FLOW GENERATED BY A SUDDENLY HEATED FLAT PLATE

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ABSTRACT

By employing the two-sided Maxwellian in Maxwell's moment method a kinetic theory description is obtained of the flow generated by a step-function increase in the temperature of an infinite flat plate. Four moments are employed in order to satisfy the three conservation equations, plus one additional equation involving the heat flux in the direction normal to the plate. For a small temperature rise the equations are linearized, and closed-form solutions are obtained for small and large time in terms of the average collision time.

Initially the disturbances propagate along two distinct characteristics, but the discontinuities across these waves damp out as time increases. At large time the main disturbance propagates with the isentropic sound speed. Solutions for mean normal velocity and temperature show the transition from the nearly collision-free regime to the Navier-Stokes-Fourier regime, which is characterized by a boundary layer near the plate surface merging into a diffuse "wave". The classical continuum equations, plus a temperature jump boundary condition, seem to be perfectly adequate to describe the flow beyond a few collision times, provided one accounts properly for the interaction between the inner thermal layer and the outer diffuse wave.

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LIST OF SYMBOLS

a_1	isentropic speed of sound = $\sqrt{(5/3)\ RT_o}$
a_1, a_2	defined by Eq. (46)
A_1, A_2	defined by Eq. (56)
a_1, a_2, a_3	propagation speed of low order waves
b_1, b_2	defined in Eq. (47)
B_1, B_2	defined in Eq. (57)
$\vec{c}$	relative particle velocity = $\vec{\xi} - \vec{u}$
c_1, c_2	defined in Eq. (48)
C_1, C_2	defined in Eq. (58)
c_1, c_2, c_3, c_4	characteristic speeds
C_1, C_2, C_3, C_4	characteristics
c_p	specific heat at constant pressure = $(\gamma R/\gamma - 1)$
D_1, D_2	defined in Eq. (59)
f_1, f_2	velocity distribution function of a particle in region I and II, respectively
F	short hand notation defined as Eq. (55)
G	defined in Eq. (111)
k	Boltzmann constant
k_o	heat conduction coefficient = $(c_p \mu_o/Pr) = (15/4)\ R\mu_o$
K_1, K_2, K_3, K_4	integration constant, defined in Eq. (43)
n_1, n_2	number densities of particles having velocity distribution function f_1, f_2, respectively

$\bar{n}_1, \bar{n}_2$	non-dimensional n_1 and n_2 , $\bar{n} = (n/n_o)$
N_1, N_2	non-dimensional perturbation of number density, $\bar{n} = 1 + N$
$\tilde{N}_1, \tilde{N}_2$	Laplace transformed N_1 and N_2
p	static pressure
p_{ii}	$p_{ii} = P_{ii} + p$
P_{ii}	normal stress = $m \int f\, c_i^{\,2}\, d\vec{\xi}$
$\vec{q}$	heat flux vector = $m \int \vec{c}\, (c^2/2)\, f\, d\vec{\xi}$
q_y	y- component of heat flux
Q	arbitrary function of particle velocity; also as defined in Eqs. (40), (41), and (42)
ΔQ	change in Q produced by collisions
$\vec{R}$	space vector
R	gas constant = k/n
Re	Reynolds number = $\left[(\rho_o R T_o \tau_f)/\mu_o\right] = (\pi/4)$
s	Laplace variable
t	time
$\bar{t}$	non-dimensional time = (t/τ_f)
T	absolute temperature, $(3/2)\, n\, k\, T = m \int f\, (c^2/2)\, d\vec{\xi}$
T_1, T_2	temperature functions appearing in the two-stream Maxwellian
$\bar{T}_1, \bar{T}_2$	non-dimensional T_1 and T_2 , $\bar{T} = (T/T_o)$
t_1, t_2	non-dimensional perturbation of T_1 and T_2
v	normal mean velocity
y	coordinate normal to surface
$\bar{y}$	non-dimensional y, $\bar{y} = \left[y/(\sqrt{RT_o}\, \tau_f\right)$

α	short hand notation of $(1/5)(18\ s^2 + 12\ Re\ s + 5\ Re^2)$
γ	ratio of specific heat, (5/3) for manatomic gas
θ	defined as $\frac{1}{2}(t_1+t_2)$
$\sqrt{\lambda_1}, \sqrt{\lambda_2}$	short hand notation defined in Eq. (44)
μ_o	viscosity coefficient
ν_o	kinematic viscosity coefficient $\nu_o = (\mu_o/\rho_o)$
$\vec{\xi}$	particle velocity vector
ρ	density
τ_f	mean free time between successive collisions
ϕ	dependent variable

Subscripts "1" and "2" denote the two components associated with the two-stream Maxwellian in general, "o" the ambient condition, "w" the conditions on the wall, "in" for inner region, and "ou" for outer region; tilde "$\sim$" denotes the Laplace transformed functions.

I. INTRODUCTION

Since it is impossible to solve the full Maxwell-Boltzmann equation exactly at the present time, various approximations have been suggested, such as the Chapman-Enskog procedure[1], Grad's thirteen moment method[2], Krook's model[3], Lees' moment method[4], etc. The last method was employed in this work. All moment methods satisfy the differential equation in an average sense rather than point-by-point. Gasdynamicists are interested in some lower moments of the distribution function, such as stresses, heat flux, and so on, but rarely in the distribution function itself. Therefore, the gross features of the problem that are obtained by the moment method are satisfactory for many purposes.

Maxwell converted the Maxwell-Boltzmann equation into an integral equation of transfer, or moment equation, for any quantity Q that is a function only of the particle velocity. The distribution function used there should be considered as a suitable weighting function which is not the exact solution of the Maxwell-Boltzmann equation in general. Lees[4] found that the distribution function employed in Maxwell's moment equation must satisfy the following basic requirements:

(1) It must have the "two-sided" character that is an essential feature of highly rarefied gas flows.

(2) It must be capable of providing a smooth transition from free molecule flows to the "continuum" regime.

(3) It should lead to the simplest possible set of differential equations and boundary conditions consistent with (1) and (2).

There are a large number of distribution functions which satisfy requirements (1) and (2). One of the simplest functions is the "two-stream" or two-sided Maxwellian[4]. One important advantage of this choice is that the surface boundary conditions are easily satisfied. The distribution functions f_1 and f_2 in the two-sided Maxwellian involve ten arbitrary functions, $\vec{u}_1$, $\vec{u}_2$, T_1 , T_2 , n_1 , and n_2 , which are determined by taking ten moments (ten equations). In some cases, one can even take less than ten functions; however, the minimum number of functions is the number of conservation equations, plus one, in order to insure that at least one of the lower moment equations(corresponding to stress or heat flux) is satisfied in addition to the conservation laws[4].

After the distribution function is chosen, the collision integral in the moment equation can be evaluated for any arbitrary inter-particle forces. For simplicity we use Maxwell's inverse fifth-power force law; however, this assumption is not essential.

The moment method has been successfully used to solve steady flow problems, such as linearized plane Couette flow[4], compressible plane Couette flow[5], and heat transfer between two concentric cylinders[6]. For unsteady flow problems, only Rayleigh's problem[4] is worked out for the case of $(\Delta T/T) << 1$ and $M^2 << 1$. The present work is to demonstrate the application of the moment method in solving unsteady problems. This work deals with an infinite flat plate resting in a monatomic, dilute gas with a uniform temperature distribution T_o initially. At time 0^+, the plate is suddenly heated to a constant temperature T_w. The flow field will be disturbed by the temperature jump, and wave motions are generated. At the beginning of the motion, $(t/\tau_f) < 1$, there are only a few collisions per molecule; it is always a rarefied gas problem regardless of the gas density. As time goes on, $(t/\tau_f) > 1$ each molecule will experience a large number of collisions, and the flow will reach the Navier-Stokes-Fourier regime. Kinetic theory has to be used in order to treat the problem over the whole range from free molecule to the continuum regime.

II. FORMULATION OF THE PROBLEM

II. 1. Basic Equations and Distribution Functions

The Maxwell Integral equation of transfer can be written in the following form[4]: *

$$(\partial/\partial t)\int f\, Q\, d\vec{\xi} + \vec{\nabla}_{\vec{R}} \cdot \int f \vec{\xi}\, Q\, d\vec{\xi} = \int f\left[(\vec{F}/m) - (\vec{\Omega} \times \vec{\xi})\right] \cdot \vec{\nabla}_{\vec{\xi}}\, Q\, d\vec{\xi} + \Delta Q\,, \tag{1}$$

where ΔQ is the total effect of changes in Q due to collisions, and is given as follows[4]:

$$\Delta Q = \iiiint (Q' - Q)\, f\, f_1\, V\, d\vec{\xi}\, d\vec{\xi}_1\, b\, db\, d\epsilon \quad . \tag{2}$$

In this problem, there is only one independent spatial variable, external forces are ignored, and there is no curvature; hence, the centrifugal force term drops out and Eq. (1) reduces to

$$(\partial/\partial t)\int f\, Q\, d\vec{\xi} + (\partial/\partial y)\int f\, \xi_y\, Q\, d\vec{\xi} = \Delta Q \quad . \tag{3}$$

The distribution function is split in such a way that particles with positive normal velocity are governed by f_1, while those directed toward the plates are described by f_2. Actually, at any point in space at a given instant (y, t), all particles which reach that point from the plate must have normal velocities equal to or larger than (y/t). Consequently, the realistic splitting of the distribution functions should be $f = f_1$ for all particles having normal velocities larger than (y/t), and $f = f_2$ for all particles having normal velocities algebraically smaller than (y/t). The realistic splitting of the distribution function will introduce many complications because of the variable coefficients in

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$$* \int \cdots\, d\vec{\xi} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots\, d\xi_x\, d\xi_y\, d\xi_z$$

the differential equations. The present choice of the distribution function will give discontinuities in the solution near the start of the motion because of the finite number of characteristics with finite wave speeds. It will not affect the "large-time" solution, since the bulk of the wave-like portion of the disturbance will propagate along the isentropic sound waves regardless of the splitting.

The distribution functions are chosen as follows:

$f = f_1$ for all particles having an upward velocity component ($\xi_y > 0$), where

$$f_1 = \frac{n_1 (y, t)}{[2\pi R T_1 (y, t)]^{3/2}} \exp\left(- \frac{\vec{\xi}^{\,2}}{2RT_1 (y, t)} \right) ; \tag{4a}$$

for all particles having $\xi_y < 0$,

$$f = f_2 = \frac{n_2 (y, t)}{[2\pi R T_2 (y, t)]^{3/2}} \exp\left(- \frac{\vec{\xi}^{\,2}}{2RT_2 (y, t)} \right) , \tag{4b}$$

where $n_1 (y, t)$, $n_2(y, t)$, $T_1(y, t)$, and $T_2(y, t)$ are four arbitrary functions to be determined by solving four differential equations obtained by taking moments. Three of the moments will give the conservation equations, and the fourth one is arbitrary. In this problem, the motion is generated by heating, so it seems appropriate to choose the fourth one to correspond to the heat flux in the direction normal to the plate surface.

The four equations are obtained as follows:

For any Q which is a function of the particle velocity only, the average of Q is evaluated as follows:

$$< n\,Q > \; = \int f\,Q\,d\vec{\xi}$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} f_1\,Q\,d\xi_x\,d\xi_y\,d\xi_z + \int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f_2\,Q\,d\xi_x\,d\xi_y\,d\xi_z \;. \tag{5}$$

As examples, let Q be m and $m\xi_y$, respectively. We find

$$\rho = < n\,m > = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} \frac{mn_1\,(y,t)}{[2\pi RT_1(y,t)]^{3/2}}\, e^{-(\xi_x^2+\xi_y^2+\xi_z^2)/2RT_1(y,t)}\, d\xi_x\,d\xi_y\,d\xi_z$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} \frac{mn_2\,(y,t)}{[2\pi RT_2(y,t)]^{3/2}}\, e^{-(\xi_x^2+\xi_y^2+\xi_z^2)/2RT_2(y,t)}\, d\xi_x\,d\xi_y\,d\xi_z$$

$$= (m/2)\,[n_1(y,t) + n_2\,(y,t)]$$

and

$$\rho v = < nm\xi_y > = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} \frac{mn_1\,(y,t)}{[2\pi RT_1(y,t)]^{3/2}}\,\xi_y\, e^{-(\xi_x^2+\xi_y^2+\xi_z^2)/2RT_1(y,t)}\, d\xi_x\,d\xi_y\,d\xi_z$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} \frac{mn_2\,(y,t)}{[2\pi RT_2(y,t)]^{3/2}}\,\xi_y\, e^{-(\xi_x^2+\xi_y^2+\xi_z^2)/2RT_2(y,t)}\, d\xi_x\,d\xi_y\,d\xi_z$$

$$= m\,n_1\sqrt{(R/2\pi)}\;\sqrt{T_1} - mn_2\;\sqrt{(R/2\pi)}$$

$$= \sqrt{(R/2\pi)}\;\; m\,(n_1\sqrt{T_1} - n_2\sqrt{T_2}\,) \;\;.$$

Knowing how to evaluate the integrals, we can derive the equations as follows:

(1) Equation of Continuity

With $Q = m$, $\Delta Q = 0$; then Eq. (3) gives

$$(\partial/\partial t)\int f\, m\, d\vec{\xi} + (\partial/\partial y)\int f\, m\, \xi_y\, d\vec{\xi} = 0 \text{ , which leads to}$$

$$(\partial/\partial t)\,(n_1 + n_2) + \sqrt{(2R/\pi)}\ (\partial/\partial y)\,(n_1\sqrt{T_1} - n_2\sqrt{T_2}\ ) = 0 \quad . \quad (6)$$

(2) Momentum Equation

With $Q = m\xi_y$, $\Delta Q = 0$; then Eq. (3) gives

$$(\partial/\partial t)\int f\, m\, \xi_y\, d\vec{\xi} + (\partial/\partial y)\int f\, m\, \xi_y\, \xi_y\, d\vec{\xi} = 0 \quad .$$

We obtain

$$\sqrt{(2R/\pi)}\ \ (\partial/\partial t)(n_1\sqrt{T_1} - n_2\sqrt{T_2}\ ) + R\,(\partial/\partial y)(n_1 T_1 + n_2 T_2) = 0 \, . \quad (7)$$

(3) Energy Equation

With $Q = m\,\xi^2/2$, $\Delta Q = 0$, and

$$(\partial/\partial t)\int f\, m\, (\xi^2/2)\, d\vec{\xi} + (\partial/\partial y)\int f\ \xi_y\ m\, (\xi^2/2)\, d\vec{\xi} = 0 \quad ,$$

or

$$(\partial/\partial t)(n_1 T_1 + n_2 T_2) + (4/3)\sqrt{(2R/\pi)}\ \ (\partial/\partial y)\,(n_1 T_1^{3/2} - n_2 T_2^{3/2}) = 0 \, . \quad (8)$$

(4) Heat Flux Equation

$$Q = m\,\xi_y\,(\xi^2/2) \quad .$$

In this case, Q is not a collisional invariant and $\Delta Q \neq 0$. For Maxwell particles, one has[4]

$$\Delta Q = (p/\mu) \left[-(2/3)\, q_y + p_{yy}\, v \right] \quad , \tag{9}$$

where μ is proportional to temperature for Maxwell particles[4], i.e., $(\mu/\mu_o) = T/T_o$, and

$$p = n\,k\,T \quad .$$

Therefore, $(p/\mu) = n\,k\,(T/\mu) = n\,k\,(T_o/\mu_o) = (\rho R T_o)/\mu_o$.

We then obtain the heat flux equation

$$\sqrt{(2R/\pi)}\;(\partial/\partial t)(n_1 T_1^{3/2} - n_2 T_2^{3/2}) + (5/4)\,R\,(\partial/\partial y)(n_1 T_1^{2} + n_2 T_2^{2})$$

$$= -(1/3)\sqrt{(2R/\pi)}\;mR(T_o/\mu_o)\left[(n_1 T_1^{3/2} - n_2 T_2^{3/2})(n_1+n_2) - (5/4)(n_1 T_1 + n_2 T_2)(n_1\sqrt{T_1} - n_2\sqrt{T_2})\right] . \tag{10}$$

Eqs. (6), (7), (8), and (10) are the four basic equations for the four unknowns $n_1(y,t)$, $n_2(y,t)$, $T_1(y,t)$, and $T_2(y,t)$.

All the mean flow quantities are defined by kinetic theory as follows:

$$\rho = \int m\,f\,d\vec{\xi}$$

$$\rho\,u_i = \int m\,f\,\xi_i\,d\vec{\xi}$$

$$\sigma = -m\int \vec{c}\,\vec{c}\,f\,d\vec{\xi} = -pI + \tau \quad ,$$

where I is the identity tensor.

$$\tau = -m\int (\vec{c}\,\vec{c} - 1/3\ c^2 I)\, f\, d\vec{\xi}$$

$$p_{ij} = -m\int f\, c_i\, c_j\, d\vec{\xi}$$

$$P_{ij} = p_{ij} \quad \text{if} \quad i \neq j$$

$$P_{ii} = -p + p_{ii}$$

$$p = n\,k\,T = (2/3)\int m\,(c^2/2)\, f\, d\vec{\xi} = -\sum_i (p_{ii}/3) = \rho\, R\, T$$

$$\vec{q} = m\int \vec{c}\,(c^2/2)\, f\, d\vec{\xi} \quad .$$

For our distribution function [Eqs. (4a) and (4b)] one can determine all the mean quantities uniquely in terms of the four unknown functions:

$$\rho(y,\ t) = \langle n\, m \rangle = \int f\, m\, d\vec{\xi} = (m/2)\ [n_1(y, t) + n_2(y, t)] \tag{11}$$

$$v(y, t) = \frac{\langle n\, m\, \xi_y \rangle}{\langle n\, m \rangle} = (1/\rho)\int f\, m\ \xi_y\ d\vec{\xi}$$

$$= \sqrt{(2R/\pi)}\ \frac{n_1\sqrt{T_1} - n_2\sqrt{T_2}}{n_1 + n_2} \tag{12}$$

$$p(y, t) = \tfrac{1}{2}\, mR(n_1 T_1 + n_2 T_2) - (1/3)(mR/\pi)\, \frac{(n_1\sqrt{T_1} - n_2\sqrt{T_2})^2}{n_1 + n_2} \tag{13}$$

$$P_{yy}(y, t) = -(mR/2)(n_1 T_1 + n_2 T_2) + (mR/\pi)\, \frac{(n_1\sqrt{T_1} - n_2\sqrt{T_2}\ )^2}{n_1 + n_2} \tag{14}$$

and

$$q(y, t) = \sqrt{(2R/\pi)}\ mR\ (n_1 T_1^{\ 3/2} - n_2 T_2^{\ 3/2})$$

$$-(5/4)\sqrt{(2R/\pi)}\ \ mR\ (n_1 T_1 + n_2 T_2)\, \frac{n_1\sqrt{T_1} - n_2\sqrt{T_2}}{n_1 + n_2} \tag{15}$$

$$+ \sqrt{(2R/\pi)}\ \ (mR/\pi)\ \frac{(n_1\sqrt{T_1} - n_2\sqrt{T_2}\ )^3}{(n_1 + n_2)^2} \quad .$$

II. 2. Boundary Conditions

The interaction of impinging particles with a solid surface is a very complicated process, and detailed knowledge of this phenomenon is not available at the present time. Hurlbut[7] suggested that most "engineering surfaces" are rough on the microscopic scale, and hence will provide diffusive reflection as far as tangential momentum is concerned. However, the energy accommodation coefficient α and normal momentum reflection coefficient σ' depend on the details of the process. For most cases they are close to one (provided that the incident particles do not have extremely high energy). We take the simplest case $\alpha = \sigma = \sigma' = 1$. One should bear in mind that the two sided distribution function was chosen so that any surface interaction can easily be incorporated into our analysis.

For completely diffusive reemission $\sigma = \sigma' = \alpha = 1$, the boundary conditions are

(1) At the plate $(y = 0)$ (16a)

$$T_1 = T_w \quad ,$$

and $v = 0$ or $n_1 \sqrt{T_1} - n_2 \sqrt{T_2} = 0$;

(2) Far from the plate $(y \longrightarrow \infty)$ (16b)

all flow quantities such as temperature, density, etc. approach the ambient state.

II. 3. Non-Dimensional Equations and Boundary Conditions

By introducing the following non-dimensional variables,

$$\bar{t} = t/\tau_f \quad , \quad \bar{y} = (y/\tau_f)\sqrt{RT_o}$$

$$\bar{T} = T/T_o \quad \text{and} \quad \bar{n} = n/n_o \quad ,$$

where τ_f is the mean free time or $(1/\tau_f)$ is then the collision frequency defined as[4]

$$\tau_f = (\pi/4)(\mu/p) = (\pi/4)\,\frac{kT/(\frac{3}{2}\,A_2\sqrt{2mK}\,)}{nkT} = (\pi/4)\,\frac{1}{\frac{3}{2}A_2\sqrt{2mK}\;n} \quad ,$$

where A_2 and K are constants. Eqs. (6), (7), (8), and (10) lead to

$$(\partial/\partial\bar{t})(\bar{n}_1 + \bar{n}_2) + \sqrt{(2/\pi)}\;(\partial/\partial\bar{y})\,(\bar{n}_1\sqrt{\bar{T}_1} - \bar{n}_2\sqrt{\bar{T}_2}\,) = 0 \tag{17}$$

$$\sqrt{(2/\pi)}\;(\partial/\partial\bar{t})(\bar{n}_1\sqrt{\bar{T}_1} - \bar{n}_2\sqrt{\bar{T}_2}\,) + (\partial/\partial\bar{y})(\bar{n}_1\bar{T}_1 + \bar{n}_2\bar{T}_2) = 0 \tag{18}$$

$$(\partial/\partial\bar{t})(\bar{n}_1\bar{T}_1+\bar{n}_2\bar{T}_2)+(4/3)\sqrt{(2/\pi)}\;(\partial/\partial\bar{y})\,(\bar{n}_1\bar{T}_1^{\,3/2} - \bar{n}_2\bar{T}_2^{\,3/2}) = 0 \tag{19}$$

and

$$\sqrt{(2/\pi)}\;(\partial/\partial\bar{t})(\bar{n}_1\bar{T}_1^{\,3/2} - \bar{n}_2\bar{T}_2^{\,3/2})+(5/4)(\partial/\partial\bar{y})(\bar{n}_1\bar{T}_1^{\,2} + \bar{n}_2\bar{T}_2^{\,2})$$

$$= -\,(1/3)\,\sqrt{(2/\pi)}\;\mathrm{Re}\Big[(\bar{n}_1\bar{T}_1^{\,3/2} - \bar{n}_2\bar{T}_2^{\,3/2})(\bar{n}_1+\bar{n}_2) - \tfrac{5}{4}(\bar{n}_1\bar{T}_1+\bar{n}_2\bar{T}_2)(\bar{n}_1\sqrt{\bar{T}_1} - \bar{n}_2\sqrt{\bar{T}_2}\,)\Big] \;. \tag{20}$$

There is no free parameter involved since the only parameter

$$\mathrm{Re} = \frac{\rho_o\sqrt{RT_o}\;(\tau_f\sqrt{RT_o})}{\mu_o} = \frac{p_o\tau_f}{\mu_o} = (\pi/4)$$

is a constant. The problem depends mainly on the gas properties through τ_f.

The boundary conditions are

(1) At the plate ($\bar{y} = 0$) (21a)

$$\bar{T}_1 = T_1/T_o = T_w/T_o = \bar{T}_w$$

$$\bar{v} = 0 \quad \text{implies} \quad \bar{n}_1\sqrt{\bar{T}_1} - \bar{n}_2\sqrt{\bar{T}_2} = 0 \; ;$$

(2) Far from the plate ($y \longrightarrow \infty$) (a1b)

all flow quantities go to the ambient value, e.g., $\bar{T}_1 = T_1/T_o = 1$.

