TRANSIENT WAVE PROPAGATION IN ELASTIC PLATES WITH CYLINDRICAL BOUNDARIES, STUDIED WITH THE AID OF MULTI-INTEGRAL TRANSFORMS

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Richard Anthony Scott

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ABSTRACT

Some mixed time dependent boundary value problems for isotropic elastic plates with circular cylindrical boundaries are studied using the linear equations of elasticity. A multi-integral transform approach is employed, necessitating the introduction of extended Hankel transforms, and formal solutions are obtained with the aid of residue theory. Some properties of the Rayleigh-Lamb frequency equation, pertinent to the inversion processes, are derived. The problem of a free infinite plate with a circular cylindrical cavity subjected to a step normal displacement is studied in detail and numerical information for the far-field, showing the effect of the cavity radius on the displacements, is obtained using stationary phase techniques.

The generation of transient elastic waves in free isotropic infinite elastic plates by time dependent body forces is also treated and the results for a radial body force, with step time-dependence, are compared with the corresponding plate-cavity results. Good agreement between the two is found in the far-field.

Similar problems for a free, transversely isotropic, semi-infinite plate (slab) are also studied and some numerical information for the far-field is obtained using the head of the pulse method. Stationary phase solutions for an isotropic slab subjected to a step edge displacement are obtained and compared with the corresponding plate-cavity results. It is found that at a fixed station the plate cavity solutions approach those for the slab, as the cavity radius goes to zero. A comparison between the head of the pulse and stationary pulse results for the isotropic slab is also made and some discrepancies between the two are found.

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INTRODUCTION

Though the equations of motion of a homogeneous linearly elastic solid--the so-called "exact" equations--present many formidable difficulties, the literature (1) in recent years reflects an increasing interest in their application. This is due in part to the occurrence of problems involving high rates of loading, and to the growth of mathematical techniques which make the equations tractable. Although a great deal of work has been done in connection with infinite media and half-spaces, much of the interest has centered on wave-guide propagation, i.e., propagation in configurations with characteristic lengths. For such problems multi-integral transform approaches have proven to be of use. The difficulties encountered increase with the complexity of the geometry and depend also on the nature of the applied boundary conditions. It is generally found that nonmixed conditions (displacements or stresses specified) are more troublesome than mixed conditions (displacement and stress specified). In fact, for geometries involving two or more perpendicular boundaries, no solutions to nonmixed problems have as yet been written.

To put the present work in perspective, a brief review of the current situation for such a geometry will now be given, taking the semi-infinite circular isotropic rod as an example. When the lateral surfaces of the rod are stress free, mixed problems can be conveniently classified into two groups, viz., those in which the normal displacement and shear stress at the rod end are specified, henceforth termed longitudinal impact type problems, and those in which the normal stress and lateral displacement are specified at the rod end, henceforth termed pressure shock type problems.

In connection with compressional waves Skalak (2) has given a solution to a problem in the first category, in which the rod end is subjected to a uniform step normal velocity and zero shear stress. Folk, Fox, Shook, and Curtis (3, 4) have given a general method for solving problems of the second class and applied it to the case in which the rod end is subjected to a uniform step normal pressure and zero radial displacement. DeVault and Curtis (5) have extended this latter type of work to problems in which both flexural and compressional waves are generated. They also incorporated variations across the rod end into their solutions. A major point of interest of the authors of References (3), (4), and (5) has been the applicability of their solutions to experiments simulating nonmixed problems. They found that, for large distances down the rod, the main features of the experimental records compared quite well with the theoretical results.

Both types of problem have been extensively discussed by Miklowitz (6), using the Mindlin-Herrmann (7) approximate equations of motion. He showed that this approximate rod theory gave the same result as the exact theory for the main features of the pulse for large distances from the source (lower mode activity). Comparisons between the theoretical predictions and his and Nisewanger's (8) experimental results not only clarified the regions of validity of the approximate equations but also showed that the Mindlin-Herrmann theory modeled certain higher mode influences in the pulse. The experiments also confirmed the Poisson's ratio coupling predicted by Skalak's solution.

In the same connection the use of time-dependent body forces should also be noted. Fox (9) has shown that his experimental results for a nonmixed problem also agreed quite well, for large distances down the rod, with theoretical solutions obtained for an infinite rod in which a concentrated body force, with step time-dependence, is acting in the axial direction.

Similar information for the flat elastic plate is lacking, i.e., no comparisons between experiments simulating nonmixed problems and theoretical solutions to analogous mixed problems have been made. Also problems in which lateral surface loads arise are of technical importance. Miklowitz (10), using Laplace and Hankel transforms, has solved a problem of this type, in which the surfaces of an infinite plate are subjected to suddenly applied concentrated normal loads. Such loadings generate cylindrically-crested waves and it is the consideration of these that led to the problems at hand.

In the present work interest is in a flat elastic plate with circular cylindrical boundaries, and solutions have been written for both flexural and compressional mixed wave problems in isotropic media. A multi-integral transform approach is adopted, necessitating the introduction of special Hankel transforms, and some theorems concerning the nature of the zeros of certain transcendental functions, pertinent to the inversion processes, have been derived. Approximations to the solutions, valid at large distances from the source, have been written and evaluated for a particular compressional wave problem in an infinite plate with a circular cylindrical cavity.

The methods and solutions have been discussed in the light of the above rod situation and in this connection the problem of an infinite plate subjected to certain time-dependent body forces has also been examined. In the final section some related problems for a semi-infinite plate, which is a limiting case of the plate with the cavity, have been discussed. There, partly to illustrate the scope of multi-integral transform techniques, the plate material has been taken as anisotropic (transversely isotropic).

Section I. TRANSIENT COMPRESSIONAL AND FLEXURAL WAVES IN FINITE AND INFINITE FLAT ELASTIC PLATES WITH CIRCULAR CYLINDRICAL BOUNDARIES

INTRODUCTION

In this section certain transient compressional and flexural wave problems for a homogeneous, isotropic, linearly elastic, flat plate of thickness 2H, with circular cylindrical boundaries, are examined. Kromm (11), Miklowitz (12), and Goodier and Jahsman (13), worked on wave problems involving such geometries, using the plane-stress theory, but, previous to the present work, these types of problems have not been approached with the exact theory, or higher order approximate theories.

Miklowitz, in the problem on cylindrically-crested waves mentioned above (10), employed a multi-integral transform technique, using ordinary Hankel transforms, i.e., those for the interval $(0, \infty)$, to suppress the spatial variable in the plane of the plate. Here also a multi-integral transform approach is adopted, but ordinary Hankel transforms cannot be employed, since the spatial interval of physical interest does not include the origin. Thus the first step in approaching the geometry at hand is to seek suitable transform pairs to suppress the spatial variable in the plane of the plate. Such transform pairs have been found and are discussed, together with some of their properties, in Appendix A.

The lateral surfaces of the plate have been assumed stress free and various time-dependent boundary conditions of the mixed type are specified on the cylindrical surfaces. The development used exhibits the natural occurrence of the mixed conditions and also illustrates the limitations of the method.

1.1. STATEMENT OF PROBLEMS AND DERIVATION OF FORMAL SOLUTIONS

The boundary and initial conditions are assumed to be such that axial symmetry prevails and the cylindrical polar coordinate system shown in Figure 1 is chosen. The stress equations of motion of a linear elastic solid, for the case of axial symmetry, are (14):

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho' F_r(r, t, z) = \rho' \frac{\partial^2 u_r}{\partial t^2} \tag{1.1}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho' F_z(r, t, z) = \rho' \frac{\partial^2 u_z}{\partial t^2} \tag{1.2}$$

where

$$\sigma_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{1.3}$$

$$\sigma_{zz} = (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) \tag{1.4}$$

$$\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r} + \lambda\frac{u_r}{r} + \lambda\frac{\partial u_z}{\partial z} \tag{1.5}$$

$$\sigma_{\theta\theta} = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{u_r}{r} \tag{1.6}$$

are the stress components, with the usual notation as regards their suffixes, u_r and u_z are the displacement components, F_r and F_z are the body force components per unit mass, λ and μ are the Lamé constants, ρ' is the material density, and t is the time variable. The strains which are of importance in the subsequent discussions are given by:

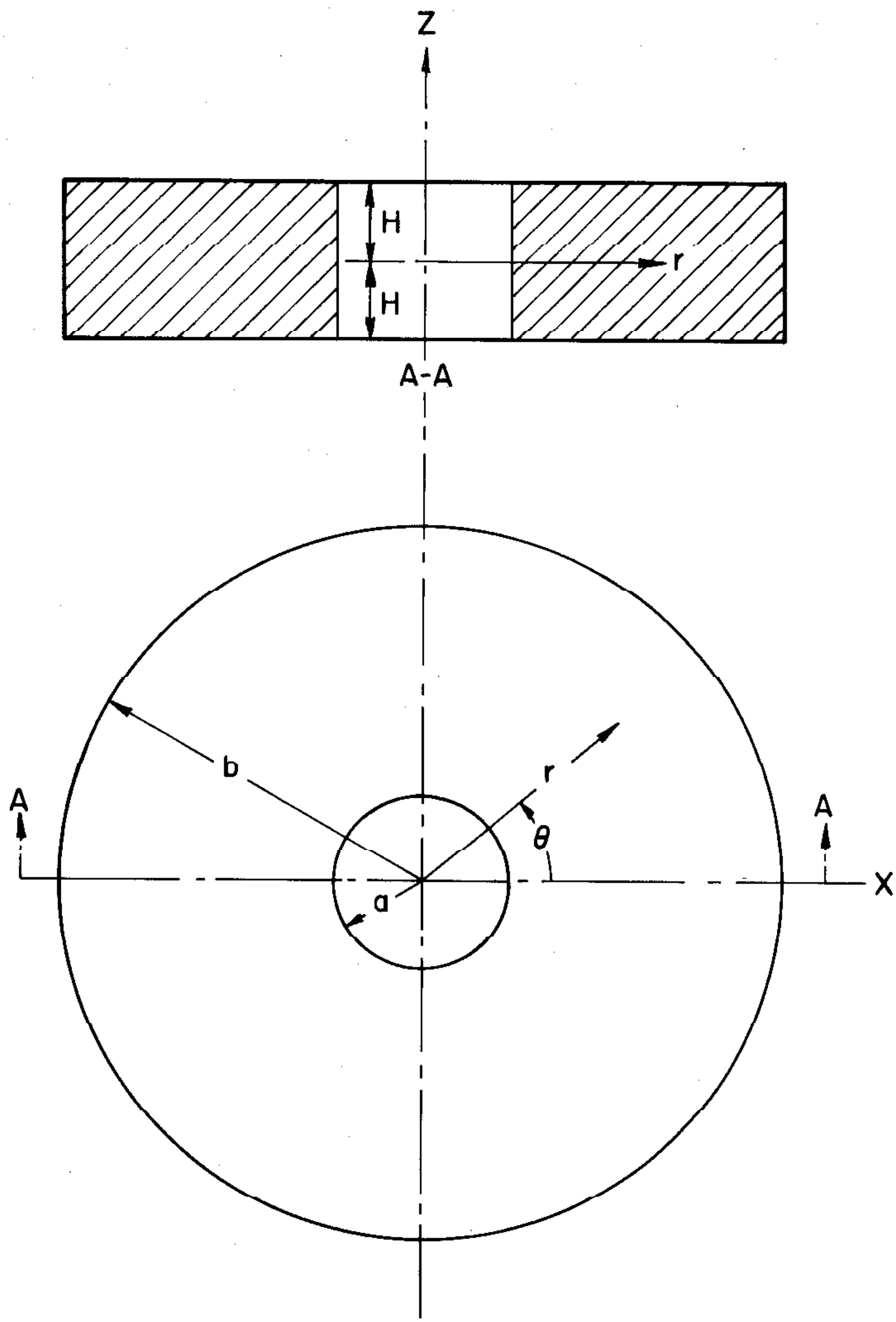

Fig. 1. Geometry of plate and coordinates used.

$$e_{rr} = \frac{\partial u_r}{\partial r} \tag{1.7}$$

$$e_{zz} = \frac{\partial u_z}{\partial z} \tag{1.8}$$

$$e_{\theta\theta} = \frac{u_r}{r} \tag{1.9}$$

Substituting equations 1.3 through 1.6 into equations 1.1 and 1.2 gives, after some rearranging, the displacement equations of motion

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \left(1 - \frac{c_s^2}{c_d^2}\right)\frac{\partial^2 u_z}{\partial z \partial r} + \frac{c_s^2}{c_d^2}\frac{\partial^2 u_r}{\partial z^2}$$

$$+ \frac{1}{c_d^2} F_r(r, t, z) = \frac{1}{c_d^2}\frac{\partial^2 u_r}{\partial t^2} \tag{1.10}$$

$$\left(1 - \frac{c_s^2}{c_d^2}\right)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + \frac{\partial^2 u_z}{\partial z^2} + \frac{c_s^2}{c_d^2}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right)$$

$$+ \frac{1}{c_d^2} F_z(r, t, z) = \frac{1}{c_d^2}\frac{\partial^2 u_z}{\partial t^2} \tag{1.11}$$

where

$$c_s^2 = \frac{\mu}{\rho},$$

$$c_d^2 = \frac{\lambda + 2\mu}{\rho}$$

are the infinite medium equivoluminal and dilatational wave speeds, respectively.

Taking the Laplace transform (w. r. t. t) of equations 1.3 through 1.11 gives:

$$\bar{\sigma}_{rz} = \mu\left(\frac{\partial \bar{u}_r}{\partial z} + \frac{\partial \bar{u}_z}{\partial r}\right) \tag{1.12}$$

$$\bar{\sigma}_{zz} = (\lambda + 2\mu)\frac{\partial \bar{u}_z}{\partial z} + \lambda\left(\frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r}\right) \tag{1.13}$$

$$\bar{\sigma}_{rr} = (\lambda + 2\mu)\frac{\partial \bar{u}_r}{\partial r} + \lambda\frac{\bar{u}_r}{r} + \lambda\frac{\partial \bar{u}_z}{\partial z} \tag{1.14}$$

$$\bar{\sigma}_{\theta\theta} = \lambda\left(\frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r} + \frac{\partial \bar{u}_z}{\partial z}\right) + 2\mu\frac{\bar{u}_r}{r} \tag{1.15}$$

$$\bar{e}_{rr} = \frac{\partial \bar{u}_r}{\partial r} \tag{1.16}$$

$$\bar{e}_{zz} = \frac{\partial \bar{u}_z}{\partial z} \tag{1.17}$$

$$\bar{e}_{\theta\theta} = \frac{\bar{u}_r}{r} \tag{1.18}$$

$$\frac{\partial^2 \bar{u}_r}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}_r}{\partial r} - \frac{\bar{u}_r}{r^2} + \left(1 - \frac{c_s^2}{c_d^2}\right)\frac{\partial^2 \bar{u}_z}{\partial z \partial r} + \frac{c_s^2}{c_d^2}\frac{\partial^2 \bar{u}_r}{\partial z^2}$$

$$+ \frac{1}{c_d^2}\bar{F}_r(r, p, z) = \frac{p^2}{c_d^2}\bar{u}_r \tag{1.19}$$

$$\left(1 - \frac{c_s^2}{c_d^2}\right)\frac{\partial}{\partial z}\left(\frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r}\right) + \frac{\partial^2 \bar{u}_z}{\partial z^2} + \frac{c_s^2}{c_d^2}\left(\frac{\partial^2 \bar{u}_z}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}_z}{\partial r}\right)$$

$$+ \frac{1}{c_d^2}\bar{F}_z(r, p, z) = \frac{p^2}{c_d^2}\bar{u}_z \tag{1.20}$$

where the bar denotes the Laplace transform of a variable, p is the transform parameter, and the following initial conditions have been assumed:

$$\left.\begin{aligned} u_r &= 0 \\ u_z &= 0 \\ \frac{\partial u_r}{\partial t} &= 0 \\ \frac{\partial u_z}{\partial t} &= 0 \end{aligned}\right\} \quad t = 0, \; a \le r \le b, \; -H \le z \le H \tag{1.21}$$

Operating on equations 1.13, 1.17, and 1.20, with the zero order Hankel transform given by equation A3.2, and on equations 1.12 and 1.19 with the first order Hankel transform given by equation A3.1, where the α and β of Appendix A have been replaced by a and b, respectively, and taking the body forces to be zero, one obtains:

$$\tilde{\tilde{\sigma}}^{1}_{rz} = \mu \frac{\partial \tilde{\tilde{u}}^{1}_{r}}{\partial z} - \mu k \tilde{\tilde{u}}^{o}_{z} + \mu \left[r\bar{u}_z C_1(k, r, a) \right]_b^a \tag{1.22}$$

$$\tilde{\tilde{\sigma}}^{o}_{zz} = (\lambda + 2\mu) \frac{\partial \tilde{\tilde{u}}^{o}_{z}}{\partial z} + \lambda k \tilde{\tilde{u}}^{1}_{r} + \lambda \left[r\bar{u}_r C_o(k, r, a) \right]_a^b \tag{1.23}$$

$$\tilde{\tilde{e}}^{o}_{zz} = \frac{\partial \tilde{\tilde{u}}^{o}_{z}}{\partial z} \tag{1.24}$$

$$\frac{d^2 \tilde{\tilde{u}}^{1}_{r}}{dz^2} - h^2 \tilde{\tilde{u}}^{1}_{r} - k\left(\frac{c_d^2}{c_s^2} - 1 \right) \frac{d\tilde{\tilde{u}}^{o}_{z}}{dz} = L(z) \tag{1.25}$$

$$\frac{d^2 \tilde{\tilde{u}}^{o}_{z}}{dz^2} - g^2 \tilde{\tilde{u}}^{o}_{z} - k\left(\frac{c_s^2}{c_d^2} - 1 \right) \frac{d\tilde{\tilde{u}}^{1}_{r}}{dz} = M(z) \tag{1.26}$$

where

$$h^2 = \frac{k^2 c_d^2 + p^2}{c_s^2} \tag{1.27}$$

$$g^2 = \frac{k^2 c_s^2 + p^2}{c_d^2}$$

$$L(z) = \frac{c_d^2}{c_s^2} \left\{ kr\bar{u}_r C_o(k, r, a) - \left[r \frac{\partial \bar{u}_r}{\partial r} + \bar{u}_r \right.\right.$$

$$\left.\left. + \left(1 - \frac{c_s^2}{c_d^2}\right) r \frac{\partial \bar{u}_z}{\partial z} \right] C_1(k, r, a) \right\}_a^b \qquad (1.29)$$

$$M(z) = - \left\{ \left[\left(1 - \frac{c_s^2}{c_d^2}\right) r \frac{\partial \bar{u}_r}{\partial z} + \frac{c_s^2}{c_d^2} r \frac{\partial \bar{u}_z}{\partial r} \right] C_o(k, r, a) \right.$$

$$\left. + \frac{c_s^2}{c_d^2} kr\bar{u}_z C_1(k, r, a) \right\}_a^b \qquad (1.30)$$

and, as specified in Appendix A, it is to be understood that if b is finite then k in the above expressions is to be replaced by k_j.

It should be noted here that transformed expressions for e_{rr} and σ_{rr} cannot be written, since application of either the zero or first order Hankel transforms to equations 1.5 and 1.7 does not lead to expressions which involve $\tilde{\tilde{u}}_z^o$ and $\tilde{\tilde{u}}_z^1$ alone. Note however that for a fluid plate (i.e., $\mu = 0$) a transformed expression can be written for the radial stress σ_{rr}, viz.,

$$\tilde{\tilde{\sigma}}_{rr}^o = \lambda \frac{\partial \tilde{\tilde{u}}_z^o}{\partial z} + \lambda k \tilde{\tilde{u}}_r^1 + \lambda \left[r\bar{u}_r C_o(k, r, a) \right]_a^b$$

This feature is not unique to the transform pairs at hand, since it is also true of the analogous Hankel transform pairs for the spatial interval $(0, \infty)$. This point will receive some further discussion later in the text. No attention has been given to $e_{\theta\theta}$, since it can be obtained directly from u_r.

The solutions to equations 1.25 and 1.26 may be written as follows:

$$\tilde{\tilde{u}}_r^1 = (\tilde{\tilde{u}}_r^1)_{ho.} + (\tilde{\tilde{u}}_r^1)_{pt.}$$

$$\tilde{\tilde{u}}_z^o = (\tilde{\tilde{u}}_z^o)_{ho.} + (\tilde{\tilde{u}}_z^o)_{pt.}$$

where $(\tilde{\tilde{u}}_r^1)_{ho.}$, $(\tilde{\tilde{u}}_z^o)_{ho.}$ and $(\tilde{\tilde{u}}_r^1)_{pt.}$, $(\tilde{\tilde{u}}_z^o)_{pt.}$ are solutions to the homogeneous equations and particular solutions of the nonhomogeneous equations, respectively. These particular integrals can be evaluated explicitly when the nonhomogeneous terms in the differential equations are known. To preserve generality, discussion of these boundary terms is reserved for later in the section, except to the extent that it is assumed that the specification of two independent quantities on the curved surfaces of the plate reduces the boundary terms in equations 1.22, 1.23, 1.25 and 1.26 to known factors. In this context only these equations need be considered, since they are the ones which arise when the conditions of stress free lateral surfaces are applied.

