AN INVESTIGATION OF SPONTANEOUS BREAKDOWN OF SU(3) SYMMETRY IN THE SYSTEM OF PSEUDOSCALAR MESON AND BARYON OCTETS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1967

(Submitted May 26, 1967)

ACKNOWLEDGMENTS

I like to thank Professor F. Zachariasen for his guidance and encouragement during my graduate studies, and Professor S. C. Frautschi for useful conversations. I am most happy to acknowledge the good wishes of these and other members of the faculty (and, too, their patience!).

I am grateful for financial support received from the California Institute, the Woodrow Wilson Foundation, the I.B.M. Foundation, the Schlumberger Foundation, and the National Science Foundation.

ABSTRACT

The system of pseudoscalar meson and baryon octets is examined for spontaneous breakdown of SU(3) symmetry in a bootstrap theory with vanishing renormalization constants. The latter are calculated in second order perturbation theory; the splittings are taken to retain SU(2) symmetry and are included to first order only. The equations are diagonalized in the dimension of the representations.

The F-D mixing parameter α is found to have the value 3/4. The system shows great instability in the $\underset{\sim}{8}$ representation, and two solutions exist for not unreasonable values of the coupling constant and the meson-baryon mass ratio; one of the solutions has the observed relative signs for the mass splittings and exhibits a coupling splitting pattern found by other workers. A solution exists in the $\underset{\sim}{27}$ representation with a coupling constant squared which is two orders of magnitude too large.

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I. INTRODUCTION

A number of investigations indicate that many of the observed features in elementary particle physics may arise from the bootstrap hypothesis,[1] which can be stated as a self-consistency requirement. Assuming a set of particles with certain forces, it is often found that the system requires, in a non-trivial way, the existence of particles which can be identified with the ones used as an input.[2] Or again, the bootstrap condition has been found to produce symmetries[3] and conservation laws[4] observed in nature.

The present work is an investigation of the latter type into whether the observed breakdown of SU(3) symmetry[5] in the pseudo-scalar meson and baryon octets can be produced by the mutual interactions among those particles themselves. We do not admit any other particles into our system; it is clear, therefore, that we are making a very drastic approximation to nature. Nevertheless, it is of interest to ask how this system behaves when isolated from everything else. In keeping with the spirit of a true bootstrap theory we shall treat the mesons and the baryons on an equal footing.[6] (Often in practical bootstrap calculations not all of the particles in the system are required to be generated self-consistently; in effect, then, some particles are treated as elementary.) The bootstrap requirement will be imposed in the form of vanishing renormalization constants,[7,8] calculated in second order perturbation theory. As an additional restriction in our problem we include the splittings of masses and coupling constants to first order only. It is evident that the results cannot be expected to be quantitatively significant, but it is hoped that the investigation may give some useful insights into the problem.

The procedure will be to calculate the vertex renormalization constants and the baryon and meson wave function renormalization constants as a function of all the independent couplings and masses in the problem. We write

$$
\begin{aligned}
Z_{\bar{i}jk} &= Z_{\bar{i}jk}(g_{\bar{r}st}, m_q, \mu_\ell) \ , \\
Z_{2i} &= Z_{2i}(g_{\bar{r}st}, m_q, \mu_\ell) \ , \qquad (1.1) \\
Z_{3k} &= Z_{3k}(g_{\bar{r}st}, m_q, \mu_\ell) \ .
\end{aligned}
$$

$Z_{\bar{i}jk}$ is the renormalization constant for the vertex at which meson k may be absorbed into baryon j to form baryon i, and Z_{2i} and Z_{3k} are the renormalization constants for baryon i and meson k respectively. $g_{\bar{r}st}$ is the coupling constant at the vertex in which meson t is absorbed by baryon s to form baryon r, m_q is the mass of baryon q, and μ_ℓ is the mass of meson ℓ. It is understood that $g_{\bar{r}st}$, m_q, and μ_ℓ represent all the independent parameters of the three types in the problem.

If the baryon and meson octets obeyed SU(3) symmetry the bootstrap requirement applied to the renormalization constants would be

$$Z_{\bar{i}jk}\Big|_s = 0 \ ,$$
$$Z_{2i}\Big|_s = 0 \ , \qquad (1.2)$$
$$Z_{3k}\Big|_s = 0 \ .$$

The symbol $|_s$ means evaluation at SU(3) symmetric values. It implies equal masses for the baryons and mesons respectively ($m_q = m$, $\mu_\ell = \mu$ for all q, ℓ), but does not specify definite ratios between all the coupling constants unless a value of the mixing parameter between the two possible SU(3) symmetric coupling patterns (called D and F) is given. It will be found that the condition $Z_{\bar{i}jk}\Big|_s = 0$, which is to be satisfied for all (ijk), fixes the mixing parameter, commonly called α, to have the value $\alpha = 3/4$. The ratios between all couplings are then determined, once a definite phase convention for the particle states has been adopted.

We are interested in whether the bootstrapped meson-baryon system admits not only a SU(3)-symmetric solution but also a solution which breaks the symmetry. It greatly simplifies the calculation to look for a solution which differs from the symmetric one by only small quantities. We may then expand the renormalization constants about the symmetric values and keep only first order differences, obtaining

$$Z_\beta = Z_\beta\Big|_s + \delta Z_\beta \ , \qquad \text{where} \qquad (1.3)$$

$$\delta Z_\beta = \sum_{rst} \frac{\partial Z_\beta}{\partial g_{\bar{r}st}}\Bigg|_s \delta g_{\bar{r}st} + \sum_q \frac{\partial Z_\beta}{\partial m_q}\Bigg|_s \delta m_q + \sum_\ell \frac{\partial Z_\beta}{\partial \mu_\ell^2}\Bigg|_s \delta\mu_\ell^2 \ ,$$

and where β stands for any of $(\bar{i}jk)$, 2i, or 3k.[9] Using $Z_\beta|_s = 0$ the conditions $Z_\beta = 0$ then become

$$\delta Z_\beta = \sum_\nu \left. \frac{\partial Z_\beta}{\partial v_\nu} \right|_s \delta v_\nu = 0 \tag{1.4}$$

for all β. Here v_ν stands for any of the variables $g_{\bar{r}st}$, m_q and μ_ℓ^2. These conditions will be satisfied only if

$$\det \left(\left. \frac{\partial Z_\beta}{\partial v_\nu} \right|_s \right) = 0 \quad . \tag{1.5}$$

As is well known, a field theoretic calculation of the Z's will give divergent integrals; we circumvent this difficulty by the common expedient of introducing cutoffs. The quantities $Z_\beta|_s$ and $(\partial Z_\beta/\partial v_\nu)|_s$ are functions of the meson-baryon mass ratio μ/m and an over-all coupling constant g. For given values of these two parameters we adjust the cutoffs such that $Z_\beta|_s = 0$ (three cutoffs will be needed, one for each of the three types of renormalization constants). Using these cutoffs we compute $\det[(\partial Z_\beta/\partial v_\nu)|_s]$ and search for a zero as μ/m and g are varied.

Let us appraise the magnitude of the problem. We restrict our investigation by looking only for solutions that retain SU(2) symmetry, as the latter is rather well satisfied by nature in comparison with the large SU(3) violations. There are then four independent baryons $(N, \Xi, \Lambda, \Sigma)$, three mesons (π, K, η), and twelve independent couplings ($NN\pi$, $\Xi\Xi\pi$, $\Lambda\Sigma\pi$, $\Sigma\Sigma\pi$, $N\Lambda K$, $\Xi\Lambda K$, $N\Sigma K$, $\Xi\Sigma K$, $NN\eta$, $\Xi\Xi\eta$, $\Lambda\Lambda\eta$, $\Sigma\Sigma\eta$). This makes a total of nineteen independent variables, so that there is a determinant of a 19 by 19 matrix to be computed. By the use of group theory, the problem can be reduced; in the next section we describe the general procedure by which this will be accomplished.

II. SIMPLIFICATION BY GROUP THEORY

We have noted that our calculation will be to first order in the splittings. In writing the equation

$$\delta Z_\beta = \sum_\nu \frac{\partial Z_\beta}{\partial v_\nu}\bigg|_s \delta v_\nu$$

we had in mind to first calculate Z_β by using a Lagrangian with arbitrary masses and coupling constants and then compute δZ_β by differentiating with respect to all independent variables and evaluating at SU(3) symmetric values as indicated. Alternatively, we could have considered an effective interaction Lagrangian which at the outset was separated into two parts. We write $\mathcal{L}_{I\,eff} = \mathcal{L}_{I\,o} + \delta\mathcal{L}$, where $\mathcal{L}_{I\,o}$ is SU(3) symmetric and $\delta\mathcal{L}$ is composed of the independent splitting terms like, for example, the lambda mass splitting term $-\delta m_\Lambda \overline{\psi}_\Lambda \psi_\Lambda$ and the SU(2) symmetric pion-nucleon coupling splitting term $\delta g_{NN\pi}(\overline{\psi}_N \gamma_5 \underline{\tau} \psi_N) \cdot \underline{\phi}$. It is not difficult to convince oneself that if we evaluate any perturbation amplitude with fixed external momenta to a given order in the couplings by using these two Lagrangians, we get identical results to first order in the splittings.(10) Now the renormalization constants Z_β, in any order in the couplings, are related to perturbation amplitudes in which the external particle lines are evaluated on the mass shell. Hence in computing δZ_β there are additional first order splitting terms arising from differentiation with respect to the external particle masses. The method of using the effective interaction Lagrangian to get δZ_β would therefore need a more careful treatment. For the purpose of making plausible our subsequent method of calculation, however, we shall ignore this point in the present section. The calculation itself will be explicit and needs no further proof.

The advantage of looking at the problem from the point of view of an effective interaction Lagrangian will emerge if we consider the transformation properties of $\delta\mathcal{L}$; in particular, we shall expand $\delta\mathcal{L}$ into irreducible representations of SU(3).

Let us define splitting operators for the couplings and masses as follows:

$$g\delta f = g\sum_{n\alpha} \delta f_{(n,\alpha)} O^{(n,\alpha)} ,$$

$$\delta m = \sum_{n\alpha} \delta m_{(n,\alpha)} P^{(n,\alpha)} , \qquad (2.1)$$

$$\delta\mu^2 = \sum_{n\alpha} \delta\mu^2_{(n,\alpha)} Q^{(n,\alpha)} .$$

Here $O^{(n,\alpha)}$, $P^{(n,\alpha)}$, and $Q^{(n,\alpha)}$ are irreducible tensor operators transforming according to the n-dimensional representation of SU(3); α supplies the additional quantum numbers necessary to specify the tensor completely. $O^{(n,\alpha)}$ is a trilinear form of field operators in the combination $\overline{\psi}_i \psi_j \phi_k$, and $P^{(n,\alpha)}$ and $Q^{(n,\alpha)}$ are bilinear forms of $\overline{\psi}_i \psi_i$ and $\phi_k \phi_k$ respectively (ψ_i and ϕ_k stand for the field operators for baryon i and meson k). $\delta f_{(n,\alpha)}$, $\delta m_{(n,\alpha)}$, and $\delta\mu^2_{(n,\alpha)}$ are constants, which we shall refer to as the irreducible splittings. For later convenience we have introduced a common coupling factor g;

the physical couplings are written $g_{\bar{i}jk} = gf_{\bar{i}jk}$, so that the f's and δf's contain the SU(3) information. Physical splittings, like the lambda mass splitting or the $\Lambda p\pi^-$ coupling splitting, would be obtained by taking matrix elements:

$$\delta m_\Lambda = \langle \Lambda | \delta m | \Lambda \rangle = \sum_{n\alpha} \delta m_{(n,\alpha)} \langle \Lambda | P^{(n,\alpha)} | \Lambda \rangle \quad ,$$

$$\delta g_{\bar{\Lambda} p\pi^-} = \langle \Lambda | g\delta f | p\pi^- \rangle = g \sum_{n\alpha} \delta f_{(n,\alpha)} \langle \Lambda | O^{(n,\alpha)} | p\pi^- \rangle \quad .$$

In terms of the splitting operators we have simply

$$\delta \mathcal{L} = g\hat{\delta f} - \delta m - \delta\mu^2 \quad , \tag{2.2}$$

where $\hat{\delta f}$ differs from δf only in that the Dirac matrix γ_5 is inserted between $\bar{\psi}_i$ and ψ_j in the expansion of $O^{(n,\alpha)}$; the SU(3) properties of $\hat{\delta f}$ and δf are the same.

Now we look at the expansion of the splitting operators in greater detail. We are interested in those splittings which keep SU(2) symmetry intact. The tensor operators $O^{(n,\alpha)}$, $P^{(n,\alpha)}$, and $Q^{(n,\alpha)}$ must therefore conserve hypercharge Y and isospin I. That is, we shall admit only the Y = 0, I = 0 components of the irreducible representations. (This condition excludes all representations which are not self-conjugate as the latter have no Y = 0, I = 0 member.)

First consider the mass splitting operators. They are composed of two octets, so that n must be a representation which is contained in $\underset{\sim}{8} \times \underset{\sim}{8} = \underset{\sim}{1} + \underset{\sim}{8}_1 + \underset{\sim}{8}_2 + \underset{\sim}{10} + \underset{\sim}{10}^* + \underset{\sim}{27}$. The $\underset{\sim}{10}$ and $\underset{\sim}{10}^*$ representations have no Y = 0, I = 0 member. We can write explicitly[11]

$$\delta m = \delta m_{(1)} P^{(1)} + \delta m_{(8_1)} P^{(8_1)} + \delta m_{(8_2)} P^{(8_2)} + \delta m_{(27)} P^{(27)} ,$$

$$\delta \mu^2 = \delta \mu^2_{(1)} Q^{(1)} + \delta \mu^2_{(8_1)} Q^{(8_1)} + \delta \mu^2_{(27)} Q^{(27)} . \tag{2.3}$$

The antisymmetric representation $\underset{\sim}{8}_2$ is not allowed for the mesons because it would split the K and $\overline{K}$ masses. These are all well-known results.

The coupling splitting operators are composed of three octets. Therefore n must be a representation contained in

$$\begin{aligned} \underset{\sim}{8} \times \underset{\sim}{8} \times \underset{\sim}{8} &= (\underset{\sim}{1} + \underset{\sim}{8}_1 + \underset{\sim}{8}_2 + \underset{\sim}{10} + \underset{\sim}{10}^* + \underset{\sim}{27}) \times \underset{\sim}{8} \\ &= \underset{\sim}{8} + 2(\underset{\sim}{1} + \underset{\sim}{8}_1 + \underset{\sim}{8}_2 + \underset{\sim}{10} + \underset{\sim}{10}^* + \underset{\sim}{27}) \\ &\quad + (\underset{\sim}{8} + \underset{\sim}{10} + \underset{\sim}{27} + \underset{\sim}{35}) + (\underset{\sim}{8} + \underset{\sim}{10}^* + \underset{\sim}{27} + \underset{\sim}{35}^*) \\ &\quad + (\underset{\sim}{8} + \underset{\sim}{10} + \underset{\sim}{10}^* + \underset{\sim}{27}_1 + \underset{\sim}{27}_2 + \underset{\sim}{35} + \underset{\sim}{35}^* + \underset{\sim}{64}) . \end{aligned}$$

We find there are two $\underset{\sim}{1}$'s, eight $\underset{\sim}{8}$'s, six $\underset{\sim}{27}$'s, and one $\underset{\sim}{64}$ or a total of seventeen coupling patterns. (The $\underset{\sim}{35}$'s have no $Y = 0$, $I = 0$ member.) In order to distinguish all these patterns (their compositions are all different) we shall keep track of which intermediate representation, obtained from the product of the first two $\underset{\sim}{8}$'s, they originated. We write (n, α) in this case as $(n_\gamma, n'_{\gamma'})$, where $n'_{\gamma'}$ specifies the intermediate representation. The last $\underset{\sim}{27}$ in the expansion of $\underset{\sim}{8} \times \underset{\sim}{8} \times \underset{\sim}{8}$, for example, would be identified by $(27_2, 27)$.

At this point we may note that we have got more independent coupling patterns (seventeen) than we ought to have, because there are twelve independent coupling constants in the SU(2) symmetric Lagrangian. The discrepancy arises because in counting the independent SU(2) couplings we automatically assumed that the system possessed hermitian symmetry, as must be the case for any physically acceptable theory. Thus, for example, no distinction was made between the $\overline{N}\Lambda K$ and $\overline{\Lambda}N\overline{K}$ couplings. (We here distinguish the antiparticles explicitly by a bar.) But mathematically there is a distinction, because the two patterns have different SU(3) quantum numbers. This situation prevails in five of the twelve previously enumerated cases, viz. $\Lambda\Sigma\pi$, $N\Lambda K$, $\Xi\Lambda K$, $N\Sigma K$, and $\Xi\Sigma K$; this accounts for the discrepancy. In practice, we shall find that if we form the irreducible coupling splitting operators by first combining the antibaryon and baryon operators $\overline{\psi}_i\psi_j$ into the representation $n'_{\gamma'}$, and then put this bilinear form together with the meson operator ϕ_k to make a $Y = 0$, $I = 0$ member of the representation n_γ, then exactly five of the operators $O^{(n_\gamma, n'_{\gamma'})}$ will violate hermiticity, and the other twelve will conserve hermiticity. (More precisely, it will be necessary to take linear combinations of those operators for which $n'_{\gamma'}$ is either $\underset{\sim}{10}$ or $\underset{\sim}{10}^*$.) Furthermore, the operators do not mix the two types of splittings. Hence to look for hermiticity conserving solutions we simply require that those irreducible splittings $\delta f_{(n_\gamma, n'_{\gamma'})}$ which go with the hermiticity violating operators be zero.[12]

We have, for baryon i,

$$\delta m_i = \langle i|\delta m|i\rangle = \sum_{n\gamma} \delta m_{(n_\gamma)} \langle i|P^{(n_\gamma)}|i\rangle \quad .$$

Since $P^{(n_\gamma)}$ is just a combination of terms $\bar{\psi}_j\psi_j$ forming the $Y = 0$, $I = 0$ member of the representation n_γ, the matrix element $\langle i|P^{(n_\gamma)}|i\rangle$ is essentially given by a Clebsch-Gordan (CG) coefficient. We write $\langle i|P^{(n_\gamma)}|i\rangle = P_i^{(n_\gamma)}$. As is expected the matrix will be orthogonal so that we also have

$$\delta m_{(n_\gamma)} = \sum_i P_i^{(n_\gamma)} \delta m_i \quad . \tag{2.4}$$

In the same way we get

$$\delta\mu^2_{(n_\gamma)} = \sum_k Q_k^{(n_\gamma)} \delta\mu_k^2 \quad , \tag{2.5}$$

where $Q_i^{(n_\gamma)} = P_i^{(n_\gamma)}$ if the label i refers to the quantum numbers $\nu_i = (Y_i, I_i, I_{iz})$ without distinguishing the baryons and the mesons. For the coupling splittings we have the analogous result

$$\begin{aligned} \delta f_{\bar{i}jk} &= \sum_{nn'\gamma\gamma'} \delta f_{(n_\gamma, n'_{\gamma'})} O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})} \quad , \\ \delta f_{(n_\gamma, n'_{\gamma'})} &= \sum_{ijk} O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})} \delta f_{\bar{i}jk} \quad , \end{aligned} \tag{2.6}$$

where $O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})} = \langle i|O^{(n_\gamma, n'_{\gamma'})}|jk\rangle$. It is essentially the product of two CG coefficients; the first coefficient combines $\bar{\psi}_i\psi_j$ into the representation $n'_{\gamma'}$ and the second coefficient couples the result with ϕ_k to form the $Y = 0$, $I = 0$ member of n_γ.

The simplification of the problem, which is achieved by using irreducible splittings, comes about because the equations can be diagonalized in the quantum number n. The matrix $(\partial Z_\beta/\partial v_\nu|_s)$ is thus transformed into diagonal block form. There will be four such blocks, corresponding to $n = \underset{\sim}{1}, \underset{\sim}{8}, \underset{\sim}{27}$, and $\underset{\sim}{64}$, each of which can be analyzed independently.

Let us look at the baryon wave function renormalization constants Z_{2i}. They are related to the S-matrix elements $\langle B_i|S|B_i\rangle$, where B_i stands for baryon i. In the case of SU(3) symmetry the S operator is a unitary singlet so that these matrix elements would transform just like $\langle i|P^{(1)}|i\rangle = P_i^{(1)}$. When the symmetry is broken the change in the matrix elements to first order in the splittings may be written as $\langle B_i|\delta S|B_i\rangle$, where δS contains $\delta\mathcal{L}$ just once. (More precisely δS contains the corresponding Hamiltonian operator $\delta\mathcal{H}$, which, however, can be put equal to $-\delta\mathcal{L}$.) The essential point is that δS will be composed of the same irreducible components as $\delta\mathcal{L}$, i.e. of $n = \underset{\sim}{1}, \underset{\sim}{8}, \underset{\sim}{27}$, and $\underset{\sim}{64}$.

If we now form the linear combination $\sum_i P_i^{(n_\gamma)}\langle B_i|\delta S|B_i\rangle$, we can show by a straightforward application of the Wigner-Eckart theorem that it transforms according to the representation n. (We prove this in detail in Appendix A4, making use of the results in Secs. III and IV.) Therefore, if the combination is expanded into irreducible splittings $\delta f_{(\mu_\alpha, \mu'_{\alpha'})}$, $\delta m_{(\mu_\alpha)}$, and $\delta\mu^2_{(\mu_\alpha)}$, only those terms will be present for which $\mu = n$ (α, however, need not equal γ).

Our procedure can now be described as follows. We compute Z_{2i} for general masses and coupling constants, find the first order change $\delta Z_{2i} = \sum_{\nu} (\partial Z_{2i}/\partial v_{\nu})|_s \delta v_{\nu}$, and form the linear combinations $\delta Z_2^{(n_\gamma)}$ defined by

$$\delta Z_2^{(n_\gamma)} = \sum_i P_i^{(n_\gamma)} \delta Z_{2i} \; . \tag{2.7}$$

The right-hand side can now be expressed in terms of $\delta f_{(n_\alpha, n'_{\alpha'})}$, $\delta m_{(n_\alpha)}$, and $\delta\mu^2_{(n_\alpha)}$, so that a diagonalization with respect to n has been accomplished for the equations $\delta Z_{2i} = 0$. Analogously the combinations

$$\delta Z_3^{(n_\gamma)} = \sum_k Q_k^{(n_\gamma)} \delta Z_{3k} \tag{2.8}$$

for the meson renormalization constants are diagonal with respect to n.

The vertex renormalization constant $Z_{\bar{i}jk}$ is related to the matrix element $\langle B_i | S | B_j M_k \rangle$, which in SU(3) symmetry transforms like $O_{\bar{i}jk}^{(1, n'_{\gamma'})}$. (The transformation property of $O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})}$ is independent of the values of $n'_{\gamma'}$ and γ, but the splitting patterns do depend on them.) The first order change in the matrix, which we may write as $\langle B_i | \delta S | B_j M_k \rangle$, has the property that the linear combination $\sum_{ijk} O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})} \langle B_i | \delta S | B_j M_k \rangle$ is diagonal in n when expressed in terms of irreducible splittings. (Proof in Appendix A4.) To find the appropriate form for the vertex constant, let us look at the exact relation between the matrix element $\langle B_i | S | B_j M_k \rangle$ and $Z_{\bar{i}jk}$. The second order part of $Z_{\bar{i}jk}$ is given by[13]

$$igf_{\bar{i}jk} Z^{(2)}_{\bar{i}jk} = - \langle B_i | S^{(3)} | B_j M_k \rangle \Big|_{\text{mass shell}} \qquad (2.9)$$

where $S^{(3)}$ is of third order in the couplings (we may count by the over-all constant g), and the matrix element is evaluated on the mass shell. Let us define a second order vertex function $\Lambda_{\bar{i}jk}$ by

$$ig\Lambda_{\bar{i}jk} = \langle B_i | S^{(3)} | B_j M_k \rangle \Big|_{\text{mass shell}} \,, \qquad (2.10)$$

and a total vertex constant $\Gamma_{\bar{i}jk}$ by

$$\Gamma_{\bar{i}jk} = f_{\bar{i}jk} Z_{\bar{i}jk} \ . \qquad (2.11)$$

Up to second order we therefore have

$$\Gamma_{\bar{i}jk} = f_{\bar{i}jk} (1 + Z^{(2)}_{\bar{i}jk}) = f_{\bar{i}jk} - \Lambda_{\bar{i}jk} \ . \qquad (2.12)$$

The vertex bootstrap condition in the many-particle case is just[14]

$$\Gamma_{\bar{i}jk} = 0 \ . \qquad (2.13)$$

This becomes

$$\Gamma_{\bar{i}jk} \Big|_{s} = 0 \qquad (2.14)$$

in the case of SU(3) symmetry; for first order splittings we have the additional requirement of

$$\delta\Gamma_{\bar{i}jk} = \delta f_{\bar{i}jk} - \delta\Lambda_{\bar{i}jk} = 0 \,, \tag{2.15}$$

where

$$\delta\Lambda_{\bar{i}jk} = \sum_{\nu} \left. \frac{\delta\Lambda_{\bar{i}jk}}{\partial v_{\nu}} \right|_{s} \delta v_{\nu} \,.$$

The connection between $\delta\Lambda_{\bar{i}jk}$ and $\langle B_i | \delta S | B_j M_k \rangle$ suggests that the useful combinations of $\delta\Gamma_{\bar{i}jk}$ are $\sum_{\bar{i}jk} O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})} \delta\Gamma_{\bar{i}jk}$, and in fact we shall find that these expressions are diagonal in n. (We may note that the terms $\delta f_{\bar{i}jk}$ in $\delta\Gamma_{\bar{i}jk}$ just make up the irreducible splitting $\delta f_{(n_\gamma, n'_{\gamma'})}$, so that this part is diagonal not only in n but in γ and $n'_{\gamma'}$ as well.)

Our procedure for the vertex equations is then summarized as follows. We compute the total vertex constant

$$\Gamma_{\bar{i}jk} = f_{\bar{i}jk} - \Lambda_{\bar{i}jk} \,,$$

calculate the first order differences by

$$\delta\Gamma_{\bar{i}jk} = \sum_{\nu} \left. \frac{\partial\Gamma_{\bar{i}jk}}{\partial v_{\nu}} \right|_{s} \delta v_{\nu} \,,$$

and form the combinations

$$\delta\Gamma^{(n_\gamma, n'_{\gamma'})} = \sum_{ijk} O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})} \delta\Gamma_{\bar{i}jk} \,. \tag{2.16}$$

The right-hand side when expanded in irreducible splittings is diagonal in n. This accomplishes the partial diagonalization of the equations $\delta\Gamma_{\overline{ijk}} = 0$.

III. NOTATION AND CONVENTIONS

We write the interaction Lagrangian as

$$\mathcal{L}_I = \sum_{ijk} g f_{\bar{i}jk} \bar{\psi}_i \gamma_5 \psi_j \phi_k + \text{renormalization terms} \quad . \qquad (3.1)$$

Here ψ_r (ϕ_r) is the field operator which destroys a baryon (pseudoscalar meson) with SU(3) quantum numbers given by $\nu_r = (Y_r, I_r, I_{rz})$ and creates its antiparticle, and $\bar{\psi}_r$ is the field operator which creates a baryon with quantum numbers ν_r and destroys its antiparticle. (It is understood that the ordering of the creation and destruction operators is in normal form.) The Lagrangian is assumed to be expanded in renormalized masses and coupling constants,[15] so that the physical coupling constant acting at a vertex at which meson k is absorbed by baryon j to form baryon i is $g f_{\bar{i}jk}$. (The constants $f_{\bar{i}jk}$ contains the SU(3) information; g is an over-all coupling constant.) In case of SU(3) symmetry we write

$$f_{\bar{i}jk} = \sum_{\gamma=1}^{2} f^{(\gamma)}_{\bar{i}jk} \quad , \text{ where}$$

$$f^{(\gamma)}_{\bar{i}jk} = -\sqrt{8}\, h_\gamma \begin{pmatrix} 8 & 8 & 8_\gamma \\ -i & j & -k \end{pmatrix} \begin{pmatrix} 8 & 8 & 1 \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk} \; . \qquad (3.2)$$

The index γ, which may have the values 1 and 2, refers to the two independent SU(3) invariant coupling schemes, called D and F. The bracketed symbols are SU(3) Clebsch-Gordan (CG) coefficients,[16] where for simplicity of notation we have written

$$j = \nu_j = (Y_j, I_j, I_{jz}) \quad . \tag{3.3}$$

The negative of ν_j (and j) represents the SU(3) quantum numbers of the antiparticle of j and is defined by

$$-j = -\nu_j = (-Y_j, I_j, -I_{jz}) \quad . \tag{3.4}$$

The first CG coefficient couples the pair $\overline{\psi}_i \psi_j$ to form a combination transforming as a member of an octet with $\nu = -\nu_k$. The second coefficient combines this with meson k to make an overall SU(3) singlet, as is required for SU(3) symmetry. The symbol 0 here stands for $\nu = (0,0,0)$. (The properties of the SU(3) CG coefficients are discussed in Appendix A1.)

The factor h_γ specifies the relative amount of D and F coupling ($-\sqrt{8}$ has been taken out for later convenience). In terms of the usual mixing parameter α the ratio[17] of h_1 and h_2 is given by

$$\frac{h_1}{h_2} = \frac{\sqrt{5}}{3} \frac{\alpha}{1-\alpha} \quad , \tag{3.5}$$

so that $\alpha = \left(\frac{\sqrt{5}}{3} \frac{h_2}{h_1} + 1 \right)^{-1}$. We shall choose the magnitudes of h_γ to be

$$\begin{aligned} h_1 &= \frac{2}{3}\sqrt{15}\,\alpha \quad , \\ h_2 &= 2\sqrt{3}\,(1-\alpha) \quad . \end{aligned} \tag{3.6}$$

With this choice the over-all coupling constant g will equal $g_{NN\pi}$ (see Table 3) if SU(3) symmetry is assumed to be valid for the couplings. (The experimental value is $g^2_{NN\pi}/4\pi \approx 15.$)

The factor $\eta_{\bar{i}jk}$ takes account of the phase difference[18] between the mathematical SU(3) states[19] and the corresponding particle states. (The CG coefficients operate on mathematical states, so a correction is necessary if the $f_{\bar{i}jk}$ are to couple physical particle states.) Let us consider the meson states $|M_i\rangle$ and the corresponding mathematical states which we write as $|M;\nu_i\rangle$. We introduce a phase factor $\eta(\nu_i) = \eta_i$ such that

$$|M;\nu_i\rangle = \eta_i |M_i\rangle = \eta_i (M_i)^\dagger |0\rangle \quad , \tag{3.7}$$

where in the last expression $(M_i)^\dagger$ stands for the creation operator for the meson M_i and $|0\rangle$ is the vacuum state. For the anti-particle of M_i, which is just $M_{\bar{i}}$, we have

$$|M;-\nu_i\rangle = \eta_{\bar{i}} |M_{\bar{i}}\rangle = \eta_{\bar{i}} (M_{\bar{i}})^\dagger |0\rangle \tag{3.8}$$

where $\eta_{\bar{i}} \equiv \eta(-\nu_i)$. We now require that[20]

$$\eta_i \eta_{\bar{i}} = (-1)^{Q_i} \quad . \tag{3.9}$$

Here $Q_i = I_{iz} + \frac{1}{2} Y_i$, the "charge" on particle i. The condition (3.9) is satisfied if we define

$$\begin{aligned} \eta_i &= -1 \quad \text{if} \quad \nu_i = (0,1,+1) \quad \text{or} \quad (-1,\tfrac{1}{2},-\tfrac{1}{2}) \; , \\ \eta_i &= +1 \quad \text{otherwise} \quad . \end{aligned} \tag{3.10}$$

This is the standard phase convention for the mesons. We choose exactly the same phase convention for the baryons and the antibaryons. The correspondence between the mathematical SU(3) states and the physical particle operators is then as given in Table 1. (Our choice of phase differs from that in some references in which the signs in front of Ξ^- and $\overline{\Xi}^-$ are reversed, but it has the advantage that all three octets can be treated uniformly. Also, the SU(2) expansions, which we turn to shortly, will have greater symmetry.) To simplify the notation we define products of η's by

$$\eta_{rst\ldots} = \eta_r \eta_s \eta_t \ldots \quad . \tag{3.11}$$

We then have

$$\begin{aligned} \eta_{ii} &= \eta_{\bar{i}\bar{i}} = 1 \quad , \\ \eta_{\bar{i}i} &= (-1)^{Q_i} \quad . \end{aligned} \tag{3.12}$$

Now we return to the phase factor required in the coupling constant $f_{\bar{i}jk}$ multiplying the field operator product $\overline{\psi}_i \psi_j \phi_k$. Since $\overline{\psi}_i$, ψ_j and ϕ_k contain the creation operators for baryon ν_i, antibaryon $-\nu_j$ and meson $-\nu_k$ respectively, we need the product $\eta_i \eta_{\bar{j}} \eta_{\bar{k}}$,(21) which we write as

$$\eta_{\bar{j}i\bar{k}} = \eta_{\bar{i}jk} (-1)^{Q_i + Q_j + Q_k} = \eta_{\bar{i}jk} \quad . \tag{3.13}$$

(In the last step we used conservation of charge at the vertex: $Q_i = Q_j + Q_k$.)

To define the SU(2)-independent coupling constants that we

Table 1. Phase relation between physical particle operators and SU(3) base states.

State	Creation operator $(X)^\dagger$ where X is:		
$\vert Y,I,I_z\rangle$	Baryon	Antibaryon	PS meson
$\vert 0,0,0\rangle$	Λ	$\overline{\Lambda}$	η
$\vert 0,1,+1\rangle$	$-\Sigma^+$	$-\overline{\Sigma^-}$	$-\pi^+$
$\vert 0,1,0\rangle$	Σ^0	$\overline{\Sigma}^0$	π^0
$\vert 0,1,-1\rangle$	Σ^-	$\overline{\Sigma^+}$	π^-
$\vert 1,\frac{1}{2},+\frac{1}{2}\rangle$	p	$\overline{\Xi^-}$	K^+
$\vert 1,\frac{1}{2},-\frac{1}{2}\rangle$	n	$\overline{\Xi}^0$	K^0
$\vert -1,\frac{1}{2},+\frac{1}{2}\rangle$	Ξ^0	$\overline{n}$	$\overline{K}^0$
$\vert -1,\frac{1}{2},-\frac{1}{2}\rangle$	$-\Xi^-$	$-\overline{p}$	$-K^-$

shall use, we express the SU(3) symmetric Lagrangian in terms of SU(2) invariant forms. The SU(2) dependence of $f_{\bar{i}jk}$ is separated out by writing (see Appendix A2)

$$f_{\bar{i}jk} = f_{\bar{A}BC}\, f^{I}_{\bar{i}jk}\, \eta_{\bar{i}jk} \quad , \tag{3.14}$$

where

$$f^{I}_{\bar{i}jk} = C\begin{matrix} I_A & I_B & I_C \\ -M_i & M_j & -M_k \end{matrix}\; C\begin{matrix} I_C & I_C & 0 \\ -M_k & M_k & 0 \end{matrix} \quad ,$$

$$f_{\bar{A}BC} = \sum_{\gamma=1}^{2} f^{(\gamma)}_{\bar{A}BC}$$

$$= \sum_{\gamma} -\sqrt{8}\, h_{\gamma} \begin{pmatrix} 8 & 8 & 8 \\ I_A\,\text{-}Y_A & I_B Y_B & I_C\,\text{-}Y_C \end{pmatrix}_{\gamma} \begin{pmatrix} 8 & 8 & 1 \\ I_C\,\text{-}Y_C & I_C Y_C & 0\ 0 \end{pmatrix} . \tag{3.15}$$

A, B and C refer to the SU(2) multiplets to which the particles i, j and k respectively belong. I_A, I_B, I_C are the isospins of the multiplets, and $M_i = (I_{Az})_i$ etc.

We now define the SU(2) invariant forms, which we call $((\bar{A}BC))$, by

$$((\bar{A}BC)) = \frac{1}{N_{\bar{A}BC}} \sum_{M_i M_j M_k} f^{I}_{\bar{i}jk}\, \eta_{\bar{i}jk}\, \bar{\Psi}_{A_i} \psi_{B_j} \phi_{C_k} . \tag{3.16}$$

The normalization factor $N_{\bar{A}BC}$ is introduced so that $((\bar{A}BC))$ may have a closer relation to the commonly written SU(2) invariant forms. The SU(3) symmetric Lagrangian is finally written as

$$\mathcal{L}_I = \sum_{ABC} g_{ABC}((\bar{A}BC)) + \text{renormalization terms}, \tag{3.17}$$

where

$$g_{ABC} = g f_{\bar{A}BC} N_{\bar{A}BC} \quad . \tag{3.18}$$

(We have suppressed the Dirac matrix γ_5.)