III. LINEARIZATION

III. 1. Linearized Equations

For a small initial temperature jump at the plate, i. e., $(T_w - T_o)/T_o << 1$,

$$\bar{n}_1 = (n_1/n_0) = 1 + N_1 + \ldots \tag{22a}$$

$$\bar{n}_2 = (n_2/n_0) = 1 + N_2 + \ldots \tag{22b}$$

$$\bar{T}_1 = T_1/T_o = 1 + t_1 + \ldots \tag{22c}$$

$$\bar{T}_2 = T_2/T_o = 1 + t_2 + \ldots \quad , \tag{22d}$$

where N_1 , N_2 , t_1 , and t_2 are non-dimensional perturbations of number density and temperature appearing in the distribution functions. Substituting Eqs. (22a-d) into Eqs. (17), (18), (19), and (20) and retaining only the first order terms, one obtains the linearized equations,

$$(\partial/\partial\bar{t})(N_1 + N_2) + \sqrt{(2/\pi)}\,(\partial/\partial\bar{y})(N_1 - N_2 + \tfrac{1}{2}\,t_1 - \tfrac{1}{2}\,t_2) = 0 \tag{23}$$

$$\sqrt{(2/\pi)}\,(\partial/\partial\bar{t})(N_1 - N_2 + \tfrac{1}{2}t_1 - \tfrac{1}{2}t_2) + (\partial/\partial\bar{y})(N_1 + N_2 + t_1 + t_2) = 0 \tag{24}$$

$$(\partial/\partial\bar{t})(N_1 + N_2 + t_1 + t_2) + (4/3)\sqrt{(2/\pi)}\,(\partial/\partial\bar{y})\left[(N_1 - N_2 + (3/2)t_1 - (3/2)t_2\right] = 0 \tag{25}$$

$$\sqrt{(2/\pi)}\,(\partial/\partial\bar{t})\left[N_1 - N_2 + (3/2)t_1 - (3/2)t_2\right] + (5/4)(\partial/\partial\bar{y})(N_1 + N_2 + 2t_1 + 2t_2)$$
$$= (1/6)\sqrt{(2/\pi)}\ \mathrm{Re}\left[\,(N_1 - N_2) - (7/2)\,(t_1 - t_2)\,\right] \ . \tag{26}$$

The linearized boundary conditions are

(1) At $\bar{y} = 0$

$$t_1 = (T_w/T_o) - 1 = \overline{T}_w - 1 = t_w$$

$$v = 0 \quad , \quad \text{or} \quad N_1 - N_2 + \tfrac{1}{2} t_1 - \tfrac{1}{2} t_2 = 0 \quad ; \tag{27a}$$

(2) All perturbations vanish as $y \longrightarrow \infty$. (27b)

The relations between the flow quantities and the unknown functions are linearized as follows:

$$\rho = \rho_o + \tfrac{1}{2} \rho_o (N_1 + N_2) = \rho_o \left[1 + (N_1/2) + (N_2/2)\right] \tag{28}$$

$$v = \sqrt{(RT_o/2\pi)} \quad (N_1 - N_2 + \tfrac{1}{2} t_1 - \tfrac{1}{2} t_2) \tag{29}$$

$$P_{yy} = - p_o - \tfrac{1}{2} p_o (N_1+N_2+t_1+t_2) = - p_o \left(1 + \frac{N_1+N_2+t_1+t_2}{2}\right) \tag{30}$$

$$= - p$$

$$T = T_o + \tfrac{1}{2} T_o (t_1+t_2) = T_o \left(1 + \frac{t_1+t_2}{2} \right) \tag{31}$$

$$q_y = (1/4) \sqrt{(2/\pi)} \quad \rho_o (RT_o)^{3/2} \left[(7/2) t_1 - (7/2) t_2 - N_1 + N_2\right]$$

$$= \sqrt{(RT_o/8\pi)} \quad p_o \left[(7/2) t_1 - (7/2) t_2 - N_1 + N_2 \right] \quad . \tag{32}$$

III. 2. Characteristics[8]

Characteristics can be defined as the loci of discontinuities in the dependent variables. Suppose the propagation speed is c for characteristics C; thus

$$(dy/dt) = c \quad \text{on } C \quad . \tag{33}$$

We can write the general form of a first order quasi-linear system for two independent variables as follows:

$$L_i\,[u_i] = A_i\,u_{iy} + B_i\,u_{it} = D_i \quad , \tag{34}$$

where L_i is a linear operator, u_i is the ith dependent variable, A_i , B_i, and D_i are the coefficients in the ith equation.

Along characteristic C, one can replace $(\partial/\partial t)$ by $-\,c(\partial/\partial y)$. Then Eq. (34) can be written in characteristic form

$$L_i\,[u] = (A_i - c\,B_i)\,u_{iy} = D_i \quad ,$$

where c is the characteristic speed which is defined by the vanishing of the determinant

$$|A - c\,B| = 0 \quad . \tag{35}$$

From Eqs. (23), (24), (25), and (26), we find the characteristics

$$\begin{vmatrix} -\bar{c} & \sqrt{(2/\pi)} & 0 & 0 \\ 0 & -\bar{c} & \sqrt{(\pi/2)} & 0 \\ 0 & 0 & -\bar{c} & \frac{4}{3}\sqrt{(2/\pi)} \\ -\frac{5}{4}\sqrt{(\pi/2)} & 0 & \frac{5}{2}\sqrt{(\pi/2)} & -\bar{c} \end{vmatrix} = 0 \quad .$$

Therefore, $\bar{c}^4 - (10/3)\,\bar{c}^2 + (5/3) = 0$

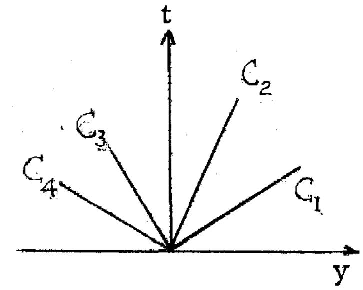
and

$$\bar{c} = (d\bar{y}/d\bar{t}) = \pm\sqrt{(5/3) \pm (\sqrt{10}\,/3\;)}$$

or

$$c = (dy/dt) = \bar{c}\,\sqrt{RT_o} = \pm\sqrt{(5/3) \pm (\sqrt{10}\,/3\;)}\,\sqrt{RT_o} \quad . \tag{36}$$

There are four characteristics because there are four equations for the unknowns N_1 , N_2 , t_1 , and t_2 . The equations are linear with



constant coefficients; hence the characteristics are straight lines. The characteristics define the domain of dependence and the proper number of boundary conditions. This problem can be solved by the characteristic method. The "jump" relations among the four variables can be obtained by integrating the differential equations across the characteristics. Across the fast wave $(d\bar{y}/d\bar{t}) = \bar{c}_1 = \sqrt{(5/3) + (\sqrt{10}/3)}$, and we can write all the discontinuities in terms of one of them, i. e.,

$$\Delta Q_i = f_i (\Delta Q_1)_{C_1} \quad , \quad i = 2, 3, 4,$$

where $(\Delta Q_i)_{C_1}$ is the finite jump of quantity Q_i crossing the fast wave c_1. Across the slow wave, conditions are not that simple since the flow field ahead of the wave is not known a priori there. However, at t = 0, one can write the jump conditions across the slow wave $(d\bar{y}/d\bar{t}) = \bar{c}_2 = \sqrt{(5/3)-(\sqrt{10}/3)}$

$$\Delta Q_i = g_i (\Delta Q_1)_{C_2} \quad , \quad i = 2, 3, 4 \quad .$$

So all quantities behind the slow wave are completely determined in terms of the two jumps in Q_1 at t = 0. But we have two boundary conditions at the plate and hence the two jumps can be obtained. In principle all the quantities can be calculated point by point for t > 0 by using the characteristic relations along the waves. We shall not solve this problem by this numerical method. However, more detailed discussion of the characteristics will be given in Section V.

III. 3. Single Equation for Disturbance

One can combine the four first order partial differential equations, Eqs. (23) - (26), into one single fourth order partial differential equation by eliminating three dependent variables.

$$\left(\frac{\partial^4 \phi}{\partial \bar{t}^4} - \frac{10}{3} \frac{\partial^4 \phi}{\partial \bar{t}^2 \partial y^2} + \frac{5}{3} \frac{\partial^4 \phi}{\partial y^4} \right) + \frac{2}{3} \mathrm{Re} \left(\frac{\partial^3 \phi}{\partial \bar{t}^3} - \frac{5}{3} \frac{\partial^3 \phi}{\partial \bar{t} \, \partial \bar{y}^2} \right) = 0 \quad , \tag{37}$$

All the dependent variables satisfy the same single equation since the problem is linear.

The basic Eq. (37) can also be written in a different form

$$\begin{aligned} &\left(\frac{\partial}{\partial \bar{t}} + \bar{c}_1 \frac{\partial}{\partial \bar{y}} \right)\left(\frac{\partial}{\partial \bar{t}} + \bar{c}_2 \frac{\partial}{\partial \bar{y}} \right)\left(\frac{\partial}{\partial \bar{t}} + \bar{c}_3 \frac{\partial}{\partial \bar{y}} \right)\left(\frac{\partial}{\partial \bar{t}} + \bar{c}_4 \frac{\partial}{\partial \bar{y}} \right) \phi \\ &+ \frac{2}{3} \mathrm{Re} \left(\frac{\partial}{\partial \bar{t}} + \bar{a}_1 \frac{\partial}{\partial \bar{y}} \right)\left(\frac{\partial}{\partial \bar{t}} \quad \bar{a}_2 \frac{\partial}{\partial \bar{y}} \right)\left(\frac{\partial}{\partial \bar{t}} + \bar{a}_3 \frac{\partial}{\partial \bar{y}} \right) \phi = 0 \end{aligned} \tag{38}$$

where

$$\bar{c}_1 = \sqrt{\frac{5}{3} + \frac{\sqrt{10}}{3}} \; , \quad \bar{c}_2 = \sqrt{\frac{5}{3} - \frac{\sqrt{10}}{3}} \; , \quad \bar{c}_3 = -\sqrt{\frac{5}{3} - \frac{\sqrt{10}}{3}} \; , \quad \bar{c}_4 = -\sqrt{\frac{5}{3} + \frac{\sqrt{10}}{3}}$$

and

$$\bar{a}_1 = \sqrt{(5/3)} \; , \quad \bar{a}_2 = 0 \; , \quad \bar{a}_3 = -\sqrt{(5/3)} \quad . \tag{39}$$

For the wave motion defined by Eq. (38), the presence of additional lower order derivatives (waves propagating with speed a_i) will produce an exponential damping, along the higher order waves while the presence of higher order waves will produce a diffusion of the lower order wave motion when $\bar{t} >> 1$. The lowest order terms describe the main disturbances for $\bar{t} >> 1$; in other words the main disturbance moves with speed a at large time. The highest derivatives define the characteristics; from the equations we see that the high order waves are indeed the characteristics with speed c's (Section III. 2). The characteristics play a fundamental role in defining the domain of dependence and the proper number of boundary conditions. An extensive discussion on general wave motion was given by G. B. Whitham[9]. More detailed discussion on this problem will be given in Section V.

IV. SOLUTION BY METHOD OF LAPLACE TRANSFORMS

IV. 1. General Solution in Terms of Transforms

Since one is interested in the problem only after the plate is suddenly heated, i. e., $t > 0$ is the region of interest, the Laplace transformation is an adequate technique to eliminate the independent variable t so that the set of partial differential equations becomes a set of ordinary differential equations. The Laplace transform with zero initial conditions is defined as[10]

$$L\{Q\} = \tilde{Q} = \int_0^\infty e^{-st}\, Q(y,t)\, dt \tag{40}$$

$$L\{(\partial^n Q/\partial t^n)\} = s^n \tilde{Q} \quad , \tag{41}$$

and the inverse transformation is given as

$$Q = (1/2\pi i) \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, \tilde{Q}(y,s)\, ds \quad , \tag{42}$$

where γ is the largest real part of all singularities.

By applying the Laplace transformation to Eq. (37), we have

$$(d^4\tilde{\phi}/d\bar{y}^4) - (2/3)s(3s+Re)(d^2\tilde{\phi}/d\bar{y}^2) + (s^3/5)(3s+2Re)\tilde{\phi} = 0 \quad . \tag{43}$$

The solution of Eq. (43) is

$$\tilde{\phi} = K_1\, e^{-\sqrt{\lambda_1}\,\bar{\eta}} + K_2\, e^{-\sqrt{\lambda_2}\,\bar{\eta}} + K_3\, e^{+\sqrt{\lambda_1}\,\bar{\eta}} + K_4\, e^{+\sqrt{\lambda_2}\,\bar{\eta}} \quad , \tag{44}$$

where

$$\begin{aligned} \sqrt{\lambda_{1,2}} &= \left[(1/3)s(3s+Re) \pm \tfrac{1}{3}\, s\sqrt{\tfrac{1}{5}\,(18s^2+12Re\,s+5Re^2)}\right]^{\frac{1}{2}} \\ &= \left[\, s^2 + (Re/3)\, s \pm (s/3)\, \sqrt{a}\, \right]^{\frac{1}{2}} \end{aligned} \tag{45}$$

and

$$\alpha = (18/5)\ s^2 + (12/5)\ \mathrm{Re}\ s + \mathrm{Re}^2 \text{ for simplicity.} \tag{46}$$

The boundary conditions far from the plate require that $K_3 = K_4 \equiv 0$. Since all the dependent variables satisfy the same equation, we may put

$$\begin{aligned}
\tilde{N}_1 + \tilde{N}_2 &= a_1\ e^{-\sqrt{\lambda_1}\,\bar{y}} + a_2\ e^{-\sqrt{\lambda_2}\,\bar{y}} \\
\tilde{N}_1 - \tilde{N}_2 &= b_1\ e^{-\sqrt{\lambda_1}\,\bar{y}} + b_2\ e^{-\sqrt{\lambda_1}\,\bar{y}} \\
\tilde{t}_1 + \tilde{t}_2 &= c_1\ e^{-\sqrt{\lambda_1}\,\bar{y}} + c_2\ e^{-\sqrt{\lambda_1}\,\bar{y}} \\
\tilde{t}_1 - \tilde{t}_2 &= K_1\ e^{-\sqrt{\lambda_1}\,\bar{y}} + K_2\ e^{-\sqrt{\lambda_1}\,\bar{y}} \quad .
\end{aligned} \tag{47}$$

The coefficients, however, are not linearly independent, and the relation between them is obtained by satisfying the system of equations (23) - (26). By applying the Laplace transformation to Eqs. (23) - (26), and substituting the expressions (47) into the transformed equations, we have

$$\begin{aligned}
& s\,a_1 + \sqrt{(2/\pi)}\ (-\sqrt{\lambda_1}\ b_1) + \tfrac{1}{2}\sqrt{(2/\pi)}\ (-\sqrt{\lambda_1}\ K_1) = 0 \\
& (-\sqrt{\lambda_1}\ a_1) + (-\sqrt{\lambda_1}\ c_1) + \sqrt{(2/\pi)}\ s\ b_1 + \tfrac{1}{2}\sqrt{(2/\pi)}\ s\ K_1 = 0 \\
& s\,a_1 + s\,c_1 + (4/3)\sqrt{(2/\pi)}\ (-\sqrt{\lambda_1} b_1) + 2\sqrt{(2/\pi)}\ (-\sqrt{\lambda_1} K_1) = 0 \\
& (5/4)\ (-\sqrt{\lambda_1}\ a_1) + (5/2)\ (-\sqrt{\lambda_1} c_1) + \sqrt{(2/\pi)}\ s\ b_1 + (3/2)\ \sqrt{(2/\pi)}\ s\ K_1 \\
& \qquad = (1/6)\sqrt{(2/\pi)}\ \mathrm{Re}\ (b_1 - \tfrac{7}{2} K_1) \quad ,
\end{aligned} \tag{48}$$

with an identical set of equations for a_2, b_2, c_2, K_2, λ_2.

By solving this set of algebraic equations for a_1, a_2, ... c_1, c_2, we obtain

$$a_1 = \frac{1}{2}\sqrt{(2/\pi)}\ \frac{-s + Re - (5/3)\sqrt{a}}{s\left(\frac{9}{8}s + \frac{1}{3}Re\right)}\ \sqrt{\lambda_1}\ K_1 \tag{49a}$$

$$a_2 = \frac{1}{2}\sqrt{(2/\pi)}\ \frac{-s + Re + (5/3)\sqrt{a}}{s\left(\frac{9}{8}s + \frac{1}{3}Re\right)}\ \sqrt{\lambda_2}\ K_2 \tag{49b}$$

$$b_1 = \frac{-(17/16)\ s + (1/3)\ Re - (5/6)\sqrt{a}}{(9/8)\ s + (1/3)\ Re}\ K_1 \tag{50a}$$

$$b_2 = \frac{-(17/16)\ s + (1/3)\ Re + (5/6)\sqrt{a}}{(9/8)\ s + (1/3)\ Re}\ K_2 \tag{50b}$$

$$c_1 = (1/6)\sqrt{(2/\pi)}\ \frac{8s + (11/3)\ Re - (5/3)\sqrt{a}}{s\left(\frac{9}{8}s + \frac{1}{3}Re\right)}\ \sqrt{\lambda_1}\ K_1 \quad , \tag{51a}$$

and finally

$$c_2 = (1/6)\sqrt{(2/\pi)}\ \frac{8\ s + (11/3)\ Re + (5/3)\sqrt{a}}{s\left(\frac{9}{8}s + \frac{1}{3}Re\right)}\ \sqrt{\lambda_2}\ K_2 \quad .. \tag{51b}$$

Now K_1 and K_2 can be determined from the boundary conditions at the surface ($y = 0$):

$$\tilde{t}_1 = \frac{1}{2}\left[(\tilde{t}_1 + \tilde{t}_2) - (\tilde{t}_1 - \tilde{t}_2)\right] = (1/s)(\overline{T}_w - 1) = (t_w/s) \tag{52a}$$

$$\tilde{v} = 0 = (\tilde{N}_1 - \tilde{N}_2) + \frac{1}{2}(\tilde{t}_1 - \tilde{t}_2) = 0 \quad . \tag{52b}$$

By utilizing the relations in Eqs. (52a) and (52b) one obtains

$$K_2 = -\left[\frac{-s + Re - (5/3)\sqrt{a}}{-s + Re + (5/3)\sqrt{a}}\right] K_1$$

and

$$K_1 = \frac{T_w - T_o}{T_o}\ \frac{2\left(\frac{9}{8}s + \frac{1}{3}Re\right)\left(-s + Re + \frac{5}{3}\sqrt{a}\right)}{s\ F} \tag{53}$$

$$K_2 = \frac{T_w - T_o}{T_o} \quad \frac{2\left(\frac{9}{8}s + \frac{1}{3}Re\right)\left(s - Re + \frac{5}{3}\sqrt{a}\right)}{s\,F} \quad , \tag{54}$$

where

$$sF = \frac{10}{3} s \left(\frac{9}{8}s + \frac{1}{3}Re\right)\sqrt{a} + \frac{1}{6}\sqrt{(2/\pi)}$$
$$\times \left\{\left[8s + \frac{11}{3}Re - \frac{5}{3}\sqrt{a}\right]\left[-s + Re + \frac{5}{3}\sqrt{a}\right]\sqrt{\lambda_1} + \left[8s + \frac{11}{3}Re + \frac{5}{3}\sqrt{a}\right]\left[s - Re + \frac{5}{3}\sqrt{a}\right]\sqrt{\lambda_2}\right\}. \tag{55}$$

With these relations we can write the complete solution in the following form:

$$\tilde{N}_1 + \tilde{N}_2 = \frac{T_w - T_o}{T_o}\sqrt{\frac{2}{\pi}}\,\frac{1}{s^2F}\Big[\left(-s + Re + \tfrac{5}{3}\sqrt{a}\right)\left(-s + Re - \tfrac{5}{3}\sqrt{a}\right)\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}\,\bar{y}}$$
$$+ \left(s - Re + \tfrac{5}{3}\sqrt{a}\right)\left(-s + Re + \tfrac{5}{3}\sqrt{a}\right)\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}\,\bar{y}}\Big] \tag{56}$$
$$= \frac{T_w - T_o}{T_o}\,\frac{1}{sF}\left[A_1 e^{-\sqrt{\lambda_1}\,\bar{y}} + A_2 e^{-\sqrt{\lambda_2}\,\bar{y}}\right]$$

$$\tilde{N}_1 - \tilde{N}_2 = \frac{T_w - T_o}{T_o}\,\frac{2}{sF}\Big[\left(-s + Re + \tfrac{5}{3}\sqrt{a}\right)\left(-\tfrac{17}{16}s + \tfrac{1}{3}Re - \tfrac{5}{6}\sqrt{a}\right) e^{\sqrt{\lambda_1}\,\bar{y}}$$
$$+ \left(s - Re + \tfrac{5}{3}\sqrt{a}\right)\left(-\tfrac{17}{16}s + \tfrac{1}{3}Re + \tfrac{5}{6}\sqrt{a}\right) e^{\sqrt{\lambda_2}\,\bar{y}}\Big] \tag{57}$$
$$= \frac{T_w - T_o}{T_o}\,\frac{1}{sF}\left[B_1 e^{\sqrt{\lambda_1}\,\bar{y}} + B_2 e^{-\sqrt{\lambda_2}\,\bar{y}}\right]$$

$$\tilde{t}_1 + \tilde{t}_2 = \frac{T_w - T_o}{T_o}\,\frac{1}{3}\sqrt{\frac{2}{\pi}}\,\frac{1}{s^2F}\Big[\left(-s + Re + \tfrac{5}{3}\sqrt{a}\right)\left(8s + \tfrac{11}{3}Re - \tfrac{5}{3}\sqrt{a}\right)\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}\,\bar{y}}$$
$$+ \left(s - Re + \tfrac{5}{3}\sqrt{a}\right)\left(8s + \tfrac{11}{3}Re + \tfrac{5}{3}\sqrt{a}\right)\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}\,\bar{y}}\Big] \tag{58}$$
$$= \frac{T_w - T_o}{T_o}\,\frac{1}{sF}\left[C_1 e^{-\sqrt{\lambda_1}\,\bar{y}} + C_2 e^{-\sqrt{\lambda_2}\,\bar{y}}\right]$$

$$\tilde{t}_1 - \tilde{t}_2 = \frac{T_w - T_o}{T_o} \; \frac{2}{sF}\Big[(-s+Re+\tfrac{5}{3}\sqrt{a}\,)(\tfrac{9}{8}\; s + \tfrac{1}{3}\,Re)\; e^{-\sqrt{\lambda_1}\;\bar{y}}$$

$$+ (s-Re+\tfrac{5}{3}\sqrt{a}\,)(\tfrac{9}{8}\; s+ \tfrac{1}{3}\, Re)\; e^{-\sqrt{\lambda_2}\;\bar{y}}\Big] \qquad (59)$$

$$= \frac{T_w - T_o}{T_o} \; \frac{1}{sF}\Big[D_1\, e^{-\sqrt{\lambda_1}\bar{y}} + D_2\, e^{-\sqrt{\lambda_2}\,\bar{y}}\Big] \quad .$$

IV. 2. Approximate Solutions

The exact inversions of the transformed variables $\tilde{N}_1$, $\tilde{N}_2$, $\tilde{t}_1$, and $\tilde{t}_2$ are impossible. One has to look for approximate solutions. For small time $(t/\tau_t) << 1$ and large time $(t/\tau_f) >> 1$, approximate solutions can be obtained:

IV. 2. 1. Small Time Solution[10]

The regime $\bar{t}$ small corresponds to s large; therefore, one can expand the transformed variables into power series in (1/s), as follows:

$$\sqrt{\lambda_1} = s\Big\{\big[1+(Re/3)(1/s)\big]+ \sqrt{(2/5)}\,\big[1+(2/3)Re(1/s)+(5/18)Re^2(1/s^2)\big]^{\frac{1}{2}}\Big\}^{\frac{1}{2}}$$

$$\cong \sqrt{1 + \sqrt{(2/5)}}\;\; s\,\big[1 + (Re/6)(1/s) + \ldots\big]$$

$$= (1/\bar{c}_1)\,\big[s+(Re/6)+\ldots\big] = 1.28s + .213Re + \ldots \quad ,$$

by retaining the first two leading terms, where

$(1/\bar{c}_1) = \sqrt{1 + \sqrt{(2/5)}} = (d\bar{t}/d\bar{y})$ on characteristic C_1 ; therefore the first term of $\sqrt{\lambda_1}$ will give a time shift, i. e., all the solutions associated with $e^{-\sqrt{\lambda_1}\bar{y}}$ will be functions of $\big[\bar{t} - (\bar{y}/\bar{c}_1)\big]$; therefore, the disturbance propagates along the characteristic $\bar{c}_1 = (d\bar{y}/d\bar{t})$.

Similarly we obtain

$$\sqrt{\lambda_2} = s\left\{\left[1+(Re/3)(1/s)\right] - \sqrt{(2/5)}\left[1+(2/3)Re(1/s)+(5/18)Re^2(1/s^2)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\cong \sqrt{1-\sqrt{(2/5)}}\ s\left[1+(Re/6)(1/s)+\ldots\right]$$

$$= (1/\bar{c}_2)\left[s+(Re/6)+\ldots\right] = .607\ s + .101\ Re + \ldots$$

where

$$(1/\bar{c}_2) = (d\bar{t}/d\bar{y}) = \sqrt{1-\sqrt{(2/5)}} \quad \text{on characteristic } C_2 .$$

The disturbances propagate along the two characteristics for small time. Figures 1 and 2 show the exact values of $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$ and their approximations for small and large time. The agreement is satisfactory.

