INVARIANT SUBSPACES IN HILBERT AND NORMED SPACES

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ABSTRACT

This dissertation concerns itself with the following question: Suppose T is a bounded linear operator from an infinite dimensional Hilbert Space into itself. What are sufficient conditions to imply the existence of a nonzero, proper subspace M of H such that $T(M) \subseteq M$? The methodology used to approach the question is in line with the methods developed by Aronzajn and Smith [1] and Bernstein and Robinson [3]. The entire thesis is exposited within the framework of nonstandard analysis as developed by Robinson [9].

Chapter 1 of the dissertation develops the necessary theory involved, and presents a necessary and sufficient condition for T to have a proper invariant subspace. The conditions involve assumptions on certain finite dimensional approximations of T.

Chapter 2 demonstrates two situations under which the conditions presented in Chapter 1 come about. The first of these, which was announced by Feldman [5] and has been published in preprint form by Gillespie [6], was proved independently by the author under more relaxed conditions. For simplicity, we state here the Feldman result.

<u>Theorem:</u> <u>If</u> T <u>is quasi-nilpotent and if the algebra generated by</u> T <u>has a nonzero compact operator in its uniform closure,</u> <u>then</u> T <u>has an invariant subspace.</u>

It is still an open question whether or not the condition "T commutes with a compact operator" implies the desired result. By insisting that C be "very compact" (to be defined) the following result

is demonstrated.

<u>Theorem:</u> <u>If</u> C <u>is a nonzero "very compact" operator, and if</u> TC = CT, <u>then</u> T <u>has an invariant subspace.</u>

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INTRODUCTION

Let H denote an infinite dimensional separable Hilbert Space, and let T be an operator from H into itself. The following question remains unsolved: Does there exist a nontrivial closed subspace M of H such that $T(M) \subseteq M$ (i.e., for every $x \in M$, $T(x) \in M$)? The trivial subspaces $M = \{0\}$ and $M = H$ always possess this invariance property. Hence, for the sake of convenience, when we say that M is an invariant subspace of T it will be understood that M is nontrivial, and $T(M) \subseteq M$. If we say that M is a subspace left invariant by T, then the cases $M = \{0\}$ and $M = H$ are allowed.

The question whether or not T has an invariant subspace appears to be a reasonable one, for in the case when H is finite dimensional, the answer is affirmative.

0.1. Theorem: *If T is a linear operator from a finite dimensional complex vector space V into itself, then there is a chain $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_{n-1} \subseteq V_n = V$ of subspaces of V such that V_i is of co-dimension 1 in V_{i+1}, $i = 0, 1, 2, \dots, n-1$, and that $T(V_i) \subseteq V_i$, $i = 0, 1, 2, \dots, n$; or, equivalently, in matrix language: T may be represented by a subdiagonal matrix with respect to a suitable basis.*

On the other hand, this theorem is essentially algebraic in nature (if one considers the fundamental theorem of algebra an algebraic theorem). It follows from the fact that every such (finite dimensional) operator possesses at least one eigenvalue λ in $\mathbb{C}$ (= the complex numbers) and a corresponding eigenvector $x \in V$, $x \neq 0$, with $T(x) = \lambda x$. But the geometry of an infinite dimensional space, the non-compactness of bounded sets, alters the situation considerably. An operator T on an infinite dimensional Hilbert Space H need not have an eigenvector. This fact is most readily seen by considering the orthonormal shift operator, which is defined on an orthonormal basis $\{e_n \mid n = 0, 1, 2, \ldots\}$ of H as follows:

$$S(e_n) = e_{n+1}, \qquad n = 0, 1, 2, \ldots \quad .$$

However, S is quite rich in invariant subspaces; in fact, theorem 0.1 holds true for S, with the finite sequence $\{V_i\}$ replaced by an infinite decreasing sequence $H = V_0 \supseteq V_1 \supseteq V_2 \supseteq \ldots \supseteq V_n \supseteq \ldots$.

The first difficult theorem concerning invariant subspaces was provided by von Neuman. He proved that every compact operator on a Hilbert Space H possesses an invariant subspace, but he never published the result, and his proof appears to be lost. Aronzajn and Smith [1] published this result in 1954 for a compact operator on a Banach space. In response to a question raised by K. T. Smith and P. R. Halmos, Bernstein and Robinson [3] published the following theorem.

0.2. Theorem: (Bernstein and Robinson [3]) If T is a bounded linear operator which is polynomially compact (i.e., $p(T)$ is compact for some nonzero polynomial p), then T has an invariant subspace.

The proof of this theorem makes use of the techniques of nonstandard analysis as developed by Robinson [9]. Bernstein proved that 0.2 holds for a Banach space [2], and Bonsall [4] demonstrated the theorem for normed spaces. In his proof, Bonsall used standard, as opposed to nonstandard, methods.

Note, incidentally, that theorem 0.2 implies the existence of more than one invariant subspace. This follows because "polynomial compactness" is a global property in the following sense: if M is a proper invariant subspace of T, then $T|M$ (the restriction of T to M) is polynomially compact, and the induced operator T on the factor space H/M is also polynomially compact.

In the current investigation, we have attempted to generalize the techniques of Bernstein and Robinson to obtain still further theorems. In the first chapter is developed the necessary machinery with which we prove a sufficient condition for an operator to possess an invariant subspace, a condition which is also necessary. In Chapter 2, this theorem is applied to the situation in which the uniform closure of the algebra generated by T possesses a nonzero compact operator. The latter section of chapter 2 proved the following statement: if $TC = CT$, and C is a "very" compact operator (to be specified) then T has an invariant subspace.

CHAPTER 1

INVARIANT SUBSPACE THEORY FOR ARBITRARY BOUNDED LINEAR OPERATORS

This dissertation shall make use of the methods of non-standard analysis as developed by Robinson [9] and augmented and advanced by Luxemburg [7, 8]. As a starting point, we shall assume knowledge of the basic concepts of nonstandard analysis. Although all proofs shall be carried out with nonstandard methods, the more important theorems will have statements in standard language. At the end of this chapter, we shall present the necessary material for generalization to normed spaces.

H will denote an infinite dimensional separable Hilbert Space over the complex numbers $\mathbb{C}$. We denote by ${}^*H$ an enlargement of H in the sense of Robinson [9] (cf. Luxemburg [8]). Where no confusion shall arise, we shall consider H to be imbedded in ${}^*H$, and standard elements $x \in H$, considered as elements of ${}^*H$, will be denoted by the same symbol, x. Similarly, standard sets E, standard functions T, and standard relations ϕ will be denoted in ${}^*H$ by E, T, ϕ, or by ${}^*E$, ${}^*T$, or ${}^*\phi$ if necessary. (The reader will note that, in this presentation, the enlargement ${}^*H$ refers not just to the model of the Hilbert Space structure H, but to all entities related to H. If necessary, one could simply speak of a model of set theory, and an enlargement thereof, and carry out all arguments in this great big model).