$N_{\bar{A}BC}$ and the expansions of $((\bar{A}BC))$ are given in Table 2; g_{ABC} is tabulated in Table 3 for a general mixing parameter α in terms of the contributions from D and F couplings, and also for the special value $\alpha = 3/4$. A few comments are in order. As is seen in Table 2 the SU(2) invariant forms are defined to have the hermitian property $((\bar{A}BC)) = ((\bar{B}A\bar{C}))$. Therefore the couplings g_{ABC} are also hermitian, i.e. $g_{ABC} = g_{BAC}$. (We do not distinguish the antiparticles by bars in g_{ABC}.) Consequently g_{ABC} has been given in Table 3 only for the twelve independent couplings. In order that the forms $((\bar{A}BC))$ may be hermitian it was necessary to choose $N_{\bar{A}BC}$ such that for the K-meson conjugate forms it had opposite signs. (See Appendix A2 for some properties of the various coupling constants.) Finally we like to point out that for the sake of greater symmetry between the expansion of the pair $((NN\pi))$ and $((\Xi\Xi\pi))$, and the pair $((\bar{\Sigma}N\bar{K}))$ and $((\bar{\Sigma}\Xi K))$, the forms $((\bar{\Xi}\Xi\pi))$ and $((\bar{\Sigma}\Xi K))$ have been defined with a negative sign relative to the common forms $(\bar{\Xi}\underset{\sim}{\tau}\Xi)\cdot\underset{\sim}{\pi}$ and $\bar{\underset{\sim}{\Sigma}}\cdot(\bar{K}_c\underset{\sim}{\tau}\Xi)$. The SU(3) symmetric values of $g_{\Xi\Xi\pi}$ and $g_{\Sigma\Xi K}$ therefore differ by a sign from those given in some references.

Table 2. The expansion of the SU(2) invariant forms $((\bar{A}BC))$ in particle operators.

$\bar{A}BC$	$((\bar{A}BC))$	$N_{\bar{A}BC}$	Common notation
$\bar{N}N\pi$	$\sqrt{2}\,\bar{p}n\pi^+ + \sqrt{2}\,\bar{n}p\pi^- + (\bar{p}p - \bar{n}n)\pi^0$	$\frac{1}{\sqrt{6}}$	$(\bar{N}\underset{\sim}{\tau}N)\cdot\underset{\sim}{\pi}$
$\bar{\Xi}\Xi\pi$	$\sqrt{2}\,\bar{\Xi}^0\Xi^-\pi^+ + \sqrt{2}\,\overline{\Xi^-}\Xi^0\pi^- + (\overline{\Xi^-}\Xi^- - \bar{\Xi}^0\Xi^0)\pi^0$	$\frac{1}{\sqrt{6}}$	$-(\bar{\Xi}\underset{\sim}{\tau}\Xi)\cdot\underset{\sim}{\pi}$
$\bar{\Lambda}\Sigma\pi$	$\bar{\Lambda}(\Sigma^-\pi^+ + \Sigma^+\pi^- + \Sigma^0\pi^0)$	$-\frac{1}{\sqrt{3}}$	$\bar{\Lambda}(\underset{\sim}{\Sigma}\cdot\underset{\sim}{\pi})$
$\bar{\Sigma}\Lambda\pi$	$(\overline{\Sigma^+}\pi^+ + \overline{\Sigma^-}\pi^- + \bar{\Sigma}^0\pi^0)\Lambda$	$-\frac{1}{\sqrt{3}}$	$(\underset{\sim}{\bar{\Sigma}}\cdot\underset{\sim}{\pi})\Lambda$
$\bar{\Sigma}\Sigma\pi$	$(\bar{\Sigma}^0\Sigma^- - \overline{\Sigma^+}\Sigma^0)\pi^+ + (\overline{\Sigma^-}\Sigma^0 - \bar{\Sigma}^0\Sigma^+)\pi^-$ $+ (\overline{\Sigma^+}\Sigma^+ - \overline{\Sigma^-}\Sigma^-)\pi^0$	$-\frac{1}{\sqrt{6}}$	$-i(\underset{\sim}{\bar{\Sigma}}\times\underset{\sim}{\Sigma})\cdot\underset{\sim}{\pi}$
$\bar{N}\Lambda K$	$(\bar{p}K^+ + \bar{n}K^0)\Lambda$	$\frac{1}{\sqrt{2}}$	$(\bar{N}K)\Lambda$
$\bar{\Lambda}N\bar{K}$	$\bar{\Lambda}(pK^- + n\bar{K}^0)$	$-\frac{1}{\sqrt{2}}$	$\bar{\Lambda}(\bar{K}N)$
$\bar{\Xi}\Lambda\bar{K}$	$(\overline{\Xi^-}K^- + \bar{\Xi}^0\bar{K}^0)\Lambda$	$-\frac{1}{\sqrt{2}}$	$(\bar{\Xi}K_c)\Lambda$
$\bar{\Lambda}\Xi K$	$\bar{\Lambda}(\Xi^-K^+ + \Xi^0K^0)$	$\frac{1}{\sqrt{2}}$	$\bar{\Lambda}(\bar{K}_c\Xi)$
$\bar{\Sigma}N\bar{K}$	$\sqrt{2}\,\overline{\Sigma^+}p\bar{K}^0 + \sqrt{2}\,\overline{\Sigma^-}nK^- + \bar{\Sigma}^0(pK^- - n\bar{K}^0)$	$\frac{1}{\sqrt{6}}$	$\underset{\sim}{\bar{\Sigma}}\cdot(\bar{K}\underset{\sim}{\tau}N)$
$\bar{N}\Sigma K$	$\sqrt{2}\,\bar{p}\Sigma^+K^0 + \sqrt{2}\,\bar{n}\Sigma^-K^+ + (\bar{p}K^+ - \bar{n}K^0)\Sigma^0$	$-\frac{1}{\sqrt{6}}$	$(\bar{N}\underset{\sim}{\tau}K)\cdot\underset{\sim}{\Sigma}$
$\bar{\Sigma}\Xi K$	$\sqrt{2}\,\overline{\Sigma^+}\Xi^0K^+ + \sqrt{2}\,\overline{\Sigma^-}\Xi^-K^0 + \bar{\Sigma}^0(\Xi^-K^+ - \Xi^0K^0)$	$\frac{1}{\sqrt{6}}$	$-\underset{\sim}{\bar{\Sigma}}\cdot(\bar{K}_c\underset{\sim}{\tau}\Xi)$
$\bar{\Xi}\Sigma\bar{K}$	$\sqrt{2}\,\bar{\Xi}^0\Sigma^+K^- + \sqrt{2}\,\overline{\Xi^-}\Sigma^-\bar{K}^0 + (\overline{\Xi^-}K^- - \bar{\Xi}^0\bar{K}^0)\Sigma^0$	$-\frac{1}{\sqrt{6}}$	$-(\bar{\Xi}\underset{\sim}{\tau}K_c)\cdot\underset{\sim}{\Sigma}$
$\bar{N}N\eta$	$(\bar{p}p + \bar{n}n)\eta$	$\frac{1}{\sqrt{2}}$	$(\bar{N}N)\eta$
$\bar{\Xi}\Xi\eta$	$(\overline{\Xi^-}\Xi^- + \bar{\Xi}^0\Xi^0)\eta$	$-\frac{1}{\sqrt{2}}$	$(\bar{\Xi}\Xi)\eta$
$\bar{\Lambda}\Lambda\eta$	$\bar{\Lambda}\Lambda\eta$	1	$\bar{\Lambda}\Lambda\eta$
$\bar{\Sigma}\Sigma\eta$	$(\overline{\Sigma^+}\Sigma^+ + \overline{\Sigma^-}\Sigma^- + \bar{\Sigma}^0\Sigma^0)\eta$	$-\frac{1}{\sqrt{3}}$	$(\underset{\sim}{\bar{\Sigma}}\cdot\underset{\sim}{\Sigma})\eta$

Table 3. The coupling constant $\frac{g_{ABC}}{g}$ in SU(3) symmetry.

ABC	D contribution ($\gamma = 1$)	F contribution ($\gamma = 2$)	$\frac{g_{ABC}}{g}$	$\frac{g_{ABC}}{g}$ for $\alpha = \frac{3}{4}$
$NN\pi$	α	$1-\alpha$	1	1
$\Xi\Xi\pi$	α	$-(1-\alpha)$	$2\alpha - 1$	$\frac{1}{2}$
$\Lambda\Sigma\pi$	$\frac{2}{3}\sqrt{3}\,\alpha$	0	$\frac{2}{3}\sqrt{3}\,\alpha$	$\frac{\sqrt{3}}{2}$
$\Sigma\Sigma\pi$	0	$2(1-\alpha)$	$2(1-\alpha)$	$\frac{1}{2}$
$N\Lambda K$	$-\frac{\sqrt{3}}{3}\alpha$	$-\sqrt{3}(1-\alpha)$	$-\frac{\sqrt{3}}{3}(3-2\alpha)$	$-\frac{\sqrt{3}}{2}$
$\Xi\Lambda K$	$-\frac{\sqrt{3}}{3}\alpha$	$\sqrt{3}(1-\alpha)$	$\frac{\sqrt{3}}{3}(3-4\alpha)$	0
$N\Sigma K$	α	$-(1-\alpha)$	$2\alpha - 1$	$\frac{1}{2}$
$\Xi\Sigma K$	α	$1-\alpha$	1	1
$NN\eta$	$-\frac{\sqrt{3}}{3}\alpha$	$\sqrt{3}(1-\alpha)$	$\frac{\sqrt{3}}{3}(3-4\alpha)$	0
$\Xi\Xi\eta$	$-\frac{\sqrt{3}}{3}\alpha$	$-\sqrt{3}(1-\alpha)$	$-\frac{\sqrt{3}}{3}(3-2\alpha)$	$-\frac{\sqrt{3}}{2}$
$\Lambda\Lambda\eta$	$-\frac{2\sqrt{3}}{3}\alpha$	0	$-\frac{2\sqrt{3}}{3}\alpha$	$-\frac{\sqrt{3}}{2}$
$\Sigma\Sigma\eta$	$\frac{2\sqrt{3}}{3}\alpha$	0	$\frac{2\sqrt{3}}{3}\alpha$	$\frac{\sqrt{3}}{2}$

In the splitting calculation we wish to compute

$$\delta g_{ABC} = g N_{\bar{A}BC} \delta f_{\bar{A}BC} \quad , \tag{3.19}$$

where $\delta f_{\bar{A}BC}$ is related to the previously introduced $\delta f_{\bar{i}jk}$ by

$$\delta f_{\bar{i}jk} = \delta f_{\bar{A}BC} \, f^{I}_{\bar{i}jk} \, \eta_{\bar{i}jk} \quad . \tag{3.20}$$

Since we are chiefly interested in splittings which satisfy hermiticity, for which $\delta g_{ABC} = \delta g_{BAC}$, only one coupling shift will be computed for each conjugate pair.

IV. THE IRREDUCIBLE SPLITTINGS

In Section II we introduced splitting operators

$$\delta f = \sum_{nn'\gamma\gamma'} \delta f_{(n_\gamma, n'_{\gamma'})} \; O^{(n_\gamma, n'_{\gamma'})} \quad ,$$

$$\delta m = \sum_{n\gamma} \delta m_{(n_\gamma)} \; P^{(n_\gamma)} \quad , \tag{4.1}$$

$$\delta\mu^2 = \sum_{n\gamma} \delta\mu^2_{(n_\gamma)} \; Q^{(n_\gamma)} \quad ,$$

where

$$O^{(n_\gamma, n'_{\gamma'})} = \sum_{ijk} O^{(n_\gamma, n'_\gamma)}_{\bar{i}jk} \bar{\psi}_i \psi_j \phi_k \quad ,$$

$$P^{(n_\gamma)} = \sum_i P_i^{(n_\gamma)} \bar{\psi}_i \psi_i \quad , \tag{4.2}$$

$$Q^{(n_\gamma)} = \sum_k Q_k^{(n_\gamma)} \frac{1}{2} \phi_k^2 \quad .$$

Following the discussion in that section we can write down the matrix elements of O, P, and Q in terms of SU(3) CG coefficients. The only additional point to be observed is the question of the possible phase difference between the particle operators and the mathematical SU(3) states. This was discussed in Section III, and the appropriate phase factor η_i was defined. We therefore have

$$O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})} = \begin{pmatrix} 8 & 8 & n'_{\gamma'} \\ -i & j & -k \end{pmatrix} \begin{pmatrix} n' & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk} \quad , \tag{4.3}$$

$$P_i^{(n_\gamma)} = Q_i^{(n_\gamma)} = \begin{pmatrix} 8 & 8 & n_\gamma \\ -i & i & 0 \end{pmatrix} \eta_{\bar{i}i} \quad . \tag{4.4}$$

We like to find the composition of the irreducible mass splittings in terms of physical splittings. Consider the baryons, Eq. (2.4):

$$\delta m_{(n_\gamma)} = \sum_i \begin{pmatrix} 8 & 8 & n_\gamma \\ -i & i & 0 \end{pmatrix} (-1)^{Q_i} \delta m_i \quad . \tag{4.5}$$

The mass splittings are to conserve SU(2). We then have $\delta m_i = \delta m_A$, where A is the SU(2) multiplet to which baryon i belongs, and i is any particle within the multiplet. Writing

$$\begin{pmatrix} 8 & 8 & n_\gamma \\ -i & i & 0 \end{pmatrix} (-1)^{Q_i} = \left(\begin{matrix} 8 & 8 \\ I_A - Y_A & I_A Y_A \end{matrix} \middle| \begin{matrix} n_\gamma \\ 0 \; 0 \end{matrix} \right) C \begin{matrix} I_A & I_A & 0 \\ -M_i & M_i & 0 \end{matrix} (-1)^{M_i + \frac{1}{2} Y_A}$$

and using[22] $C \begin{matrix} I_A & I_A & 0 \\ -M_i & M_i & 0 \end{matrix} = \dfrac{(-1)^{I_A + M_i}}{\sqrt{2I_A + 1}}$ we can sum over M_i to get

$$\delta m_{(n_\gamma)} = \sum_A (-1)^{I_A - \frac{1}{2} Y_A} \sqrt{2I_A + 1} \left(\begin{matrix} 8 & 8 \\ I_A - Y_A & I_A Y_A \end{matrix} \middle| \begin{matrix} n_\gamma \\ 0 \; 0 \end{matrix} \right) \delta m_A \, . \tag{4.6}$$

(There are $(2I_A + 1)$ values of M_i, each contributing the same amount.) Evaluation of the sum using de Swart's[16] tables of isoscalar factors, gives,

$$\delta m_{(27)} = -\frac{\sqrt{30}}{20}\,(2\,\delta m_N + 2\,\delta m_\Xi - 3\,\delta m_\Lambda - \delta m_\Sigma)\ ,$$

$$\delta m_{(8_1)} = -\frac{\sqrt{5}}{5}\,(\delta m_N + \delta m_\Xi + \delta m_\Lambda - 3\,\delta m_\Sigma)\ ,$$

$$\delta m_{(8_2)} = \delta m_N - \delta m_\Xi\ , \tag{4.7}$$

$$\delta m_{(1)} = -\frac{\sqrt{2}}{4}\,(2\,\delta m_N + 2\,\delta m_\Xi + \delta m_\Lambda + 3\,\delta m_\Sigma)\ .$$

For the mesons we impose the additional requirement that $\delta\mu^2_K = \delta\mu^2_{\bar{K}}$ and get

$$\delta\mu^2_{(27)} = -\frac{\sqrt{30}}{20}\,(4\,\delta\mu^2_K - 3\,\delta\mu^2_\eta - \delta\mu^2_\pi)\ ,$$

$$\delta\mu^2_{(8)} = -\frac{\sqrt{5}}{5}\,(2\,\delta\mu^2_K + \delta\mu^2_\eta - 3\,\delta\mu^2_\pi)\ , \tag{4.8}$$

$$\delta\mu^2_{(1)} = -\frac{\sqrt{2}}{4}\,(4\,\delta\mu^2_K + \delta\mu^2_\eta + 3\,\delta\mu^2_\pi)\ ,$$

where it is understood that $\delta\mu^2_{(8)} = \delta\mu^2_{(8_1)}$, $\delta\mu^2_{(8_2)}$ being zero. The inverse relations can be read off from these equations provided we take note of the multiplicity of the isostates. The expansion for the nucleon splitting, for example, would be

$$\delta m_N = -\frac{\sqrt{30}}{20}\,\delta m_{(27)} - \frac{\sqrt{5}}{10}\,\delta m_{(8_1)} + \frac{1}{2}\,\delta m_{(8_2)} - \frac{\sqrt{2}}{4}\,\delta m_{(1)}\ .$$

The irreducible coupling splittings are

$$\delta f_{(n_\gamma, n'_{\gamma'})} = \sum_{ijk} \begin{pmatrix} 8 & 8 & n'_{\gamma'} \\ -i & j & k \end{pmatrix} \begin{pmatrix} n' & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk}\, \delta f_{\bar{i}jk}\ . \tag{4.9}$$

Let us first investigate the symmetry of the expansion under charge conjugation. Relabeling the dummy summation indices by letting $i \leftrightarrow j$ and $k \leftrightarrow \bar{k}$ we transform as follows

$$\begin{pmatrix} 8 & 8 & n'_{\gamma'} \\ -i & j & -k \end{pmatrix} \rightarrow \begin{pmatrix} 8 & 8 & n'_{\gamma'} \\ -j & i & k \end{pmatrix} = \xi_1(n'_{\gamma'})\xi_3(n'_{\gamma'})\begin{pmatrix} 8 & 8 & n'^*_{\gamma'} \\ -i & j & -k \end{pmatrix} ,$$

$$\begin{pmatrix} n' & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \rightarrow \begin{pmatrix} n' & 8 & n_\gamma \\ k & -k & 0 \end{pmatrix} = \xi_3(n' 8 n_\gamma)\begin{pmatrix} n'^* & 8 & n^*_\gamma \\ -k & k & 0 \end{pmatrix} ,$$

$$\eta_{\bar{i}jk} \rightarrow \eta_{\bar{j}i\bar{k}} = \eta_{\bar{i}jk}(-1)^{Q_i+Q_j+Q_k} = \eta_{\bar{i}jk} .$$

The phase factors ξ, defined by de Swart[16], are discussed in Appendix A1; our simplified notation for $\xi_k(\mu_1 \mu_2 \mu_{3\alpha})$ in the case $\mu_1 = \mu_2 = \underset{\sim}{8}$ is $\xi_k(\mu_{3\alpha})$. By (A1.14) we may write

$$\xi_1(n'_{\gamma'})\xi_3(n'_{\gamma'})\xi_3(n' \; 8 \; n_\gamma) = \xi_1(8 \; n' \; n_\gamma) .$$

Since n_γ has to be a self-conjugate representation in order to contain a $I = 0$, $Y = 0$ component we may drop the star on n^*_γ. Eq. (4.9) can therefore be re-expressed in the form

$$\delta f_{(n_\gamma, n'_{\gamma'})} = \xi_1(8 \; n' \; n_\gamma) \sum_{ijk} \begin{pmatrix} 8 & 8 & n'^*_{\gamma'} \\ -i & j & -k \end{pmatrix}\begin{pmatrix} n'^* & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk} \, \delta f_{\bar{j}i\bar{k}} . \tag{4.10}$$

If the splitting conserves charge conjugation we must have

$$\delta f_{(n_\gamma, n'_{\gamma'})} = \xi_1(8\ n'\ n_\gamma)\, \delta f_{(n_\gamma, n'^*_\gamma)} \quad . \qquad (4.11)$$

Now $n'_{\gamma'}$ can be anything reached by $\underset{\sim}{8} \times \underset{\sim}{8}$, i.e. $\underset{\sim}{1}$, $\underset{\sim}{8}_1$, $\underset{\sim}{8}_2$, $\underset{\sim}{10}$, $\underset{\sim}{10}^*$, and $\underset{\sim}{27}$. For the self-conjugate representations $\underset{\sim}{1}$, $\underset{\sim}{8}_1$, $\underset{\sim}{8}_2$, and $\underset{\sim}{27}$ the above equation gives the selection rule that $\xi_1(8\ n'\ n_\gamma) = 1$ for a charge conjugation conserving splitting $\delta f_{(n_\gamma, n'_{\gamma'})}$. This condition is violated for the splittings with $(n_\gamma, n'_{\gamma'}) = (27_2, 27)$, $(8_2, 8_1)$, and $(8_2, 8_2)$. For the $\underset{\sim}{10}$ representations we shall form the linear combinations $\underset{\sim}{10}_+$ and $\underset{\sim}{10}_-$, which are symbolically defined by

$$|10_\pm\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle \pm |10^*\rangle\right) \quad . \qquad (4.12)$$

We may then write

$$\delta f_{(n_\gamma, 10_\pm)} = \pm\, \xi_1(8\ \ 10\ \ n_\gamma)\, \delta f_{(n_\gamma, 10_\pm)} \ .$$

Here we have used the relation $\xi_1(8\ \ 10\ \ n) = \xi_1(8\ \ 10^*\ \ n^*)$; only the values $n_\gamma = \underset{\sim}{8}, \underset{\sim}{27}$ will be of interest so we may remove the star (and the subscript γ if we like). Evidently the linear combinations represented by $n'_{\gamma'} = \underset{\sim}{10}_\pm$ have definite symmetry under charge conjugation; we find that the splittings with $(n_\gamma, n'_{\gamma'}) = (27, 10_+)$, and $(8, 10_-)$ are zero.

Eq. (4.11) can be cast into a more convenient form by defining a sign factor σ as follows:

$$\sigma(n'_{\gamma'}) = -1 \quad \text{if} \quad n'_{\gamma'} = \underset{\sim}{10}_- \quad ,$$
$$\sigma(n'_{\gamma'}) = +1 \quad \text{if} \quad n'_{\gamma'} = \underset{\sim}{1}, \underset{\sim}{8}_1, \underset{\sim}{8}_2, \underset{\sim}{10}_+, \underset{\sim}{27} \quad . \qquad (4.13)$$

For any $n'_{\gamma'}$ we then have

$$\delta f_{(n_\gamma, n'_{\gamma'})} = \sigma(n'_{\gamma'})\xi_1(8\ n'\ n_\gamma)\,\delta f_{(n_\gamma, n'_{\gamma'})} \quad . \tag{4.14}$$

(It is understood that whenever σ appears in an expression, the linear combinations $\underset{\sim}{10}$ must be used. If a phase factor ξ is present, it will have the same value for $\underset{\sim}{10}$ and $\underset{\sim}{10}^*$.) In summary we list the splittings which violate charge conjugation symmetry;[23] they are those $\delta f_{(n_\gamma, n'_{\gamma'})}$ for which

$$(n_\gamma, n'_{\gamma'}) = (27_2, 27), (27, 10_+), (8, 10_-), (8_2, 8_1), \text{ and } (8_2, 8_2) \quad . \tag{4.15}$$

We now proceed to evaluate the irreducible coupling splittings in Eq. (4.9). Because of the assumed SU(2) symmetry of $\delta f_{\bar{i}jk}$ we may write, by Eq. (3.20),

$$\delta f_{\bar{i}jk} = \delta f_{\bar{A}BC}\, f^I_{\bar{i}jk}\, \eta_{\bar{i}jk} \quad , \text{ where}$$

$$f^I_{\bar{i}jk} = C\begin{matrix} I_A & I_B & I_C \\ -M_i & M_j & -M_k \end{matrix} \; C\begin{matrix} I_C & I_C & 0 \\ -M_k & M_k & 0 \end{matrix}$$

Separating out the isospin CG coefficients from the SU(3) CG coefficients in Eq. (4.9) gives another factor of $f^I_{\bar{i}jk}$. Since $(\eta_{\bar{i}jk})^2 = 1$, the sum over M_i, M_j, and M_k can be performed:

$$\sum_{M_i M_j M_k} (f^I_{\bar{i}jk})^2 = \frac{1}{2I_C+1} \sum_{M_i M_j M_k} \left(C\begin{matrix} I_A & I_B & I_C \\ -M_i & M_j & -M_k \end{matrix} \right)^2 = 1 \ .$$

The result is

$$\delta f_{(n_\gamma, n'_{\gamma'})} = \sum_{ABC} O_{\bar{A}BC}^{(n_\gamma, n'_{\gamma'})} \delta f_{\bar{A}BC} \quad , \text{ where}$$

$$O_{\bar{A}BC}^{(n_\gamma, n'_{\gamma'})} = \begin{pmatrix} 8 & 8 & \Big| & n'_{\gamma'} \\ I_A\text{-}Y_A & I_B Y_B & \Big| & I_C\text{-}Y_C \end{pmatrix} \begin{pmatrix} n' & 8 & \Big| & n_\gamma \\ I_C\text{-}Y_C & I_C Y_C & \Big| & 0\ 0 \end{pmatrix} . \qquad (4.16)$$

The inverse is also true:

$$\delta f_{\bar{A}BC} = \sum_{nn'\gamma\gamma'} \delta f_{(n_\gamma, n'_{\gamma'})} O_{\bar{A}BC}^{(n_\gamma, n'_{\gamma'})} \quad . \qquad (4.17)$$

The expansion coefficients are given in Table 4. The linear combinations $\underset{\sim}{10}_\pm$ of the representsentations $\underset{\sim}{10}$ and $\underset{\sim}{10}^*$ have been used, so that all the $\delta f_{(n_\gamma, n'_{\gamma'})}$ are either symmetric or anti-symmetric under charge conjugation. From the previous discussion it is clear that the charge symmetry violating combinations contain particle coupling splittings in the form $(\delta f_{\bar{i}jk} - \delta f_{\bar{j}i\bar{k}})$. The expansion has been made in terms of the isoscalar splittings $\delta f_{\bar{A}BC}$ defined by $\delta f_{\bar{i}jk} = \delta f_{\bar{A}BC} f^I_{\bar{i}jk} \eta_{\bar{i}jk}$. If we note that $f^I_{\bar{j}i\bar{k}} = (-1)^{2I_C} f^I_{\bar{i}jk}$ [see Eq. (A2.6)] , it is clear that the K meson isoscalar couplings will occur in the form $(\delta f_{\bar{A}BK} + \delta f_{\bar{B}A\bar{K}})$. In the charge symmetric combinations the situation is reversed. The particle coupling splittings appear as $(\delta f_{\bar{i}jk} + \delta f_{\bar{j}i\bar{k}})$, but the K meson isoscalar splitting are in the form $(\delta f_{\bar{A}BK} - \delta f_{\bar{B}A\bar{K}})$.

Table 4. The expansion coefficients of $\delta f_{(n_\gamma, n'_\gamma, \iota)} = \sum_{\bar{A}BC} O^{(n_\gamma, n'_\gamma, \iota)}_{\bar{A}BC} \delta f_{\bar{A}BC}$. We tabulate $\kappa^{(n_\gamma, n'_\gamma, \iota)}$ and $\bar{O}^{(n_\gamma, n'_\gamma, \iota)}_{\bar{A}BC}$ in terms of which $O^{(n_\gamma, n'_\gamma, \iota)}_{\bar{A}BC} = \kappa^{(n_\gamma, n'_\gamma, \iota)} \bar{O}^{(n_\gamma, n'_\gamma, \iota)}_{\bar{A}BC}$. The second row gives the symmetry of $\delta f_{(n_\gamma, n'_\gamma, \iota)}$ under charge conjugation C.

	$(n_\gamma, n'_\gamma, \iota)$	(64,27)	$(27_1,27)$	$(27_2,27)$	$(27,10_+)$	$(27,10_-)$	$(27,8_1)$	$(27,8_2)$	(8,27)	$(8,10_+)$	$(8,10_-)$	$(8_1,8_1)$	$(8_1,8_2)$	$(8_2,8_1)$	$(8_2,8_2)$	(8,1)	$(1,8_1)$	$(1,8_2)$
	C	+	+	-	-	+	+	+	+	+	-	+	+	-	-	+	+	+
	$\kappa^{(n_\gamma, n'_\gamma, \iota)}$	$\frac{\sqrt{105}}{105}$	$\frac{\sqrt{42}}{140}$	$\frac{\sqrt{10}}{20}$	$\frac{\sqrt{30}}{20}$	$\frac{\sqrt{30}}{20}$	$\frac{\sqrt{3}}{20}$	$\frac{\sqrt{15}}{20}$	$\frac{\sqrt{3}}{15}$	$\frac{\sqrt{5}}{10}$	$\frac{\sqrt{5}}{10}$	$\frac{\sqrt{2}}{20}$	$\frac{\sqrt{10}}{20}$	$\frac{\sqrt{10}}{20}$	$\frac{\sqrt{2}}{4}$	$\frac{1}{4}$	$\frac{\sqrt{5}}{20}$	$\frac{1}{4}$
$\bar{A}BC$	$\bar{N}N\pi$	-1	-8	0	0	$-\frac{4}{3}$	1	$\frac{1}{3}$	2	-2	0	6	2	0	0	0	-3	-1
	$\bar{\Xi}\Xi\pi$	-1	-8	0	0	$\frac{4}{3}$	1	$-\frac{1}{3}$	2	2	0	6	-2	0	0	0	-3	1
	$\bar{\Lambda}\Sigma\pi$	$-\frac{\sqrt{6}}{2}$	$-4\sqrt{6}$	0	$\frac{2\sqrt{6}}{3}$	0	$-\frac{\sqrt{6}}{3}$	0	$\sqrt{6}$	0	$\sqrt{6}$	$-2\sqrt{6}$	0	0	0	0	$\sqrt{6}$	0
	$\bar{\Sigma}\Lambda\pi$	$-\frac{\sqrt{6}}{2}$	$-4\sqrt{6}$	0	$-\frac{2\sqrt{6}}{3}$	0	$-\frac{\sqrt{6}}{3}$	0	$\sqrt{6}$	0	$-\sqrt{6}$	$-2\sqrt{6}$	0	0	0	0	$\sqrt{6}$	0
	$\bar{\Sigma}\Sigma\pi$	0	0	0	0	$-\frac{4}{3}$	0	$-\frac{2}{3}$	0	-2	0	0	-4	0	0	0	0	2
	$\bar{N}\Lambda K$	-3	-3	3	-1	-1	1	1	-3	1	1	1	1	-1	-1	0	1	1
	$\bar{\Lambda}N\bar{K}$	3	3	3	-1	1	-1	-1	3	-1	1	-1	-1	-1	-1	0	-1	-1
	$\bar{\Xi}\Lambda\bar{K}$	3	3	3	1	-1	-1	1	3	1	-1	-1	1	-1	1	0	-1	1
	$\bar{\Lambda}\Xi K$	-3	-3	3	1	1	1	-1	-3	-1	-1	1	-1	-1	1	0	1	-1
	$\bar{N}\Sigma K$	1	1	-1	-1	-1	3	-1	1	1	1	3	-1	-3	1	0	3	-1
	$\bar{\Sigma}N\bar{K}$	-1	-1	-1	-1	1	-3	1	-1	-1	1	-3	1	-3	1	0	-3	1
	$\bar{\Xi}\Sigma\bar{K}$	1	1	1	-1	1	3	1	1	-1	1	3	1	3	1	0	3	1
	$\bar{\Sigma}\Xi K$	-1	-1	1	-1	-1	-3	-1	-1	1	1	-3	-1	3	1	0	-3	-1
	$\bar{N}N\eta$	-3	4	0	0	0	-3	3	$\frac{3}{2}$	0	0	2	-2	0	0	-2	1	-1
	$\bar{\Xi}\Xi\eta$	3	-4	0	0	0	3	3	$-\frac{3}{2}$	0	0	-2	-2	0	0	2	-1	-1
	$\bar{\Lambda}\Lambda\eta$	$\frac{9\sqrt{2}}{2}$	$-6\sqrt{2}$	0	0	0	$-3\sqrt{2}$	0	$-\frac{9\sqrt{2}}{4}$	0	0	$2\sqrt{2}$	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	0
	$\bar{\Sigma}\Sigma\eta$	$-\frac{\sqrt{6}}{2}$	$\frac{2\sqrt{6}}{3}$	0	0	0	$-3\sqrt{6}$	0	$\frac{\sqrt{6}}{4}$	0	0	$2\sqrt{6}$	0	0	0	$\sqrt{6}$	$\sqrt{6}$	0

V. DYNAMICAL QUANTITIES

We need to find the derivatives of the renormalization constants with respect to all independent coupling constants and masses. The dependence on coupling constants will be dealt with in the next section; in the present section we shall calculate the dependence on masses only and it will be unnecessary to carry along the SU(3) particle indices. (We do, however, have to distinguish the various particle masses associated with all the lines in the perturbation theory diagrams, because in the SU(3) theory they will in general be different.)

The interaction Lagrangian for a spin 1/2 fermion interacting with a pseudoscalar meson can be written as[15]

$$\mathcal{L}_I = g\overline{\psi}\gamma_5\psi\phi + g(Z_1 - 1)\overline{\psi}\gamma_5\psi\phi + \frac{1}{4}\Delta\lambda Z_3^2\phi^4$$
$$+ \Delta m Z_2\overline{\psi}\psi + (Z_2 - 1)\overline{\psi}(i\nabla\!\!\!/ - m)\psi$$
$$+ \frac{1}{2}\Delta\mu^2 Z_3\phi^2 + \frac{1}{2}(Z_3 - 1)[(\nabla_\mu\phi)^2 - \mu^2\phi^2] \quad . \tag{5.1}$$

Here the field operators are all renormalized Heisenberg operators, and the coupling constant g, the spinor mass m, and the meson mass μ are the renormalized (physical) quantities. Δm and $\Delta\mu^2$ are the mass renormalizations for the spinor and meson respectively (more commonly written as δm and $\delta\mu^2$), and $\Delta\lambda$ is the coupling renormalization for the four-point meson interaction. The factors 1/2 and 1/4 arise from the Bose-Einstein statistics of the mesons. [Our convention for the Dirac matrices γ_μ is $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$, $\mu = 1,2,3;$

$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$, where the metric tensor $g_{\mu\nu}$ is diagonal with $g_{oo} = 1$, $g_{ii} = -1$ for $i = 1,2,3$; $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_o$ (it has the properties $\gamma_5^\dagger = \gamma_o \gamma_5 \gamma_o$ and $\gamma_5^2 = -1$). The slash symbol on a four vector b_μ stands for a contraction with γ_μ: $\not{b} = \gamma_\mu b_\mu = \gamma_o b_o - \underset{\sim}{\gamma} \cdot \underset{\sim}{b}$; the components of the differential operator are given by $\nabla_\mu = (\partial/\partial t, -\underset{\sim}{\nabla})$. We use the units $c = 1$, $\not{h} = 1$.]