On inserting into the homogeneous equations expressions of the type:

$$\tilde{\tilde{u}}_r^1 = A \cosh \eta z + B \sinh \eta z$$

$$\tilde{\tilde{u}}_z^o = E \cosh \eta z + D \sinh \eta z$$

where A, B, D, E, and η are not functions of z, it is found that solutions are obtained if η is chosen to be one of the values

$$\pm\left(k^2 + \frac{p^2}{c_s^2}\right)^{1/2}, \quad \pm\left(k^2 + \frac{p^2}{c_d^2}\right)^{1/2}$$

and if

$$\frac{D}{A} = \frac{E}{B} = \frac{\eta^2 - h^2}{\left(\frac{c_d^2}{c_s^2} - 1\right) k \eta}$$

Due to the odd and even nature of the hyperbolic functions sinh and cosh, only two of the above η values give independent solutions. Thus, using the method of variation of parameters to obtain the particular integrals, the general solutions to equations 1.25 and 1.26 can be shown to be:

$$\tilde{\tilde{u}}_r^1 = \sum_{j=1}^{2} (A_j \cosh \eta_j z + B_j \sinh \eta_j z) + (\tilde{\tilde{u}}_r^1)_{pt.} \qquad (1.31)$$

$$\tilde{\tilde{u}}_z^o = \sum_{j=1}^{2} (\alpha_j A_j \sinh \eta_j z + \alpha_j B_j \cosh \eta_j z) + (\tilde{\tilde{u}}_z^o)_{pt.} \qquad (1.32)$$

where

$$(\tilde{\tilde{u}}_r^1)_{pt.} = \frac{1}{\eta_1 \alpha_2 - \alpha_1 \eta_2} \sum_{j=1}^{2} (-1)^j \alpha_{2/j} \left\{ \cosh \eta_j z \int^z L(\xi) \sinh \eta_j \xi \, d\xi - \sinh \eta_j z \int^z L(\xi) \cosh \eta_j \xi \, d\xi \right\}$$
$$+ \frac{1}{\eta_1 \alpha_1 - \eta_2 \alpha_2} \sum_{j=1}^{2} (-1)^{j+1} \left\{ \cosh \eta_j z \int^z M(\xi) \cosh \eta_j \xi \, d\xi - \sinh \eta_j z \int^z M(\xi) \sinh \eta_j \xi \, d\xi \right\}$$

(1.33)

$$(\tilde{\tilde{u}}_z^o)_{pt.} = \frac{\alpha_1\alpha_2}{\eta_1\alpha_2-\eta_2\alpha_1}\sum_{j=1}^{2}(-1)^j\Big\{\sinh\eta_j z\int^z L(\xi)\sinh\eta_j\xi\,d\xi - \cosh\eta_j z\int^z L(\xi)\cosh\eta_j\xi\,d\xi\Big\}$$

$$+\frac{1}{\alpha_1\eta_1-\alpha_2\eta_2}\sum_{j=1}^{2}(-1)^{j+1}\alpha_j\Big\{\sinh\eta_j z\int^z M(\xi)\cosh\eta_j\xi\,d\xi - \cosh\eta_j z\int^z M(\xi)\sinh\eta_j\xi\,d\xi\Big\}$$

(1.34)

$$\alpha_j = \frac{\eta_j^2 - h^2}{\left(\frac{c_d^2}{c_s^2} - 1\right)k\eta_j}\quad,\quad j = 1, 2 \tag{1.35}$$

$$\eta_1 = +\left(k^2 + \frac{p^2}{c_s^2}\right)^{1/2} \tag{1.36}$$

$$\eta_2 = +\left(k^2 + \frac{p^2}{c_d^2}\right)^{1/2} \tag{1.37}$$

The boundary conditions on the lateral surfaces of the plate could now be applied to equations 1.31 and 1.32, giving four algebraic equations for the four unknowns A_1, A_2, B_1, and B_2. However it is more convenient to assume that the boundary conditions on the curved surfaces of the plate are such that either compressional or flexural waves are generated, i.e., it is assumed that L and M in equations 1.25 and 1.26 are such that solutions which are either symmetric or antisymmetric w.r.t. the middle plane of the plate are generated; the two cases are distinguished in what follows by the affixes C and F, respectively.

For the compression case the general solutions are:

$$ {}_C\overset{\approx}{u}{}^1_r = \sum_{j=1}^{2} A_j \cosh \eta_j z + ({}_C\overset{\approx}{u}{}^1_r)_{pt.} \tag{1.38} $$

$$ {}_C\overset{\approx}{u}{}^o_z = \sum_{j=1}^{2} \alpha_j A_j \sinh \eta_j z + ({}_C\overset{\approx}{u}{}^o_z)_{pt.} \tag{1.39} $$

where $({}_C\overset{\approx}{u}{}^1_r)_{pt.}$ and $({}_C\overset{\approx}{u}{}^o_z)_{pt.}$ are the expressions obtained from equations 1. 33 and 1. 34 on insertion of the appropriate values of L and M. The general solutions in the flexural case are:

$$ {}_F\overset{\approx}{u}{}^1_r = \sum_{j=1}^{2} B_j \sinh \eta_j z + ({}_F\overset{\approx}{u}{}^1_r)_{pt.} \tag{1.40} $$

$$ {}_F\overset{\approx}{u}{}^o_z = \sum_{j=1}^{2} \alpha_j B_j \cosh \eta_j z + ({}_F\overset{\approx}{u}{}^o_z)_{pt.} \tag{1.41} $$

where $({}_F\overset{\approx}{u}{}^1_r)_{pt.}$ and $({}_F\overset{\approx}{u}{}^o_z)_{pt.}$ are the expressions obtained from equations 1. 33 and 1. 34 on insertion of the values of L and M appropriate to this case.

Substituting equations 1. 38 through 1. 41 into equations 1. 22 and 1. 23 gives:

$$ {}_C\overset{\approx}{\sigma}{}^1_{rz} = \mu \sum_{j=1}^{2} A_j(\eta_j^2 - k\alpha_j) \sinh \eta_j z + S_C(z) \tag{1.42} $$

$$ {}_F\overset{\approx}{\sigma}{}^o_{zz} = \sum_{j=1}^{2} A_j \left[(\lambda + 2\mu)\alpha_j \eta_j + \lambda k \right] \cosh \eta_j z + T_C(z) \tag{1.43} $$

$$ {}_F\overset{\approx}{\sigma}{}^1_{rz} = \mu \sum_{j=1}^{2} B_j(\eta_j - k\alpha_j) \cosh \eta_j z + S_F(z) \tag{1.44} $$

$$ {}_F\tilde{\tilde{\sigma}}^o_{zz} = \sum_{j=1}^{2} B_j \left[(\lambda + 2\mu)\alpha_j\eta_j + \lambda k \right] \sinh \eta_j z + T_F(z) \tag{1.45} $$

where

$$ S_C(z) = \mu \frac{\partial}{\partial z} ({}_C\tilde{\tilde{u}}^1_r)_{pt.} - \mu k ({}_C\tilde{\tilde{u}}^o_z)_{pt.} + \mu \left[r\, {}_C\bar{u}_z C_1(k, r, a) \right]_a^b \tag{1.46} $$

$$ T_C(z) = (\lambda + 2\mu) \frac{\partial}{\partial z} ({}_C\tilde{\tilde{u}}^o_z)_{pt.} + \lambda k ({}_C\tilde{\tilde{u}}^1_r)_{pt.} + \lambda \left[r\, {}_C\bar{u}_r C_o(k, r, a) \right]_a^b \tag{1.47} $$

$$ S_F(z) = \mu \frac{\partial}{\partial z} ({}_F\tilde{\tilde{u}}^1_r)_{pt.} - \mu k ({}_F\tilde{\tilde{u}}^o_z)_{pt.} + \mu \left[r\, {}_F\bar{u}_z C_1(k, r, a) \right]_a^b \tag{1.48} $$

$$ T_F(z) = (\lambda + 2\mu) \frac{\partial}{\partial z} ({}_F\tilde{\tilde{u}}^o_z)_{pt.} + \lambda k ({}_F\tilde{\tilde{u}}^1_r)_{pt.} + \lambda \left[r\, {}_F\bar{u}_r C_o(k, r, a) \right]_a^b \tag{1.49} $$

Transforming appropriately the conditions of stress free lateral surfaces gives:

$$ {}_C\tilde{\tilde{\sigma}}^1_{zz} = 0, \quad z = \pm H \tag{1.50} $$

$$ {}_C\tilde{\tilde{\sigma}}^o_{zz} = 0, \quad z = \pm H \tag{1.51} $$

$$ {}_F\tilde{\tilde{\sigma}}^1_{rz} = 0, \quad z = \pm H \tag{1.52} $$

$$ {}_F\tilde{\tilde{\sigma}}^o_{zz} = 0, \quad z = \pm H \tag{1.53} $$

Inserting equations 1.42 and 1.43 into equations 1.50 and 1.51 gives the following pair of linear algebraic equations for the unknown A_j's;

$$ \mu \sum_{j=1}^{2} A_j(\eta_j - k\alpha_j) \sinh \eta_j H = - S_C(H) $$

$$\sum_{j=1}^{2} A_j \left[(\lambda + 2\mu)\alpha_j \eta_j + \lambda k\right] \cosh \eta_j H = - T_C(H)$$

It is shown in Appendix B that the determinant of the coefficients of A_1 and A_2 in the above equations is not zero when the parameter p is real, and so the solutions for arbitrary p may be evaluated using Cramer's rule ((15), page 42). Substituting the resulting expressions into equations 1.38 and 1.39, and using the identities

$$\eta_1 - k\alpha_1 = \frac{1}{\eta_1}(\eta_1 + k^2)$$

$$\eta_2 - k\alpha_2 = 2\eta_2$$

$$(\lambda + 2\mu)\alpha_1\eta_1 + \lambda k = - 2\mu k$$

$$(\lambda + 2\mu)\alpha_2\eta_2 + \lambda k = - \frac{\mu}{k}(\eta_1^2 + k^2)$$

gives:

$$\mu\, {}_C\tilde{\tilde{u}}_r^1 D_C(k,p) = k\left[2k\eta_1 S_C(H)\cosh \eta_1 H + (\eta_1^2+k^2)T_C(H)\sinh \eta_1 H\right]\cosh \eta_2 z$$
$$- \eta_1\left[2k\eta_2 T_C(H)\sinh \eta_2 H + (\eta_1^2+k^2)S_C(H)\cosh \eta_2 H\right]\cosh \eta_1 z$$
$$+ \mu D_C(k,p)({}_C\tilde{\tilde{u}}_r^1)_{pt.} \qquad (1.54)$$

$$\mu\, {}_C\tilde{\tilde{u}}_z^o D_C(k,p) = k\left[2k\eta_2 T_C(H)\sinh \eta_2 H + (\eta_1^2+k^2)S_C(H)\cosh \eta_2 H\right]\sinh \eta_1 z$$
$$- \eta_2\left[2k\eta_1 S_C(H)\cosh \eta_1 H + (\eta_1^2+k^2)T_C(H)\sinh \eta_1 H\right]\sinh \eta_2 z$$
$$+ \mu D_c(k,p)({}_C\tilde{\tilde{u}}_z^o)_{pt.} \qquad (1.55)$$

where

$$D_C(k,p) = (\eta_1^2 + k^2)^2 \sinh \eta_1 H \cosh \eta_2 H - 4k^2 \eta_1 \eta_2 \cosh \eta_1 H \sinh \eta_2 H \tag{1.56}$$

Similarly, inserting equations 1.44 and 1.45 into equations 1.52 and 1.53, evaluating theresulting linear algebraic equations for B_1 and B_2, and substituting their values into equations 1.40 and 1.41, gives:

$$\begin{aligned}\mu\, {}_F\tilde{\tilde{u}}_r^1\, D_F(k,p) - k\Big[(\eta_1^2 + k^2) T_F(H) \cosh \eta_1 H + 2k\eta_1 S_F(H) \sinh \eta_1 H\Big] \sinh \eta_2 z \\ - \eta_1\Big[(\eta_1^2 + k^2) S_F(H) \sinh \eta_2 H + 2k\eta_2 T_F(H) \cosh \eta_2 H\Big] \sinh \eta_1 z \\ + \mu\, D_F(k,p)({}_F\tilde{\tilde{u}}_r^1)_{pt.}\end{aligned} \tag{1.57}$$

$$\begin{aligned}\mu\, {}_F\tilde{\tilde{u}}_z^o\, D_F(k,p) = k\Big[(\eta_1^2 + k^2) S_F(H) \sinh \eta_2 H + 2k\eta_2 T_F(H) \cosh \eta_2 H\Big] \cosh \eta_1 z \\ - \eta_2\Big[(\eta_1^2 + k^2) T_F(H) \cosh \eta_1 H + 2k\eta_1 S_F(H) \sinh \eta_1 H\Big] \cosh \eta_2 z \\ + \mu\, D_F(k,p)({}_F\tilde{\tilde{u}}_z^o)_{pt.}\end{aligned} \tag{1.58}$$

where

$$D_F(k,p) - (\eta_1^2 + k^2)^2 \cosh \eta_1 H \sinh \eta_2 H - 4k^2 \eta_1 \eta_2 \cosh \eta_2 H \sinh \eta_1 H \tag{1.59}$$

Using equations 1.24, 1.55, and 1.58, the axial strains may be written:

$$\begin{aligned}\mu\, {}_C\tilde{\tilde{e}}_{zz}^o\, D_C(k,p) = \eta_1 k\Big[2k\eta_2 T_C(H) \sinh \eta_2 H + (\eta_1^2 + k^2) S_C(H) \cosh \eta_2 H\Big] \cosh \eta_1 z \\ - \eta_2^2\Big[2k\eta_1 S_C(H) \cosh \eta_1 H + (\eta_1^2 + k^2) T_C(H) \sinh \eta_1 H\Big] \cosh \eta_2 z \\ + \mu D_C(k,p)\, \frac{\partial}{\partial z}\, ({}_C\tilde{\tilde{u}}_z^o)_{pt.}\end{aligned} \tag{1.60}$$

$$\mu_F \tilde{\tilde{e}}^{o}_{zz} D_F(k,p) = \eta_1 k \Big[(\eta_1^2 + k^2) S_F(H) \sinh \eta_2 H + 2k\eta_2 T_F(H) \cosh \eta_2 H \Big] \sinh \eta_1 z$$

$$- \eta_2^2 \Big[(\eta_1^2 + k^2) T_F(H) \cosh \eta_1 H + 2k\eta_1 S_F(H) \sinh \eta_1 H \Big] \sinh \eta_2 z$$

$$+ \mu D_F(k,p) \frac{\partial}{\partial z} ({}_F\tilde{\tilde{u}}^{o}_z)_{pt.} \qquad (1.61)$$

At this stage discussion of the boundary conditions at the curved surfaces of the plate is appropriate. As mentioned before, these conditions must be such that the specification of two independent quantities reduces the boundary terms in equations 1.22, 1.23, 1.25, and 1.26, to known factors. It appears to be impossible to achieve this by any choice of nonmixed conditions, even for a fluid plate. If the kernals $C_0(k, r, b)$, $C_1(k, r, b)$ had been used, no basic change in the situation occurs. This is not a characteristic of the geometry at hand since, as shown in Section III, the same difficulty arises with slab problems. It is also a feature of the rod solutions given by Skalak (2), Folk et al. (3), and DeVault and Curtis (5).

On using a transform technique to suppress a spatial variable, which has boundaries associated with it, certain naturally occurring boundary terms of the mixed type arise. From this it would appear that boundary value problems of the nonmixed type, involving two or more perpendicular boundaries, are intractable under a multi-integral transform approach. This is further evidenced by the fact that a multi-integral transform technique is essentially a separation procedure and, as Mindlin ((16), page 2.55) has pointed out, these procedures have so far failed to yield even the modes of wave transmission for such problems (though Mindlin and Fox (17) have given a set of discrete points and associated slopes for the modes of a rectangular bar, for particular ratios of width to thickness).

Another point of interest in this connection is that dispersion through the characteristic lengths a and b will not be a feature of any solutions obtained using the above methods, including possible solutions to problems with nonmixed conditions. This is readily seen on noting that a and b appear only in the nonhomogeneous terms of the algebraic equations which determine the A's and B's, and hence will never occur in the denominators (through the zeros of which dispersion arises) of the expressions for $\tilde{u}_r^1$ and $\tilde{u}_z^o$, as given by equations 1.54, 1.55, 1.57, and 1.58. Thus if it were found from some approximate theory that solutions corresponding to nonmixed conditions were dispersive through a and b, then this would be further evidence that the above method is basically unsuited to problems of the nonmixed type. The solutions obtained by Kromm (11) and Miklowitz (12) for an infinite plate with a circular cavity are nondispersive (intrinsically), since they were obtained using the plane-stress theory. No work in this connection has been done using higher order approximate theories, such as those given by Kane and Mindlin (18), and Mindlin and Medick (19), for compressional waves, and by Mindlin (20) for flexural waves.

Sets of mixed conditions on the cylindrical surfaces of the plate which do satisfy the basic requirements will now be described. Four cases will be given, corresponding to compressional and flexural waves in finite and infinite plates.

Case (i). Compressional waves in a finite plate.

The conditions specified are:

$$
\begin{aligned}
&C\dot{u}_r = U_o f(t)\,, && r = a \\
&C\sigma_{rz} = 0, && r = a \\
&C\dot{u}_r = 0\,, && r = b \\
&C\sigma_{rz} = 0\,, && r = b
\end{aligned}
\qquad (1.62)
$$

where U_o is a constant, $f(t)$ is an arbitrary function of t, and the dot denotes differentiation w.r.t. t. The conditions at $r = a$ correspond to the plate surface being in lubricated contact with an expanding mass, whereas the conditions at $r = b$ correspond to the plate surface being in lubricated contact with a rigid layer.

Taking the Laplace transform of equation 1.62, and using equations 1.12 and 1.21, gives:

$$C\bar{u}_r = \frac{U_o}{p}\bar{f}(p)\,, \qquad r = a$$

$$\frac{\partial}{\partial z}\, C\bar{u}_r + \frac{\partial}{\partial r}\, C\bar{u}_z = 0, \qquad r = a$$

$$C\bar{u}_r = 0\,, \qquad r = b$$

$$\frac{\partial}{\partial z}\, C\bar{u}_r + \frac{\partial}{\partial r}\, C\bar{u}_z = 0, \qquad r = b$$

It follows from the first and third of these equations that

$$\frac{\partial}{\partial z}\, C\bar{u}_r = 0\,, \qquad r = a, b$$

and so an equivalent set of conditions is

$$
\left.
\begin{aligned}
&C\bar{u}_r = \frac{U_o}{p}\bar{f}(p) \\
&\frac{\partial}{\partial z}\, C\bar{u}_r - \frac{\partial}{\partial r}\, C\bar{u}_z = 0
\end{aligned}
\right\}\,, \quad r = a \qquad (1.63)
$$

$$\left.\begin{array}{l} {}_C\bar{u}_r = 0 \\[2ex] \frac{\partial}{\partial z}\ {}_C\bar{u}_r = \frac{\partial}{\partial r}\ {}_C\bar{u}_z = 0 \end{array}\right\} \quad , \quad r = b \tag{1.64}$$

Substituting these conditions into equations 1.22, 1.23, 1.29, and 1.30, and noting that $C_1(k, a, a) \equiv 0$, gives:

$${}_C\tilde{\tilde{\sigma}}^1_{rz} = \mu \frac{\partial}{\partial z}\ {}_C\tilde{\tilde{u}}^1_r - \mu k\ {}_C\tilde{\tilde{u}}^o_z + \mu \left[r\ {}_C\bar{u}_z C_1(k, r, a) \right]_b$$

$${}_C\tilde{\tilde{\sigma}}^o_{zz} = (\lambda + 2\mu) \frac{\partial}{\partial z}\ {}_C\tilde{\tilde{u}}^o_z + \lambda k\ {}_C\tilde{\tilde{u}}^1_r - \lambda a \frac{U_o}{p} \bar{f}(p)\, C_o(k, a, a)$$

$$L_C(z) = -\frac{c_d^2}{c_s^2} \left\{ \left[r \frac{\partial}{\partial r}\ {}_C\bar{u}_r + \left(1 - \frac{c_s^2}{c_d^2}\right) r \frac{\partial}{\partial z}\ {}_C\bar{u}_z \right] C_1(k, r, a) \right\}_b$$

$$- a \frac{c_d^2}{c_s^2}\, k \frac{U_o}{p}\, \bar{f}(p) C_o(k, a, a)$$

$$M_c(z) = -k \left[r\ {}_C\bar{u}_z C_1(k, r, a) \right]_b$$

These equations still contain unknown terms but these can be eliminated by choosing k to be a root of $C_1(k, b, a) = 0$, showing that the specified conditions do satisfy the basic requirements. With this choice of k, and noting that ((21), page 79)

$$C_o(k, a, a) = -\frac{2}{\pi k a}$$

the above equations reduce to:

$${}_C\tilde{\tilde{\sigma}}^1_{rz} = \mu \frac{\partial}{\partial z}\ {}_C\tilde{\tilde{u}}^1_r - \mu k\ {}_C\tilde{\tilde{u}}^o_z \tag{1.65}$$

$${}_C\tilde{\tilde{\sigma}}^o_{zz} = (\lambda + 2\mu) \frac{\partial}{\partial z}\ {}_C\tilde{\tilde{u}}^o_z + \lambda k\ {}_C\tilde{\tilde{u}}^1_r + \frac{2\lambda U_o \bar{f}(p)}{\pi k p} \tag{1.66}$$

$$L_C(z) = \frac{2c_d^2 U_o \bar{f}(p)}{\pi c_s^2 p}$$

$$M_C(z) = 0$$

Inserting these values of L_C and M_C into equations 1.33 and 1.34, and using equations 1.27 and 1.37, it is found that

$$\left({}_C\tilde{\bar{u}}_r^1\right)_{pt.} = -\frac{2U_o \bar{f}(p)}{\pi \eta_2^2 p} \tag{1.67}$$

$$\left({}_C\tilde{\bar{u}}_z^o\right)_{pt.} = 0 \tag{1.68}$$

Substituting these values into equations 1.46 and 1.47 gives:

$$S_C = 0 \tag{1.69}$$

$$T_C = \frac{2\lambda U_o \bar{f}(p)}{\pi c_d^2 \eta_2^2 k} \tag{1.70}$$

Case (ii). Compressional waves in an infinite plate.

The conditions at $r = a$ are assumed to be the same as Case (i). Instead of boundary conditions at $r = b$, the following "radiation" conditions, which stem from the hyperbolic nature of the governing partial differential equations, are assumed:

$$\lim_{r \to \infty} \left. \begin{array}{l} r\, {}_C u_r\, C_o(k, r, a) \\ r\, {}_C u_z\, C_1(k, r, a) \\ r \frac{\partial}{\partial r}\, {}_C u_r\, C_1(k, r, a) \\ r \frac{\partial}{\partial z}\, {}_C u_z\, C_1(k, r, a) \\ r \frac{\partial}{\partial z}\, {}_C u_r\, C_o(k, r, a) \\ r \frac{\partial}{\partial r}\, {}_C u_z\, C_o(k, r, a) \end{array} \right. = 0, \quad t \geq 0, \quad -H \leq z \leq H \qquad (1.71)$$

Using these relations it is seen that equations 1.67 through 1.70 are obtained again, the only difference between the two cases being in the interpretation of k, a fact which only becomes important when the spatial transforms are being inverted.

Case (iii). Flexural waves in a finite plate.

In this case the following conditions are specified:

$$\begin{array}{ll} {}_F u_r = 0, & r = a \\ {}_F \sigma_{rz} = \sigma_o Q(t), & r = a \\ {}_F u_r = 0, & r = b \\ {}_F \sigma_{rz} = 0, & r = b \end{array} \qquad (1.72)$$

where σ_o is a constant and $Q(t)$ is an arbitrary function of t. The conditions at $r = a$ correspond to a rigid insert, with rough contact, being moved perpendicular to the plane of the plate, or to a transverse ring load being applied at the plate surface (at $r = a$), with the cavity replaced by a rigid insert. The conditions at $r = b$ correspond to the plate surface being in lubricated contact with a rigid layer.