If a number $\lambda \in {}^*\mathbb{C}$ is infinitesimal (i.e., if $|\lambda| < \epsilon$ for every $\epsilon > 0$, $\epsilon \in \mathbb{R}$) then we write $\lambda =_1 0$.

The symbol st shall denote the standard part operation, which is defined as follows. For any ξ in ${}^*H$ which is near standard (or, in other words, for a $\xi \in {}^*H$ for which there exists a standard element $x \in H$ such that $\|\xi - x\| =_1 0$) $st(\xi)$ is defined, and $st(\xi) = x$. x is called the standard part of ξ. st( ) has the following properties. st is, first of all, not an internal function. st is linear in the following sense: if $\xi, \eta \in {}^*H$ are both near standard, and if $\lambda, \mu \in {}^*\mathbb{C}$ are both near standard, then

$$st(\lambda\xi + \mu\eta) = st(\lambda)\, st(\xi) + st(\mu)\, st(\eta) \quad .$$

Note that $st(\lambda)$ is a different function from $st(\xi)$, for the domains are different. But the concept is analogous, and no confusion should result.

The weak topology on ${}^*H$ provides another standard part operation, namely the weak standard part, denoted by stw( ) and defined as follows: if $\xi \in {}^*H$ and if there is a standard element x of H such that $(x - \xi, y) =_1 0$ ((a, b) denotes the inner product) for all $y \in H$, then $stw(\xi) = x$. Since the unit ball of H is weakly compact it follows that every finite element $\xi \in {}^*H$ is weakly near standard [9, p. 93], so stw is defined for those elements ξ of ${}^*H$ which have finite norm. Its full domain is even larger.

Given a set $E \subseteq {}^*H$, we denote by st(E) the set $\{st(\xi) \mid \xi \in E,\ st(\xi) \text{ is defined}\}$ and by stw(E) the set $\{stw(\xi) \mid \xi \in E,\ stw(\xi) \text{ is defined}\}$.

1.1 <u>Definition:</u> Suppose E is an internal subspace of ${}^*H$, and T is a standard bounded linear operator on H. Denote by Π_E the orthogonal projection onto E. An internal subspace M of E is called invariant under T in E if $\Pi_E T \Pi_E(M) \subseteq M$, or equivalently, if $\|\Pi_E T \Pi_E \Pi_M - \Pi_M \Pi_E T \Pi_E \Pi_M\| = 0$. M is called almost invariant under T in E, or almost invariant under T_E, if $\|\Pi_E T \Pi_E \Pi_M - \Pi_M \Pi_E T \Pi_E \Pi_M\| =_1 0$. Equivalently, M is almost invariant under T_E if, for any $x \in M$ of norm 1, there is an element $y \in M$ such that $T_E(x) =_1 y$.

It is immediate that invariance implies almost invariance.

1.2 Lemma: Suppose T is a standard bounded linear operator from H into itself. Suppose E is an internal subspace of $^*H$ such that $st(E) = H$ and that A is an internal subspace of E which is almost invariant under T_E. Then $st(A)$ is a possibly trivial subspace of H left invariant by T.

Proof: The proof is quite simple and follows from the fact that, since $st(E) = H$, then for every $x \in H$, $\Pi_E(x) =_1 x$. Define $y = \Pi_E(x)$. Then $y \in E$ is that element such that

$$\|x - y\| \leq \|x - z\| \qquad 1.3$$

for each $z \in E$. Since there is an element $\xi \in H$ such that $\|\xi - x\| =_1 0$, then $\|x - y\| =_1 0$, or $x =_1 \Pi_E(x)$.

Now, suppose $x \in st(A)$. Then $x =_1 \xi \in A$. $T(x)$ is standard, so $T(x) =_1 \Pi_E\, T(x) =_1 \Pi_E(T\,\Pi_E(x)) =_1 \Pi_E\, T\,\Pi_E(y)$ because $\Pi_E\, T\, \Pi_E$ is S-continuous (cf. 9, Ch. 4.4). Finally, because A is almost invariant under $\Pi_E\, T\, \Pi_E$, there is an element $\zeta \in A$ such that

$$\Pi_E\, T\, \Pi_E(y) =_1 \zeta \quad .$$

Hence $T(x) = st(\zeta) \in st(A)$. Hence $st(A)$ is left invariant by T.

That $st(A)$ is a linear space follows from the linearity of st, and that $st(A)$ is closed is not difficult and can be found in Robinson's book [9] or Luxemburg's paper [8].

1.4 Lemma: Suppose E is an internal subspace of $^*H$ and that A and B are two subspaces of E with $A \subseteq B$, and such that A is of co-dimension 1 in B; then $st(A)$ is of co-dimension 1 in $st(B)$.

Proof: (Robinson [3, 9]) Suppose σ_1 and $\sigma_2 \in st(B)$, so there exist τ_1, $\tau_1 \in B$ such that $\sigma_1 =_1 \tau_1$, $\sigma_2 =_1 \tau_2$. Then, for some $\lambda \in {}^*\mathbb{C}$, $\tau \in A$, $\tau_1 = \lambda\, \tau_2 + \tau$, since A is of co-dimension 1 in B. We wish to show that σ_1 and σ_2 are linearly dependent over $st(A)$. We argue by contradiction, and assume independence of σ_1 and σ_2 over $st(A)$.

If $\lambda =_1 0$, then $\tau_1 =_1 \tau \in A$, so that, in this case, $\sigma_1 \in st(A)$, contrary to assumption.

If λ is infinite, then $\tau_2 = \frac{1}{\lambda}\tau_1 - \frac{1}{\lambda}\tau =_1 - \frac{1}{\lambda}\tau$, since τ_1 has finite norm. In this case, $\sigma_2 = st - \frac{1}{\lambda}\tau \in st(A)$, contrary to assumption.

We now may assume λ to be finite, so $st(\lambda)$ exists. $\tau = \tau_2 - \lambda\tau_1 =_1 \sigma_2 - st(\lambda)\,\sigma_1$, so τ is near standard, and $\sigma_2 - st(\lambda)\,\sigma_1 = st(\tau) \in st(A)$, contrary to assumption. This completes the proof of 1.4. Note that the theorem and the proof are both valid for normed space structures.

1.5 Definition: If E is an internal subspace of ${}^*H$ such that $st(E) = H$, we shall call E an H-approximating subspace of ${}^*H$, and we shall say that E approximates H. If $st(E) = K$, a subspace of H, then we call E a K-approximating subspace of ${}^*H$.

A complete decomposition of E is an ascending sequence $\{0\} = A_0 \subseteq A_1 \subseteq \dots \subseteq A_n = E$ of internal subspaces of E such that A_i is of co-dimension 1 in A_{i+1} for each $i = 0, 1, \dots, n-1$. That is to say, $\dim(A_{i+1}/A_i) = 1$.

