-tip like r^{-1}. According to (5.11), $T_{22} > 0$, and hence τ_{22} is tensile for all small enough values of r.

We now determine the dominant character of the true stresses when the latter are referred to the spatial coordinates y_α. Since such a representation of the stresses depends on the availability of an invertible estimate for the local deformation we confine our attention at present to $n \geq 1$. One draws from (3.34) that

$$r_* \sim r^{(2n-1)/2n} \quad \text{if} \quad r_*^2 = y_1^2 + y_2^2 \quad . \tag{5.14}$$

Equations (5.9), (5.14) imply that for $1 \leq n < 3/2$,

$$\tau_{11} = O\left(r_*^{\frac{2(n-1)}{2n-1}}\right), \quad \tau_{12} = O\left(r_*^{\frac{-1}{2n-1}}\right), \quad \tau_{22} = O\left(r_*^{\frac{-2n}{2n-1}}\right) \quad . \tag{5.15}$$

For $n > 3/2$, (5.10) yield

$$\tau_{11} = o\left(r_*^{\frac{-2(n-1)}{2n-1}}\right), \quad \tau_{12} = 0\left(r_*^{\frac{-2(n-1)}{2n-1}}\right), \quad \tau_{22} = 0\left(r_*^{\frac{-2n}{2n-1}}\right) . \qquad (5.16)$$

We observe that the singularity of τ_{22} in (5.15), (5.16) becomes progressively more severe with increasing values of the hardening parameter and — for the range of n under consideration — is always stronger than that predicted by the linear theory.

From (1.18), (3.34) we gather that

$$\overset{\circ}{W}(I) = Aa^{2n}G^{n}(\theta)r^{-1} + o(r^{-1}), \quad r \to 0, \ n > 1/2 \quad , \qquad (5.17)$$

so that the strain-energy density has a singularity near the crack-tips which is of the same order as its counterpart in the linear theory.

Finally we supply an estimate for the amplitude parameter "a", valid for small loads at infinity. In this connection we recall a conservation law appropriate to the equilibrium theory of finite plane strain: for every curve $\mathcal{C}$ that is the boundary of a finite regular subregion of Π,

$$\int_{\mathcal{C}} (W\underset{\sim}{n} - \underset{\sim}{F}^{T}\underset{\sim}{s})\, d\ell = \underset{\sim}{0} \quad , \qquad (5.18)^{1}$$

provided W and $\underset{\sim}{F}$ are the strain-energy density and the deformation gradient tensor, whereas $\underset{\sim}{n}$ is the unit outward normal of $\mathcal{C}$ and $\underset{\sim}{s}$ is the Piola traction,

[1] (5.18) is an adaptation to plane strain of a conservation law originally due to Eshelby [16]. The importance of this conservation law in fracture mechanics was first recognized by Rice [17].

$$\underset{\sim}{s} = \underset{\sim}{\sigma}\,\underset{\sim}{n} \quad \text{on} \quad \mathcal{C} \ . \tag{5.19}$$

The conservation law (5.18) holds for both compressible and incompressible materials, and is readily confirmed with the aid of the divergence theorem and the appropriate field equations.

For the present purpose we consider first the Mode I loading case and assume that the solution to the nonlinear problem for sufficiently small loads is uniformly approximated by the corresponding solution from the linear theory, at all material points a finite distance from the crack-tips. A limit process depending on this assumption enables one to deduce from (5.18) and from the available solution of the linearized problem the desired small-load amplitude estimate. Since the required calculation is strictly parallel to one spelled out in detail in [3], we cite the ensuing estimate directly:

$$a^{2n} = \frac{\sigma^2 b n^{n-2}}{4\mu A m^{2n-1}} + o(\sigma^2) \quad \text{as} \quad \sigma \to 0 \ . \tag{5.20}$$

Here σ is the applied normal stress at infinity, $2b$ is the crack length, while μ is the shear modulus at infinitesimal deformations.