Taking the Laplace transform of these equations, and using equation 1.12, gives:

$$F\bar{u}_r = 0, \quad r = a$$

$$\frac{\partial}{\partial z} F\bar{u}_r + \frac{\partial}{\partial r} F\bar{u}_z = \frac{\sigma_o}{\mu} \bar{Q}(p), \quad r = a$$

$$F\bar{u}_r = 0, \quad r = b$$

$$\frac{\partial}{\partial z} F\bar{u}_r + \frac{\partial}{\partial r} F\bar{u}_z = 0, \quad r = b$$

From the first and third of these equations it follows that

$$\frac{\partial}{\partial z} F\bar{u}_r = 0, \quad r = a, b$$

and so an equivalent set of conditions is:

$$\left.\begin{aligned} F\bar{u}_r &= \frac{\partial}{\partial z} F\bar{u}_r = 0 \\ \frac{\partial}{\partial z} F\bar{u}_z &= \frac{\sigma_o}{\mu} \bar{Q}(p) \end{aligned}\right\} \quad r = a \qquad (1.73)$$

$$\left.\begin{aligned} F\bar{u}_r &= 0 \\ \frac{\partial}{\partial z} F\bar{u}_r &= \frac{\partial}{\partial r} F\bar{u}_z = 0 \end{aligned}\right\} \quad r = b \qquad (1.74)$$

Substituting these equations into equations 1.22, 1.23, 1.29, and 1.30, and choosing k to be a root of $C_1(k, b, a) = 0$, gives:

$$F\tilde{\tilde{\sigma}}^1_{rz} = \mu \frac{\partial}{\partial z} F\tilde{\tilde{u}}^1_r - \mu k\, F\tilde{\tilde{u}}^o_z$$

$$F\tilde{\tilde{\sigma}}^o_{zz} = (\lambda + 2\mu) \frac{\partial}{\partial z} F\tilde{\tilde{u}}^o_z + \lambda k\, F\tilde{\tilde{u}}^1_r$$

$$L_F = 0$$

$$M_F = -\frac{2c_s^2\sigma_o\overline{Q}(p)}{\pi\mu c_d^2 k}$$

Inserting these values of L_F and M_F into equations 1.33 and 1.34, and using equations 1.28 and 1.36, one obtains:

$$\left({}_F\tilde{\tilde{u}}_r^1\right)_{pt.} = 0 \tag{1.75}$$

$$\left({}_F\tilde{\tilde{u}}_z^o\right)_{pt.} = \frac{2\sigma_o\overline{Q}(p)}{\pi\mu k\eta_1^2} \tag{1.76}$$

and hence, from equations 1.48 and 1.49,

$$S_F = -\frac{2\sigma_o\overline{Q}(p)}{\pi\eta_1^2} \tag{1.77}$$

$$T_F = 0 \tag{1.78}$$

Case (iv). Flexural waves in an infinite plate.

In this case the conditions at $r = a$ are assumed the same as in Case (iii) and the radiation conditions given by equation 1.71, with ${}_Cu_r$ replaced by ${}_Fu_r$, etc., replace boundary conditions at $r = b$. Hence equations 1.75 through 1.78 are again obtained.

In all of the above cases the previously imposed symmetry requirements are satisfied by the assumed boundary conditions. Although uniformity of these conditions across the thickness of the plate has been postulated, in common with the analogous rod and slab solutions (though in the semi-infinite rod problem treated by Curtis and DeVault (5) non-uniform conditions are specified at the rod end), this restriction is unnecessary. For instance, in the compressional case, the conditions

$$\left.\begin{aligned} C^{u}{}_{r} &= V(z)f(t) \\ C^{\sigma}{}_{zz} &= 0 \end{aligned}\right\} \quad r = a$$

where $V(z)$ is an arbitrary function of z, could have been specified. These conditions are equivalent to

$$\left.\begin{aligned} C^{u}{}_{r} &= V(z)f(t) \\ \frac{\partial}{\partial z} C^{u}{}_{r} &= f(t)\frac{dV}{dz} \\ \frac{\partial}{\partial r} C^{u}{}_{r} &= -\ f(t)\frac{dV}{dz} \end{aligned}\right\} \quad r = a$$

The rest of the calculations then go through as before. A similar situation exists for the flexural case.

Mixed problems of the pressure shock type for the above geometries cannot be solved using the present technique, even though similar slab problems are tractable, as pointed out in Section III. As shown there, the slab solutions to longitudinal impact type problems are obtained by appropriately applying Fourier sine and cosine transforms to the governing equations of motion. These sine and cosine transforms can be interchanged and the resulting expressions again contain only two transformed variables, but the boundary terms are now of the pressure shock type. Thus it appears that the difficulty with the present geometry arises due to the lack of interchangeability of the zero and first order Hankel transforms, when applied to the axially symmetric equations of motion. A consequence of this is that no information about the non-mixed pressure shock problem can be obtained from the corresponding mixed problem, as opposed to the semi-infinite rod and plate equations.

It should be noted here that the equivalence of problems of transient compressional waves in an infinite flat plate of thickness 2H to those for an infinite flat plate of thickness H, lying on a rigid half-space with a lubricated interface, as established by Miklowitz (10), can also be shown to be true for the present geometries. The transformed boundary conditions for the rigid half-space problems are (taking the origin of z at the interface):

$$C\tilde{\tilde{\sigma}}^{1}_{rz} = C\tilde{\tilde{\sigma}}^{o}_{zz} = 0, \quad z = H \tag{1.79}$$

$$C\tilde{\tilde{\sigma}}^{1}_{rz} = C\tilde{\tilde{u}}^{o}_{z} = 0, \quad z = 0 \tag{1.80}$$

The boundary conditions on the cylindrical surfaces are the same as before. From Equations 1.38, 1.39, 1.65, 1.67, and 1.68, it follows that equation 1.80 is identically satisfied. From equations 1.38, 1.39, 1.65, 1.67, 1.68, and 1.79, it follows that equations 1.54 and 1.55 are obtained again and so the problems are equivalent.

With the convention on k given above, the solutions to both finite and infinite plate problems can be developed simultaneously. Substituting equations 1.67, 1.68, 1.69, and 1.70, into equations 1.54, 1.55, and 1.60, gives:

$$C\tilde{\tilde{u}}^{1}_{r} = \frac{2\lambda U_o p\bar{f}(p) F_{1C}(k, p, z)}{\pi\mu c_d^2 \eta_2^2 D_C(k, p)} - \frac{2U_o \bar{f}(p)}{\pi p \eta_2^2} \tag{1.81}$$

$$C\tilde{\tilde{u}}^{o}_{z} = \frac{2\lambda U_o p\bar{f}(p) F_{2C}(k, p, z)}{\pi\mu c_d^2 \eta_2^2 D_C(k, p)} \tag{1.82}$$

$$C\tilde{\tilde{e}}^{o}_{zz} = \frac{2\lambda U_o p\bar{f}(p) F_{3C}(k, p, z)}{\pi\mu c_d^2 \eta_2^2 D_C(k, p)} \tag{1.83}$$

where

$$F_{1C}(k,p,z) = (\eta_1^2 + k^2)\sinh \eta_1 H \cosh \eta_2 z - 2\eta_1\eta_2 \sinh \eta_2 H \cosh \eta_1 z \tag{1.84}$$

$$F_{2C}(k,p,z) = 2k\eta_2 \sinh \eta_2 H \sinh \eta_1 z - \frac{\eta_2}{k}(\eta_1^2 + k^2)\sinh \eta_1 H \sinh \eta_2 z \tag{1.85}$$

$$F_{3C}(k,p,z) = 2k\eta_1\eta_2 \sinh \eta_2 H \cosh \eta_1 z - \frac{\eta_2^2}{k}(\eta_1^2 + k^2)\sinh \eta_1 H \cosh \eta_2 z \tag{1.86}$$

Substituting equations 1.75, 1.76, 1.77, and 1.78, into equations 1.57, 1.58, and 1.61, gives:

$$F\tilde{\tilde{u}}_r^1 = -\frac{2\sigma_o k\overline{Q}(p)F_{1F}(k,p,z)}{\pi\mu\eta_1^2 D_F(k,p)} \tag{1.87}$$

$$F\tilde{\tilde{u}}_z^o = -\frac{2\sigma_o k\overline{Q}(p)F_{2F}(k,p,z)}{\pi\mu\eta_1^2 D_F(k,p)} + \frac{2\sigma_o\overline{Q}(p)}{\pi\mu k\eta_1^2} \tag{1.88}$$

$$F\tilde{\tilde{e}}_{zz}^o = -\frac{2\sigma_o k\overline{Q}(p)F_{3F}(k,p,z)}{\pi\mu\eta_1^2 D_F(k,p)} \tag{1.89}$$

where

$$F_{1F}(k,p,z) = 2k\eta_1 \sinh \eta_1 H \sinh \eta_2 z - \frac{\eta_1}{k}(\eta_1^2 + k^2)\sinh \eta_2 H \sinh \eta_1 z \tag{1.90}$$

$$F_{2F}(k,p,z) = (\eta_1^2 + k^2)\sinh \eta_2 H \cosh \eta_1 z - 2\eta_1\eta_2 \sinh \eta_1 H \cosh \eta_2 z \tag{1.91}$$

$$F_{3F}(k,p,z) = \eta_1(\eta_1^2 + k^2)\sinh \eta_2 H \sinh \eta_1 z - 2\eta_1\eta_2^2 \sinh \eta_1 H \sinh \eta_2 z \tag{1.92}$$

On using the inversion formula for the Laplace transform

(McLachlan (22)), and the inversion formulas for the Hankel transforms given by equations Al. 7 and Al. 10 (with $n = 1$, $\alpha = a$, $\beta = b$), A2. 9 and A2. 14, then the inverses corresponding to equations 1. 81, 1. 82, 1. 83, 1. 87, 1. 88, and 1. 89, may be written as follows, on noting that here, in the inversion of the spatial transforms, the precise nature of k must be taken into account:

$$C^u{}_r = \pi U_o \sum_j \frac{k_j^2 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)} \times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{\lambda p F_{1C}(k_j, p, z)}{\mu c_d^2 \eta_2^2(k_j) D_C(k_j, p)} - \frac{1}{p \eta_2^2(k_j)} \right] \overline{f}(p) e^{pt} \, dp \right\} \tag{1.93}$$

$$\frac{\partial}{\partial r} C^u{}_z = -\frac{\pi U_o \lambda}{\mu c_d^2} \sum_j \frac{k_j^3 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)} \times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{p F_{2C}(k_j, p, z)}{\eta_2^2(k_j) D_C(k_j, p)} \right] \overline{f}(p) e^{pt} \, dp \right\} \tag{1.94}$$

$$\frac{\partial}{\partial r} C^e{}_{zz} = -\frac{\pi \lambda U_o}{\mu c_d^2} \sum_j \frac{k_j^3 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)} \times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{p F_{3C}(k_j, p, z)}{\eta_2^2(k_j) D_C(k_j, p)} \right] \overline{f}(p) e^{pt} \, dp \right\} \tag{1.95}$$

$$F^u{}_r = -\frac{\pi \sigma_o}{\mu} \sum_j \frac{k_j^3 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)} \times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{F_{1F}(k_j, p, z)}{\eta_1^2(k_j) D_F(k_j, p)} \right] \overline{Q}(p) e^{pt} \, dp \right\} \tag{1.96}$$

$$\frac{\partial}{\partial r} F^{u}{}_{z} = \frac{\pi\sigma_o}{\mu} \sum_j \frac{k_j^3 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{k_j F_{2F}(k_j, p, z)}{\eta_1^2(k_j) D_F(k_j, p)} - \frac{1}{k_j \eta_1^2(k_j)} \right] \overline{Q}(p) e^{pt} \, dp \right\} \tag{1.97}$$

$$\frac{\partial}{\partial r} F^{e}{}_{zz} = \frac{\pi\sigma_o}{\mu} \sum_j \frac{k_j^4 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{F_{3F}(k_j, p, z)}{\eta_1^2(k_j) D_F(k_j, p)} \right] \overline{Q}(p) e^{pt} \, dp \right\} \tag{1.98}$$

where Br_1 denotes the well-known Bromwich contour in the right half of the p-plane, and it is to be noted that possible difficulty with the constant term in equation A1.7 has been avoided by inverting $\partial u_z/\partial r$ and $\partial e_{zz}/\partial r$ instead of u_z and e_{zz}.

Infinite plate.

$$C^{u}{}_{r} = \frac{2U_o}{\pi} \int_o^{\infty} \frac{k C_1(k, r, a)}{J_1^2(ka) + Y_1^2(ka)}$$

$$\times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{\lambda p F_{1C}(k, p, z)}{\mu c_d^2 \eta_2^2 D_c(k, p)} - \frac{1}{p \eta_2^2} \right] \overline{f}(p) e^{pt} \, dp \right\} dk \tag{1.99}$$

$$C^{u}{}_{z} = \frac{2\lambda \mathbf{U}_o}{\pi \mu c_d^2} \int_o^{\infty} \frac{k C_o(k, r, a)}{J_1^2(ka) + Y_1^2(ka)}$$

$$\times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{p F_{2C}(k, p, z)}{\eta_2^2 D_c(k, p)} \right] \overline{f}(p) e^{pt} \, dp \right\} dk \tag{1.100}$$

$$C^{e}{}_{zz} = \frac{2\lambda U_o}{\pi\mu c_d^2}\int_o^{\infty} \frac{kC_o(k, r, a)}{J_1^2(ka) + Y_1^2(ka)}$$

$$\times \left\{ \frac{1}{2\pi i}\int_{Br_1}\left[\frac{pF_{3C}(k, p, z)}{\eta_2^2 D_C(k, p)} \right] \bar{f}(p) e^{pt}\, dp \right\} dk \qquad (1.101)$$

$$F^{u}{}_{r} = -\frac{2\sigma_o}{\pi\mu}\int_o^{\infty} \frac{k^2 C_1(k, r, a)}{J_1^2(ka) + Y_1^2(ka)}$$

$$\times \left\{ \frac{1}{2\pi i}\int_{Br_1}\left[\frac{F_{1F}(k, p, z)}{\eta_1^2 D_F(k, p)} \right] \bar{Q}(p) e^{pt}\, dp \right\} dk \qquad (1.102)$$

$$F^{u}{}_{z} = -\frac{2\sigma_o}{\pi\mu}\int_o^{\infty} \frac{kC_o(k, r, a)}{J_1^2(ka) + Y_1^2(ka)}$$

$$\times \left\{ \frac{1}{2\pi i}\int_{Br_1}\left[\frac{kF_{2F}(k, p, z)}{\eta_1^2 D_F(k, p)} - \frac{1}{k\eta_1^2} \right] \bar{Q}(p) e^{pt}\, dp \right\} dk$$

$$(1.103)$$

$$F^{e}{}_{zz} = -\frac{2\sigma_o}{\pi\mu}\int_o^{\infty} \frac{k^2 C_o(k, r, a)}{J_1^2(ka) + Y_1^2(ka)}$$

$$\times \left\{ \frac{1}{2\pi i}\int_{Br_1}\left[\frac{F_{3F}(k, p, z)}{\eta_1^2 D_F(k, p)} \right] \bar{Q}(p) e^{pt}\, dp \right\} dk \qquad (1.104)$$

It has already been noted that transformed expressions for the radial strain and stress cannot be written. However expressions for these quantities can be obtained on differentiating through the integrals, or series, as the case may be, in equations 1.93, 1.96, 1.99, and 1.102, or, as done by Eason _et al._ (23), using equation 1.1 and integrating through the integrals. Expressions for the strains can be of importance, since many of the experimental devices in current use measure strains, or the sum of the strains. Formally differentiating through the summation and integral signs, the following expressions are obtained for the radial strain:

Finite plate.

$$C^{e}{}_{rr} = \pi U_o \sum_j \frac{k_j^2 J_1^2(k_j b) \frac{\partial}{\partial r} C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)} \times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{\lambda p F_{1C}(k_j, p, z)}{\mu c_d^2 \eta_2^2(k_j) D_c(k_j, p)} - \frac{1}{p\eta_2^2(k_j)} \right] \overline{f}(p) e^{pt} \, dp \right\} \tag{1.105}$$

$$F^{e}{}_{rr} = -\frac{\pi\sigma_o}{\mu} \sum_j \frac{k_j^3 J_1^2(k_j b) \frac{\partial}{\partial r} C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)} \times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{F_{1F}(k_j, p, z)}{\eta_1^2(k_j) D_F(k_j, p)} \right] \overline{Q}(p) e^{pt} \, dp \right\} \tag{1.106}$$

Infinite plate.

$$C^{e}{}_{rr} = \frac{2U_o}{\pi} \int_o^{\infty} \frac{k \frac{\partial}{\partial r} C_1(k, r, a)}{J_1^2(ka) + Y_1^2(ka)} \times \left\{ \frac{1}{2\pi i} \int_{Br_1} \left[\frac{\lambda p F_{1C}(k, p, z)}{\mu c_d^2 \eta_2^2 D_C(k, p)} - \frac{1}{p\eta_2^2} \right] \overline{f}(p) e^{pt} \, dp \right\} dk \tag{1.107}$$

$$F^e{}_{rr} = -\frac{2\sigma_o}{\pi\mu}\int_o^\infty \frac{k^2 \frac{\partial}{\partial r} C_1(k, r, a)}{J_1^2(ka) + Y_1^2(ka)}$$

$$\times \left\{ \frac{1}{2\pi i}\int_{Br_1}\left[\frac{F_{1F}(k, p, z)}{\eta_1^2 D_F(k, p)} \right]\overline{Q}(p)e^{pt}\, dp \right\} dk \qquad (1.108)$$

1.2. EXACT INVERSIONS

Reduction of the solutions to algebraic expressions can be achieved by inverting the Laplace transforms. Since the Laplace inversion process does not hinge on the nature of k, it can be done for both finite and infinite plate problems simultaneously. Use of the convolution theorem is planned and so the integrands are separated into two parts, viz., $\overline{f}(p)$ and $\overline{Q}(p)$ and their multipliers (the term in the square brackets in equations 1.93 through 1.108), and inversion of these parts is undertaken separately.

Provided certain conditions are met, the Bromwich integrals yield solutions which are identically zero for times less than the arrival times of the wave fronts, and, for later times, representations of these Bromwich integrals can be obtained in forms more readily evaluated. These conditions are:

(i) $\lim_{|p|\to\infty}$ (integrands) $= O(p^{-\epsilon})$, $\epsilon > 0$, uniformly in arg p and z.

(ii) Singularities of the integrands must lie to the left of the Bromwich contour.

Henceforth it is assumed that both $\overline{f}(p)$ and $\overline{Q}(p)$ satisfy these conditions, so that they invert to f(t) and Q(t) respectively. On expanding the multipliers of $\overline{f}(p)$ and $\overline{Q}(p)$ for large $|p|$, the order condition (i) is found to hold in all cases. Inspection of the integrands shows that possible singularities are branch points at $p = \pm ikc_d, \pm ikc_s$ (the zeros of η_1 and η_2) and poles at those values of p which are zeros of the denominators, i.e., at the zeros of η_1, η_2, $D_C(k,p)$, $D_F(k,p)$. The point $p = 0$ is also examined, for the two-fold purpose of determining whether it is singular or not, and to obtain the long-time, or static, solution.

Since the integrands are even functions of η_1 and η_2, no branch points arise in the present problems. For the compressional case, drawing upon the related rigid half-space problem, this is in agreement with Jardetsky's findings ((24), page 244) that in an n-layered half-space all the branch line integrals vanish except those corresponding to the nth layer, i.e., to the underlying half-space, which in this case, being rigid, does not contribute to the solutions. The function $[\overline{f}(p)]^{-1}\, {}_C\tilde{\tilde{u}}_r^1 e^{pt}$ has a simple pole at $p = 0$ and its residue there is $2U_o/\pi k^2$. The other functions are well behaved at this point. Expansions in the vicinity of the zeros of η_1 and η_2 show that no contributions to the solutions arise from these points. It is shown in Appendix B that, for real k, the zeros of $D_C(k,p)$ and $D_F(k,p)$ are complex conjugate, pure imaginary and simple. Exceptions to the proof given there were the points $p = 0$, $p = \infty$, and the zeros of η_1 and η_2. Note that all these exceptions have been discussed above.

The nature of the zeros of $D_C(k,p)$ and $D_F(k,p)$ has usually been deduced using the physical arguments that no multiple zeros, or zeros with positive real parts, are permissible, since the corresponding portions of the solutions increase with time; zeros with negative real parts are excluded on the grounds that the basic physical model has no dissipative mechanism. Denoting the zeros of $D_C(k, p)$ and $D_F(k, p)$ by $p = \pm i\omega_{nC}(k)$, $p = \pm i\omega_{nF}(k)$, respectively, and substituting these expressions into equations 1.56 and 1.59 set equal to zero, gives:

$$\frac{\tanh[k^2 - \frac{1}{c_s^2}\omega_{nC}^2(k)]^{\frac{1}{2}}H}{\tanh[k^2 - \frac{1}{c_d^2}\omega_{nC}^2(k)]^{\frac{1}{2}}H} = \frac{4k^2[k^2 - \frac{1}{c_s^2}\omega_{nC}^2(k)^2]^{\frac{1}{2}}[k^2 - \frac{1}{c_d^2}\omega_{nC}^2(k)]^{\frac{1}{2}}}{[2k^2 - \frac{1}{c_s^2}\omega_{nC}^2(k)]^2} \tag{1.109}$$

and

$$\frac{\tanh[k^2 - \frac{1}{c_s^2}\omega_{nF}^2(k)]^{\frac{1}{2}}H}{\tanh[k^2 - \frac{1}{c_d^2}\omega_{nF}^2(k)]^{\frac{1}{2}}H} = \frac{[2k^2 - \frac{1}{c_s^2}\omega_{nF}^2(k)]^2}{4k^2[k^2 - \frac{1}{c_s^2}\omega_{nF}^2(k)]^{\frac{1}{2}}[k^2 - \frac{1}{c_d^2}\omega_{nF}^2(k)]^{\frac{1}{2}}} \tag{1.110}$$

which are the well-known Rayleigh-Lamb frequency equations for symmetric (equation 1.109) and antisymmetric (equation 1.110) straight-crested waves in an infinite flat plate of thickness 2H ((24), page 283).