In anticipation of the renormalization procedure we have separated out the lowest order term in the vertex constant gZ_1, by writing it as $g + g(Z_1 - 1)$; the expansion of the last term in orders of g is therefore

$$g(Z_1 - 1) = g(Z_1^{(2)} + Z_1^{(4)} + Z_1^{(6)} + \dots)$$

The vertex renormalization procedure, to lowest order, is now described as follows. We calculate the total vertex operator $ig\Gamma_5^*$ up to third order in g by summing the graphs in Fig. 1a.[24] Then we require that this total vertex when evaluated with the external particle momenta on the mass shell be equal to the basic vertex $ig\gamma_5$ (g is the physical coupling constant). If we write the value of the triangle diagram on the mass shell as $ig\gamma_5 \Lambda$, the renormalization requirement then gives

$$Z_1^{(2)} = - \Lambda \quad . \tag{5.2}$$

Up to second order, therefore, we have

$$Z_1 = 1 - \Lambda \quad . \tag{5.3}$$

The mass and wave function renormalization procedure for the

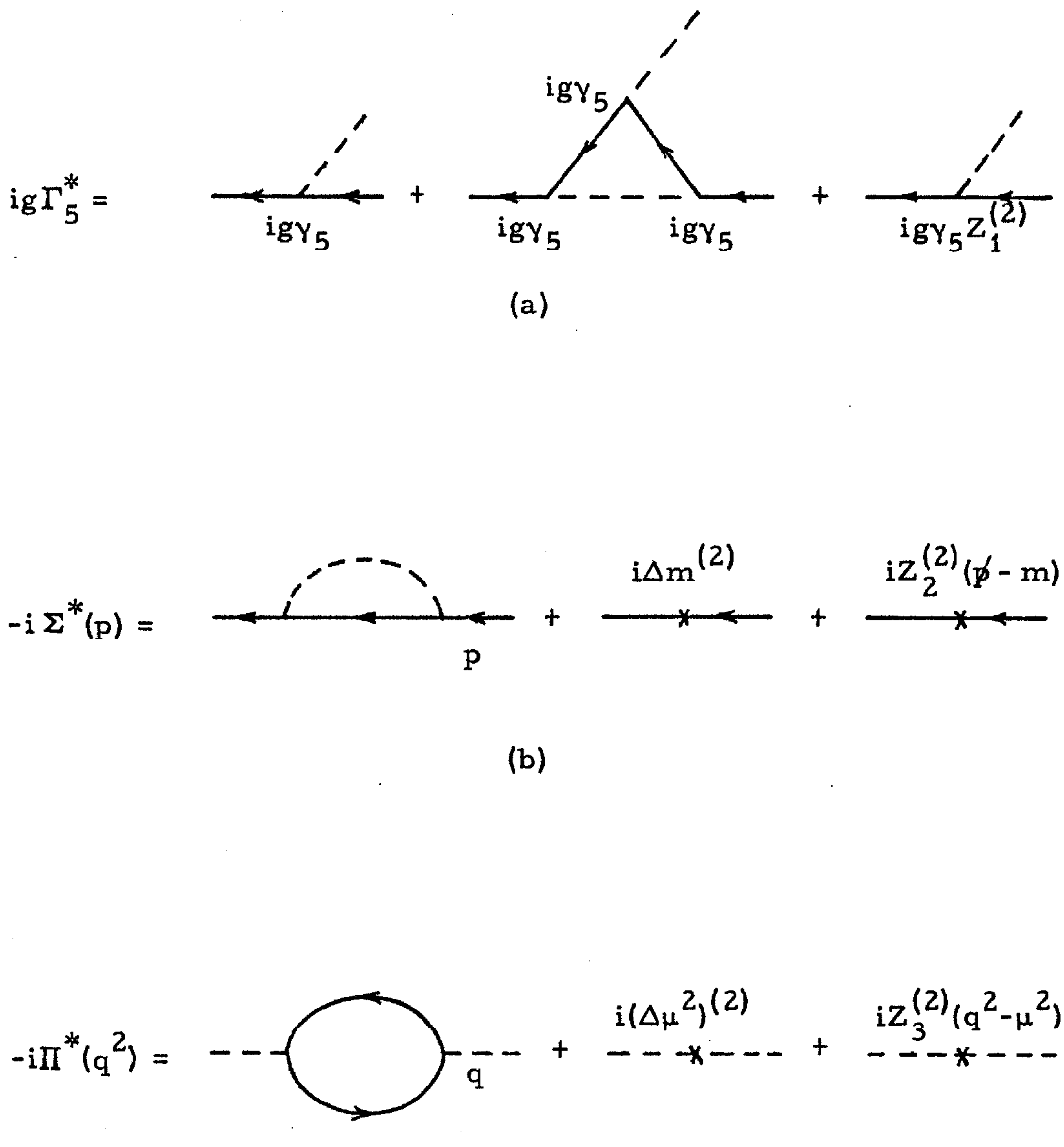


Figure 1. The renormalization procedure in lowest order

spinor, in lowest order, consists in calculating the second order self-energy operator $\Sigma^*(p)$ as given in Fig. 1b. [To second order we have $Z_2 - 1 = Z_2^{(2)}$, and $(Z_2 \Delta m)^{(2)} = \Delta m^{(2)}$; the operator $(i\not\nabla - m)$ in momentum space reads $(\not p - m)$.] Expanding the result in powers of $(\not p - m)$ we may write

$$\Sigma^*(p) = \Sigma^*(p)\Big|_{\not p = m} + (\not p - m)\,\frac{\partial \Sigma^*(p)}{\partial \not p}\Big|_{\not p = m} + (\not p - m)^2 \Sigma_c^*(p) \quad . \tag{5.4}$$

The mass and wave function renormalization requirement now is that the first two terms vanish.[25] Let us denote, as is customary, the first diagram in the expansion of $-i\Sigma^*(p)$ in Fig. 1b by $-i\Sigma(p)$. In terms of $\Sigma(p)$ the renormalization conditions are just

$$\begin{aligned} \Delta m^{(2)} &= \Sigma(p)\Big|_{\not p = m} \quad , \\ Z_2^{(2)} &= \frac{\partial \Sigma(p)}{\partial \not p}\Big|_{\not p = m} \quad . \end{aligned} \tag{5.5}$$

Up to second order we therefore have

$$Z_2 = 1 + \frac{\partial \Sigma(p)}{\partial \not p}\Big|_{\not p = m} \quad . \tag{5.6}$$

The meson renormalization procedure is parallel to the one just described. The result is

$$Z_3 = 1 + \frac{\partial \Pi(q^2)}{\partial q^2}\Big|_{q^2 = \mu^2} \quad , \tag{5.7}$$

where $-i\Pi(q^2)$ is obtained from the first graph in Fig. 1c.

We begin by evaluating Z_3. Defining masses and momenta through Fig. 2c we get, by the rules for Feynman diagrams,

$$-i\Pi(q^2;m_1,m_2) = (ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(-1)\ \mathrm{Tr}\,\{\gamma_5(\not{k}+m_1)\ \gamma_5(\not{k}-\not{q}+m_2)\}i^2}{(k^2-m_1^2)[(k-q)^2-m_2^2]} \quad . \qquad (5.8)$$

Let us first ignore the trace factor in the numerator and define a function Σ_o by the integral

$$-i\Sigma_o(q^2;m_1,m_2) = (ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{(k^2-m_1^2)[(k-q)^2-m_2^2]} \qquad (5.9)$$

(Σ_o would be the second order self-energy operator in the case all the particles were spinless.) We like to express this as a dispersion relation in the variable $q^2 = s$. Σ_o has a cut along the positive s axis in the complex plane. The discontinuity across the cut is obtained by substituting for the propagators δ-functions according to the prescription given by Cutkosky:[(26)] $i/(p^2-m^2) \rightarrow 2\pi\delta(p^2-m^2)$. We then get for the imaginary part of Σ_o (equal to $1/2i$ times the discontinuity of Σ_o)

$$\mathrm{Im}\ \Sigma_o(q^2;m_1,m_2) = -\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^4k\ \delta(k^2-m_1^2)\,\delta[(q-k)^2-m_2^2] \quad . \qquad (5.10)$$

This expression is evaluated in Appendix B with the result

$$\mathrm{Im}\ \Sigma_o(q^2;m_1,m_2) = -\pi\left(\frac{g}{4\pi}\right)^2 \frac{1}{s}\zeta(s,m_1^2,m_2^2)\theta(s-(m_1+m_2)^2) \quad . \qquad (5.11)$$

We have substituted $q^2 = s$, and written

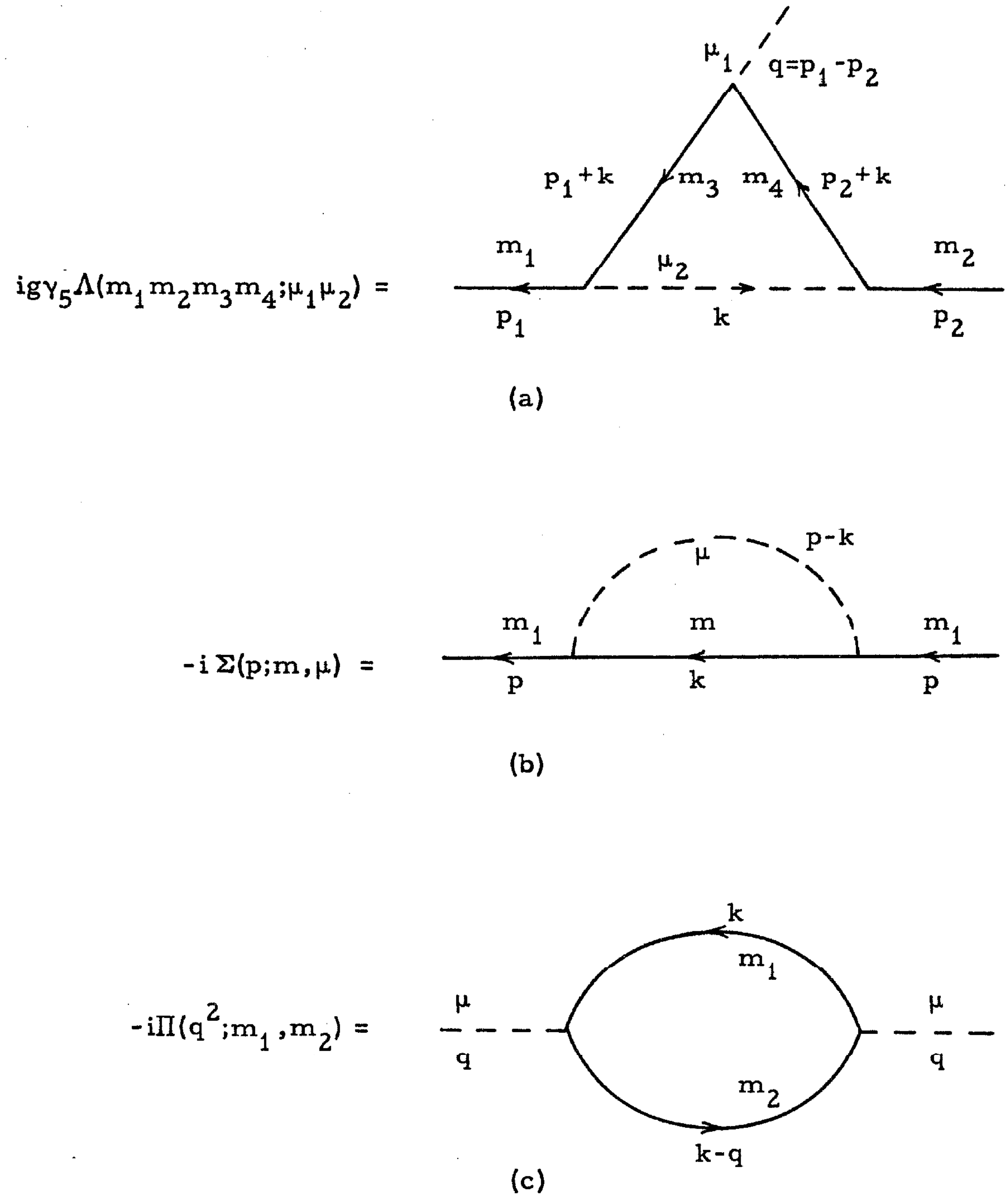


Figure 2. Identification of masses and momenta. The vertex graph in (a) is evaluated with external momenta on the mass shell.

$$\zeta(s,m_1^2,m_2^2) = \sqrt{[s-(m_1-m_2)^2][s-(m_1+m_2)^2]} \quad ; \tag{5.12}$$

θ is the unit step function defined by

$$\begin{aligned} \theta(x) &= 1 \quad \text{for} \quad x \geq 0 \quad , \\ \theta(x) &= 0 \quad \text{for} \quad x < 0 \quad . \end{aligned} \tag{5.13}$$

Now we turn back to Π. Evaluating the trace results in the factor $-4[k\cdot(q-k)+m_1m_2]$. Using Cutkosky's rule we get for the imaginary part of Π

$$\operatorname{Im}\Pi(q^2) = -\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^4k \; 4[k\cdot(q-k)+m_1m_2]\,\delta(k^2-m_1^2)\,\delta[(q-k)^2-m_2^2] \quad . \tag{5.14}$$

(The mass arguments in Π have been suppressed.) Using the first δ-function, we set $k^2 = m_1^2$ in the factor from the trace and in the second δ-function. The latter becomes $\delta(-2q\cdot k+q^2+m_1^2-m_2^2)$. We may thus put $k\cdot q = \frac{1}{2}(q^2+m_1^2-m_2^2)$ in the trace factor, which now can be taken outside the integral as it no longer depends on k. The integral is just the one we had for $\operatorname{Im}\Sigma_0$ so we get

$$\operatorname{Im}\Pi(s) = 2[s-(m_1-m_2)^2]\,\operatorname{Im}\Sigma_0(s) \quad . \tag{5.15}$$

For large s, $\operatorname{Im}\Pi(s)$ goes like s; hence the unsubtracted dispersion relation

$$\Pi(s) = \frac{1}{\pi}\int_{(m_1+m_2)^2}^{\infty} \frac{\operatorname{Im}\Pi(s')\,ds'}{s'-s}$$

does not exist since the integral is infinite. We therefore introduce a cutoff, at $s = \lambda_3^2$, in terms of which the meson renormalization constant becomes

$$Z_3 - 1 = \frac{\partial \Pi(s)}{\partial s}\bigg|_{s=\mu^2} = \frac{1}{\pi}\int_{(m_1+m_2)^2}^{\lambda_3^2} \frac{\mathrm{Im}\,\Pi(s')\,ds'}{(s'-\mu^2)^2} \quad . \tag{5.16}$$

Our final expression for Z_3 is

$$Z_3(m_1 m_2;\mu) = 1 - I_3(m_1 m_2;\mu) \quad , \tag{5.17a}$$

where

$$I_3(m_1 m_2;\mu) = \left(\frac{g}{4\pi}\right)^2 \int_{(m_1+m_2)^2}^{\lambda_3^2} \frac{ds'\,\zeta(s',m_1^2,m_2^2)\,2\,[s'-(m_1-m_2)^2]}{s'(s'-\mu^2)^2} . \tag{5.17b}$$

Next we evaluate Z_2. Defining masses and momenta as in Fig. 2b we obtain

$$-i\Sigma(p;m,\mu) = (ig)^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{\gamma_5(\not k + m)\gamma_5 i^2}{(k^2-m^2)[(p-k)^2-\mu^2]} \quad . \tag{5.18}$$

Commuting one γ_5 through the factor $(\not k + m)$ and using $\gamma_5^2 = -1$ gives $\gamma_5(\not k + m)\gamma_5 = \not k - m$. We see that Σ has one part which is composed of the matrices γ_μ in a Lorentz invariant form and one part which is proportional to the four-dimensional unit matrix. Since p_μ is the only physical four vector given, relativistic invariance requires that the γ_μ matrices occur in the form $\not p$. We can therefore write

$$\Sigma(p) = mA(p^2) + \not{p}B(p^2) \tag{5.19}$$

(the mass arguments have been suppressed), and identify

$$A(p^2) = -(-ig^2)\int \frac{d^4k}{(2\pi)^4} \frac{i^2}{(k^2-m^2)[(p-k)^2-\mu^2]}$$

$$\not{p}B(p^2) = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}i^2}{(k^2-m^2)[(p-k)^2-\mu^2]} \quad . \tag{5.20}$$

The imaginary part of $A(p^2)$ is just

$$\text{Im } A(p^2) = -\text{ Im } \Sigma_o(p^2;m,\mu)$$

so we get

$$A(p^2) = \left(\frac{g}{4\pi}\right)^2 \int_{(m+\mu)^2}^{\lambda_2^2} \frac{ds'\zeta(s',m^2,\mu^2)}{s'(s'-p^2)} \quad . \tag{5.21}$$

As before, we introduce a cutoff, at $s = \lambda_2^2$. In the equation for $\not{p}B(p^2)$ we multiply through by $\not{p}$, take the trace, and evaluate the imaginary part of the expression $p^2B(p^2)$. The result is

$$p^2 \text{ Im } B(p^2) = -\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^4k(p\cdot k)\delta(k^2-m^2)\delta[(p-k)^2-\mu^2]$$

where the factor p^2 has been taken outside the imaginary sign (it is real on the real axis). Using the δ-functions gives $p\cdot k = \frac{1}{2}(p^2+m^2-\mu^2)$, and proceeding as before gives

$$B(p^2) = -\left(\frac{g}{4\pi}\right)^2 \int_{(m+\mu)^2}^{\lambda_2^2} \frac{ds'\,\zeta(s',m^2,\mu^2)}{s'(s'-p^2)} \;\; \frac{1}{2}\left(1 + \frac{m^2-\mu^2}{s'}\right) \quad . \tag{5.22}$$

Having evaluated $A(p^2)$ and $B(p^2)$ we identify Z_2 by expanding $\Sigma(p)$ in powers of $(\not{p} - m_1)$; the coefficient of the first power of $(\not{p} - m_1)$, symbolically written as $\partial\Sigma(p)/\partial\not{p}\,|_{\not{p}=m_1}$, equals $Z_2 - 1$. A and B are both functions of p^2, so we shall first expand in powers of $(p^2 - m_1^2)$. We get

$$\Sigma(p) = mA(p^2) + \not{p}B(p^2)$$

$$= m[A(m_1^2) + (p^2 - m_1^2)A'(m_1^2)]$$

$$+ \not{p}[B(m_1^2) + (p^2 - m_1^2)B'(m_1^2)] + \text{order } (p^2 - m_1^2)^2 ,$$

where the primes denote first order derivatives with respect to p^2. The expansion in powers of $(\not{p} - m_1)$ is now accomplished by writing $\not{p} = m_1 + (\not{p} - m_1)$, and $(p^2 - m_1^2) = (\not{p} + m_1)(\not{p} - m_1) = 2m_1(\not{p} - m_1) + (\not{p} - m_1)^2$. Substituting and collecting terms gives

$$\Sigma(p) = mA(m_1^2) + m_1B(m_1^2)$$

$$+ (\not{p} - m_1)[B(m_1^2) + 2m_1^2B'(m_1^2) + 2mm_1A'(m_1^2)]$$

$$+ \text{order } (\not{p} - m_1)^2 .$$

The final result for Z_2 can be expressed in the form

$$Z_2(m_1m;\mu) = 1 - I_2(m_1m;\mu) , \qquad (5.23a)$$

where

$I_2(m_1m;\mu) =$

$$\left(\frac{g}{4\pi}\right)^2\int_{(m+\mu)^2}^{\lambda_2^2}\frac{ds'\zeta(s',m^2,\mu^2)}{2s'^2(s'-m_1^2)^2}\left[(s'-mm_1)^2-\mu^2(s'+m_1^2)+s'(m-m_1)^2\right] . \tag{5.23b}$$

For the vertex we wish to evaluate the operator corresponding to Fig. 2a on the mass shell. First we state precisely what this means. For arbitrary external momenta p_1 and p_2 (momentum conservation requires that $q = p_1 - p_2$) let us write this graph as $ig\Lambda_5(p_1,p_2)$. Then by definition the desired constant Λ (suppressing the mass arguments) is

$$\gamma_5\Lambda = \Lambda_5(p_1,p_2)\Big|_{\text{mass shell}} . \tag{5.24}$$

The operation of evaluating Λ_5 on the mass shell is now defined by

$$(\bar{u}_{p_1m_1}\Lambda_5(p_1,p_2)u_{p_2m_2})\Big|_{q^2=\mu_1^2} = \Lambda(\bar{u}_{p_1m_1}\gamma_5u_{p_2m_2}) , \tag{5.25}$$

where $\bar{u}_{p_1m_1}$ and $u_{p_2m_2}$ are free particle spinors with the property $\bar{u}_{p_1m_1}(\not{p}_1 - m_1) = 0$, $(\not{p}_2 - m_2)u_{p_2m_2} = 0$. Hence if a $\not{p}_1$ appearing in $\Lambda_5(p_1,p_2)$ is commuted through to the left it may be replaced by m_1; similarly a p_2, when commuted through to the right, may be replaced by m_2. (The common notation for this operation is $\Lambda_5(p_1,p_2)\Big|_{\substack{\not{p}_1=m_1\\ \not{p}_2=m_2}}$.) We have

$$ig\Lambda_5(p_1, p_2) = (ig)^3 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_5(\not{p}_1 + \not{k} + m_3)\gamma_5(\not{p}_2 + \not{k} + m_4)\gamma_5 \; i^3}{[(p_1+k)^2 - m_3^2][(p_2+k)^2 - m_4^2](k^2 - \mu_2^2)} \; . \quad (5.26)$$

Bringing the two γ_5 matrices at the ends through to the one at the center, the factor in the numerator may be written as

$$\gamma_5(\not{p}_1 + \not{k} + m_3)\gamma_5(\not{p}_2 + \not{k} + m_4)\gamma_5 = -(\not{p}_1 + \not{k} - m_3)\gamma_5(\not{p}_2 + \not{k} - m_4) \; .$$

The terms $\not{p}_1$ and $\not{p}_2$ when evaluated on the mass shell will just give m_1 and m_2 respectively; the factor in the numerator then becomes

$$\gamma_5[k^2 - \not{k}(m_1 - m_2 - m_3 + m_4) - (m_1 - m_3)(m_2 - m_4)] \; .$$

Let us separate the resulting Λ_5 into three parts, writing

$$\Lambda_5 = \gamma_5(\Lambda_1 + \not{\Lambda}_2 + \Lambda_3) \; , \quad (5.27)$$

where

$$\Lambda_1 = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{D} k^2 \; ,$$

$$\not{\Lambda}_2 = -(m_1 - m_2 - m_3 + m_4)(-ig^2) \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{D} \not{k} , \quad (5.28)$$

$$\Lambda_3 = -(m_1 - m_3)(m_2 - m_4)(-ig^2) \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{D} \; ,$$

and where $D = [(k+p_1)^2 - m_3^2][(k+p_2)^2 - m_4^2](k^2 - \mu_2^2)$. We like to express these functions as dispersion integrals in the variable

$q^2 = (p_1 - p_2)^2 = t$. We "cut" the lines with the masses m_3 and m_4, putting them on the mass shell by Cutkosky's rule:

$$\frac{i^2}{[(k+p_1)^2 - m_3^2][(k+p_2)^2 - m_4^2]} \rightarrow (2\pi)^2 \delta[(k+p_1)^2 - m_3^2]\delta[(k+p_2)^2 - m_4^2] \ .$$

Making this substitution in Λ_1 the imaginary part is

$$\mathrm{Im}\ \Lambda_1(q^2) = -\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^4k\ \delta[(k+p_1)^2 - m_3^2]\ \delta[(k+p_2)^2 - m_4^2]\ \frac{k^2}{k^2 - \mu_2^2} \ . \tag{5.29}$$

Writing $k^2/(k^2 - \mu_2^2) = 1 + \mu_2^2/(k^2 - \mu_2^2)$ we notice that the first term gives an integral over two δ-functions very similar to the expression for $\mathrm{Im}\ \Sigma_o(q^2)$ in Eq. (5.10). In fact, making a change of the variable of integration by $k \rightarrow k - p_1$, it is just equal to $\mathrm{Im}\ \Sigma_o(q^2; m_3, m_4)$. The second term we write as $\mu_2^2\ \mathrm{Im}\ \Lambda_o(q^2)$ where

$$\mathrm{Im}\ \Lambda_o(q^2) = -\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^4k\ \frac{\delta[(k+p_1)^2 - m_3^2]\ \delta[(k+p_2)^2 - m_4^2]}{k^2 - \mu_2^2} \ . \tag{5.30}$$

(Λ_o would be the vertex function in the case when all the particles are spinless.) The integral is evaluated in Appendix B with the result

$$\mathrm{Im}\ \Lambda_o(t) = \pi\left(\frac{g}{4\pi}\right)^2 \frac{1}{\zeta(t, m_1^2, m_2^2)} \log\left(\frac{\alpha+\beta}{\alpha-\beta}\right) \theta(t - (m_3+m_4)^2) \ , \tag{5.31}$$

where

$$\alpha = (t - m_1^2 + m_2^2)(t - m_3^2 + m_4^2) - 2t(m_2^2 + m_4^2 - \mu_2^2) ,$$

$$\beta = \zeta(t, m_1^2, m_2^2)\zeta(t, m_3^2, m_4^2) .$$

The function ζ was defined in Eq. (5.12), and we have written $p_1^2 = m_1^2$, $p_2^2 = m_2^2$. Hence

$$\Lambda_1 = \frac{1}{\pi}\int_{(m_3+m_4)^2}^{\lambda_1^2} \frac{dt'}{t'-\mu_1^2} [\,\mathrm{Im}\,\Sigma_o(t';m_3,m_4) + \mu_2^2\,\mathrm{Im}\,\Lambda_o(t')] , \quad (5.32)$$

where we have introduced a cutoff at $t' = \lambda_1^2$, and evaluated at the mass shell value $t = \mu_1^2$.

Now consider $\not{X}_2$ which we write as

$$\not{X}_2 = -(m_1 - m_2 - m_3 + m_4) L_\mu \gamma_\mu ,$$

where (5.33)

$$L_\mu = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{D} k_\mu .$$

Since p_1 and p_2 are the only independent momenta given, L_μ must have the form

$$L_\mu = p_{1\mu} L_1 + p_{2\mu} L_2 . \quad (5.34)$$

Multiplying this equation by $p_{1\mu}$ and $p_{2\mu}$ respectively we get two equations. These can be solved for L_1 and L_2 with the result

$$L_1 = \frac{1}{d}[\,p_2^2(p_1 \cdot L) - (p_1 \cdot p_2)(p_2 \cdot L)] ,$$

(5.35)

$$L_2 = \frac{1}{d}[\,-(p_1 \cdot p_2)(p_1 \cdot L) + p_1^2(p_2 \cdot L)] ,$$

where $d = p_1^2 p_2^2 - (p_1 \cdot p_2)^2$. Before starting to evaluate $(p_1 \cdot L)$ and $(p_2 \cdot L)$ let us look at the mass shell contribution of $\not{L}$. $\not{L}$ appears in Λ_5 in the form $\gamma_5 \not{L} = \gamma_5(\not{p}_1 L_1 + \not{p}_2 L_2)$. We commute $\not{p}_1$ with γ_5 to the left, which gives a minus sign; $\not{p}_2$ is already to the right. The result is

$$\gamma_5 \not{L} \Big|_{\substack{\not{p}_1 = m_1 \\ \not{p}_2 = m_2}} = \gamma_5(-m_1 L_1 + m_2 L_2) \quad , \tag{5.36}$$

where it is understood that we also put $q^2 = \mu_1^2$. Hence the mass shell contribution of the $\not{X}_2$ term to the quantity Λ, defined by Eq. (5.25), is

$$\Lambda_2 = (m_1 - m_2 - m_3 + m_4)(m_1 L_1 - m_2 L_2) \quad . \tag{5.37}$$

Now what we eventually want to compute are the first order derivatives of Λ evaluated at equal baryon masses and equal meson masses. Because of the factor $(m_1 - m_2 - m_3 + m_4)$ in Λ_2, we therefore only need the equal-mass (eq.m) value of the second factor, i.e. $m(L_1 - L_2)\Big|_{eq.m}$. At equal masses $p_1^2 = p_2^2 = m^2$ so that

$$(L_1 - L_2)\Big|_{eq.m} = \frac{1}{d}(m^2 + p_1 \cdot p_2)(p_1 - p_2)\cdot L\Big|_{eq.m} \quad . \tag{5.38}$$

We have

$$(p_1 - p_2)\cdot L\Big|_{eq.m} = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{(p_1 - p_2)\cdot k \;\; i^2}{(k^2 + 2k\cdot p_1)(k^2 + 2k\cdot p_2)(k^2 - \mu^2)} \quad . \tag{5.39}$$

Putting the first two propagators on the mass shell to find the discontinuity gives the δ-functions

$$\delta(k^2 + 2k\cdot p_1)\delta(k^2 + 2k\cdot p_2) = \delta(k^2 + 2k\cdot p_1)\delta(2k\cdot(p_2 - p_1)) \quad .$$

Hence $k\cdot(p_1 - p_2) = 0$ so that the discontinuity vanishes. Therefore Λ_2 gives no contribution to Λ and its first order derivatives evaluated at equal masses.

The term Λ_3, which equals $-(m_1 - m_3)(m_2 - m_4)\Lambda_0$, clearly gives no contribution to Λ and its first order derivatives evaluated at equal masses either.

The effective value of Λ, therefore, is just Λ_1 as given in Eq. (5.32). Evaluating at equal spinor masses and equal meson masses gives, dropping the prime on the integration variable,

$$\Lambda\Big|_{eq.m} = -\left(\frac{g}{4\pi}\right)^2 \int_{4m^2}^{\lambda_1^2} \frac{dt}{(t-\mu^2)\sqrt{t(t-4m^2)}} \left[(t-4m^2) - \mu^2 \log\left(1 + \frac{t-4m^2}{\mu^2}\right)\right]. \tag{5.40}$$

The expression within square brackets is always positive, so that Λ is negative.[27] Since $Z_1 = 1 - \Lambda$ this means that Z_1 cannot be zero for a single spin $1/2$ fermion interacting with a pseudoscalar meson. If the particles belong to octets, however, the SU(3) coefficients will change the sign so that a bootstrap situation is possible.

We finally list in explicit integral form all the dynamical quantities that we shall need. They are the integrals that we have just derived, and their first order derivatives, evaluated at equal spinor masses and equal meson masses (we shall again refer to the evaluation at equal masses by the symbol $\big|_s$ introduced in Sec. I).

Let us define, for Z_1, the positive integral I_1 by

$$I_1(m_1m_2m_3m_4;\mu_1\mu_2) = -\Lambda \tag{5.41}$$

A straightforward calculation now gives the following vertex quantities:

$$V = I_1\Big|_s = \left(\frac{g}{4\pi}\right)^2\int_{4m^2}^{\lambda_1^2} \frac{dt}{(t-\mu^2)}\sqrt{\frac{t-4m^2}{t}}\left[1 - \frac{1}{y}\log(1+y)\right], \tag{5.42}$$

$$a_2 = \frac{\partial I_1}{\partial m_1}\Big|_s = \frac{\partial I_1}{\partial m_2}\Big|_s = -2m\left(\frac{g}{4\pi}\right)^2\int_{4m^2}^{\lambda_1^2} \frac{dt}{(t-\mu^2)\sqrt{t(t-4m^2)}}$$

$$\times\left[\frac{1}{y}\log(1+y) - \frac{1}{1+y}\right], \tag{5.43}$$

$$a_2' = \frac{\partial I_1}{\partial m_3}\Big|_s = \frac{\partial I_1}{\partial m_4}\Big|_s = -2m\left(\frac{g}{4\pi}\right)^2\int_{4m^2}^{\lambda_1^2} \frac{dt}{(t-\mu^2)}\sqrt{\frac{t-4m^2}{t}}\left(\frac{1}{t-4m^2+\mu^2}\right), \tag{5.44}$$

$$a_3 = \frac{\partial I_1}{\partial \mu_1^2}\Big|_s = \left(\frac{g}{4\pi}\right)^2\int_{4m^2}^{\lambda_1^2} \frac{dt}{(t-\mu^2)^2}\sqrt{\frac{t-4m^2}{t}}\left[1 - \frac{1}{y}\log(1+y)\right], \tag{5.45}$$

$$a_3' = \frac{\partial I_1}{\partial \mu_2^2}\Big|_s = -\left(\frac{g}{4\pi}\right)^2\int_{4m^2}^{\lambda_1^2} \frac{dt}{(t-\mu^2)\sqrt{t(t-4m^2)}}\left[\log(1+y) - \frac{y}{1+y}\right], \tag{5.46}$$

where $y = (t-4m^2)/\mu^2$. Subscripts 2 and 3 refer to differentiation with respect to baryon and meson masses respectively, while a prime is added to denote differentiation with respect to an internal particle mass.

With a similar notation we write the dynamical quantities for Z_2 as:

$$b = I_2\Big|_s = \frac{1}{2}\left(\frac{g}{4\pi}\right)^2 \int_{(m+\mu)^2}^{\lambda_2^2} \frac{ds\,\zeta}{s^2(s-m^2)^2}\,[\,(s-m^2)^2 - \mu^2(s+m^2)\,]\,, \tag{5.47}$$

$$b_2 = \frac{\partial I_2}{\partial m_1}\Big|_s = m\left(\frac{g}{4\pi}\right)^2 \int_{(m+\mu)^2}^{\lambda_2^2} \frac{ds\,\zeta}{s^2(s-m^2)^3}\,[\,(s-m^2)^2 - \mu^2(3s+m^2)]\,, \tag{5.48}$$

$$b_2' = \frac{\partial I_2}{\partial m}\Big|_s = -m\left(\frac{g}{4\pi}\right)^2 \int_{(m+\mu)^2}^{\lambda_2^2} \frac{ds}{s^2(s-m^2)^2}\left\{\frac{(s-m^2+\mu^2)[(s-m^2)^2-\mu^2(s+m^2)]}{\zeta} - (s-m^2)\,\zeta\right\}\,, \tag{5.49}$$

$$b_3 = \frac{\partial I_2}{\partial \mu^2}\Big|_s = -\frac{1}{2}\left(\frac{g}{4\pi}\right)^2 \int_{(m+\mu)^2}^{\lambda_2^2} \frac{ds}{s^2(s-m^2)^2}\left\{\frac{(s+m^2-\mu^2)[\,(s-m^2)^2-\mu^2(s+m^2)]}{\zeta} + (s+m^2)\,\zeta\right\}\,, \tag{5.50}$$

where $\zeta = \zeta(s,m^2,\mu^2) = \sqrt{[\,s-(m-\mu)^2][\,s-(m+\mu)^2]}$.

Finally, the quantities for Z_3 are:

$$c = I_3\Big|_s = 2\left(\frac{g}{4\pi}\right)^2 \int_{4m^2}^{\lambda_3^2} \frac{ds\sqrt{s(s-4m^2)}}{(s-\mu^2)^2}\,, \tag{5.51}$$

$$c_2 = \frac{\partial I_3}{\partial m_1}\Big|_s = \frac{\partial I_3}{\partial m_2}\Big|_s = -4m\left(\frac{g}{4\pi}\right)^2 \int_{4m^2}^{\lambda_3^2} \frac{ds}{(s-\mu^2)^2}\sqrt{\frac{s}{s-4m^2}}\,, \tag{5.52}$$

$$c_3 = \frac{\partial I_3}{\partial \mu^2}\bigg|_s = 4\left(\frac{g}{4\pi}\right)^2 \int_{4m^2}^{\lambda_3^2} \frac{ds\,\sqrt{s(s-4m^2)}}{(s-\mu^2)^3} \quad . \tag{5.53}$$

We have deleted the prime on b_3 and c_3 because there is no ambiguity in what mass is the variable of differentiation. The integrals I_1, I_2, and I_3 are dimensionless quantities like the Z's and depend on g, the meson-baryon mass ratio μ/m, and the dimensionless cutoffs $(\lambda/m)^2$. The derivatives, like $c_2 = (\partial I_3/\partial m_1)|_s$ and $c_3 = (\partial I_3/\partial \mu^2)|_s$, occur in our equations in the forms $c_2 \delta m$ and $c_3 \delta\mu^2$; these we write in terms of dimensionless factors by $(mc_2)(\delta m/m)$ and $(m^2 c_3)(\delta\mu^2/m^2)$, choosing the average baryon mass as our unit of mass. In evaluating the dynamical quantities we put $m = 1$; μ and the cutoff masses λ_1, λ_2, and λ_3 are then understood to be in units of m. Our system is independent of the scale of mass, which is a feature considered desirable in a bootstrap theory.