According to a conclusion reached earlier in this section, the finite theory — in contrast to linearized elastostatics — predicts that the crack opens in the vicinity of its tips also for a Mode II loading, at least for a certain range of the hardening parameter. Clearly, if the solution to the linearized Mode II problem is to approximate its counterpart in the finite theory uniformly at small loads on every bounded and closed material point set that excludes the crack-tips, the lengths of the opening crack-segments would have to tend to zero with the load intensity. Making this assumption, invoking the conservation law (5.18) once again, and appealing to the Mode II results of the linear theory summarized in (2.12), (2.13), (2.14), one is led to precisely the estimate obtained for Mode I:

$$L \equiv a^{2n} = \frac{\sigma^2 b n^{n-2}}{4\mu A m^{2n-1}} + o(\sigma^2) \quad , \tag{5.21}$$

in which σ is now the shear stress applied at infinity. The length parameter L, in view of (5.3), governs the scale of the crack-opening, which is thus found to be a second-order effect at small shearing loads.

Appendix: Representation of functions biharmonic in a neighborhood of infinity.

In this paper a function U is called biharmonic on an open plane region $\mathcal{R}$, which need not be simply connected, if it is four times continuously differentiable on $\mathcal{R}$ and $\nabla^4 U = 0$ on $\mathcal{R}$.

Let $\mathcal{N}$ be a neighborhood of infinity in the (x_1, x_2)-plane, so that $\mathcal{N}$ is the exterior of some circle centered at the origin, and let $\mathcal{N}'$ be the simply-connected domain obtained by deleting from $\mathcal{N}$ its intersection with the positive x_1-axis. As shown by Muskhelishvili [14] in his analysis of the structure of the Airy stress function on multiply connected domains, any function U biharmonic on $\mathcal{N}'$, whose partial derivatives of the second and higher orders admit continuous extensions to $\mathcal{N}$, may be represented in the form

$$U(x_1, x_2) = \mathrm{Re}\{\bar{z}\Psi(z) + \chi(z)\} \quad , \tag{1}$$

where

$$\left.\begin{aligned} \Psi(z) &= Az\log z + B\log z + \Psi^*(z) \quad , \\ \chi(z) &= Cz\log z + D\log z + \chi^*(z) \quad . \end{aligned}\right\} \tag{2}$$

Here $z = x_1 + ix_2 = \rho e^{i\varphi}$, $\log z$ refers to the principal branch of the logarithm, while $\Psi^*(z)$ and $\chi^*(z)$ are functions analytic on $\mathcal{N}$.

Let A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 denote the real and imaginary parts of A, B, C, D, respectively. Invoking the Laurent expansions on $\mathcal{N}$ of $\Psi^*(z)$ and $\chi^*(z)$, one readily deduces from (1) and

(2) the representation

$$U(x_1,x_2) = A_1\rho^2\log\rho - A_2\rho^2\varphi$$

$$+ [(B_1+C_1)\cos\varphi + (B_2+C_2)\sin\varphi]\rho\log\rho$$

$$+ [(B_1-C_2)\sin\varphi - (B_2+C_1)\cos\varphi]\rho\varphi$$

$$+ D_1\log\rho - D_2\varphi + \sum_{j=-\infty}^{\infty} [a_j\cos j\varphi + b_j\sin j\varphi$$

$$+ c_j\cos(j-2)\varphi + d_j\sin(j-2)\varphi]\rho^{-j} \quad , \tag{3}$$

valid on $\mathcal{N}'$, the coefficients a_j, b_j, c_j, d_j being real constants.