These equations have recently been studied in great detail by Holden (25), Mindlin and Onoe (26), Tolstoy and Usdin (27), Sherwood (28), and others. These studies have shown that there are an infinite number of $\omega_{nC}(k)$ and $\omega_{nF}(k)$ versus k branches, or modes of wave transmission. A few of the symmetric modes are shown in Figure 2. They have also shown that there are real ω's associated with complex k's, but these need be of no concern in the present problems, since the Hankel inversion formulas restrict k to be real. However it should

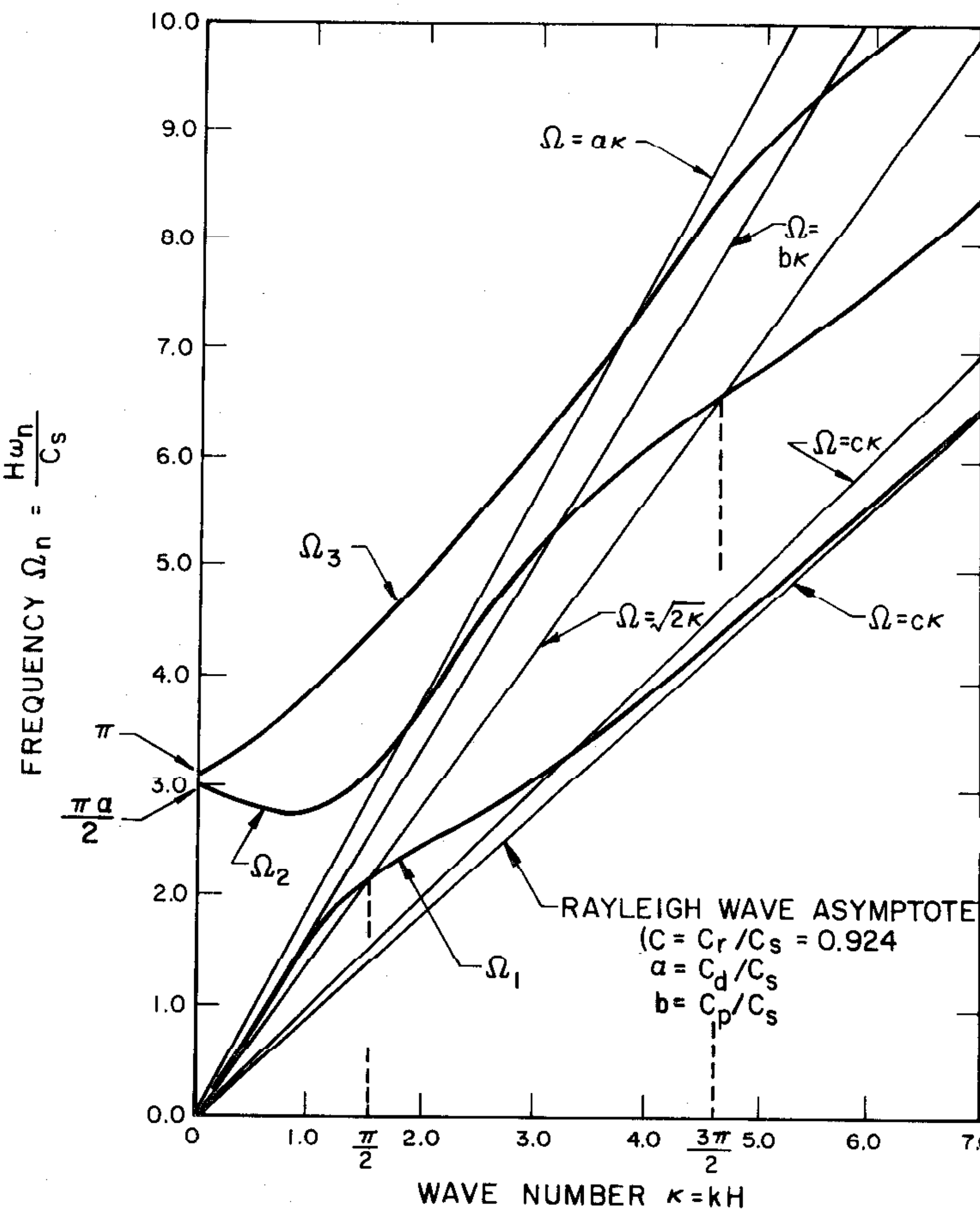

Fig. 2. Symmetric wave frequency spectrum ($\sigma = .31$).

be pointed out that in the scheme of inversion where the spatial transforms are inverted first, given by Lloyd and Miklowitz (29) and others, use may be made of these complex branches to afford different representations of the solutions.

The residues corresponding to the zeros of $D_C(k,p)$ and $D_F(k,p)$ are now evaluated. Then, using Cauchy's theorem and residue theory, the terms in the square brackets in equations 1.93 through 1.104 are inverted, the results being in the form of infinite series. Using these results and the convolution theorem of the Laplace transform, equations 1.93 through 1.108 may be written:

Finite plate.

$$C^{u}{}_{r} = \pi U_o \sum_j \frac{k_j^2 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \int_{t_w}^{t} f(t-\xi) \left[\frac{2\lambda}{\mu} \Phi_{1C}(k_j, \xi, z) - \frac{1}{k_j^2} \right] d\xi \right\} \qquad (1.111)$$

$$\frac{\partial}{\partial r} C^{u}{}_{z} = - \frac{2\pi\lambda U_o}{\mu} \sum_j \frac{k_j^3 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \int_{t_w}^{t} f(t-\xi) \Phi_{2C}(k_j, \xi, z) \, d\xi \right\} \qquad (1.112)$$

$$\frac{\partial}{\partial r} C^{e}{}_{zz} = - \frac{2\pi\lambda U_o}{\mu} \sum_j \frac{k_j^3 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \int_{t_w}^{t} f(t-\xi) \Phi_{3C}(k_j, \xi, z) \, d\xi \right\} \qquad (1.113)$$

$$C^e{}_{rr} = \pi U_o \sum_j \frac{k_j^2 J_1^2(k_j b)\ \frac{\partial}{\partial r}\ C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \int_{t_w}^{t} f(t-\xi) \left[\frac{2\lambda}{\mu}\ \Phi_{1C}(k_j, \xi, z) - \frac{1}{k_j^2} \right] d\xi \right\} \qquad (1.114)$$

$$F^u{}_r = - \frac{2\pi c_s^2 \sigma_o}{\mu} \sum_j \frac{k_j^3 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \int_{t_w}^{t} Q(t-\xi) \Theta_{1F}(k_j, \xi, z)\ d\xi \right\} \qquad (1.115)$$

$$\frac{\partial}{\partial r}\ F^u{}_z = \frac{2\pi c_s^2 \sigma_o}{\mu} \sum_j \frac{k_j^4 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \int_{t_w}^{t} Q(t-\xi) \Theta_{2F}(k_j, \xi, z)\ d\xi \right\} \qquad (1.116)$$

$$\frac{\partial}{\partial r}\ F^e{}_{zz} = \frac{2\pi c_s^2 \sigma_o}{\mu} \sum_j \frac{k_j^4 J_1^2(k_j b) C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \int_{t_w}^{t} Q(t-\xi) \Theta_{3F}(k_j, \xi, z)\ d\xi \right\} \qquad (1.117)$$

$$F^e{}_{rr} = - \frac{2\pi c_s^2 \sigma_o}{\mu} \sum_j \frac{k_j^3 J_1^2(k_j b)\ \frac{\partial}{\partial r}\ C_1(k_j, r, a)}{J_1^2(k_j a) - J_1^2(k_j b)}$$

$$\times \left\{ \int_{t_w}^{t} Q(t-\xi) \Theta_{1F}(k_j, \xi, z)\ d\xi \right\} \qquad (1.118)$$

where

$$\Phi_{mC}(k,t,z) = \sum_{n=1}^{\infty} \frac{F_{mC}[k, i\omega_{nC}(k), z] \ \cos \omega_{nC}(k)t}{[k^2 c_d^2 - \omega_{nC}^2(k)]\, N_C[k, i\omega_{nC}(k)]} \quad , \ m = 1, 2, 3 \qquad (1.119)$$

$$\Theta_{mF}(k,t,z) = \sum_{n=1}^{\infty} \frac{F_{mF}[k, i\omega_{nF}(k)z] \ \sin \omega_{nF}(k)t}{[k^2 c_s^2 - \omega_{nF}^2(k)]\, \omega_{nF}(k) N_F[k, i\omega_{nF}(k)]} \quad , \ m = 1, 2, 3$$

(1.120)

$$N_C[k, i\omega_{nC}(k)] = \left(\frac{1}{p} \frac{\partial D_C}{\partial p} \right)_{p=i\omega_{nC}(k)}$$

$$= H \left[\frac{\left(\gamma_{sn}^C\right)^2}{c_s^2 \delta_{sn}^C} - \frac{4k^2 \delta_{sn}^C}{c_d^2} \right] \cosh \delta_{sn}^C H \cosh \delta_{dn}^C H$$

$$+ H \left[\frac{\left(\gamma_{sn}^C\right)^2}{c_d^2 \delta_{dn}^C} - \frac{4k^2 \delta_{dn}^C}{c_s^2} \right] \sinh \delta_{sn}^C H \sinh \delta_{dn}^C H$$

$$- 4k^2 \left[\frac{\delta_{dn}^C}{c_s^2 \delta_{sn}^C} + \frac{\delta_{sn}^C}{c_d^2 \delta_{dn}^C} \right] \cosh \delta_{sn}^C H \sinh \delta_{dn}^C H$$

$$+ \frac{4}{c_s^2} \gamma_{sn}^C \cosh \delta_{dn}^C H \sinh \delta_{sn}^C H \qquad (1.121)$$

$$N_F[k, i\omega_{nF}(k)] = \left(\frac{1}{p}\ \frac{\partial D_F}{\partial p}\right)_{p=i\omega_{nF}(k)}$$

$$= H\left[\frac{\left(\gamma_{sn}^F\right)^2}{c_s^2\delta_{sn}^F} - \frac{4k^2\delta_{sn}^F}{c_d^2}\right]\sinh\delta_{dn}^F H \sinh\delta_{sn}^F H$$

$$+ H\left[\frac{\left(\gamma_{sn}^F\right)^2}{c_d^2\delta_{dn}^F} - \frac{4k^2\delta_{dn}^F}{c_s^2}\right]\cosh\delta_{dn}^F H \cosh\delta_{sn}^F H$$

$$- 4k^2\left[\frac{\delta_{dn}^F}{c_s^2\delta_{sn}^F} + \frac{\delta_{sn}^F}{c_d^2\delta_{dn}^F}\right]\cosh\delta_{dn}^F H \sinh\delta_{sn}^F H$$

$$+ \frac{4}{c_s^2}\gamma_{sn}^F \cosh\delta_{sn}^F H \sinh\delta_{dn}^F H \qquad (1.122)$$

$$\gamma_{sn} = [2k^2 - \frac{1}{c_s^2}\omega_n^2(k)] \qquad (1.123)$$

$$\delta_{sn} = [k^2 - \frac{1}{c_s^2}\omega_n^2(k)]^{\frac{1}{2}} \qquad (1.124)$$

$$\delta_{dn} = [k^2 - \frac{1}{c_d^2}\omega_n^2(k)]^{\frac{1}{2}} \qquad (1.125)$$

and t_w denotes the arrival time of the wave front at the station (r, z).

Infinite plate.

$$C^{u_r} = \frac{2U_o}{\pi}\int_o^\infty \frac{kC_1(k, r, a)}{J_1^2(ka) + Y_1^2(ka)}\left\{\int_{t_w}^t f(t-\xi)\left[\frac{2\lambda}{\mu}\Phi_{1C}(k, \xi, z) - \frac{1}{k^2}\right]d\xi\right\}dk \qquad (1.126)$$

$$C^{u_z} = \frac{4\lambda U_o}{\pi\mu}\int_o^\infty \frac{kC_o(k, r, a)}{J_1^2(ka) + Y_1^2(ka)}\left\{\int_{t_w}^t f(t-\xi)\Phi_{2C}(k, \xi, z)\,d\xi\right\}dk \qquad (1.127)$$

$$C^e{}_{zz} = \frac{4\lambda U_o}{\pi\mu}\int_o^\infty \frac{kC_o(k,r,a)}{J_1^2(ka)+Y_1^2(ka)}\left\{\int_{t_w}^t f(t-\xi)\Phi_{3C}(k,\xi,z)\,d\xi\right\}dk \qquad (1.128)$$

$$C^e{}_{rr} = \frac{2U_o}{\pi}\int_o^\infty \frac{k\frac{\partial}{\partial r}C_1(k,r,a)}{J_1^2(ka)+Y_1^2(ka)}\left\{\int_{t_w}^t f(t-\xi)\left[\frac{2\lambda}{\mu}\Phi_{1C}(k,\xi,z)-\frac{1}{k^2}\right]d\xi\right\}dk \qquad (1.129)$$

$$F^u{}_r = -\frac{4c_s^2\sigma_o}{\pi\mu}\int_o^\infty \frac{kC_1(k,r,a)}{J_1^2(ka)+Y_1^2(ka)}\left\{\int_{t_w}^t Q(t-\xi)\Theta_{1F}(k,\xi,z)\,d\xi\right\}dk \qquad (1.130)$$

$$F^u{}_z = -\frac{4c_s^2\sigma_o}{\pi\mu}\int_o^\infty \frac{kC_o(k,r,a)}{J_1^2(ka)+Y_1^2(ka)}\left\{\int_{t_w}^t Q(t-\xi)\Theta_{2F}(k,\xi,z)\,d\xi\right\}dk \qquad (1.131)$$

$$F^e{}_{zz} = -\frac{4c_s^2\sigma_o}{\pi\mu}\int_o^\infty \frac{kC_o(k,r,a)}{J_1^2(ka)+Y_1^2(ka)}\left\{\int_{t_w}^t Q(t-\xi)\Theta_{3F}(k,\xi,z)\,d\xi\right\}dk \qquad (1.132)$$

$$F^e{}_{rr} = -\frac{4c_s^2\sigma_o}{\pi\mu}\int_o^\infty \frac{k\frac{\partial}{\partial r}C_1(k,r,a)}{J_1^2(ka)+Y_1^2(ka)}\left\{\int_{t_w}^t Q(t-\xi)\Theta_{1F}(k,\xi,z)\,d\xi\right\}dk \qquad (1.133)$$

The above expressions are quite general but they can be made more general still by working with certain classes of anisotropic media which are transversely isotropic, the axis of symmetry being perpendicular to the plane of the plate. Morse (30), Eason (31), and Anderson (32), have recently worked with such media, and some slab problems involving them are discussed in Section III. The transverse isotropy indicates that axially symmetric motions are possible; it can readily be shown that the Hankel transform pairs introduced above are applicable to the equations of motion of such materials.

Further generalizations can be obtained by including layered media of the type considered in Appendix B, and linear viscoelastic solids can be treated on letting λ and μ be certain linear operators involving partial derivatives w.r.t. t, as illustrated recently by Miklowitz (33). However generalizations obtained by relaxing the conditions of axial symmetry are not readily forthcoming in that appropriate Hankel transform pairs do not appear to exist, even for plates without the cylindrical boundaries. The trouble lies in the fact that the separation solutions of the elastic equations of motion in cylindrical coordinates involve Bessel functions and also their derivatives, when there is a θ dependence (see, for example, Reference 5). This fact would have to be reflected in the kernals of any possible transform pairs. A possible approach in such cases would be to suppress the z variable instead of the r variable, by means of appropriate finite Fourier sine and cosine transforms, thereby having to work with mixed conditions on the flat surfaces of the plate.

Other restrictions on the generality of the solutions occur because of the given form. It is known that the form of solution given above is convenient only when far-field information is required. If near-field data, or data on high frequency narrow-bandwidth pulses, is sought, then, in the former case, ray theory and Cagniards inversion technique give more suitable representations (for restricted times), as shown recently by Rosenfeld and Miklowitz (34), whereas, in the latter case, the methods given by Redwood ((35), Chapter 9) would be more readily applicable.

Attention will now be restricted to a particular problem in compressional wave propagation in an infinite plate. If the longitudinal impact problem is chosen, i.e., if $f(t)$ in equation 1.62 is chosen to be the

Heaviside step function, then it is found from the resulting solutions that the radial displacement and strain grow linearly with time. This result differs from Skalak's rod solution (2), in which the axial displacement grows with time, but the axial strain remains bounded. A physically reasonable problem, with somewhat simpler algebra, is obtained on taking $f(t)$ to be the Dirac delta function, $\delta(t)$, which is equivalent to specifying a step normal displacement at the inner cylindrical surface of the plate.

Substituting $f(t) = \delta(t)$ in equations 1.126 and 1.127, interchanging summation and integration, and using the result ((36), page 352)

$$\int_0^{\infty} \frac{C_n(k, r, a)}{k[J_n^2(ka) + Y_n^2(ka)]} \, dk = -\frac{\pi a^n}{2r^n}$$

one obtains:

$$\frac{{}_C u_r}{U_o} = \frac{a}{r} + \frac{4\lambda}{\pi\mu} \sum_{n=1}^{\infty} \int_0^{\infty} kC_1(k, r, a)[W_n(k)]^{-1} F_{1C}[k, i\omega_{nC}(k), z] \cos \omega_{nC}(k)t \, dk \tag{1.134}$$

$$\frac{{}_C u_z}{U_o} = \frac{4\lambda}{\pi\mu} \sum_{n=1}^{\infty} \int_0^{\infty} kC_o(k, r, a)[W_n(k)]^{-1} F_{2C}[k, i\omega_{nC}(k), z] \cos \omega_{nC}(k)t \, dk \tag{1.135}$$

where

$$W_n(k) = [J_1^2(ka) + Y_1^2(ka)][k^2 c_d^2 - \omega_{nC}^2(k)] N[k, i\omega_{nC}(k)] \tag{1.135a}$$

and where it is to be understood that ${}_C u_r$ and ${}_C u_z$ are zero for times less than the arrival time of the wave front. Note that the first term on the right hand side of equation 1.134 stems from the evaluation of the residue at $p = 0$, and so is the long-time solution.

A discussion of the possible singularities of the integrands in equations 1.134 and 1.135 is given in Appendix C; it is shown there that the integrands are well behaved throughout the region of integration. It is also shown (formally) there that ${}_C u_r$ and ${}_C u_z$, as given by these equations, satisfy the differential equations and the boundary and initial conditions.

1.3. FAR-FIELD APPROXIMATIONS

In general computation directly from equations 1.134 and 1.135 presents many formidable numerical difficulties, particularly if near-field information is required. For inputs with an arbitrary time-dependence, many propagation modes are excited, but, as yet, no theory exists which predicts the relative strengths of the various modes. Even if the input is such that only one mode is excited, the numerical problem involved in calculating the Fourier Bessel type integrals in the solutions can still be quite complicated. For the near-field, approximations to the integrals are not very successful, because of the difficulty in giving a physical interpretation to the results. In fact, as was mentioned earlier, the form of solution given here is not a convenient representation when data close to the source is required. However, for the far-field, the disturbance may be thought of as having separated into an aggregate of wave groups, and then group velocity ideas can be employed in physically interpreting approximations to the integrals.

Prior to approximating the integrals, the solutions will be written in terms of certain dimensionless variables, so as to utilize some recent similar work of Miklowitz (10) on a related problem. Also, the affix C will henceforth be deleted, since the remaining work in this section is solely concerned with equations 1.134 and 1.135. Letting $r = \rho H$, $z = \zeta H$, $\omega_s = c_s/H$, $\tau = \omega_s t$, $\alpha^2 = c_d^2/c_s^2$, $R = a/H$, $k = K/H$ and $\omega_n(k) = \omega_s \Omega_n(K)$, these equations may be written*:

$$\frac{\pi(1-2\sigma)u_\rho}{8\sigma U_o} = \frac{\pi(1-2\sigma)}{8\sigma}\frac{R}{\rho} + \sum_{n=1}^{\infty}\int_o^{\infty} C_1(K,\rho,R)[W_n(K)]^{-1}F_1[K, i\Omega_n(K), \zeta] \cos \Omega_n(K)\tau \, dK \tag{1.136}$$

$$\frac{\pi(1-2\sigma)u_\zeta}{8\sigma U_o} = \sum_{n=1}^{\infty}\int_o^{\infty} C_o(K,\rho,R)[W_n(K)]^{-1}F_2[K, i\Omega_n(K), \zeta] \cos \Omega_n(K)\tau \, dK \tag{1.137}$$

where

σ = Poisson's ratio

$$C_1(K,\rho,R) = J_1(K\rho)Y_1(KR) - J_1(KR)Y_1(K\rho) \tag{1.138}$$

$$C_o(K,\rho,R) = J_o(K\rho)Y_1(KR) - J_1(KR)Y_o(K\rho) \tag{1.139}$$

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*The notation has been chosen to coincide as far as possible with that used by Miklowitz. Note however that, because of a difference in choice of origins, ζ here is equivalent to $(1-\zeta)$ in his work.

$$F_1[K, i\Omega_n(K), \zeta] = \psi_n \sin Kk_{sn} \cosh Kk_{dn}\zeta - 2k_{sn}k_{dn} \sinh Kk_{dn} \cos Kk_{sn}\zeta \tag{1.140}$$

$$F_2[K, i\Omega_n(K), \zeta] = 2k_{dn} \sinh Kk_{dn} \sin Kk_{sn}\zeta - k_{dn}\psi_n \sin Kk_{sn} \sinh Kk_{dn}\zeta \tag{1.141}$$

$$W_n(K) = [J_1^2(KR) + Y_1^2(KR)][K^2a^2 - \Omega_n^2(K)]\, N[K, i\Omega_n(K)] \tag{1.141a}$$

$$N[K, i\Omega_n(K), \zeta] = \left[\frac{\psi_n^2}{a^2k_{dn}} - 4k_{dn}\right] \sin Kk_{sn} \sinh Kk_{dn}$$
$$+ \frac{4\psi_n}{K} \cosh Kk_{dn} \sin Kk_{sn} - \left[\frac{\psi_n^2}{k_{sn}} + \frac{4k_{sn}}{a^2}\right] \cos Kk_{sn} \cosh Kk_{dn}$$
$$- \frac{4}{K}\left(\frac{k_{sn}}{a^2k_{dn}} - \frac{k_{dn}}{k_{sn}}\right) \cos Kk_{sn} \sinh Kk_{dn} \tag{1.142}$$

$$\psi_n = 2 - \frac{\Omega_n^2(K)}{K^2} \tag{1.143}$$

$$k_{sn} = \left[\frac{\Omega_n^2(K)}{K^2} - 1\right]^{1/2} \tag{1.144}$$

$$k_{dn} = \left[1 - \frac{\Omega_n^2(K)}{a^2K^2}\right]^{1/2} \tag{1.145}$$

The following asymptotic representations of equations 1.136 and 1.137 for large ρ are obtained on replacing the Bessel functions containing ρ by the leading terms in their large-argument asymptotic expansions ((21), page 85):

$$\frac{\pi(1-2\sigma)u_\rho}{8\sigma U_o} = \frac{\pi(1-2\sigma)}{8\sigma}\frac{R}{\rho}$$

$$+\left(\frac{2}{\pi\rho}\right)^{1/2}\sum_{n=1}^{\infty}\int_o^{\infty}[Y_1(KR)\cos(K\rho-\frac{3\pi}{4}) - J_1(KR)\sin(K\rho-\frac{3\pi}{4})][\sqrt{K}\,W_n(K)]^{-1}$$

$$\times F_1[K, i\Omega_n(K),\zeta]\cos\Omega_n(K)\tau\, dK + O(\rho^{-3/2}) \qquad (1.146)$$

$$\frac{\pi(1-2\sigma)u}{8\sigma U_o} = \left(\frac{2}{\pi\rho}\right)^{1/2}\sum_{n=1}^{\infty}\int_o^{\infty}[Y_1(KR)\cos(K\rho-\frac{\pi}{4}) - J_1(KR)\sin(K\rho-\frac{\pi}{4})][\sqrt{K}\,W_n(K)]^{-1}$$

$$\times F_2[K, i\Omega_n(K),\zeta]\cos\Omega_n(K)\tau\, dK + O(\rho^{-3/2}) \qquad (1.147)$$

That these are asymptotic representations can be seen on noting the theorem ((37), page 16) which states that the asymptotic expansion of an integral containing a large parameter can be obtained by replacing the integrand with its asymptotic series in terms of this parameter, providing the resulting integrals exist. Note that it has been assumed that ρ is so large that the suitable asymptotic form is always that for large argument, even when the order n of the Bessel functions gets large. This is not a necessary assumption since representations of the Bessel functions for large n, and transition regions, can readily be incorporated. However, since attention will finally be restricted to the lowest mode, this point will not be enlarged upon here. Using arguments similar to those given in Appendix C, it can be shown that the integrals in equations 1.146 and 1.147 do exist, and hence the above procedure is valid. The only difficulty which arises is that, for the lowest mode, the integrand in equation 1.146 behaves like $(K)^{-1/2}$ at $K = 0$, but this singularity is integrable.