VI. THE SU(3) SYMMETRIC BOOTSTRAP CONDITIONS

We now turn to the full SU(3) calculation. It proceeds as in Section V except that we shall use the Lagrangian given in Sec. III:

$$\mathcal{L}_I = \sum_{ijk} g f_{\bar{i}jk} \overline{\psi}_i \gamma_5 \psi_j \phi_k + \text{renormalization terms} \quad .$$

In the graphs from which the renormalization constants are derived we now need to sum the internal particles over all the members of the octets. This is illustrated in Fig. 3. In terms of the integrals I_1, I_2, and I_3, which were introduced in the preceding section, we can immediately write down the required expressions. For the wave function renormalization constants we get, in the notation of Figs. 3b and 3c,

$$Z_{2i} = 1 - \sum_{jk} f_{\bar{i}jk} f_{\bar{j}i\bar{k}} I_2(m_i m_j; \mu_k) \quad , \tag{6.1}$$

$$Z_{3k} = 1 - \sum_{ij} f_{\bar{j}i\bar{k}} f_{\bar{i}jk} I_3(m_i m_j; \mu_k) \quad . \tag{6.2}$$

According to the discussion of Sec. II the vertex bootstrap condition, in the case of many particles, will be applied on the total vertex constant $\Gamma_{\bar{i}jk}$. We had $\Gamma_{\bar{i}jk} = f_{\bar{i}jk} - \Lambda_{\bar{i}jk}$, where $\Lambda_{\bar{i}jk}$ is the many-particle generalization of the vertex constant Λ introduced in the previous section. We found that Λ was negative and defined the positive integral I_1 by $I_1 = -\Lambda$. In terms of I_1 we therefore have, using Fig. 3a,

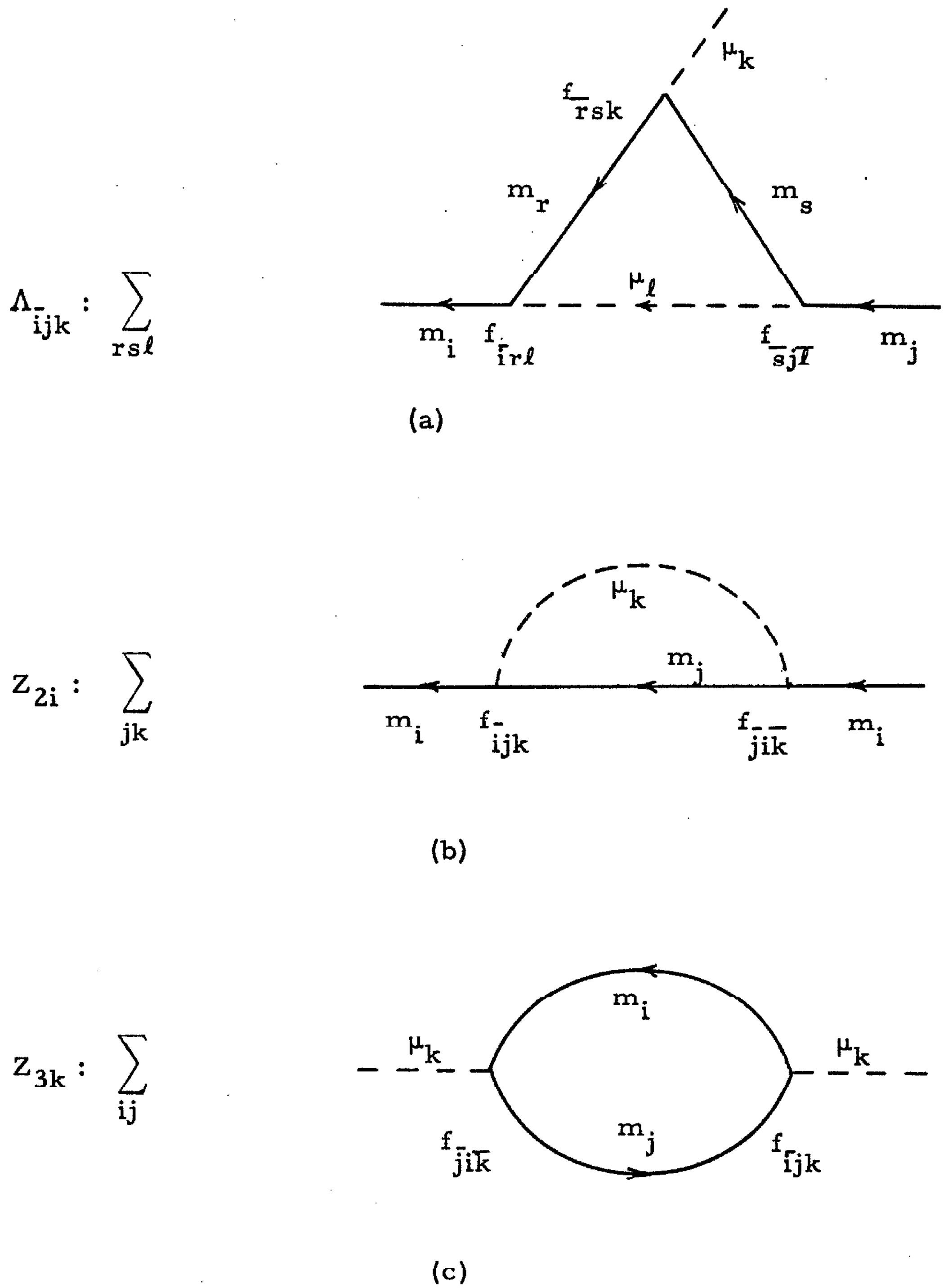


Figure 3. SU(3) structure of the lowest order renormalization graphs.

$$\Gamma_{\bar{i}jk} = f_{\bar{i}jk} - \Lambda_{\bar{i}jk} \quad ,$$

where (6.3)

$$\Lambda_{\bar{i}jk} = - \sum_{rs\ell} f_{\bar{i}s\ell}\, f_{\bar{r}sk}\, f_{\bar{s}j\bar{\ell}}\, I_1(m_i m_j m_r m_s ; \mu_k \mu_\ell) \ .$$

These then are the quantities on which we shall apply the bootstrap conditions; first at the SU(3) symmetric values, and then on the lowest order variations in the couplings and masses from those values. In this section we solve the first part of the problem, leaving the second part to Sec. VII.

Consider the vertex conditions

$$\Gamma_{\bar{i}jk}\Big|_s = 0 \quad \text{for all } (ijk) \quad . \tag{6.4}$$

We have

$$\Lambda_{\bar{i}jk}\Big|_s = - \Big(\sum_{rs\ell} f_{\bar{i}r\ell}\, f_{\bar{j}s\ell}\, f_{\bar{r}sk} \Big) V \quad , \tag{6.5}$$

where $V = I_1|_s$ and the f's are the SU(3) constants defined in Sec. II (we have written $f_{\bar{s}j\bar{\ell}} = f_{\bar{j}s\ell}$ from the requirement of hermitian symmetry). Expressing the f's in the form of Eq. (A2.2) in Appendix A2,

$$f_{\bar{i}jk} = \sum_\gamma h_\gamma \eta_{ijk} (-1)^{Q_j} \begin{pmatrix} 8 & 8 & 8_\gamma \\ -i & j & -k \end{pmatrix},$$

gives

$$\sum_{rs\ell} f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk}$$

$$= \sum_{\substack{rs\ell \\ \gamma_1\gamma_2\gamma_3}} h_{\gamma_1} h_{\gamma_2} h_{\gamma_3} \eta_{ijk} (-1)^{Q_r} \begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -i & r & -\ell \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_{\gamma_2} \\ -j & s & -\ell \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_{\gamma_3} \\ -r & s & -k \end{pmatrix} .$$

The three CG coefficients can be summed to form a SU(3) Racah coefficient[28] (the CG coefficients and Racah coefficients are discussed in Appendices A1 and A3 respectively). Using Eqs. (A1.5) and (A1.6) we may write

$$\begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -i & r & -\ell \end{pmatrix} = \xi(\gamma_1)(-1)^{Q_i} \begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -i & \ell & -r \end{pmatrix} ,$$

$$\begin{pmatrix} 8 & 8 & 8_{\gamma_2} \\ -j & s & -\ell \end{pmatrix} = \xi(\gamma_2)(-1)^{Q_s} \begin{pmatrix} 8 & 8 & 8_{\gamma_2} \\ \ell & s & j \end{pmatrix} .$$

Here the phase factor $\xi(\gamma)$ stands for $\xi_\alpha(\mu_1\mu_2\mu_{3\gamma})$ in the case that $\mu_1 = \mu_2 = \mu_3 = \underset{\sim}{8}$; it is independent of α and has the value $\xi(\gamma) = (-1)^{\gamma+1}$. By charge conservation at the vertices the factors $(-1)^Q$ become

$$(-1)^{Q_r}(-1)^{Q_i}(-1)^{Q_s} = (-1)^{(Q_r - Q_i)}(-1)^{Q_s} = (-1)^{(Q_\ell + Q_s)} = (-1)^{Q_j} .$$

The sum over $(rs\ell)$ can now be performed:

$$\sum_{rs\ell} \begin{pmatrix} \overset{1}{8} & \overset{2}{8} & \overset{12}{8}_{\gamma_1} \\ -i & \ell & -r \end{pmatrix} \begin{pmatrix} \overset{2}{8} & \overset{3}{8} & \overset{23}{8}_{\gamma_2} \\ \ell & s & j \end{pmatrix} \begin{pmatrix} \overset{12}{8} & \overset{3}{8} & \overset{\mu}{8}_{\gamma_3} \\ -r & s & -k \end{pmatrix}$$

$$= \sum_{\gamma} \langle 8_{\gamma_1}\, 8_{\gamma_3} | 8_{\gamma_2}\, 8_{\gamma} \rangle \begin{pmatrix} \overset{1}{8} & \overset{23}{8} & \overset{\mu}{8}_{\gamma} \\ -i & j & -k \end{pmatrix} \quad ,$$

where we have written out the symbols over the four CG coefficients to make explicit the correspondence with Eq. (A3.5); for notation, see also (A3.7). Hence we have

$$\sum_{\ell rs} f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\overline{rs}k} = \sum_{\gamma_1\gamma_2\gamma_3\gamma} h_{\gamma_1} h_{\gamma_2} h_{\gamma_3} (-1)^{Q_j} \begin{pmatrix} 8 & 8 & 8_{\gamma} \\ -i & j & -k \end{pmatrix}$$

$$\times\, \xi(\gamma_1)\xi(\gamma_2) \langle 8_{\gamma_1}\, 8_{\gamma_3} | 8_{\gamma_2}\, 8_{\gamma} \rangle \quad . \qquad (6.6)$$

From Table 16 it is found that the Racah symbol in which all the μ's are $\underset{\sim}{8}$'s is non-zero only if the γ indices are equal in pairs; furthermore, when it is non-zero it has the value 1/2 except in the case that all the γ's are 1 (for which it equals -3/10). Using these properties it is easily shown that for fixed γ and γ_1 the sum over γ_2 and γ_3 vanishes if $\gamma \neq \gamma_1$. We may therefore put $\gamma = \gamma_1$ and consequently also $\gamma_3 = \gamma_2$; the result can be written as

$$\sum_{rs\ell} f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} = \sum_{\gamma_1} f_{\bar{i}jk}^{(\gamma_1)} C^{(\gamma_1)} , \tag{6.7}$$

where

$$C^{(\gamma_1)} = \sum_{\gamma_2} h_{\gamma_2}^2 \xi(\gamma_1)\xi(\gamma_2)\langle 8_{\gamma_1} 8_{\gamma_2} | 8_{\gamma_2} 8_{\gamma_1} \rangle . \tag{6.8}$$

We therefore have

$$\Gamma_{\bar{i}jk} \Big|_s = \sum_{\gamma_1} f_{\bar{i}jk}^{(\gamma_1)} (1 + C^{(\gamma_1)} V) , \tag{6.9}$$

where, evaluating,

$$C^{(1)} = -\frac{3}{10} h_1^2 - \frac{1}{2} h_2^2 , \tag{6.10a}$$

$$C^{(2)} = -\frac{1}{2} h_1^2 + \frac{1}{2} h_2^2 . \tag{6.10b}$$

Now the SU(3) symmetric constants $f_{\bar{i}jk}^{(1)}$ and $f_{\bar{i}jk}^{(2)}$ represent independent coupling patterns, and we may choose (ijk) such that either $f_{\bar{i}jk}^{(1)}$ or $f_{\bar{i}jk}^{(2)}$ vanishes. The bootstrap requirement (6.4) therefore gives the two conditions

$$1 + C^{(1)} V = 0 , \tag{6.11a}$$

$$1 + C^{(2)} V = 0 , \tag{6.11b}$$

and thus $C^{(1)} = C^{(2)}$. This gives $h_1 = h_2\sqrt{5}$ and consequently $\alpha = 3/4$, which is the result quoted in Sec. I. A useful form for the vertex condition is

$$1 - 2h_2^2 V = 0 \quad , \tag{6.12}$$

which further reduces to

$$V = \frac{2}{3} \; . \tag{6.13}$$

We might also satisfy Eq. (6.4) by pure D coupling ($\alpha = 1$, $f^{(2)}_{\bar{i}jk} = 0$), with a V such that

$$1 + C^{(1)} V = 1 - \frac{3}{10} h_1^2 V = 1 - 2V = 0 \; .$$

Since the indications[29] are that α is closer to 3/4 than to 1 (usually α is found to be between 0.60 and 0.75), we do not consider the latter case further. (Pure F coupling is excluded because it would require V to be negative.) We therefore take Eq. (6.13) as the condition which will determine the cutoff $(\lambda_1/m)^2$ for the vertex integrals as a function of the over-all coupling constant g and the mass ratio μ/m.

The wave function renormalization constants, Eqs. (6.1) and (6.2), are easily evaluated at the SU(3) symmetric values. We write $I_2|_s = b$ and $I_3|_s = c$ as in Sec. V, use the hermitian property $f_{\bar{j}i\bar{k}} = f_{\bar{i}jk}$ and express $f_{\bar{i}jk}$ by Eqs. (A2.3) and (A2.2) in Z_{2i} and Z_{3k} respectively. The orthogonality property (A1.2) of the CG coefficients then gives for the sums

$$\sum_{jk\gamma_1\gamma_2} f^{(\gamma_1)}_{\bar{i}jk} f^{(\gamma_2)}_{\bar{i}jk} = \sum_{ij\gamma_1\gamma_2} f^{(\gamma_1)}_{\bar{i}jk} f^{(\gamma_2)}_{\bar{i}jk} = \sum_{\gamma_1\gamma_2} h_{\gamma_1} h_{\gamma_2} \delta_{\gamma_1,\gamma_2} = h_1^2 + h_2^2 \; . \tag{6.14}$$

The conditions $Z_{2i}|_s = 0$ and $Z_{3k}|_s = 0$ therefore give

$$b = \frac{2}{9} \quad , \tag{6.15}$$

$$c = \frac{2}{9} \quad , \tag{6.16}$$

where we have used $\alpha = 3/4$. These equations determine the cutoffs $(\lambda_2/m)^2$ and $(\lambda_3/m)^2$ respectively in terms of g and μ/m.

Let us return for a moment to our result $\alpha = 3/4$, which was essentially a consequence of the properties of the SU(3) symmetric coupling constants (the only dynamical requirement was that V be positive). In a more complete calculation we would expect the detailed dynamics to play a role, because the expression for $\Gamma_{\bar{i}jk}|_s$ in Eq. (6.9) would acquire more terms. For example, Cutkosky and Lin[30] have considered the system of the baryon octet and the $3/2^+$ resonance decuplet in a static bootstrap model using the Bethe-Salpeter equation. Their vertex bootstrap condition is similar to ours, and they get $\alpha = 0.63$. The difference is directly attributable to the inclusion of the decuplet, and the result is dependent on dynamical assumptions. It is interesting to know what would happen if we were to include fifth order terms in our calculation. We could still maintain our dynamics-independent result provided the SU(3) factors turned out right; the requirement is that for any fifth order graph W the two factors $C_W^{(1)}$ and $C_W^{(2)}$, corresponding to the $C^{(\gamma)}$'s for the third order graph, be equal for $\alpha = 3/4$. We find, however, that this condition is not satisfied for one fifth order vertex graph, namely the one in which the two internal meson lines cross; the two SU(3) factors for this graph are equal for

$\alpha = 0.68$. (The details are worked out in Appendix C.) In conclusion, then, we have to say that our value for α, though satisfactory and obtained in a simple way in our lowest order calculation, is not expected to remain unchanged in a more complete model.

VII. PARTIAL DIAGONALIZATION OF DETERMINANT

We use the general expressions for $\Gamma_{\bar{i}jk}$, Z_{2i}, and Z_{3k} in Eqs. (6.1), (6.2), and (6.3) to compute their first order variations from the SU(3) symmetric values. A straightforward calculation using the dynamical quantities defined in Sec. V gives

$$\delta\Gamma_{\bar{i}jk} = \delta f_{\bar{i}jk} + V\sum_{rs\ell}(\delta f_{\bar{r}sk} f_{\bar{i}r\ell} f_{\bar{j}s\ell} + \delta f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} + \delta f_{\bar{s}\bar{j}\bar{\ell}} f_{\bar{i}r\ell} f_{\bar{r}sk})$$

$$+ a_2(\delta m_i + \delta m_j)\sum_{rs\ell} f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} + a_2'\sum_{rs\ell}(\delta m_r + \delta m_s) f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk}$$

$$+ a_3\delta\mu_k^2\sum_{rs\ell} f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} + a_3'\sum_{rs\ell}\delta\mu_\ell^2 f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} , \qquad (7.1)$$

$$-\delta Z_{2i} = b\sum_{jk}(\delta f_{\bar{i}jk} f_{\bar{i}jk} + \delta f_{\bar{j}\bar{i}\bar{k}} f_{\bar{i}jk}) + b_2\delta m_i\sum_{jk} f_{\bar{i}jk}^2$$

$$+ b_2'\sum_{jk}\delta m_j f_{\bar{i}jk}^2 + b_3\sum_{jk}\delta\mu_k^2 f_{\bar{i}jk}^2 , \qquad (7.2)$$

$$-\delta Z_{3k} = c\sum_{ij}(\delta f_{\bar{i}jk} f_{\bar{i}jk} + \delta f_{\bar{j}\bar{i}\bar{k}} f_{\bar{i}jk}) + c_2\sum_{ij}(\delta m_i + \delta m_j) f_{\bar{i}jk}^2$$

$$+ c_3\delta\mu_k^2\sum_{ij} f_{\bar{i}jk}^2 . \qquad (7.3)$$

Here all the $f_{\bar{i}jk}$ are the SU(3) symmetric constants and we have used their hermitian property by writing $f_{\bar{j}\bar{i}\bar{k}} = f_{\bar{i}jk}$ in a few places. For a splitting pattern which preserves charge conjugation symmetry we also have $\delta f_{\bar{j}i\bar{k}} = \delta f_{\bar{i}jk}$. In the remainder of this section we shall make this substitution. The above expressions, however, have their general form, for we shall come back later to investigate the nature of the hermiticity violating solutions.

According to the discussion in Sec. II we now form the combinations

$$\delta\Gamma^{(n_\gamma, n'_{\gamma'})} = \sum_{ijk} \begin{pmatrix} 8 & 8 & n'_{\gamma'} \\ -i & j & -k \end{pmatrix} \begin{pmatrix} n' & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk}\, \delta\Gamma_{\bar{i}jk}\,, \quad (7.4)$$

$$\delta Z_2^{(n_\gamma)} = \sum_i \begin{pmatrix} 8 & 8 & n_\gamma \\ -i & i & 0 \end{pmatrix} \eta_{\bar{i}i}\, \delta Z_{2i}\,, \quad (7.5)$$

$$\delta Z_3^{(n_\gamma)} = \sum_k \begin{pmatrix} 8 & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \eta_{\bar{k}k}\, \delta Z_{3k}\,, \quad (7.6)$$

where we have written out the explict forms of $O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})}$, $P_i^{(n_\gamma)}$, and $Q_k^{(n_\gamma)}$ using the result of Sec. IV. These combinations, when expressed in terms of the irreducible splittings $\delta f_{(n_\gamma, n'_\gamma)}$, $\delta m_{(n_\gamma)}$, and $\delta\mu^2_{(n_\gamma)}$, are diagonal in the quantum number n.

The explicit evaluation will be facilitated if we first investigate the symmetry properties of the above expressions under charge conjugation. First consider $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$. Redefining the dummy summation indices by $i \leftrightarrow j$, $k \leftrightarrow \bar{k}$ gives [compare with the analysis in getting Eq. (4.10)]

$$\delta\Gamma^{(n_\gamma, n'_{\gamma'})} = \xi_1(8n'n_\gamma) \sum_{ijk} \begin{pmatrix} 8 & 8 & n'^*_{\gamma'} \\ -i & j & -k \end{pmatrix} \begin{pmatrix} n'^* & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk}\, \delta\Gamma_{\bar{j}i\bar{k}}\,. \quad (7.7)$$

It is easy to show that $\delta\Gamma_{\bar{j}i\bar{k}} = \delta\Gamma_{\bar{i}jk}$, if the splittings are hermitian. Then, just as in Sec. IV, we may write

$$\delta\Gamma^{(n_\gamma, n'_{\gamma'})} = \sigma(n'_{\gamma'})\xi_1(8\ n'\ n_\gamma)\delta\Gamma^{(n_\gamma, n'_{\gamma'})} \quad . \tag{7.8}$$

(The presence of $\sigma(n'_{\gamma'})$ means that we are using the linear combinations $\underset{\sim}{10}_\pm$ instead of the representations $\underset{\sim}{10}$ and $\underset{\sim}{10}^*$.) This may be expressed as

$$\delta\Gamma^{(n_\gamma, n'_{\gamma'})} = \phi\,\delta\Gamma^{(n_\gamma, n'_{\gamma'})} \quad , \tag{7.9a}$$

where

$$\phi = \frac{1}{2}\left[\, 1 + \sigma(n'_{\gamma'})\xi_1(8\ n'\ n_\gamma)\right] \quad . \tag{7.9b}$$

ϕ is zero for the values of $(n_\gamma, n'_{\gamma'})$ given in (4.15), viz. $(27_2, 27)$ $(27, 10_+)$, $(8, 10_-)$, $(8_2, 8_1)$, and $(8_2, 8_2)$. Therefore we do not need to evaluate these combinations explicitly. Furthermore, in deriving the relation $\delta\Gamma_{\bar{j}i\bar{k}} = \delta\Gamma_{\bar{i}jk}$ it is found that $\delta\Gamma_{\bar{i}jk}$ contains three pairs of terms whose members turn into each other under charge conjugation. The members of each pair therefore contribute equally in those $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$ for which $\phi = 1$, and subtract to bring about the cancellation in the others for which $\phi = 0$. The result is that we can use an effective $\delta\Gamma_{\bar{i}jk}$ given by

$$(\delta\Gamma_{\bar{i}jk})_{\text{eff}} = \delta f_{\bar{i}jk} + V\sum_{rs\ell}\left(\delta f_{\bar{r}sk} f_{\bar{i}r\ell} f_{\bar{j}s\ell} + 2\delta f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk}\right)$$

$$+ 2\sum_{rs\ell}(a_2\delta m_i + a_2'\delta m_r) f_{\bar{i}s\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} + \sum_{rs\ell}(a_3\delta\mu_k^2 + a_3'\delta\mu_\ell^2) f_{\bar{i}s\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} \quad , \tag{7.10}$$

in terms of which

$$\delta\Gamma^{(n_\gamma, n'_{\gamma'})} = \phi \sum_{ijk} O_{\bar{i}jk}^{(n_\gamma, n'_{\gamma'})} (\delta\Gamma_{\bar{i}jk})_{eff} \ . \tag{7.11}$$

Now consider the forms $Z_3^{(n_\gamma)}$. Because the meson octet contains its own antiparticles we may put $\delta\mu_k^2 = \delta\mu_{\bar{k}}^2$. (For the baryons we have implicitly used the equality of masses for a particle and its antiparticle. In writing m_i we always mean the mass of the baryon with quantum numbers $\nu_i = (Y_i, I_i, I_{iz})$ and baryon number +1; note that $m_{\bar{i}}$ does not equal m_i.) Evidently, then, $\delta Z_{3\bar{k}} = \delta Z_{3k}$. Relabeling the summation index k by $k \longleftrightarrow \bar{k}$ in Eq. (7.6) gives

$$\delta Z_3^{(n_\gamma)} = \sum_k \begin{pmatrix} 8 & 8 & n_\gamma \\ k & -k & 0 \end{pmatrix} \eta_{k\bar{k}} \, \delta Z_{3\bar{k}}$$

$$= \xi_1(n_\gamma) \sum_k \begin{pmatrix} 8 & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \eta_{\bar{k}k} \, \delta Z_{3k} \ . \tag{7.12}$$

Thus

$$\delta Z_3^{(n_\gamma)} = \frac{1}{2}[1 + \xi_1(n_\gamma)] \, \delta Z_3^{(n_\gamma)} \quad . \tag{7.13}$$

The factor $\frac{1}{2}[1 + \xi_1(n_\gamma)]$ vanishes for $n_\gamma = 8_2$, which is not surprising since $\delta\mu_{(8_2)}^2 = 0$. The two terms $c_2 \sum_{ij} \delta m_i f_{\bar{i}jk}^2$ and $c_2 \sum_{ij} \delta m_j f_{\bar{i}jk}^2$ will contribute equally to $\delta Z_3^{(n_\gamma)}$ (if $n_\gamma \neq 8_2$) so that we can use an effective δZ_{3k} given by

$$-(\delta Z_{3k})_{eff} = 2c\sum_{ij} f_{\bar{i}jk}\,\delta f_{\bar{i}jk} + 2c_2\sum_{ij}\delta m_i f^2_{\bar{i}jk} + c_3\delta\mu_k^2\sum_{ij} f^2_{\bar{i}jk} \tag{7.14}$$

in terms of which

$$\delta Z_3^{(n_\gamma)} = \tfrac{1}{2}[1+\xi_1(n_\gamma)]\sum_k Q_k^{(n_\gamma)}(\delta Z_{3k})_{eff} \quad . \tag{7.15}$$

No similar simplification can be made for $\delta Z_2^{(n_\gamma)}$; however, we note that the terms containing the b coefficient become just $2b\sum_{ij} f_{\bar{i}jk}\,\delta f_{\bar{i}jk}$ when we use $\delta f_{\bar{j}i\bar{k}} = \delta f_{\bar{i}jk}$.

The actual computation of the forms $\delta\Gamma^{(n_\gamma n'_{\gamma'})}$, $\delta Z_2^{(n_\gamma)}$, and $\delta Z_3^{(n_\gamma)}$ in terms of the irreducible splittings $\delta f_{(n_\gamma, n'_{\gamma'})}$, $\delta m_{(n_\gamma)}$, and $\delta\mu^2_{(n_\gamma)}$ is a somewhat tedious application of the recoupling (Racah) formalism. We give three explicit examples in Appendix D, together with some comments. Here we merely quote the results. We get

$$\delta\Gamma^{(n_\gamma, n'_\gamma)} = \delta f_{(n_\gamma, n'_{\gamma'})} + V\sum_{\gamma_1\gamma_2\beta} h_{\gamma_1}h_{\gamma_2}\xi(\gamma_1)\xi(\gamma_2)\langle 8_{\gamma_1} n'_{\gamma'}|8_{\gamma_2} n'_\beta\rangle\,\delta f_{(n_\gamma, n'_\beta)}$$

$$+\phi 2V\sum_{\substack{\gamma_1\gamma_2\mu\mu'\\ \alpha\alpha'\beta\beta'\epsilon}} h_{\gamma_1}h_{\gamma_2}\xi(\gamma_1)\xi_1(8\mu n_\beta)\xi_1(\mu'_{\alpha'})\xi_1(8\mu' n_{\beta'}) \times\langle n'_\gamma, n_\gamma|\mu_\alpha n_\beta\rangle\langle 8_{\gamma_1}\mu_\alpha|8_{\gamma_2}\mu_\epsilon\rangle\langle\mu_\epsilon n_\beta|\mu'_\alpha n_{\beta'}\rangle\,\delta f_{(n_{\beta'}, \mu'_{\alpha'})}$$

$$+\phi 2a_2\sum_{\gamma_1\beta} h_{\gamma_1} C^{(\gamma_1)}\langle n'_\gamma, n_\gamma|8_{\gamma_1} n_\beta\rangle\,\delta m_{(n_\beta)}$$

$$+\phi 2a'_2\sum_{\substack{\gamma_1\gamma_2\gamma_3\\ \alpha\beta}} h_{\gamma_1}h_{\gamma_2}h_{\gamma_3}\xi(\gamma_1)\xi(\gamma_2)\langle 8_{\gamma_1} n'_{\gamma'}|8_{\gamma_2} n'_\alpha\rangle\langle n'_\alpha n_\gamma|8_{\gamma_3} n_\beta\rangle\,\delta m_{(n_\beta)} +$$

$$+ a_3 h_{\gamma'} C^{(\gamma')} \delta_{n',8} \delta\mu^2_{(n_\gamma)}$$

$$+ a_3' \sum_{\substack{\gamma_1\gamma_2\gamma_3\mu \\ \alpha\alpha'\beta\beta'}} h_{\gamma_1} h_{\gamma_2} h_{\gamma_3} \xi(\gamma_1)\xi(\gamma_2)\xi_1(8\mu n_\beta)\xi_1(n_{\beta'}) \times \langle n'_\gamma n_\gamma | \mu_\alpha \mu_\beta\rangle \langle 8_{\gamma_2}\mu_\alpha | 8_{\gamma_3}\mu_{\alpha'}\rangle \langle \mu_{\alpha'} n_\beta | 8_{\gamma_1} n_{\beta'}\rangle \delta\mu^2_{(n_{\beta'})} , \tag{7.16}$$

$$-\delta Z_2^{(n_\gamma)} = 2b \sum_{\gamma_1\mu\alpha\beta} h_{\gamma_1} h_{\gamma_2} \langle 8_{\gamma_1} n_\gamma | \mu_\alpha n_\beta\rangle \sigma(\mu_\alpha)\delta f_{(n_\beta,\mu_\alpha)} + b_2(h_1^2 + h_2^2)\delta m_{(n_\gamma)}$$

$$+ b_2' \sum_{\gamma_1\gamma_2\beta} h_{\gamma_1} h_{\gamma_2} \xi(\gamma_1)\xi(\gamma_2) \langle 8_{\gamma_1} n_\gamma | 8_{\gamma_2} n_\beta\rangle \delta m_{(n_\beta)}$$

$$+ b_3 \sum_{\gamma_1\gamma_2\beta} h_{\gamma_1} h_{\gamma_2} \langle 8_{\gamma_1} n_\gamma | 8_{\gamma_2} n_\beta\rangle \delta\mu^2_{(n_\beta)} , \tag{7.17}$$

$$-\delta Z_3^{(n_\gamma)} = 2c \sum_{\gamma_1} h_{\gamma_1} \delta f_{(n_\gamma, 8_{\gamma_1})}$$

$$+ \tfrac{1}{2}[1 + \xi_1(n_\gamma)]\, 2c_2 \sum_{\gamma_1\gamma_2\beta} h_{\gamma_1} h_{\gamma_2} \langle 8_{\gamma_1} n_\gamma | 8_{\gamma_2} n_\beta\rangle \delta m_{(n_\beta)}$$

$$+ c_3(h_1^2 + h_2^2)\delta\mu^2_{(n_\gamma)} . \tag{7.18}$$

$C^{(\gamma)}$ and ϕ have been defined in Eqs. (6.8) and (7.9b). The factors ϕ and $\frac{1}{2}[1 + \xi_1(n_\gamma)]$ have been written out only where necessary; they might have been written in front of each term in (7.16) and (7.18) respectively. It is understood that $\delta\mu^2_{(8_2)}$ and the hermiticity violating coupling splittings given in (4.15) are zero.

The evaluation of these expressions is easiest done by matrix algebra. For example, let us define matrices C_1 and C_2 by[31]

$$\langle n'_{\gamma'} n_\gamma | C_1 | \mu_\alpha n_\beta \rangle = \xi_1(\mu_\alpha)\xi_1(8\mu n_\beta)\langle n'_{\gamma'} n_\gamma | \mu_\alpha n_\beta \rangle \quad , \tag{7.19}$$

$$\langle \mu_\alpha n_\beta | C_2 | \mu_{\alpha'} n_{\beta'} \rangle = \sum_{\gamma_2 \gamma_3} h_{\gamma_2} h_{\gamma_3} \xi(\gamma_2) \langle 8_{\gamma_2} \mu_\alpha | 8_{\gamma_3} \mu_{\alpha'} \rangle \delta_{\beta,\beta'} \; . \tag{7.20}$$

(n_β and $n_{\beta'}$ have been introduced in the definition of C_2 so that it may operate in the same space as C_1.) Then the two expressions in $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$ which involve triple products of Racah coefficients can be written rather simply as

$$\phi\, 2V \sum_{\mu'\alpha'\beta} \langle n'_{\gamma'} n_\gamma | C_1 C_2 C_1 | \mu'_{\alpha'} n_{\beta'} \rangle \, \delta f^{(n_{\beta'}, \mu'_{\alpha'})}$$

and

$$a'_3 \sum_{\gamma_1 \beta'} \langle n'_{\gamma'} n_\gamma | C_1 C_2 C_1 | 8_{\gamma_1} n_{\beta'} \rangle h_{\gamma_1} \delta\mu^2_{(n_{\beta'})} \; .$$

The calculation is straightforward and we merely quote the results in Table 5. This table, then, gives the elements of the matrix $(\partial Z_\beta / \partial v_\nu)|_s$ in a partially diagonalized form; the diagonalization is with respect to the representation n (= $\underset{\sim}{1}$, $\underset{\sim}{8}$, $\underset{\sim}{27}$, $\underset{\sim}{64}$) of the irreducible splittings. In Table 5 the elements have been written for general α, implicit in h_1 and h_2. The value $\alpha = 3/4$, for which we want to

Table 5. The expansion of $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$, $\delta Z_2^{(n_\gamma)}$, and $\delta Z_3^{(n_\gamma)}$ in irreducible splittings $\delta f_{(\mu_\alpha, \mu'_{\alpha'})}$, $\delta m_{(\mu_\alpha)}$, and $\delta\mu^2_{(\mu_\alpha)}$.