The following lemma demonstrates the role which compact operators play in investigations of this type.

1.6 Lemma: Suppose E is a *-finite dimensional subspace of ${}^*H$ which approximates H, and that C is a nonzero compact operator on H. If $E_0 \subseteq E_1 \subseteq \dots \subseteq E_\mu = E$ is a complete decomposition of E such that each E_i is almost invariant under $\Pi_E\, C\, \Pi_E$, then for some integer λ_0, $st(E_{\lambda_0})$ is a nonzero proper subspace of H.

Proof: Note that $st(E_{\lambda_0})$ in the theorem will also be an invariant subspace of C.

It was remarked in the proof of 1.2 that $st(E_\lambda)$ is a closed subspace for each E_λ. Without loss of generality, suppose $||C|| = 1$. Define the sequence β_λ of non-negative real numbers via

$$\beta_\lambda = ||\Pi_E \, C \, \Pi_{E_\lambda}|| \quad . \tag{1.7}$$

It is clear that $\beta_0 = 0$, and because E is H-approximating,

$$\beta_\mu = ||\Pi_E \, C \, \Pi_E|| =_1 ||C|| = 1 \quad .$$

β_λ is an internal, increasing, *-finite sequence of real numbers, $0 \le \lambda \le \mu$, so there is some integer λ' such that $\beta_{\lambda'} \le \frac{1}{2} \le \beta_{\lambda'+1}$.

First, we claim that $st(E_{\lambda'}) \neq H$. For if it were, then $E_{\lambda'}$ would be H-approximating, and then

$$||\Pi_E \, C \, \Pi_{E_{\lambda'}}|| =_1 ||C|| = 1 \quad .$$

Since $\beta_{\lambda'} \le \frac{1}{2}$, this cannot be.

Next, we claim that $st(E_{\lambda'+1}) \neq \{0\}$. Since $\beta_{\lambda'+1} \ge \frac{1}{2}$, then there is an element $\xi \in E$, $||\xi|| = 1$, such that

$$||\Pi_E \, C \, \Pi_{E_{\lambda'+1}}(\xi)|| \ge \frac{1}{4} \quad .$$

Since E_λ is almost invariant under C in E, then there is a $\zeta \in E_{\lambda'+1}$ such that

$$\zeta =_1 C \, \Pi_{E_{\lambda'+1}}(\xi) \quad ,$$

and because C is compact, ζ is near standard [9, pp. 66ff.]. But

$$||\zeta|| = ||\Pi_E \zeta|| =_1 ||\Pi_E C \Pi_{E_{\lambda'+1}}(\xi)|| \geq \frac{1}{4} ,$$

so $st(\zeta)$ exists and is nonzero, so $st(E_{\lambda'+1}) \neq 0$.

Lemma 1.4 states that $st(E_{\lambda'})$ is of co-dimension 1 in $st(E_{\lambda'+1})$. Since $st(E_{\lambda'}) \neq H$ and $st(E_{\lambda'+1}) \neq 0$, we have that either $st(E_{\lambda'})$, or $st(E_{\lambda'+1})$, or both, are proper subspaces of H. This completes the proof of 1.6.

We shall now digress momentarily and introduce some standard (as opposed to non-standard) concepts which will be analogous to the definitions already presented.

1.8 <u>Definition:</u> Suppose T is a bounded linear operator on H, and M and N are subspaces of H with $N \subseteq M$. If $y \in H$, $||y - N|| = \inf_{x \in N} ||y - x||$. Define the function $\rho_T(N, M)$ via

$$\rho_T(N, M) = \sup \left\{ ||\Pi_M T \Pi_N(x) - N|| \,\middle|\, x \in H, \ ||x|| = 1 \right\} .$$

$\rho_T(N, M)$ yields a number which states how close the subspace N of M is to being an invariant subspace of the operator $\Pi_M T \Pi_M$.

1.9 <u>Definition:</u> A sequence M_n of finite dimensional subspaces of H is called an H-approximating sequence if, for every $x \in H$, there is a sequence $\{x_n\}$, $x_n \in M_n$, such that $\lim_{n\to\infty} ||x_n - x|| = 0$. In other notation, $\lim_{n\to\infty} M_n = H$.

If K is a subspace of H and $\lim_{n\to\infty} M_n = K$, then M_n is called a K-approximating sequence.

1.10 <u>Definition:</u> If M_n is an H-approximating sequence and N_n is a sequence of subspaces of M_n, then the sequence N_n is called almost invariant under T in M_n provided $\lim_{n\to\infty} \rho_T(N_n, M_n) = 0$.

We now continue as we were before by formulating a somewhat geometric condition for an operator T to have an invariant subspace. The condition, though somewhat unnatural, is necessary as well as sufficient. In section 2, we shall show how this condition arises under more natural circumstances.

1.11 <u>Theorem:</u> <u>Suppose that T is a bounded linear operator and C is a compact operator, and that E_n is an H-approximating sequence of finite dimensional subspaces of H. If there is a complete decomposition of E_n into subspaces $0 = A_0^n \subseteq A_1^n \subseteq \ldots \subseteq A_{k_n}^n = E_n$ such that every sequence $A_{j(n)}^n \subseteq E_n$ is almost invariant under C and under T in E_n, then T has an invariant subspace.</u>

<u>Proof:</u> Most of the work has already been done, and we need only to transfer the above concepts to their nonstandard counterpart, and apply the previous lemmas. Let μ be an infinitely large integer. Since E_n is an H-approximating sequence, then $st(E_\mu) = H$.

To see this, pick $x \in H$. There is a sequence $x_n \in E_n$ such that $\lim_{n\to\infty} x_n = x$. Hence $st(x_\nu) = x$ for each infinite ν, in particular for $\nu = \mu$. But $x_\mu \in H_\mu$ so $x \in st(E_\mu)$.

For every integer ν, $1 \le \nu \le k_\mu = \dim(E_\mu)$, the subspace A^μ_ν of E_μ is almost invariant under $\Pi_{E_\mu} T \Pi_{E_\mu}$ and $\Pi_{E_\mu} C \Pi_{E_\mu}$. In other words,

$$\rho_C(A^\mu_\nu, E_\mu) =_1 \rho_T(A^\mu_\nu, E_\mu) =_1 0 \quad .$$

We shall argue on T, the same proof applying to C as well.

Suppose β is a standard positive real number, and $\rho_T(A^\mu_\nu, E_\mu) \ge \beta$. Then for every standard integer M, the following sentence is true in $^*H$. (Recall that $k_m = \dim(E_m)$).

"There exists an integer m, there exists an integer n, $m \ge M$, $1 \le n \le k_m$, such that $\rho_T(A^m_n, E_m) \ge \beta$."