All the other quantities [Eqs. (56)-(59)] can be expanded as follows for only retaining the first two leading terms:

$$sF \cong 12.5\ s\,(s^2 + .78\ Re\ s + \ldots)$$

$$\cong 12.5\ s\left[(s+.39\ Re)^2 - (.39\ Re)^2\right]$$

$$A_1 \cong -s\left[9.22\ s + 10.4\ Re + \ldots\right]$$

$$A_2 \cong s\left[4.37\ s + 4.92\ Re + \ldots\right]$$

$$B_1 \cong s\left[11.4\ s + 11.73\ Re + \ldots\right]$$

$$B_2 \cong s\left[4.32\ s + 7.24\ Re + \ldots\right]$$

$$C_1 \cong s\left[3.56\ s + 5.92\ Re + \ldots\right]$$

$$C_2 \cong s\left[7.53\ s + 4.55\ Re + \ldots\right]$$

$$D_1 \cong s\left[4.75\, s + 5.97\, Re + \ldots\right]$$

$$D_2 \cong s\left[9.16\, s + 2.91\, Re + \ldots\right] ,$$

where A_1, A_2, ... D_1, D_2 are defined in Eqs. (56), (57), (58), and (59). Hence for the small time regime, the transform solutions are of the form

$$\tilde{g}(s, y) = \frac{\lambda\,(s+\eta) + \mu}{(s+\eta)^2 - \beta^2}\; e^{-\alpha s} ,$$

where α, λ, μ, and η are constants. The inversion for this transform can be easily found to be[11]

$$g(t, y) = e^{-\eta(t-\alpha)}\left\{\lambda \cosh\left[\beta(t-\alpha)\right] + (\mu/\beta)\sinh\left[\beta(t-\alpha)\right]\right\} H(t-\alpha)$$

where

$$H(t-\alpha) = \begin{matrix} 1, \\ 0, \end{matrix} \begin{cases} t > \alpha \\ t < \alpha \end{cases} .$$

Therefore, we have for small time:

$$N_1 + N_2 = \frac{T_w - T_o}{T_o}\left\{ e^{-\frac{Re}{6}\frac{y}{\bar{c}_1}}\, e^{-.39Re(\bar{t} - \frac{y}{\bar{c}_1})}\left[-.74 \cosh(.39Re)(\bar{t} - \frac{y}{\bar{c}_1}) - 1.4 \sinh(.39Re)(\bar{t} - \frac{y}{\bar{c}_1})\right]\right.$$

$$\left. + e^{-\frac{Re}{6}\frac{y}{\bar{c}_2}}\, e^{-.39Re(\bar{t} - \frac{y}{\bar{c}_2})}\left[.35\cosh(.39Re)(\bar{t} - \frac{y}{\bar{c}_2}) + .67 \sinh(.39Re)(\bar{t} - \frac{y}{\bar{c}_2})\right]\right\}$$

$$N_1 - N_2 = \frac{T_w - T_o}{T_o}\left\{ e^{-\frac{Re}{6}\frac{y}{\bar{c}_1}}\, e^{-.39Re(\bar{t} - \frac{y}{\bar{c}_1})}\left[.91\cosh(.39Re)(\bar{t} - \frac{y}{\bar{c}_1}) - 1.5\sinh(.39Re)(\bar{t} - \frac{y}{\bar{c}_1})\right]\right.$$

$$\left. + e^{-\frac{Re}{6}\frac{y}{\bar{c}_2}}\, e^{-.39Re(\bar{t} - \frac{y}{\bar{c}_2})}\left[.35\cosh(.39Re)(\bar{t} - \frac{y}{\bar{c}_2}) + 1.14\sinh(.39Re)(\bar{t} - \frac{y}{\bar{c}_2})\right]\right\}$$

$$t_1+t_2 = \frac{T_w-T_o}{T_o}\left\{e^{-\frac{Re}{6}\frac{\bar{y}}{\bar{c}_1}} e^{-.39Re(\bar{t}-\frac{\bar{y}}{\bar{c}_1})}[.28\cosh(.39Re)(\bar{t}-\frac{\bar{y}}{\bar{c}_1})+.94\sinh(.39Re)(\bar{t}-\frac{\bar{y}}{\bar{c}_1})]\right.$$

$$\left.+ e^{-\frac{Re}{6}\frac{\bar{y}}{\bar{c}_2}} e^{-.39Re(\bar{t}-\frac{\bar{y}}{\bar{c}_2})}[.6\cosh(.39Re)(\bar{t}-\frac{\bar{y}}{\bar{c}_2})+.34\sinh(.39Re)(\bar{t}-\frac{\bar{y}}{\bar{c}_2})]\right\}$$

$$t_1-t_2 = \frac{T_w-T_o}{T_o}\left\{e^{-\frac{Re}{6}\frac{\bar{y}}{\bar{c}_1}} e^{-.39Re(\bar{t}-\frac{\bar{y}}{\bar{c}_1})}[.38\cosh(.39Re)(\bar{t}-\frac{\bar{y}}{\bar{c}_1})+.86\sinh(.39Re)(\bar{t}-\frac{\bar{y}}{\bar{c}_1})]\right.$$

$$\left.+ e^{-\frac{Re}{6}\frac{\bar{y}}{\bar{c}_2}} e^{-.39Re(\bar{t}-\frac{\bar{y}}{\bar{c}_2})}[.74\cosh(.39Re)(\bar{t}-\frac{\bar{y}}{\bar{c}_2})-.14\sinh(.39Re)(\bar{t}-\frac{\bar{y}}{\bar{c}_2})]\right\} .$$

From these relations, one can easily deduce all the flow quantities such as density, velocity, etc. by using Eqs. (28)-(32). The results are

$$\rho = \rho_o + \frac{P_o}{2}\frac{T_w-T_o}{T_o}\left\{ e^{\frac{P_o}{6\mu_o}\frac{y}{c_2}} e^{-.39\frac{P_o}{\mu_o}(t-\frac{y}{c_2})}[.35\cosh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_2})+.67\sinh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_2})]\right.$$

$$\left. - e^{-\frac{P_o}{6\mu_o}\frac{y}{c_1}} e^{-.39\frac{P_o}{\mu_o}(t-\frac{y}{c_1})}[.74\cosh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_1})+1.4\sinh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_1})]\right\} \qquad (60)$$

$$T = T_o + \frac{T_w-T_o}{T_o}\left\{ e^{-\frac{P_o}{6\mu_o}\frac{y}{c_2}} e^{-.39\frac{P_o}{\mu_o}(t-\frac{y}{c_2})}[.6\cosh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_2})+.34\sinh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_2})]\right.$$

$$\left. + e^{-\frac{P_o}{6\mu_o}\frac{y}{c_1}} e^{-.39\frac{P_o}{\mu_o}(t-\frac{y}{c_1})}[.28\cosh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_1})+.94\sinh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_1})]\right\} \qquad (61)$$

$$-P_{\bar{y}\bar{y}} = p = p_o + \frac{P_o}{2}\frac{T_w-T_o}{T_o}\left\{ e^{-\frac{P_o}{\mu_o}\frac{y}{c_2}} e^{-.39\frac{P_o}{\mu_o}(t-\frac{y}{c_2})}[.95\cosh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_2})+\sinh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_2})]\right.$$

$$\left. - e^{-\frac{P_o}{6\mu_o}\frac{y}{c_1}} e^{-.39\frac{P_o}{\mu_o}(t-\frac{y}{c_1})}[.46\cosh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_1})+.46\sinh(.39\frac{P_o}{\mu_o})(t-\frac{y}{c_1})]\right\} \qquad (62)$$

$$v = \sqrt{\frac{RT_o}{2\pi}}\ \frac{T_w - T_o}{T_o}\left\{e^{-\frac{p_o}{6\mu_o}\frac{y}{c_2}}\ e^{-.39\frac{p_o}{\mu_o}(t-\frac{y}{c_2})}\left[.72\cosh(.39\frac{p_o}{\mu_o})(t-\frac{y}{c_2})+1.07\sinh(.39\frac{p_o}{\mu_o})(t-\frac{y}{c_2})\right]\right.$$

$$\left. -e^{-\frac{p_o}{6\mu_o}\frac{y}{c_1}}\ e^{-.39\frac{p_o}{\mu_o}(t-\frac{y}{c_1})}\left[.72\cosh(.39\frac{p_o}{\mu_o})(t-\frac{y}{c_1})+1.07\sinh(.39\frac{p_o}{\mu_o})(t-\frac{y}{c_1})\right]\right\} \quad (63)$$

and

$$q_y = \sqrt{\frac{RT_o}{8\pi}}\ p_o\ \frac{T_w - T_o}{T_o}\left\{e^{-\frac{p_o}{6\mu_o}\frac{y}{c_2}}\ e^{-.39\frac{p_o}{\mu_o}(t-\frac{y}{c_2})}\left[2.24\cosh(.39\frac{p_o}{\mu_o})(t-\frac{y}{c_2})-1.63\sinh(.39\frac{p_o}{\mu_o})(t-\frac{y}{c_2})\right]\right.$$

$$\left. + e^{-\frac{p_o}{6\mu_o}\frac{y}{c_1}}\ e^{-.39\frac{p_o}{\mu_o}(t-\frac{y}{c_1})}\left[2.24\cosh(.39\frac{p_o}{\mu_o})(t-\frac{y}{c_1})+4.51\sinh(.39\frac{p_o}{\mu_o})(t-\frac{y}{c_1})\right]\right\} . \quad (64)$$

At the plate, $y = 0$, Eq. (63) gives $v = 0$ for all time. The heat flux on the surface is then

$$(q_y)_{y=0} = \sqrt{\frac{RT_o}{8\pi}}\ p_o\ \frac{T_w - T_o}{T_o}\left[4.48\cosh(.39\frac{p_o}{\mu_o}t)+2.88\sinh(.39\frac{p_o}{\mu_o}t)\right]e^{-.39\frac{p_o}{\mu_o}t} . \quad (65)$$

The normal stress on the plate is

$$(P_{yy})_{y=0} = -p_o - \frac{p_o}{2}\ \frac{T_w - T_o}{T_o}\left[e^{-.39\frac{p_o}{\mu_o}t}\ \left[.49\cosh(.39\frac{p_o}{\mu_o}t)+.54\sinh(.39\frac{p_o}{\mu_o}t)\right]\right] . \quad (66)$$

At the plate, the normal stress has a finite value at $t = 0^+$. As time goes on, $(P_{yy})_{y=0}$ increases first for a very short time and then decreases. This increase in the normal stress is simply an indication of the acceleration of the gas away from the plate which can be seen from the momentum equation.

The equations show that all the disturbances propagate along the two characteristics $(dy/dt) = c_1$ and $(dy/dt) = c_2$, and the magnitudes of the "jumps" damp out exponentially.

Let us examine the normal velocity. The particles have zero normal velocity at the plate, and they are accelerated to a finite value by crossing the first wave (characteristics), and then decelerated to zero by crossing the other wave. The first wave can be considered as a compression wave which accelerates the gas, and the other wave corresponds to an expansion wave which decelerates the gas. It is clear that the two waves are necessary in order to satisfy the boundary conditions both at the plate and at infinity. In other words, the four moments we took are the minimum number of moments so that meaningful results can be obtained.

The discontinuous behavior of the solutions for small times is caused by the finite number of characteristics on which the disturbances propagate. Smooth solutions are expected either by taking infinite numbers of moments or by a more realistic splitting of the distribution functions.

Some of the quantities are plotted in Figures 3, 4, 5, and 6 for different values of time.

IV. 2. 2. Large Time Solution[10]

The largest real part of the singularities in the expressions for the Laplace transforms is at the origin. For fixed y, large time corresponds to s small, and hence one can expand the transformed variables in a power series in s. We have by retaining only the highest order term

$$\sqrt{\lambda_1} \cong \sqrt{(2/3)\ \mathrm{Re}}\ \sqrt{s} + \ldots$$
$$\sqrt{\lambda_2} \cong \sqrt{(3/5)}\ s + \ldots \qquad .$$

Since $e^{-\sqrt{\lambda_1}\,\bar{y}} \longrightarrow e^{-\sqrt{(2/3)\mathrm{Re}}\,\sqrt{s}\,\bar{y}}$ this transform leads to a solution that is diffusive in nature. On the other hand,

$e^{-\sqrt{\lambda_2}\,\bar{y}} \cong e^{-\sqrt{(3/5)}\,s\,\bar{y}}$ gives a shift of the form

$$\left[\bar{t} - (\bar{y} / \sqrt{(5/3)}\ )\right] = (1/\tau_f)\left[t - \left(\frac{y}{\sqrt{\frac{5}{3}}\,RT_o}\right)\right] = (1/\tau_f)\,(t - \frac{y}{a_1}\ ) ,$$

where $a = \sqrt{(5/3)\ RT_o}$ = isentropic speed of sound. Therefore, for large values of time, the solution has a diffusive part and a wave part; and the main disturbance will propagate along the isentropic sound waves.

Similarly neglecting higher order terms in s, we can obtain the expansions for sF and the other quantities appearing in the transforms.

$$sF \cong 1.48\ \mathrm{Re}^2 \sqrt{s}\,(\sqrt{s} + .39\sqrt{\mathrm{Re}}\,) + \ldots$$

$$A_1 \cong -1.16\ \mathrm{Re}^{5/2}\,(1/\sqrt{s}\,) + \ldots$$

$$A_2 \cong -2.667\ \mathrm{Re}^2 + \ldots$$

$$B_1 \cong 1.111\ \mathrm{Re}^2 + \ldots$$

$$B_2 \cong 1.556\ \mathrm{Re}^2 + \ldots$$

$$C_1 \cong 1.16\ \mathrm{Re}^{5/2}\,(1/\sqrt{s}\,) + \ldots$$

$$C_2 \cong 1.8\ \mathrm{Re}^2 + \ldots$$

$$D_1 \cong 0.738\ \mathrm{Re}^2 + \ldots$$

$$D_2 \cong 0.442\ \mathrm{Re}^2 + \ldots ,$$

where $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, sF , A_1 , $A_2 \ldots D_1$, D_2 are defined in Eqs. (56)-(59), (45), and (55).

The inverse transformations[11] required are as follows:

$$\frac{1}{s(\sqrt{s}+\ell)}\ e^{-a\sqrt{s}} \longrightarrow \frac{1}{\ell}\ \left[\operatorname{erfc}\left(\frac{1}{2}\frac{\alpha}{\sqrt{t}}\right) - e^{\ell\alpha+\ell^2 t}\ \operatorname{erfc}\left(\frac{1}{2}\frac{\alpha}{\sqrt{t}}+\ell\sqrt{t}\right)\right]$$

$$\frac{1}{\sqrt{s}(\sqrt{s}+\ell)}\ e^{-a\sqrt{s}} \longrightarrow e^{\ell\alpha+\ell^2 t}\ \operatorname{erfc}\left(\frac{1}{2}\frac{\alpha}{\sqrt{t}}+\ell\sqrt{t}\right)$$

$$\frac{1}{\sqrt{s}(\sqrt{s}+\ell)}\ e^{-bs} \longrightarrow e^{\ell^2(t-b)}\ \operatorname{erfc}\left(\ell\sqrt{t-b}\right)$$

where

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = (2/\sqrt{\pi}) \int_x^\infty e^{-y^2}\, dy \quad .$$

Then for large values of time and fixed y we have

$$N_1+N_2 = \frac{T_w-T_o}{T_o}\left[-2\operatorname{erfc}\left(\frac{y}{\sqrt{6\nu_o t}}\right)+2\, e^{.32\frac{p_o}{\mu_o}\frac{y}{\sqrt{RT_o}}+.152\frac{p_o}{\mu_o}t}\operatorname{erfc}\left(\frac{y}{\sqrt{6\nu_o t}}+.39\sqrt{\frac{p_o}{\mu_o}t}\right)\right.$$

$$\left.+\ .75\, e^{.152\frac{p_o}{\mu_o}(t-\frac{y}{a_1})}\operatorname{erfc}\left(.39\sqrt{\frac{p_o}{\mu_o}}\sqrt{t-\frac{y}{a_1}}\right)\right]$$

$$t_1+t_2 = \frac{T_w-T_o}{T_o}\left[2\operatorname{erfc}\left(\frac{y}{\sqrt{6\nu_o t}}\right) - 2\, e^{.32\frac{p_o}{\mu_o}\frac{y}{\sqrt{RT_o}}+.152\frac{p_o}{\mu_o}t}\operatorname{erfc}\left(\frac{y}{\sqrt{6\nu_o t}}+.39\sqrt{\frac{p_o}{\mu_o}t}\right)\right.$$

$$\left.+\ .5\, e^{.152\frac{p_o}{\mu_o}(t-\frac{y}{a_1})}\operatorname{erfc}\left(.39\sqrt{\frac{p_o}{\mu_o}}\sqrt{t-\frac{y}{a_1}}\right)\right]$$

$$N_1-N_2 = \frac{T_w-T_o}{T_o}\left[-1.8\, e^{.32\frac{p_o}{\mu_o}\frac{y}{\sqrt{RT_o}}+.152\frac{p_o}{\mu_o}t}\operatorname{erfc}\left(\frac{y}{\sqrt{6\nu_o t}}+.39\sqrt{\frac{p_o}{\mu_o}t}\right)\right.$$

$$\left.+\ 1.05\, e^{.152\frac{p_o}{\mu_o}(t-\frac{y}{a_1})}\operatorname{erfc}\left(.39\sqrt{\frac{p_o}{\mu_o}}\sqrt{t-\frac{y}{a_1}}\right)\right]$$

$$t_1-t_2 = \frac{T_w-T_o}{T_o}\left[1.2\, e^{.32\frac{p_o}{\mu_o}\frac{y}{\sqrt{RT_o}}+.152\frac{p_o}{\mu_o}t}\operatorname{erfc}\left(\frac{y}{\sqrt{6\nu_o t}}+.39\sqrt{\frac{p_o}{\mu_o}t}\right)\right.$$

$$\left.+\ 0.3\, e^{.152\frac{p_o}{\mu_o}(t-\frac{y}{a_1})}\operatorname{erfc}\left(.39\sqrt{\frac{p_o}{\mu_o}}\sqrt{t-\frac{y}{a_1}}\right)\right]$$

All the physical quantities are obtained as follows:

$$\rho = \rho_o + \rho_o \left(\frac{N_1 + N_2}{2} \right)$$

$$= \rho_o + \rho_o \frac{T_w - T_o}{T_o} \Big[- \operatorname{erfc}\left(\frac{y}{\sqrt{6}\nu_o t} \right) + e^{.32 \frac{p_o}{\mu_o} \frac{y}{\sqrt{RT_o}} + .152 \frac{p_o}{\mu_o} t} \operatorname{erfc}\left(\frac{y}{\sqrt{6}\nu_o t} + .39 \sqrt{\frac{p_o}{\mu_o} t} \right)$$

$$+ .375\, e^{.152 \frac{p_o}{\mu_o}\left(t - \frac{y}{a_1}\right)} \operatorname{erfc}\left(.39 \sqrt{\frac{p_o}{\mu_o}} \sqrt{t - \frac{y}{a_1}} \right) \Big] \tag{67}$$

$$v = \sqrt{(RT_o/2\pi)} \left(N_1 - N_2 + \tfrac{1}{2} t_1 - \tfrac{1}{2} t_2 \right)$$

$$= \sqrt{(RT_o/2\pi)} \frac{T_w - T_o}{T_o} \Big[- 1.2\, e^{.32 \frac{p_o}{\mu_o} \frac{y}{\sqrt{RT_o}} + .152 \frac{p_o}{\mu_o} t} \operatorname{erfc}\left(\frac{y}{\sqrt{6}\nu_o t} + .39 \sqrt{\frac{p_o}{\mu_o}} t \right) \tag{68}$$

$$+ 1.2\, e^{.152 \frac{p_o}{\mu_o}\left(t - \frac{y}{a_1}\right)} \operatorname{erfc}\left(.39 \sqrt{\frac{p_o}{\mu_o}} \sqrt{t - \frac{y}{a_1}} \right) \Big]$$

$$p = - P_{yy} = p_o + p_o \left(\frac{N_1 + N_2 + t_1 + t_2}{2} \right)$$

$$= p_o + p_o \frac{T_w - T_o}{T_o} \Big[.625\, e^{.152 \frac{p_o}{\mu_o}\left(t - \frac{y}{a_1}\right)} \operatorname{erfc}\left(.39 \sqrt{\frac{p_o}{\mu_o}} \sqrt{t - \frac{y}{a_1}} \right) \Big] \tag{69}$$

$$T = T_o + T_o \left(\frac{t_1 + t_2}{2} \right)$$

$$= T_o + T_o \frac{T_w - T_o}{T_o} \Big[\operatorname{erfc}\left(\frac{y}{\sqrt{6}\nu_o t} \right) - e^{.32 \frac{p_o}{\mu_o} \frac{y}{\sqrt{RT_o}} + .152 \frac{p_o}{\mu_o} t} \operatorname{erfc}\left(\frac{y}{\sqrt{6}\nu_o t} + .39 \sqrt{\frac{p_o}{\mu_o} t} \right) \tag{70}$$

$$+ .25\, e^{.152 \frac{p_o}{\mu_o}\left(t - \frac{y}{a_1}\right)} \operatorname{erfc}\left(.39 \sqrt{\frac{p_o}{\mu_o}} \sqrt{t - \frac{y}{a_1}} \right) \Big]$$

$$q_y = -\sqrt{(RT_o/8\pi)}\; p_o\left[(N_1 - N_2) - (7/2)(t_1 - t_2)\right]$$

$$= \sqrt{(RT_o/8\pi)}\; p_o \frac{T_w - T_o}{T_o}\left[6\, e^{.32\frac{p_o}{\mu_o}\frac{y}{\sqrt{RT_o}} + .152\frac{p_o}{\mu_o}t}\, \operatorname{erfc}\left(\frac{y}{\sqrt{6\nu_o t}} + .39\sqrt{\frac{p_o}{\mu_o}t}\right)\right] . \tag{71}$$

From the above equations we see that the large time solutions are composed of two parts -- a diffusive part and a wave part. These two effects give a net zero normal velocity at the plate. The diffusive part vanishes rapidly away from the plate, and the solution becomes purely inviscid. The viscous effect of the wave part can be obtained by retaining more terms in the expansion procedure. One can think of this problem as if there were a boundary layer near the wall, and the flow field becomes inviscid away from the plate. Eq. (69) shows that the pressure is a pure wave type (inviscid in this case). This result is expected because the pressure is constant across the boundary layer in the first order approximation; in other words, viscosity will not effect the pressure directly but through the "induced velocity" (Section V.). Eq. (71) gives the heat flux a purely diffusive solution.

The large time solutions obtained here are valid only when y is held fixed. Solutions near the wave front can be deduced by the method of steepest descent keeping (y/t) fixed. However, the complicated expressions in this problem make it difficult to do. In the next section we deduce the solutions near the wave front by a more direct method. At the same time the solutions in the whole field for large time are improved by taking into account the interaction between the thermal boundary layer and the "outer" wave motion. Figures 7, 8, and 10 are the pressure and heat flux on the plate for both small and large values of time.

V. INTERACTION BETWEEN THERMAL BOUNDARY LAYER AND WAVE MOTION

V.I. Existence of a Boundary Layer

According to Section III. 3, the basic equation for the motion is [Eq. (38)]

$$\left(\frac{\partial}{\partial \bar{t}} + \bar{c}_1 \frac{\partial}{\partial \bar{y}}\right)\left(\frac{\partial}{\partial \bar{t}} + \bar{c}_2 \frac{\partial}{\partial \bar{y}}\right)\left(\frac{\partial}{\partial \bar{t}} + \bar{c}_3 \frac{\partial}{\partial \bar{y}}\right)+\left(\frac{\partial}{\partial \bar{t}} + \bar{c}_4 \frac{\partial}{\partial \bar{y}}\right)\phi$$

$$+(2/3)\,\mathrm{Re}\left(\frac{\partial}{\partial \bar{t}} + \bar{a}_1 \frac{\partial}{\partial \bar{y}}\right)\left(\frac{\partial}{\partial \bar{t}} + \bar{a}_2 \frac{\partial}{\partial \bar{y}}\right)\left(\frac{\partial}{\partial \bar{t}} + \bar{a}_3 \frac{\partial}{\partial \bar{y}}\right)\phi = 0 \quad ,$$

or in dimensional form

$$\left(\frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial t} + c_2 \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial t} + c_3 \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial t} + c_4 \frac{\partial}{\partial y}\right)\phi$$

$$+ (2/3)\frac{\mathcal{P}_o}{\mu_o}\left(\frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial t} + a_2 \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial t} + a_3 \frac{\partial}{\partial y}\right)\phi = 0 \quad ,$$

where the c's are the high order wave speeds which define the characteristics, and the a's are the low order wave speeds at which the main disturbances propagate when $(t/\tau_f) >> 1$.

The behavior of the various wave motions can be found using the principle that along a wave front moving with speed v, the derivatives $(\partial/\partial t)$ and $-v(\partial/\partial y)$ of any quantity are approximately equal. The wave motions corresponding to c_1 and c_2 are then found as follows:

For wave c_1

$$\left(\frac{\partial}{\partial \bar{t}} + \bar{c}_1 \frac{\partial}{\partial \bar{y}} \right)(\bar{c}_2 - \bar{c}_1)(\bar{c}_3 - \bar{c}_1)(\bar{c}_4 - \bar{c}_1)\, (\partial^3\phi/\partial\bar{y}^3)$$

$$+ (2/3)\ \mathrm{Re}\ (\bar{a}_1 - \bar{c}_1)(\bar{a}_2 - \bar{c}_1)(\bar{a}_3 - \bar{c}_1)\, (\partial^3\phi/\partial\bar{y}^3) = 0 \quad .$$

Therefore,

$$(\partial\phi/\partial\bar{t}) + \bar{c}_1\ (\partial\phi/\partial\bar{y}) + (2/3)\ \mathrm{Re}\ \frac{(\bar{a}_1 - \bar{c}_1)(\bar{a}_2 - \bar{c}_1)(\bar{a}_3 - \bar{c}_1)}{(\bar{c}_2 - \bar{c}_1)(\bar{c}_3 - \bar{c}_1)(\bar{c}_4 - \bar{c}_1)}\ \phi = 0$$

and

$$\phi = g_1\left(\bar{t} - \frac{\bar{y}}{\bar{c}_1}\right) \exp\left[- (2/3)\mathrm{Re}\ \frac{(\bar{a}_1 - \bar{c}_1)(\bar{a}_2 - \bar{c}_1)(\bar{a}_3 - \bar{c}_1)}{(c_2 - c_1)(c_3 - c_1)(c_4 - c_1)}\ \frac{\bar{y}}{\bar{c}_1} \right] \quad . \tag{72}$$

Similarly for wave c_2 ,

$$\phi = g_2\ \left(\bar{t} - \frac{\bar{y}}{\bar{c}_2}\right) \exp\left[- (2/3)\mathrm{Re}\ \frac{(\bar{a}_1 - \bar{c}_2)(\bar{a}_2 - \bar{c}_2)(\bar{a}_3 - \bar{c}_2)}{(\bar{c}_1 - \bar{c}_2)(\bar{c}_3 - \bar{c}_2)(\bar{c}_4 - \bar{c}_2)}\ \frac{\bar{y}}{\bar{c}_2} \right] , \tag{73}$$

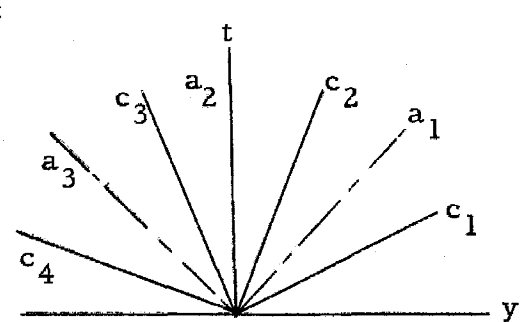
where g_1 and g_2 are determined by the initial and boundary conditions. The wave speeds $\bar{c}_3$ and $\bar{c}_4$ are negative, and these waves do not propagate into the region of interest.