$n = \underset{\sim}{8}$	$\delta f_{(8,27)}$	$\delta f_{(8,10_+)}$	$\delta f_{(8_1,8_1)}$	$\delta f_{(8_1,8_2)}$	$\delta f_{(8,1)}$	$\delta m_{(8_1)}$	$\delta m_{(8_2)}$	$\delta\mu^2_{(8)}$
$\delta\Gamma^{(8,27)}$	$1+\left(\frac{63}{100}h_1^2-\frac{11}{12}h_2^2\right)V$	0	$-\frac{\sqrt{6}}{4}\left(\frac{39}{25}h_1^2-h_2^2\right)V$	$\frac{\sqrt{6}}{10}h_1h_2V$	$\frac{\sqrt{3}}{4}\left(\frac{3}{5}h_1^2-h_2^2\right)V$	$\frac{3\sqrt{6}}{20}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_2$ $-\frac{\sqrt{6}}{20}h_1\left(\frac{3}{5}h_1^2-h_2^2\right)a'_2$	$-\frac{\sqrt{6}}{4}h_2\left(h_1^2-h_2^2\right)a_2$ $+\frac{\sqrt{6}}{6}h_2\left(\frac{3}{5}h_1^2-h_2^2\right)a'_2$	$-\frac{7\sqrt{6}}{40}h_1\left(\frac{39}{35}h_1^2-h_2^2\right)a'_3$
$\delta\Gamma^{(8,10_+)}$	0	$1-\frac{6}{5}h_1^2V$	0	$-\frac{\sqrt{2}}{2}\left(\frac{1}{5}h_1^2-h_2^2\right)V$	0	$-\frac{\sqrt{2}}{2}h_2\left(h_1^2-h_2^2\right)a_2$ $-\frac{2\sqrt{2}}{5}h_1^2h_2a'_2$	$\frac{\sqrt{2}}{2}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_2$ $+\frac{2\sqrt{2}}{5}h_1^3a'_2$	$-\frac{\sqrt{2}}{4}h_2\left(\frac{1}{5}h_1^2-h_2^2\right)a'_3$
$\delta\Gamma^{(8_1,8_1)}$	$-\frac{\sqrt{6}}{4}\left(\frac{39}{25}h_1^2-h_2^2\right)V$	0	$1+\left(\frac{1}{50}h_1^2+\frac{1}{2}h_2^2\right)V$	$-\frac{9}{5}h_1h_2V$	$\frac{\sqrt{2}}{4}\left(\frac{3}{5}h_1^2-h_2^2\right)V$	$\frac{3}{10}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_2$ $+h_1\left(\frac{9}{50}h_1^2-\frac{13}{10}h_2^2\right)a'_2$	$-\frac{1}{2}h_2\left(h_1^2-h_2^2\right)a_2$ $-\frac{1}{2}h_2\left(\frac{13}{5}h_1^2-h_2^2\right)a'_2$	$-\frac{1}{2}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_3$ $+\frac{2}{5}h_1\left(\frac{2}{5}h_1^2-h_2^2\right)a'_3$
$\delta\Gamma^{(8_1,8_2)}$	$\frac{\sqrt{6}}{10}h_1h_2V$	$-\frac{\sqrt{2}}{2}\left(\frac{1}{5}h_1^2-h_2^2\right)V$	$-\frac{9}{5}h_1h_2V$	$1+\frac{1}{2}\left(h_1^2+h_2^2\right)V$	$-\frac{\sqrt{2}}{2}h_1h_2V$	$-\frac{1}{2}h_2\left(h_1^2-h_2^2\right)a_2$ $+\frac{1}{2}h_2\left(\frac{11}{5}h_1^2+h_2^2\right)a'_2$	$-\frac{1}{2}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_2$ $+\frac{1}{2}h_1\left(h_1^2-h_2^2\right)a'_2$	$-\frac{1}{2}h_2\left(h_1^2-h_2^2\right)a_3$ $-\frac{2}{5}h_1^2h_2a'_3$
$\delta\Gamma^{(8,1)}$	$\frac{\sqrt{3}}{4}\left(\frac{3}{5}h_1^2-h_2^2\right)V$	0	$\frac{\sqrt{2}}{4}\left(\frac{3}{5}h_1^2-h_2^2\right)V$	$-\frac{\sqrt{2}}{2}h_1h_2V$	$1+\frac{5}{4}\left(h_1^2+h_2^2\right)V$	$\frac{\sqrt{2}}{4}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_2$ $-\frac{\sqrt{2}}{2}h_1\left(h_1^2+h_2^2\right)a'_2$	$\frac{\sqrt{2}}{4}h_2\left(h_1^2-h_2^2\right)a_2$ $-\frac{\sqrt{2}}{2}h_2\left(h_1^2+h_2^2\right)a'_2$	$\frac{3\sqrt{2}}{8}h_1\left(\frac{1}{5}h_1^2-h_2^2\right)a'_3$
$-\delta Z_2^{(8_1)}$	$-\frac{3\sqrt{6}}{10}h_1b$	$\sqrt{2}\,h_2b$	$-\frac{3}{5}h_1b$	h_2b	$-\frac{\sqrt{2}}{2}h_1b$	$\left(h_1^2+h_2^2\right)b_2$ $-\frac{1}{2}\left(\frac{3}{5}h_1^2-h_2^2\right)b'_2$	$-h_1h_2b'_2$	$-\frac{1}{2}\left(\frac{3}{5}h_1^2-h_2^2\right)b_3$
$-\delta Z_2^{(8_2)}$	$\frac{\sqrt{6}}{2}h_2b$	$-\sqrt{2}\,h_1b$	h_2b	h_1b	$-\frac{\sqrt{2}}{2}h_2b$	$-h_1h_2b'_2$	$\left(h_1^2+h_2^2\right)b_2$ $+\frac{1}{2}\left(h_1^2+h_2^2\right)b'_2$	$h_1h_2b_3$
$-\delta Z_3^{(8_1)}$	0	0	$2h_1c$	$2h_2c$	0	$-\left(\frac{3}{5}h_1^2-h_2^2\right)c_2$	$2h_1h_2c_2$	$\left(h_1^2+h_2^2\right)c_3$

Table 5 (Continued)

$n = \underset{\sim}{27}$	$\delta f_{(27_1,27)}$	$\delta f_{(27,10\_)}$	$\delta f_{(27,8_1)}$	$\delta f_{(27,8_2)}$	$\delta m_{(27)}$	$\delta \mu^2_{(27)}$
$\delta\Gamma^{(27_1,27)}$	$1+\left(\frac{9}{50}h_1^2-\frac{5}{6}h_2^2\right)V$	$-\frac{\sqrt{7}}{3}h_1h_2V$	$-\frac{2\sqrt{14}}{25}h_1^2V$	$-\frac{2\sqrt{14}}{15}h_1h_2V$	$-\frac{\sqrt{14}}{5}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_2+\frac{2\sqrt{14}}{15}h_1\left(\frac{3}{5}h_1^2-h_2^2\right)a_2'$	$-\frac{\sqrt{14}}{15}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_3'$
$\delta\Gamma^{(27,10\_)}$	$-\frac{\sqrt{7}}{3}h_1h_2V$	$1-\left(\frac{1}{30}h_1^2-\frac{5}{18}h_2^2\right)V$	$\frac{2\sqrt{2}}{3}h_1h_2V$	$\frac{2\sqrt{2}}{9}\left(\frac{6}{5}h_1^2-h_2^2\right)V$	$-\frac{\sqrt{2}}{3}h_2\left(h_1^2-h_2^2\right)a_2-\frac{4\sqrt{2}}{15}h_1^2h_2a_2'$	$\frac{\sqrt{2}}{9}h_2\left(\frac{21}{5}h_1^2-h_2^2\right)a_3'$
$\delta\Gamma^{(27,8_1)}$	$-\frac{2\sqrt{14}}{25}h_1^2V$	$\frac{2\sqrt{2}}{3}h_1h_2V$	$1+\left(\frac{11}{50}h_1^2+\frac{5}{6}h_2^2\right)V$	$-\frac{7}{15}h_1h_2V$	$-\frac{1}{5}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_2-\frac{1}{5}h_1\left(\frac{3}{5}h_1^2-\frac{13}{3}h_2^2\right)a_2'$	$-\frac{1}{2}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_3+\frac{13}{30}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)a_3'$
$\delta\Gamma^{(27,8_2)}$	$-\frac{2\sqrt{14}}{15}h_1h_2V$	$\frac{2\sqrt{2}}{9}\left(\frac{6}{5}h_1^2-h_2^2\right)V$	$-\frac{7}{15}h_1h_2V$	$1+\left(\frac{5}{6}h_1^2+\frac{19}{18}h_2^2\right)V$	$\frac{1}{3}h_2\left(h_1^2-h_2^2\right)a_2-\frac{1}{3}h_2\left(\frac{11}{5}h_1^2+h_2^2\right)a_2'$	$-\frac{1}{2}h_2\left(h_1^2-h_2^2\right)a_3+\frac{1}{6}h_2\left(\frac{13}{5}h_1^2+\frac{5}{3}h_2^2\right)a_3'$
$-\delta Z_2^{(27)}$	$\frac{2\sqrt{14}}{5}h_1b$	$\frac{2\sqrt{2}}{3}h_2b$	$\frac{2}{5}h_1V$	$-\frac{2}{3}h_2b$	$\left(h_1^2+h_2^2\right)b_2+\left(\frac{1}{5}h_1^2-\frac{1}{3}h_2^2\right)b_2'$	$\left(\frac{1}{5}h_1^2-\frac{1}{3}h_2^2\right)b_3$
$-\delta Z_3^{(27)}$	0	0	$2h_1c$	$2h_2c$	$\frac{2}{3}\left(\frac{3}{5}h_1^2-h_2^2\right)c_2$	$\left(h_1^2+h_2^2\right)c_3$

$n = \underset{\sim}{1}$	$\delta f_{(1,8_1)}$	$\delta f_{(1,8_2)}$	$\delta m_{(1)}$	$\delta \mu^2_{(1)}$
$\delta\Gamma^{(1,8_1)}$	$1-\left(\frac{9}{10}h_1^2+\frac{1}{2}h_2^2\right)V$	$-h_1h_2V$	$-h_1\left(\frac{3}{5}h_1^2+h_2^2\right)\left(a_2+a_2'\right)$	$-\frac{1}{2}h_1\left(\frac{3}{5}h_1^2+h_2^2\right)\left(a_3+a_3'\right)$
$\delta\Gamma^{(1,8_2)}$	$-h_1h_2V$	$1-\frac{1}{2}\left(h_1^2-3h_2^2\right)V$	$-h_2\left(h_1^2-h_2^2\right)\left(a_2+a_2'\right)$	$-\frac{1}{2}h_2\left(h_1^2-h_2^2\right)\left(a_3+a_3'\right)$
$-\delta Z_2^{(1)}$	$2h_1b$	$2h_2b$	$\left(h_1^2+h_2^2\right)\left(b_2+b_2'\right)$	$\left(h_1^2+h_2^2\right)b_3$
$-\delta Z_3^{(1)}$	$2h_1c$	$2h_2c$	$2\left(h_1^2+h_2^2\right)c_2$	$\left(h_1^2+h_2^2\right)c_3$

$n = \underset{\sim}{64}$	$\delta f_{(64,27)}$
$\delta\Gamma^{(64,27)}$	$1+\left(\frac{3}{5}h_1^2+\frac{1}{3}h_2^2\right)V$

compute the determinant, is equivalent to $h_1 = h_2\sqrt{5}$; making this substitution in Table 5, using $1 = 2h_2^2V$, and dividing out suitable factors[32] we get the result presented in Table 6. We have not included there the equation for the splitting according to the representation $n = \underset{\sim}{64}$. It reads

$$\delta\Gamma^{(64,27)} = \frac{16}{3}\, h_2^2\, V\, \delta f_{(64,27)} = 0 \quad .$$

The coefficient has the value $8/3$, and we must have $\delta f_{(64,27)} = 0$; there is no splitting for $n = \underset{\sim}{64}$.

The quantities $X_V = V/h_2$, $X_b = b/h_2$, and $X_c = c/h_2$ appearing in the coupling splitting columns of Table 6 are constant factors by virtue of the SU(3) symmetric bootstrap conditions. (The ratios are $X_b/X_V = X_c/X_V = 1/3$.) If we diagonalize the symmetric square matrices involving X_V, i.e. solve for $\delta f_{(n_\gamma, n'_{\gamma'})}$ in terms of $\delta m_{(n_\alpha)}$ and $\delta\mu^2_{(n_\alpha)}$, we can then use the diagonal elements to get rid of the terms in X_b and X_c. Thus we reduce the determinant to that of a three by three matrix for $n = \underset{\sim}{8}$ and two by two matrix for $n = \underset{\sim}{27}$ and $\underset{\sim}{1}$. In Table 7 we give the solution to the first step, that of solving the coupling splittings in terms of the mass splittings.[33] Using these relations we eliminate the coupling splittings from the remaining equations and are thus left with equations connecting mass splittings only. These we present in matrix form in Table 8. (We have taken a linear combination of the equations coming from $\delta Z_2^{(8_1)}$ and $\delta Z_2^{(8_2)}$ in Table 6 to get rid of the b_3 terms in the first row of Table 8.) This is the limit of our algebraic diagonalization of the original 19 by 19 determinant-matrix.

Table 6. Expansions of the equations $\delta\Gamma^{(n_\gamma, n'_{\gamma'})} = 0$, $\delta Z_2^{(n_\gamma)} = 0$, and $\delta Z_3^{(n_\gamma)} = 0$ for $\alpha = 3/4$. The normalization differs from that in Table 5 by a suitably chosen quantity. The columns have been divided through by the factors appearing at the top. We have defined $X_V = V/h_2$, $X_b = b/h_2$, and $X_c = c/h_2$.

$\underset{\sim}{n} = 8$

	$e_{(n_\gamma,n'_{\gamma'})}\delta f_{(n_\gamma,n'_{\gamma'})}$							
$(n_\gamma, n'_{\gamma'})$	(8, 27)	$(8, 10_+)$	$(8_1, 8_1)$	$(8_1, 8_2)$	(8, 1)	$\delta m_{(8_1)}$	$\sqrt{5}\,\delta m_{(8_2)}$	$\delta\mu^2_{(8)}$
$e_{(n_\gamma, n'_\gamma)}$	$\frac{\sqrt{30}}{20}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{5}}{10}$	$\frac{1}{2}$	$\frac{\sqrt{10}}{4}$			
$\delta\Gamma^{(8,27)}$	$\frac{127}{9}X_V$	0	$-17X_V$	X_V	X_V	$3a_2 - a'_2$	$-a_2 + \frac{1}{3}a'_2$	$-4a'_3$
$\delta\Gamma^{(8,10_+)}$	0	$-2X_V$	0	0	0	$-a_2 - a'_2$	$a_2 + a'_2$	0
$\delta\Gamma^{(8_1,8_1)}$	$-17X_V$	0	$13X_V$	$-9X_V$	X_V	$3a_2 - a'_2$	$-a_2 - 3a'_2$	$-5a_3 + a'_3$
$\delta\Gamma^{(8_1,8_2)}$	X_V	0	$-9X_V$	$5X_V$	$-X_V$	$-a_2 + 3a'_2$	$-a_2 + a'_2$	$-a_3 - a'_3$
$\delta\Gamma^{(8,1)}$	X_V	0	X_V	$-X_V$	$\frac{19}{5}X_V$	$a_2 - 3a'_2$	$\frac{1}{5}a_2 - \frac{3}{5}a'_2$	0
$\delta Z_2^{(8_1)}$	$-6X_b$	$2X_b$	$-6X_b$	$2X_b$	$-2X_b$	$6b_2 - b'_2$	$-b'_2$	$-b_3$
$\delta Z_2^{(8_2)}$	$2X_b$	$-2X_b$	$2X_b$	$2X_b$	$-\frac{2}{5}X_b$	$-b'_2$	$\frac{6}{5}b_2 + \frac{3}{5}b'_2$	b_3
$\delta Z_3^{(8_1)}$	0	0	$10X_c$	$2X_c$	0	$-c_2$	c_2	$3c_3$

Table 6. Continued

$n = \underset{\sim}{27}$	$e_{(n_\gamma,n'_{\gamma'})}\delta f_{(n_\gamma,n'_{\gamma'})}$					
$(n_\gamma,n'_{\gamma'})$	$(27_1,27)$	$(27,10_-)$	$(27,8_1)$	$(27,8_2)$	$\delta m_{(27)}$	$\delta\mu^2_{(27)}$
$e_{(n_\gamma,n'_{\gamma'})}$	$\frac{3\sqrt{70}}{5}$	$5\sqrt{2}$	$\frac{3\sqrt{5}}{10}$	$\frac{1}{2}$		
$\delta\Gamma^{(27_1,27)}$	$\frac{31}{168}X_V$	$-\frac{1}{8}X_V$	$-X_V$	$-X_V$	$-3a_2+a'_2$	$-a'_3$
$\delta\Gamma^{(27,10_-)}$	$-\frac{1}{8}X_V$	$\frac{19}{200}X_V$	X_V	X_V	$-\frac{3}{5}a_2-\frac{3}{5}a'_2$	a'_3
$\delta\Gamma^{(27,8_1)}$	$-X_V$	X_V	$\frac{59}{3}X_V$	$-7X_V$	$-6a_2+2a'_2$	$-15a_3+13a'_3$
$\delta\Gamma^{(27,8_2)}$	$-X_V$	X_V	$-7X_V$	$65X_V$	$6a_2-18a'_2$	$-9a_3+11a'_3$
$\delta Z_2^{(27)}$	X_b	$\frac{1}{5}X_b$	$2X_b$	$-2X_b$	$9b_2+b'_2$	b_3
$\delta Z_3^{(27)}$	0	0	$10X_c$	$6X_c$	$2c_2$	$9c_3$

$n = \underset{\sim}{1}$	$\sqrt{5}\,\delta f^{(1,8_1)}$	$\delta f^{(1,8_2)}$	$4\delta m_{(1)}$	$2\delta\mu^2_{(1)}$
$\delta\Gamma^{(1,8_1)}$	$-\frac{3}{5}X_V$	$-X_V$	$-(a_2+a'_2)$	$-(a_3+a'_3)$
$\delta\Gamma^{(1,8_2)}$	$-X_V$	X_V	$-(a_2+a'_2)$	$-(a_3+a'_3)$
$\delta Z_2^{(1)}$	$2X_b$	$2X_b$	$\frac{3}{2}(b_2+b'_2)$	$3b_3$
$\delta Z_3^{(1)}$	$2X_c$	$2X_c$	$3c_2$	$3c_3$

Table 7. Coupling constant splittings in terms of mass splittings. We give the expression of $e_{(n_\gamma,n'_\gamma)} X_V \delta f_{(n_\gamma,n'_\gamma)}$, where $e_{(n_\gamma,n'_\gamma)}$ and X_V are the quantities used in Table 6. ($X_V = V/h_2 = 4\sqrt{3}/9$.)

$$\frac{\sqrt{30}}{20} X_V \delta f_{(8,27)} = \frac{3}{680}\left\{(9a_2 + 51a_2')\delta m_{(8_1)} - (33a_2 + 17a_2')\sqrt{5}\,\delta m_{(8_2)} - 87a_3\delta\mu^2_{(8)}\right\}$$

$$\frac{\sqrt{2}}{2} X_V \delta f_{(8,10_+)} = \frac{1}{2}\left\{-(a_2 + a_2')\delta m_{(8_1)} + (a_2 + a_2')\sqrt{5}\,\delta m_{(8_2)}\right\}$$

$$\frac{\sqrt{5}}{10} X_V \delta f_{(8_1,8_1)} = \frac{1}{340}\left\{(79a_2 + 51a_2')\delta m_{(8_1)} - (63a_2 + 17a_2')\sqrt{5}\,\delta m_{(8_2)} - (112a_3 + 85a_3')\delta\mu^2_{(8)}\right\}$$

$$\frac{1}{2} X_V \delta f_{(8_1,8_2)} = \frac{1}{68}\left\{(39a_2 - 17a_2')\delta m_{(8_1)} - (7a_2 + 17a_2')\sqrt{5}\,\delta m_{(8_2)} - (20a_3 + 17a_3')\delta\mu^2_{(8)}\right\}$$

$$\frac{\sqrt{10}}{4} X_V \delta f_{(8,1)} = \frac{1}{136}\left\{-(25a_2 - 85a_2')\delta m_{(8_1)} + (a_2 + 17a_2')\sqrt{5}\,\delta m_{(8_2)} + 15a_3\delta\mu^2_{(8)}\right\}$$

$$\frac{3\sqrt{70}}{5} X_V \delta f_{(27_1,27)} = \frac{63}{10}\left\{-(22a_2 + \frac{4}{3}a_2')\delta m_{(27)} + 11a_3\delta\mu^2_{(27)}\right\}$$

$$5\sqrt{2}\, X_V \delta f_{(27,10_-)} = \frac{5}{2}\left\{-(158a_2 + \frac{4}{3}a_2')\delta m_{(27)} + 55a_3\delta\mu^2_{(27)}\right\}$$

$$\frac{3\sqrt{5}}{10} X_V \delta f_{(27,8_1)} = \frac{3}{20}\left\{(102a_2 - 2a_2')\delta m_{(27)} - (21a_3 + 5a_3')\delta\mu^2_{(27)}\right\}$$

$$\frac{1}{2} X_V \delta f_{(27,8_2)} = \frac{1}{4}\left\{(22a_2 + \frac{2}{3}a_2')\delta m_{(27)} - (5a_3 + a_3')\delta\mu^2_{(27)}\right\}$$

$$\sqrt{5}\, X_V \delta f_{(1,8_1)} = -\frac{5}{2}\left\{2(a_2 + a_2')\delta m_{(1)} + (a_3 + a_3')\delta\mu^2_{(1)}\right\}$$

$$\delta f_{(1,8_2)} = \frac{1}{\sqrt{5}}\,\delta f_{(1,8_1)}$$

Table 8. The reduced equations connecting mass splittings

$\delta m_{(8_1)}$	$\sqrt{5}\ \delta m_{(8_2)}$	$\delta\mu^2_{(8)}$
$\frac{14}{51} a_2 - \frac{2}{3} a'_2 + 3b_2 - b'_2$	$\frac{38}{255} a_2 - \frac{2}{15} a'_2 + \frac{3}{5} b_2 - \frac{1}{5} b'_2$	$\frac{4}{17} a_3$
$-\frac{19}{51} a_2 - \frac{5}{3} a'_2 + 6b_2 - b'_2$	$\frac{47}{51} a_2 + \frac{1}{3} a'_2 - b'_2$	$\frac{59}{51} a_3 + \frac{1}{3} a'_3 - b_3$
$\frac{59}{51} a_2 + \frac{1}{3} a'_2 - c_2$	$-\frac{35}{51} a_2 - \frac{1}{3} a'_2 + c_2$	$-\frac{22}{17} a_3 - a'_3 + 3c_3$

$\delta m_{(27)}$	$\delta\mu^2_{(27)}$
$-66a_2 - \frac{10}{3} a'_2 + 9b_2 + b'_2$	$31a_3 - \frac{1}{3} a'_3 + b_3$
$62a_2 - \frac{2}{3} a'_2 + 2c_2$	$-13a_3 - 3a'_3 + 9c_3$

$2\delta m_{(1)}$	$\delta\mu^2_{(1)}$
$(a_2 + a'_2) - \frac{3}{2}(b_2 + b'_2)$	$(a_3 + a'_3) - 3b_3$
$(a_2 + a'_2) - 3c_2$	$(a_3 + a'_3) - 3c_3$

VIII. DISCUSSION OF RESULTS; STABILITY

The determinants of the matrices in Table 8 were evaluated numerically, using a computer, for selected values of μ/m as a function of g. The experimental mass ratio μ/m is 0.356, where μ is the root mean square of the meson masses and m is the average baryon mass.[9] In the case that the coupling splittings are small, $g^2/4\pi$ would be about 15. We have plotted the determinants for $\mu/m = 0.356$ as a function of the coupling constant in Fig. 4; the normalization is as in Table 5.[34]

We would have hoped to find a zero in the $n = \underset{\sim}{8}$ determinant at about the experimental value of g, since the measured mass splittings are dominantly $n = \underset{\sim}{8}$,[35] but the determinant never reaches zero although it has a rather sharp minimum at $g^2/4\pi = 43$. The determinant for $n = \underset{\sim}{27}$ does go through zero but at a value of $g^2/4\pi$ which is about two orders of magnitude too large; since the experimental mass splittings according to the $\underset{\sim}{27}$ representation are small[35] it is encouraging that our model predicts no spontaneous symmetry breakdown for any "reasonable" values of g in $n = \underset{\sim}{27}$. (Although the $n = \underset{\sim}{27}$ curve approaches the axis for small g, it is very unlikely that it will actually cross; both terms in the expansion of the determinant are negative and about the same magnitude.) The $n = \underset{\sim}{1}$ determinant does not have a zero in the calculated region,[36] but a zero for some lower value of g is probable because the two terms in the expansion of the determinant tend to cancel.

If μ/m is varied we find that the curve for $n = \underset{\sim}{8}$ is quite

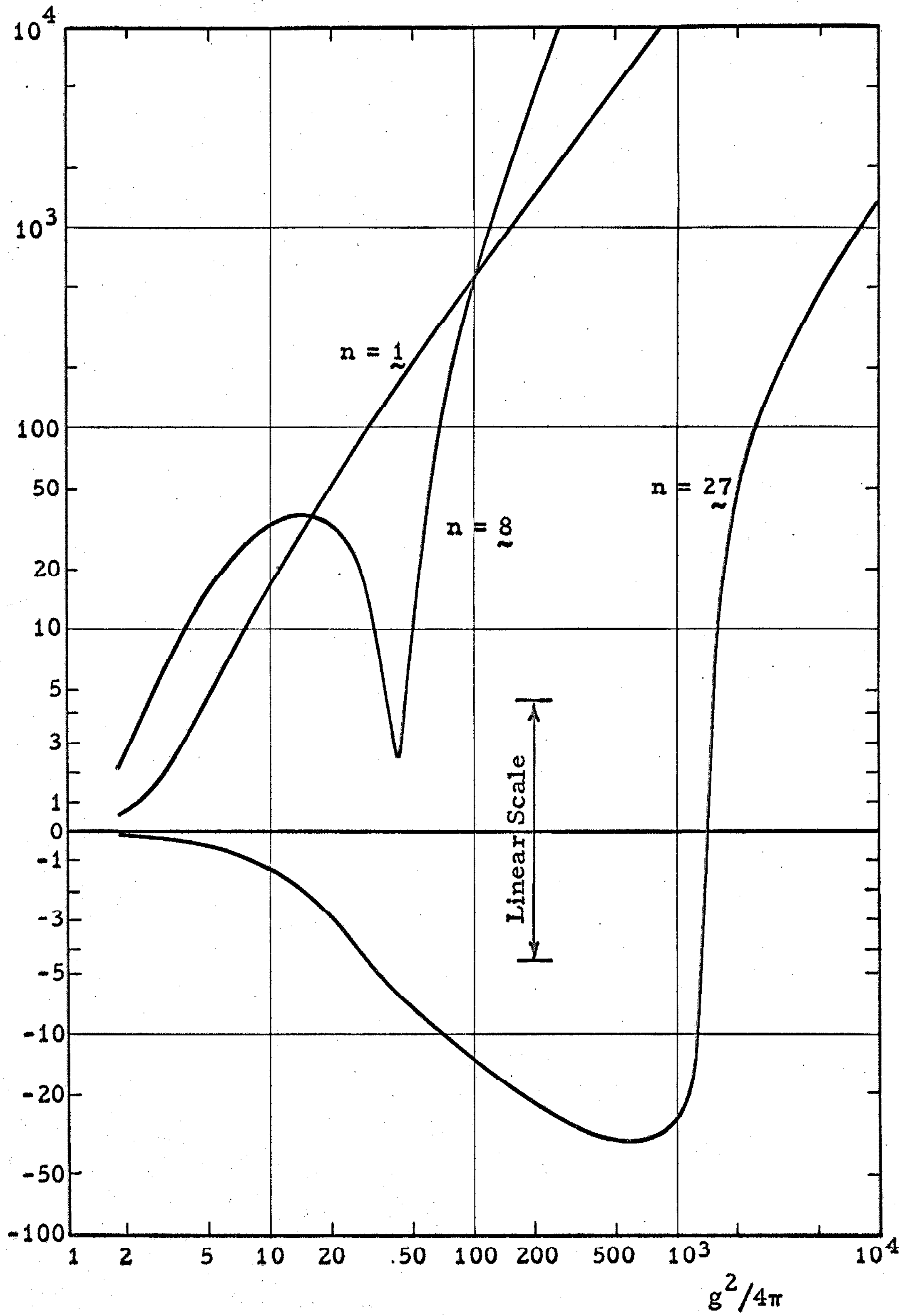


Figure 4. Determinants for $\mu/m = 0.356$.

unstable; for $\mu/m = 0.250$ the minimum has dipped below the axis and we have two zeros, at $g^2/4\pi = 46$ and 103. We have plotted in Fig. 5 the curves for some values of μ/m to illustrate the instability in $n = \underset{\sim}{8}$; in all cases there is a rather sharp minimum, which suggests that even when no actual spontaneous breakdown occurs, the system is potentially unstable against perturbing influences which have been disregarded in our simple model. This is in agreement with other studies,[37] and, in particular, is consistent with the theory of octet enhancement by Dashen and Frautschi.[38] The zeros for $n = \underset{\sim}{27}$ still occur for very high values of the coupling constant. In Table 9 we present the location of the zeros for $n = \underset{\sim}{8}$ and $\underset{\sim}{27}$ for a few values of the mass ratio, together with the approximate position and magnitude of the minimum in the curve for $n = \underset{\sim}{8}$.

TABLE 9. The determinants for $n = \underset{\sim}{8}$ and $\underset{\sim}{27}$ as a function of μ/m and $g^2/4\pi$

	$g^2/4\pi$				
	Zeros for $n = \underset{\sim}{8}$		Minimum for $n = \underset{\sim}{8}$		Zero for $n = \underset{\sim}{27}$
$\frac{\mu}{m}$			Position	Value	
.147	50.5	529	350	-10^4	1310
.250	45.9	103	75	-130	1320
.356	...		43	2.5	1420
.500	...		28	0.8	1682
.712	...		20	0.3	2430

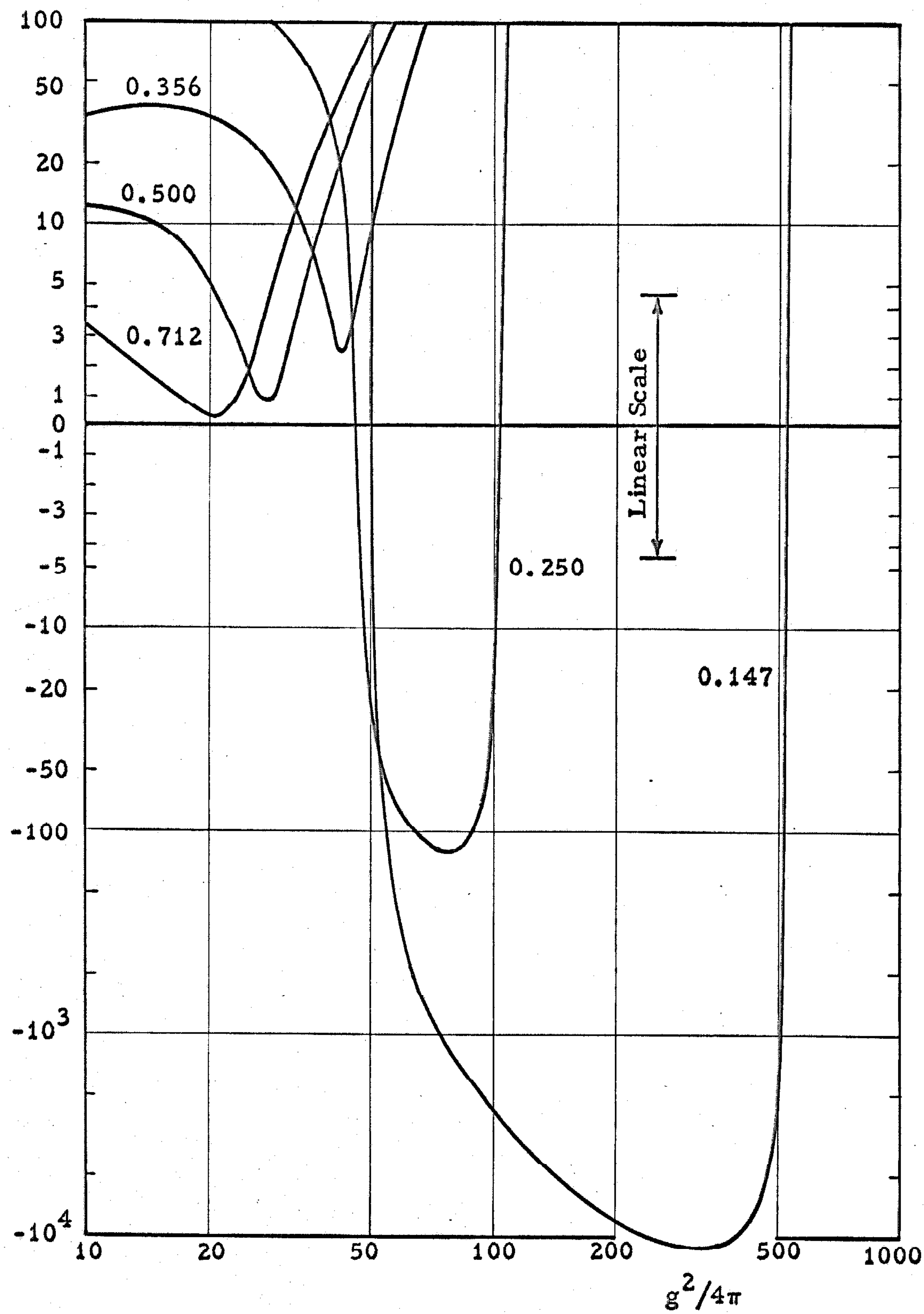


Figure 5. The $n = \underset{\sim}{8}$ determinant for some μ/m.

Before turning our attention to the splittings we like to test our solutions for stability in a different way. We wish to find out how sensitive our results are to the way in which the renormalization constants are calculated. If, for example, we had included fourth order terms in Z_{3k}, it would have been of the form

$$Z_{3k} = 1 - g^2 J_{3k}^{(2)} - g^4 J_{3k}^{(4)} \quad . \tag{8.1}$$

In a simple investigation of stability we substitute for the last term a parametric constant ϵ_3 independent of couplings and masses. We may then write Eq. (8.1) as

$$Z_{3k} = (1 - \epsilon_3)(1 - g_3^2 J_{3k}^{(2)}) \quad , \tag{8.2}$$

where we have defined $g_3^2 = g^2/(1 - \epsilon_3)$. Our bootstrap conditions $Z_{3k}|_s = 0$ and $\delta Z_{3k} = 0$ are then just like what we had before (we may divide out the factor $(1 - \epsilon_3)$ in front), except that the coupling constant g_3 is used in place of g in the dynamical quantities c, c_2, and c_3. In the same way we write

$$\begin{aligned} Z_{2i} &= 1 - g^2 J_{2i}^{(2)} - \epsilon_2 \\ &= (1 - \epsilon_2)(1 - g_2^2 J_{2i}^{(2)}) \end{aligned} \tag{8.3}$$

and take into account the effect of ϵ_2 by using the coupling constant $g_2^2 = g^2/(1 - \epsilon_2)$ rather than the given g^2 in the dynamical quantities b, b_2, b_2', and b_3. For the vertex constant $\Gamma_{\bar{i}jk}$ we introduce ϵ_1 by

$$\Gamma_{\bar{i}jk} = f_{\bar{i}jk} - \Lambda_{\bar{i}jk} - f_{\bar{i}jk}\,\epsilon_1 \quad . \tag{8.4}$$

Writing $\Lambda_{\bar{i}jk} = g^2 J_{\bar{i}jk}$ and defining $g_1^2 = g^2/(1 - \epsilon_1)$ gives

$$\Gamma_{\bar{i}jk} = (1 - \epsilon_1)(f_{\bar{i}jk} - g_1^2 J^{(2)}_{\bar{i}jk}) \quad .$$

The factor $(1 - \epsilon_1)$ in front is again irrelevant for the bootstrap conditions $\Gamma_{\bar{i}jk}\big|_s = 0$ and $\delta\Gamma_{\bar{i}jk} = 0$, so that the only effective change is $g^2 \rightarrow g_1^2$ in the dynamical quantities c, c_2, c_2', c_3, and c_3'.

We now test the stability of the solutions by assigning non-zero values to the ϵ-parameters[39] and find the zeros of the determinants. The results of such an analysis for $\mu/m = 0.356$ is presented in Table 10, where we give the location of the zeros for $n = \underset{\sim}{8}$ and $n = \underset{\sim}{27}$. It is interesting that although previously the $\underset{\sim}{8}$ determinant for $\mu/m = 0.356$ did not reach zero, it does so for some not-unreasonable value for any one of the ϵ-parameters. We tabulate for one of the ϵ's non-zero with magnitude 0.5, and also, for $n = \underset{\sim}{8}$, the smallest amount in steps of 0.1 that an ϵ has to be changed from zero in order to produce a solution.

TABLE 10. Stability test for location of zero for $\mu/m = 0.356$.

ϵ_1	ϵ_2	ϵ_3	$g^2/4\pi$		
			$n = \underset{\sim}{8}$		$n = \underset{\sim}{27}$
0	0	0	(Minimum at 43)		1420
-0.5	0	0	15.4	16.8	1190
0.5	0	0	210	734	1800
0	0.5	0	5.4	16.7	574
0	0	0.5	9.0	45	1270
0.2	0	0	73	88	
0	0.2	0	23	28	
0	0	0.1	33	43	

We note in passing that a zero was also found for the $n = \underset{\sim}{1}$ determinant with $\epsilon_1 = 0.5$, $\epsilon_2 = \epsilon_3 = 0$ at $g^2/4\pi = 8.2$. This seems to confirm our suspicion that a solution exists for some low value of g in the unperturbed case, since the effect of a positive ϵ_1 is consistently to shift the position of the zero to a larger g value. The sequence of the solutions in $n = \underset{\sim}{1}, \underset{\sim}{8}$, and $\underset{\sim}{27}$ conforms with the observation which has been made that the forces tend to be stronger in a channel with a lower-dimensional representation.[40]

It is evident from Table 10 that the solutions are quite unstable; the values of $g^2/4\pi$ for $n = \underset{\sim}{8}$ tend, however, to be the right order of magnitude, whereas for $n = \underset{\sim}{27}$ they remain quite large. We shall not concern ourselves with the latter representation any further, but limit our attention to the case $n = \underset{\sim}{8}$, which is the more interesting one experimentally anyway.