Then this sentence also holds true in H, for every integer M. Hence, we may construct a sequence $A^n_{j(n)} \subseteq E^n$ such that $\rho_T(A^n_{j(n)}, E^n)$ does not converge to 0, contrary to the hypotheses of the theorem.

Lemma 1.2 states that $st(A^\mu_\nu)$ is a possibly trivial subspace of H left invariant by T, for all ν, $1 \le \nu \le k_\mu$. Since $\{A^\mu_\nu\}$ is a complete decomposition of E and each A^μ_ν is almost invariant under C in E, we may apply lemma 1.6 to assert that for some integer ν_0, $1 \le \nu_0 \le k_\mu$, $st(A^\mu_{\nu_0})$ is a nonzero proper subspace of H. Hence, $st(A^\mu_{\nu_0})$ is an invariant subspace for T.

1.12 Note: It happens that the conditions in theorem 1.11 are necessary as well as sufficient. That is, the following statement holds true.

1.13 Theorem: Suppose T is a bounded linear operator on H which has an invariant subspace. Then there is an H-approximating sequence E_n of finite dimensional subspaces of H, and a compact operator C with the following properties: there is a complete decomposition $0 = A_0 \subseteq A_1^n \subseteq \ldots \subseteq A_{k_n} = E_n$ of E_n such that every sequence $A_{j(n)}^{(n)}$ is almost invariant under T and under C in E_n.

The proof of 1.13 is actually trivial and sheds no insight into the problem. For if M is an invariant subspace, let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of M, and $\{d_i\}_{i-n}^{\infty}$ be an orthonormal basis of $M^{\perp}$. Let $E_n = \{e_i, \ldots, e_n, d_i, \ldots, d_n\}$, the linear span of $e_1, e_2, \ldots, e_n$, $d_1, d_2, \ldots, d_n$; E_n is H-approximating and $A_n^n = \{e_1, \ldots, e_n\}$ is invariant under T_{E_n}, and A_n^n is M-approximating. A compact operator satisfying the conditions need be no more complicated than the following: $C(d_1) = e_1$, $C(x) = 0$ for all $x \perp d_1$. Indeed, C cannot be too much more complicated, or it would imply the existence of more invariant subspaces. The construction of the complete decomposition A_k^n, $1 \leq k \leq 2n$ such that each A_k^n is invariant under T and under C in E_n is quite easy. Merely apply theorem 0.1 to T restricted to A_n^n, and to the induced operator $T\colon E_n/A_n^n \to E_n/A_n^n$.

Actually, we have shown that the sequence E_n need not be so dependent upon the knowledge of M. In fact, given an arbitrary

H-approximating sequence E_n', there is a subsequence E_n of co-dimension 1 in E_n' which satisfies the hypotheses of 1.13. The proof, however, is tedious, and conceptually quite analogous to the above.

1.14 Note: The hypotheses of 1.13 assumes the existence of only one invariant subspace, and the conclusion of 1.11 yields only one invariant subspace. Actually, we are interested more in operators T with a dense invariant subspace property, defined as follows: for any two distinct subspaces $M \subseteq N \subseteq H$ which are left invariant by T, there is a subspace P of H, $M \subseteq P \subseteq N$, $M \neq P$, $P \neq N$, such that P is left invariant by T. For if T does not have this property, say, for two distinct subspaces M and N as above, there is no such P, then the induced operator $\hat{T}$: $N/M \rightarrow N/M$ has no invariant subspaces.

Hence, the truly valuable theorem would be one which places necessary and sufficient conditions for T to have the dense invariant subspace property. In fact, if the compact operator in theorem 1.11 has its null space equal to $\{0\}$, then T has this density property. Proving the necessity of this condition would be quite involved, even if possible. Such a theorem will probably have conditions more complicated than simple algebraic relationships between T and a compact operator C, such as $p(T) = C$ for some polynomial p, or $TC = CT$. The orthonormal shift operator (Introd.) for example, has the dense invariant subspace property, but S commutes with no compact operator C, except $C = 0$. This implies, in particular, that no sequence $p_n(S)$ can converge to a nonzero compact operator in any topology. To see this, suppose C is compact, and $SC = CS$. Then $S^nC = CS^n$. Let

$\{e_n\}$ be the orthonormal basis along which S shifts.

$$||CS^n(e_j)|| = ||C(e_{j+n})|| = ||S^nC(e_j)|| = ||C(e_j)||\,;$$

but $\lim_{n\to\infty} ||C(e_{n+j})|| = 0$, so $||C(e_j)|| = 0$, $j=1,2,\ldots$.

Hence $C = 0$.

1.15 Note: We wish to indicate how the foregoing sequence of theorems can be stated and proved in the case that the structure at hand is a normed space. The method we present here will make use of the techniques of F. Bonsall [4] who first generalized the Bernstein-Robinson theorem to normed spaces. Conceptually the statements and proofs are the same. We only need to circumvent the use of orthogonality properties, and in particular, the use of projections.

We shall, as before, restrict ourselves to the case in which the normed space N is separable.

Instead of discussing arbitrary N-approximating sequences, we shall construct a specific N-approximating sequence E_n using Bonsall's technique.

Pick $e_0 \in N$ of norm 1. Let $E_0 = \{e_0\}$. Let $E_1 = \{e_0, T(e_0)\}$. Similarly, let $E_n = \{e_0, T(e_0), \ldots, T^n(e_0)\}$. Note that $T(E_n) \subseteq E_{n+1}$. If $E_n = E_m$ for all n greater than some m, then E_m is an invariant subspace of T. We are aiming in the direction of theorem 1.11, in which we conclude that T has an invariant subspace. Hence, we shall assume that E_n is strictly increasing, and that

$$\lim_{n\to\infty} E_n = \{y \in N \mid \text{there exists } x_n \in E_n,\ ||x_n - y|| \to 0\}$$

is equal to N.

The essential feature of Hilbert Spaces, which, in general, is absent in a normed space, is the concept of orthogonality. We circumvent this difficulty by constructing a basis $e_0, \ldots, e_n, \ldots$ for N which has the desired properties of an orthonormal basis. This is done via the following lemma.

1.16 Lemma: If A and B are two finite dimensional subspaces of N, $A \subseteq B$ and A is of co-dimension 1 in B, then there is an element $b \in B$ of norm 1 such that the distance from b to any element of A is greater than or equal to 1; that is, $\inf_{a \in A} \|b - a\| = 1$.

Proof: The finite dimensionality of A implies that bounded sets in A are compact. Pick $b' \in B$, $b' \notin A$ such that $\|b'\| = 1$. It follows that $\inf_{a \in A} \|b' - a\|$ is attained at some $a_0 \in A$. $\|b' - a_0\| \neq 0$ since $b' \notin A$. The element $b = b' - a_0 / \|b' - a_0\|$ satisfies the requirements of the theorem.