The stability condition requires the exponential functions to be negative. Since $\mathrm{Re} > 0$, we require that

$$\frac{(\bar{a}_1 - \bar{c}_1)(\bar{a}_2 - \bar{c}_1)(\bar{a}_3 - \bar{c}_1)}{(\bar{c}_2 - \bar{c}_1)(\bar{c}_3 - \bar{c}_1)(\bar{c}_4 - \bar{c}_1)} > 0$$

or[9]

$$\bar{c}_4 < \bar{a}_3 < \bar{c}_3 < \bar{a}_2 < \bar{c}_2 < \bar{a}_1 < \bar{c}_1 \quad .$$



For convenience we repeat the $\bar{a}$'s and $\bar{c}$'s here.

$$\bar{c}_1 = \sqrt{(5/3) + (\sqrt{10}/3)}\ ,\quad \bar{c}_2 = \sqrt{(5/3) - (\sqrt{10}/3)},\quad \bar{c}_3 = -\bar{c}_2\ ,\quad \bar{c}_4 = -\bar{c}_1$$

$$\bar{a}_1 = \sqrt{(5/3)}\ ,\quad \bar{a}_2 = 0\ ,\quad a_3 = -a_1\ .$$

Evidently they do indeed satisfy the stability condition.

It is clear from Eqs. (72) and (73) that the presence of the low order waves gives an exponential damping to the high order waves.

If $(2/3)\,\mathrm{Re}\,(\bar{y}/\bar{c}_1) >> 1$, or $(2/3)\,\mathrm{Re}\,\bar{t} >> 1$ $(\mathrm{Re}\,\bar{t} = \frac{p_0}{\mu_0}\,t)$, the exponential decay of the high order waves is accentuated, and the high order waves can be neglected; in other words, for large values of time, the wave motion is dominated by the low order waves.

The equation corresponding to the wave motion at speed a_1 is obtained by replacing $(\partial/\partial t)$ by $-a_1(\partial/\partial y)$ in Eq. (38). We have

$$(\bar{c}_1 - \bar{a}_1)(\bar{c}_2 - \bar{a}_1)(\bar{c}_3 - \bar{a}_1)(\bar{c}_4 - \bar{a}_1)\,(\partial^4\phi/\partial\bar{y}^4)$$

$$+ (2/3)\,\mathrm{Re}\,(\bar{a}_2 - \bar{a}_1)(\bar{a}_3 - \bar{a}_1)\left(\frac{\partial}{\partial\bar{t}} + \bar{a}_1\frac{\partial}{\partial\bar{y}}\right)(\partial^2\phi/\partial y^2) = 0\ ,$$

$$(\partial\phi/\partial\bar{t}) + \bar{a}_1\,(\partial\phi/\partial\bar{y}) = (1/2\mathrm{Re})\,(\partial^2\phi/\partial\bar{y}^2) \qquad (74)$$

or in dimensional form,

$$(\partial\phi/\partial t) + a_1\,(\partial\phi/\partial y) = (RT_0/2)\,\frac{1}{(p_0/\mu_0)}\,\frac{\partial^2\phi}{\partial y^2} = \frac{\nu_0}{2}\,\frac{\partial^2\phi}{\partial y^2}$$

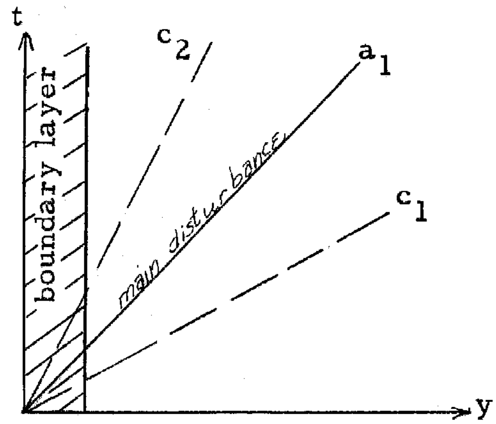
where

$$\nu_0 = (\mu_0/\rho_0) \quad \text{and} \quad a_1 = \sqrt{(5/3)\,RT_0}\ .$$

This equation represents diffusion of the wave with the diffusion coefficient

$$\frac{RT_0}{2\,(p_0/\mu_0)} = (\nu_0/2)\ .$$

When $(t/\tau_f) > 1$, but not too large, diffusion is unimportant except at the initial wave front $y = a_1 t$, but for $(t/\tau_f) >> 1$ diffusion spreads out from the wave front and is responsible for the ultimate decay of the disturbance.



Only a_1 is positive; therefore, a_1 is the only low order wave propagating into the fluid. Consequently we have only one equation for large values of time. But we have two boundary conditions (the number of boundary conditions equal to the number of characteristics pointing into the fluid, i. e., with positive speed); therefore, a boundary layer at the plate surface is required, and this boundary layer grows with time. The growing thermal boundary layer at the plate surface produces an expansion of the gas "outside" the layer, and this boundary layer-induced velocity must be matched to the external wave motion, i. e.,

$$\lim_{\eta \to \infty} v_{inner} (\eta , t) = \lim_{y \to 0} v_{outer} (y, t) \quad , \qquad (75)$$

where $\eta \sim (y/\sqrt{\kappa_0 t})$ is the "proper" distance from the plate surface in the inner (boundary layer) solution. In other words, the induced velocity serves as an effective piston motion at y = 0 for the outer solution. However, the interaction is not unidirectional; the wave generated by the thermal layer preheats the gas, and the thermal layer-induced velocity depends on the difference between the plate temperature and the temperature behind the outgoing wave. The circle is closed by recognizing that the amount of pre-heating itself depends on the induced velocity; this dependence is contained in Eq. (75) and the relation between temperature and velocity in the outer wave-like solution.

V. 2. Boundary Layer Solution (Inner Solution)

Inside the boundary layer $(\partial/\partial\bar{y}) >> (\partial/\partial\bar{t})$, and one is tempted to drop all the derivatives with respect to time in Eq. (38). In that case Eq. (38) becomes

$$\bar{c}_1 \, \bar{c}_2 \, \bar{c}_3 \, \bar{c}_4 \, (\partial^4 \phi/\partial\bar{y}^4) + (2/3) \, \mathrm{Re} \, \bar{a}_1 \, \bar{a}_3 \, (\partial^3 \phi/\partial\bar{t}\partial\bar{y}^2) = 0$$

$$(\partial\phi/\partial\bar{t}) = \left[3/(2\mathrm{Re})\right](\partial^2\phi/\partial\bar{y}^2) \tag{76}$$

or

$$(\partial\phi/\partial t) = (3/2) \, \nu_0 \, (\partial^2\phi/\partial y^2) \quad , \tag{77}$$

corresponding to diffusion with the ordinary thermal diffusivity $\chi_0 = (3/2)\,\nu_0$ where $\mathrm{Pr} = 2/3$. Now the question arises as to the nature of the boundary layer approximations in Eqs. (23) - (26) that lead to an equation of the form of Eq. (76). Evidently in the continuity equation [Eq. (23)] one cannot simply drop the $\bar{t}$- derivative because the quantity $\left[(N_1 - N_2) + \frac{1}{2}(t_1 - t_2)\right]$ is of a smaller order of magnitude than $(N_1 + N_2)$; in fact the terms involving $(\partial/\partial\bar{y})$ and $(\partial/\partial\bar{t})$ are of the same order, as expected. However, in the y-momentum equation [Eq. (24)]

$$(\partial/\partial\bar{y})\,(N_1 + N_2 + t_1 + t_2) \cong 0 \tag{78}$$

or

$$N_1 + N_2 + t_1 + t_2 = f(\bar{t}) \quad . \tag{79}$$

From Eq. (30) we see that $(N_1 + N_2 + t_1 + t_2)$ is the pressure perturbation. Therefore, the physical meaning of Eq. (79) is clear; the static pressure is independent of y inside the boundary layer.

Similarly, the heat flux equation [Eq. (26)] leads to the relation

$$(5/4)(\partial/\partial\bar{y})(N_1+N_2+2t_1+2t_2) = (1/6)\ \ (2/\pi)\ \mathrm{Re}\left[(N_1-N_2)-(7/2)(t_1-t_2)\right]. \quad (80)$$

Substituting Eq. (78) in Eq. (80), we have

$$(5/4)(\partial/\partial\bar{y})(t_1+t_2) = (1/6)\ \ (2/\pi)\ \mathrm{Re}\left[(N_1-N_2)-(7/2)(t_1-t_2)\right], \quad (81)$$

or in dimensional form

$$(5/4)(\partial/\partial y)(t_1+t_2) = (1/6)\ \ (2/\pi)\ \frac{p_o}{\mu_o\sqrt{RT_o}}\left[(N_1-N_2)-(7/2)(t_1-t_2)\right].$$

But we have from Eqs. (31) and (32) that

$$T = T_o + \tfrac{1}{2}T_o\,(t_1+t_2)$$

and

$$q_y = -\,p_o\ \ (RT_o/8\pi)\ \left[(N_1-N_2)-(7/2)(t_1-t_2)\right].$$

Therefore, $\underline{q_y = -\,k_o(\partial T/\partial y)}$, (82)

where

$$k_o = (15/4)\ \mu_o R = (3/2)\ c_p\mu_o ,$$

corresponding to Pr = 2/3 for a monatomic gas. Thus Fourier's law holds inside the boundary layer.

By carefully examining the continuity and energy equations [Eqs. (23) and (25)] it becomes clear that so far as the boundary layer solutions are concerned the variation of static pressure with time is of higher order compared with the time variations of density and temperature. To be specific, suppose that

$$(T/T_o) = 1 + \tfrac{1}{2}(t_1 + t_2) = 1 + \theta = 1 + \theta_{in} + \theta_{ou} \quad ,$$

where the subscripts "in" and "ou" denote the inner (boundary layer) and outer (wave-like) solutions, respectively. Then the boundary layer approximation amounts to stating that

$$(\partial/\partial\bar{t})\left[N_1 + N_2 + t_1 + t_2\right]_{in} \cong 0 \quad ,$$

or [Eq. (79)],

$$(N_1 + N_2)_{in} \cong -(t_1 + t_2)_{in} = -2\,\theta_{in} \quad . \tag{83}$$

In other words the time history of the static pressure is entirely contained in the outer solution. Of course Fourier's law [Eq. (82)] also applies only to the inner solutions.

By recognizing that the quantity $\left[N_1 - N_2 + (3/2)(t_1 - t_2)\right]_{in}$ appearing in the energy equation* can be rewritten as

$$(5/4)\left[(N_1 - N_2) + \tfrac{1}{2}(t_1 - t_2)\right]_{in} - (1/4)\left[(N_1 - N_2) - (7/2)(t_1 - t_2)\right]_{in} \quad ,$$

eliminating the first fracket ($\sim$ v) between the energy and continuity equations, and making use of Eqs. (81) and (83), we obtain

$$(\partial\theta_{in}/\partial t) = (3/2Re)(\partial^2\theta_{in}/\partial\bar{y}^2) \quad , \tag{84}$$

corresponding exactly to Eq. (76). By applying the Laplace transform to Eq. (84) and requiring that θ_{in} be finite as $y \longrightarrow \infty$, we get the solution

---

* This quantity corresponds to the term $(q_y + 5/2\ p_o\ v)$.

$$\tilde{\theta}_{in}(\bar{y}, s) = \tilde{\theta}_{in}(0, s) \exp\left\{-\sqrt{(2/3)\,Re}\,\sqrt{s}\,\bar{y}\right\} . \qquad (85)$$

The boundary conditions on $\tilde{\theta}_{in}(\bar{y}, s)$ are derived as follows: At the plate surface $v = 0$, or $\left[(N_1 - N_2) + \frac{1}{2}(t_1 - t_2)\right] = 0$, and the heat flux equation becomes

$$(5/4)(\partial/\partial\bar{y})(t_1 + t_2) = (1/6)\sqrt{(2/\pi)}\,Re\left[(N_1 - N_2 + \tfrac{1}{2}t_1 - \tfrac{1}{2}t_2) - 4(t_1 - t_2)\right]$$

$$= -(2/3)\sqrt{(2/\pi)}\;Re\,(t_1 - t_2) .$$

Therefore,

$$(\partial/\partial\bar{y})\,(t_1 + t_2)_{y=0} = -(8/15)\sqrt{(2/\pi)}\;Re\,(t_1 - t_2)_{y=0} . \qquad (86)$$

But from the other boundary condition $T_1 = T_w$ at the plate surface, we have

$$T_1 = T_o\,(1 + t_1) = T_w \quad \text{at} \quad \bar{y} = 0 .$$

Therefore,

$$t_1 = \left[(T_w - T_o)/T_o\right] = t_w = \tfrac{1}{2}(t_1 + t_2) + \tfrac{1}{2}(t_1 - t_2) ;$$

so

$$(t_1 - t_2)_{\bar{y}=0} = 2\,t_w - (t_1 + t_2)_{\bar{y}=0} . \qquad (87)$$

By combining Eqs. (86) and (87) one obtains

$$(\partial/\partial\bar{y})(t_1 + t_2) = -(8/15)\sqrt{(2/\pi)}\;Re\left[2\,t_w - (t_1 + t_2)\right] ,$$

or

$$\tfrac{1}{2}(t_1 + t_2) - (15/8Re)\sqrt{(\pi/2)}\;(\partial/\partial\bar{y})\left[\tfrac{1}{2}(t_1 + t_2)\right] = t_w , \qquad (88)$$

at $y = 0$, corresponding to the usual "temperature jump" condition.

Writing this relation in terms of inner and outer temperatures, we have

$$(\theta_{in} + \theta_{ou})_{\bar{y}=0} - (15/8\mathrm{Re})\sqrt{(\pi/2)}\,(\partial/\partial\bar{y})\,(\theta_{in} + \theta_{ou})_{\bar{y}=0} = t_w\,. \tag{89}$$

But $(\partial\theta_{ou}/\partial y)_{y=0} << (\partial\theta_{in}/\partial y)_{y=0}$, so that

$$\theta_{in}(0, \bar{t}) - (15/8\mathrm{Re})\sqrt{(\pi/2)}\,(\partial\theta_{in}/\partial\bar{y})_{\bar{y}=0} = t_w - \theta_{ou}(0, \bar{t}). \tag{90}$$

By applying the Laplace transform to Eq. (90) one obtains

$$\tilde{\theta}_{in}(0, s) - (15/8\mathrm{Re})\sqrt{(\pi/2)}\,(d\tilde{\theta}_{in}/dy)_{y=0} = (t_w/s) - \tilde{\theta}_{ou}(0, s). \tag{91}$$

Substituting $\tilde{\theta}_{in}(y, s)$ from Eq. (85) we obtain

$$\begin{aligned}\tilde{\theta}_{in}(\bar{y}, s) &= \tfrac{1}{2}(\tilde{t}_1 + \tilde{t}_2)_{in} = -\tfrac{1}{2}(\tilde{N}_1 + \tilde{N}_2)_{in} \\ &= (8/5)\sqrt{(\mathrm{Re}/3\pi)}\,\frac{t_w - s\,\tilde{\theta}_{ou}(0, s)}{s(\sqrt{s} + \frac{8}{5}\sqrt{(\mathrm{Re}/3\pi)})}\exp\left\{-\sqrt{\tfrac{2}{3}\mathrm{Re}}\,\sqrt{s}\,\bar{y}\right\}.\end{aligned} \tag{92}$$

By utilizing Eq. (92), integrating the continuity equation, and imposing the condition that

$$\lim_{y\to\infty} \tilde{v}_{in} = 0 = \lim_{y\to\infty} \left[(\tilde{N}_1 - \tilde{N}_2) + \tfrac{1}{2}(\tilde{t}_1 - \tilde{t}_2)\right]_{in},$$

we get

$$\left[(\tilde{N}_1 - \tilde{N}_2) + \tfrac{1}{2}(\tilde{t}_1 - \tilde{t}_2)\right]_{in} = -\frac{8}{5}\,\frac{t_w - s\,\tilde{\theta}_{ou}(0, s)}{\sqrt{s}\,(\sqrt{s} + \frac{8}{5}\sqrt{(\mathrm{Re}/3\pi)})}\exp\left\{-\sqrt{\tfrac{2}{3}\mathrm{Re}}\,\sqrt{s}\,\bar{y}\right\}. \tag{93}$$

Also, from Eq. (81),

$$\left[(\tilde{N}_1 - \tilde{N}_2) - \tfrac{7}{2}(\tilde{t}_1 - \tilde{t}_2)\right]_{in} = -8\left[\frac{t_w - s\,\tilde{\theta}_{ou}(0, s)}{\sqrt{s}\,(\sqrt{s} + \frac{8}{5}\sqrt{\frac{\mathrm{Re}}{3\pi}}}\right]\exp\left[-\sqrt{(2/3)\,\mathrm{Re}}\,\sqrt{s}\,\bar{y}\right].$$

Hence we have

$$(\tilde{t}_1-\tilde{t}_2)_{in} = \frac{8}{5} \frac{t_w - s\ \tilde{\theta}_{ou}(0, s)}{\sqrt{s}\ (\sqrt{s} + \frac{8}{5}\sqrt{\frac{Re}{3\pi}})} \exp(-\sqrt{(2/3)Re}\ \sqrt{s}\ \bar{y}) \tag{94}$$

$$(\tilde{N}_1-\tilde{N}_2)_{in} = -\frac{12}{5} \frac{t_w - s\ \tilde{\theta}_{ou}(0, s)}{\sqrt{s}\ (\sqrt{s} + \frac{8}{5}\sqrt{\frac{Re}{3\pi}})} \exp(-\sqrt{(2/3)Re}\ \sqrt{s}\ \bar{y}) \ . \tag{95}$$

Evidently the downwash on the plate surface ($y = 0$) given by Eq. (93) must be counterbalanced by an equal and opposite upwash furnished by the outer solution.

V. 3. Outer Solution and Matching of Boundary Layer and Wave-Like Solutions

Applying the Laplace transformation to Eq. (74) we have

$$s\tilde{\phi} + \sqrt{(5/3)}\ (d\tilde{\phi}/d\bar{y}) = (1/2Re)(d^2\tilde{\phi}/dy^2) \ . \tag{96}$$

Let $\tilde{\phi} = \frac{1}{2}(\tilde{t}_1 + \tilde{t}_2)_{ou} = \tilde{\theta}_{ou}$, then we have

$$\tilde{\theta}_{ou}(y, s) = \frac{1}{2}(\tilde{t}_1 + \tilde{t}_2)_{ou} = \tilde{\theta}_{ou}(0,s)\ e^{-(\sqrt{2Re}\sqrt{s+(5/6)Re} - \sqrt{(5/3)}Re)\bar{y}} \ . \tag{97}$$

All the other variables can be determined from the differential equations [Eqs. (23) - (26)] .

From Eqs. (24) and (26), we have

$$(\tilde{N}_1 - \tilde{N}_2)_{ou} - \frac{7}{2}(\tilde{t}_1 - \tilde{t}_2)_{ou} = \frac{5}{\sqrt{(2/\pi)}} \frac{1}{s + (2/3)\ Re} (d/d\bar{y})(\tilde{t}_1 + \tilde{t}_2)_{ou}$$

$$= -\frac{10}{\sqrt{(2/\pi)}}\ \tilde{\theta}_{ou}(0, s)\ \frac{\sqrt{2Re}\sqrt{s+(5/6)Re} - \sqrt{(5/3)}\ Re}{s + (2/3)\ Re}\ e^{-(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}Re)\bar{y}} \ .$$

Combining Eqs. (23) and (25), we have

$$(\tilde{N}_1 + \tilde{N}_2)_{ou} = (3/2)(\tilde{t}_1 + \tilde{t}_2)_{ou} - \tfrac{1}{2}\sqrt{(2/\pi)}\ (1/s)(d/d\bar{y})\left[(\tilde{N}_1 - \tilde{N}_2) - \tfrac{1}{2}(\tilde{t}_1 - \tilde{t}_2)\right]_{ou} .$$

Therefore,

$$(\tilde{N}_1 + \tilde{N}_2)_{ou} = \left[3 - 5\frac{\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}\,Re\right)^2}{s(s+\frac{2}{3}Re)}\right]\tilde{\theta}_{ou}(0, s)\ e^{-\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}Re\right)\bar{y}} \tag{98}$$

From Eq. (24) we have

$$(\tilde{t}_1 - \tilde{t}_2)_{ou} = -\frac{1}{4}\frac{1}{\sqrt{\frac{2}{\pi}}}\frac{1}{s}\frac{d}{d\bar{y}}(\tilde{N}_1 + \tilde{N}_2 + \tilde{t}_1 + \tilde{t}_2)_{ou} - \frac{1}{4}\left[(\tilde{N}_1 - \tilde{N}_2) - \frac{7}{2}(\tilde{t}_1 - \tilde{t}_2)\right]_{ou}$$

$$= \frac{5}{4}\frac{1}{\sqrt{\frac{2}{\pi}}}\left[3\frac{(s+\frac{2}{9}Re)\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}\,Re\right)}{s(s+\frac{2}{3}Re)} - \frac{\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}\,Re\right)^3}{s^2(s+\frac{2}{3}Re)}\right]\tilde{\theta}_{ou}(0, s)\ e^{-\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}Re\right)\bar{y}} \tag{99}$$

and finally

$$(\tilde{N}_1 - \tilde{N}_2)_{ou} = \frac{5}{8}\frac{1}{\sqrt{\frac{2}{\pi}}}\left[5\frac{(s+\frac{14}{15}Re)\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}\,Re\right)}{s(s+\frac{2}{3}Re)} - 7\frac{\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}\,Re\right)^3}{s^2(s+\frac{2}{3}Re)}\right]\tilde{\theta}_{ou}(0, s)\ e^{-\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}Re\right)\bar{y}} \tag{100}$$

Since the resultant normal velocity must vanish on the plate

$$\tilde{v}(0, s) = \tilde{v}(0, s)_{in} + \tilde{v}(0, s)_{ou} = 0$$

or

$$(\tilde{N}_1 - \tilde{N}_2 + \tfrac{1}{2}\tilde{t}_1 - \tfrac{1}{2}\tilde{t}_2)_{in} + (\tilde{N}_1 - \tilde{N}_2 + \tfrac{1}{2}\tilde{t}_1 - \tfrac{1}{2}\tilde{t}_2)_{ou} = 0 . \tag{101}$$

Therefore,

$$\frac{5}{\sqrt{\frac{2}{\pi}}}\left[\frac{\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re}\ -\sqrt{\frac{5}{3}}Re\right)}{s} - \frac{\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re}\ -\sqrt{\frac{5}{3}}Re\right)^3}{s^2\ (s+\frac{2}{3}Re)}\right]\tilde{\theta}_{ou}(0,s)$$

$$= \frac{8}{5}\ \frac{t_w - s\,\tilde{\theta}_{ou}(0,s)}{\sqrt{s}\left(\sqrt{s}+\frac{8}{5}\sqrt{\frac{Re}{3\pi}}\right)}$$

$$\tilde{\theta}_{ou}(0,s) = \frac{t_w\ s^{3/2}\ (s+\frac{2}{3}Re)}{s^{5/2}(s+\frac{2}{3}Re)+\frac{25}{8}\sqrt{\frac{\pi}{2}}\left(\sqrt{s}+\frac{8}{5}\sqrt{\frac{Re}{3\pi}}\right)\left[s(s+\frac{2}{3}Re)\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re}-\sqrt{\frac{5}{3}}Re\right)-\left(\sqrt{2Re}\sqrt{s+\frac{5}{6}Re}-\sqrt{\frac{5}{3}}Re\right)^3\right]} \qquad (102)$$

The inverse transformation of Eq. (102) is hopeless; however, some limiting results can be obtained for large time far from the wave front and near the wave front, by means of the approximation method.

V. 4. Nature of Solutions Far from Wave Front and Near Wave Front

V. 4. 1. Far from Wave Front

$$t - (y/a_1) >> \tau_f \ \text{or}\ (\bar{t} - \frac{\bar{y}}{\bar{a}_1}) >> 1$$

Large time far from the wave front corresponds to s small; hence, we can expand all the functions in power series in s.