Suppose now U is biharmonic on $\mathcal{N}$ and hence continuous on $\mathcal{N}$ together with all its partial derivatives up to the fourth order. Then (3) demands that

$$A_2 = 0;\ B_1 - C_2 = 0;\ B_2 + C_1 = 0,\ D_2 = 0 \quad . \tag{4}$$

Suppose further that

$$U(x_1,x_2) = O(1) \quad \text{as} \quad \rho\to\infty \quad . \tag{5}$$

This requirement evidently yields the additional restrictions

$$A_1 = 0,\ B_1 + C_1 = 0,\ B_2 + C_2 = 0,\ D_1 = 0,\ a_j = b_j = c_j = d_j = 0 \quad \text{for} \quad j < 0 \quad . \tag{6}$$

In view of (4), (6), the representation (3) finally reduces to

$$U(x_1,x_2) = \sum_{j=0}^{\infty} [a_j \cos j\varphi + b_j \sin j\varphi + c_j \cos(j-2)\varphi + d_j \sin(j-2)\varphi]\rho^{-j}. \quad (7)$$

This expression is valid on $\mathcal{R}$, the infinite series being uniformly and absolutely convergent on every closed subset of $\mathcal{R}$. Thus any function biharmonic on $\mathcal{R}$ and bounded at infinity necessarily admits the representation (7).

References

[1] Rice, J.R., Mathematical analysis in the mechanics of fracture, Fracture, Vol.II, Chapter 3, Academic Press, New York, 1968.

[2] Wong, F.S. and Shield, R.T., Large plane deformations of thin elastic sheets of neo-Hookean material, Zeitschrift für angewandte Mathematik und Physik, 20 (1969) 2, p.176.

[3] Knowles, J.K. and Sternberg, E., An asymptotic finite-deformation analysis of the elastostatic field near the tip of a crack, Journal of Elasticity, 3 (1973) 2, p.67.

[4] Knowles, J.K. and Sternberg, E., Finite-deformation analysis of the elastostatic field near the tip of a crack: Reconsideration and higher order results, Journal of Elasticity, 4 (1974) 3, p.201.

[5] Knowles, J.K., The finite anti-plane shear field near the tip of a crack for a class of incompressible materials, International Journal of Fracture, 13 (1977) 5, p.61.

[6] Lo, K.K., Finite-deformation crack in an infinite body under anti-plane simple shear, International Journal of Solids and Structures, 13 (1977), p.1219.

[7] Knowles, J.K. and Sternberg, E., Discontinuous deformation gradients near the tip of a crack in finite anti-plane shear: an example, Journal of Elasticity, 10 (1980) 1, p.81.

[8] Knowles, J.K. and Sternberg, E., Anti-plane shear fields with discontinuous deformation gradients near the tip of a crack in finite elastostatics, Technical Report No.43, ONR Contract N00014-75-C-0196, September 1979. To appear in Journal of Elasticity.

[9] Knowles, J.K., Finite elastostatic fields with unbounded deformation gradients, in Finite Elasticity, Applied Mechanics Division, ASME, Vol.27 (1977), p.23.

[10] Knowles, J.K., Crack problems in finite elastostatics, Proceedings, Symposium on Mathematical Problems in Fracture, AMS and SIAM, New York, March 1978.

[11] Knowles, J.K., On some inherently nonlinear singular problems in finite elastostatics, Proceedings, Eighth U.S.National Congress of Applied Mechanics, UCLA, June 1978.

[12] Sternberg, E., Some recent advances in the application of nonlinear elastostatics to singular problems, in Trends in Solid Mechanics 1979, Proceedings of the Symposium dedicated to the 65th Birthday of W.T.Koiter, Delft University Press, Sijthoff and Noordhoff International Publishers, 1979.

[13] Abeyaratne, R.C., Discontinuous deformation gradients in plane finite elastostatics of incompressible materials, Technical Report No.41, ONR Contract N00014-75-C-0196, June 1979. To appear in Journal of Elasticity.

[14] Muskhelishvili, N.I., Some basic problems of the mathematical theory of elasticity, Second English edition, P.Noordhoff Ltd., Groningen, 1963.

[15] Abromowitz, M. and Stegun, I.A., eds., Handbook of mathematical functions, National Bureau of Standards, Applied Mathematics Series 55, 1964.

[16] Eshelby, J.D., The continuum theory of lattice defects, Solid State Physics, Vol.3, Academic Press, New York, 1956.