The equations may be written:

$$\frac{\pi(1-2\sigma)u_\rho}{8\sigma U_o} = \frac{\pi(1-2\sigma)}{8\sigma}\frac{R}{\rho} + \frac{1}{2\sqrt{\pi\rho}}\sum_{n=1}^{\infty}\Big\{(\mathrm{Im} - \mathrm{Re})$$

$$\times\int_0^\infty Y_1(KR)[\sqrt{K}\ W_n(K)]^{-1}F_1[(K, i\Omega_n(K), \zeta][\exp i\rho f_+(K) + \exp i\rho f_-(K)]\ dK$$

$$+ (\mathrm{Im} + \mathrm{Re})\int_0^\infty J_1(KR)[\sqrt{K}\,W_n(K)]^{-1}F_1[K, i\Omega_n(K), \zeta][\exp i\rho f_+(K) + \exp i\rho f_-(K)]\ dK\Big\}$$

$$+ O(\rho^{-3/2}) \tag{1.148}$$

$$\frac{\pi(1-2\sigma)u_\zeta}{8\sigma U_o} = \frac{1}{2\sqrt{\pi\rho}}\sum_{n=1}^{\infty}\Big\{(\mathrm{Im} + \mathrm{Re})$$

$$\times\int_0^\infty Y_1(KR)[\sqrt{K}\ W_n(K)]^{-1}F_2[K, i\Omega_n(K), \zeta][\exp i\rho f_+(K) + \exp i\rho f_-(K)]\ dK$$

$$- (\mathrm{Im} - \mathrm{Re})\int_0^\infty J_1(KR)[\sqrt{K}\,W_n(K)]^{-1}F_2[K, i\Omega_n, \zeta][\exp i\rho f_+(K) + \exp i\rho f_-(K)]\,dK\Big\}$$

$$+ O(\rho^{-3/2}) \tag{1.149}$$

where

$$f_+ = K + \frac{\tau}{\rho}\,\Omega_n(K) \tag{1.150}$$

$$f_- = K - \frac{\tau}{\rho}\,\Omega_n(K) \tag{1.151}$$

and Re and Im denote real and imaginary part, respectively. The solutions are now in a form to which the method of stationary phase is applicable.

The principle of stationary phase ((38), page 506) states that, given two continuous real-valued functions $f(K)$ and $\varphi(K)$, then, for large ρ,

$$\int_{o}^{\infty} \varphi(K) \exp[\, i\rho f(K)]\, dK = \left[\frac{2\pi}{\rho\, |f''(S)|} \right]^{1/2} \varphi(S) \exp i[\, \rho f(S) + \frac{\pi}{4} \operatorname{sgn} f''(S)] + O(\rho^{-1}) \tag{1.152}$$

where the primes denote differentiation, and the points of stationary phase S are given by

$$[\, f'(K)]_{K=S} = 0 \tag{1.153}$$

From equations 1.150 and 1.151 it is seen that, for the problem at hand, these points are given by:

$$C_g(K) \pm \frac{\rho}{\tau} = 0 \tag{1.154}$$

where

$$C_g(K) = \Omega_n'(K) \tag{1.155}$$

is the group velocity. It is well known that stationary phase methods and the concept of group velocity are closely connected. The signal is assumed to consist of a series of wave groups, or packets, each travelling with the speed (the group velocity) proper to the mean wavenumber of the packet. At a given station ρ the major contribution to the response at time τ comes from the group which has the speed ρ/τ. The component waves of this group are thought of as additively interfering, whereas all other waves are thought of as destructively interfering. The validity of these ideas hinges on whether sufficient time has elapsed for the initial disturbance to have dispersed into such a series of groups. In this connection, recent studies by Whitham (39) on group velocity, from a kinematic viewpoint, can be of interest. The kinematic approach utilizes the principle of conservation of wavenumber, and not only affords further physical insight into the nature of group velocity, but, when used in conjunction with the theory of characteristics, enables some estimates to be made of the times for which the group concept is valid.

There is a continuous distribution of stationary phase points which are solutions of equations 1.154. Since τ and ρ are positive quantities, the equation

$$C_g(K) - \frac{\rho}{\tau} = 0$$

can be satisfied only by those K's which are associated with positive values of C_g, and similarly, as noted by Lamb ((40), page 396), the equation

$$C_g(K) + \frac{\rho}{\tau} = 0$$

can be satisfied only by those K's which are associated with negative values of C_g. Tolstoy and Usdin (27) showed that portions of several modes of the Rayleigh-Lamb frequency equations have negative group velocities associated with them (see, for example, the second mode in Figure 2), and so these portions can contribute to the stationary phase solutions. Mindlin ((26), page 25) has shown that the necessary and sufficient condition for the existence of a frequency minimum, and hence of negative group velocities, is that the curvature of the mode at $K = 0$ be negative, a condition which is often realized in practice. Note that the phase velocities ($c = \Omega_n(K)/K$) associated with these portions are positive.

The presence of negative group velocities, though disquieting at first sight, does not present any particular problems, and several systems have been considered by Lamb (41), Crandall (42), and others, in which they arise. The explanation given by these authors, and by Tolstoy and Usdin, is that their essential implication is that the phase and group velocities are oppositely directed, and so if the group as a whole moves in one direction then its component "phase" waves move in the opposite direction, i.e., the component waves appear at the front of the group and disappear at the rear. Biot's proof (43) of the identity of group velocity with the velocity of energy transport, for dissipationless processes in very general classes of media, applies here also. However this does not mean that energy is being carried towards the source, as can be seen from the fact that the component waves of the groups having negative velocities have the phase $[K\rho + \Omega_n(K)\tau]$. These waves are travelling towards the source, since the associated phase velocity is positive. Hence, using the interpretation of negative group velocities given above, the group as a whole moves away from the source. An interesting point here is that if an analogous situation existed for viscoelastic plates, then the above explanations are not so apparent, i.e., the group velocity is not necessarily the velocity of energy transport. The techniques discussed by Brillouin (44) may be more suitable in this case.

Apart from the conceptual restrictions on equation 1.152 discussed above, other restrictions occur because implicit in its derivation is the assumption that the higher derivatives are small in comparison with $f''(S)$. This is not necessarily true in the vicinity of points S_m for which $f''(S_m) = 0$, i.e., near the maxima and minima of the group velocity in the present case. For the case of a minimum (a case of practical interest later), the following approximation (45)* can be used:

$$\int_{S_m-\epsilon}^{S_m+\epsilon} \varphi(K) \exp i[K\rho \pm \Omega_n(K)\tau]\, dK$$

$$= 2\pi\varphi(S_m)\left[\frac{2\Omega_n'(S_m)}{\rho\,|\Omega_n'''(S_m)|}\right]^{1/3} Ai(-\nu_\pm) \exp i[\rho S_m \pm \Omega_n(S_m)\tau] + O(\rho^{-2/3}) \tag{1.156}$$

where ϵ is a small quantity, $\varphi(K)$ is a continuous real-valued function of K, $Ai(\nu)$ is the Airy function given by (38)

$$Ai(\nu) = \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}t^3 + \nu t)\, dt \tag{1.157}$$

and

$$\nu_\pm = \left[\frac{2}{\tau\,|\Omega_n'''(S_m)|}\right]^{1/3} [\rho \pm \Omega_n'(S_m)\tau] \tag{1.158}$$

In the derivation of this expression it has been assumed that the higher derivatives are small in comparison with $\Omega_n'''(S_m)$, a point which has been discussed by Pekeris (45) and Newlands (47). It has been termed the "Airy phase" by Pekeris, and is of importance in that, as pointed out by Ewing, Jardetsky, and Press ((24), page 145), it becomes relatively stronger with increasing distance from the source.

These approximations will now be applied to equations 1.148 and 1.149. Attention will be restricted to the lowest mode, the justifications for which are discussed in Section IV. There are no frequency minima in the lowest mode, and hence no associated negative group velocities.

*This, and similar expressions, have also been discussed in considerable detail by Cerrillo (46), using the techniques of saddle-point integration.

Thus, for the lowest mode, the negative travelling harmonic wave trains do not contribute to the stationary phase solutions. There is a group velocity maximum at $K = 0$ and a minimum at $K = S_m \cong 2$, for $\sigma = 0.31$ (10). Hence equation 1.152 is not applicable at these points. Note that $K = 0$ must also be excluded for the reason that the integrand in equation 1.148 is not continuous there, as mentioned above. Taking these factors into account, application of equations 1.152 through 1.158 to equations 1.148 and 1.149 gives, on considering only the lowest mode:

$$\frac{\pi(1-2\sigma)u_\rho}{8\sigma U_o} = \frac{\pi(1-2\sigma)R}{8\sigma\rho} + \frac{1}{\rho}\left[\frac{\Omega_1(S)}{S\,|\Omega_1''(S)|}\right]^{1/2}[W_1(S)]^{-1}F_1[S, i\Omega_1(S), \zeta]$$

$$\times[J_1(SR)\cos\rho f_-(S) + Y_1(SR)\sin\rho f_-(S)] + O(\rho^{-3/2}), \quad \epsilon_1 < S \leq S_m - \epsilon \tag{1.165}$$

$$= \frac{\pi(1-2\sigma)R}{8\sigma\rho} + \frac{1}{\rho}\left[\frac{\Omega_1(S)}{S\,|\Omega_1''(S)|}\right]^{1/2}[W_1(S)]^{-1}F_1[S, i\Omega_1(S), \zeta]$$

$$\times[J_1(SR)\sin\rho f_-(S) - Y_1(SR)\cos\rho f_-(S)] + O(\rho^{-3/2}), \quad S \geq S_m + \epsilon \tag{1.166}$$

$$= \left(\frac{1}{\rho}\right)^{5/6}\left[\frac{2\Omega_1'(S_m)}{|\Omega_1'''(S_m)|}\right]^{1/3}\left(\frac{\pi}{S_m}\right)^{1/2}[W_1(S_m)]^{-1}F_1[S_m, i\Omega_1(S_m), \zeta]$$

$$\times\Big\{Ai(-\nu_-)Y_1(S_m, R)[\sin\rho f_-(S_m) - \cos\rho f_-(S_m)]$$

$$+ Ai(-\nu_-)J_1(S_m R)[\sin\rho f_-(S_m) + \cos\rho f_-(S_m)]$$

$$+ Ai(-\nu_+)Y_1(S_m R)[\sin\rho f_+(S_m) - \cos\rho f_+(S_m)]$$

$$+ Ai(-\nu_+)J_1(S_m R)[\sin\rho f_+(S_m) + \cos\rho f_+(S_m)]\Big\} + O(\rho^{-1}),$$

$$S_m - \epsilon \leq S \leq S_m + \epsilon \tag{1.167}$$

$$\frac{\pi(1-2\sigma)u_\zeta}{8\sigma U_o} = \frac{1}{\rho}\left[\frac{\Omega_1'(S)}{S|\Omega_1''(S)|}\right]^{1/2}[W_1(S)]^{-1}F_2[S, i\Omega_1(S), \zeta]$$

$$\times[Y_1(SR)\cos\rho f_-(S) - J_1(SR)\sin\rho f_-(S)] + O(\rho^{-3/2}), \quad \epsilon_1 < S \leq S_m - \epsilon$$

(1.168)

$$= \frac{1}{\rho}\left[\frac{\Omega_1'(S)}{S|\Omega_1''(S)|}\right]^{1/2}[W_1(S)]^{-1}F_2[S, i\Omega_1(S), \zeta]$$

$$\times[Y_1(SR)\sin\rho f_-(S) + J_1(SR)\cos\rho f_-(S)] + O(\rho^{-3/2}), \quad S \geq S_m + \epsilon \qquad (1.169)$$

$$= \left(\frac{1}{\rho}\right)^{5/6}\left[\frac{2\Omega_1'(S_m)}{|\Omega_1'''(S_m)|}\right]^{1/3}\left(\frac{\pi}{S_m}\right)^{1/2}[W_1(S_m)]^{-1}F_2[S_m, i\Omega_1(S_m)\zeta]$$

$$\times\Big\{Ai(-\nu_-)Y_1(S_mR)[\sin\rho f_-(S_m) + \cos\rho f_-(S_m)]$$

$$- Ai(-\nu_-)J_1(S_mR)[\sin\rho f_-(S_m) - \cos\rho f_-(S_m)]$$

$$+ Ai(-\nu_+)Y_1(S_mR)[\sin\rho f_+(S_m) + \cos\rho f_+(S_m)]$$

$$- Ai(-\nu_+)J_1(S_mR)[\sin\rho f_+(S_m) - \cos\rho f_+(S_m)]\Big\} + O(\rho^{-7/6}),$$

$$S_m - \epsilon \leq S \leq S_m + \epsilon \qquad (1.170)$$

where

$$\nu_\pm = \frac{\sqrt{2}\,\rho f_\pm(S_m)}{[\tau|\Omega'''(S_m)|]^{1/2}} \qquad (1.171)$$

ϵ and ϵ_1 are small quantities, and $f_\pm(S_m)$ as are given in equations 1.150 and 1.151 (with n replaced by 1).

Representations of the solutions, which are valid in the vicinity of K = 0, can be obtained using Skalak's method (2), given for the rod. On the basis of the lowest mode alone, this region describes the earliest arriving portions of the disturbance--the so-called head of the pulse--since the lowest mode group velocity has its maximum value at $K = 0$. The procedure is as follows. The analytic behavior of the low frequency, long wavelength portion of the lowest mode is first determined by finding the corresponding leading terms of a series expansion. This expansion is then inserted into the various integrands, and only the dominant terms are retained, except in the trigonometric functions, where the first and second terms are kept. Using the result (10)

$$\Omega_1(K) = bK(1 - \delta K^2), \qquad K \ll 1 \tag{1.172}$$

where

$$b = c_P/c_s = \left[\frac{4(\lambda+\mu)}{\lambda+2\mu} \right]^{1/2} \tag{1.173}$$

$$\delta = \frac{(\alpha^2 - 2)^2}{6\alpha^4} \tag{1.174}$$

and following the above procedure, one obtains:

$$\frac{\pi(1-2\sigma)u_\rho}{8\sigma U_o} = \frac{\pi(1-2\sigma)R}{8\sigma\rho} + \frac{\pi(1-2\sigma)R}{8\sigma\sqrt{\rho}} \frac{1}{\pi^{3/2}} \int_o^{\epsilon_1} (K)^{-1/2}$$

$$\times \Big\{ \sin\left[K(\rho+\beta\tau) - K^2\beta\tau\delta\right] + \sin\left[K(\rho-\beta\tau) + K^3\beta\tau\delta\right]$$

$$- \cos\left[K(\rho+\beta\tau) - K^3\beta\tau\delta\right] - \cos\left[K(\rho-\beta\tau) + K^3\beta\tau\delta\right] \Big\} dK,$$

$$0 \leq K \leq \epsilon_1 \tag{1.175}$$

$$\frac{\pi(1-2\sigma)u_\zeta}{8\sigma U_o} = -\frac{\pi(1-2\sigma)R\zeta}{8(1-\sigma)\sqrt{\rho}}\ \frac{1}{\pi^{3/2}}\int_o^{\epsilon_1}\sqrt{K}\left\{\sin[K(\rho+\beta\tau)-K^3\beta\tau\delta]\right.$$

$$+\sin[K(\rho-\beta\tau)+K^3\beta\tau\delta]+\cos[K(\rho+\beta\tau)-K^3\beta\tau\delta]$$

$$\left.+\cos[K(\rho-\beta\tau)+K^3\beta\tau\delta]\right\}dK, \quad 0\le K\le\epsilon_1 \tag{1.176}$$

The integrals appearing in equations 1.175 and 1.176 cannot be written in closed form (even if the limits of integration are extended), and numerical computation seems to be necessary. In particular, they cannot be written in terms of Airy functions and their integrals, or generalizations of Airy functions ((48), page 380), as opposed to the corresponding situation in problems involving straight-crested waves (cf. Section III).

Another possible procedure for obtaining representations of the solutions near $K = 0$ will now be outlined. The various functions in the integrands of equations 1.136 and 1.137 (considering only the lowest mode), except the Bessel functions containing ρ and the trigonometric functions, are replaced by their limiting values (as K goes to zero), and the limits of integration in the equations are taken to be from 0 to ϵ_1. For large ρ and small K, such that $K\rho$ is large, C_o and C_1 may be approximated by

$$C_o(K,\rho,R) = (KR)^{-1}J_o(K\rho) + O(K^{-1/2})$$

$$C_1(K,\rho,R) = (KR)^{-1}J_1(K\rho) + O(K^{-1/2})$$

On using the result (21)

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp i(-n\theta + z\sin\theta)\, d\theta$$

integral representations of the Bessel functions are then inserted into the integrands, and the order of integration in the equations is interchanged. The inner range of integration (0 to ϵ_1) can then be extended to infinity, since the integrals so added are of a higher order. The inner integrals can then be expressed in terms of Airy functions and their derivatives. A typical expression arising from this procedure is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mp i\theta} \, Ai[(3\beta\tau\delta)^{-1/3}(\rho \sin\theta - \beta\tau)] \, d\theta$$

and it appears that numerical evaluation is necessary in this case also.

The Airy phase contribution may be assessed more readily with the aid of the following limiting forms of the Airy functions (38):

$$Ai(x) = \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp(-\tfrac{2}{3}x^{3/2})[1 + O(x^{-3/2}] \qquad (1.177)$$

$$Ai(-x) = \frac{1}{\sqrt{\pi}} x^{-1/4}[\sin(\tfrac{2}{3}x^{3/2} + \tfrac{\pi}{4}) + O(x^{-3/2})] \qquad (1.178)$$

for x large, real and positive;

$$Ai(x) = \frac{3^{-2/3}}{\pi}\left[\frac{\Gamma(\frac{1}{3})}{\Gamma(1)} \sin\frac{2\pi}{3} + \frac{\Gamma(\frac{2}{3})}{\Gamma(2)} 3^{1/3} x \sin\frac{4}{3}\pi + O(x^2)\right] \qquad (1.179)$$

where Γ denotes the Gamma function, for x small and real. For times less than $\rho/\Omega_1'(S_m)$, $-\nu_-$ and $-\nu_+$ are large and negative, and so, from equation 1.178, the Airy functions in equations 1.167 and 1.170 have an oscillatory behavior, with amplitudes of order $\rho^{-1/4}$. For times much greater than $\rho/\Omega_1'(S_m)$, $-\nu_-$ is large and positive, whereas

$-\nu_+$ is large and negative. Hence $Ai(-\nu_+)$ has an oscillatory behavior, with amplitude of order $\rho^{-1/4}$, and $Ai(-\nu_-)$ is exponentially small. For times close to $\rho/\Omega_1'(S_m)$, ν_- is small whereas ν_+ is still large (for large ρ). In this range the dominant terms are those involving $Ai(-\nu_-)$, which, from equation 1.179, are of order unity. As expected, (since the corresponding terms do not contribute appreciably in the stationary phase solutions), the terms involving $Ai(-\nu_+)$ do not contribute appreciably in this range. Equation 1.170 shows that the solution in this region oscillates with a constant period--a characteristic of the Airy phase--with an amplitude modulated by the Airy functions.

Miklowitz, in Reference (10), has computed the functions

$$V_1(\rho, S, \zeta) = \frac{1}{\rho}\left[\frac{S\Omega_1'(S)}{|\Omega_1''(S)|}\right]^{1/2} \frac{F_1[S, i\Omega_1(S), \zeta]\ \sin \rho f_-(S)}{2\Omega_1^2(S)N[S, i\Omega_1(S)]}, \quad \epsilon_1 < S \leq S_m - \epsilon \tag{1.180}$$

$$V_2(\rho, S, \zeta) = \frac{1}{\rho}\left[\frac{S\Omega_1'(S)}{|\Omega_1''(S)|}\right]^{1/2} \frac{F_2[S, i\Omega_1(S), \zeta]\ \cos \rho f_-(S)}{2\Omega_1^2(S)N[S, i\Omega_1(S)]}, \quad \epsilon_1 < S \leq S_m - \epsilon \tag{1.181}$$

for several values of ζ. In terms of these functions, equations 1.165 and 1.168 may be written:

$$\frac{\pi(1-2\sigma)u_\rho}{8\sigma U_0} = \frac{\pi(1-2\sigma)R}{8\sigma\rho} + V_1(\rho, S, \zeta)M_1(\rho, S) + O(\rho^{-3/2}), \quad \epsilon_1 < S \leq S_m - \epsilon \tag{1.182}$$

$$\frac{\pi(1-2\sigma)u_\zeta}{8\sigma U_0} = V_2(\rho, S, \zeta)M_2(\rho, S) + O(\rho^{-3/2}), \quad \epsilon_1 < S \leq S_m - \epsilon \tag{1.183}$$

where

$$M_1(\rho, S) = \frac{2\Omega_1^2(S)[J_1(SR) \cot \rho f_-(S) + Y_1(SR)]}{S[J_1^2(SR) + Y_1^2(SR)][S^2\alpha^2 - \Omega_1^2(S)]} \tag{1.184}$$

$$M_2(\rho, S) = \frac{2\Omega_1^2(S)[Y_1(SR) - J_1(SR) \tan \rho f_-(S)]}{S[J_1^2(SR) + Y_1^2(SR)][S^2\alpha^2 - \Omega_1^2(S)]} \tag{1.185}$$

Finally it should be noted that the long-time (static) solution to equation 1.145 is of the same order as the stationary phase contribution, as opposed to Miklowitz's solution (10) to the related problem on cylindrically-crested waves, where it is of a higher order. Also note that the stationary phase solutions are of higher order than the Airy phase contributions, showing that the latter becomes relatively stronger with increasing distance from the source.