Now define the sequence $\{e_n\}$ $n = 0, 1, \ldots$ as follows. e_0 is already defined. Let $e_1 \in E_1$ be an element of norm 1 such that $\|e_1 - E_0\| = 1$. Similarly, let $e_n \in E_n$ such that $\|e_n\| = 1$, $\|e_n - E_{n-1}\| = 1$. This defines a sequence e_n. Note that $\{e_0, \ldots, e_n\} = E_n$ for $n = 0, 1, 2, \ldots$.

For each n, $T(e_n) \in E_{n+1}$, so there is a number $\alpha_{n+1} \in \mathbb{C}$ and an element $y_n \in E_n$ such that

$$T(e_n) = \alpha_{n+1} e_{n+1} + y_n \quad . \tag{1.17}$$

Define the linear transformation T_n on E_n as follows. If $m < n$, $T_n(e_m) = T(e_m) \in E_{m+1} \subseteq E_n$; if $m = n$, $T_n(e_n) = y_n \in E_n$, where y_n is defined by 1.17.

The concepts of invariance and almost invariance are quite the same as before; the linear transformation T_n takes the place of $\Pi_{E_n} T \Pi_{E_n}$ in the earlier definition. Also lemma 1.2 holds as stated. That is, if μ is an infinitely large integer and A is an internal subspace of E_μ (the μ^{th} entry in the sequence E_n defined above) and A is almost invariant under T_μ, then st(A) is a possibly trivial subspace of N which is left invariant by T. The proof of this statement is the same as the proof of 1.2, once we have verified the following: if y is a standard element of N, then $T(y) =_1 T_\mu(y)$. But since $st(E_{\mu-1}) = N_1$ then there is an element $\xi \in E_{\mu-1}$ such that $\xi =_1 y$. $T_\mu(\xi) = T(\xi) \in E_\mu$. $T(\xi) =_1 T(y)$, and $T_\mu(\xi) =_1 T(y)$, so $T(y) =_1 T_\mu(y)$.

As noted earlier, lemma 1.4 holds as stated and as proved for normed spaces. Definitions 1.5 a) and b) also hold for normed spaces.

The sequence E_n of subspaces of N was defined with respect to T. Hence, for a different operator C, it is slightly more difficult to define the analog for $\Pi_{E_n} C \Pi_{E_n}$. We take an easier route. Since C will always be a compact operator, it is possible to define a concept of almost invariance with respect to C, rather than with respect to some C_n related to C. For, if E_μ is H-approximating, then E_μ is almost invariant under C in H; that is, if $x \in E_\mu$, $||x|| = 1$, then, since $C(x)$ is near standard, and E_μ is H-approximating, then there is a $z \in E_\mu$ such that $C(x) =_1 z$. Because of this, we may define, for any subspace A of E_μ, the following concept of almost invariance: A is almost invariant under C in E_μ if, for any $x \in A$ of norm 1, there is a $y \in A$ such that $C(x) =_1 y$. With this definition, lemma 1.6 holds true in N as stated. The proof is essentially the same, except for the definition of the sequence β_λ. In this case, we define β_λ as follows:

$$\beta_\lambda = \max_{\substack{x \in E_\lambda \\ ||x|| = 1}} ||C(x)|| \quad . \tag{1.7'}$$

With β_λ so defined, the proof of 1.6 is essentially as stated earlier.

We shall omit the reduction to nonstandard terminology, such reduction being quite the same as before. Theorem 1.11 holds as stated and essentially as proved, given the above modifications of the necessary lemmas. The introduction of the ersatz orthonormal sequence $\{e_n\}$ is not really of primary importance here, but is considerably more necessary in Chapter 2. Where appropriate, we shall insert a sentence or two in Chapter 2 which indicates how a particular proof would be carried out in normed spaces.

CHAPTER 2

INVARIANT SUBSPACES FOR OPERATORS SATISFYING CERTAIN ALGEBRAIC CONDITIONS

We now proceed to demonstrate how the hypotheses of Theorem 1.11 come about under more natural situations. Of course, if the operator T in question is itself compact, the hypotheses are immediately satisfied, since the matrix $\Pi_{E_n} T \Pi_{E_n}$ always has a complete decomposition of invariant subspaces (theorem 0.1). If T is polynomially compact, then the hypotheses of theorem 1.11 are satisfied; this is the Bernstein-Robinson theorem. We shall prove the most general result in this direction. The statement of the theorem was communicated by Feldman in the notices of the American Mathematical Society [5]. The proof of his theorem was proved independently by the author in a more general setting, and was subsequently proved by Gillespie [6] for normed spaces, using the methods developed by Feldman and Bonsall.

Thus far, there have been no requirements on the way in which the sequence of operators $\Pi_{E_n} T \Pi_{E_n}$ approximate T. The following definition proposes a stronger type of approximation.

2.1 <u>Definition:</u> The bounded linear operator T is called quasi-compact if there is a sequence E_n of finite dimensional subspaces of H such that $\lim_{n\to\infty} E_n \neq \{0\}$ and such that $\lim_{n\to\infty} || \Pi_{E_n} T \Pi_{E_n} - T \Pi_{E_n} || = 0.$

For normed spaces, we require

$$\lim_{n\to\infty} \left[\sup_{x \in E_n, ||x|| = 1} \left(|| T_n(x) - T(x) || \right) \right] = 0$$

We digress momentarily to justify the intrusion of this definition. Suppose, for the sake of convenience, that we call a backward operator one which has a complete invariant subspace decomposition of the form $0 \subseteq E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n \subseteq \ldots$ with $\dim(E_n/E_{n-1}) = 1$ for $n \geq 1$, and with $\lim_{n\to\infty} E_n = H$. A forward operator will be one whose adjoint is backward; equivalently, a forward operator may be defined as above, *replacing the increasing chain by a decreasing chain.* The easiest example of a backward operator is the adjoint S^* of the orthonormal shift operator. The orthonormal shift operator, S, is a forward operator.

Stampfli [10] has shown that perturbations of S^* (or of S), that is, operators of the form $S^* + C$, with C compact, possess invariant subspaces. In fact, such operators $S^* + C$ even possess eigenvalues. It would be interesting to inquire whether or not perturbations of a backward operator B (i.e., operators of the form $B + C$ with C compact) always possess invariant subspaces.

Note that such operators $B + C$ are quasi-compact. The following theory does not answer the above question, but does yield an answer in the event that the uniformly closed algebra of $B + C$ contains a nonzero compact operator.

2.2 Lemma: If T is compact, T is quasi-compact.

Proof: For any H-approximating sequence E_n of finite dimensional subspaces of H, $\lim_{n\to\infty} ||\Pi_{E_n} T \Pi_{E_n} - T|| = 0.$

2.3 Lemma: If T is quasi-nilpotent, then T is quasi-compact.

Proof: Let $e_1 \epsilon H$, $||e_1|| = 1$.