The exponential function which appears in the outer solutions becomes

$$\sqrt{2Re}\sqrt{s+\frac{5}{6}Re} - \sqrt{\frac{5}{3}}Re = \sqrt{\frac{5}{3}}Re\left(1+\frac{s}{\frac{5}{6}Re}\right)^{\frac{1}{2}} - \sqrt{\frac{5}{3}}Re \cong \frac{s}{\sqrt{\frac{5}{3}}} + \ldots ,$$

corresponding to a wave with propagation speed $\sqrt{(5/3)}$, which is the non-dimensional speed of sound.

The inner solution becomes

$$(\tilde{N}_1+\tilde{N}_2)_{in} = -(\tilde{t}_1+\tilde{t}_2)_{in} \cong - \frac{-\frac{16}{5}\sqrt{\frac{Re}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}} \; \frac{t_w}{s\left(\sqrt{s}+\dfrac{\frac{8}{5}\sqrt{\frac{Re}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}}\right)} \; e^{-\sqrt{\frac{2}{3}Re}\,\sqrt{s}\,\bar{y}}$$

$$(\tilde{t}_1-\tilde{t}_2)_{in} \cong \frac{\frac{8}{5}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}} \; \frac{t_w}{\sqrt{s}\left(\sqrt{s}+\dfrac{\frac{8}{5}\sqrt{\frac{Re}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}}\right)} \; e^{-\sqrt{\frac{2}{3}Re}\,\sqrt{s}\,\bar{y}}$$

and

$$(\tilde{N}_1-\tilde{N}_2)_{in} \cong - \frac{\frac{12}{5}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}} \; \frac{t_w}{\sqrt{s}\left(\sqrt{s}+\dfrac{\frac{8}{5}\sqrt{\frac{Re}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}}\right)} \; e^{-\sqrt{\frac{2}{3}Re}\,\sqrt{s}\,\bar{y}} \; .$$

The outer solutions become

$$(\tilde{t}_1+\tilde{t}_2)_{ou} \cong \frac{\frac{16}{25}\sqrt{\frac{10}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}} \; \frac{t_w}{\sqrt{s}\left(\sqrt{s}+\dfrac{\frac{8}{5}\sqrt{\frac{Re}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}}\right)} \; e^{-(\bar{y}/\sqrt{5/3})s}$$

$$(\tilde{N}_1+\tilde{N}_2)_{ou} \cong \frac{\frac{24}{25}\sqrt{\frac{10}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}} \; \frac{t_w}{\sqrt{s}\left(\sqrt{s}+\dfrac{\frac{8}{5}\sqrt{\frac{Re}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}}\right)} \; e^{-(\bar{y}/\sqrt{5/3})s}$$

$$(\tilde{t}_1-\tilde{t}_2)_{ou} \cong \frac{\frac{2}{5}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}} \; \frac{t_w}{\sqrt{s}\left(\sqrt{s}+\dfrac{\frac{8}{5}\sqrt{\frac{Re}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}}\right)} \; e^{-(\bar{y}/\sqrt{5/3})s}$$

and

$$(\tilde{N}_1-\tilde{N}_2)_{ou} \cong \frac{\frac{7}{5}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}} \; \frac{t_w}{\sqrt{s}\left(\sqrt{s}+\dfrac{\frac{8}{5}\sqrt{\frac{Re}{3\pi}}}{1+\frac{8}{25}\sqrt{\frac{10}{3\pi}}}\right)} \; e^{-(\bar{y}/\sqrt{5/3})s} \; .$$

The appropriate transformations for these equations are

$$\frac{1}{s(\sqrt{s}+\ell)}\, e^{-\alpha\sqrt{s}} \longrightarrow \frac{1}{\ell}\left[\operatorname{erfc}\left(\frac{1}{2}\frac{\alpha}{\sqrt{t}}\right) - e^{\ell\alpha+\ell^2 t}\operatorname{erfc}\left(\frac{1}{2}\frac{\alpha}{\sqrt{t}}+\ell\sqrt{t}\right)\right]$$

$$\frac{1}{\sqrt{s}\,(\sqrt{s}+\ell)}\, e^{-\alpha\sqrt{s}} \longrightarrow e^{\ell\alpha+\ell^2 t}\operatorname{erfc}\left(\frac{1}{2}\frac{\alpha}{\sqrt{t}}+\ell\sqrt{t}\right)$$

and

$$\frac{1}{\sqrt{s}\,(\sqrt{s}+\ell)}\, e^{-bs} \longrightarrow e^{\ell^2(t-b)}\operatorname{erfc}\left(\ell\sqrt{t-b}\right) .$$

In our case $\ell = \dfrac{(8/5)\sqrt{(Re/3\pi)}}{1+(8/25)\sqrt{(10/3\pi)}} = .39\,Re$,

$\alpha = \sqrt{(2/3)Re}\ \bar{y}$, and $b = \bar{y}/\sqrt{(5/3)}$; hence we have

$$(t_1+t_2)_{in} = -(N_1+N_2)_{in}$$

$$= 2\,t_w\left[\operatorname{erfc}\left(\frac{Re\,\bar{y}}{\sqrt{6Re\,\bar{t}}}\right) - e^{.32Re\bar{y}+.152Re\bar{t}}\operatorname{erfc}\left(\frac{Re\,\bar{y}}{\sqrt{6Re\,\bar{t}}}+.39\sqrt{Re\,\bar{t}}\right)\right]$$

$$= 2\,t_w\left[\operatorname{erfc}\left(\sqrt{\frac{y^2}{6\nu_o t}}\right) - e^{.32\frac{\sqrt{RT_o}}{\nu_o}y+.152\frac{p_o}{\mu_o}t}\operatorname{erfc}\left(\sqrt{\frac{y^2}{6\nu_o t}}+.39\sqrt{\frac{p_o}{\mu_o}t}\right)\right]$$

$$(t_1-t_2)_{in} = 1.2\,t_w e^{.32Re\bar{y}+.152Re\bar{t}}\operatorname{erfc}\left(\frac{Re\,\bar{y}}{\sqrt{6Re\,\bar{t}}}+.39\sqrt{Re\,\bar{t}}\right)$$

$$= 1.2\,t_w\, e^{.32\frac{\sqrt{RT_o}}{\nu_o}y+.152\frac{p_o}{\mu_o}t}\operatorname{erfc}\left(\frac{y^2}{6\nu_o t}+.39\sqrt{\frac{p_o}{\mu_o}t}\right)$$

$$(N_1-N_2)_{in} = -1.8\,t_w e^{.32Re\bar{y}+.152Re\bar{t}}\operatorname{erfc}\left(\frac{Re\,\bar{y}}{\sqrt{6Re\,\bar{t}}}+.39\sqrt{Re\,\bar{t}}\right)$$

$$= -1.8\,t_w e^{.32\frac{\sqrt{RT_o}}{\nu_o}y+.152\frac{p_o}{\mu_o}t}\operatorname{erfc}\left(\sqrt{\frac{y^2}{6\nu_o t}}+.39\sqrt{\frac{p_o}{\mu_o}t}\right)$$

The outer solutions are

$$(t_1+t_2)_{ou} = .5\, t_w e^{.152\, Re\, (\bar{t} - \frac{\bar{y}}{\sqrt{5/3}})} \quad \text{erfc}\, (.39\sqrt{Re}\sqrt{\bar{t} - \frac{\bar{y}}{\sqrt{5/3}}}\,)$$

$$= .5\, t_w e^{.152 \frac{p_0}{\mu_0}(t - \frac{y}{a_1})} \quad \text{erfc}\, (.39\sqrt{\frac{p_0}{\mu_0}}\sqrt{t - \frac{y}{a_1}}\,)$$

$$(N_1+N_2)_{ou} = .75\, t_w e^{.152 \frac{p_0}{\mu_0}(t - \frac{y}{a_1})} \quad \text{erfc}\, (.39\sqrt{\frac{p_0}{\mu_0}}\sqrt{t - \frac{y}{a_1}}\,)$$

$$(t_1-t_2)_{ou} = .3\, t_w e^{.152 \frac{p_0}{\mu_0}(t - \frac{y}{a_1})} \quad \text{erfc}\, (.39\sqrt{\frac{p_0}{\mu_0}}\sqrt{t - \frac{y}{a_1}}\,)$$

$$(N_1-N_2)_{ou} = 1.05\, t_w e^{.152 \frac{p_0}{\mu_0}(t - \frac{y}{a_1})} \quad \text{erfc}\, (.39\sqrt{\frac{p_0}{\mu_0}}\sqrt{t - \frac{y}{a_1}}\,)\,,$$

where one must keep in mind that these solutions are valid only when $(t - \frac{y}{a_1}) >> \tau_f$, and not near the wave front.

The mean values can be deduced from the above relations according to Eqs. (28) - (32). We then have (adding inner and outer solutions)

$$\rho = \rho_0 + \rho_0 \left(\frac{N_1 + N_2}{2}\right)$$

$$= \rho_0 + \rho_0 \frac{T_w - T_0}{T_0}\Big[.375\, e^{.152 \frac{p_0}{\mu_0}(t - \frac{y}{a_1})} \text{erfc}\, (.39\sqrt{\frac{p_0}{\mu_0}}\sqrt{t - \frac{y}{a_1}}\,) \tag{103}$$

$$- \text{erfc}\left(\frac{y}{\sqrt{6\nu_0 t}}\right) + e^{.32 \frac{\sqrt{RT_0}}{\nu_0} y + .152 \frac{p_0}{\mu_0} t} \text{erfc}\left(\frac{y}{\sqrt{6\nu_0 t}} + .39\sqrt{\frac{p_0}{\mu_0}}\, t\right)$$

$$v = \sqrt{\frac{RT_0}{2\pi}}\; \frac{T_w - T_0}{T_0}\left(N_1 - N_2 + \tfrac{1}{2} t_1 - \tfrac{1}{2} t_2\right)$$

$$= 1.2\sqrt{\frac{RT_0}{2\pi}}\; \frac{T_w - T_0}{T_0}\Big[e^{.152 \frac{p_0}{\mu_0}(t - \frac{y}{a_1})} \quad \text{erfc}\, (.39\sqrt{\frac{p_0}{\mu_0}}\sqrt{t - \frac{y}{a_1}}\,) \tag{104}$$

$$- e^{.32 \frac{\sqrt{RT_0}}{-\nu_0} y + .152 \frac{p_0}{\mu_0} t} \quad \text{erfc}\left(\frac{y}{\sqrt{6\nu_0 t}} + .39\sqrt{\frac{p_0}{\mu_0}}\; t\right)\Big]$$

$$p = p_o + \frac{1}{2} p_o \frac{T_w - T_o}{T_o} (N_1 + N_2 + t_1 + t_2)$$

$$= p_o + .625\, p_o \frac{T_w - T_o}{T_o} \left[e^{.152 \frac{\rho_o}{\mu_o}(t - \frac{y}{a_1})} \operatorname{erfc} \left(.39 \sqrt{\frac{\rho_o}{\mu_o}} \sqrt{t - \frac{y}{a_1}}\right) \right]. \tag{105}$$

$$T = T_o + \frac{1}{2} T_o \frac{T_w - T_o}{T_o} (t_1 + t_2)$$

$$= T_o + T_o \frac{T_w - T_o}{T_o} \left[.25\; e^{.152 \frac{\rho_o}{\mu_o}(t - \frac{y}{a_1})} \operatorname{erfc} \left(.32 \frac{\rho_o}{\mu_o} \sqrt{t - \frac{y}{a_1}}\right) \right. \tag{106}$$

$$\left. + \operatorname{erfc} \frac{y}{\sqrt{6 \nu_o t}} - e^{.32 \frac{\sqrt{RT_o}}{\nu_o} y + .152 \frac{\rho_o}{\mu_o} t} \operatorname{erfc} \left(\frac{y}{\sqrt{6 \nu_o t}} + .39 \sqrt{\frac{\rho_o}{\mu_o}}\, t \right) \right]$$

and

$$q_y = - \sqrt{(RT_o/8\pi)}\, \rho_o \left[N_1 - N_2 - (7/2)(t_1 - t_2) \right]$$

$$= 6 \sqrt{\frac{RT_o}{8\pi}}\, p_o \frac{T_w - T_o}{T_o} \left[e^{.32 \frac{\sqrt{RT_o}}{\nu_o} y + .152 \frac{\rho_o}{\mu_o} t} \operatorname{erfc} \left(\frac{y}{\sqrt{6 \nu_o t}} + .39 \sqrt{\frac{\rho_o}{\mu_o}}\, t \right) \right]. \tag{107}$$

When both $\bar{t}$ and $(\bar{t} - \frac{\bar{y}}{\bar{a}_1})$ are very large, the outer solutions behave like

$$\frac{1}{\sqrt{Re\,(\bar{t} - \frac{\bar{y}}{\bar{a}_1})}} .$$

This is the inviscid solution of the wave. The viscous effect appears near the wave front.

V. 4. 2. Behavior Near the Wave Front

Near the wave front, the solution can be obtained by using the method of steepest descent.[16] The detailed evaluation is given in the Appendix. The results are

$$(t_1+t_2)_{ou} \cong \frac{T_w-T_0}{T_0}\,\frac{.718}{\left(\frac{p_0}{\mu_0}t\right)^{\frac{1}{4}}}\left[\frac{\left(\frac{5}{3}\right)^2\left(\frac{p_0}{\mu_0}\right)^2\left(t-\frac{y}{a_1}\right)^2}{8\left(5/6\right)\left(p_0/\mu_0\right)t}\right]^{\frac{1}{4}}\exp\left[-\frac{\left(\frac{5}{3}\right)^2\left(\frac{p_0}{\mu_0}\right)^2\left(t-\frac{y}{a_1}\right)^2}{8\left(\frac{5}{6}\right)\left(\frac{p_0}{\mu_0}\right)t}\right]$$
$$\left\{I_{-\frac{1}{4}}\left[\frac{\left(\frac{5}{3}\right)^2\left(\frac{p_0}{\mu_0}\right)^2\left(t-\frac{y}{a_1}\right)^2}{8\left(\frac{5}{6}\right)\left(\frac{p_0}{\mu_0}\right)t}\right]+\operatorname{sign}\left(t-\frac{y}{a_1}\right)I_{\frac{1}{4}}\left[\frac{\left(\frac{5}{3}\right)^2\left(\frac{p_0}{\mu_0}\right)^2\left(t-\frac{y}{a_1}\right)^2}{8\left(\frac{5}{6}\right)\left(\frac{p_0}{\mu_0}\right)t}\right]\right\} \quad (108)$$

provided $(t - \frac{y}{a_1})$ is bounded as $t \longrightarrow \infty$.

Near the wave front, i. e.,

$$\frac{p_0}{\mu_0}\,\frac{\left(t-\frac{y}{a_1}\right)^2}{t} << 1,$$

the Bessel functions can be expanded in power series. The solution then becomes

$$(t_1+t_2)_{ou} \cong \frac{T_w-T_0}{T_0}\left[\frac{\left(\frac{4}{15}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)}\,\frac{1}{\left(\frac{p_0}{\mu_0}t\right)^{\frac{1}{4}}}+\frac{\left(\frac{5}{108}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)}\,\frac{\frac{p_0}{\mu_0}\left(t-\frac{y}{a_1}\right)}{\left(\frac{p_0}{\mu_0}t\right)^{3/4}}\right]\exp\left[-\frac{\left(\frac{5}{3}\right)^2\left(\frac{p_0}{\mu_0}\right)^2\left(t-\frac{y}{a_1}\right)^2}{8\left(\frac{5}{6}\right)\left(\frac{p_0}{\mu_0}t\right)}\right] \quad (109)$$

"Far" from the wave front,

$$\left(\frac{p_0}{\mu_0}\,\frac{\left(t-y/a_1\right)^2}{t} >> 1\right)$$

the solution becomes

$$(t_1+t_2)_{ou} \cong 2\sqrt{\frac{2}{5\pi}}\,\frac{1}{\sqrt{\frac{p_0}{\mu_0}\left(t-\frac{y}{a_1}\right)}}$$

Thus it merges into the inviscid wave solution. However, Eq. (108) is not valid too far from the wave front; it bridges the gap between the wave front Eq. (109) and the inviscid solution obtained in Section V. 4. 1.

The flow quantities are thus

$$\rho = \rho_0 + \rho_0 \frac{T_w - T_0}{T_0} \frac{.54}{(\frac{p_0}{\mu_0})^{\frac{1}{4}}} G^{\frac{1}{4}} e^{-G}\left[I_{-\frac{1}{4}}(G) + \mathrm{sign}(t - \frac{y}{a_1}) I_{\frac{1}{4}}(G)\right] \qquad (110)$$

where

$$G = \frac{(\frac{5}{3})^2 (\frac{p_0}{\mu_0})^2 (t - \frac{y}{a_1})^2}{8 (\frac{5}{6})(\frac{p_0}{\mu_0} t)} \qquad (111)$$

$$v = 2.88 \frac{T_w - T_0}{T_0} \sqrt{\frac{R T_0}{2\pi}} \left(\frac{G}{\frac{p_0}{\mu_0} t}\right)^{\frac{1}{4}} e^{-G}\left[I_{-\frac{1}{4}}(G) + \mathrm{sign}(t - \frac{y}{a_1}) I_{\frac{1}{4}}(G)\right] \qquad (112)$$

$$p = p_0 + p_0 \frac{T_w - T_0}{T_0} (.9) \left(\frac{G}{\frac{p_0}{\mu_0} t}\right)^{\frac{1}{4}} e^{-G}\left[I_{-\frac{1}{4}}(G) + \mathrm{sign}(t - \frac{y}{a_1}) I_{\frac{1}{4}}(G)\right] \qquad (113)$$

$$T = T_0 + (T_w - T_0)(.36) \left(\frac{G}{\frac{p_0}{\mu_0} t}\right)^{\frac{1}{4}} e^{-G}\left[I_{-\frac{1}{4}}(G) + \mathrm{sign}(t - \frac{y}{a_1}) I_{\frac{1}{4}}(G)\right] \qquad (114)$$

The heat flux behaves slightly differently. The result is (See Appendix):

$$\dot{q}_y = -\frac{5}{4} \sqrt{\frac{5}{2\pi}} \sqrt{\frac{R T_0}{8\pi}}\, p_0 \frac{T_w - T_0}{T_0} \left(\frac{G}{\frac{5}{48} \frac{p_0}{\mu_0} t}\right)^{\frac{3}{4}} e^{-2G}$$
$$\left[\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{2})} {}_1F_1(-\tfrac{1}{4}; \tfrac{1}{2}; 2G) - \mathrm{sign}(t - \tfrac{y}{a_1}) {}_1F_1(\tfrac{1}{4}; \tfrac{3}{2}; 2G)\right] \qquad (115)$$

where ${}_1F_1(\frac{\nu}{2} - \frac{\mu}{2} + 1; \nu + 1; \frac{a^2}{4p^2})$ is the hypergeometric function.[17]

Near the wave front, more precisely

$$\frac{p_0}{\mu_0}\frac{\left(t-\frac{y}{a_1}\right)^2}{t} << 1$$

the above relations become

$$\rho = \rho_0 + \rho_0 \frac{T_w - T_0}{T_0}(.75)\left[\frac{\left(\frac{4}{15}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\left(\frac{p_0}{\mu_0}t\right)^{\frac{1}{4}}} + \frac{\left(\frac{5}{108}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)}\frac{\frac{p_0}{\mu_0}\left(t-\frac{y}{a_1}\right)}{\left(\frac{p_0}{\mu_0}t\right)^{3/4}}\right]e^{-G} \tag{116}$$

$$v = 4\,\frac{T_w - T_0}{T_0}\sqrt{\frac{RT_0}{2\pi}}\left[\frac{\left(\frac{4}{15}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\left(\frac{p_0}{\mu_0}t\right)^{\frac{1}{4}}} + \frac{\left(\frac{5}{108}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)}\frac{\frac{p_0}{\mu_0}\left(t-\frac{y}{a_1}\right)}{\left(\frac{p_0}{\mu_0}t\right)^{3/4}}\right]e^{-G} \tag{117}$$

$$p = P_0 + P_0 \frac{T_w - T_0}{T_0}(1.25)\left[\frac{\left(\frac{4}{15}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\left(\frac{p_0}{\mu_0}t\right)^{\frac{1}{4}}} + \frac{\left(\frac{5}{108}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)}\frac{\frac{p_0}{\mu_0}\left(t-\frac{y}{a_1}\right)}{\left(\frac{p_0}{\mu_0}t\right)^{3/4}}\right]e^{-G} \tag{118}$$

$$T = T_0 + \frac{T_w - T_0}{2}\left[\frac{\left(\frac{4}{15}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\left(\frac{p_0}{\mu_0}t\right)^{\frac{1}{4}}} + \frac{\left(\frac{5}{108}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)}\frac{\frac{p_0}{\mu_0}\left(t-\frac{y}{a_1}\right)}{\left(\frac{p_0}{\mu_0}t\right)^{3/4}}\right]e^{-G} \tag{119}$$

On the wave front, $G = 0$, the solutions decrease with time like $t^{-1/4}$. However, the disturbance is not symmetric about the wave front. Ahead of the wave front the amplitudes all decrease much faster than behind the wave front.

VI. DISCUSSION AND CONCLUSION

The solutions obtained here show that the motion is of the wave type initially $[(t/\tau_f) < 1]$, and later $[(t/\tau_f) >> 1]$ is the result of the interaction between heat diffusion near the plate and the wave motion. The disturbances propagate along the characteristics for small values of time. The presence of low order waves provides diffusion of the "discontinuities" across the high order waves (characteristics). The "jumps" eventually damp out to zero as time increases. At large values of time, the main disturbance propagates on the low order wave, which is the isentropic sound wave in the linear case. The wave is practically inviscid everywhere, except near the wave front, where diffusion dominates. Near the plate there is a boundary layer that makes it possible to satisfy the boundary conditions, and also provides an interaction between the viscous and inviscid flows.

The viscous parts of the solutions inside the boundary layer damp out very fast and leave the motion inviscid. For large values of time and far from the wave front, the inviscid solution goes like $1/\sqrt{t - (y/a_1)}$. However, near the wave front, diffusion is important. On the wave front, the amplitudes decrease like $t^{-1/4}$. The damping is not symmetric about the wave front; the disturbance damps out faster ahead of the wave and slower behind the wave.

As mentioned before, the kinetic theory is capable of describing the flow field for all densities, from the free molecule flow at very low density to the Navier-Stokes continuum regime. Rayleigh's problem for $(t/\tau_f) << 1$ has been worked out by Yang and Lees[13] by using the collisionless Boltzmann equation. The free molecule solutions obtained there can be used as a basis for comparison for our solutions at the beginning of the motion.

As $\bar{t} \longrightarrow 0$, Eqs. (60), (61), (62), and (64) give the following results on the plate:

$$\lim_{\bar{t}\to 0} \rho\,(0,\ \bar{t}) = \rho_o - 0.2\left[(T_w - T_o)/T_o\right]\rho_o$$

$$\lim_{\bar{t}\to 0} T\,(0,\ \bar{t}) = T_o + .44\,(T_w - T_o)$$

$$\lim_{\bar{t}\to 0} p\,(0,\ \bar{t}) = -P_{yy}\,(0,\ \bar{t}) = p_o + (1/4)\,p_o\left[(T_w - T_o)/T_o\right]$$

and

$$\lim_{\bar{t}\to 0} q_y\,(0, \bar{t}) = 1.1\,p_o\,\sqrt{(2RT_o)/\pi}\,\left[(T_w - T_o)/T_o\right] .$$

Eqs. (3.5), (3.16), (3.27), and (3.35) from Reference 13 give the results for diffusive reemission:

i.e., $\alpha = 0$, and for $\left[(T_w - T_o)/T_o\right] << 1$

$$\lim_{\bar{t}\to 0} \rho\,(0,\ \bar{t}) = \rho_o\left[1 + \tfrac{1}{2}\left(\sqrt{T_o/T_w} - 1\right)\right]$$

$$= \rho_o - .25\,\rho_o\left[(T_w - T_o)/T_o\right]$$

$$\lim_{\bar{t}\to 0} T\,(0,\ \bar{t}) = T_o\,\frac{1 + \sqrt{T_w/T_o}}{1 + \sqrt{T_o/T_w}}$$

$$= T_o + \tfrac{1}{2}\,(T_w - T_o)$$

$$\lim_{\bar{t}\to 0} p(0, \bar{t}) = p_o \left[1 + \tfrac{1}{2} \left(\sqrt{T_w/T_o} - 1\right)\right]$$

$$= p_o + (1/4)\, p_o \left[(T_w - T_o)/T_o\right]$$

$$\lim_{\bar{t}\to 0} q_y(0, \bar{t}) = p_o \sqrt{(2kT_o)/\pi} \left[(T_w - T_o)/T_o\right] .$$

The present results agree fairly well with those obtained by Yang and Lees, the numerical differences occur because of the crudeness of the four moment method and the presence of finite "jump" across characteristics[4].