Further discussion of the solutions is given in Section IV.

Section II. THE GENERATION OF TRANSIENT COMPRESSIONAL AND FLEXURAL WAVES IN FLAT ELASTIC PLATES BY VARIABLE BODY FORCES

INTRODUCTION

The generation of elastic waves by means of body forces has received considerable attention in the literature, but most of the work has been done in connection with infinite media and half-spaces. In this section solutions are written for certain problems involving variable body forces in a flat, homogeneous, isotropic, linear elastic plate, of thickness 2H, with stress-free lateral surfaces. Far-field approximations are derived for a particular compressional wave problem. Apart from any intrinsic interest, these solutions could be of use in describing the mixed and nonmixed pressure shock type problems for the infinite plate with the circular cylindrical cavity, which, as will be recalled from Section I, appear to be intractable. It was pointed out in the general introduction that body force solutions for the infinite rod agreed quite well with experimental results (obtained under nonmixed conditions) for large distances down the rod. It is hoped that this is equally true in the present case.

Multi-integral transform techniques are used, but in this case the pertinent Hankel transform pairs are those for the interval $(0, \infty)$. The notation used is that of Section I, unless otherwise specified.

2.1. STATEMENT OF PROBLEMS AND DERIVATION OF FORMAL SOLUTIONS

The body forces, boundary and initial conditions, are assumed to be such that axial symmetry prevails. Hence the governing equations of motion are given by equations 1.3, 1.4, 1.7, 1.8, 1.10, and 1.11. Taking the Laplace transform of these equations, and assuming that the initial conditions given by equation 1.21 hold in this case also, equations 1.12, 1.13, 1.16, 1.17, 1.19, and 1.20, are obtained again. Taking the Hankel transform of order one--for the interval $(0, \infty)$--of equations 1.12 and 1.19, and the Hankel transform of order zero of equations 1.13, 1.17, and 1.20, gives, on using certain properties of transforms of derivatives, and combinations of derivatives (49):

$$\frac{d^2\tilde{\bar{u}}_r^1}{dz^2} - h^2\tilde{\bar{u}}_r^1 - k\left(\frac{c_d^2}{c_s^2} - 1\right)\frac{d\tilde{\bar{u}}_z^o}{dz} = -\frac{1}{c_s^2}\tilde{\bar{F}}_r^1 \qquad (2.1)$$

$$\frac{d^2\tilde{\bar{u}}_z^o}{dz^2} - g^2\tilde{\bar{u}}_z^o - k\left(\frac{c_s^2}{c_d^2} - 1\right)\frac{d\tilde{\bar{u}}_r^1}{dz} = -\frac{1}{c_d^2}\tilde{\bar{F}}_z^o \qquad (2.2)$$

$$\tilde{\bar{e}}_{zz}^o = \frac{d\tilde{\bar{u}}_z^o}{dz} \qquad (2.3)$$

$$\tilde{\bar{\sigma}}_{rz}^1 = \mu\left(\frac{d\tilde{\bar{u}}_r^1}{dz} - k\tilde{\bar{u}}_z^o\right) \qquad (2.4)$$

$$\tilde{\bar{\sigma}}_{zz}^o = (\lambda + 2\mu)\frac{d\tilde{\bar{u}}_z^o}{dz} + \lambda k\tilde{\bar{u}}_r^1 \qquad (2.5)$$

where

$$\tilde{\bar{u}}_r^1(k, p, z) = \int_o^\infty \bar{u}_r(r, p, z)J_1(kr)\, dr$$

$$\tilde{\bar{u}}_z^o(k, p, z) = \int_o^\infty \bar{u}_z(r, p, z)J_o(kr)\, dr$$

In deriving the above equations, the radiation conditions given by equation 1.71, with $C_o(k, r, a)$ and $C_1(k, r, a)$ replaced by $J_o(kr)$ and $J_1(kr)$, respectively, have been assumed.

The general solutions to equations 2.1 and 2.2 are given by equations 1.31, 1.32, 1.33, and 1.34, with L(z) and M(z) replaced by $-(c_s^2)^{-1}\tilde{\bar{F}}_r^1$ and $-(c_d^2)^{-1}\tilde{\bar{F}}_z^o$, respectively. As in Section I, separation into symmetric and antisymmetric problems is now effected. Taking

$$\left.\begin{aligned} F_r(r,t,z) &= V(r)f(t) \\ F_z(r,t,z) &= 0 \end{aligned}\right\} \qquad (2.6)$$

where V(r) and f(t) are arbitrary functions of r and t, respectively, compressional waves are excited, and so the general solutions of equations 2.1 and 2.2 are given by equations 1.38 and 1.39, with appropriate expressions for the particular integrals. Taking the Laplace and Hankel transforms of equations 2.6 gives:

$$\tilde{\bar{F}}_r^1 = \tilde{V}'(k)\bar{f}(p)$$

$$\tilde{\bar{F}}_z^o = 0$$

Substituting these expressions into equations 1.33 and 1.34 gives the particular integrals, and hence, on using equations 1.27 and 1.37, equations 1.38 and 1.39 may be written:

$$C\tilde{\bar{u}}_r^1 = \sum_{j=1}^{\infty} A_j \cosh \eta_j z + (c_d^2\eta_2^2)^{-1}\tilde{V}'(k)\bar{f}(p) \qquad (2.7)$$

$$C\tilde{\bar{u}}_z^o = \sum_{j=1}^{2} \alpha_j A_j \sinh \eta_j z \qquad (2.8)$$

Taking

$$F_r(r, t, z) = 0$$
$$F_z(r, t, z) = w(r)Q(t) \tag{2.9}$$

where $w(r)$ and $Q(t)$ are arbitrary functions of r and t, respectively, flexural waves are excited, and so the general solutions of equations 2.1 and 2.2 are given by equations 1.40 and 1.41, with appropriate expressions for the particular integrals. Transforming equation 2.9 gives:

$$\tilde{\tilde{F}}{}^1_r = 0$$

$$\tilde{\tilde{F}}{}^o_z = \tilde{w}^o(k)\bar{Q}(p)$$

Substituting these expressions into equations 1.33 and 1.34 gives the particular integrals, and hence, on using equations 1.28 and 1.34, equations 1.40 and 1.41 may be written:

$${}^{F}\tilde{\tilde{u}}{}^1_r = \sum_{j=1}^{2} B_j \sinh \eta_j z \tag{2.10}$$

$${}^{F}\tilde{\tilde{u}}{}^o_z = \sum_{j=1}^{2} \alpha_j B_j \cosh \eta_j z + (c_s^2 \eta_1^2)^{-1} \tilde{w}^o(k)\bar{Q}(p) \tag{2.11}$$

Inserting equations 2.7, 2.8, 2.10, and 2.11, into equations 2.4 and 2.5, and applying the transformed boundary conditions of stress free lateral surfaces, expressions are obtained for the A's and B's, and hence, from equations 2.3, 2.7, 2.8, 2.10, and 2.11, for the transformed displacements and axial strain. However, in effect, this has been done in Section I (§1.1), and the final results can be obtained from there on setting

$$S_C = 0$$

$$T_C = \lambda k(c_d^2\eta_2^2)^{-1}\tilde{V}'(k)\bar{f}(p)$$

$$S_F = -\mu k(c_s^2\eta_1^2)^{-1}\tilde{w}^o(k)\bar{Q}(p)$$

$$T_F = 0$$

in equations 1.54, 1.55, 1.57, 1.58, 1.60, and 1.61. There results:

$$ {}_C\tilde{\bar{u}}_r^1 = \left[\frac{\lambda k^2\tilde{V}'(k)F_{1C}(k,p,z)}{\mu c_d^2\eta_2^2 D_C(k,p)} + \frac{\tilde{V}'(k)}{c_d^2\eta_2^2}\right]\bar{f}(p) \qquad (2.12)$$

$$ {}_C\tilde{\bar{u}}_z^o = \left[\frac{\lambda k^2\tilde{V}'(k)\,F_{2C}(k,p,z)}{\mu c_d^2\eta_2^2 D_C(k,p)}\right]\bar{f}(p) \qquad (2.13)$$

$$ {}_C\tilde{\bar{e}}_{zz}^o = \left[\frac{\lambda k^2\tilde{V}'(k)\,F_{3C}(k,p,z)}{\mu c_d^2\eta_2^2 D_C(k,p)}\right]\bar{f}(p) \qquad (2.14)$$

$$ {}_F\tilde{\bar{u}}_r^1 = -\left[\frac{k^2\tilde{w}^o(k)F_{1F}(k,p,z)}{c_s^2\eta_1^2 D_F(k,p)}\right]\bar{Q}(p) \qquad (2.15)$$

$$ {}_F\tilde{\bar{u}}_z^o = -\left[\frac{k^2\tilde{w}^o(k)F_{2F}(k,p,z)}{c_s^2\eta_1^2 D_F(k,p)}\right]\bar{Q}(p) \qquad (2.16)$$

$$ {}_F\tilde{\bar{e}}_{zz}^o = -\left[\frac{k^2\tilde{w}^o(k)F_{3F}(k,p,z)}{c_s^2\eta_1^2 D_F(k,p)}\right]\bar{Q}(p) \qquad (2.17)$$

2.2. EXACT INVERSIONS

As in Section I (§1.2), the Laplace transforms are inverted first, and use of the convolution theorem will again be made. It is assumed that $\bar{f}(p)$ and $\bar{Q}(p)$ are such that they invert to $f(t)$ and $Q(t)$, respectively. It can be shown that the order conditions of the Laplace inversion theorem are satisfied by the multipliers of $\bar{f}(p)$ and $\bar{Q}(p)$ (the terms in the square brackets) in equations 2.12 through 2.17. The discussion of the singularities given in Section I is relevant here also, with some slight modifications, and, using the information given there, inversion of the multipliers is achieved through Cauchy's theorem and residue theory. Using the results together with the convolution theorem, and applying the Hankel transform inversion formulas (19), the inverses of equations 2.12 through 2.17 are found to be:

$$_{C}u_{r} = \frac{2\lambda}{\mu}\int_{o}^{\infty} k^{3}J_{1}(kr)\tilde{V}'(k)\left[\int_{t_w}^{t} f(t-\xi)\Theta_{1C}(k,\xi,z)\, d\xi\right] dk \qquad (2.18)$$

$$_{C}u_{z} = \frac{2\lambda}{\mu}\int_{o}^{\infty} k^{3}J_{o}(kr)\tilde{V}'(k)\left[\int_{t_w}^{t} f(t-\xi)\Theta_{2C}(k,\xi,z)d\xi\right] dk \qquad (2.19)$$

$$_{C}e_{zz} = \frac{2\lambda}{\mu}\int_{o}^{\infty} k^{3}J_{o}(kr)\tilde{V}'(k)\left[\int_{t_w}^{t} f(t-\xi)\Theta_{3C}(k,\xi,z)d\xi\right] dk \qquad (2.20)$$

$$_{F}u_{r} = -2\int_{o}^{\infty} k^{3}J_{1}(kr)\tilde{w}^{o}(k)\left[\int_{t_w}^{t} Q(t-\xi)\Psi_{1F}(k,\xi,z)d\xi\right] dk \qquad (2.21)$$

$$_{F}u_{z} = -2\int_{o}^{\infty} k^{3}J_{o}(kr)\tilde{w}^{o}(k)\left[\int_{t_w}^{t} Q(t-\xi)\Psi_{2F}(k,\xi,z)d\xi\right] dk \qquad (2.22)$$

$$_{F}e_{zz} = -2\int_{o}^{\infty} k^{3}J_{o}(kr)\tilde{w}^{o}(k)\left[\int_{t_w}^{t} Q(t-\xi)\Psi_{3F}(k,\xi,z)d\xi\right] dk \qquad (2.23)$$

where

$$\Theta_{mC}(k,t,z) = \sum_{n=1}^{\infty} \frac{F_{mC}[k, i\omega_{nC}(k), z] \sin \omega_{nC}(k)t}{[k^2 c_d^2 - \omega_{nC}^2(k)]\omega_{nC}(k) N_C[k, i\omega_{nC}(k)]}, \quad m = 1, 2, 3 \tag{2.24}$$

$$\Psi_{mF}(k,t,z) = \sum_{n=1}^{\infty} \frac{F_{mF}[k, i\omega_{nF}(k), z] \sin \omega_{nF}(k)t}{[k^2 c_s^2 - \omega_{nF}^2(k)]\omega_{nF}(k) N_F[k, i\omega_{nF}(k)]}, \quad m = 1, 2, 3 \tag{2.25}$$

The radial strains are given by

$$_C e_{rr} = \frac{2\lambda}{\mu} \int_0^{\infty} k^3 \frac{\partial}{\partial r} J_1(kr) \tilde{V}'(k) \left[\int_{t_w}^{t} f(t-\xi) \Theta_{1C}(k, \xi, z) d\xi \right] dk \tag{2.26}$$

$$_F e_{rr} = -2 \int_0^{\infty} k^3 \frac{\partial}{\partial r} J_1(kr) \tilde{w}^o(k) \left[\int_{t_w}^{t} Q(t-\xi) \Psi_{1F}(k, \xi, z) d\xi \right] dk \tag{2.27}$$

Application of these quite general expressions to a particular compressional wave problem will now be discussed. The body force per unit mass in the r direction is chosen to be

$$F_r(r,t,z) = V(r)f(t) = \frac{P_o \delta(r-a)}{4\pi r H} f(t) \tag{2.28}$$

where $\delta(r)$ is the Dirac delta function, a and P_o are constants, and the choice has been dictated by the fact that the above expressions are being used to simulate the plate with the circular cavity. Note that

$$\int_0^{\infty} \int_{-H}^{H} \int_0^{2\pi} \rho' F_r(r,t,z) r \, dr \, dz \, d\theta = \rho' P_o f(t)$$

so that the total force acting is finite. Taking the first order Hankel transform of $V(r)$, as given by equation 2.28, one obtains:

$$\tilde{V}'(k) = \frac{P_o J_1(ka)}{4\pi H} \qquad (2.29)$$

On letting a go to zero, equation 2.28 represents a radial line source (at $r = 0$) in the plate. However equation 2.29 shows that in this case $\tilde{V}'(k)$ is zero. The difficulty appears to lie in the fact that a radial line source violates the conditions of continuity which are built into the equations of motion.

With the above choice of $V(r)$, and taking $f(t)$ to be the Heaviside step function $H(t)$, one obtains, on inverting directly from equations 2.12 and 2.13 (so as to obtain a more suitable representation of the long time solution), the following expressions for the displacements:

$$_C u_r = R_{0r}(r, z) - \frac{\lambda P_o}{2\pi\mu H} \sum_{n=1}^{\infty} \times \int_0^{\infty} \frac{k_3 J_1(kr) J_1(ka) F_{1C}[k, i\omega_{nC}(k), z] \cos \omega_{nC}(k)t}{[k^2 c_d^2 - \omega_{nC}^2(k)]\, \omega_{nC}^2(k) N_C[k, i\omega_{nC}(k)]}\, dk + \frac{P_o}{8\pi H c_d^2} \begin{cases} \frac{r}{a}, & r \leq a \\ \frac{a}{r}, & r \geq a \end{cases} \qquad (2.30)$$

$$_C u_z = R_{0z}(r, z) - \frac{\lambda P_o}{2\pi\mu H} \sum_{n=1}^{\infty} \times \int_0^{\infty} \frac{k^3 J_o(kr) J_1(ka) F_{2C}[k, i\omega_{nC}(k), z] \cos \omega_{nC}(k)t}{[k^2 c_d^2 - \omega_{nC}^2(k)]\, \omega_{nC}^2(k) N_C[k, i\omega_{nC}(k)]}\, dk \qquad (2.31)$$

where

$$R_{0r}(r,z) = \frac{\lambda P_o}{8\pi\mu c_d^2 H} \int_o^\infty \frac{kJ_1(kr)J_1(ka)E(k,z)}{kH + \sinh kH \cosh kH}\, dk \qquad (2.32)$$

$$R_{0z}(r,z) = \frac{\lambda P_o}{8\pi\mu c_d^2 H} \int_o^\infty \frac{kJ_o(kr)J_1(ka)G(k,z)}{kH + \sinh kH \cosh kH}\, dk \qquad (2.33)$$

$$E(k,z) = \frac{1}{k}(H \cosh kz \cosh kH - z \sinh kz \sinh kH) - \frac{c_s^2 \sinh kH \cosh kz}{(c_d^2 - c_s^2)k^2} \qquad (2.34)$$

$$G(k,z) = \frac{1}{k}(z \cosh kz \sinh kH - H \cosh kH \sinh kz) - \frac{c_d^2 \sinh kH \sinh kz}{(c_d^2 - c_s^2)k^2} \qquad (2.35)$$

Note that it is to be understood that ${}_C u_r$ and ${}_C u_z$ are zero for times less than the arrival time of the wave front. In arriving at these equations use has been made of the result ((36), page 47)

$$\int_o^\infty \frac{J_\nu(ka)J_\nu(kr)}{k}\, dk = \frac{1}{2\nu}\left(\frac{r}{a}\right)^\nu, \quad 0 < r \leq a$$

$$= \frac{1}{2\nu}\left(\frac{a}{r}\right)^\nu, \quad a \leq r < \infty$$

The R_{0r}, R_{0z} terms arise from the evaluation of the residues at $p = 0$ (use of L'Hospital's rule is necessary in these calculations). The strains have not been inverted, since they do not play a role in the subsequent discussions.

A discussion of the possible singularities of the integrands in in equations 2.30 and 2.31 is given in Appendix C; it is shown there that these integrands are well behaved in the domain of integration. It is also shown (formally) there that ${}_Cu_r$ and ${}_Cu_z$, as given by equations 2.30 and 2.31, satisfy the boundary and initial conditions.

2.3. FAR-FIELD APPROXIMATIONS

Henceforth the affix C will be deleted. Changing to the dimensionless variables introduced in Section I (§1.3), and replacing the Bessel functions containing ρ by the leading terms of their large-argument asymptotic expansions, equations 2.30 and 2.31 give, on retaining only the lowest mode contribution:

$$\frac{2\pi\mu Hc_s^2 u_\rho}{\lambda P_o} = \frac{2\pi\mu Hc_s^2}{\lambda P_o} R_{0\rho}(\rho,\zeta) + \frac{(1-2\sigma)^2 R}{16\sigma(1-\sigma)\rho} - \frac{1}{2\sqrt{\pi\rho}} (\mathrm{Im} - \mathrm{Re})$$

$$\times \int_0^\infty \frac{K^2 J_1(KR) F_1[K, i\Omega_1(K), \zeta][\exp i\rho f_+(K) + \exp i\rho f_-(K)]}{\sqrt{K}\,[K^2 a^2 - \Omega_1^2(K)]\,\Omega_1^2(K) N[K, i\Omega_1(K)]} dK + O(\rho^{-3/2}),$$

$$\rho > R \quad (r > a) \qquad (2.36)$$

$$\frac{2\pi\mu Hc_s^2 u_\zeta}{\lambda P_o} = \frac{2\pi\mu Hc_s^2}{\lambda P_o} R_{0\zeta}(\rho,\zeta) - \frac{1}{2\sqrt{\pi\rho}} (\mathrm{Im} + \mathrm{Re})$$

$$\times \int_0^\infty \frac{K^2 J_1(KR) F_2[K, i\Omega_1(K), \zeta][\exp i\rho f_+(K) + \exp i\rho f_-(K)]}{\sqrt{K}\,[K^2 a^2 - \Omega_1^2(K)]\,\Omega_1^2(K) N[K, i\Omega_1(K)]} dK + O(\rho^{-3/2})$$

$$(2.37)$$

where $R_{0\rho}$ and $R_{0\zeta}$ are the expressions obtained on substituting the dimensionless variables into equations 2.32 through 2.35, with the Bessel functions containing ρ replaced by the leading terms of their large-argument asymptotic expansions.

Using arguments similar to those given in Appendix C, it can be shown that the integrands in equations 2.33 (with $J_o(kr)$ replaced by the leading term in its asymptotic expansion) and 2.37 are well behaved, whereas the integrands in equations 2.32 (with $J_1(kr)$ replaced by the leading term in its asymptotic expansion) and 2.36 have integrable singularities of the type $(K)^{-1/2}$ at $K = 0$. Hence the above procedure is valid.

The terms $R_{0\rho}$ and $R_{0\zeta}$ are essentially Fourier integrals, and, using the Riemann-Lebesque lemma (38), they can be shown to be order $\rho^{-3/2}$ (cf. Reference (10) for details of this proof). It is of interest that these terms, which are part of the long-time solution (the term in R/ρ is also a part), are of higher order than the other terms in the equations.

Applying equations 1.152, 1.180, and 1.181, to equations 2.36 and 2.37, and retaining only those portions which will be evaluated subsequently, gives:

$$\frac{2\pi\mu Hc_s^2}{\lambda P_o} u_\rho = \frac{(1-2\sigma)^2 R}{16\sigma(1-\sigma)\rho} - M_3(S)V_1(\rho, S, \zeta) + O(\rho^{-3/2}), \quad \epsilon_1 \leq S \leq S_m - \epsilon \quad (2.38)$$

$$\frac{2\pi\mu Hc_s^2}{\lambda P_o} u_\zeta = - M_3(S)V_2(\rho, S, \zeta) + O(\rho^{-3/2}), \quad \epsilon_1 \leq S \leq S_m - \epsilon \quad (2.39)$$

where

$$M_3(S) = \frac{2SJ_1(SR)}{[S^2 a^2 - \Omega_1^2(S)]} \qquad (2.40)$$

Further discussion of these results is given in Section IV.

Section III. RELATED SLAB PROBLEMS

INTRODUCTION

In this section some mixed time-dependent boundary value problems for a homogeneous, linearly elastic, semi-infinite flat plate (slab), will be examined, for both isotropic and certain classes of anisotropic media. Apart from intrinsic interest in such problems, the slab is a limiting case of the plate with a circular cylindrical cavity (as the radius a goes to infinity) and so is important in the context of the present work. For the isotropic slab several authors have contributed transient solutions, and some of this work is reproduced here for the two-fold purpose of having solutions on which to base further numerical work, and to clarify some of the issues raised in Section I.

The slab has attracted attention in that it is the simplest geometry involving two or more perpendicular boundaries. A solution to a pressure shock type problem (step time-dependence) has been derived by Folk (50), who discussed its applicability to nonmixed cases. A similar problem for a wide rectangular bar has been solved by Jones and Ellis (51, 52), using the generalized plane-stress theory. Their results can be utilized in the plane-strain, or slab, case, on applying the well-known transformation of elastic constants. Rosenfeld and Miklowitz (34) have contributed solutions to both longitudinal impact and pressure shock problems, with step time-dependencies, and have derived the amplitudes and location of all the wave fronts. In this section far-field approximations have been derived for a problem in which the slab end is subjected to a uniform step normal displacement.