Let $E_1 = \{e_1\}$, $E_2 = \{e_1, T(e_1)\}, \ldots, E_n = \{e_1, \ldots, T^{n-1}(e_1)\}$ all $n \epsilon N$. If $E_m = E_{n_0}$ for all $m \geq n_0$, some n_0, then T is quasi-compact, since

$$||\Pi_{E_{n_0}} T \Pi_{E_{n_0}} - T \Pi_{E_{n_0}}|| = 0 \ .$$

Hence, assume E_n is strictly increasing, and define $e_n \epsilon E_n$ to be an element of norm 1 such that $e_n \perp E_{n-1}$.

Since $T(e_n) \epsilon E_{n+1}$, then $T(e_n) = \alpha_{n+1} e_{n+1} + y_n$, $y_n \epsilon E_n$, and this relation defines a sequence α_n of complex numbers.

$$T^{n+1}(e_1) \equiv \alpha_2 \alpha_3 \ldots \alpha_{n+1} e_{n+1} \bmod E_n \ . \tag{2.4}$$

Hence

$$|| T^{n+1}(e_1)|| \geq |\alpha_2| \ |\alpha_3| \ \ldots \ |\alpha_{n+1}| \ . \tag{2.5}$$

$$0 =_1 \ ||T^{\omega+1}||^{1/\omega+1} \geq |\alpha_2| \ \ldots \ |\alpha_{\omega+1}|$$

$$\geq \left[\left[\min_{2 \leq i \leq \omega+1} \alpha_i\right]^{\omega+1}\right]^{1/\omega+1} = \min_{2 \leq i \leq \omega} [\alpha_i] \tag{2.6}$$

for infinitely large ω. Hence, $\alpha_\mu =_1 0$, some $\mu \epsilon {}^*N$, $2 \leq \mu \leq \omega$.

For this μ,

$$||\Pi_{E_{\mu-1}} T \Pi_{E_{\mu-1}} - T \Pi_{E_{\mu-1}}|| = |\alpha_\mu| =_1 0 \;;$$

or T is quasi-compact. Note that this theorem is true for normed spaces, with the definitions as stated in note 1.15.

2.7 Lemma: (Berstein-Robinson) If T is polynomially compact, then T is quasi-compact.

Proof: Generate a sequence $\{e_n\}$ of H and a sequence α_n of complex numbers as in the proof of 2.3. Let

$$p(x) = \sum_{k=0}^{n} \beta_k x^k$$

be a polynomial such that p(T) is compact, and such that $\beta_n \neq 0$.

For each j,

$$p(T)\,(e_j) = \sum_{k=0}^{n} \beta_k T^k(e_j)$$

$$\equiv \beta_n \alpha_{j+1} \cdots \alpha_{j+n+1}\, e_{j+n+1} \bmod E_{j+n} \quad .$$

Take j to be infinite, so $stw(e_j) = 0$. p(T) compact implies $||p(T)(e_j)|| =_1 0.$*

*If $\xi \in {}^*H$ is of finite norm and $stw(\xi) = 0$, and C is a standard compact operator, then $C(\xi)$ is near standard. But since C is a bounded linear operator, C is weakly continuous, so $stw(C(\xi)) = 0$. $C(\xi)$ being near standard, $stw(C(\xi)) = st(C(\xi)) = 0$.

$$0 =_1 p(T)(e_j) \geq |\beta_n| \, |\alpha_{j+1} \cdots \alpha_{j+n+1}|$$
$$\geq \beta_n \min_{j+1 \leq k \leq j+n+1} |\alpha_k| \quad . \qquad (2.8)$$

This implies that $\alpha_k =_1 0$ some k, and for this k,

$$\|\Pi_{E_{k-1}} T \Pi_{E_{k-1}} - T \Pi_{E_{k-1}}\| =_1 0 \quad ,$$

as in the proof of 2.3.

Note that 2.7 also holds for normed spaces, with appropriate modifications of the proof.

We now state the theorem alluded to in the beginning of this section.

2.9 <u>Theorem</u>: <u>If</u> T <u>is quasi-compact and if the algebra generated by</u> T <u>contains a nonzero compact operator in its norm closure, then</u> T <u>has an invariant subspace.</u>

<u>Note</u>: Feldman's statement was for quasi-nilpotent operators.

<u>Proof</u>: Let E_n be a sequence of finite dimensional subspaces of H such that

$$\lim_{n\to\infty} \|\Pi_{E_n} T \Pi_{E_n} - T \Pi_{E_n}\| = 0 \quad . \qquad (2.10)$$

If $st(E_\mu) \neq H$ ($st(E_\mu) \neq 0$ by definition) for an infinitely large $\mu \in {}^*N$, then $st(E_\mu)$ is an invariant subspace, by Lemma 1.2. Hence we assume that E_μ is H-approximating.

Let $x \in E_\mu$, $||x|| = 1$. 2.10 assures us that

$$\Pi_{E_\mu} T \Pi_{E_\mu}(x) =_1 T \Pi_{E_\mu}(x) = T(x) \quad .$$

We wish to show that

$$\left[\Pi_{E_\mu} T \Pi_{E_\mu}\right]^n (x) =_1 T^n(x) \tag{2.11}$$

for all standard $n \in N$. We proceed by induction on n, and assume that 2.11 holds for a given n. Then

$$T\left[\Pi_{E_\mu} T \Pi_{E_\mu}\right]^n (x) =_1 T\left(T^n(x)\right) \quad ,$$

since T is continuous. But $\left[\Pi_{E_\mu} T \Pi_{E_\mu}\right]^n(x) \in E_\mu$, so 2.10 implies that

$$\Pi_{E_\mu} T\left[\Pi_{E_\mu} T \Pi_{E_\mu}\right]^n(x) =_1 T\left[\Pi_{E_\mu} T \Pi_{E_\mu}\right]^n(x) \quad ,$$

and consequently

$$\left[\Pi_{E_\mu} T \Pi_{E_\mu}\right]^{n+1}(x) =_1 T^{n+1}(x) \quad .$$

Since the uniform closure of $A(T)$, the algebra generated by T, contains a nonzero compact operator C, then there is a sequence of polynomials p_n such that

$$\lim_{n\to\infty} ||p_n(T) - C|| = 0 \quad . \tag{2.12}$$

By 2.11, we have

$$d_n = ||p_n\left(\Pi_{E_\mu} T \Pi_{E_\mu}\right) - p_n(T)\, \Pi_{E_\mu}|| =_1 0 \tag{2.13}$$

for all standard integers n.