The classical continuum limit gives

$$(T - T_o)/T_w - T_o) = \text{erfc}\left(y/\sqrt{6\nu_o t} \right) .$$

Eq. (70) (as well as Eq. (88)) gives

$$\frac{T - T_o}{T_w - T_o} = \text{erfc}\left(\frac{y}{\sqrt{6\nu_o t}}\right) - e^{\left(.32\frac{p_o}{\mu_o}\frac{y}{\sqrt{RT_o}} + .152\frac{p_o}{\mu_o}t\right)} \text{erfc}\left(\frac{y}{\sqrt{6\nu_o t}} + .39\sqrt{\frac{p_o}{\mu_o}t}\right)$$

$$+ .25\, e^{.152\frac{p_o}{\mu_o}\left(t - \frac{y}{a_1}\right)} \text{erfc}\left[\sqrt{.152\frac{p_o}{\mu_o}\left(t - \frac{y}{a_1}\right)}\right] .$$

For t and also $[t - (y/a_1)]$ ver large, or , in other words, for large values of time and far from the wave front, we obtain the classical result, since the complementary error function can be expanded into a power series[12].

$$\text{erfc}\, x = 1 - \text{erf}\, x = \frac{e^{-x^2}}{\sqrt{\pi}\, x} \left[1 - \frac{2!}{1!\,(2x)^2} + \frac{4!}{2!\,(2x)^4} - \dots\right] .$$

As we have shown in Section V, Fourier's law holds inside the boundary layer. Also, when t becomes very large and $\bar{y}$ is finite, in other words, for very large time and far from the wave front, the heat flux reduces to

$$q_y \cong \sqrt{\frac{RT_o}{8\pi}}\; p_o \frac{T_w - T_o}{T_o} \left[6\, e^{.32\frac{p_o}{\mu_o}\frac{y}{\sqrt{RT_o}} + .152\frac{p_o}{\mu_o} t} \;\; \frac{e^{-\left(\frac{y}{\sqrt{6\nu_o t}} + .39\sqrt{\frac{p_o}{\mu_o} t}\right)^2}}{\sqrt{\pi}\left(\frac{y}{\sqrt{6\nu_o t}} + .39\sqrt{\frac{p_o}{\mu_o} t}\right)} \right.,$$

$$\cong 8.7 \sqrt{\frac{RT_o}{8\pi}}\; p_o \frac{1}{\sqrt{\frac{p_o}{\mu_o} t}}\; e^{-y^2/6\nu_o t}$$

$$= - k_o\,(\partial T/\partial y)$$

where

$$k_o = (15/4)\, R\mu_o \quad .$$

Figure 9 shows the comparison of the actual heat flux with the classical Fourier's law, and Figure 11 shows the temperature profiles and the classical and free molecule limit on the plate.

The isothermal, low-speed Rayleigh problem was worked out by L. Lees[4] as a demonstration example for the moment method. There is a close similarity between that problem and the present one. At first, the solutions have a wave-like behavior. Collisions between particles are relatively infrequent when $[(t/\tau_f) < 1]$; thus the diffusive effects are secondary. The over-simplified version of the two-stream Maxwellian employed here introduces a certain averaging process over the particle velocities, and the choice of four moments results in two characteristics propagating into the fluid with speed $c_1 = \sqrt{(5/3)+(\sqrt{10}/3)}\,\sqrt{RT_o}$ and $c_2 = \sqrt{(5/3)-(\sqrt{10}/3)}\,\sqrt{RT_o}$, respectively. The characteristic speeds have no physical meaning at all; they depend entirely on the unknown functions that we choose and on the moment equations we use (since we can take any moment of the distribution function in order to get a set of differential equations). If more moments are taken, more characteristics appear, and in the limit the physical quantities are "smooth" even for $(t/\tau_f) << 1$.

The discontinuous behavior in the solutions associated with the finite number of characteristics with finite speeds can also be removed by replacing the distribution function employed here by a more realistic one (discussed in Section II. 1); that is, $f = f_1$ for $\xi_y > (y/t)$, $f = f_2$ for $\xi_y < (y/t)$. Hence, one may expect two characteristics with speed varying with (y/t), and approaching infinite speed far from the plate (see also Reference 4).

It is remarkable that even the rather crude splitting of the distribution function employed here leads to solutions for mean normal velocity and temperature that show very clearly the transition from the nearly collision-free regime to the Navier-Stokes regime, which is characterized by a boundary layer merging into a diffuse "wave" (Figure 15). This simple example shows that the classical picture of a thin thermal boundary layer at the plate surface plus a sharp acoustic (inviscid) wave front "far" from the plate surface is never really correct. In fact the older iteration schemes (Van Dyke[14]) in powers of $1/\sqrt{t}$ that start with the Rayleigh-type thermal layer always lead to an artificial singularity of the type $(t - \frac{y}{a})^{-\frac{1}{2}}$ at the wave front. They can never be linked up properly with the correct behavior for $(1/\tau_f)(y - \frac{y}{a}) << 1$, which determines the solution near the wave front.

However, the Navier-Stokes-Fourier equations, plus a temperature jump boundary condition, seem to be perfectly adequate to describe the flow when $(t/\tau_f) \gtrsim 2\text{-}3$, provided one properly accounts for the interaction between the diffuse wave and the inner thermal layer.

For a large temperature jump on the plate, the problem is non-linear, and the mean normal velocity is no longer small compared to the ambient sound speed. In that case the distribution functions employed here have to be modified; at least the normal velcotiy component has to be taken into account. The simplest formulation is

$$f_1 = \frac{n_1}{(2\pi R T_1)^{3/2}} \exp\left[- \frac{\xi_x^2 + (\xi_y - v)^2 + \xi_y^2}{2RT_1}\right] \quad \text{in region I}$$

$$f_2 = \frac{n_2}{(2\pi R T_2)^{3/2}} \exp\left[- \frac{\xi_x^2 + (\xi_y - v)^2 + \xi_y^2}{2RT_2}\right] \quad \text{in region II}$$

where n_1, n_2, T_1, T_2 and v are five unknown functions. Five moments must now be taken, corresponding to the three conservation equations, plus one moment equation for q_y and one for p_{yy}. When $(t/\tau_f) << 1$ it should be possible to find solutions in the form of power series in (t/τ_f). On the other hand, when $(t/\tau_f) >> 1$ the problem reduces to a viscous interaction problem with a shock wave in the "external" flow, and the finite viscous stress and heat flux just behind the shock must be taken into account.

In the present problem, the stress p_{yy} vanishes identically because of the choice of four moments and the linearization. Therefore we have only four mean quantities (density, temperature, velocity, and heat flux) which are uniquely determined by the four unknown functions n_1 , n_2 , T_1 , and T_2 . All the diffusion is accomplished by heat conduction when $(t/\tau_f) > 1$, and not by viscosity. In a linear problem this idealization is adequate, but in a non-linear problem at least one additional moment is required to give an independent status to p_{yy} and provide for viscosity.

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APPENDIX

EVALUATION AND ASYMPTOTIC EXPANSIONS OF SOLUTIONS NEAR THE WAVE FRONT

The outer solution for the temperature perturbation is, in the transform plane,

$$\frac{1}{t_W}\tilde{\theta}_{ou}(\bar{y},s) = F(s)\exp\left[-\left(\sqrt{2Re}\sqrt{s+\tfrac{5}{6}Re} - \sqrt{\tfrac{5}{3}}Re\right)\bar{y}\right]$$

where $F(s) \cong \sqrt{\frac{2}{5}}\frac{1}{\sqrt{Re}\,s}$ when only the highest order term in s is retained as $s \to 0$. The inverse transform is

$$\frac{1}{t_W}\theta_{ou}(\bar{y},\bar{t}) = \frac{1}{2\pi i}\int_C F(s)\exp\left[-\sqrt{2Re}\sqrt{s+\tfrac{5}{6}Re} - \sqrt{\tfrac{5}{3}}Re\right]\bar{y}\,\exp s\bar{t}\;ds$$

We remove the branch point at $s = -(5/6)$ Re by the conformal transformation[15]

$$\varsigma^2 = \frac{6}{5}\frac{s}{Re} + 1, \qquad \text{or} \qquad \frac{s}{Re} = \frac{5}{6}(\varsigma^2 - 1) \quad .$$

Then we obtain

$$\frac{1}{t_W}\theta_{ou}(\bar{y},\bar{t}) = \frac{5Re}{6\pi i}\int_{C_1}\varsigma F_1(\varsigma)\exp\left[\tfrac{5}{6}t'(\varsigma^2 - 2\omega\varsigma + 2\omega - 1)\right]d\varsigma$$

where

$$F_1(\varsigma) = F\left(\tfrac{5}{6}Re(\varsigma^2 - 1)\right)$$

$$t' = Re\,\bar{t}\;, \quad y' = \frac{Re\,\bar{y}}{\sqrt{5/3}} \quad .$$

$$\omega = \frac{y'}{t'}$$

Let

$$f(\zeta) = \zeta^2 - 2\omega\zeta + 2\omega - 1$$

Then the saddle points occur at the zeros of f'( ).

$$f'(\zeta) = 2\zeta - 2\omega = 0$$

$$\zeta_0 = \omega$$

At the saddle point

$$f(\zeta_0) = \omega^2 - 2\omega^2 + 2\omega - 1 = -(\omega - 1)^2$$

Then the integral contains the term $\exp[-\frac{6}{5}(\omega-1)^2 t']$.

$(w-1)^2$ is minimum (zero) at w = 1 (ζ_0 = 1). This result indicates that for large t', the disturbance is concentrated near w = 1, i.e., $\bar{y} = \sqrt{(5/3)}\ \bar{t}$, the sonic wave front. Now write the exponential function as

$$\exp[\tfrac{5}{6} t'(\zeta-1)^2] \exp[\tfrac{5}{3}(t'-y')(\zeta-1)]$$

and assume $|(t' - y')|$ is bounded as $t' \to \infty$. Then the path of the steepest descent[16] passes through the saddle point ζ = 1 with

$$I_m (\zeta - 1)^2 = 0$$

Hence the steepest descent path is

$$\zeta = 1 + i\eta$$

Along this path the exponential function is

$$\exp[-\tfrac{6}{5} t'\eta^2 + i\,\tfrac{5}{3}(t'-y')\eta] .$$

Near $\zeta = 1$, the function $\zeta F_1(\zeta)$ becomes

$$\zeta F_1(\zeta) \cong \frac{\sqrt{6}}{5\,Re}\,\frac{1}{(\zeta-1)^{\frac{1}{2}}} .$$

Hence the outer solution becomes

$$\frac{1}{t_w}\theta_{ou}(\bar{y},\bar{t}) \cong \frac{1}{\sqrt{6}\,\pi i}\int_{1-i\infty}^{1+i\infty}\frac{1}{(\zeta-1)^{\frac{1}{2}}}\exp\left[\tfrac{5}{6}t'(\zeta-1)^2+\tfrac{5}{3}(t'-y')(\zeta-1)\right]d\zeta$$

$$= \frac{2}{2\pi\sqrt{3}}\left\{\int_0^{\infty} e^{-\frac{5}{6}t'\lambda^2}\cos\left[\tfrac{5}{3}(t'-y')\lambda\right]\frac{d\lambda}{\sqrt{\lambda}} + \int_0^{\infty} e^{-\frac{5}{6}t'\lambda^2}\sin\left[\tfrac{5}{3}(t'-y')\lambda\right]\frac{d\lambda}{\sqrt{\lambda}}\right\}$$

where $\lambda e^{i\theta} = \zeta - 1$. The integrals may be evaluated in terms of Bessel functions[17], giving

$$\int_0^{\infty} e^{-\frac{5}{6}t'\lambda^2}\cos\left[\tfrac{5}{3}(t'-y')\lambda\right]\frac{d\lambda}{\sqrt{\lambda}}$$

$$= \frac{\pi}{2}\sqrt{\frac{3}{5t'}}\exp\left[-\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]\sqrt{\tfrac{5}{3}|t'-y'|}\; I_{-\frac{1}{4}}\left[\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]$$

since $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

$$\int_0^{\infty} e^{-\frac{5}{6}t'\lambda^2}\sin\left[\tfrac{5}{3}(t'-y')\lambda\right]\frac{d\lambda}{\sqrt{\lambda}}$$

$$= \mathrm{sign}(t'-y')\,\frac{\pi}{2}\sqrt{\frac{3}{5t'}}\exp\left[-\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]\sqrt{\tfrac{5}{3}|t'-y'|}\; I_{\frac{1}{4}}\left[\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]$$

since $\mathrm{sign}\,x\; J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos(x-\pi/2) = \sqrt{\frac{2}{\pi x}}\sin x$

Hence

$$\frac{1}{t_w}\theta_{ou}(\bar{y},\bar{t}) \cong \frac{1}{2\sqrt{5t'}} \exp\left[-\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]\sqrt{\frac{5}{3}|t'-y'|}$$

$$\left\{I_{-\frac{1}{4}}\left[\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right] + \operatorname{sign}(t'-y')\, I_{\frac{1}{4}}\left[\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]\right\}$$

Near the wave front

$$I_{-\frac{1}{4}}\left[\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right] \cong \frac{1}{\Gamma(\frac{3}{4})}\left[\frac{8(5/6)t'}{(\frac{5}{3})^2(t'-y')^2}\right]^{\frac{1}{4}} + \ldots$$

$$I_{\frac{1}{4}}\left[\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right] \cong \frac{1}{\Gamma(\frac{5}{4})}\left[\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]^{\frac{1}{4}} + \ldots$$

Hence

$$\frac{1}{t_w}\theta_{ou}(\bar{t},\bar{y}) \cong \frac{1}{2\sqrt{5t'}} \exp\left[-\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]$$

$$\left\{\frac{1}{\Gamma(\frac{3}{4})}\left[8(\tfrac{5}{6})t'\right]^{\frac{1}{4}} + \frac{1}{\Gamma(\frac{5}{4})}\,\frac{\frac{5}{3}(t'-y')}{\left[8(\frac{5}{6})t'\right]^{\frac{1}{4}}}\right\}$$

$$= \frac{1}{2}\left[\frac{(\frac{4}{15})^{\frac{1}{4}}}{\Gamma(\frac{3}{4})}\frac{1}{t'^{\frac{1}{4}}} + \frac{(\frac{5}{108})^{\frac{1}{4}}}{\Gamma(\frac{5}{4})}\frac{t'-y'}{t'^{3/4}}\right]\exp\left[-\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]$$

The main part is symmetric with respect to the wave front and decays as $t^{-\frac{1}{4}}$. The second part is anti-symmetric with respect to the wave front, negative ahead of the wave, and thus it gives skewness to the wave shape, but it decays faster than the first part (as $t^{-3/4}$). Asymptotically the Bessel functions behave as follows

$$I_{\frac{1}{4}}(\zeta),\ I_{-\frac{1}{4}}(\zeta) \sim \frac{1}{\sqrt{2\pi\zeta}}\, e^{\zeta}\left[1 - O\left(\frac{1}{\zeta}\right)\right]$$

+ (exponentially small terms) .

These two differ only in the exponentially small terms. Thus far away from the wave front;

(1) $\Theta_{0u}(\bar{t}, \bar{y}) \sim$ exponentially small, for $t' - y' < 0$, or ahead of the wave

(2) $\frac{1}{t_w}\Theta_{0u}(\bar{t}, \bar{y}) \sim \sqrt{\frac{2}{5\pi}}\, \frac{1}{\sqrt{t' - y'}}$,

for $t' - y' > 0$, and thus it merges into the inviscid wave field.

All the transformed variables behave like $\tilde{\Theta}_{0u}(\bar{y}, s)$ except the heat flux which is evaluated as follows

$$\tilde{q}_y(\bar{y}, s) = \sqrt{\frac{RT_0}{8\pi}}\, p_0 \left(-\frac{10}{\sqrt{\frac{2}{\pi}}}\right) \tilde{\Theta}_{0u}(0, s)\, \frac{\sqrt{2Re}\sqrt{s + \frac{5}{6}Re} - \sqrt{\frac{5}{3}}\, Re}{s + \frac{2}{3}Re}\, \exp\left[-\left(\sqrt{2Re}\sqrt{s + \frac{5}{6}Re} - \sqrt{\frac{5}{3}}\, Re\right)\bar{y}\right]$$

$$\cong \frac{6\sqrt{3\pi}}{Re}\sqrt{\frac{RT_0}{8\pi}}\, p_0\, \frac{T_w - T_0}{T_0}\sqrt{\frac{s}{Re}}\, \exp\left[-\left(\sqrt{2Re}\sqrt{s + \frac{5}{6}Re} - \sqrt{\frac{5}{3}}\, Re\right)\bar{y}\right]$$

Following the steps as evaluating of $\tilde{\Theta}_{0u}$, we have as $\zeta \to 1$

$$\zeta F_1(\zeta) \cong -\frac{6\sqrt{5\pi}}{Re}\sqrt{\frac{RT_0}{8\pi}}\, p_0\, \frac{T_w - T_0}{T_0}\sqrt{\lambda}\, e^{i\theta/2}$$

$$\zeta - 1 = \lambda e^{i\theta}$$

Then the integral becomes

$$q_y(\bar{y},\bar{t}) \cong -\frac{5\sqrt{5\pi}}{\pi i}\sqrt{\frac{RT_0}{8\pi}}\, p_0\, \frac{T_w - T_0}{T_0}\int_{1-i\infty}^{1+i\infty}(\varsigma-1)^{\frac{1}{2}}\exp\left[\frac{5}{6}t'(\varsigma-1)^2+\frac{5}{3}(t'-y')(\varsigma-1)\right]d\varsigma$$

$$= -5\sqrt{\frac{5}{\pi}}\sqrt{\frac{RT_0}{8\pi}}\, p_0\, \frac{T_w-T_0}{T_0}\left\{\int_0^\infty \lambda e^{-\frac{5}{6}t'\lambda^2}\left[\frac{\cos\frac{5}{3}(t'-y')\lambda}{\sqrt{\lambda}}\right]d\lambda\right.$$

$$\left. -\int_0^\infty \lambda e^{-\frac{5}{6}t'\lambda^2}\left[\frac{\sin\frac{5}{3}(t'-y')\lambda}{\sqrt{\lambda}}\right]d\lambda\right\}$$

$$= -\frac{5}{4}\sqrt{\frac{5}{2\pi}}\sqrt{\frac{RT_0}{8\pi}}\, p_0\, \frac{T_w-T_0}{T_0}\left(\frac{1}{\frac{5}{48}t'}\right)^{3/4}\left[\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]^{3/4}\exp\left[-2\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right]$$

$$\left\{\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{2})}\,{}_1F_1\left(-\frac{1}{4};\frac{1}{2};2\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right)-\mathrm{sign}(t'-y')\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{2})}\,{}_1F_1\left(\frac{1}{4};\frac{3}{2};2\frac{(\frac{5}{3})^2(t'-y')^2}{8(5/6)t'}\right)\right\}$$

where ${}_1F_1\left(\frac{\nu}{2}-\frac{\mu}{2}+1;\ \nu+1;\ \frac{a^2}{4p^2}\right)$ is the hypergeometric function.[17]