Solutions to wave problems in anisotropic media are considerably more difficult than corresponding cases in isotropic media, because of the greater number of elastic constants involved. The situation is further complicated by the fact that in an infinite anisotropic medium there are, in general, three speeds of propagation corresponding to the three possible types of plane waves in any given direction. Also, there is no sharp distinction between distortional and dilatational waves. Despite these difficulties some work has been done on steady wave propagation problems. Buckwald (53), and others, have treated such problems in infinite media. For certain classes of materials, studies have been made by Sato (54), Stoneley (55), and others, on Rayleigh and Love type waves, and this type of work has been recently extended by Anderson (32) to layered, transversely isotropic media. Steady wave guide propagation has also received some attention. Newman and Mindlin (56) have examined in detail the frequency equation for an infinite plate of monoclinic crystal, the digonal axis of the crystal being parallel to the plate surfaces. Morse (30) has determined the frequency equation for an infinite circular rod of transversely isotropic material, the axis of the rod being parallel to the axis of material symmetry. However no attempt was made to evaluate the modes, although the small and large wavelength limits of the lowest mode were calculated. Mindlin and his group have further developed the approximate theories given in Reference (16), and applied them to problems of vibration of crystal plates (cf. (35), Chapter 13, for references on this topic).

As yet, very few solutions have been written for transient problems in anisotropic media. Of note in this connection are the solutions recently given by Eason (31) to problems of step loaded spherical and cylindrical cavities in infinite media. The material he treated had the property that it was isotropic in all planes perpendicular to the radial direction. The lack of transient solutions for anisotropic wave-guides generated interest in the present work, in which the plate material is taken to be transversely isotropic, with the axis of material symmetry parallel to the lateral surfaces. This has been done to illustrate the scope of multi-integral transform techniques, and to survey the difficulties inherent in such problems. Far-field approximations, which describe the head of the pulse, have been derived for certain longitudinal impact and pressure shock type problems.

3.1. DERIVATION OF FORMAL SOLUTIONS

Several substances have the property of transverse isotropy, i.e., isotropy in all planes perpendicular to a given direction. For instance, many crystalline solids of the hexagonal system satisfy this requirement. Of the noncrystalline substances, bedded sediments are a notable example. The difficulty of a given problem for such a medium depends on the orientation of the axis of symmetry of the material w.r.t. the free surfaces. In the present case the axis of material symmetry is taken parallel to the lateral surfaces of the plate, but it should be noted that another possible simple choice would be to take this axis perpendicular to the surfaces (this latter choice being of greater interest in geophysics).

The cartesian coordinate system shown in Figure 3 is chosen,* and, with the usual notation, the matrix of elastic constants for the transversely isotropic system is (57):

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{vmatrix}$$

where $c_{66} = \frac{1}{2}(c_{11} - c_{12})$. Thus there are five independent elastic constants. Reduction to the isotropic case is achieved by setting

$$\begin{aligned} c_{66} &= c_{44} \\ c_{11} &= c_{33} \\ c_{12} &= c_{13} \end{aligned} \tag{3.1}$$

The Lamé constants are given by:

$$\lambda = c_{12}, \quad \mu = c_{44} \tag{3.2}$$

and hence

$$c_{11} = \lambda + 2\mu \tag{3.3}$$

*Note the difference between z here and in Sections I and II. The present choice conforms to the standard use of z as the axis of material symmetry.

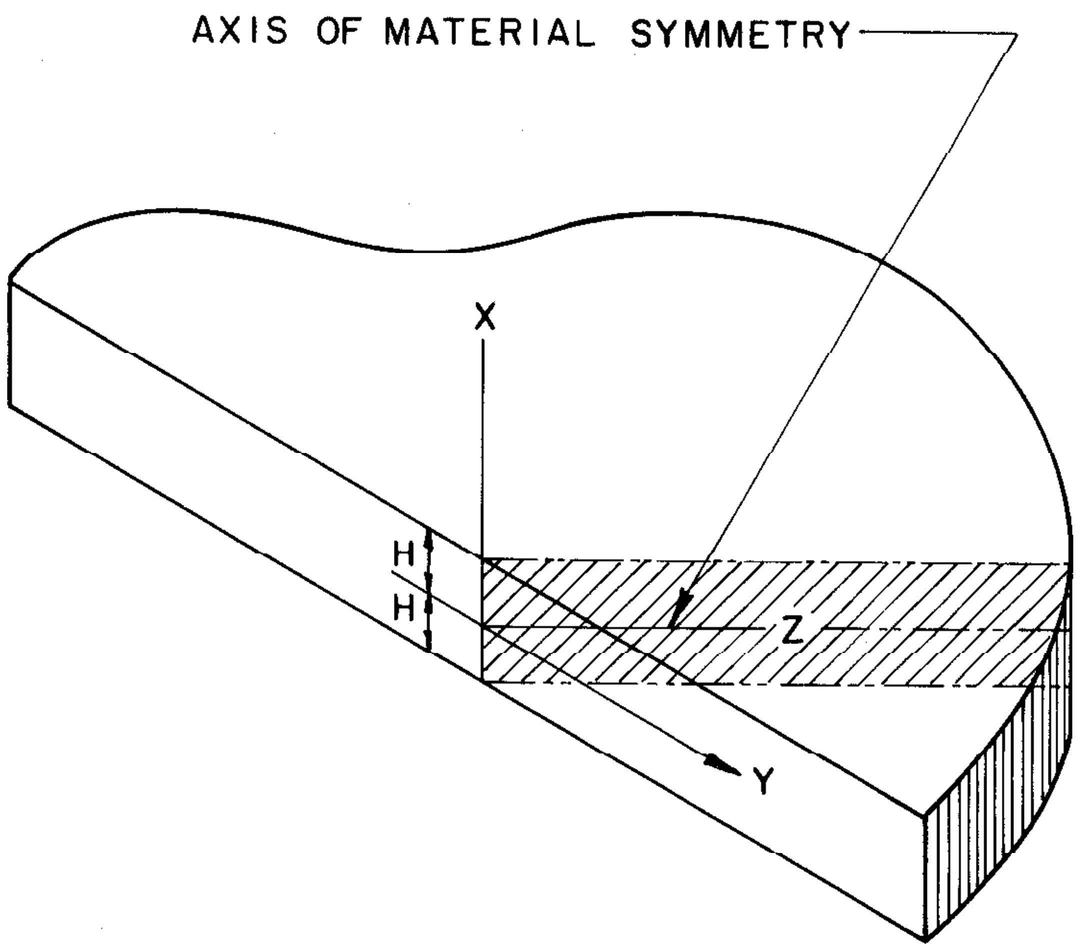

Fig. 3. Geometry of slab and coordinates used.

Since the plate material is isotropic in all planes perpendicular to the z axis, solutions independent of y, and for which $u_y = 0$, are possible, and the boundary and initial conditions are assumed to be such that these are generated. In this case, and in the absence of body forces, the displacement equations of motion are:

$$\frac{\partial^2 u_x}{\partial x^2} + \frac{c_{13} + c_{44}}{c_{11}} \frac{\partial^2 u_z}{\partial x \partial z} + \frac{c_{44}}{c_{11}} \frac{\partial^2 u_x}{\partial z^2} = \frac{\rho'}{c_{11}} \frac{\partial^2 u_x}{\partial t^2} \tag{3.4}$$

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{c_{13} + c_{44}}{c_{44}} \frac{\partial^2 u_x}{\partial x \partial z} + \frac{c_{33}}{c_{44}} \frac{\partial^2 u_z}{\partial z^2} = \frac{\rho'}{c_{44}} \frac{\partial^2 u_z}{\partial t^2} \tag{3.5}$$

The stresses and strains, which are of importance to the problems to be considered, are given by:

$$e_{zz} = \frac{\partial u_z}{\partial z} \tag{3.6}$$

$$e_{xx} = \frac{\partial u_x}{\partial x} \tag{3.7}$$

$$\sigma_{xz} = c_{44}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \tag{3.8}$$

$$\sigma_{xx} = c_{11} \frac{\partial u_x}{\partial x} + c_{13} \frac{\partial u_z}{\partial z} \tag{3.9}$$

$$\sigma_{zz} = c_{13} \frac{\partial u_x}{\partial x} + c_{33} \frac{\partial u_z}{\partial z} \tag{3.10}$$

Taking the Laplace (w. r. t. t) and Fourier sine (w. r. t. z) transforms of equations 3.5 and 3.8, and the Laplace and Fourier cosine transforms of equations 3.4 and 3.9, and using the properties of transforms of derivatives ((49), page 27), gives:

$$\frac{d^2\tilde{\tilde{u}}_x^c}{dx^2} - \frac{k^2c_{44} + \rho' p^2}{c_{11}} \tilde{\tilde{u}}_x^c + \frac{c_{13} + c_{44}}{c_{11}} k \frac{d\tilde{\tilde{u}}_z^s}{dx} = \sqrt{\frac{2}{\pi}} \frac{c_{44}}{c_{11}} \left(\frac{\partial \bar{u}_x}{\partial z}\right)_{z=0} \tag{3.11}$$

$$\frac{d^2\tilde{\tilde{u}}_z^s}{dx^2} - \frac{k^2c_{33} + \rho' p^2}{c_{44}} \tilde{\tilde{u}}_z^s - \frac{c_{13} + c_{44}}{c_{44}} k \frac{d\tilde{\tilde{u}}_x^c}{dx} = -\sqrt{\frac{2}{\pi}} \frac{c_{33}}{c_{44}} k(\bar{u}_z)_{z=0} \tag{3.12}$$

$$\tilde{\tilde{\sigma}}_{xz}^s = c_{44}\left(\frac{d\tilde{\tilde{u}}_z^s}{dx} - k\tilde{\tilde{u}}_x^c\right) \tag{3.13}$$

$$\tilde{\tilde{\sigma}}_{xx}^c = c_{11} \frac{d\tilde{\tilde{u}}_x^c}{dx} + c_{13}k\tilde{\tilde{u}}_z^s - \sqrt{\frac{2}{\pi}}\, c_{13}(\bar{u}_z)_{z=0} \tag{3.14}$$

where

$$\tilde{\tilde{u}}_x^c(k, p, x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{u}_x(z, p, x) \cos kz \, dz$$

$$\tilde{\tilde{u}}_z^s(k, p, x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{u}_z(z, p, x) \sin kz \, dz$$

and initial and radiation conditions analogous to those given by equations 1.21 and 1.71 have been assumed.

The boundary conditions are taken to be:

$$\sigma_{xx} = \sigma_{xz} = 0, \quad x = \pm H \tag{3.15}$$

$$\left.\begin{aligned} u_z &= U_o H(t) \\ \sigma_{xz} &= 0 \end{aligned}\right\} z = 0 \tag{3.16}$$

Using equation 3.16, the boundary terms in equations 3.11 through 3.14 can be evaluated. Symmetric solutions to equations 3.11 and 3.12 involving two arbitrary "constants" can then be written, in the manner of Section I. On substituting these solutions into appropriately-transformed equations 3.15, the arbitrary constants are determined, and hence expressions can be written for the transformed displacements. These can be shown to be:

$$\tilde{\tilde{u}}_z^s = -\sqrt{\frac{2}{\pi}}\,\frac{U_o\rho' c_{13} kp F_1(k,p,x)}{c_{44}(c_{33}k^2+\rho' p^2)D(k,p)} + \sqrt{\frac{2}{\pi}}\,\frac{c_{33}kU_o}{p(c_{33}k^2+\rho' p^2)} \tag{3.17}$$

$$\tilde{\tilde{u}}_x^c = -\sqrt{\frac{2}{\pi}}\,\frac{U_o\rho' c_{13} kp F_2(k,p,x)}{c_{44}(c_{33}k^2+\rho' p^2)D(k,p)} \tag{3.18}$$

where

$$F_1(k,p,x) = \eta_1(\eta_1 - k\alpha_1)\sinh \eta_1 H \cosh \eta_2 H - \eta_1(\eta_2 - k\alpha_2)\sinh \eta_2 H \cosh \eta_1 x \tag{3.19}$$

$$F_2(k,p,x) = \alpha_2\eta_1(\eta_1 - k\alpha_1)\sinh \eta_1 H \sinh \eta_2 x - \alpha_1\eta_1(\eta_2 - k\alpha_2)\sinh \eta_2 H \sinh \eta_1 x \tag{3.20}$$

$$c_{44}(c_{13}+c_{44})^2 D(k,p) = (\eta_2)^{-1}\eta_1 f_2 h_1 \cosh \eta_1 H \sinh \eta_2 H - f_1 h_2 \sinh \eta_1 H \cosh \eta_2 H \tag{3.21}$$

$$(c_{13}+c_{44})k\eta_j\alpha_j = c_{44}\eta_j^2 - (k^2 c_{33} + \rho' p^2), \quad j = 1, 2 \tag{3.22}$$

$$f_j = c_{13}\eta_j^2 + (k^2 c_{33} + \rho' p^2)\,, \quad j = 1, 2 \tag{3.23}$$

$$h_j = c_{11}c_{44}\eta_j^2 + k^2[c_{13}(c_{13}+c_{44}) - c_{11}c_{33}] - \rho' c_{11} p^2 \tag{3.24}$$

$$\sqrt{2c_{11}c_{44}}\,\eta_j = \left\{k^2[\,c_{11}c_{33} - c_{13}(c_{13} + 2c_{44})] + \rho'(c_{11} + c_{44})p^2 - (-1)^j\beta(k,p)\right\}^{1/2}, \quad j = 1, 2 \qquad (3.25)$$

$$\beta(k,p) = \Big\{k^4(c_{11}c_{33} - c_{13}^2)[\,(c_{11}c_{33} - c_{13}^2) - 4c_{44}(c_{13} + c_{44})] + (\rho')^2p^4(c_{11} - c_{44})^2 + k^2p^2\rho'[\,2(c_{11} + c_{44})(c_{11}c_{33} - c_{13}^2) - 4c_{44}^2(c_{13} + c_{11}) - 4c_{11}c_{44}(c_{13} + c_{33})]\Big\}^{1/2} \qquad (3.26)$$

It can readily be shown that, for the isotropic case, equations 3.19, 3.20, 3.21, 3.22, and 3.25, reduce to equations 1.84, 1.85, 1.56, 1.35, 1.36, and 1.37, respectively.

The Fourier sine and cosine transforms will now be interchanged. Assuming zero initial and radiation conditions, taking the Laplace and Fourier cosine transforms of equations 3.5 and 3.8, and the Laplace and Fourier sine transforms of equations 3.4 and 3.9, gives:

$$\frac{d^2\tilde{\tilde{u}}_x^s}{dx^2} - \frac{c_{44}k^2 + \rho'p^2}{c_{11}}\tilde{\tilde{u}}_x^s - \frac{c_{13} + c_{44}}{c_{11}}\,k\,\frac{d\tilde{\tilde{u}}_z^c}{dx} = -\sqrt{\frac{2}{\pi}}\,\frac{c_{44}k}{c_{11}}\left(\bar{u}_x\right)_{z=0} \qquad (3.27)$$

$$\frac{d^2\tilde{\tilde{u}}_z^c}{dx^2} - \frac{c_{33}k^2 + \rho'p^2}{c_{44}}\tilde{\tilde{u}}_z^c + \frac{c_{13} + c_{44}}{c_{44}}\,k\,\frac{d\tilde{\tilde{u}}_x^s}{dx} = \frac{1}{c_{44}}\sqrt{\frac{2}{\pi}}\left[(c_{13} + c_{44})\frac{\partial\bar{u}_x}{\partial x} + c_{33}\frac{\partial\bar{u}_z}{\partial z}\right] \qquad (3.28)$$

$$\tilde{\tilde{\sigma}}_{xz}^c = c_{44}\frac{d\tilde{\tilde{u}}_z^c}{dx} + c_{44}k\tilde{\tilde{u}}_x^s - \sqrt{\frac{2}{\pi}}\,c_{44}\left(\bar{u}_x\right)_{z=0} \qquad (3.29)$$

$$\tilde{\tilde{\sigma}}^{s}_{xx} = c_{11}\frac{d\tilde{\tilde{u}}^{s}_{x}}{dx} - c_{13}k\tilde{\tilde{u}}^{c}_{z} \tag{3.30}$$

Note that these equations still contain only two transformed variables, and the boundary terms correspond to mixed conditions of the pressure shock type (this will be shown explicitly below). Thus, as mentioned in Section I, both types of mixed problem are tractable for the slab, in contrast to the situation for the plate with the cavity.

The boundary conditions are taken to be:

$$\left.\begin{aligned} u_x &= 0 \\ \sigma_{zz} &= -P_o\delta(t) \end{aligned}\right\}^{*},\quad z = 0 \tag{3.31}$$

$$\sigma_{xx} = \sigma_{xz} = 0, \quad x = \pm H \tag{3.32}$$

where P_o is a constant and $\delta(t)$ is the Dirac Delta function. It is seen from equation 3.10 that a set of conditions equivalent to equation 3.31 is:

$$\left.\begin{aligned} u_x &= 0 \\ \frac{\partial u_x}{\partial x} &= 0 \\ \frac{\partial u_z}{\partial z} &= -\frac{P_o}{c_{33}}\delta(t) \end{aligned}\right. ,\quad z = 0 \tag{3.33}$$

The remaining algebra is now very similar to that used in the first example and so will not be discussed here. The following expressions

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*The choice of the time dependence in the boundary conditions has been partly dictated by the fact that step loads have received considerable attention in the isotropic case.

are obtained for the transformed displacements:

$$\tilde{\tilde{u}}_z^c = \sqrt{\frac{2}{\pi}}\left[\frac{P_o C_{13} k^2 F_1(k, p, x)}{C_{44}(C_{33}k^2 + \rho' p^2) D(k, p)} + \frac{P_o}{C_{33}k^2 + \rho' p^2}\right] \tag{3.34}$$

$$\tilde{\tilde{u}}_x^s = -\sqrt{\frac{2}{\pi}} \; \frac{P_o C_{13} k^2 F_2(k, p, x)}{C_{44}(C_{33}k^2 + \rho' p^2) D(k, p)} \tag{3.35}$$

3.2. EXACT INVERSIONS

Inversion of the Laplace transforms for the two problems is now undertaken. It can readily be shown that the expressions given by equations 3.17, 3.18, 3.34, and 3.35, are even functions of η_1 and η_2, and consequently no branch point singularities arise. Possible pole type singularities occur at $p = 0$, $\sqrt{\rho'}\, p = \pm\sqrt{C_{33}}\, k$, and at the zeros of $D(k, p)$. On expanding the various expressions around the points $\sqrt{\rho'}\, p = \pm i\sqrt{C_{33}}\, k$, it is found that these points are not singularities. The point $p = 0$ is a simple pole of the second term on the right-hand side of equation 3.17. It is shown in Appendix B that the zeros of $D(k, p)$, for real k, are complex conjugate, pure imaginary, and simple, given by $p = \pm\, i\omega_n(k)$, say (note that the exceptions to the proof given there receive special attention here). Substituting these values of p into equation 3.21, set equal to zero, gives:

$$\frac{\tanh \eta_1[k, i\omega_n(k)]H}{\tanh \eta_2[k, i\omega_n(k)]H} = \frac{\eta_1[k, i\omega_n(k)]\, f_2[k, i\omega_n(k)]\, h_1[k, i\omega_n(k)]}{\eta_2[k, i\omega_n(k)]\, f_1[k, i\omega_n(k)]\, h_2[k, i\omega_n(k)]} \tag{3.36}$$

This last equation can readily be shown to be the frequency equation for the propagation of straight-crested symmetric waves in an infinite plate of thickness 2H. For the isotropic case it reduces to the Rayleigh-Lamb frequency equation, given by equation 1.109. The roots of equation 3.36 are presumably infinite in number.

On expanding the various expressions for large p it is seen that the order condition of the Laplace inversion theorem is satisfied in all cases. Hence, using Cauchy's theorem and residue theory, the Laplace transforms may be inverted. Then applying the inverse Fourier transforms (49), the inverses of equations 3.17, 3.18, 3.34, and 3.35, can be shown to be:

(i) Displacement input

$$u_z = U_o - \frac{4U_o\rho' C_{13}}{\pi C_{44}} \sum_{n=1}^{\infty} \int_o^{\infty} \frac{k \sin kz\, F_1[k, i\omega_n(k), x] \cos \omega_n(k)t}{[C_{33}k^2 - \rho'\omega_n^2(k)]\, N[k, i\omega_n(k)]}\, dk \qquad (3.37)$$

$$u_x = - \frac{4U_o\rho' C_{13}}{\pi C_{44}} \sum_{n=1}^{\infty} \int_o^{\infty} \frac{k \cos kz\, F_2[k, i\omega_n(k), x] \cos \omega_n(k)t}{[C_{33}k^2 - \rho'\omega_n^2(k)]\, N[k, i\omega_n(k)]}\, dk \qquad (3.38)$$

where

$$\begin{aligned} C_{44}(C_{13}+C_{44})^2 N(k,p) = {} & H[(\eta_2)^{-1}\eta_1 f_2 h_1 \delta_1 - f_1 h_2 \delta_2] \sinh \eta_1 H \sinh \eta_2 H \\ & + (\eta_2)^{-1}[f_2 h_1 \delta_1 - (\eta_2)^{-1}\eta_1 f_2 h_1 \delta_2 + 2\eta_1 h_1 (C_{13}\eta_2\delta_2 + \rho') \\ & \qquad + 2C_{11}\eta_1 f_2 (C_{44}\eta_1\delta_1 - \rho'] \cosh \eta_1 H \sinh \eta_2 H \\ & + H[(\eta_2)^{-1}\eta_1 f_2 h_1 \delta_2 - f_1 h_2 \delta_1] \cosh \eta_1 H \cosh \eta_2 H \\ & - 2[h_2(C_{13}\eta_1\delta_1 + \rho') + C_{11} f_1 (C_{44}\eta_2\delta_2 - \rho')] \sinh \eta_1 H \cosh \eta_2 H \end{aligned} \qquad (3.39)$$

$$\delta_j = (4C_{11}C_{44}\eta_j)^{-1}[2\rho'(C_{11}+C_{44}) - (-1)^j(2\beta)^{-1}\gamma], \quad j = 1, 2 \tag{3.40}$$

$$\gamma = \Big\{4(\rho')^2(C_{11}-C_{44})^2p^2 + 2\rho' k^2[2(C_{11}+C_{44})(C_{11}C_{33}-C_{13}^2) - 4C_{44}^2(C_{11}+C_{13}) - 4C_{11}C_{44}(C_{13}+C_{33})]\Big\} \tag{3.41}$$

For the isotropic case equation 3.39 reduces to equation 1.111.