Having found the zeros of the determinant we can evaluate the splittings in the corresponding solutions. Since the equations connecting the splittings are homogeneous, we can at most find the ratios. Before we present the numerical solutions, we derive some relations between the coupling splittings which rest solely on the assumption that they transform according to the $\underset{\sim}{8}$-representation. This we can do because there are only five distinct hermiticity-conserving irreducible splittings for $n = \underset{\sim}{8}$, in terms of which the twelve independent δg_{ABC} are to be calculated. We therefore expect seven relations[41] among the δg_{ABC}. To find these we use Table 4. It gives the expansion of

$$\delta f_{\bar{A}BC} = \sum_{\gamma n'\gamma'} \delta f_{(8_\gamma, n'_{\gamma'})} O_{\bar{A}BC}^{(n_\gamma, n'_{\gamma'})} ,$$

where on the right-hand side we now include only the five hermiticity-conserving ($C = +$) splittings for $n = \underset{\sim}{8}$. Eliminating the irreducible splittings among the twelve independent equations (if the conjugate forms $\bar{A}BC$ and $\bar{B}A\bar{C}$ are different we need consider only one of them) we get seven relations between the $\delta f_{\bar{A}BC}$'s. These are expressed in terms of δg_{ABC} by using Eq. (3.19); the normalization constant $N_{\bar{A}BC}$ is given in Table 2. The result is presented in Table 11. We shall prefer to express our calculated splittings in terms of the fractional values $(\delta g_{ABC}/g_{ABC}) = \Delta(ABC)$. The sum rules for the fractional splittings are given in Table 12. (For $\alpha = 3/4$ we have $g_{\Xi\Lambda K} = g_{NN\eta} = 0$; for purposes of normalization we have put them both equal to $g\sqrt{3}/2$.) It should be noted that the relations in Table 12 depend on α, but those in Table 11 do not.

The splittings corresponding to the zeros in $n = 8$ which have been given in Tables 9 and 10 are presented in Table 13. We have calculated Δm_1 and Δm_2 which are the ratios of the baryon mass splittings to the meson mass splitting defined by

$$\Delta m_1 = \left(\frac{\delta m_{(8_1)}}{m}\right) \Big/ \left(\frac{\delta \mu^2_{(8)}}{m^2}\right) , \text{ exp. value} = -0.32 ;$$

$$\Delta m_2 = \left(\frac{\delta m_{(8_2)}}{m}\right) \Big/ \left(\frac{\delta \mu^2_{(8)}}{m^2}\right) , \text{ exp. value} = 1.33 .$$

The coupling splittings $\Delta(ABC)$ are fractional splittings,

Table 11. Coupling splitting sum rules for symmetry breakdown according to $n = \underset{\sim}{8}$.

$$\delta g_{\Xi\Sigma K} = -\frac{1}{2}\,\delta g_{NN\pi}$$

$$\delta g_{N\Sigma K} = -\frac{1}{2}\,\delta g_{\Xi\Xi\pi}$$

$$\delta g_{N\Lambda K} + \delta g_{\Xi\Lambda K} = -\frac{\sqrt{3}}{2}(\delta g_{NN\pi} + \delta g_{\Xi\Xi\pi}) + 2\delta g_{\Lambda\Sigma\pi}$$

$$\delta g_{N\Lambda K} - \delta g_{\Xi\Lambda K} = -\frac{1}{2\sqrt{3}}(\delta g_{NN\pi} - \delta g_{\Xi\Xi\pi}) + \frac{2}{\sqrt{3}}\,\delta g_{\Sigma\Sigma\pi}$$

$$\delta g_{\Lambda\Lambda\eta} - \delta g_{\Sigma\Sigma\eta} = 2\delta g_{\Lambda\Sigma\pi}$$

$$\delta g_{\Xi\Xi\eta} - \delta g_{NN\eta} = \frac{1}{\sqrt{3}}(\delta g_{NN\pi} - \delta g_{\Xi\Xi\pi} + 2\delta g_{\Sigma\Sigma\pi})$$

$$\delta g_{\Xi\Xi\eta} + \delta g_{NN\eta} - 2\delta g_{\Sigma\Sigma\eta} = \sqrt{3}\,(\delta g_{NN\pi} + \delta g_{\Xi\Xi\pi})$$

Table 12. The relations in Table 11 expressed in fractional forms for $\alpha = 3/4$. We write $\Delta(ABC) = \delta g_{ABC}/g_{ABC}$; $g_{\Xi\Lambda K}$ and $g_{NN\eta}$ have been assigned the value $g\sqrt{3}/2$.

$$\Delta(\Xi\Sigma K) = -\frac{1}{2}\,\Delta(NN\pi)$$

$$\Delta(N\Sigma K) = -\frac{1}{2}\,\Delta(\Xi\Xi\pi)$$

$$\Delta(N\Lambda K) - \Delta(\Xi\Lambda K) = \Delta(NN\pi) + \frac{1}{2}\,\Delta(\Xi\Xi\pi) - 2\Delta(\Lambda\Sigma\pi)$$

$$3\Delta(N\Lambda K) + 3\Delta(\Xi\Lambda K) = \Delta(NN\pi) - \frac{1}{2}\Delta(\Xi\Xi\pi) - 2\Delta(\Sigma\Sigma\pi)$$

$$\Delta(\Lambda\Lambda\eta) + \Delta(\Sigma\Sigma\eta) = -2\Delta(\Lambda\Sigma\pi)$$

$$3\Delta(NN\eta) + 3\Delta(\Xi\Xi\eta) = -2\Delta(NN\pi) + \Delta(\Xi\Xi\pi) - 2\Delta(\Sigma\Sigma\pi)$$

$$\Delta(NN\eta) - \Delta(\Xi\Xi\eta) - 2\Delta(\Sigma\Sigma\eta) = 2\Delta(NN\pi) + \Delta(\Xi\Xi\pi)$$

Table 13. The splittings for the $n = \underset{\sim}{8}$ solutions in Tables 9 and 10. We tabulate the ratio of the baryon mass splittings to the meson mass splitting (experimentally $\Delta m_1 = -0.32$, $\Delta m_2 = 1.33$), and the fractional coupling splittings $\Delta(ABC) = \delta g_{ABC}/g_{ABC}$ (see text for details). The λ's are the *cutoff masses of the integrals.*

$\frac{\mu}{m}$	ϵ_1	ϵ_2	ϵ_3	$\frac{g^2}{4\pi}$	$\frac{\lambda_1}{m}$	$\frac{\lambda_2}{m}$	$\frac{\lambda_3}{m}$	Δm_1	Δm_2	Δ(ABC) NNπ	ΛΣπ	ΣΣπ	ΞΞπ	NΛK	NΣK	ΞΛK	ΞΣK	NNη	ΛΛη	ΣΣη	ΞΞη	
0.147	0	0	0	50.5	2.492	1.416	2.124	-0.22	0.47	-0.42	-0.20	-0.10	-0.03	-0.06	0.01	-0.01	0.21	0.00	0.13	0.27	0.33	FIRST ZERO
0.250	0	0	0	45.9	2.567	1.554	2.131	-0.18	0.31	-0.26	-0.12	-0.05	-0.02	-0.03	0.01	-0.02	0.13	0.00	0.08	0.17	0.19	
0.356	0.2	0	0	73.2	2.385	1.595	2.093	-0.13	0.22	-0.26	-0.12	-0.04	-0.01	-0.05	0.00	-0.02	0.13	0.00	0.07	0.17	0.20	
	0	0.2	0	23.5	2.999	1.800	2.206	0.60	0.59	-0.12	-0.06	-0.13	0.11	0.04	-0.05	-0.01	0.06	0.00	0.16	-0.04	0.20	
	0	0	0.1	33.4	2.764	1.761	2.149	-0.29	0.16	-0.15	-0.08	-0.01	-0.04	-0.03	0.02	-0.01	0.07	0.01	0.03	0.13	0.08	
	-0.5	0	0	15.4	4.046	2.071	2.280	-0.59	0.06	-0.07	-0.04	0.01	-0.04	-0.02	0.02	-0.01	0.04	0.01	0.00	0.09	0.02	
	0.5	0	0	210	2.157	1.478	2.045	0.01	0.25	-0.55	-0.20	-0.11	0.12	-0.11	-0.06	-0.02	0.27	-0.03	0.18	0.22	0.51	
	0	0.5	0	5.4	6.095	2.305	2.622	5.78	2.95	-0.03	-0.02	-0.28	0.29	0.13	-0.14	-0.01	0.02	-0.01	0.31	-0.27	0.32	
	0	0	0.5	9.0	4.326	2.470	2.249	-0.17	0.18	-0.06	-0.04	-0.02	-0.01	0.00	0.01	-0.01	0.03	0.01	0.03	0.05	0.04	
0.147	0	0	0	529	2.104	1.209	2.025	-7.31	-2.63	-2.44	-1.15	8.36	-9.30	-4.81	4.65	-0.02	1.22	0.75	-9.09	-11.4	-7.80	SECOND ZERO
0.250	0	0	0	103	2.327	1.430	2.075	-2.26	-0.61	-0.48	-0.23	0.74	-0.91	-0.49	0.43	-0.02	0.24	0.07	-0.78	1.25	-0.55	
0.356	0.2	0	0	88.0	2.343	1.569	2.082	-0.43	0.09	-0.30	-0.14	0.07	-0.13	-0.11	0.06	-0.02	0.15	0.01	-0.04	0.32	0.10	
	0	0.2	0	28.0	2.872	1.748	2.182	0.08	0.36	-0.14	-0.07	-0.06	0.03	0.00	-0.02	-0.01	0.07	0.00	0.09	0.05	0.14	
	0	0	0.1	43.0	2.638	1.696	2.125	-0.88	-0.10	-0.18	-0.10	0.11	-0.18	-0.09	0.09	-0.02	0.09	0.02	-0.10	0.30	-0.03	
	-0.5	0	0	16.8	3.844	2.025	2.263	-0.95	-0.11	-0.08	-0.05	0.04	-0.08	-0.03	0.04	-0.01	0.04	0.01	-0.03	0.13	-0.02	
	0.5	0	0	734	2.081	1.412	2.019	-1.18	-0.29	-1.22	-0.50	1.44	-1.68	-1.07	0.84	-0.02	0.61	0.23	-1.64	2.64	-0.94	
	0	0.5	0	16.8	3.313	1.760	2.263	-0.10	0.34	-0.11	-0.06	-0.03	0.01	-0.01	0.00	-0.01	0.06	0.00	0.05	0.06	0.09	
	0	0	0.5	44.9	2.618	1.687	2.081	-4.49	-1.76	-0.22	-0.15	0.85	-1.03	-0.46	0.52	-0.02	0.11	0.10	-0.92	1.22	-0.86	

$\delta g_{ABC}/g_{ABC}$ (with $g_{\Xi\Lambda K} = g_{NN\pi} = g\sqrt{3}/2$), evaluated by using the experimental value of $\delta\mu^2_{(8)}/m^2$. We have also tabulated the cutoff masses λ_1, λ_2, and λ_3 in units of the average baryon mass (the threshold masses are $m_1/m = 2$, $m_2/m = 1 + \mu/m$, $m_3/m = 2$).

The mass splittings for the zero with the lower g tend to have the correct sign, but the magnitudes vary considerably and a comparison with experimental values does not appear meaningful. For the zero with the higher g the dominant mass splitting Δm_2 tends to have the wrong sign.

The coupling splittings are fairly unstable also, both with respect to changes in μ/m and the ϵ-parameters. Nevertheless there are a few features worth pointing out.

The splittings which are consistently smallest in both solutions are those whose SU(3) symmetric couplings vanish ($g_{\Xi\Lambda K}$ and $g_{NN\eta}$); the $\Xi\Lambda K$ coupling in particular stays remarkably small throughout.

The coupling splittings for the zero with the lower g in those cases where the ϵ's are small exhibit a behavior similar to that observed by Dashen, Dothan, Frautschi, and Sharp (DDFS) in their calculation of coupling shifts[42]; in the table the baryon-antibaryon channels have been arranged in order of increasing mass for each meson, and it is apparent that the coupling shift tends to increase as we progress from low to high mass channels. The result of DDFS was exactly opposite to this. (It is amusing that if we change all the signs in the coupling splittings for the two cases $\mu/m = 0.250$ and

$\mu/m = 0.356$ with $\epsilon_1 = 0.2$ we reproduce their values within ± 0.02.) They consider the reciprocal bootstrap model(43) for the baryon octet and $3/2^+$ baryon decuplet and investigate the dynamical breakdown of SU(3) symmetry by an S-matrix method for calculating perturbations.(44) Aside from technical differences between our approach and theirs, a conceptually simpler difference is that they include the decuplet in their bootstrap system and treat the meson octet as a fixed input (with regard to the meson masses(45)), whereas we admit only the baryon and meson octets but treat them both equally. Since the same pattern of relative coupling shifts emerges in both calculations it is suggestive to think that the main effect of the decuplet is to supply an over-all factor (of order -1) for the meson-baryon coupling splittings. In any case the decuplet does have an important effect on the coupling shifts, as was noticed(46) in the work of DDFS, and should be taken into account in any realistic calculation.

Empirically, "hard" information about the coupling constants is scarce.(47) Theoretical analyses of experiments are usually based on simple models, and it is difficult to estimate the reliability of the answers. Roughly speaking, SU(3) symmetry of the couplings seems to work fairly well. (As mentioned by DDFS, however, their result that the η-couplings are reduced is an attractive one experimentally since η-production is relatively rare; also the reduction of K-couplings relative to π-couplings was reported(48) to have some empirical evidence.) We shall only make one comment about our results with reference to experiment. In our model there seems to be a correlation between the pi-nucleon coupling splitting and the pi-lambda-sigma coupling splitting, which exists in both solutions. Approximately, the fractional splittings are related by $\Delta(\Lambda\Sigma\pi) = \frac{1}{2}\Delta(NN\pi)$.[49] The $\Lambda\Sigma\pi$ coupling constant has been estimated to be about equal to or larger than the $NN\pi$ coupling.[50] The SU(3) symmetric $NN\pi$ and $\Lambda\Sigma\pi$ couplings, for $\alpha = 3/4$, are g and $g\sqrt{3}/2$ respectively; assuming $\Delta(\Lambda\Sigma\pi) = \frac{1}{2}\Delta(NN\pi)$ the total coupling constants can only be equal if both fractional splittings are negative. This is in agreement with our results, and contrary to those of DDFS (their result for the total couplings is $g^2_{\Lambda\Sigma\pi}: g^2_{NN\pi} = 0.45$, which is outside the "empirical" range). Although we feel that the latter work is a realistic one (for one thing the reciprocal bootstrap model upon which it is based is among the most successful ones there is), this example serves to illustrate the uncertainty in the calculation of coupling constants.

One can discover other properties of our model; for example, in the case of $\mu/m = 0.356$ the magnitude of the pi-nucleon coupling constant increases fairly consistently with increasing g in both solutions as the ϵ's are varied. However, we do not wish to dwell further on the detailed properties of our model, the significance of which would be quite unclear. We stress again that our calculation is quite sensitive to the parameters we have introduced into the problem; the same feature was found in an investigation of the self-interacting vector meson system by Cutkosky and Leon[51] using an approach somewhat similar to ours.

To the extent that our simple model has quantitative validity, the solution with the higher value of g does not seem to reflect itself in nature, whereas the one with the lower value of g does. It is an interesting question theoretically to ask if there is a reason for the system to select one of the two solutions in $n = \underset{\sim}{8}$ instead of the other; we can find no criterion that would make such a choice.

IX. SOME RELATED PROBLEMS; CONCLUSION

We like to make some comments on other matters related to our investigation.

In their calculation of coupling shifts, DDFS found solutions which violated charge conjugation. This difficulty, based on the lack of "vertex symmetry" in most N/D-type calculations, has been discussed by DDFS and others.[52] It may be interesting to see what happens in the present work when the hermiticity requirement on the coupling splittings is relaxed.

First, however, we find the conditions which ensured the existence of hermiticity conserving solutions. When the requirement that the hermiticity violating coupling splittings be zero was imposed, it was essential that the number of vertex equations effectively be reduced from the original seventeen to twelve. This reduction was taken care of by the phase factor ϕ in Eq. (7.9a); it was made possible by the hermitian-type property $\delta\Gamma_{\bar{j}i\bar{k}} = \delta\Gamma_{\bar{i}jk}$. In deriving the latter relation we used, beside the condition $\delta f_{\bar{j}i\bar{k}} = \delta f_{\bar{i}jk}$, symmetry properties of the dynamical vertex quantity $I_1(m_i m_j m_r m_s; \mu_k \mu_\ell)$, namely $\partial I_1/\partial m_i|_s = \partial I_1/\partial m_j|_s$ and $\partial I_1/\partial m_r|_s = \partial I_1/\partial m_s|_s$. These follow from the general condition of vertex symmetry, which is that I_1 be invariant under the simultaneous exchange of $m_i \leftrightarrow m_j$ and $m_r \leftrightarrow m_s$; this general property is easily shown to be satisfied by I_1.

Without imposing the hermitian condition on coupling splittings in the expressions for $\delta\Gamma_{\bar{i}jk}$, δZ_{2i}, and δZ_{3k}, the resulting equations in the irreducible splittings have the following properties. The hermiticity violating splittings do not couple to the hermiticity conserving splittings or to the mass splittings. The equations connecting the hermiticity violating splittings are linearly dependent in both the $\underset{\sim}{8}$ and $\underset{\sim}{27}$ representations (for $\alpha = 3/4$) so that a non-trivial solution exists for all g in both representations.[53] We show this in detail in Appendix E.

The SU(3) symmetric vertex bootstrap condition $\Gamma_{\bar{i}jk}|_s = 0$ gave us, without much effort, the rather attractive result $\alpha = 3/4$ (another possibility was $\alpha = 1$). We like to apply this condition to a few other simple cases. First consider a pseudoscalar singlet ϕ in a Yukawa coupling with the baryon octet. The basic interaction Lagrangian is

$$\mathcal{L}_I = \sum_i g f_i (\bar{\psi}_i \gamma_5 \psi_i)\phi \quad , \tag{9.1}$$

where

$$f_i = \begin{pmatrix} 8 & 8 & 1 \\ -i & i & 0 \end{pmatrix} \eta_{\bar{i}i} \quad .$$

The total vertex constant is

$$\Gamma_i = f_i - \Lambda_i \quad , \quad \text{where}$$
$$\Lambda_i = -f_i^3 I_1(m_i m_i m_i m_i; \mu\, \mu) \quad . \tag{9.2}$$

$I_1|_s$ is positive; hence the vertex bootstrap condition $\Gamma_i|_s = 0$ cannot be satisfied with a pseudoscalar singlet coupled to a baryon octet.

Next we look at the system of scalar mesons interacting with a baryon octet. The basic dynamical quantity Λ (introduced in Sec.V; for pseudoscalar mesons we defined $I_1 = -\Lambda$) has the same sign for scalar and pseudoscalar mesons interacting with baryons, if we consider the high-energy dominant term in the integral. These two types of mesons, therefore, behave in the same way with regard to the vertex bootstrap condition. Finally we look at the case of vector mesons interacting with baryons (by an electric type coupling). The vertex integral has a behavior similar to that for a vertex in electrodynamics; in the latter case $Z_1 = Z_2$ by Ward's identity and therefore Z_1 is one minus a positive quantity. Hence for a single vector meson interacting with a single spinor field it is possible to satisfy the vertex bootstrap requirement (Λ is positive); the same is evidently true for a vector meson singlet interacting with a baryon octet. If both particles belong to octets the SU(3) symmetric vertex bootstrap conditions are, for a general F-D ratio,

$$\sum_{\gamma_1} f_{\bar{i}jk}^{(\gamma_1)} (1 - C^{(\gamma_1)} \Lambda) = 0 \qquad \text{for all } (ijk) , \tag{9.3}$$

where the coefficients $C^{(\gamma)}$ are given in (6.10). With Λ positive, Eq. (9.3) can be satisfied only for pure F coupling. This is a theoretically attractive result because in the limit of SU(3) symmetry the ρ meson would then be coupled to the conserved isospin current.[54] We summarize in Table 14 the results of our simple application of the vertex bootstrap condition.

Table 14. The vertex bootstrap condition for some mesons interacting with a baryon octet

Meson		Bootstrap Condition Satisfied	Remarks
Scalar	$\underset{\sim}{1}$	No	
	$\underset{\sim}{8}$	Yes	$\alpha = 3/4$ or pure D
Pseudo-scalar	$\underset{\sim}{1}$	No	
	$\underset{\sim}{8}$	Yes	$\alpha = 3/4$ or pure D
Vector	$\underset{\sim}{1}$	Yes	
	$\underset{\sim}{8}$	Yes	Pure F

One of the features in the numerical calculation was that the splittings for those couplings whose SU(3) symmetric values vanished were consistently small. In the case of the vector meson octet coupled to the baryon octet with pure F we have a situation where three SU(3) symmetric coupling constants are zero (see Table 3), and it is interesting to see what would happen in a spontaneous breakdown of symmetry in this case. From Table 4 it is seen that the couplings for $\Lambda\Sigma\rho$, $\Lambda\Lambda\phi$, and $\Sigma\Sigma\phi$, which are zero in pure F, connect to the irreducible splittings $\delta f_{(n_\gamma, n'_{\gamma'})}$ for which $(n_\gamma, n'_{\gamma'}) = (64, 27), (27_1, 27), (27, 8_1), (8, 27), (8_1, 8_1), (8, 1)$, and $(1, 8_1)$. Evaluating the bootstrap conditions in Table 5 for $h_1 = 0$ we find the following. Of the above-mentioned irreducible splittings the ones belonging to $n = \underset{\sim}{64}, \underset{\sim}{27}$, and $\underset{\sim}{1}$ must be zero; the three with $n = \underset{\sim}{8}$ couple only to $\delta m_{(8_2)}$. In the reduced determinant for $n = \underset{\sim}{8}$ the splitting $\delta m_{(8_2)}$ does not couple to the other mass splittings; hence if there is a solution in the $\underset{\sim}{8}$ representation it is either such that $\delta m_{(8_2)} = 0$ and the others are non-zero, or vice versa (it is very unlikely that all the mass splittings are different from zero). Although we can say nothing conclusive without calculating numerically, it is interesting that there is the possibility of a symmetry breakdown in which the vanishing symmetric couplings remain identically zero.

In conclusion we like to summarize the main results of our simple model of the bootstrapped pseudoscalar meson and baryon octets. The value of the mixing parameter was found to be $\alpha = 3/4$, which is surprisingly good. A symmetry breaking solution is found in the $\underset{\sim}{27}$ representation but for a coupling constant squared which is two orders of magnitude too large. In the $\underset{\sim}{8}$ representation the system shows great instability; two solutions can easily be produced for not unreasonable values of the coupling constant and the mass ratio. The solution with the lower value of the coupling constant has the experimentally observed relative signs of the mass splittings; in addition, it exhibits the same coupling splitting pattern found by other workers in a calculation based on the static reciprocal bootstrap model, but with the relative signs of the coupling shifts and mass shifts reversed. We feel that a spontaneous symmetry breakdown has been proved to be possible, but our model is too crude to reflect anything but a few remnants of the real situation.

APPENDIX A

SU(3) COEFFICIENTS

1. Clebsch-Gordan Coefficients

We summarize here some properties of the SU(3) CG coefficients, following the work of de Swart.[16] We may combine the eigenstates $\phi_{\nu_1}^{(\mu_1)}$ and $\phi_{\nu_2}^{(\mu_2)}$ of the irreducible representations μ_1 and μ_2 of SU(3) into an eigenstate $\psi\begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma \\ & & \nu\end{pmatrix}$ of the representation μ. The expansion coefficients are the CG coefficients $\begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu\end{pmatrix}$:

$$\psi\begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma \\ & & \nu\end{pmatrix} = \sum_{\nu_1 \nu_2} \begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu\end{pmatrix} \phi_{\nu_1}^{(\mu_1)} \phi_{\nu_2}^{(\mu_2)} \quad . \qquad \text{(A1.1)}$$

Here the ν's stand for the quantum numbers (Y, I, I_z) of the eigenstates, the μ's denote the representations, and γ is any additional quantum number needed to specify the representation in the product space. The coefficients have the orthogonality properties

$$\sum_{\nu_1 \nu_2} \begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu\end{pmatrix}\begin{pmatrix}\mu_1 & \mu_2 & \mu'_{\gamma'} \\ \nu_1 & \nu_2 & \nu'\end{pmatrix} = \delta_{\mu\mu'}\delta_{\nu\nu'}\delta_{\gamma\gamma'} \quad , \qquad \text{(A1.2)}$$

$$\sum_{\mu\gamma\nu} \begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu\end{pmatrix}\begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma \\ \nu'_1 & \nu'_2 & \nu\end{pmatrix} = \delta_{\nu_1\nu'_1}\,\delta_{\nu_2\nu'_2} \quad . \qquad \text{(A1.3)}$$

The dependence on the z component of the isospin is just as in SU(2). We may make this explicit by writing

$$\begin{pmatrix} \mu_1 \; \mu_2 \; \mu_\gamma \\ \nu_1 \; \nu_2 \; \nu \end{pmatrix} = C^{\,I_1 \; I_2 \; I}_{\,I_{1z} \; I_{2z} \; I_z} \left(\begin{matrix} \mu_1 & \mu_2 \\ I_1Y_1 & I_2Y_2 \end{matrix} \middle| \begin{matrix} \mu_\gamma \\ IY \end{matrix} \right) \quad , \qquad \text{(A1.4)}$$

where the isoscalar coefficient $\left(\begin{matrix} \mu_1 & \mu_2 \\ I_1Y_1 & I_2Y_2 \end{matrix} \middle| \begin{matrix} \mu_\gamma \\ IY \end{matrix} \right)$ does not depend on the I_z's. The SU(2) CG coefficients $C^{\,I_1 \; I_2 \; I}_{\,I_{1z} \; I_{2z} \; I_z}$ and the isoscalar coefficients both satisfy orthogonality conditions similar to (A1.2) and (A1.3).

We shall frequently use the symmetry properties of the coefficients. These are written as

$$\begin{pmatrix} \mu_1 \; \mu_2 \; \mu_{3\gamma} \\ \nu_1 \; \nu_2 \; \nu_3 \end{pmatrix} = \xi_1(\mu_1 \; \mu_2 \; \mu_{3\gamma}) \begin{pmatrix} \mu_2 \; \mu_1 \; \mu_{3\gamma} \\ \nu_2 \; \nu_1 \; \nu_3 \end{pmatrix} \quad , \qquad \text{(A1.5)}$$

$$\begin{pmatrix} \mu_1 \; \mu_2 \; \mu_{3\gamma} \\ \nu_1 \; \nu_2 \; \nu_3 \end{pmatrix} = \xi_2(\mu_1 \; \mu_2 \; \mu_{3\gamma}) \, (-1)^{Q_1} \sqrt{\frac{N_3}{N_2}} \begin{pmatrix} \mu_1 \; \mu_3^* \; \mu_{2\gamma'}^* \\ \nu_1 \; -\nu_3 \; -\nu_2 \end{pmatrix} \quad , \qquad \text{(A1.6)}$$

$$\begin{pmatrix} \mu_1 \; \mu_2 \; \mu_\gamma \\ \nu_1 \; \nu_2 \; \nu_3 \end{pmatrix} = \xi_3(\mu_1 \; \mu_2 \; \mu_{3\gamma}) \begin{pmatrix} \mu_1^* \; \mu_2^* \; \mu_{3\gamma}^* \\ -\nu_1 \; -\nu_2 \; -\nu_3 \end{pmatrix} \quad . \qquad \text{(A1.7)}$$

Here ξ_1, ξ_2, and ξ_3 are phase factors equal to ± 1; $Q_1 = I_{1z} + \frac{1}{2} Y_1$ = the "charge" of the state ν_1; N_2 and N_3 are the dimensions of the representations μ_2 and μ_3; and μ^* is the representation conjugate to μ. The negative of the quantum number $\nu = (Y, I, I_z)$ is defined by $-\nu = (-Y, I, -I_z)$. In our application of Eq. (A1.6) the indices γ'

and γ may be equated.

In the present work we shall need two kinds of CG coefficients. The first is for the case $\mu_1 = \mu_2 = \underset{\sim}{8}$ and μ_3 any representation contained in $\underset{\sim}{8} \times \underset{\sim}{8}$. To simplify the notation the phase factors ξ are then written with μ_1 and μ_2 deleted:

$$\xi_k(\mu_{3\gamma}) \equiv \xi_k(8\ \ 8\ \ \mu_{3\gamma}) \quad . \tag{A1.8}$$

The second case is where one of μ_1 and μ_2 is $\underset{\sim}{8}$, the other of μ_1 and μ_2 is anything contained in $\underset{\sim}{8} \times \underset{\sim}{8}$, and μ_3 is a self-conjugate representation contained in $\mu_1 \times \mu_2$, viz. one of $\underset{\sim}{1}$, $\underset{\sim}{8}_1$, $\underset{\sim}{8}_2$, $\underset{\sim}{27}_1$, $\underset{\sim}{27}_2$, and $\underset{\sim}{64}$. The ξ factors in the case where $\mu_1 = \underset{\sim}{8}$ are given in Table 15. (If $\mu_2 = \underset{\sim}{8}$ the ξ's may be derived by using symmetry relations.) We note that in the special case when all three representations are $\underset{\sim}{8}$'s, we may define

$$\xi(\gamma) = \xi_k(8\ \ 8\ \ 8_\gamma) = (-1)^{\gamma+1} \quad , \tag{A1.9}$$

where the phase is independent of the index $k = 1, 2, 3$. γ can here take on the two values 1 and 2 corresponding to the symmetric and antisymmetric representation respectively. (In fact we also have $\xi_k(8\ \ 27\ \ 27_\gamma) = (-1)^{\gamma+1}$.)

We sometimes need to perform the double operation of changing the order of μ_1 and μ_2 and changing the sign of all the ν's. For the SU(2) coefficients this operation is very simple; we have just

$$C^{I_1\ I_2\ I}_{I_{1z}\ I_{2z}\ I_z} = C^{I_2\ \ I_1\ \ I}_{-I_{2z}\ -I_{1z}\ -I_z} \quad . \tag{A1.10}$$

Table 15. The phase factors ξ for $\mu_1 = \underset{\sim}{8}$

μ_1	μ_2	μ_3	ξ_1	ξ_2	ξ_3
		27	1	-1	1
		10	-1	-1	1
		10^*	-1	1	1
8	8	8_1	1	1	1
		8_2	-1	-1	-1
		1	1	-1	1
		35	1	-1	1
8	10	27	-1	1	1
		10	-1	-1	-1
		8	1	1	-1
		35^*	1	-1	1
8	10^*	27	-1	-1	1
		10^*	-1	-1	-1
		8	1	-1	-1
		64	1	-1	1
		35	-1	-1	1
		35^*	-1	1	1
8	27	27_1	1	1	1
		27_2	-1	-1	-1
		10	1	-1	-1
		10^*	1	1	-1
		8	1	-1	1

For the SU(3) coefficients we write

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} = \xi_{13}(\mu_1\ \mu_2\ \mu_\gamma) \begin{pmatrix} \mu_2^* & \mu_1^* & \mu_\gamma^* \\ -\nu_2 & -\nu_1 & -\nu \end{pmatrix} \tag{A1.11}$$

where

$$\xi_{13}(\mu_1\ \mu_2\ \mu_\gamma) = \xi_1(\mu_1\ \mu_2\ \mu_\gamma)\xi_3(\mu_2\ \mu_1\ \mu_\gamma)$$

$$= \xi_3(\mu_1\ \mu_2\ \mu_\gamma)\xi_1(\mu_1^*\ \mu_2^*\ \mu_\gamma^*) \quad .$$

For the two cases mentioned above which we shall encounter we have (n_γ here stands for a self-conjugate representation),

$$\xi_{13}(\mu_\gamma) = \xi_{13}(8\ \mu\ n_\gamma) = -1 \text{ if } \mu = \underset{\sim}{10}\text{, or } \underset{\sim}{10}^* \ ,$$

$$+1 \text{ if } \mu = \underset{\sim}{1}, \underset{\sim}{8}_1, \underset{\sim}{8}_2, \underset{\sim}{27} \ . \tag{A1.12}$$

Therefore

$$\xi_{13}(\mu_\alpha)\xi_{13}(8\ \mu\ n_\gamma) = +1 \quad . \tag{A1.13}$$

We may evidently put stars on either of the μ's, and we may also switch the order of $\underset{\sim}{8}$ and μ. This enables us to simplify some triple products of ξ's by writing, for example,

$$\xi_{13}(\mu_\alpha)\xi_3(\mu\ 8\ n_\gamma) = \xi_1(8\ \mu\ n_\gamma) \ . \tag{A1.14}$$

2. Coupling Coefficients for Two Octets

In Sec. III we derived the SU(3) symmetric coupling coefficients $f_{\bar{i}jk}$. They were written as

$$f_{\bar{i}jk} = \sum_{\gamma=1}^{2} f_{\bar{i}jk}^{(\gamma)} \text{ , where}$$

$$f_{\bar{i}jk}^{(\gamma)} = -\sqrt{8}\, h_\gamma \begin{pmatrix} 8 & 8 & 8_\gamma \\ -i & j & -k \end{pmatrix} \begin{pmatrix} 8 & 8 & 1 \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk} \quad . \tag{A2.1}$$

Here ν_r has been written as r in the CG coefficients. The expression for $f_{\bar{i}jk}^{(\gamma)}$ is put into more convenient form by evaluating the last coefficient [see Ref. 16, Eq. (14.10)]:

$$\begin{pmatrix} 8 & 8 & 1 \\ -k & k & 0 \end{pmatrix} = -\frac{(-1)^{Q_k}}{\sqrt{8}} \quad .$$

(This explains why the factor $-\sqrt{8}$ was introduced at the outset.) Writing $\eta_{\bar{i}jk} = \eta_{ijk}(-1)^{Q_i}$, and using $Q_i = Q_j + Q_k$ (charge conservation at the vertex) we get

$$f_{\bar{i}jk}^{(\gamma)} = h_\gamma \eta_{ijk} (-1)^{Q_j} \begin{pmatrix} 8 & 8 & 8_\gamma \\ -i & j & -k \end{pmatrix} \quad . \tag{A2.2}$$

By applying symmetry relations of the SU(3) CG coefficients we may write this in several forms. Another useful form is

$$f_{\bar{i}jk}^{(\gamma)} = h_\gamma \eta_{ijk} \begin{pmatrix} 8 & 8 & 8_\gamma \\ -k & -j & -i \end{pmatrix} \quad . \tag{A2.3}$$

The $f_{\bar{i}jk}$ are the coefficients in front of the triple product of particle field operators $(\bar{\psi}_i \psi_j \phi_k)$ in a Lagrangian. They must therefore possess symmetry under hermitian conjugation, i.e. they must satisfy

$$f_{\bar{i}jk} = f_{\bar{j}i\bar{k}} \quad . \tag{A2.4}$$

Let us check this explicitly using the form (A2.1). We have

$$f^{(\gamma)}_{\bar{j}i\bar{k}} = -\sqrt{8}\, h_\gamma \begin{pmatrix} 8 & 8 & 8_\gamma \\ -j & i & k \end{pmatrix} \begin{pmatrix} 8 & 8 & 1 \\ k & -k & 0 \end{pmatrix} \eta_{\bar{j}i\bar{k}}$$

$$= -\sqrt{8}\, h_\gamma \xi_{13}(8_\gamma) \begin{pmatrix} 8 & 8 & 8_\gamma \\ -i & j & -k \end{pmatrix} \xi_1(1) \begin{pmatrix} 8 & 8 & 1 \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk}$$

$$= f^{(\gamma)}_{\bar{i}jk} \quad .$$

Here we used $\eta_{\bar{j}i\bar{k}} = \eta_{\bar{i}jk}(-1)^{Q_i+Q_j+Q_k} = \eta_{\bar{i}jk}$ (by charge conservation at a vertex), and $\xi_{13}(8_\gamma) = +1$ by Eq. (A1.12). (We write $\underset{\sim}{8}^* = \underset{\sim}{8}$ throughout.) Since in our calculation we preserve SU(2) symmetry, we sometimes like to separate out the SU(2) dependence explicitly. Using (A1.4) in the basic form (A2.1) we get

$$f^{(\gamma)}_{\bar{i}jk} = f^{(\gamma)}_{\bar{A}BC}\, f^{I}_{\bar{i}jk}\, \eta_{\bar{i}jk} \quad , \quad \text{where}$$

$$f^{I}_{\bar{i}jk} = C\begin{matrix} I_A & I_B & I_C \\ -M_i & M_j & -M_k \end{matrix} \; C\begin{matrix} I_C & I_C & 0 \\ -M_k & M_k & 0 \end{matrix} \quad , \tag{A2.5}$$

$$f^{(\gamma)}_{\bar{A}BC} = -\sqrt{8}\, h_\gamma \left(\begin{matrix} 8 \\ I_A\text{-}Y_A \end{matrix} \;\; \begin{matrix} 8 \\ I_B Y_B \end{matrix} \middle| \begin{matrix} 8 \\ I_C\text{-}Y_C \end{matrix}\right) \left(\begin{matrix} 8 \\ I_C\text{-}Y_C \end{matrix} \;\; \begin{matrix} 8 \\ I_C Y_C \end{matrix} \middle| \begin{matrix} 1 \\ 0\,0 \end{matrix}\right) .$$

Here A, B, and C stand for the SU(2) multiplets to which particles i, j, and k belong; to simplify the notation we have written $(I_{Az})_i = M_i$, and similarly for particles j and k. To investigate the hermiticity property of $f^{I}_{\bar{i}jk}$ we write

$$f^{I}_{\bar{j}i\bar{k}} = C\begin{matrix} I_B & I_A & I_C \\ -M_j & M_i & M_k \end{matrix} \; C\begin{matrix} I_C & I_C & 0 \\ M_k & -M_k & 0 \end{matrix}$$

$$= C\begin{matrix} I_A & I_B & I_C \\ -M_i & M_j & -M_k \end{matrix} (-1)^{2I_C} \; C\begin{matrix} I_C & I_C & 0 \\ -M_k & M_k & 0 \end{matrix}$$

$$= (-1)^{2I_C} f^{I}_{\bar{i}jk} \qquad \text{(A2.6)}$$

Since $\eta_{\bar{j}i\bar{k}} = \eta_{\bar{i}jk}$ and $f^{(\gamma)}_{\bar{j}i\bar{k}} = f^{(\gamma)}_{\bar{i}jk}$ we get

$$f^{(\gamma)}_{\bar{B}A\bar{C}} = (-1)^{2I_C} f^{(\gamma)}_{\bar{A}BC} \quad . \qquad \text{(A2.7)}$$

In Sec. III we formed SU(2) invariant forms of the multiplets A, B, and C, written as $((\bar{A}BC))$ and defined by

$$((\bar{A}BC)) = \frac{1}{N_{\bar{A}BC}} \sum_{M_i M_j M_k} f^{I}_{\bar{i}jk} \, \eta_{\bar{i}jk} \, \bar{\psi}_{A_i} \psi_{B_i} \phi_{C_k} \; .$$

We required that $((\bar{A}BC))$ have the hermitian property $((\bar{A}BC)) = ((\bar{B}A\bar{C}))$. Evidently, then, the normalization constant $N_{\bar{A}BC}$ must be chosen such that

$$N_{\bar{A}BC} = (-1)^{2I_C} N_{\bar{B}A\bar{C}} \quad . \qquad \text{(A2.8)}$$

This accounts for the sign difference in $N_{\bar{A}BC}$ for the conjugate forms involving the K meson.