[17] Rice, J.R., A path-independent integral and the approximate analysis of stress concentration by notches and cracks, Journal of Applied Mechanics, 35 (1968) 2, p.379.

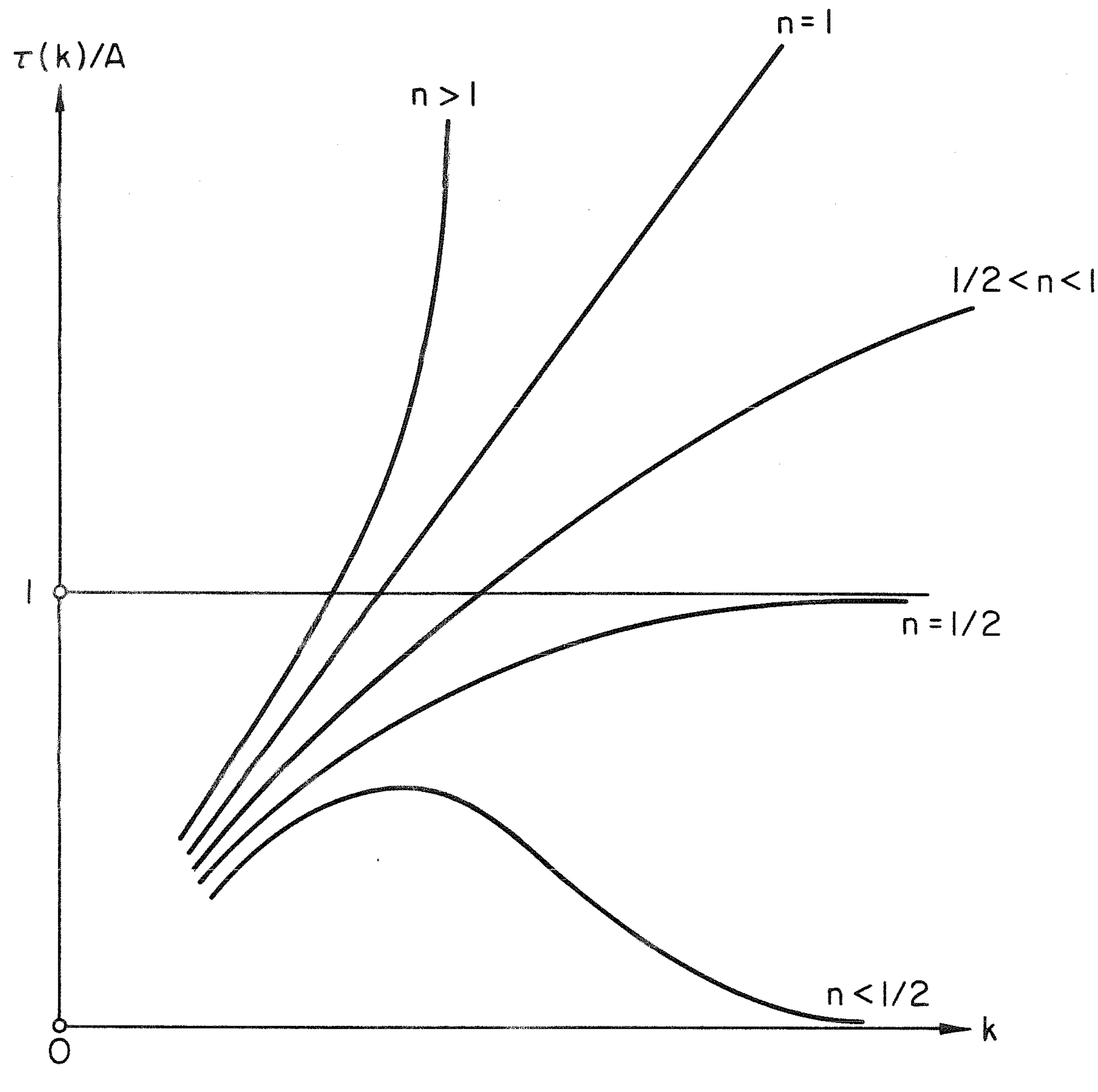

FIGURE 1. POWER-LAW MATERIAL RESPONSE CURVES FOR SIMPLE SHEAR.