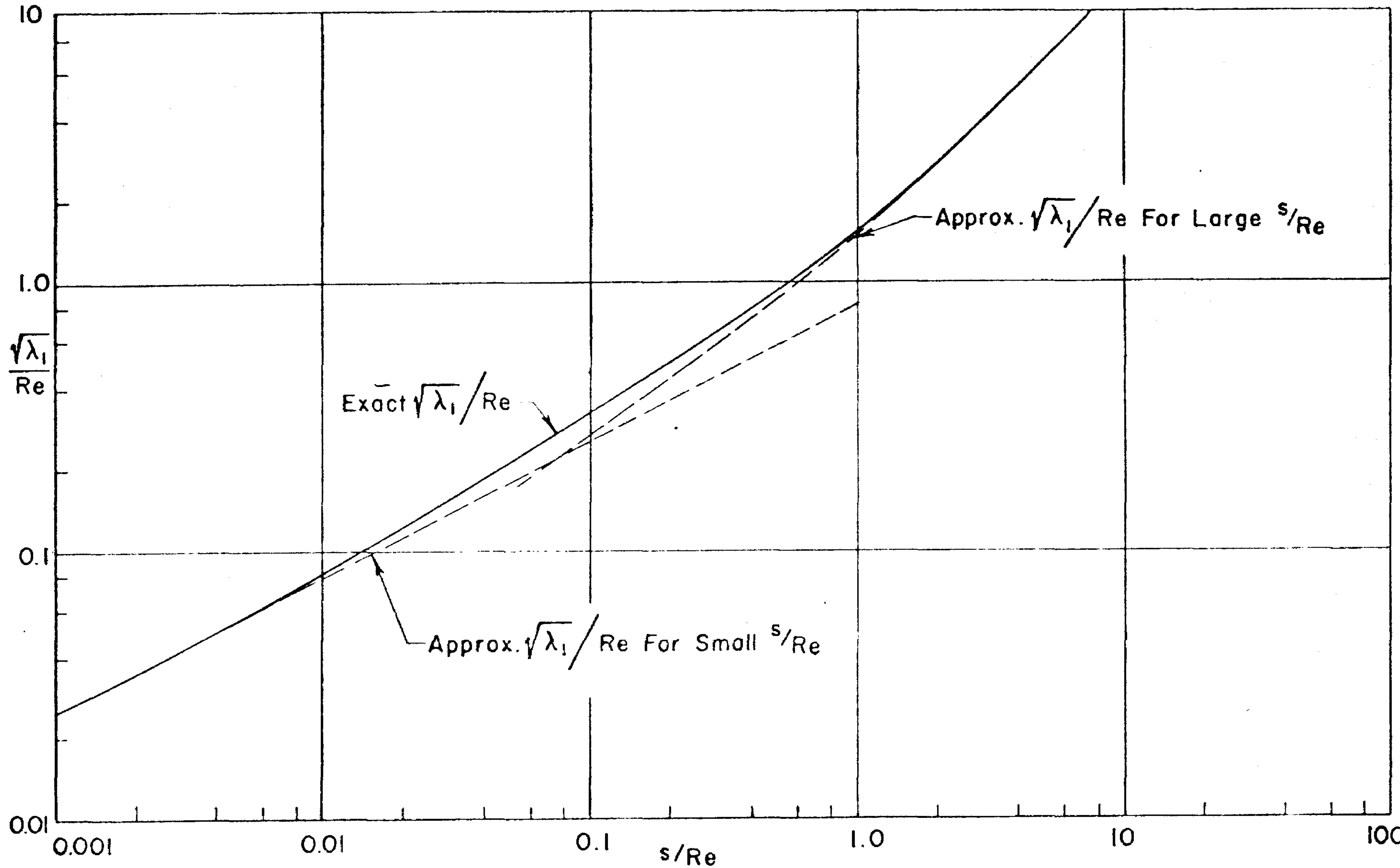

FIG. 1 – COMPARISON OF THE EXACT $\sqrt{\lambda_1}$ AND ITS APPROXIMATE EQUATIONS

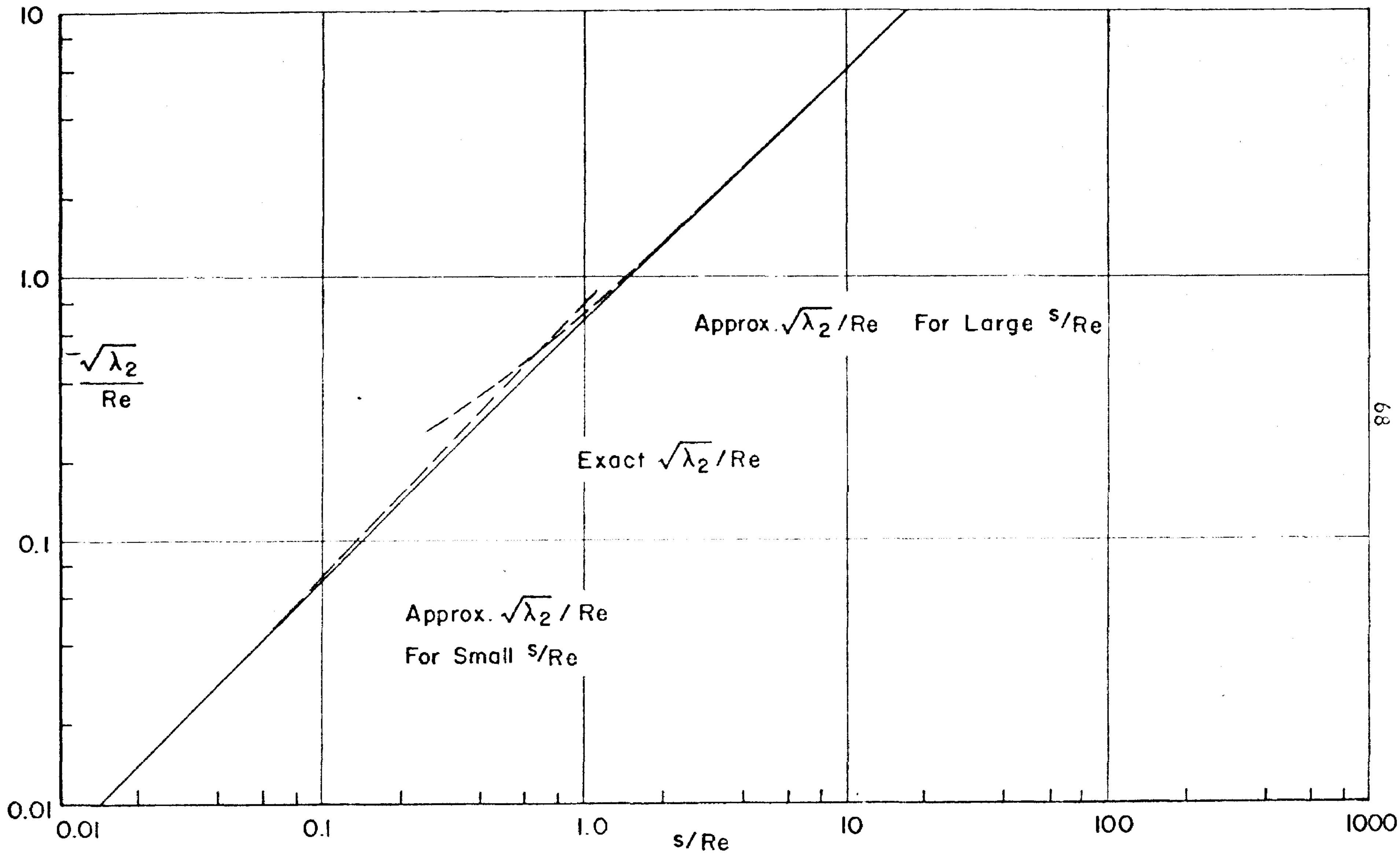

FIG. 2 - COMPARISON OF THE EXACT FUNCTION $\sqrt{\lambda_2}$ AND ITS APPROXIMATE EQUATIONS

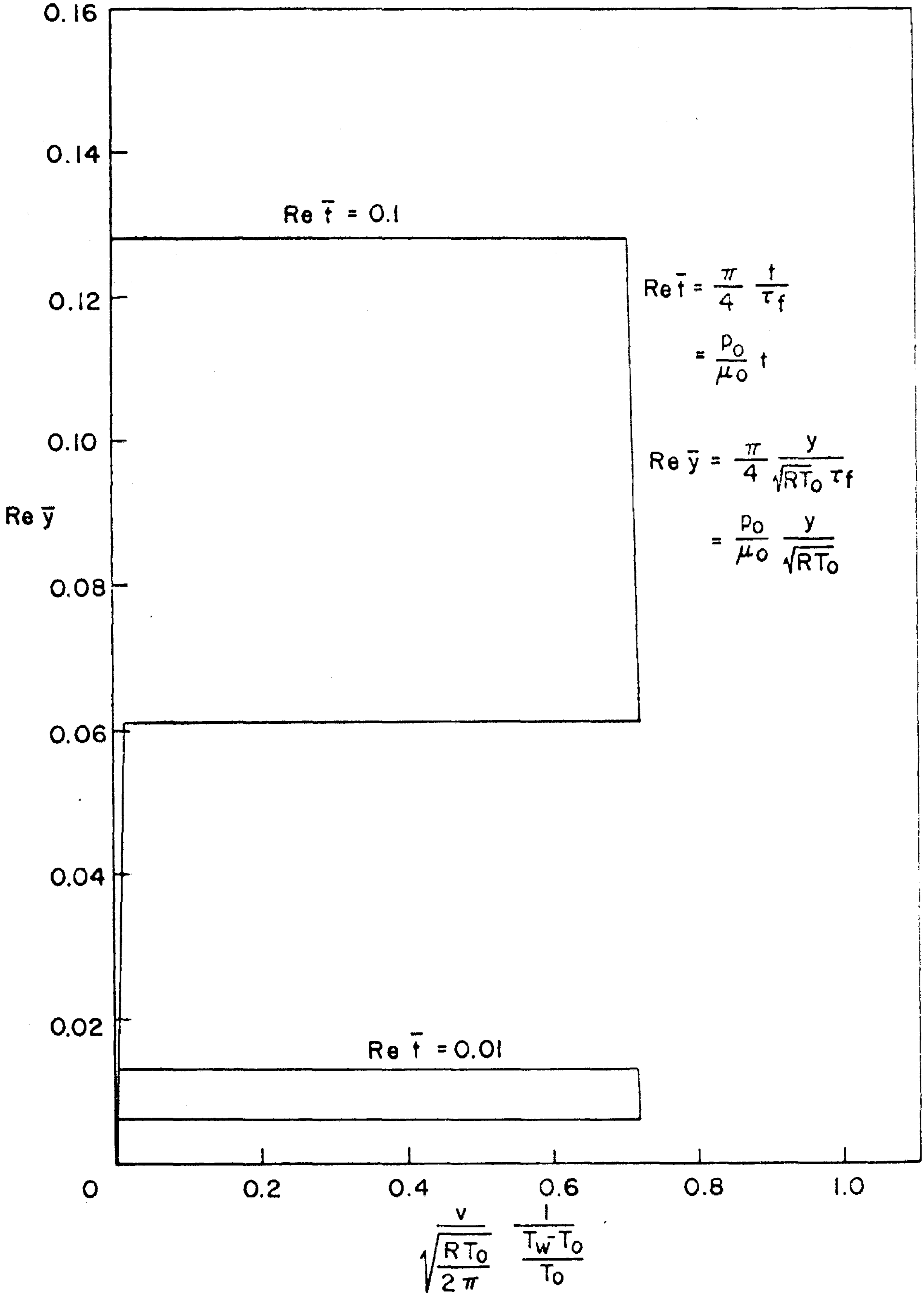

FIG.3–NORMAL VELOCITY PROFILES FOR SMALL VALUES OF TIME ($Re\,\bar{t} = \frac{p_0}{\mu_0}t$ = 0.01 AND 0.1)