But d_n is an internal sequence, and $d_n =_1 0$ for all $n \in N$. For such sequences, Robinson first observed that there is an integer $\nu_0 \in {}^*N-N$ such that $d_n =_1 0$ for all $n \le \nu_0$. This follows by considering the sequence $n d_n$, which also has the property that $n d_n =_1 0$ all $n \in N$. Hence the set $\{n \in {}^*N \mid 0 \le |n d_n| \le 1\}$ is internal and contains N, so it must contain an interval $[1, \nu_0]$, some $\nu_0 \in {}^*N-N$ [9]. Hence, 2.13 holds for all $n \le \nu_0$.

Since $$\|p_{\nu_0}(T) - C\| =_1 0 \quad , \qquad \text{then}$$

$$\| p_{\nu_0}(T)\, \Pi_{E_\mu} - C\, \Pi_{E_\mu} \| =_1 0 \quad , \qquad \text{and so}$$

$$\|p_{\nu_0}\left(\Pi_{E_\mu}\, T\, \Pi_{E_\mu}\right) - C\, \Pi_{E_\mu} \| =_1 0 \quad , \qquad \text{by (2.13).}$$

The rest is easy. Let $E_0 \subseteq E_1 \subseteq \ldots \subseteq E_\mu$ be a complete decomposition of E_μ into invariant subspaces of $\Pi_{E_\mu}\, T\, \Pi_{E_\mu}$. Then each of these subspaces is invariant under $p_{\nu_0}\left(\Pi_{E_\mu}\, T\, \Pi_{E_\mu}\right)$, and hence, (2.13) states that they are almost invariant under $C\, \Pi_{E_\mu}$, and hence under $\Pi_{E_\mu}\, C\, \Pi_{E_\mu}$. This is precisely the necessary requirement to apply theorem 1.11, and so the proof of theorem 2.9 is complete.

The orthonormal shift operator S on $\{e_n \mid n = 1, 2, \ldots\}$, an orthonormal basis for H, is not quasi-compact, but the adjoint of S is quasi-compact. Thus, in a sense which we shall leave imprecise, an operator which is very "forward" may have an adjoint which is quasi-compact. Such a condition will be sufficient to yield a conclusion as in theorem 2.9.

2.14 Theorem: If the uniformly closed algebra generated by T contains a nonzero compact operator, and if T^*, the adjoint of T, is quasi-compact, then T has an invariant subspace.

Proof: We shall use theorem 2.9 to show that T^* has an invariant subspace, which in turn implies that T has an invariant subspace.

If p_n is a sequence of polynomials such that $\|p_n(T) - C\| \to 0$, where C is a nonzero compact operator, then $\|[p_n(T) - C]^*\| = \|p_n(T) - C\|$ converges to zero. But $[p_n(T) - C]^* = p_n^*(T^*) - C^*$, where p_n^* is the polynomial optained from p_n by taking the complex conjugate of its coefficients. Since C is compact, C^* is also compact (and nonzero) so 2.9 implies that T^* has an invariant subspace.

2.15 Note: It is an open question whether or not one may weaken the condition of quasi-compactness, or to consider a convergence of $p_n(T)$ in a weaker topology.

In a slightly different direction one may ask this question: if T commutes with a nonzero compact operator C, then does T possess an invariant subspace. The question seems justified in view of the fact that two commuting $n \times n$ matrices over $\mathbb{C}$ possess a common complete decomposition $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = V$ of invariant subspaces. As in the case of the existence of a complete decomposition of invariant subspaces for a single matrix, the theorem is an algebraic one, following from the fact that commutivity implies the existence of a common eigenvector for the two finite dimensional operators.

The following lemma generalizes the above theorem for use in the present investigation.

2.16 Lemma: Let $\varepsilon > 0$ be a real number, and n be an integer. Then there is a real number $\delta = \delta(\varepsilon, n) > 0$ with the following property: for any two complex $n \times n$ matrices defining linear transformations of operator norm ≤ 1 on a complex n-dimensional vector space E, then the relation $\|AB - BA\| < \delta$ implies the existence of a complete decomposition

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = E$$

such that $\rho_A(M_i, E) < \varepsilon$, and $\rho_B(M_i, E) < \varepsilon$ for $i = 0, 1, 2, \dots, n$.

[Recall (Def. 1.8) that

$$\rho_T(N, M) = \sup\{\|\Pi_M T \Pi_N(x) - N\|, x \in N, \|x\| = 1\}.]$$

Proof: Suppose the statement is not true. Then there is an ε_0, n_0 for which it fails: that is, for every $\delta > 0$, there exist two $n \times n$ matrices A, B of norm 1, such that $\|AB - BA\| < \delta$, and such that for every complete decomposition $0 \subseteq M_1 \subseteq \dots \subseteq M_n = E$ of E, either

$$\rho_A(M_i, E) \geq \varepsilon_0 \quad \text{or} \quad \rho_B(M_i, E) \geq \varepsilon_0 \tag{2.17}$$

for some $i = 1, 2, \dots, n$.

Then this statement remains valid in the enlargement for some $\delta =_1 0$: that is, there are two matrices A, B of operator norm 1, with

$$\|AB - BA\| =_1 0 \tag{2.18}$$

which satisfy (2.17).

Since $\|A\| = \|B\| = 1$, then the coefficients (α_{ij}) of A and (β_{ij}) of B must be finite, so $(st(\alpha_{ij}))$ and $(st(\beta_{ij}))$ exist, and since a matrix depends continuously on its coefficients, we have

$$\|A - (st(\alpha_{ij}))\| =_1 0 =_1 \|B - (st(\beta_{ij}))\| \tag{2.19}$$

$$\begin{aligned}
&\|(st(\alpha_{ij}))(st(\beta_{ij})) - (st(\beta_{ij}))(st(\alpha_{ij}))\| \\
&\le \|(st(\alpha_{ij}))(st(\beta_{ij})) - (st(\alpha_{ij}))B\| + \|(st(\alpha_{ij})B) - AB\| \\
&+ \|AB - BA\| + \|BA - B(st(\alpha_{ij}))\| + \|B(st(\alpha_{ij})) - (st(\beta_{ij}))(st(\alpha_{ij}))\| \\
&\le \|(st(\alpha_{ij}))\| \, \|(st(\beta_{ij})) - B\| + \|(st(\alpha_{ij})) - A\| \, \|B\| \\
&+ \|AB - BA\| + \|B\| \, \|A - (st(\alpha_{ij}))\| \\
&\qquad + \|B - (st(\beta_{ij}))\| \, \|(st(\alpha_{ij}))\| =_1 0
\end{aligned} \tag{2.20}$$

by (2.18) and (2.19).