3. Racah Coefficients

The eigenstates $\phi_{\nu_1}^{(\mu_1)}$, $\phi_{\nu_2}^{(\mu_2)}$, and $\phi_{\nu_3}^{(\mu_3)}$ of the representations μ_1, μ_2, and μ_3 of SU(3) can be combined to form an eigenstate of the representation μ in two distinct ways. We may first combine μ_1 and μ_2 into a representation μ_{12} which is then put together with μ_3 to make μ; or we may form μ_{23} from μ_2 and μ_3, and then join μ_1 and μ_{23} to make μ. In terms of the SU(3) CG coefficients these two eigenstates of μ are

$$\psi\begin{pmatrix} \mu_{12\alpha} & \mu_3 & \mu_\gamma \\ & & \nu \end{pmatrix} = \sum_{\substack{\nu_1\nu_2 \\ \nu_{12}\nu_3}} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{12\alpha} \\ \nu_1 & \nu_2 & \nu_{12} \end{pmatrix} \begin{pmatrix} \mu_{12} & \mu_3 & \mu_\gamma \\ \nu_{12} & \nu_3 & \nu \end{pmatrix} \phi_{\nu_1}^{(\mu_1)} \phi_{\nu_2}^{(\mu_2)} \phi_{\nu_3}^{(\mu_3)} , \tag{A3.1}$$

$$\psi\begin{pmatrix} \mu_1 & \mu_{23\alpha'} & \mu_{\gamma'} \\ & & \nu \end{pmatrix} = \sum_{\substack{\nu_1\nu_2 \\ \nu_3\nu_{23}}} \begin{pmatrix} \mu_2 & \mu_3 & \mu_{23\alpha'} \\ \nu_2 & \nu_3 & \nu_{23} \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_{23} & \mu_{\gamma'} \\ \nu_1 & \nu_{23} & \nu \end{pmatrix} \phi_{\nu_1}^{(\mu_1)} \phi_{\nu_2}^{(\mu_2)} \phi_{\nu_3}^{(\mu_3)} . \tag{A3.2}$$

They are connected by a real orthogonal transformation whose matrix elements are called recoupling (or Racah) coefficients; we write

$$\psi\begin{pmatrix} \mu_{12\alpha} & \mu_3 & \mu_\gamma \\ & & \nu \end{pmatrix} = \sum_{\mu_{23}\alpha'\gamma'} \langle (\mu_1\mu_2)\mu_{12\alpha}\mu_3\mu_\gamma | \mu_1(\mu_2\mu_3)\mu_{23\alpha'}\mu_{\gamma'} \rangle \times \psi\begin{pmatrix} \mu_1 & \mu_{23\alpha'} & \mu_{\gamma'} \\ & & \nu \end{pmatrix} . \tag{A3.3}$$

The notation is basically that of Krammer[27]; however, we have chosen to expand $\psi\begin{pmatrix}\mu_{12\alpha} & \mu_3 & \mu_\gamma \\ & & \nu\end{pmatrix}$ in terms of $\psi\begin{pmatrix}\mu_1 & \mu_{23\alpha'} & \mu_\gamma \\ & & \nu\end{pmatrix}$, rather than vice versa, with the same symbol for the Racah coefficient (the advantage of this is that our expansion formulas can be read in a more natural way from left to right). Substituting (A3.1) and (A3.2) into (A3.3) and making use of the orthogonal property of the eigenstates $\phi^{(\mu_1)}_{\nu_1}$, $\phi^{(\mu_2)}_{\nu_2}$, and $\phi^{(\mu_3)}_{\nu_3}$ gives

$$\sum_{\nu_{12}} \begin{pmatrix}\mu_1 & \mu_2 & \mu_{12\alpha} \\ \nu_1 & \nu_2 & \nu_{12}\end{pmatrix}\begin{pmatrix}\mu_{12} & \mu_3 & \mu_\gamma \\ \nu_{12} & \nu_3 & \nu\end{pmatrix}$$

$$= \sum_{\substack{\mu_{23}\nu_{23} \\ \alpha'\gamma'}} \langle (\mu_1\,\mu_2)\mu_{12\alpha}\,\mu_3\,\mu_\gamma | \mu_1\,(\mu_2\mu_3)\,\mu_{23\alpha'}\mu_{\gamma'}\rangle$$

$$\times \begin{pmatrix}\mu_2 & \mu_3 & \mu_{23\alpha'} \\ \nu_2 & \nu_3 & \nu_{23}\end{pmatrix}\begin{pmatrix}\mu_1 & \mu_{23} & \mu_{\gamma'} \\ \nu_1 & \nu_{23} & \nu\end{pmatrix} \quad . \qquad \text{(A3.4)}$$

Using the orthogonality of the CG coefficients we can remove first one and then the other of the two coefficients on the right-hand side:

$$\sum_{\nu_{12}\nu_2\nu_3} \begin{pmatrix}\mu_1 & \mu_2 & \mu_{12\alpha} \\ \nu_1 & \nu_2 & \nu_{12}\end{pmatrix}\begin{pmatrix}\mu_{12} & \mu_3 & \mu_\gamma \\ \nu_{12} & \nu_3 & \nu\end{pmatrix}\begin{pmatrix}\mu_2 & \mu_3 & \mu_{23\alpha'} \\ \nu_2 & \nu_3 & \nu_{23}\end{pmatrix}$$

$$= \sum_{\gamma'} \langle (\mu_1\,\mu_2)\,\mu_{12\alpha}\,\mu_3\,\mu_\gamma | \mu_1\,(\mu_2\mu_3)\,\mu_{23\alpha'}\,\mu_{\gamma'}\rangle \begin{pmatrix}\mu_1 & \mu_{23} & \mu_{\gamma'} \\ \nu_1 & \nu_{23} & \nu\end{pmatrix},$$

(A3.5)

$$\sum_{\substack{\nu_1\nu_2\nu_3\\ \nu_{12}\nu_{23}}} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{12\alpha} \\ \nu_1 & \nu_2 & \nu_{12} \end{pmatrix} \begin{pmatrix} \mu_{12} & \mu_3 & \mu_\gamma \\ \nu_{12} & \nu_3 & \nu \end{pmatrix} \begin{pmatrix} \mu_2 & \mu_3 & \mu_{23\alpha'} \\ \nu_2 & \nu_3 & \nu_{23} \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_{23} & \mu_{\gamma'} \\ \nu_1 & \nu_{23} & \nu \end{pmatrix}$$

$$= \langle (\mu_1\ \mu_2)\mu_{12\alpha}\ \mu_3\ \mu_\gamma \,|\, \mu_1\ (\mu_2\mu_3)\,\mu_{23\alpha'}\ \mu_{\gamma'} \rangle \ . \qquad \text{(A3.6)}$$

In the present work only Racah coefficients for the case $\mu_1 = \mu_2 = \mu_3 = \underset{\sim}{8}$ will be needed. We introduce the simplified notation

$$\langle \mu_{12\alpha}\ \mu_\gamma \,|\, \mu_{23\alpha'}\ \mu_{\gamma'} \rangle = \langle (8\ 8)\mu_{12\alpha}\ 8\ \mu_\gamma \,|\, 8\ (8\ 8)\ \mu_{23\alpha'} \mu_{\gamma'} \rangle \ . \qquad \text{(A3.7)}$$

Eq. (A3.4) will be frequently used for the case that $\nu = 0$ (by this is meant $Y = 0$, $I = 0$). Since $\nu = 0$ can be formed from ν' and ν'' only if $\nu' = -\nu''$, the summation over ν_{12} and ν_{23} may be dropped:

$$\begin{pmatrix} 8 & 8 & \mu_\alpha \\ \nu_1 & \nu_2 & -\nu_3 \end{pmatrix} \begin{pmatrix} \mu & 8 & n_\gamma \\ -\nu_3 & \nu_3 & 0 \end{pmatrix}$$

$$= \sum_{\mu'\alpha'\gamma'} \langle \mu_\alpha\ n_\gamma \,|\, \mu'_{\alpha'}\ n_{\gamma'} \rangle \begin{pmatrix} 8 & 8 & \mu'_{\alpha'} \\ \nu_2 & \nu_3 & -\nu_1 \end{pmatrix} \begin{pmatrix} 8 & \mu' & n_{\gamma'} \\ \nu_1 & -\nu_1 & 0 \end{pmatrix} \ . \qquad \text{(A3.8)}$$

(We have redefined some of the symbols.) Similarly, for $\nu = 0$ and $\mu_1 = \mu_2 = \mu_3 = \underset{\sim}{8}$ (A3.5) becomes

$$\sum_{\nu_2 \nu_3} \begin{pmatrix} 8 & 8 & \mu_\alpha \\ \nu_1 & \nu_2 & -\nu_3 \end{pmatrix} \begin{pmatrix} \mu & 8 & n_\gamma \\ -\nu_3 & \nu_3 & 0 \end{pmatrix} \begin{pmatrix} 8 & 8 & \mu'_{\alpha'} \\ \nu_2 & \nu_3 & -\nu_1 \end{pmatrix}$$

$$= \sum_{\gamma'} \langle \mu_\alpha n_\gamma | \mu'_{\alpha'} n_{\gamma'} \rangle \begin{pmatrix} 8 & \mu' & n_{\gamma'} \\ \nu_1 & -\nu_1 & 0 \end{pmatrix} . \qquad \text{(A3.9)}$$

In Sec. IV it is shown that the useful irreducible coupling splittings are not those obtained directly from the representations $\underset{\sim}{10}$ and $\underset{\sim}{10}^*$ but rather the linear combinations $\underset{\sim}{10}_\pm$ defined symbollically by

$$|10_\pm\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |10^*\rangle) \quad . \qquad \text{(A3.10)}$$

It will be advantageous to express the Racah coefficients directly in terms of these linear combinations, not only because they will give the desired answers directly but also because the evaluation of products of Racah coefficients will often be simplified. Using the tables of Krammer and the orthogonal transformation matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ connecting the representations $\underset{\sim}{10}$ and $\underset{\sim}{10}^*$ with the combinations $\underset{\sim}{10}_+$ and $\underset{\sim}{10}_-$, we compile in Table 16 the coefficients that will be needed. Those are the coefficients $\langle \mu_\alpha \chi_\beta | \mu'_{\alpha'} \chi_{\beta'} \rangle$ for which $\chi = \underset{\sim}{1}, \underset{\sim}{8}, \underset{\sim}{10}_\pm, \underset{\sim}{27}$ and $\underset{\sim}{64}$ (μ_α and $\mu'_{\alpha'}$ are representations contained in $\underset{\sim}{8} \times \underset{\sim}{8}$); in the case $\chi = \underset{\sim}{10}_\pm$ we only need $\mu = \mu' = \underset{\sim}{8}$.

Table 16. The Racah coefficients $\langle \mu_\alpha \chi_\beta | \mu'_{\alpha'} \chi_{\beta'} \rangle$

$\chi = \underset{\sim}{8}$

$\mu_{\alpha,\beta}$ \ $\mu'_{\alpha',\beta'}$	27	10_+	10_-	$8_{1,1}$	$8_{2,1}$	$8_{1,2}$	$8_{2,2}$	1
27	$\frac{7}{40}$	0	$\frac{\sqrt{15}}{20}$	$-\frac{3\sqrt{6}}{20}$	0	0	$\frac{\sqrt{6}}{4}$	$\frac{3\sqrt{3}}{8}$
10_+	0	0	0	0	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	0	0
10_-	$-\frac{\sqrt{15}}{20}$	0	$-\frac{1}{2}$	$\frac{\sqrt{10}}{5}$	0	0	0	$\frac{\sqrt{5}}{4}$
$8_{1,1}$	$-\frac{3\sqrt{6}}{20}$	0	$-\frac{\sqrt{10}}{5}$	$-\frac{3}{10}$	0	0	$\frac{1}{2}$	$-\frac{\sqrt{2}}{4}$
$8_{2,1}$	0	$\frac{\sqrt{2}}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$8_{1,2}$	0	$-\frac{\sqrt{2}}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$8_{2,2}$	$\frac{\sqrt{6}}{4}$	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$-\frac{\sqrt{2}}{4}$
1	$\frac{3\sqrt{3}}{8}$	0	$-\frac{\sqrt{5}}{4}$	$-\frac{\sqrt{2}}{4}$	0	0	$-\frac{\sqrt{2}}{4}$	$\frac{1}{8}$

$\chi = \underset{\sim}{1}$

μ_α \ $\mu'_{\alpha'}$	8_1	8_2
8_1	1	0
8_2	0	1

$\chi = \underset{\sim}{64}$

μ \ μ'	27
27	1

Table 16 Continued

$\chi = \underset{\sim}{27}$

$\mu_{\alpha,\beta}$ \ $\mu'_{\alpha',\beta'}$	$27_{,1}$	$27_{,2}$	10_+	10_-	8_1	8_2
$27_{,1}$	$\frac{3}{10}$	0	$\frac{\sqrt{35}}{10}$	0	$\frac{\sqrt{14}}{5}$	0
$27_{,2}$	0	$\frac{1}{2}$	0	$\frac{\sqrt{3}}{6}$	0	$\frac{\sqrt{6}}{3}$
10_+	$\frac{\sqrt{35}}{10}$	0	$\frac{1}{2}$	0	$-\frac{\sqrt{10}}{5}$	0
10_-	0	$-\frac{\sqrt{3}}{6}$	0	$-\frac{5}{6}$	0	$\frac{\sqrt{2}}{3}$
8_1	$\frac{\sqrt{14}}{5}$	0	$-\frac{\sqrt{10}}{5}$	0	$\frac{1}{5}$	0
8_2	0	$\frac{\sqrt{6}}{3}$	0	$-\frac{\sqrt{2}}{3}$	0	$-\frac{1}{3}$

$\chi = \underset{\sim}{10}_\pm$

$\mu_{\alpha,\beta}$ \ $\mu'_{\alpha',\beta'}$	$8_{1,+}$	$8_{2,+}$	$8_{1,-}$	$8_{2,-}$
$8_{1,+}$	$-\frac{2}{5}$	0	0	$-\frac{\sqrt{5}}{5}$
$8_{2,+}$	0	0	$\frac{\sqrt{5}}{5}$	0
$8_{1,-}$	0	$-\frac{\sqrt{5}}{5}$	$-\frac{2}{5}$	0
$8_{2,-}$	$\frac{\sqrt{5}}{5}$	0	0	0

4. The Wigner-Eckart theorem, two applications

If we have a tensor operator $T_{\nu_2}^{(\mu_2)}$ belonging to the irreducible representation μ_2, the matrix element $\left(\phi_{\nu_3}^{(\mu_3)}, T_{\nu_2}^{(\mu_2)}\phi_{\nu_1}^{(\mu_1)}\right)$, where $\phi_{\nu_1}^{(\mu_1)}$ and $\phi_{\nu_3}^{(\mu_3)}$ are base states of the irreducible representations μ_1 and μ_3, can be expressed in the form

$$\left(\phi_{\nu_3}^{(\mu_3)}, T_{\nu_2}^{(\mu_2)}\phi_{\nu_1}^{(\mu_1)}\right) = \sum_{\gamma} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} (\mu_3 \,||T^{(\mu_2)}||\, \mu_1)_\gamma \, . \qquad \text{(A4.1)}$$

This is the Wigner-Eckart theorem for SU(3); its usefulness lies in that the ν dependence of the matrix element is contained entirely in the CG coefficient.

We use the theorem to prove two statements made in Sec. II. It was stated that the forms

$$X_1 = \sum_i P_i^{(n_\gamma)} \langle B_i | \delta S | B_i \rangle \quad ,$$

$$X_2 = \sum_{ijk} O_{\bar{i}jk}^{(n_\gamma,\mu_\beta)} \langle B_i | \delta S | B_i M_k \rangle \quad , \qquad \text{(A4.2)}$$

when expressed in terms of irreducible splittings are diagonal in n. The coefficients $P_i^{(n_\gamma)}$ and $O_{\bar{i}jk}^{(n_\gamma,\mu_\beta)}$ are given in Eqs. (4.4) and (4.3). The operator δS may be expanded in irreducible components (with $\nu = 0$) by

$$\delta S = \sum_{\chi} T_o^{(\chi)} \quad , \qquad \text{(A4.3)}$$

where $\chi = \underset{\sim}{1}, \underset{\sim}{8}, \underset{\sim}{27}$, and $\underset{\sim}{64}$; except for $\chi = \underset{\sim}{64}$ the operator $T_o^{(\chi)}$ stands for several inequivalent irreducible operators of dimension χ. The particle states $|B_i\rangle$ and $|M_k\rangle$ are identified with $\eta_i \phi_{\nu_i}^{(8)}$ and $\eta_k \phi_{\nu_k}^{(8)}$ respectively.

The first form can thus be written

$$X_1 = \sum_i \begin{pmatrix} 8 & 8 & n_\gamma \\ -i & i & 0 \end{pmatrix} (-1)^{Q_i} \sum_{\chi \epsilon} \begin{pmatrix} 8 & \chi & 8_\epsilon \\ i & 0 & i \end{pmatrix} (8 \,||T^{(\chi)}||\, 8)_\epsilon \eta_{ii} \, . \tag{A4.4}$$

Using (A1.5) and (A1.6) we write

$$\begin{pmatrix} 8 & 8 & n_\gamma \\ -i & i & 0 \end{pmatrix} = \xi_1(n_\gamma) \begin{pmatrix} 8 & 8 & n_\gamma \\ i & -i & 0 \end{pmatrix} \quad ,$$

$$\begin{pmatrix} 8 & \chi & 8_\epsilon \\ i & 0 & i \end{pmatrix} = \xi_2(8 \;\chi\; 8_\epsilon)(-1)^{Q_i} \sqrt{\frac{8}{N_\chi}} \begin{pmatrix} 8 & 8 & \chi_\epsilon \\ i & -i & 0 \end{pmatrix} .$$

The factors $(-1)^{Q_i}$ cancel out; the sum over i gives $\delta_{n,\chi}\,\delta_{\gamma,\epsilon}$ so that only $\chi = n$ survives. This establishes the desired result for the first form.

In the second form we first note that the state $|B_i M_k\rangle$ does not belong to an irreducible representation, so that the Wigner-Eckart theorem cannot be applied to $\langle B_i | T_o^{(\chi)} | B_i M_k\rangle$ directly. We need to take a combination of $|B_i M_k\rangle$ of the form

$$\phi_{\nu'}^{(\mu'_{\beta'})} = \sum_{jk} \begin{pmatrix} 8 & 8 & \mu'_{\beta'} \\ j & k & \nu' \end{pmatrix} \eta_{jk} |B_j M_k\rangle \tag{A4.5}$$

in order to get an irreducible base state. But $O_{\bar{i}jk}^{(n_\gamma, \mu_\beta)}$ can be expressed (by A3.8) as

$$O_{\bar{i}jk}^{(n_\gamma, \mu_\beta)} = \sum_{\mu'\beta'\gamma'} \langle \mu_\beta n_\gamma | \mu'_{\beta'} n_{\gamma'} \rangle \begin{pmatrix} 8 & 8 & \mu'_{\beta'} \\ j & k & i \end{pmatrix} \begin{pmatrix} 8 & \mu' & n_{\gamma'} \\ -i & i & 0 \end{pmatrix} \eta_{\bar{i}jk} , \qquad \text{(A4.6)}$$

which contains the type of CG coefficient we need in (A4.5). The second form may now be written as

$$X_2 = \sum_{\substack{i \\ \mu'\beta'\gamma'}} \langle \mu_\beta n_\gamma | \mu'_{\beta'} n_{\gamma'} \rangle \begin{pmatrix} 8 & \mu' & n_{\gamma'} \\ -i & i & 0 \end{pmatrix} \eta_{\bar{i}} \sum_{\chi\epsilon} \begin{pmatrix} \mu' & \chi & 8_\epsilon \\ i & 0 & i \end{pmatrix} (8 \| T^{(\chi)} \| \mu')_\epsilon \, \eta_i . \qquad \text{(A4.7)}$$

Expressing the CG coefficients in the form

$$\begin{pmatrix} 8 & \mu' & n_{\gamma'} \\ -i & i & 0 \end{pmatrix} = \xi_1(8\ \mu'\ n_{\gamma'}) \begin{pmatrix} \mu' & 8 & n_{\gamma'} \\ i & -i & 0 \end{pmatrix} ,$$

$$\begin{pmatrix} \mu' & \chi & 8_\epsilon \\ i & 0 & i \end{pmatrix} = \xi_2(\mu'\ \chi\ 8_\epsilon)(-1)^{Q_i} \sqrt{\frac{8}{N_\chi}} \begin{pmatrix} \mu' & 8 & \chi_\epsilon \\ i & -i & 0 \end{pmatrix} ,$$

the phase factors become $\eta_{\bar{i}} \eta_i (-1)^{Q_i} = 1$; again we can sum over i giving $\delta_{\chi, n} \delta_{\gamma', \epsilon}$, which establishes the desired result $\chi = n$.

APPENDIX B

TWO INTEGRALS IN THE EVALUATION OF WEIGHT FUNCTIONS

We calculate the basic integrals encountered in Sec. V for the imaginary parts of the second order self-energy diagram and the third order vertex diagram. For the self-energy diagram we derived

$$\mathrm{Im}\,\Sigma_o(p^2) = -\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^4k\,\delta(k^2 - m_1^2)\,\delta[(p-k)^2 - m_2^2] \;. \tag{B1}$$

The integral is evaluated in a special coordinate system after which the general result is obtained by relativistic invariance. Let us choose our coordinate system such that $p = (w, \underset{\sim}{0})$; w is then given by $w^2 = p^2 = s$. The δ-functions become

$$\delta(k^2 - m_1^2)\,\delta(k^2 - 2k\cdot p + p^2 - m_2^2)$$

$$= \delta(k_o^2 - \underset{\sim}{k}^2 - m_1^2)\,\delta(-2k_o w + s + m_1^2 - m_2^2)$$

$$= \frac{1}{2|w|}\,\delta(\underset{\sim}{k}^2 - k_o^2 + m_1^2)\,\delta[k_o - \frac{1}{2w}(s + m_1^2 - m_2^2)] \;.$$

In terms of $\underset{\sim}{k}^2$ and k_o we have

$$\int d^4k = \int dk_o \int \frac{1}{2}\,|\underset{\sim}{k}|\,d(\underset{\sim}{k})^2 \int d\Omega_{\underset{\sim}{k}} \quad ,$$

where the integration over the direction of $\underset{\sim}{k}$ is $\int d\Omega_{\underset{\sim}{k}} = 4\pi$. Using the δ-functions gives

$$\text{Im}\ \Sigma_o(p^2) = -\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \frac{2\pi|\underset{\sim}{k}|}{|w|} \quad .$$

$\underset{\sim}{k}$ is obtained from

$$\underset{\sim}{k}^2 = k_o^2 - m_1^2 = \frac{1}{4w^2}(s + m_1^2 - m_2^2)^2 - m_1^2$$

$$= \frac{1}{4s}\zeta^2(s, m_1^2, m_2^2) \quad ,$$

where we have defined

$$\zeta(s, m_1^2, m_2^2) = \sqrt{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]} \quad . \tag{B2}$$

The integral exists for $s \geq (m_1 + m_2)^2$; therefore

$$\text{Im}\ \Sigma_o(s) = -\pi\left(\frac{g}{4\pi}\right)^2 \frac{1}{s}\zeta(s, m_1^2, m_2^2)\theta(s - (m_1 + m_2)^2) \quad , \tag{B3}$$

where θ is the unit step function defined by

$$\theta(x) = 1 \quad \text{for} \quad x \geq 0 \ ,$$

$$\theta(x) = 0 \quad \text{for} \quad x < 0 \ .$$

The basic integral to be evaluated for the vertex diagrams is

$$\text{Im}\ \Lambda_o(q^2) = -\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^4k \ \frac{\delta[(p_1 + k)^2 - m_3^2]\ \delta[(p_2 + k)^2 - m_4^2]}{k^2 - \mu_2^2} \ . \tag{B4}$$

Changing the variable of integration by $k \to k - p_2$ gives

$$\operatorname{Im} A_o(q^2) = -\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^4k \; \frac{\delta[(q+k)^2 - m_3^2]\,\delta(k^2 - m_4^2)}{(k-p_2)^2 - \mu_2^2} \tag{B5}$$

(we have written $p_1 - p_2 = q$). There are two independent four-vectors in the problem. We choose a coordinate system such that $q = (w, \underset{\sim}{0})$ and $p_2 = (p_o, 0, 0, p)$ where $p \geq 0$. Evidently $w^2 = q^2 = t$, and p_o and p are related by $p_o^2 - p^2 = p_2^2 = m_2^2$ (p_1 and p_2 are the momenta of external particle lines so that $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$.) The integrand now becomes

$$\frac{\delta(k_o^2 - \underset{\sim}{k}^2 - m_4^2)\,\delta(2wk_o + t + m_4^2 - m_3^2)}{-2k_op_o + 2|\underset{\sim}{k}|p\cos\theta + m_4^2 + m_2^2 - \mu_2^2}$$

$$= \frac{1}{2|w|2|\underset{\sim}{k}|p} \; \frac{\delta[k_o + \frac{1}{2w}(t + m_4^2 - m_3^2)]\,\delta(\underset{\sim}{k}^2 - k_o^2 + m_4^2)}{(\cos\theta + Q)} \; .$$

Here θ is the angle between $\underset{\sim}{k}$ and the z axis and

$$Q = \frac{1}{2p|\underset{\sim}{k}|}\left(-2p_ok_o + m_4^2 + m_2^2 - \mu_2^2\right) \; . \tag{B6}$$

Making use of the azimuthal symmetry of the integrand we write

$$\int d^4k = \int dk_o \int \frac{1}{2}|\underset{\sim}{k}|\, d(\underset{\sim}{k}^2) \int 2\pi\, d(\cos\theta) \; .$$

Performing the integral now results in

$$\mathrm{Im}\,\Lambda_o(t) = -\frac{\pi}{2}\left(\frac{g}{2\pi}\right)^2 \frac{|\underset{\sim}{k}|}{4|w|\,|\underset{\sim}{k}|p}\ \log\left(\frac{Q+1}{Q-1}\right) . \tag{B7}$$

With $w = \sqrt{t}$ the δ-functions give

$$k_o = -\frac{1}{2\sqrt{t}}\,(t + m_4^2 - m_3^2) \quad ,$$

$$|\underset{\sim}{k}| = \frac{1}{2\sqrt{t}}\,\zeta\,(t, m_3^2, m_4^2) \quad .$$

Using $m_1^2 = p_1^2 = (q + p_2)^2 = t + m_2^2 + 2wp_o$ we get

$$p_o = -\frac{1}{2\sqrt{t}}\,(t + m_2^2 - m_1^2) \quad ,$$

$$p = \frac{1}{2\sqrt{t}}\,\zeta\,(t, m_1^2, m_2^2) \quad .$$

Substituting into (B6) and (B7) gives finally

$$\mathrm{Im}\,\Lambda_o(t) = \pi\left(\frac{g}{4\pi}\right)^2 \frac{1}{\zeta(t, m_1^2, m_2^2)}\ \log\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\ \theta(t - (m_3 + m_4)^2) \quad , \tag{B8}$$

where

$$\alpha = (t - m_1^2 + m_2^2)(t - m_3^2 + m_4^2) - 2t(m_2^2 + m_4^2 - \mu_2^2) \ ,$$

$$\beta = \zeta(t, m_1^2, m_2^2)\zeta(t, m_3^2, m_4^2) \quad .$$

The θ-function implies that $\mathrm{Im}\,\Lambda_o(t) = 0$ for $t < (m_3 + m_4)^2$.

In the above derivations we have not been entirely rigorous with regard to signs; implicit in the application of Cutkosky's rule is the selection of the "proper" root of the δ-function arguments. Our answers, however, are in agreement with the results of a rigorous calculation of the discontinuities directly from the Feynman integrals.

Since our work depends crucially on the sign of the vertex constant Λ, effectively equal to Λ_1 as given in Eq. (5.32), we like to point out that the dominant term in the integrand, which is just $\mathrm{Im}\,\Sigma_o(t';m_3,m_4)$, has necessarily the correct sign; the opposite sign for $\mathrm{Im}\,\Sigma_o$ would have given the results $Z_2^{(2)} > 0$ and $Z_3^{(2)} > 0$, which are unacceptable from more basic principles (see remark in footnote 39).

APPENDIX C

SU(3) FACTORS IN FIFTH ORDER VERTEX GRAPHS

In Sec. VI it was shown that the third order vertex graph in SU(3) symmetry can be written as

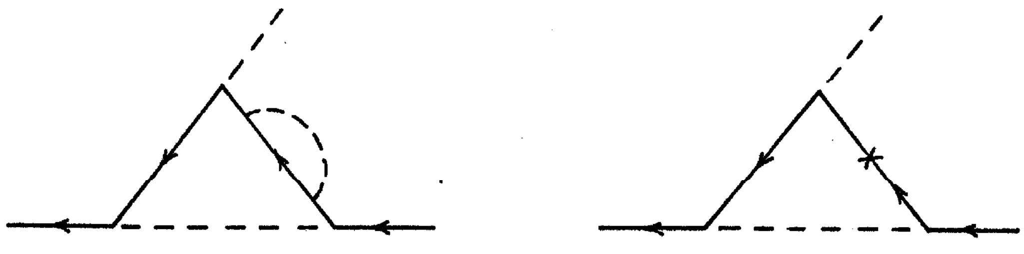
$$ig\Lambda_{\overline{i}jk}\Big|_s = ig\sum_{\gamma} f^{(\gamma)}_{\overline{i}jk}\, C^{(\gamma)}\Lambda \quad . \tag{C1}$$

For $\alpha = 3/4$ the constants $C^{(1)}$ and $C^{(2)}$ are equal and the vertex function is proportional to the total SU(3) coupling constant $f_{\overline{i}jk}$. We like to know if this rule holds for fifth order graphs also, i.e. if for any fifth order graph W the two constants $C^{(\gamma)}_w$ corresponding to $C^{(\gamma)}$ in the third order graph are equal at $\alpha = 3/4$.

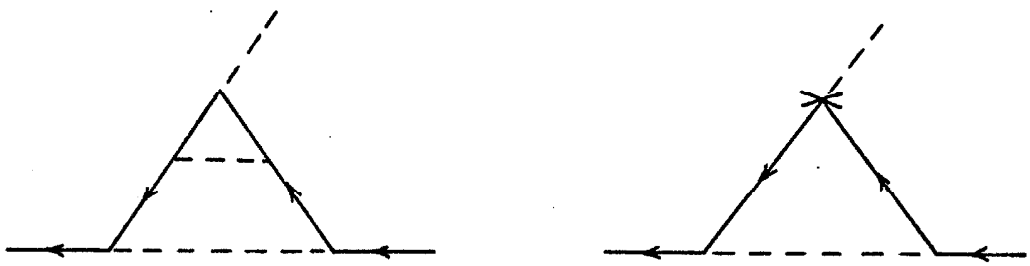
Fifth order graphs which are obtained from the third order graph by making second order insertions in the internal lines, like the ones in Fig. 6a, satisfy the rule because the magnitude of the insertion is independent of the SU(3) quantum numbers of the line.

Making a second order vertex insertion into the third order graph, as in Fig. 6b, will, for $\alpha = 3/4$, just have the effect of multiplying the total SU(3) coupling constant at that vertex by a factor; by induction, the $\alpha = 3/4$ rule is maintained.