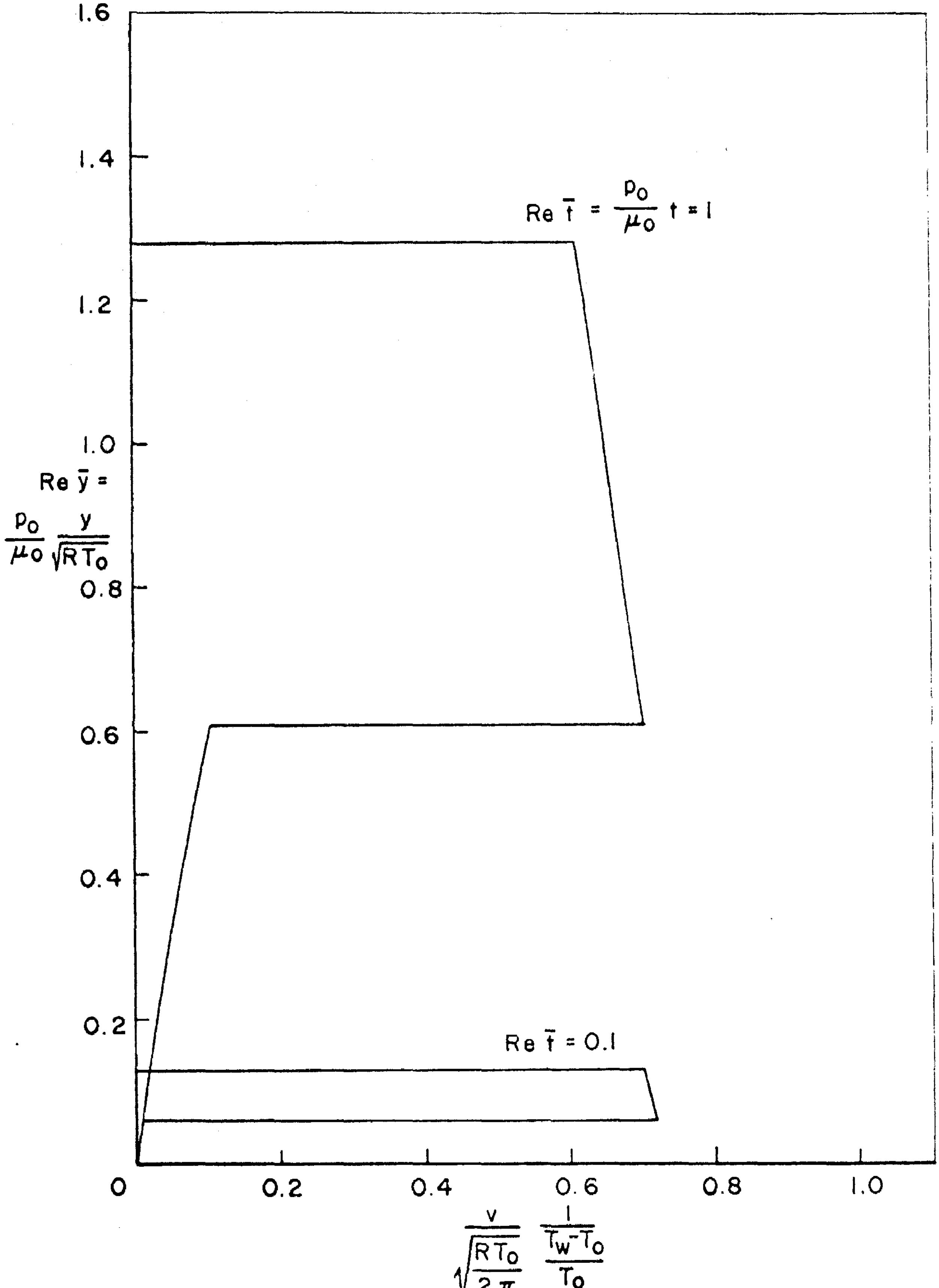

FIG. 4 - NORMAL VELOCITY PROFILES FOR SMALL VALUES OF TIME

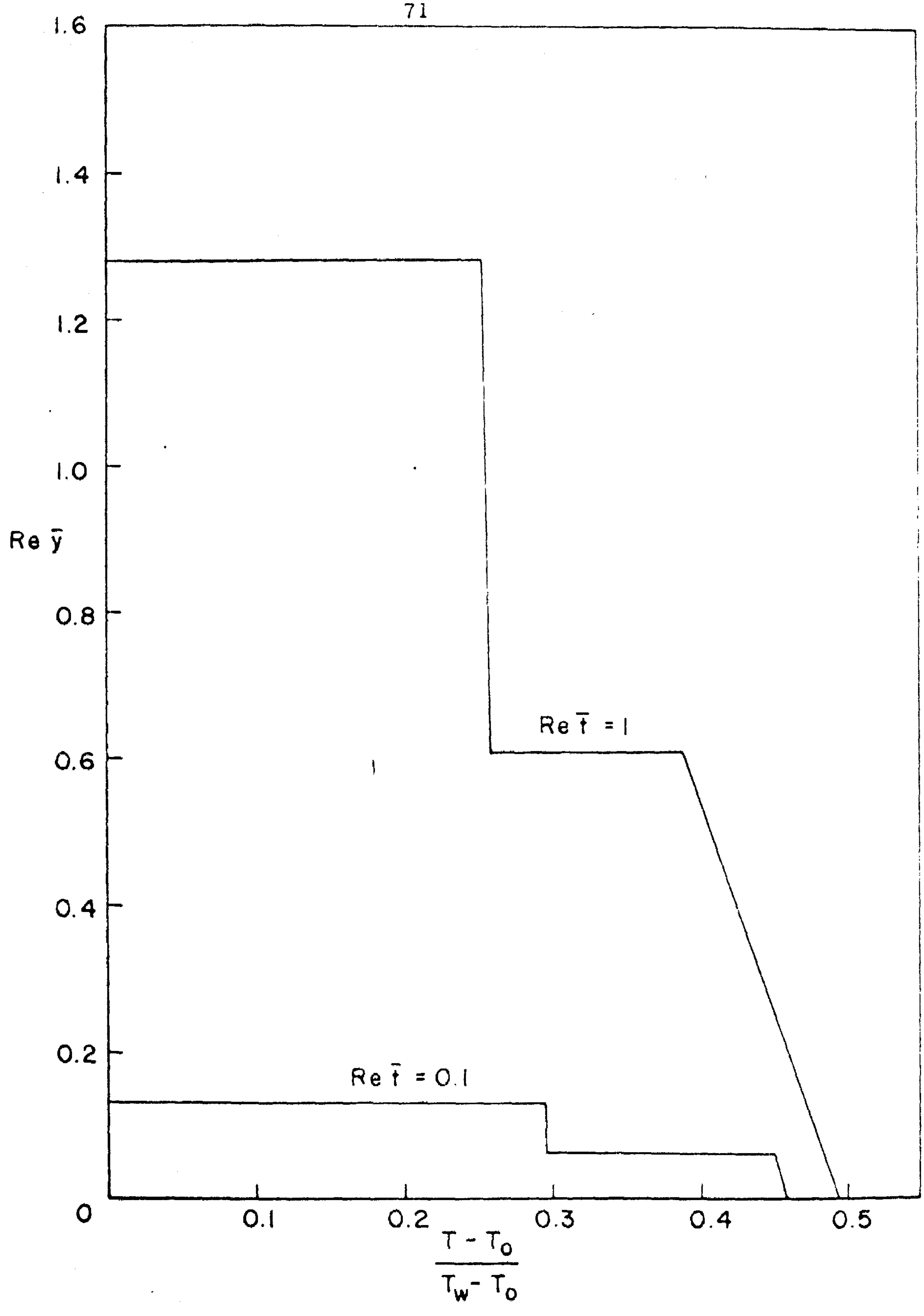

FIG. 5 – TEMPERATURE PROFILES FOR SMALL VALUES OF TIME

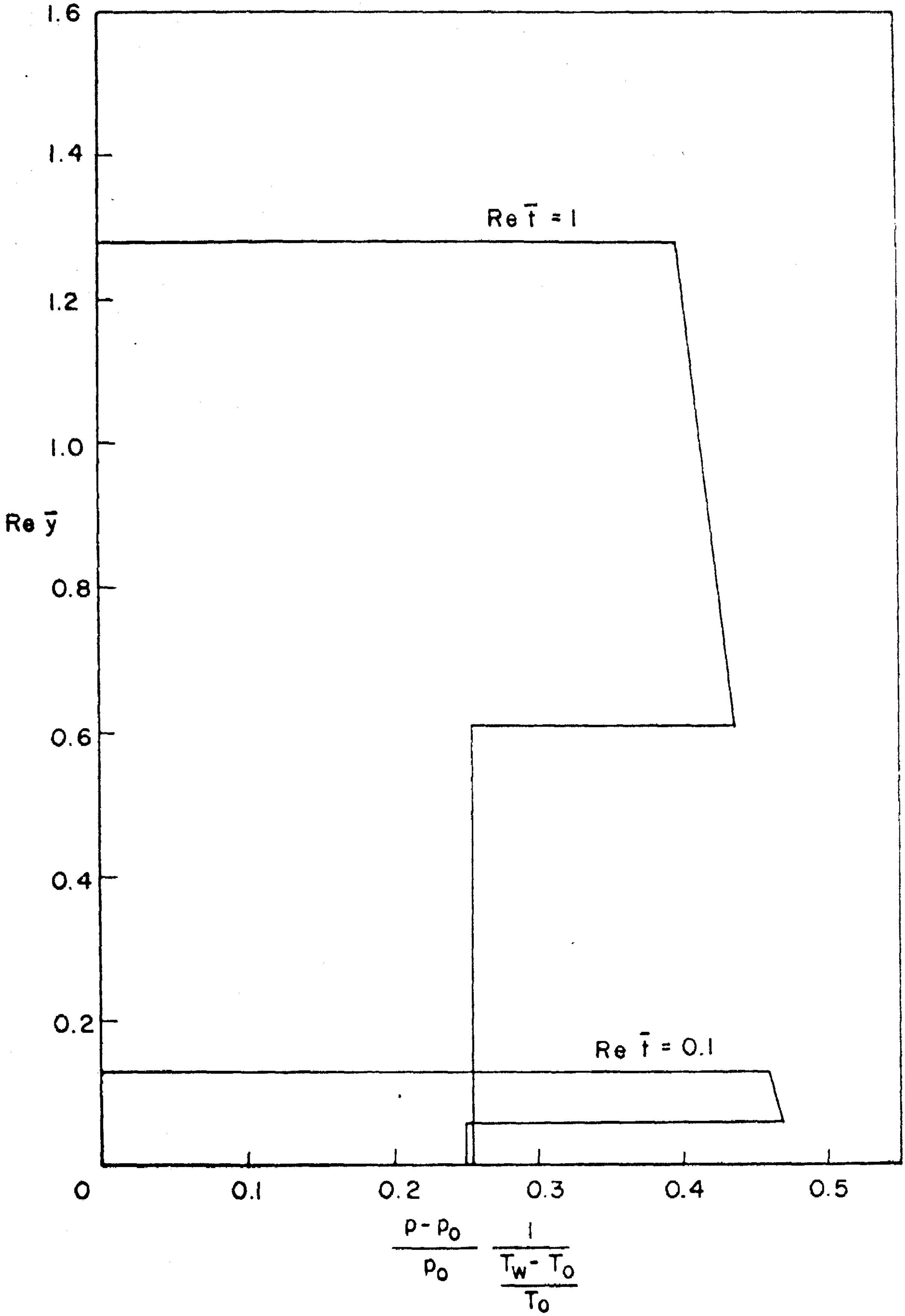

FIG. 6 - NORMAL STRESS PROFILES FOR SMALL VALUES OF TIME

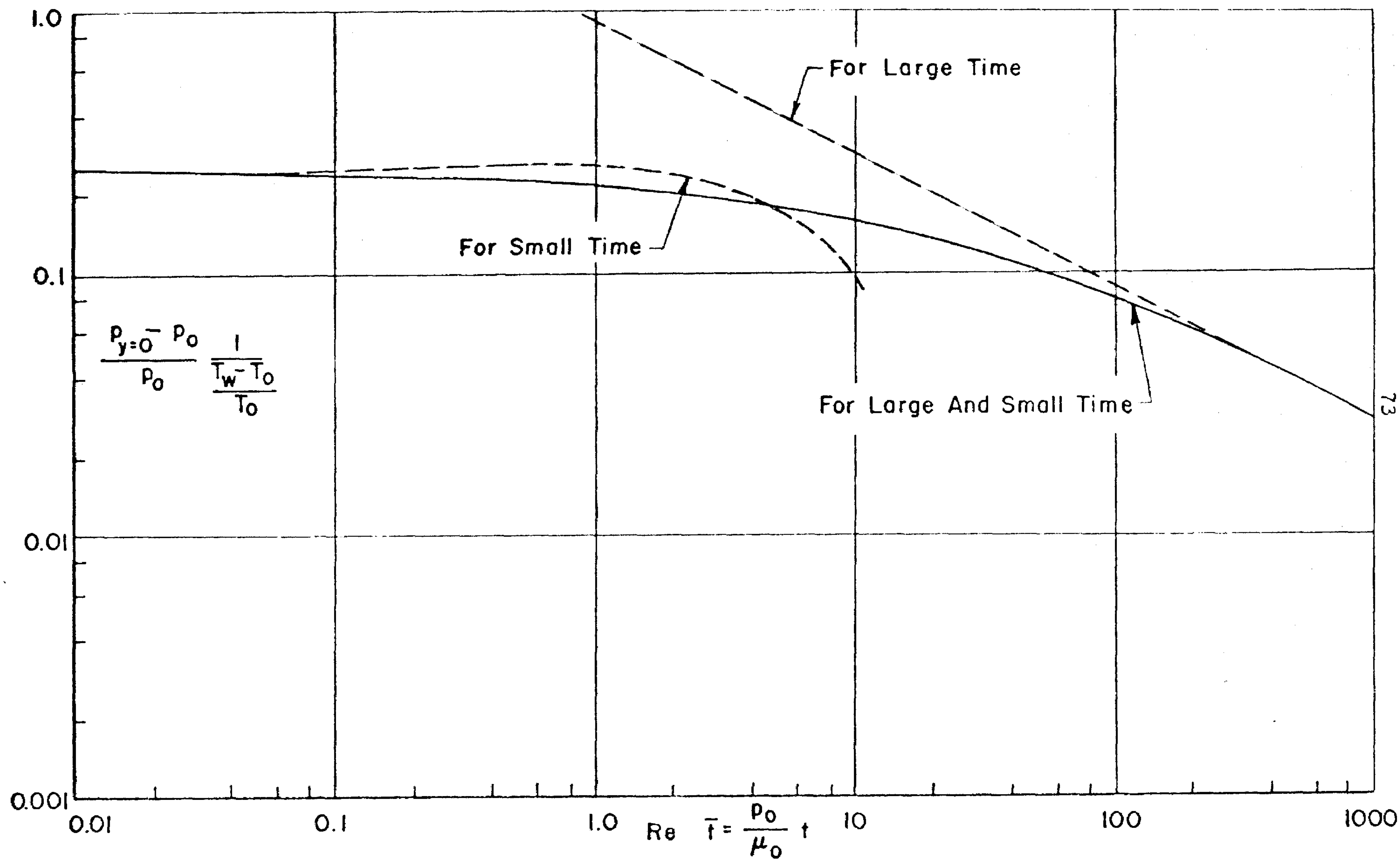

FIG. 7 - PRESSURE ON PLATE

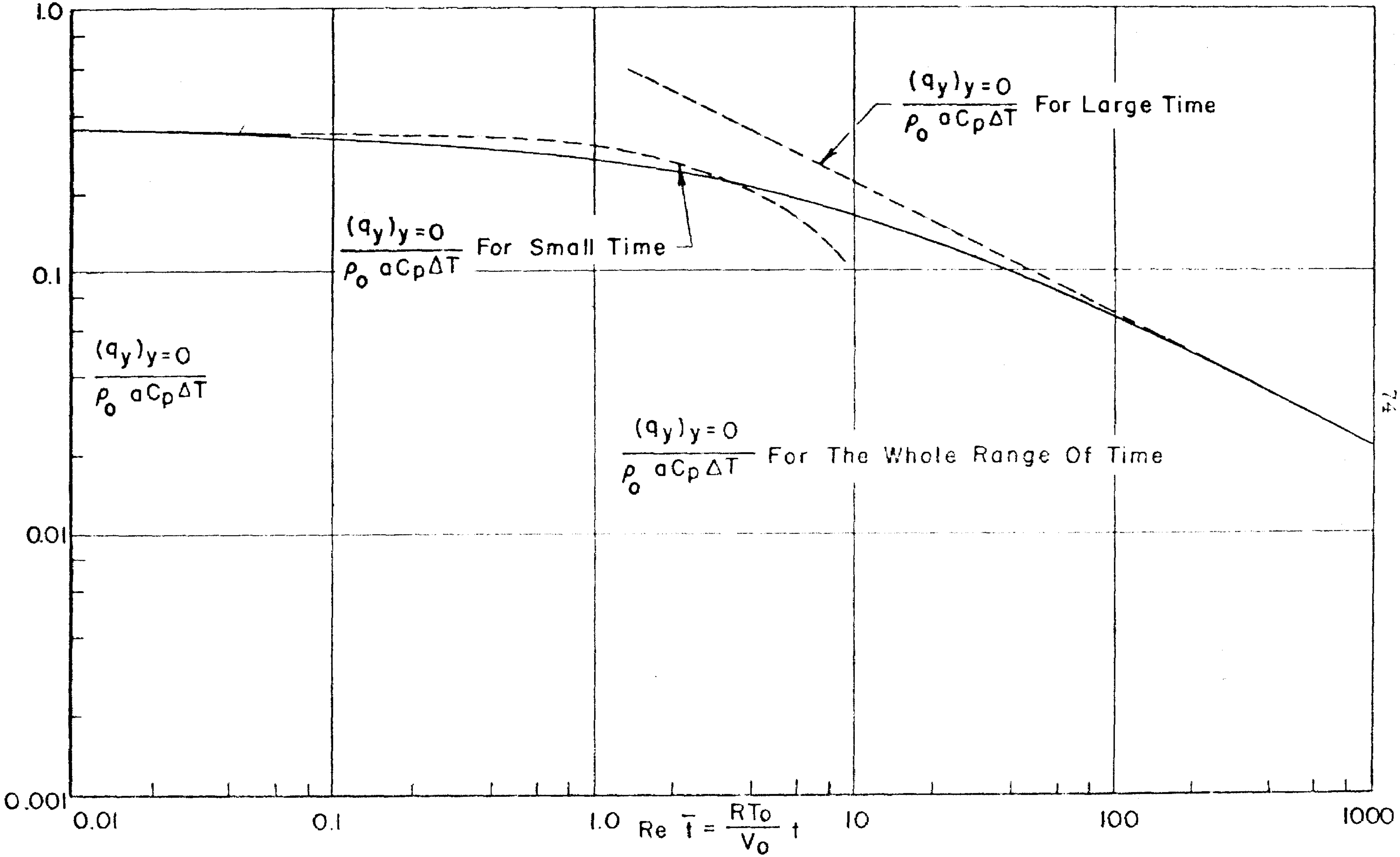

FIG. 8 - NORMAL HEAT FLUX ON PLATE

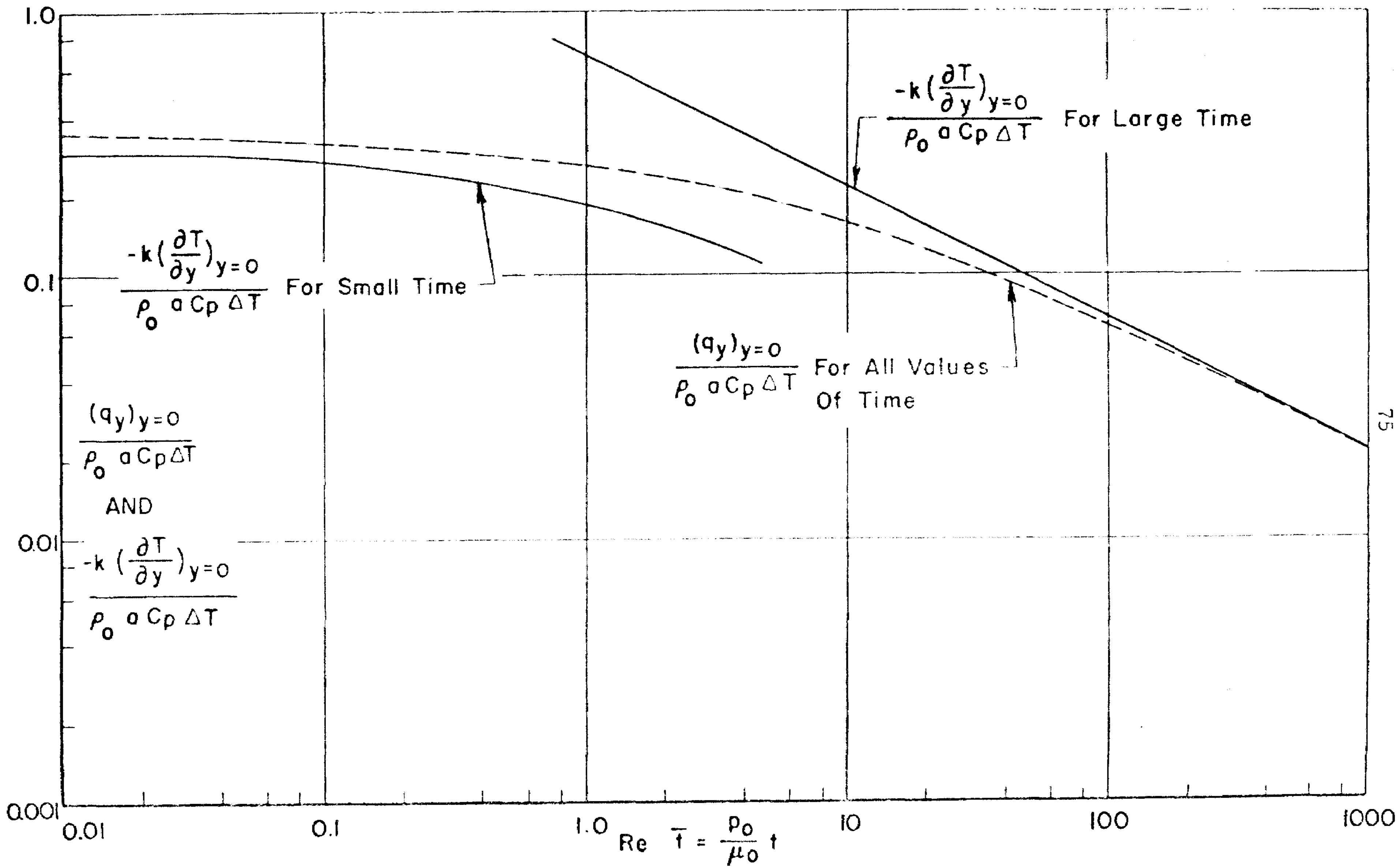

FIG. 9 - COMPARISON OF NORMAL HEAT FLUX AND THE CLASSICAL FOURIER'S LAW

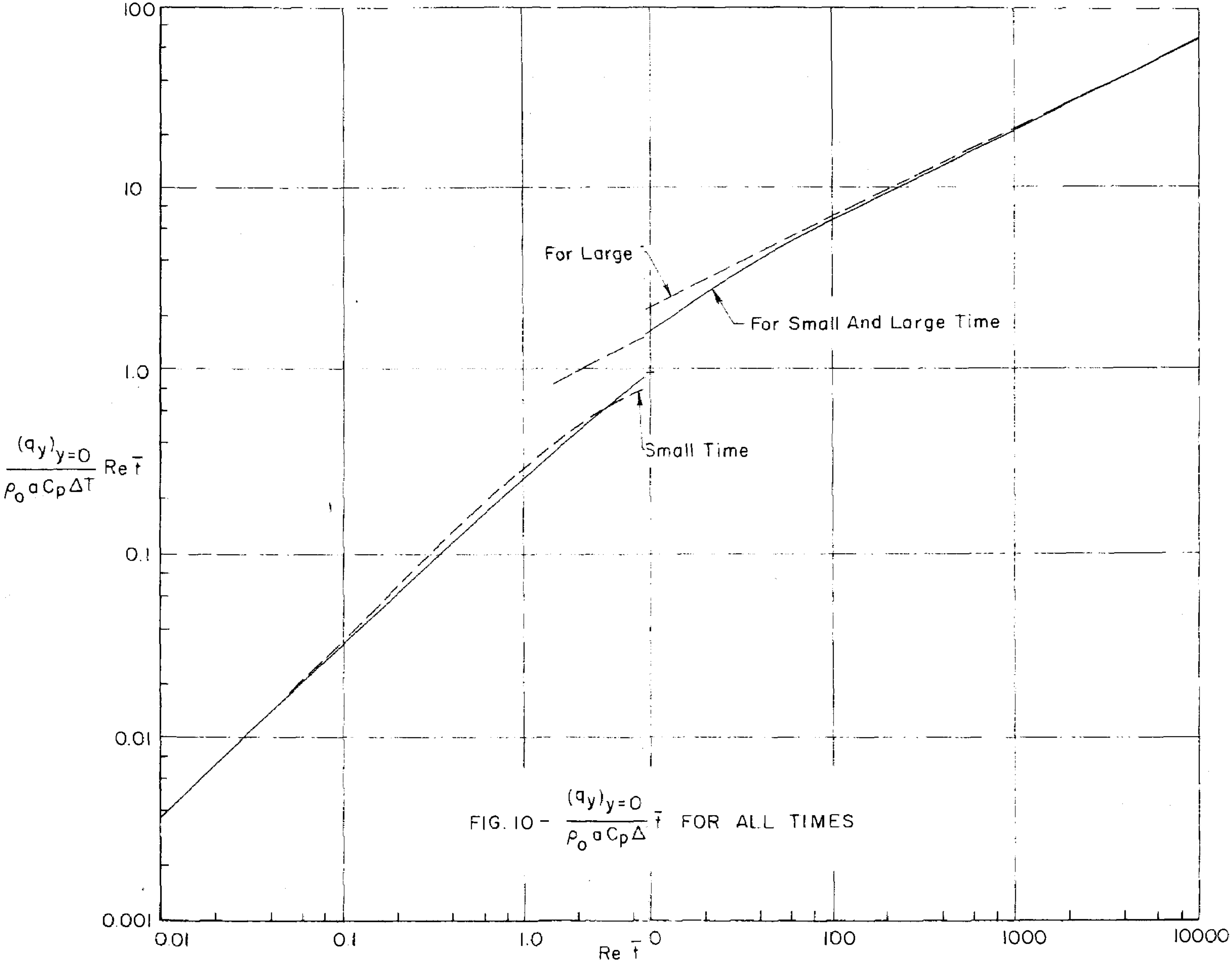

FIG. 10 - $\frac{(q_y)_{y=0}}{\rho_o a C_p \Delta} \bar{t}$ FOR ALL TIMES

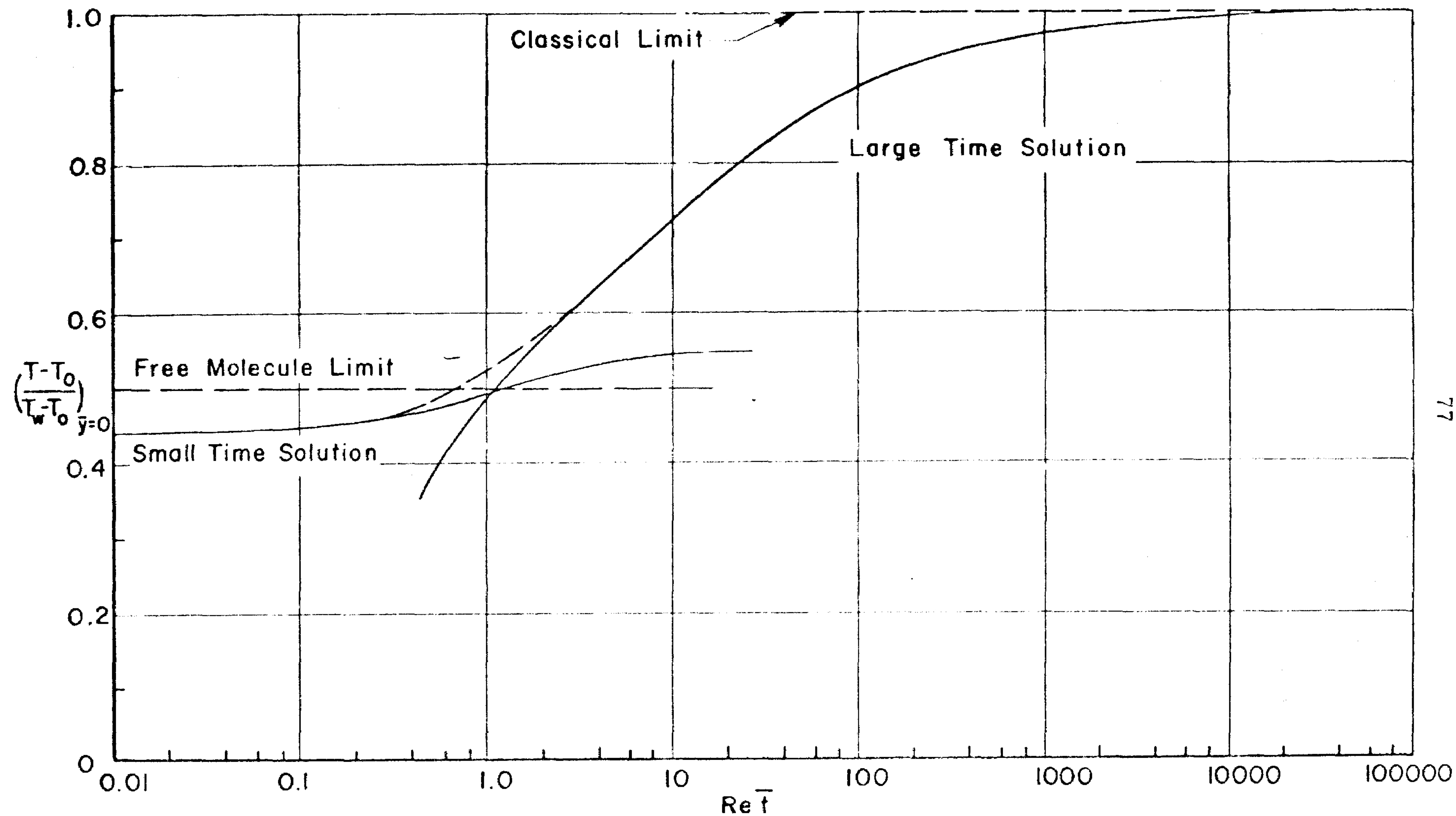

FIG. 11 ABSOLUTE TEMPERATURE OF GAS ON WALL

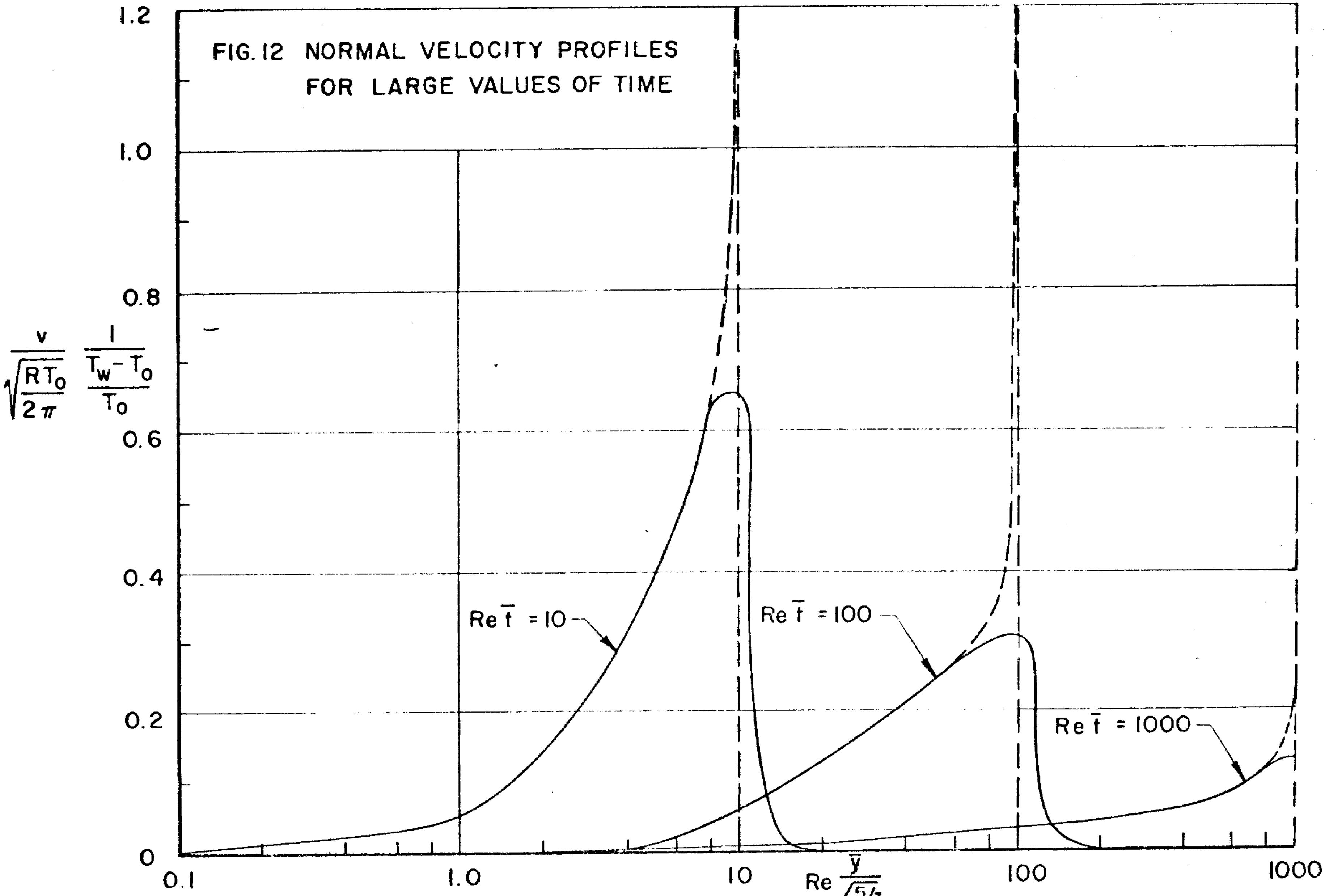

FIG. 12 NORMAL VELOCITY PROFILES FOR LARGE VALUES OF TIME

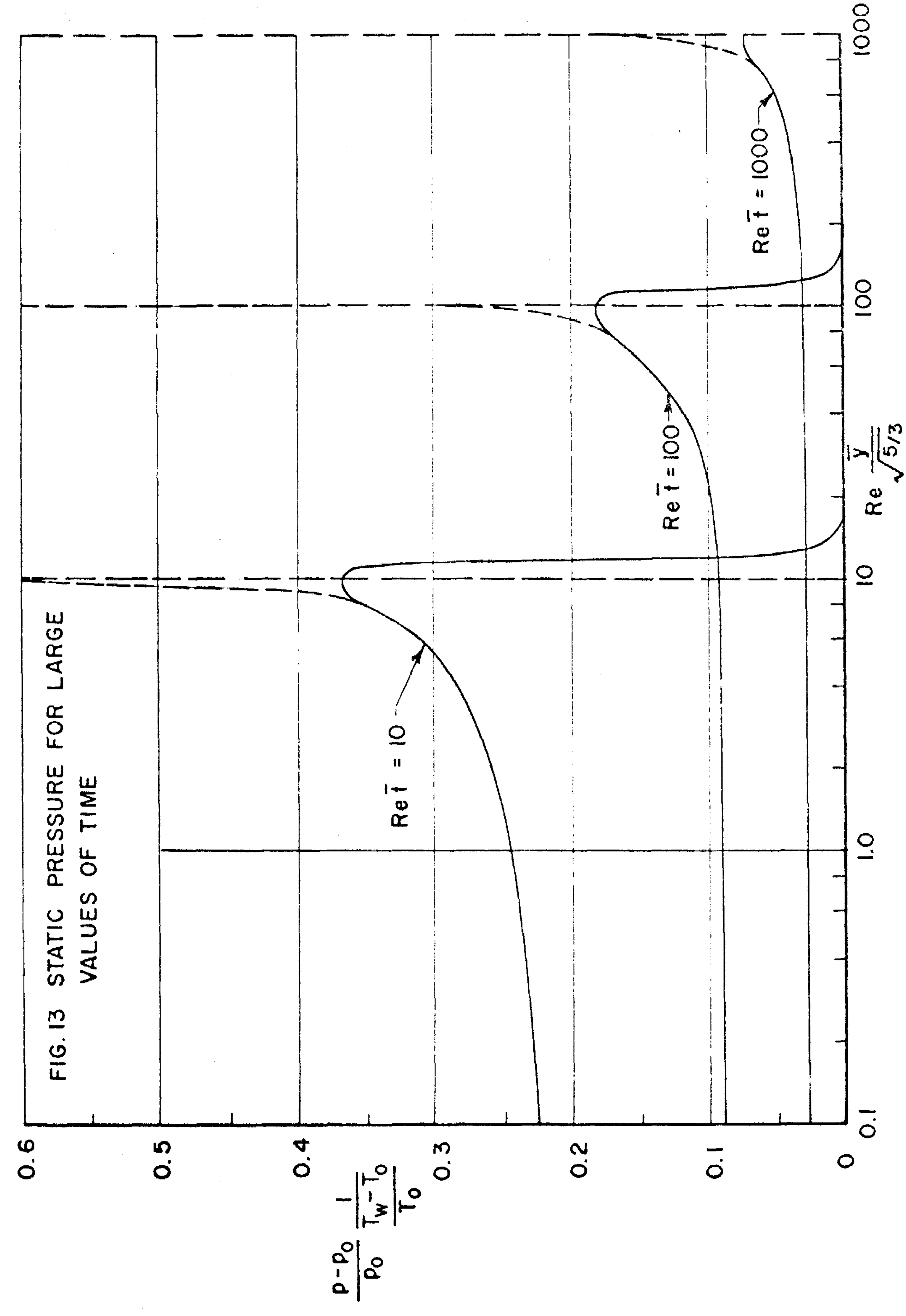

FIG. 13 STATIC PRESSURE FOR LARGE VALUES OF TIME

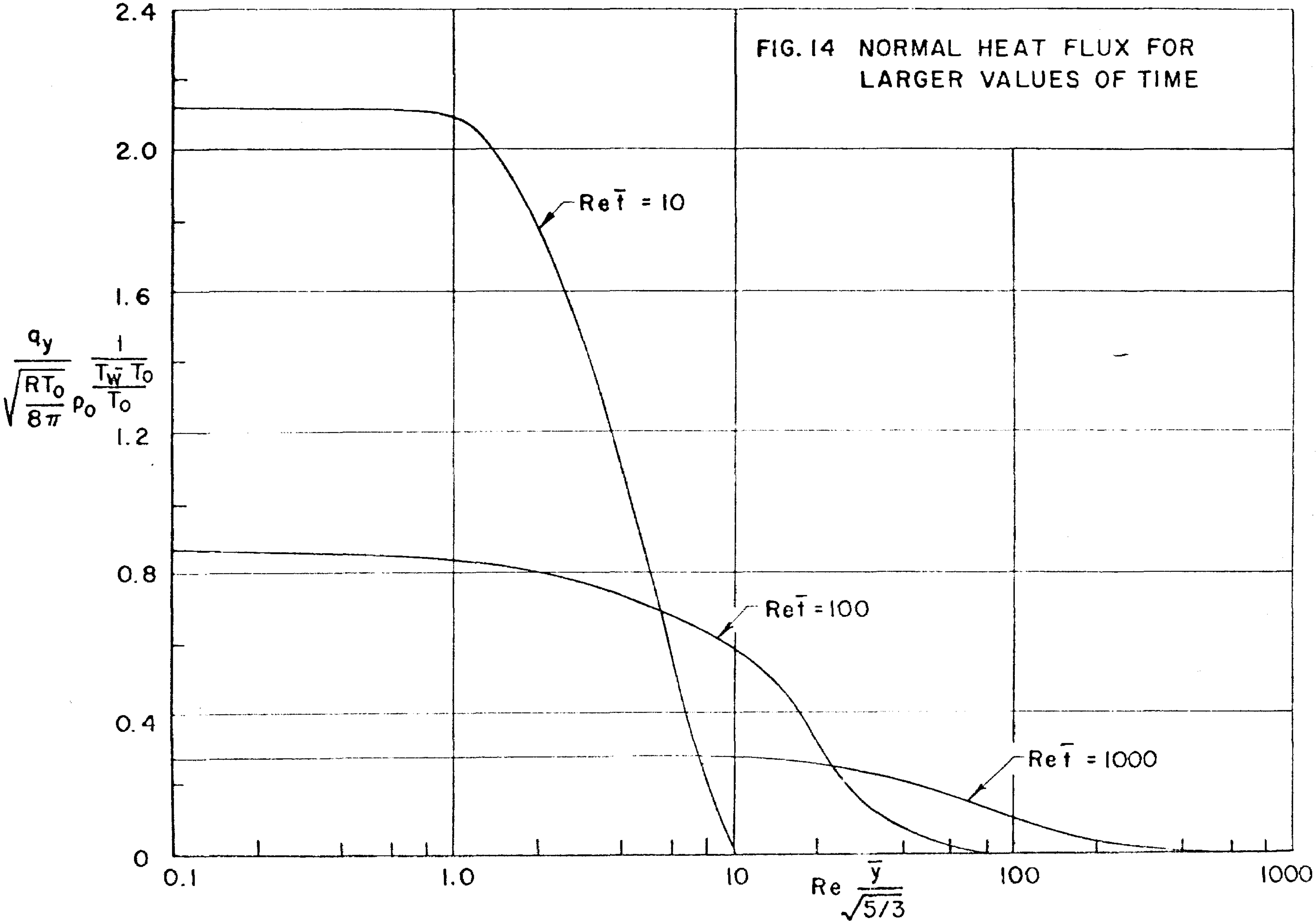

FIG. 14 NORMAL HEAT FLUX FOR LARGER VALUES OF TIME

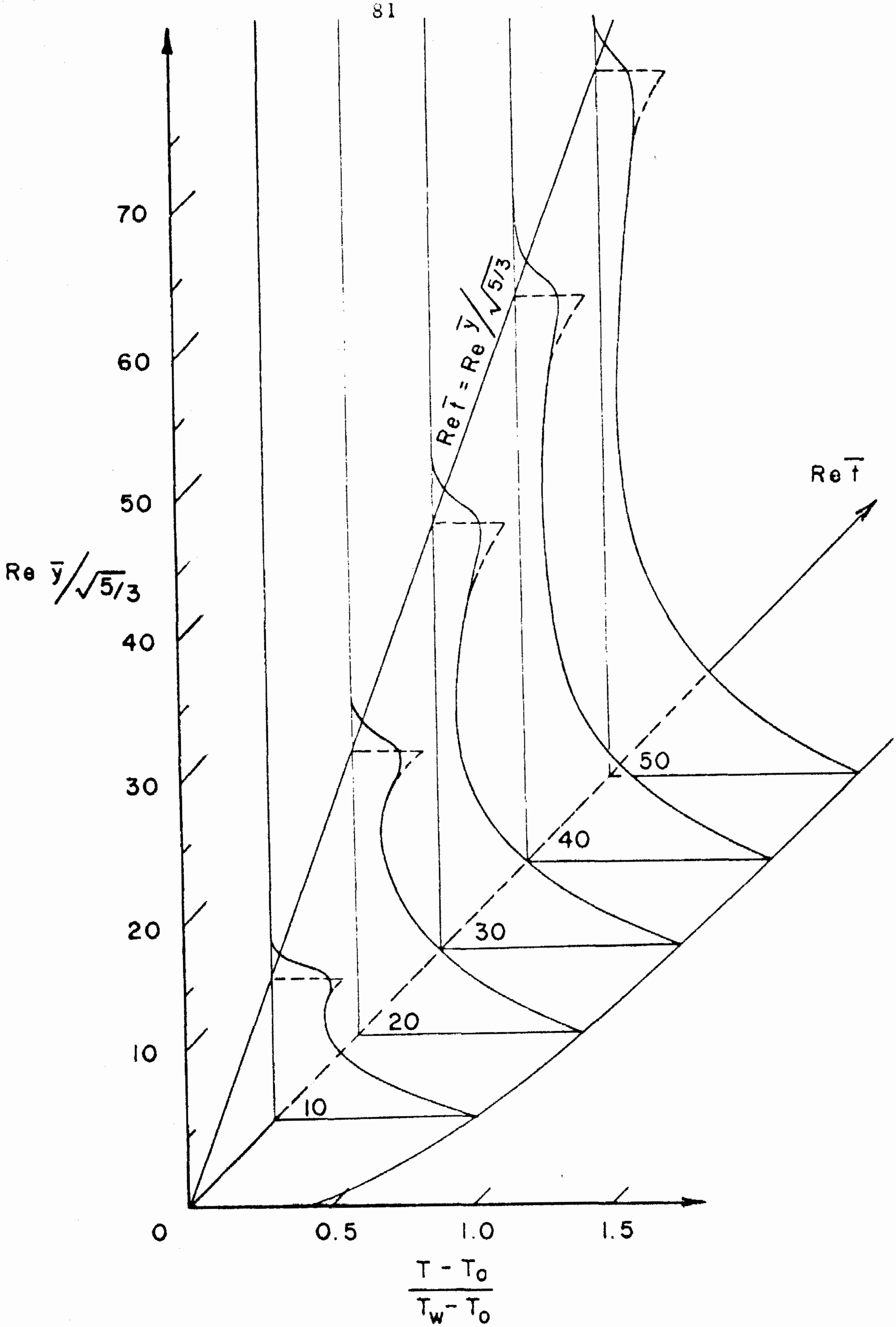

FIG. 15 TEMPERATURE PROFILES IN yt - PLANE