But $(st(\alpha_{ij}))$ and $(st(\beta_{ij}))$ being standard matrices, the norm of their commutator is standard, and hence 0. Hence there is a complete decomposition $0 \subseteq M_1 \subseteq \ldots \subseteq M_n = E$ of subspaces of E which are invariant under $(st(\alpha_{ij}))$ and $(st(\beta_{ij}))$. By (2.19), it must follow that $\rho_A(M_i, E) =_1 0$ and $\rho_B(M_i, E) =_1 0$, $i=1,\ldots,n$ which contradicts (2.17). This completes the proof of 2.16.

We shall make use of this function $\delta(\epsilon, n)$ in a later proof. It's dependence on n is its crucial feature. Calculations to determine

$\delta(\varepsilon, n)$ are very difficult, and involve very rough estimates; such results seem to indicate $\delta \sim \varepsilon^n/n$.

2.20 Lemma: If T and C are commuting bounded linear operators on H, and if E is a subspace of H, then

$$||\Pi_E T \Pi_E \Pi_E C \Pi_E - \Pi_E C \Pi_E \Pi_E T \Pi_E||$$
$$\leq ||\Pi_E C(I-\Pi_E)|| \; ||(I-\Pi_E)T|| + ||\Pi_E T(I-\Pi_E)|| \; ||(I-\Pi_E)C|| \quad (2.21)$$

Proof: For simplicity of notation, we denote Π_E, the orthogonal projection onto E, by Π, and $I-\Pi$ by $\Pi^\perp$.

$$||\Pi T \Pi C \Pi - \Pi C \Pi T \Pi|| \leq ||\Pi T \Pi C \Pi - \Pi T C \Pi|| + ||\Pi C T \Pi - \Pi C \Pi T \Pi|| \quad . \qquad (2.22)$$

$$||\Pi T \Pi C \Pi - \Pi T C \Pi|| \leq ||\Pi T(\Pi C - C|| = ||\Pi T \Pi^\perp \Pi^\perp C|| \leq ||\Pi T \Pi^\perp|| \; ||\Pi^\perp C|| \quad . \qquad (2.23)$$

Similarly

$$||\Pi C T \Pi - \Pi C \Pi T \Pi|| \leq ||\Pi C \Pi^\perp|| \; ||\Pi^\perp T|| \quad . \qquad (2.24)$$

Combining (2.22, 23, and 24), we obtain (2.21).

We require two more inequalities.

2.25 Lemma: $||\Pi C - C|| \leq ||\Pi C \Pi - C||$.

2.26 Lemma: $||\Pi C \Pi^\perp|| \leq ||\Pi C \Pi - C||$.

Proof of 2.25: Let $\xi \in H$, $||\xi|| = 1$, $\xi = \Pi(\xi) + \Pi^\perp(\xi) = x + y$.

$$\|\Pi C \Pi - C(x+y)\|^2 = \|\Pi C \Pi(x) + \Pi C \Pi(y) - C(x) - C(y)\|^2$$
$$= \|-\Pi^{\perp} C(x) - C(y)\|^2 = \|\Pi^{\perp} C(x) + \Pi^{\perp} C(y) + \Pi C(y)\|^2$$
$$= \|\Pi^{\perp} C(x) + \Pi^{\perp} C(y)\|^2 + \|\Pi C(y)\|^2$$
$$\geq \|\Pi^{\perp} C(x+y)\|^2 = \|\Pi^{\perp} C(\xi)\|^2 \quad .$$

Therefore,

$$\|\Pi C \Pi - C\| = \sup_{\|\xi\|=1} \|(\Pi C \Pi - C)(\xi)\|$$
$$\geq \|(\Pi C \Pi - C)(\xi)\| \geq \|\Pi^{\perp} C(\xi)\|$$

for each $\xi \in H$, $\|\xi\| = 1$. Taking the sup on the right, we obtain (2.25).

Proof of 2.26: $\|\Pi C \Pi^{\perp}\| = \|\Pi C - \Pi C \Pi\|$
$$\leq \|\Pi\| \, \|C - \Pi C \Pi\| = \|\Pi C \Pi - C\| \quad .$$

Let C be a compact operator on H. Then for any H-approximating sequence E_n of finite dimensional subspaces of H, $\|\Pi_{E_n} C \Pi_{E_n} - C\| \to 0$ as $n \to \infty$. This justifies the following definition.

2.27 Definition: We say that a compact operator C is δ-compact (see 2.15) if there is an H-approximating sequence E_n of finite dimensional subspaces of H such that

$$\delta\left(\|\Pi_{E_n} C \Pi_{E_n} - C\| \ , \ \dim E_n\right) \to 0 \quad \text{as} \quad n \to \infty \ .$$

2.28 Theorem: *If T is a bounded linear transformation on H which commutes with a δ-compact operator C, then T has an invariant subspace.*

Proof: Without loss of generality, assume $||T|| = \frac{1}{2}$. We may assume that the sequence E_n is H-approximating, for if $\lim_{n\to\infty} E_n = K \subseteq H$, the properties of the function δ imply that $C = 0$ on $K^{\perp}$. But the set $\{y \mid C(y) = 0\}$ is a subspace left invariant by T, and if $C \neq 0$, and $K^{\perp} \neq 0$, then it is a proper nonzero invariant subspace for T.

Inequalities 2.21, 2.25, and 2.26 yield

$$||\Pi_{E_\omega} T \Pi_{E_\omega} C \Pi_{E_\omega} - \Pi_{E_\omega} C \Pi_{E_\omega} T \Pi_{E_\omega}|| \leq ||\Pi_{E_\omega} C \Pi_{E_\omega} - C||$$

for $\omega \in {}^*N$.

Then

$$\delta\ ||\Pi_{E_\omega} T \Pi_{E_\omega} C \Pi_{E_\omega} - \Pi_{E_\omega} C \Pi_{E_\omega} T \Pi_{E_\omega},\ \dim E_\omega|| =_1 0\ .$$

Lemma 2.16 demonstrates the existence of a complete decomposition $0 \subseteq M_1 \subseteq \ldots \subseteq M_\omega = E_\omega$ of E_ω such that each M_λ is almost invariant under $\Pi_{E_\omega} T \Pi_{E_\omega}$ and under $\Pi_{E_\omega} C \Pi_{E_\omega}$. Hence, theorem 1.11 applies, and T has an invariant subspace.

We state the following easily demonstrated theorem to demonstrate a possible direction to go in quest of the goal mentioned in Note 1.17 of section 1.

2.29 Theorem: Suppose T is a bounded linear operator on H and E_n is an H-approximating sequence of finite dimensional subspaces, and p_n is a sequence of polynomials such that, for some compact operator C, $\lim_{n\to\infty} ||p_n(\Pi_{E_n} T \Pi_{E_n}) - C|| = 0$. Then T has an invariant subspace.

It might be remarked that the standard theory for self adjoint operators can be obtained in the manner herein described, but the machinery required for the proof is essentially that which is required for the usual type of proofs [cf. Bernstein].

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