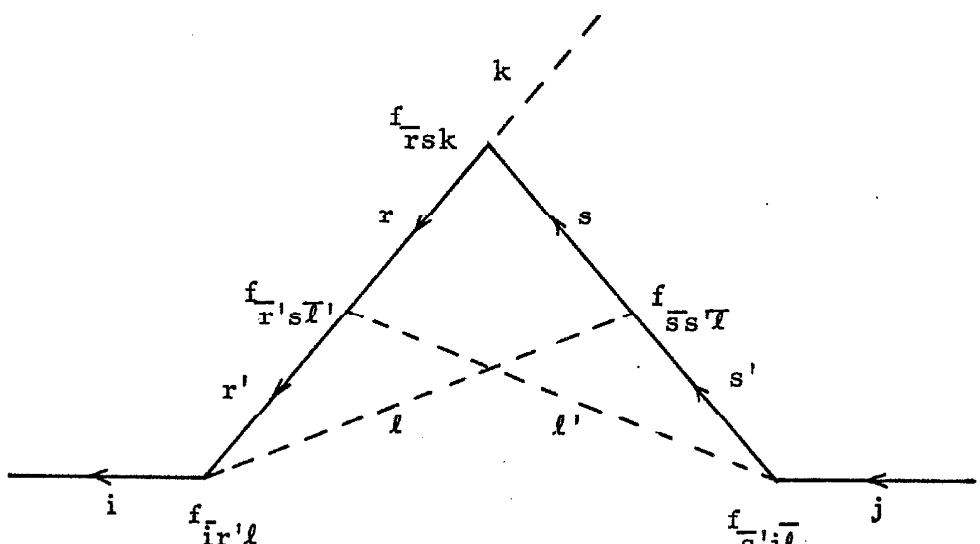
There remains the graph in Fig. 6c to be investigated. We need to evaluate the product of the coupling constants summed over all internal particle indices; we write



(a)



(b)



(c)

Figure 6. Some fifth order vertex graphs.

$$w = \sum_{\substack{\ell r s \\ \ell' r' s'}} f_{\bar{i} r' \ell} \, f_{\bar{r}' r \ell'} \, f_{\bar{r} s k} \, f_{\bar{s}' s \ell} \, f_{\bar{j} s' \ell'}$$

$$= \sum_{\substack{\ell r s (\gamma_t) \\ \ell' r' s'}} \left(\prod_{t=1}^{5} h_{\gamma_t} \right) \eta_{ijk} \begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -\ell & -r' & -i' \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_{\gamma_2} \\ -\ell' & -r & -r' \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_{\gamma_3} \\ -k & -s & -r \end{pmatrix}$$

$$\times \begin{pmatrix} 8 & 8 & 8_{\gamma_4} \\ -\ell & -s & -s' \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_{\gamma_5} \\ -\ell' & -s' & -j \end{pmatrix} . \qquad \text{(C2)}$$

Here we have made use of the hermitian property of the couplings and expressed them in the form (A2.3); (γ_t) stands for $\gamma_1, \gamma_2, \ldots \gamma_5$. By recoupling the two internal meson lines can effectively be uncrossed; changing the order of the first two rows in the first CG coefficient the sum over r' in the first two CG coefficients can be performed by (A3.5):

$$\sum_{r'} \xi(\gamma_1) \begin{pmatrix} 8 & 8 & 8_{\gamma_2} \\ -\ell' & -r & -r' \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -r' & -\ell & -i \end{pmatrix}$$

$$= \sum_{\mu\alpha\beta\nu} \xi(\gamma_1) \langle 8_{\gamma_2} \; 8_{\gamma_1} | \mu_\alpha \; 8_\beta \rangle \begin{pmatrix} 8 & 8 & \mu_\alpha \\ -r & -\ell & \nu \end{pmatrix} \begin{pmatrix} 8 & \mu & 8_\beta \\ -\ell' & \nu & -i \end{pmatrix} . \qquad \text{(C3)}$$

The states ℓ, r, and s are now coupled basically as in the third order vertex graph; summing over these indices gives

$$\sum_{rs\ell}\begin{pmatrix} 8 & 8 & \mu_\alpha \\ -r & -\ell & \nu \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_{\gamma_3} \\ -k & s & -r \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_{\gamma_4} \\ -\ell & -s & -s' \end{pmatrix}$$

$$= \sum_{\alpha'} \xi(\gamma_4)\langle 8_{\gamma_3}\ \mu_\alpha | 8_{\gamma_4}\ \mu_{\alpha'}\rangle \begin{pmatrix} 8 & 8 & \mu_{\alpha'} \\ -k & -s' & \nu \end{pmatrix} \ . \qquad \text{(C4)}$$

We are left with three CG coefficients, namely

$$\begin{pmatrix} 8 & 8 & \mu_{\alpha'} \\ -k & -s' & \nu \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_{\gamma_5} \\ -\ell' & -s' & -j \end{pmatrix}\begin{pmatrix} 8 & \mu & 8_\beta \\ -\ell' & \nu & -i \end{pmatrix} \ .$$

Changing the order of the first two elements in the last two CG coefficients the product can be summed over $(\ell' s' \nu)$ with the result

$$\sum \xi(\gamma_5)\xi_1(8\ \mu\ 8_\beta)\langle \mu_{\alpha'}\ 8_\beta | 8_{\gamma_5}\ 8_\epsilon\rangle \begin{pmatrix} 8 & 8 & 8_\epsilon \\ -k & -j & -i \end{pmatrix} \ . \qquad \text{(C5)}$$

The last CG coefficient multiplied by the phase factor η_{ijk} in (C2) is just $h_\epsilon^{-1} f_{\bar{i}jk}^{(\epsilon)}$. Collecting all the factors our final result is

$$w = \sum_\epsilon f_{\bar{i}jk}^{(\epsilon)} C_w^{(\epsilon)} \ , \text{ where} \qquad \text{(C6)}$$

$$C_w^{(\epsilon)} = h_\epsilon^{-1} \sum_{\substack{\mu\alpha\alpha' \\ (\gamma_t)\beta}} \left(\prod_t h_{\gamma_t}\right) \xi(\gamma_1)\xi(\gamma_4)\xi(\gamma_5)\xi_1(8\ \mu\ 8_\beta)$$

$$\langle 8_{\gamma_2}\ 8_{\gamma_1} | \mu_\alpha 8_\beta\rangle \langle 8_{\gamma_3}\mu_\alpha | 8_{\gamma_4}\mu_{\alpha'}\rangle \langle \mu_{\alpha'} 8_\beta | 8_{\gamma_5} 8_\epsilon\rangle \ . \qquad \text{(C7)}$$

Evaluating gives

$$C_w^{(1)} = \frac{4}{25} h_1^4 + \frac{8}{5} h_1^2 h_2^2 \ ,$$

$$C_w^{(2)} = \frac{4}{5} h_1^4 \ . \qquad (C8)$$

Putting $C_w^{(1)}$ and $C_w^{(2)}$ equal we find

$$h_1 = \sqrt{\frac{5}{2}}\ h_2 \ , \quad \text{or} \quad h_1 = 0 \ . \qquad (C9)$$

The first value corresponds to $\alpha = 0.68$. Hence the dynamics-independent result $\alpha = 3/4$ in lowest order of consistency is not maintained in higher orders.

APPENDIX D

EXAMPLES OF THE CALCULATION USING THE RECOUPLING FORMALISM

As a first example we evaluate the contribution of the a_2 mass term in Eq. (7.10) to $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$:

$$Y_1 = 2a_2 \sum_{ijk} \begin{pmatrix} 8 & 8 & n'_{\gamma'} \\ -i & j & -k \end{pmatrix} \begin{pmatrix} n' & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk} \sum_{rs\ell} \delta m_i f_{\bar{i}s\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} \quad . \tag{D1}$$

The sum over $(rs\ell)$ is just what we had in Sec. VI; by Eq. (6.7), using (A2.3),

$$\sum_{rs} f_{\bar{i}s\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} = \sum_{\gamma'} C^{(\gamma_1)} h_{\gamma_1} \eta_{ijk} \begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -k & -j & -i \end{pmatrix} \quad . \tag{D2}$$

The indices (ijk) do not occur in a form suitable for summing. Eventually the mass term δm_i must give an irreducible mass splitting like

$$\delta m_{(n_\beta)} = \sum_i \begin{pmatrix} 8 & 8 & n_\beta \\ -i & i & 0 \end{pmatrix} \eta_{\bar{i}i}\, \delta m_i \quad . \tag{D3}$$

Combining the phase factors in (D1) and (D2) gives the desired form $\eta_{\bar{i}i}$. A CG coefficient which has the required "magnetic quantum numbers" can be obtained by recoupling of the two CG coefficients in (D1); using (A3.8) we get

$$\begin{pmatrix} 8 & 8 & n'_{\gamma'} \\ -i & j & -k \end{pmatrix}\begin{pmatrix} n' & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} = \sum_{\mu\alpha\beta} \langle n'_{\gamma'}, n_\gamma | \mu_\alpha \mu_\beta \rangle \begin{pmatrix} 8 & 8 & \mu_\alpha \\ i & k & i \end{pmatrix}\begin{pmatrix} 8 & \mu & n_\beta \\ -i & i & 0 \end{pmatrix} . \tag{D4}$$

Since the CG coefficient in (D2) is just equal to $\begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ j & k & i \end{pmatrix}$ we can sum over (jk) obtaining $\delta_{\mu,8}\,\delta_{\alpha,\gamma_1}$ by orthogonality. The remaining CG coefficient in (D4) now has the same form as the one in (D3). The final result is

$$Y_1 = 2a_2 \sum_{\gamma_1} h_{\gamma_1} C^{(\gamma_1)} \langle n'_{\gamma'}, n_\gamma | \mu_\alpha n_\beta \rangle \ \delta m_{(n_\beta)} \ . \tag{D5}$$

Next we consider the coupling term in $\delta Z_2^{(n_\gamma)}$:

$$Y_2 = \sum_{ijk} \begin{pmatrix} 8 & 8 & n_\gamma \\ -i & i & 0 \end{pmatrix} \eta_{\bar{i}i} \sum_{\gamma_1} h_{\gamma_1} \eta_{ijk} \begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -k & -j & -i \end{pmatrix} \delta f_{\bar{i}jk} \ , \tag{D6}$$

where $f_{\bar{i}jk}$ has been expressed in the form (A2.3). An irreducible coupling splitting is obtained from a sum of the type

$$\delta f_{(n_\beta,\mu_\alpha)} = \sum_{ijk} \begin{pmatrix} 8 & 8 & \mu_\alpha \\ -i & j & -k \end{pmatrix}\begin{pmatrix} \mu & 8 & n_\beta \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk} \ \delta f_{\bar{i}jk} \ . \tag{D7}$$

The two η-factors in (D6) become $\eta_{\bar{i}jk}$, which is just what we need; to get the needed magnetic quantum numbers we again recouple:

$$\begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -k & -j & -i \end{pmatrix}\begin{pmatrix} 8 & 8 & n_\gamma \\ -i & i & 0 \end{pmatrix} = \sum_{\mu\alpha\beta} \langle 8_{\gamma_1} n_\gamma | \mu_\alpha n_\beta \rangle \begin{pmatrix} 8 & 8 & \mu_\alpha \\ -j & i & k \end{pmatrix}\begin{pmatrix} 8 & \mu & n_\beta \\ -k & k & 0 \end{pmatrix} . \tag{D8}$$

Switching the order of the first two components of both CG coefficients and changing all the signs gives the ξ-factors $\xi_{13}(\mu_\alpha)\xi_{13}(8\,\mu\, n_\beta)$ the product of which is unity by (A1.13); the sum over (ijk) gives the irreducible splitting $\delta f_{(n_\beta, \mu_\alpha^*)}$. The result is

$$Y_2 = \sum_{\gamma_1 \mu\alpha\beta} h_{\gamma_1} \langle 8_{\gamma_1} n_\gamma | \mu_\alpha\, n_\beta \rangle\, \delta f_{(n_\beta, \mu_\alpha^*)} \quad . \tag{D9}$$

The sum of μ_α over the representations $\underset{\sim}{10}$ and 10^* looks like

$$|10\, n_\beta\rangle\, \delta f_{(n_\beta, 10^*)} + |10^* n_\beta\rangle\, \delta f_{(n_\beta\, 10)} \quad ; \tag{D10}$$

expressed in terms of $|10_\pm\, n_\beta\rangle$ and $\delta f_{(n_\beta, 10_\pm)}$ it reads

$$|10_+ n_\beta\rangle\, \delta f_{(n_\beta, 10_+)} - |10_- n_\beta\rangle\, \delta f_{(n_\beta, 10_-)} \quad . \tag{D11}$$

The minus sign for the term in $\underset{\sim}{10}_-$ can be introduced by the sign factor $\sigma(\mu_\alpha)$ defined in (4.13). Therefore

$$Y_2 = \sum_{\gamma_1 \mu\alpha\beta} h_{\gamma_1} \langle 8_{\gamma_1} n_\gamma | \mu_\alpha\, n_\beta \rangle\, \sigma(\mu_\alpha)\, \delta f_{(n_\beta, \mu_\alpha)} \quad . \tag{D12}$$

As our final example we evaluate one of the more complex terms in $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$:

$$Y_3 = \sum_{\substack{ijk \\ rs\ell}} \begin{pmatrix} 8 & 8 & n'_{\gamma'} \\ -i & j & -k \end{pmatrix} \begin{pmatrix} n' & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk}\, \delta f_{\bar{i}r\ell}\, f_{\bar{j}s\ell}\, f_{\bar{r}sk} \quad . \tag{D13}$$

The summation over the indices of $\delta f_{\bar{i}r\ell}$ will be done in the last step to form an irreducible splitting of the type

$$\delta f_{(n_{\beta'},\mu'_{\alpha'})} = \sum_{ir\ell} \begin{pmatrix} 8 & 8 & \mu'_{\alpha'} \\ -i & r & -\ell \end{pmatrix} \begin{pmatrix} \mu' & 8 & n_{\beta'} \\ -\ell & \ell & 0 \end{pmatrix} \eta_{\bar{i}r\ell}\, \delta f_{\bar{i}r\ell} \,. \tag{D14}$$

We therefore begin the summation with the indices (jsk). Writing the SU(3) symmetric couplings in the form (A2.2),

$$f_{\bar{j}s\ell}\, f_{\overline{rsk}} = \sum_{\gamma_1\gamma_2} h_{\gamma_1} h_{\gamma_2} \eta_{jr\ell k} \begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -j & s & -\ell \end{pmatrix} \begin{pmatrix} 8 & 8 & 8_{\gamma_2} \\ -r & s & -k \end{pmatrix}, \tag{D15}$$

we note that the phase factor $\eta_{jr\ell k}$ together with $\eta_{\bar{i}jk}$ in (D13) will give $\eta_{\bar{i}r\ell}$ which is what is needed in (D14). The four coefficients in (D15) and (D13) are not of the form required in (D14), so the indices have to be reshuffled by applying (A3.9). Eventually we need a coefficient like $\begin{pmatrix} \mu' & 8 & n_{\beta'} \\ -\ell & \ell & 0 \end{pmatrix}$ which has to come from $\begin{pmatrix} n' & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix}$ by recoupling. This cannot be done directly, because we would need a CG coefficient which contains both the indices k and ℓ, and there is no such coefficient; we have to apply (A3.9) more than once. The way to proceed is not unique, and each way will give the answer in a different form. One criterion which we like to apply to all our final expressions is that they contain Racah coefficients for the expansion $\underset{\sim}{8} \times \underset{\sim}{8} \times \underset{\sim}{8}$ only (these have been tabulated by Krammer). No simplification of the forms of the answers occurs if coefficients of other expansions are admitted; in practice the method of trial and error was often resorted to for finding the simplest form. We outline the procedure used to calculate the expression in (D13). Using (A3.9) we recouple the two CG coefficients appearing in (D13).

$$\begin{pmatrix} 8 & 8 & n'_{\gamma'} \\ -i & j & -k \end{pmatrix}\begin{pmatrix} n' & 8 & n_\gamma \\ -k & k & 0 \end{pmatrix} = \sum_{\mu\alpha\beta} \langle n'_{\gamma'}, n_\gamma | \mu_\alpha n_\beta \rangle \begin{pmatrix} 8 & 8 & \mu_\alpha \\ j & k & i \end{pmatrix}\begin{pmatrix} 8 & \mu & n_\beta \\ -i & i & 0 \end{pmatrix} . \tag{D16}$$

Our immediate goal is to sum over (jsk) by (A3.5). Writing

$$\begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ -j & s & -\ell \end{pmatrix} = \xi(\gamma_1)(-1)^{Q_s} \begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ \ell & s & j \end{pmatrix} ,$$

$$\begin{pmatrix} 8 & 8 & 8_{\gamma_2} \\ -r & s & -k \end{pmatrix} = (-1)^{Q_s} \begin{pmatrix} 8 & 8 & 8_{\gamma_2} \\ s & k & r \end{pmatrix} ,$$

the total phase factor is independent of (jsk); hence by (A3.5)

$$\sum_{jsk} \begin{pmatrix} 8 & 8 & 8_{\gamma_1} \\ \ell & s & j \end{pmatrix}\begin{pmatrix} 8 & 8 & \mu_\alpha \\ j & k & i \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_{\gamma_2} \\ s & k & r \end{pmatrix}$$

$$= \sum \langle 8_{\gamma_1} \mu_\alpha | 8_{\gamma_2} \mu_\epsilon \rangle \begin{pmatrix} 8 & 8 & \mu_\epsilon \\ \ell & r & i \end{pmatrix} .$$

Collecting all the factors, we have so far

$$Y_3 = \sum_{\substack{ir\ell\gamma_1\gamma_2\\ \mu\alpha\beta\epsilon}} h_{\gamma_1} h_{\gamma_2} \xi(\gamma_1)\eta_{\bar{i}r\ell} \langle n'_{\gamma'} n_\gamma | \mu_\alpha n_\beta\rangle \langle 8_{\gamma_1}\mu_\alpha | 8_{\gamma_2}\mu_\epsilon\rangle$$

$$\times \begin{pmatrix} 8 & 8 & \mu_\epsilon \\ \ell & r & i \end{pmatrix} \begin{pmatrix} 8 & \mu & n_\beta \\ -i & i & 0 \end{pmatrix} \delta f_{\bar{i}r\ell} \quad . \tag{D17}$$

two CG coefficients in (D17) are now recoupled to get the form (D14); switching the order of $\underset{\sim}{8}$ and μ in the last coefficient they may be written

$$\xi_1(8\ \mu\ n_\beta) \begin{pmatrix} 8 & 8 & \mu_\epsilon \\ \ell & r & i \end{pmatrix} \begin{pmatrix} \mu & 8 & n_\beta \\ i & -i & 0 \end{pmatrix}$$

$$= \xi_1(8\ \mu\ n_\beta) \sum_{\mu'\alpha'\beta'} \langle \mu_\epsilon n_\beta | \mu'_{\alpha'} n_{\beta'}\rangle \begin{pmatrix} 8 & 8 & \mu'_{\alpha'} \\ r & -i & \ell \end{pmatrix} \begin{pmatrix} 8 & \mu' & n_{\beta'} \\ \ell & -\ell & 0 \end{pmatrix} .$$

Switching the first two components in both CG coefficients gives the desired form (D14); our final result is

$$Y_3 = \sum_{\substack{\gamma_1\gamma_2\mu\mu'\\ \alpha\alpha'\beta\beta'\epsilon}} h_{\gamma_1} h_{\gamma_2} \xi(\gamma_1)\xi_1(8\ \mu\ n_\beta)\xi_1(\mu'_{\alpha'})\xi_1(8\ \mu'\ n_{\beta'})$$

$$\times \langle n'_{\gamma'} n_\gamma | \mu_\alpha n_\beta\rangle \langle 8_{\gamma_1}\mu_\alpha | 8_{\gamma_2}\mu_\epsilon\rangle \langle \mu_\epsilon n_\beta | \mu'_{\alpha'} n_{\beta'}\rangle \delta f_{(n_{\beta'}, \mu'_{\alpha'})} \quad . \tag{D18}$$

APPENDIX E

HERMITICITY VIOLATING SOLUTIONS

We examine the forms $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$, $\delta Z_2^{(n_\gamma)}$, and $\delta Z_3^{(n_\gamma)}$ given by Eqs. (7.4 - 7.6) using the general expressions for $\delta\Gamma_{\bar{i}jk}$, δZ_{2i}, and δZ_{3k} as given in (7.1 - 7.3). The only difference from our previous work is that the hermitian condition $\delta f_{\bar{j}i\bar{k}} = \delta f_{\bar{i}jk}$ is not imposed on the coupling splittings. It is clear that the irreducible mass splitting terms in the expansions of $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$, $\delta Z_2^{(n_\gamma)}$, and $\delta Z_3^{(n_\gamma)}$ in Eqs. (7.16 - 7.18) are unchanged; in particular, by (7.9a), the mass splittings will not appear in those combinations of $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$ for which the phase factor ϕ is zero.

Before considering the coupling terms we like to denote, for easier reference, those values of $(n_\gamma, n'_{\gamma'})$ for which ϕ equals 1 and 0 as belonging to sets I and II respectively. The irreducible splittings $\delta f^{(n_\gamma, n'_{\gamma'})}$ of type I thus conserve hermiticity, whereas those of type II do not.

The expansions of δZ_{2i} and δZ_{3k} contain the coupling splittings in the hermitian form $(\delta f_{\bar{i}jk} + \delta f_{\bar{j}i\bar{k}})$; the combinations $\delta Z_2^{(n_\gamma)}$ and $\delta Z_3^{(n_\gamma)}$, therefore, do not mix in splittings of type II.

When we previously showed that the vertex combinations $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$ of type II vanish [Eq. (7.9a)] , we used the hermitian property $\delta f_{\bar{j}i\bar{k}} = \delta f_{\bar{i}jk}$. What we really proved, therefore, was that the forms $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$ of type II do not contain splittings of type I. We can test the presence of type II splittings by imposing the anti-hermitian property $\delta f_{\bar{j}i\bar{k}} = -\delta f_{\bar{i}jk}$. Consider only the coupling splitting terms in $\delta\Gamma_{\bar{i}jk}$:

$$\delta F_{\bar{i}jk} = \delta f_{\bar{i}jk} + V \sum_{rs\ell} (\delta f_{\bar{r}sk} f_{\bar{i}r\ell} f_{\bar{j}s\ell} + \delta f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk} + \delta f_{\bar{s}j\ell} f_{\bar{i}r\ell} f_{\bar{r}sk}) \quad . \tag{E1}$$

For anti-hermitian splittings we have $\delta F_{\bar{j}i\bar{k}} = - \delta F_{\bar{i}jk}$; denoting by $\delta\Gamma_f^{(n_\gamma, n'_{\gamma'})}$ the contribution of $\delta F_{\bar{i}jk}$ to $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$ we obtain by an analysis similar to that in getting (7.8)

$$\delta\Gamma_f^{(n_\gamma, n'_{\gamma'})} = - \sigma(n'_{\gamma'})\xi_1(8\ n'\ n_\gamma)\delta\Gamma_f^{(n_\gamma, n'_{\gamma'})} \quad . \tag{E2}$$

Therefore

$$\phi\, \delta\Gamma_f^{(n_\gamma, n'_{\gamma'})} = 0 \quad , \tag{E3}$$

which proves that type II splittings are not contained in type I combinations of $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$.

In summary, we have just shown that type II coupling splittings appear only in type II forms of $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$ and do not connect to the mass splittings.

To compute the expansion coefficients we note first that the last two terms in $\delta F_{\bar{i}jk}$ go into the negative of each other under hermitian conjugation ($i \longleftrightarrow j$, $k \longleftrightarrow \bar{k}$) for anti-hermitian splittings. In type II forms of $\delta\Gamma_f^{(n_\gamma, n'_{\gamma'})}$ they therefore add equally, so we may use an effective $\delta F_{\bar{i}jk}$ given by

$$(\delta F_{\bar{i}jk})_{eff} = \delta f_{\bar{i}jk} + V \sum_{rs\ell} (\delta f_{\bar{r}sk} f_{\bar{i}r\ell} f_{\bar{j}s\ell} + 2\delta f_{\bar{i}r\ell} f_{\bar{j}s\ell} f_{\bar{r}sk}) \quad . \tag{E4}$$

This is the same form that was used in Sec. VII. Its contribution to $\delta\Gamma_f^{(n_\gamma, n'_{\gamma'})}$ has the same algebraic expression as before, because in the derivation of the latter no further hermiticity property of the splittings was used; we just drop the factor ϕ in the third term of Eq. (7.16) and evaluate only the coefficients of type II splittings in the type II forms of $\delta\Gamma_f^{(n_\gamma, n'_{\gamma'})}$. The result is given for general α in Table 17.

Table 17. The hermiticity violating forms of $\delta\Gamma^{(n_\gamma, n'_{\gamma'})}$

$n = \underset{\sim}{8}$	$\delta f_{(8,10_-)}$	$\delta f_{(8_2,8_1)}$	$\delta f_{(8_2,8_2)}$
$\delta\Gamma^{(8,10_-)}$	$1 + (\frac{1}{5} h_1^2 - h_2^2)V$	$\frac{\sqrt{10}}{5} h_1 h_2 V$	$-\frac{\sqrt{10}}{5} h_1^2 V$
$\delta\Gamma^{(8_2,8_1)}$	$\frac{\sqrt{10}}{5} h_1 h_2 V$	$1 - (\frac{3}{10} h_1^2 - \frac{1}{2} h_2^2)V$	$-h_1 h_2 V$
$\delta\Gamma^{(8_2,8_2)}$	$-\frac{\sqrt{10}}{5} h_1^2 V$	$-h_1 h_2 V$	$1 + \frac{1}{2}(h_1^2 + h_2^2)V$

$n = \underset{\sim}{27}$	$\delta f_{(27_2,27)}$	$\delta f_{(27,10_+)}$
$\delta\Gamma^{(27_2,27)}$	$1 - (\frac{3}{10} h_1^2 - \frac{1}{2} h_2^2)V$	$\frac{\sqrt{15}}{15} h_1 h_2 V$
$\delta\Gamma^{(27,10_+)}$	$\frac{\sqrt{15}}{15} h_1 h_2 V$	$1 - (\frac{3}{10} h_1^2 + \frac{1}{6} h_2^2)V$

Evaluating with $h_1 = h_2\sqrt{5}$ (equivalent to $\alpha = 3/4$) and putting $1 = 2h_2^2 V$ gives three identical equations for $n = \underset{\sim}{8}$ and two identical equations for $n = \underset{\sim}{27}$:

$$\sqrt{2}\ \delta f_{(8,10_-)} + \delta f_{(8_2,8_1)} - \sqrt{5}\ \delta f_{(8_2,8_2)} = 0 \quad , \tag{E5}$$

$$\delta f_{(27_2,27)} + \frac{1}{\sqrt{3}}\,\delta f_{(27,10_+)} = 0 \quad . \tag{E6}$$

We therefore have the curious result that the mathematics admits non-trivial hermiticity violating solutions for all g in both the $\underset{\sim}{8}$ and $\underset{\sim}{27}$ representations.

REFERENCES

1. G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 7, 395 (1961).
2. Among the first papers which clearly and simply displayed the bootstrap mechanism by a relativistic calculation were: F. Zachariasen, Phys. Rev. Letters 7, 112 (1961), erratum p.268; F. Zachariasen and C. Zemach, Phys. Rev. 128, 849 (1962).
3. R. E. Cutkosky, Phys. Rev. 131, 1888 (1963); R. H. Capps, Phys. Rev. Letters 10, 312 (1963); E. Abers, F. Zachariasen, and C. Zemach, Phys. Rev. 132, 1831 (1963).
4. F. Zachariasen and C. Zemach 138, B441 (1965); T. Sakuma and N. Tokuda, Prog. Theor. Phys. 35, 331 (1966); N. Tokuda, Prog. Theor. Phys. 35, 1043 (1966).
5. For a comprehensive treatment of SU(3) symmetry in particle physics see M. Gell-Mann and Y. Ne'eman, The Eightfold Way (W. A. Benjamin, Inc., New York, 1964).
6. R. P. Feynman has proposed the following criterion for "the correct theory": It ought to be such that no one particle can be considered more fundamental than any other.
7. A. Salam, Nuovo Cimento 25, 224 (1962).
8. The connection between the condition of vanishing renormalization constants and the frequently-used N/D method in bootstrap calculations is discussed by, for example, B. W. Lee, K. T. Mahanthappa, I. S. Gerstein, and M. L. Whippman in Ann. Phys. (N.Y.) 28, 466 (1964).

9. We follow the common practice of calculating with the first power of the baryon mass and the square of the meson mass, as is suggested by the forms in which they appear in the Lagrangian.

10. This is easily seen if the perturbation amplitude is expressed in integral form by the usual Feynman rules. For a coupling constant variable the operation $\delta g_{\bar{r}st}(\partial/\partial g_{\bar{r}st})$ just replaces any $g_{\bar{r}st}$ appearing in the graph by $\delta g_{\bar{r}st}$. For an internal mass variable, say a baryon mass variable m_q, the operation $\delta m_q(\partial/\partial m_q)$ is equivalent to inserting a mass term $-i\delta m_q$ into the propagator line:

$$\delta m_q \frac{\partial}{\partial m_q}\left(\frac{i}{\not{k} - m_q}\right) = \left(\frac{i}{\not{k} - m_q}\right)(-i\delta m_q)\left(\frac{i}{\not{k} - m_q}\right).$$

11. The splittings which transform according to $n = \underset{\sim}{1}$ do not violate SU(3) symmetry; we carry them along for the sake of completeness, and also because the singlet coupling splittings might produce a change in the F-D mixing parameter α.

12. It may be noted that another set of irreducible coupling splitting operators can be formed by first combining the meson-baryon pair $\phi_k \psi_j$ into a representation $n'_{\gamma'}$ and then use the resulting form together with $\bar{\psi}_i$ to make a $Y = 0$, $I = 0$ member of n_γ. This set is connected by a unitary transformation to the first one, but is unsatisfactory because most of the splittings will contain both hermiticity conserving and hermiticity violating parts.

13. Our definition of $Z_{\bar{i}jk}$ is the many-particle generalization of the vertex renormalization constant Z_1 for a single meson field interacting with a single spinor field (see Sec. V); in lowest order Z_1 and $Z_{\bar{i}jk}$ are both unity.

14. In Sec. I we applied the bootstrap condition to $Z_{\bar{i}jk}$ to maintain a uniformity of expression.

15. P. T. Matthews and A. Salam, Phys. Rev. 94, 185 (1954).

16. J. J. deSwart, Rev. Mod. Phys. 35, 916 (1963), erratum 37, 326 (1965).

17. Sometimes the relative amount of D and F coupling is specified by a mixing angle θ. It is related to the ratio of our h parameters by $\tan\theta = h_2/h_1$.

18. See, for example, P. Carruthers, Introduction to Unitary Symmetry (Interscience Publishers, New York, 1966).

19. By "mathematical SU(3) states" we mean eigenstates of irreducible representations of SU(3) whose relative phases follow the standard Condon and Shortley phase convention. See Ref. 16, Secs. 7 and 8.

20. See, for example, Ref. 16, Eq. (8.2).

21. Since in field theory the emission of a particle and the absorption of the anitparticle are on the same footing, the destruction operators in $\bar{\psi}_i$, ψ_j and ϕ_k couple with the same phase as the creation operators.

22. See, for example, A. R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, New Jersey, 1957).

23. Compare with R. F. Dashen, Y. Dothan, S. C. Frautschi, and D. H. Sharp, Phys. Rev. 151, 1127 (1966), footnote 28.

24. We write out explictily the factors i which come from the rules of the perturbation expansion of the S-matrix.

25. The mass and wave function renormalization conditions are more fundamentally stated by the properties of the renormalized propagator: The propagator must have a pole at the physical mass with unit residue. In the renormalized Heisenberg representation the propagator summed to all orders is $(\not{p} - m - \Sigma^*(p))^{-1}$, where $\Sigma^*(p) = \Sigma(p) - Z_2\Delta m - (Z_2-1)(\not{p} - m)$. Choosing Δm and Z_2 such that $Z_2\Delta m = \Sigma(p)\big|_{\not{p}=m}$ and $(Z_2 - 1) = (\partial\Sigma(p)/\partial\not{p})\big|_{\not{p}=m}$ will result in $\Sigma^*(p)$ being of the form $(\not{p} - m)^2\Sigma_c^*(p)$, where $\Sigma_c^*(p)$ is finite and non-singular at $\not{p} = m$. (We have defined the self-energy operators in Figs. 1a and 1b with the factor $-i$ so that they may appear in the propagators as additions to the mass terms without extra factors.)

26. R. E. Cutkosky, J. Math.Phys. 1, 429 (1960).

27. We do not agree with R. Delbourgo, Nuovo Cimento 27, 1431 (1963), whose result for Λ differs from ours by a (crucial) sign. See Appendix B.

28. M. Krammer, Acta Physica Austriaca, Suppl. 1, 183 (1964).

29. A. Martin and K. Wali, Phys. Rev. 130, 2455 (1963); S. Glashow and A. Rosenfeld, Phys. Rev. Letters 10, 192 (1963).

30. K. Y. Lin and R. E. Cutkosky, Phys. Rev. 140, B205 (1965).

31. $\langle n'_\gamma, n_\gamma | C_1 | \mu_\alpha n_\beta \rangle$ is the transformation matrix which connects the set $\delta f_{(n_\gamma, n'_{\gamma'})}$ with the set $\delta f_{(n_\beta, \mu_\alpha)}$ obtained by the method described in footnote 12.

32. The factors are easily found by comparing the entries in Tables 5 and 6 in the mass columns $\delta m_{(8_1)}$, $\delta m_{(27)}$, and $4\delta m_{(1)}$.

33. It was mentioned in footnote 11 that the $n = \underset{\sim}{1}$ splittings do not violate SU(3) symmetry but that they might change the F-D mixing. The last equation in Table 7, which reads $\delta f_{(1,8_2)} = \frac{1}{\sqrt{5}} \delta f_{(1,8_1)}$, now tells us that the F-D mixing remains unchanged with $\alpha = 3/4$. We may show this by considering only the $n_\gamma = \underset{\sim}{1}$ terms in Eq. (2.6), and write

$$\delta f_{\bar{i}jk} = \sum_{\gamma'} \begin{pmatrix} 8 & 8 & 8_{\gamma'} \\ -i & j & -k \end{pmatrix} \begin{pmatrix} 8 & 8 & 1 \\ -k & k & 0 \end{pmatrix} \eta_{\bar{i}jk}\, \delta f_{(1,8_{\gamma'})}$$

$$= \sum_{\gamma'} -(\sqrt{8}\, h_{\gamma'})^{-1} f^{(\gamma')}_{\bar{i}jk}\, \delta f_{(1,8_{\gamma'})} \; .$$

Using $h_1 = h_2\sqrt{5}$ and $\delta f_{(1,8_1)} = \sqrt{5}\, \delta f_{(1,8_2)}$ it is seen that the quantity $R = (\sqrt{8}\, h_{\gamma'})^{-1} \delta f_{(1,8_{\gamma'})}$ is independent of γ'. The total coupling constant is therefore

$$f_{\bar{i}jk} + \delta f_{\bar{i}jk} = (1 - R) \sum_{\gamma'} f^{(\gamma')}_{\bar{i}jk} = (1-R)\, f_{\bar{i}jk} \; ,$$

so that the whole effect is just a change in the over-all constant g.

34. The determinants of the matrices in Table 8 are multiplied by the factors -765, -4/81, and -9 for $n = \underset{\sim}{8}, \underset{\sim}{27}$, and $\underset{\sim}{1}$ respectively to recover the normalization of Table 5.

35. The experimental mass splittings, in units of the average baryon mass $m = 1151$ MeV, are: $\delta m_{(8_2)}/m = -0.329$, $\delta m_{(8_1)}/m = 0.080$, $\delta m_{(27)}/m = 0.006$, $\delta\mu^2_{(8)}/m^2 = -0.248$, and $\delta\mu^2_{(27)}/m^2 = 0.013$.

36. For $\mu/m = 0.356$ we extended the calculation down to $g^2/4\pi = 1.8$, at which value the cutoff masses are approximately $\lambda_1/m = 30$, $\lambda_2/m = 9.5$, and $\lambda_3/m = 3.7$.

37. P. Tarjanne and R. E. Cutkosky, Phys. Rev. 133, B1292 (1964).

38. R. Dashen and S. C. Frautschi, Phys. Rev. Letters 13, 497 (1964); and Phys. Rev. 137, B1331 (1965).

39. In an integral representation, $(Z_2^{-1} - 1)$ and $(Z_3^{-1} - 1)$ have weight functions which are sums of positive definite quantities; we therefore consider only positive values of ϵ_2 and ϵ_3. No similar statement can be made for Z_1, so ϵ_1 is assigned either sign. See, for example, An Introduction to Relativistic Quantum Field Theory (Row, Peterson and Co., New York, 1961).

40. D. Neville [Phys. Rev. Letters 13, 118 (1964)] upon considering the properties of crossing matrices makes the following comment: "For all groups only lowest dimensional representations should be of dynamical significance." (I am indebted to Professor

Frautschi for drawing my attention to this reference.)

41. M. Muraskin and S. L. Glashow [Phys. Rev. 132, 482 (1963)] have given five sum rules for the total couplings g_{ABC}. We get two more by requiring that the two singlet splittings vanish.

42. R. F. Dashen, Y. Dothan, S. C. Frautschi, and D. H. Sharp, Phys. Rev. 143, 1185 (1966); 151, 1127 (1966).

43. R. F. Dashen, Phys. Letters 11, 89 (1964).

44. R. F. Dashen and S. C. Frautschi, Phys. Rev. 137, B1318 (1965).

45. In the approach of DDFS the meson mass splittings are not expected to have a significant effect. See Ref. 42.

46. S. C. Frautschi, private communication.

47. For a summary see P. Carruthers, Introduction to Unitary Symmetry (Interscience Publishers, New York, 1966), pp. 118-120, and the references quoted therein.

48. C. Peck, quoted in Ref. 42.

49. This is not a consequence of the exact relations, which give $2\Delta(\Lambda\Sigma\pi) - \Delta(NN\pi) = \Delta(\Xi\Lambda K) - \Delta(N\Lambda K) - \Delta(N\Sigma K)$. With $\Delta(\Xi\Lambda K)$ small, $\Delta(\Lambda\Sigma\pi) \approx \frac{1}{2}\Delta(NN\pi)$ suggests $\Delta(N\Sigma K) \approx -\Delta(N\Lambda K)$ but the latter relation is not quite as well satisfied.

50. J. J. deSwart and C. K. Iddings, Phys. Rev. 128, 2810 (1963), and Phys. Rev. 130, 319 (1963).

51. R. E. Cutkosky and M. Leon, Phys. Rev. 138, B667 (1965).

52. See, for example, Ref. 51.

53. If the SU(2) system of pions and nucleons is investigated for spontaneous symmetry breakdown, it is interesting that the equation for the hermiticity-violating coupling splitting (there is only one) is still identically satisfied, because the coefficient multiplying the splitting variable vanishes due to the vertex bootstrap requirement.

54. See, for example, P. Carruthers, Ref. 47, pp. 126 f.