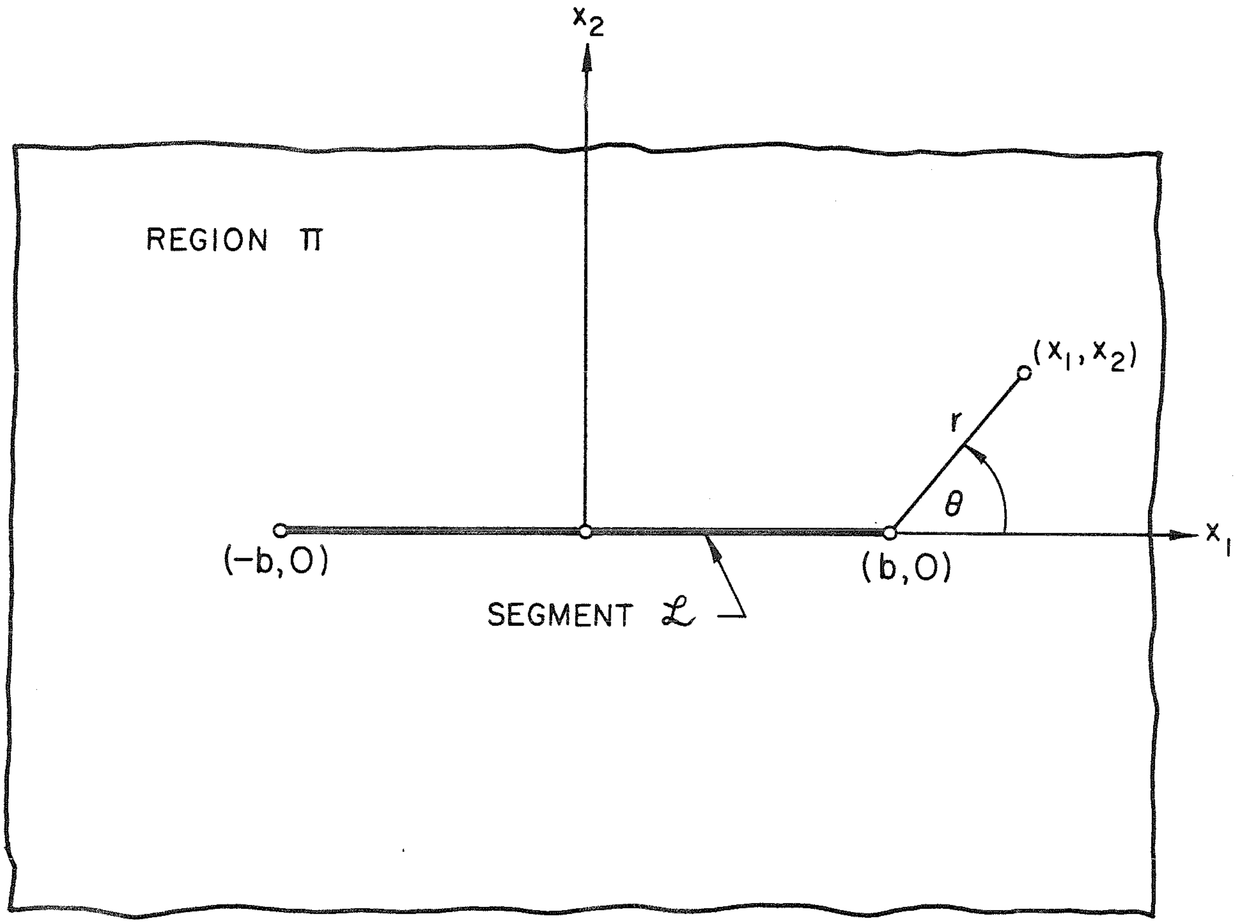

FIGURE 2. SLAB WITH CRACK AND COORDINATES.