(ii) Stress input

$$u_z = \frac{4P_oC_{13}}{\pi C_{44}} \sum_{n=1}^{\infty} \int_o^{\infty} \frac{k^2 \cos kzF_1[k, i\omega_n(k), x] \sin \omega_n(k)t}{[C_{33}k^2 - \omega_n^2(k)]\,\omega_n(k)N[k, i\omega_n(k)]}\,dk \tag{3.42}$$

$$u_x = -\frac{4P_oC_{13}}{\pi C_{44}} \sum_{n=1}^{\infty} \int_o^{\infty} \frac{k^2 \sin kzF_2[k, i\omega_n(k), x] \sin \omega_n(k)t}{[C_{33}k^2 - \omega_n^2(k)]\,\omega_n(k)N[k, i\omega_n(k)]}\,dk \tag{3.43}$$

It is to be understood that u_z and u_x in both cases are zero for times less than the arrival time of the wave front. Further, it will be assumed that the integrals in equations 3.37, 3.38, 3.42, and 3.43, are well behaved (in this case this is difficult to prove, since the modes are not known explicitly), and that the expressions for the displacements as given by these equations satisfy the boundary value problems.

3.3. FAR-FIELD APPROXIMATIONS

Two types of far-field approximations will now be derived. The first type, which describes the head of the pulse, will be derived for both isotropic and anisotropic cases, whereas the second, which is of the stationary phase type, will be determined only in the case of the displacement input in an isotropic slab.

The techniques for deriving head of the pulse approximations have been discussed in Section 1 (§1.3) and they will now be applied here. Though the roots of equation 3.36 are unknown, except in the isotropic case, the analytic behavior of the low frequency, large wavelength portion of the lowest mode may be determined on substituting an expression of the type $\omega_1(k) = \ell k + nk^3$,* where ℓ and n are constants, into the equation. However, this not only leads to very cumbersome algebra, but also involves making certain assumptions regarding the magnitudes of the various parameters. An alternative method, given by Chree (58) for the analogous rod problem, will be adopted here.

Solutions of the form

$$u_x = \Psi(x) \cos kz \cos \omega t$$

$$u_z = \Theta(x) \sin kz \cos \omega t$$

are substituted into equations 3.4 and 3.5, and from the resulting equations two uncoupled, fourth order, ordinary differential equations for Ψ and Θ may be derived. Series solutions are then written for these differential equations, and, on substituting these series into the boundary conditions, an expansion of the frequency equation in powers of H is obtained. Thus the problem of suitably expanding η_j is circumvented.

*That the expression is of this type follows from the fact that $\omega_n^2(k)$ is an even function of k.

After considerable algebra, this procedure gives, to the order of H^2:

$$\omega_1(k) = kc_p(1 - \delta k^2) \tag{3.44}$$

where

$$\rho' C_{11} c_p^2 = C_{11} C_{33} - C_{13}^2 \tag{3.45}$$

$$6 C_{11}^2 \delta = C_{13}^2 H^2 \tag{3.46}$$

Using equations 3.2 and 3.3, it can readily be seen that, in the isotropic case, equations 3.44 and 1.172 are equivalent. Positiveness of the strain energy function requires that ((57), page 14)

$$C_{11} C_{33} - C_{13}^2 > 0$$

so that the limiting phase velocity, as given by equation 3.44, is real. An alternative form of the equation can be written in terms of E, E', σ, σ', Young's modulus and Poisson's ratio parallel and perpendicular to the axis of material symmetry, respectively (these are the parameters most readily measured). Using the results (58)

$$C_{33} = E^2(1 - \sigma')[E(1 - \sigma') - 2E'\sigma^2]^{-1}$$

$$C_{13} = EE'\sigma[E(1 - \sigma') - 2E'\sigma^2]^{-1}$$

$$C_{11} = E'(E - E'\sigma^2)\left\{(1 + \sigma')[E(1 - \sigma') - 2E'\sigma^2]\right\}^{-1}$$

equations 3.45 and 3.46 may be written:

$$c_p = \left[\frac{E^2}{\rho'(E - E'\sigma^2)} \right]^{1/2}$$

$$\delta = \frac{H^2}{6}\left[\frac{E\sigma(1+\sigma')}{E - E'\sigma^2}\right]^2$$

These alternative forms exhibit more clearly an interesting feature of the results, viz., their dependence on the elastic properties perpendicular to the axis of symmetry. This is in contrast to the analogous rod expression (58), where E and σ alone are involved. This however is not unexpected, since in the isotropic case the plate velocity depends on σ whereas the rod velocity $(\sqrt{E/\rho'})$ does not.

Differentiating equation 3.44 gives:

$$c_g = \frac{d\omega_1(k)}{dk} = c_P - 3\delta k^2$$

Since the second term on the right hand side of this equation is always positive, the group velocity has a maximum at $k = 0$. It is henceforth assumed that this maximum greater than any other possible maxima of the first mode, i.e., on the basis of the first mode alone, it is assumed that the contribution from the vicinity of $k = 0$ describes the head of the pulse. The plausability of this assumption follows from the fact that it is true in the limiting case of the isotropic plate and it also holds for the more complex problem of the monoclinic plate (56), of which the present example is a limiting case.

On substituting the first term of equation 3.44 into equations 3.19, 3.20, and 3.39, the following limiting values are obtained:*

*In thedetermination of these limiting values set $\eta_j = k\psi_j$, $\eta_j\delta_j = \varphi_j$, $j=1, 2$, where the ψ_j's and φ_j's are constants. Considerable algebra may be avoided on noting that an explicit determination of these constants is unnecessary, since they cancel in the final results.

$$\frac{F_1[k, i\omega_1(k), x]}{[C_{33}k^2 - \rho'\omega_1^2(k)]\,N[k, i\omega_1(k)]} \xrightarrow[k \to 0]{} \frac{C_{44}}{2\rho' C_{13} k^2}$$

$$\frac{F_2[k, i\omega_1(k), x]}{[C_{33}k^2 - \rho'\omega_1^2(k)]\,N[k, i\omega_1(k)]} \xrightarrow[k \to 0]{} -\frac{C_{44}x}{2\rho' C_{11} k}$$

Substituting these values into equations 3.37, 3.38, 3.42, and 3.43, and retaining the second term of equation 3.44 in the trigonometric functions, one finds, on extending the upper limit of integration to infinity (which is permissible, since the integrals so added are of order $z^{-1/2}$):

(i) Displacement input

$$\frac{u_z}{U_o} = 1 + [\tfrac{1}{3} + \int_o^{\beta'} Ai(-\xi)\,d\xi] - [\tfrac{1}{3} + \int_o^{\beta''} Ai(-\xi)\,d\xi] + O(z^{-1/2}),$$

$$0 \leq k \leq \epsilon_1 \qquad (3.47)$$

$$\frac{u_x}{U_o} = \frac{C_{13}x}{C_{11}(3c_P t\,\delta)^{1/3}}[Ai(-\beta') + Ai(-\beta'')] + O(z^{-1/2}), \quad 0 \leq k \leq \epsilon_1$$

(3.48)

where

$$\beta' = \frac{(c_P t - z)}{(3c_P t\delta)^{1/3}} \qquad (3.49)$$

$$\beta'' = \frac{(c_P t + z)}{(3c_P t\delta)^{1/3}} \qquad (3.50)$$

(ii) Stress input

$$u_z = \frac{P_o}{\rho' c_P} \left\{ [\frac{1}{3} + \int_0^{\beta''} Ai(-\xi)\, d\xi] + [\frac{1}{3} + \int_0^{\beta'} Ai(-\xi)\, d\xi] - 1 \right\} + O(z^{-1/2}),$$

$$0 \leq k \leq \epsilon_1 \qquad (3.51)$$

$$u_x = \frac{P_o C_{13} x[Ai(-\beta') - Ai(-\beta'')]}{\rho' C_{11} c_P (3 c_P t \delta)^{1/3}} + O(z^{-1/2}), \quad 0 \leq k \leq \epsilon_1 \qquad (3.52)$$

Some interesting features of these expressions should be noted. If the k^3 term in the trigonometric functions is deleted in the limiting processes, then, as in the rod case, expressions are obtained for the solutions far ahead and far behind the wave front. Far ahead of the wave front $(z >> c_P t)$ one finds that u_z, as given by both equation 3.47 and 3.51, is zero. Far behind the wave front $(z << c_P t)$ one finds that

$$u_z = U_o$$

in the case of the displacement input, and

$$u_z = \frac{P_o}{\rho' c_P}$$

in the case of the stress input. Thus far ahead and far behind the wave front the above results are in agreement with the elementary theory* predictions, which can readily be shown to be:

(i) Displacement input

$$u_z = U_o H(t - \frac{z}{c_P})$$

*By elementary theory is meant the theory described by the equations $\partial^2 u_z/\partial z^2 = (1/c_P^2)(\partial^2 u_z/\partial t^2)$, $\sigma_{zz} = \rho' c_P^2 (\partial u_z/\partial z)$.

where H denotes the Heaviside step function.

(ii) Stress input

$$u_z = \frac{P_o}{\rho' c_P} H(t - \frac{z}{c_P})$$

Since the roots of equation 3.36 have not been evaluated, approximations of the stationary phase type cannot be written for equations 3.37, 3.38, 3.42, and 3.43, except in the isotropic case. Assuming isotropy, application of equation 1.158 to equations 3.37 and 3.38 -- the example of most interest in the context of the present work -- gives, on retaining only the lowest mode contribution, and using equations 1.180 and 1.181:

$$\frac{\pi(1-2\sigma)}{8\sigma U_o}(u_\rho - U_o) = - M_4(\rho, S)V_1(\rho, S, \zeta) + O(\rho^{-1}), \quad \epsilon_1 \leq S \leq S_m - \epsilon$$

(3.53)

$$\frac{\pi(1-2\sigma)u_\zeta}{8\sigma U_o} = - M_5(\rho, S)V_2(\rho, S, \zeta) + O(e^{-1}), \quad \epsilon_1 \leq S \leq S_m - \epsilon \quad (3.54)$$

where

$$M_4(\rho, S) = (\frac{\rho\pi}{S})^{1/2} \frac{\Omega_1^2(S)}{[a^2S^2 - \Omega_1^2(S)]} [1 + \text{cotan}\, \rho f_-(S)] \quad (3.55)$$

$$M_5(\rho, S) = (\frac{\rho\pi}{S})^{1/2} \frac{\Omega_1^2(S)}{[a^2S^2 - \Omega_1^2(S)]} [1 - \tan \rho f_-(S)] \quad (3.56)$$

$$\rho = z/H$$

$$\zeta = x/H$$

and the rest of the notation is as in Section I (§1.3).

Note that the first term on the right hand side of equation 3.53 (and 3.54) is of order $\rho^{-1/2}$, which is of higher order than the U_o term. This is in contrast to the situation for the plate with the cavity, where both these terms are of the same order. It means that, for the far-field, the U_o term dominates in this portion of the response. This representation of the solution, though somewhat unsatisfactory, has more validity than extensions of equations 3.47 and 3.48 into this region $(0 < S \leq S_m - \epsilon)$ since they are only valid in the neighborhood of $t = z/c_P$.

Further discussion of the above results will be given in the next section.

Section IV. NUMERICAL RESULTS

In the problems discussed in the previous sections various approximations to the exact solutions were given. Specifically, stationary phase approximations to portions of the lowest mode were made for the displacements in the problems of the displacement input in the plate with the cavity (Section I, equations 1.182 and 1.183), the plate with the body force (Section II, equations 2.38 and 2.39), and the displacement input in the isotropic slab (Section III, equations 3.53 and 3.54). In this section some numerical evaluations of these approximate solutions are given and several comparisons of the results are made. In these calculations the parameters ρ and σ are held fixed, with values 20 and 0.31, respectively, and the parameters R and ζ are varied. The head of the pulse approximations to the displacements in the slab problems (Section III, equations 3.47, 3.48, 3.51, and 3.52) are also evaluated (in both isotropic and transversely isotropic cases) and the results are compared with the corresponding stationary phase results (for the displacement input in the isotropic slab).

A physical interpretation of the assumptions under which equations 1.182, 1.183, 2.38, 2.39, 3.53 and 3.54 are valid, is that frequencies above a certain value, viz., that value corresponding to the lowest mode group velocity, do not arise, as can be seen on inspection of the modes, Fig. 2. Thus any numerical work based on these equations is suitable for comparison with measurements of a recorder having a low frequency bandpass, or, when used in conjunction with the superposition integral, for input functions which do not excite the higher frequencies.

Another insight into the restrictions on these equations is afforded by Fig. 4, taken from (10), which shows the predominant period versus time after arrival of the "wave front" curves, for the three lowest modes of the Rayleigh-Lamb symmetric frequency equation. The predominant period $2\pi/\Omega_n(K)$ is the central period of the dominant group at a given time τ at station ρ and $1/Cg(K) - 1/b = \frac{1}{\rho}(\tau - \frac{\rho}{b})$ is the time after arrival of the head of the pulse, which, on the basis of the first mode alone, is the time after arrival of the wave front (if all modes are admitted, the wave front speed is of course c_d/c_s). Note that the portions of $2\pi/\Omega_2$, corresponding to negative group velocities are not shown.

It is seen from Fig. 4 that for times out to the point A the disturbance is governed by the low frequency, long wavelength portion of the lowest mode, though it should be pointed out that portions of the higher modes also contribute in this region. [In fact some portions contribute for times less than $\frac{1}{\rho}(\tau - \frac{\rho}{b}) = 0$, since the higher mode group velocities approach $a = c_d/c_s < b = c_P/c_s$ for large values of the wavenumber.] For arrival times greater than A the second and third modes become operative. If the minimum of Ω_2 (= 2.75) is chosen as the upper bound on permissable frequency, then periods down to 2.28 -- the long time maximum of $2\pi/\Omega_2$ -- are admitted. With this restriction the region in which only first mode contributions arise can be extended to the point B (the point corresponding to the minimum group velocity). No changes in these conclusions occur if the negative group velocities are taken into account. Note that in the time region DB (D is the intersection of Ω_1 with the vertical line through C) two portions of the lowest mode contribute. If only times out to E are admitted -- corresponding to an upper bound on frequency well below that of possible coupling between the lowest and higher modes -- then the shorter wavelength, higher frequency portions of the lowest mode do not contribute. As pointed out by Miklowitz (10), it is the portion out to E that represents the strongest approximation in work of this nature. These various time regions are indicated on the figures given at the end of the section. For the region DBC only the contributions from the portion DB are shown. To assess the contribution from the portion BC, use would have to be made of equations 1.166, 1.169, and the corresponding equations in the body force and slab problems. Also, the Airy phase solutions (§1.3) would have to be employed in the immediate vicinity of B.

The results of numerical computations based on equations 1.182, 1.183, 2.38, 2.39, 3.53, 3.54, 3.47, 3.48, 3.51, and 3.52 are shown in Figures 5 through 18. Figures 5 through 8 give the radial and vertical displacements in the plate cavity problem; the corresponding results for the body force problem are shown in Figures 9 through 12. Comparisons of the results are shown in Figures 13 and 14. The results for the horizontal and vertical (modified) displacements in the transversely isotropic slab, as given by the head of the pulse approximation, are shown in Figures 15 and 16. The displacements in the isotropic slab (displacement input), as given by both the head of the pulse and stationary phase approximations, are given in Figures 17 and 18. Following

Miklowitz (10), the wave front terms in the above stationary phase solutions were obtained on letting $S \rightarrow 0$ (excluding $S = 0$) in equations 1.182, 1.183, 2.38, 2.39, 3.53, 3.54. This point will be discussed later.

Fig. 5 shows that, for the radial displacement, as R increases the ratios of the successive amplitudes of oscillation (around the long time, or static, value) to the maximum amplitude decreases. For instance, for $R = 0.01$, $\zeta = 1$, the ratio of the second to the first (maximum) amplitude is about 0.43, whereas for $R = 2$, $\zeta = 1$, this same ratio is about 0.28. The former result should be compared with the corresponding result for the horizontal displacement in the slab, Fig. 17, where this ratio is about 0.48. Thus the height of the second peak w.r.t. the initial peak is smaller in the plate cavity problem than in the slab problem and this effect becomes more pronounced as R increases. Note that this trend with increasing R is as expected. The calculations were for fixed ρ and so as R increases the distance between source and observation station decreases. Hence the results should take on more far-field character as R gets smaller, i.e., they should get closer to the slab solutions.

No such effect is evident in the vertical displacements, Fig. 6. The ratio of the second to the first amplitude is about 0.85 for both plate-cavity (all R) and slab problems. However the value of R does effect the phases in both radial and vertical displacements. As R increases the initial peak in the radial displacements arrives earlier, whereas in the vertical displacements it arrives later.

The ζ-dependence of the solutions can be seen from Figures 7 and 8. It is seen that the radial displacements are a maximum at the plate center and decrease towards the lateral plate surfaces. The vertical displacements are zero at the plate center and reach their maximum values at the lateral surfaces. This behavior is as expected in view of the nature of the applied boundary conditions.

Inspection of Figures 9 through 12 shows that the above comments on amplitudes and ζ-dependence also hold in the case of the body force solutions. In this case however the phase of the solutions appears to be independent of R. One of the major points of interest in connection with the body force solutions is how closely they agree with the plate cavity solutions. Examination of the curves shows that, in general, there is good agreement in the broad overall features. One of the difficulties in making this comparison is the fact that the static solutions (about which the oscillations occur) in the two problems are different.

To facilitate such comparisons, and to have solutions more suitable for possible experimental analysis, the static solutions are taken to be the same. Inspection of equations 1.182, 1.183, 2.38, and 2.39, shows that the ratio of the static solutions is:

$$\frac{\text{static solution in plate cavity problem}}{\text{static solution in body force problem}} = \frac{2\pi(1-\sigma)}{1-2\sigma}$$

$$= 11.41, \text{ for } \sigma = 0.31$$

Hence, on multiplying equations 2.38 and 2.39 by this numerical factor, solutions having the same static value are obtained. These solutions, viz.,

$$\frac{2\pi\mu H c_s^2\, 11.41}{\lambda P_o} u_\rho \quad \text{and} \quad \frac{2\pi\mu H c_s^2\, 11.41}{\lambda P_o} u_\zeta$$

are termed the modified displacements; in an experiment the necessary modification can be achieved by suitably adjusting P_o (the load factor).

Comparisons between these modified displacements and the displacements in the plate cavity problem are shown in Figures 13 and 14. Inspection of these figures shows that on the whole the agreement is quite good, particularly for small R. The discrepancies which arise for the large value of R are not surprising in view of the fact that an increase in R holding ρ fixed corresponds to a decrease in the distance between source and observer. The similarity of the results (for small R) out to moderately large arrival times, i.e., for moderately large wavenumbers, should be noted. It should also be noted that in the light of the rod situation discussed earlier (general introduction), viz., the good agreement for large distances down the rod between Fox's (9) body force solutions and experiments simulating nonmixed conditions, the present work suggests that in the plate-cavity problem the mixed case is a good model for the nonmixed case, for large source-observer distances.

In calculating the head of the pulse approximations to the displacements in the transversely isotropic slab, Figures 15 and 16, values of the Airy function were obtained from Miller (59) and of the integral of the Airy function from Jones (60). For large values of z (or ρ), and hence t (or τ), β'', as given by equation 3.50, is large and positive. Equation 1.178 shows that in this case $Ai(-\beta'')$ is of order $\rho^{-1/4}$ and so, in equations 3.48 and 3.52 is small in comparison to $Ai(-\beta')$, for small β'. Also $\frac{1}{3} + \int^{\beta''} Ai(-\xi)\,d\xi$ approaches unity as β'' gets large (60) and so the first and third terms on the right sides of equations 3.47 and 3.51 cancel. The fact that the terms involving β'' do not contribute appreciably for large ρ is as expected, since their counterparts in the stationary phase approximations do not arise either. For purposes of generality, the ordinates and abscissae in Figures 15 and 16 are chosen differently from those in the other figures. Reduction to displacements and arrival times can readily be achieved in a specific case.

Note that in these head of the pulse solutions disturbances are shown which arrive earlier than $\tau = \rho/b$. These arise mathematically in that the various functions involved have nonzero values for small negative values of β'. However these portions of the response are difficult to interpret physically, since the approximations as a whole stem from a region in which the maximum group velocity is b. They are retained here for the reason that the corresponding portions of the solutions of Fox et al. (4) to a similar rod problem agree with their experimental results.

The surface horizontal and vertical displacements in the isotropic slab (displacement input), as given by both head of the pulse and stationary phase approximations, are shown in Figures 17 and 18. These comparisons not only throw some light on the question of the regions of validity of the two approximations, but also give a measure of the validity of the procedure of letting $S \to 0$ in the stationary phase solutions. It is seen that, in general, the two approximations are reasonably close. The head of the pulse solutions oscillate more rapidly and have larger amplitudes. In the case of the horizontal displacement the maximum amplitudes differ by about 12% (at the plate surface; the difference is more pronounced at the plate center). Differences in the immediate vicinity of $S = 0$, i.e., $\tau = \rho/b$, are more marked, particularly in the case of the vertical displacements, but the head of the pulse solutions should be more accurate in this region. As yet no procedures exist whereby the regions of validity of the two representations can be precisely demarcated and the final criterion would have to be experimental. However a reasonable appraisal can be made on the basis of the figures. Thus for $\tau/\rho - 1/b < 0.025$ the head of the pulse approximation would appear to be the more accurate. In any event the results indicate that both representations are necessary to obtain an overall picture of the response, particularly in the region out to the initial peak. Though this has only been shown for the slab it indicates that a similar situation exists for the plate with the cavity and other axially symmetric plate problems. This suggests that numerical computation of the head of the pulse solutions in the plate-cavity problem (equations 1.175 and 1.176) would be desirable.

Some comments on the accuracy of the numerical results should be made. Miklowitz (10) used a simple central difference technique in calculating $\Omega_1'(S)$ and $\Omega_1''(S)$ from the $\Omega_1(S)$ versus S curve shown in Fig. 2. One of his numerical checks was a comparison of the maxima of the functions V_1 and V_2 (equations 1.180 and 1.181) obtained from the difference technique and those obtained using exact expressions for the derivatives Ω_1' and Ω_1''. The results obtained were:

	S	$\Omega_1''(S)$	V_1	V_2
Exact	1.15	-1.3641		0.1757×10^{-2}
Approx.	1.15	-1.2410		0.1873×10^{-2}
	1.27	-1.7895	0.8803×10^{-3}	
	1.27	-1.3948	1.016×10^{-3}	
	1.30	-1.5761		0.1739×10^{-2}
	1.30	-1.4055		0.2054×10^{-2}
	1.41	-1.3263		0.2272×10^{-2}
	1.41	-1.3018		0.2267×10^{-2}

The agreement is seen to be quite good. Another check used by him was a comparison between the numerical results and those obtained through the leading terms in "wave front" ($S \rightarrow 0$) expansions. Again good agreement was obtained. In the present numerical work the results were obtained using an IBM 7090 computer and ample checks were obtained using independent calculations.

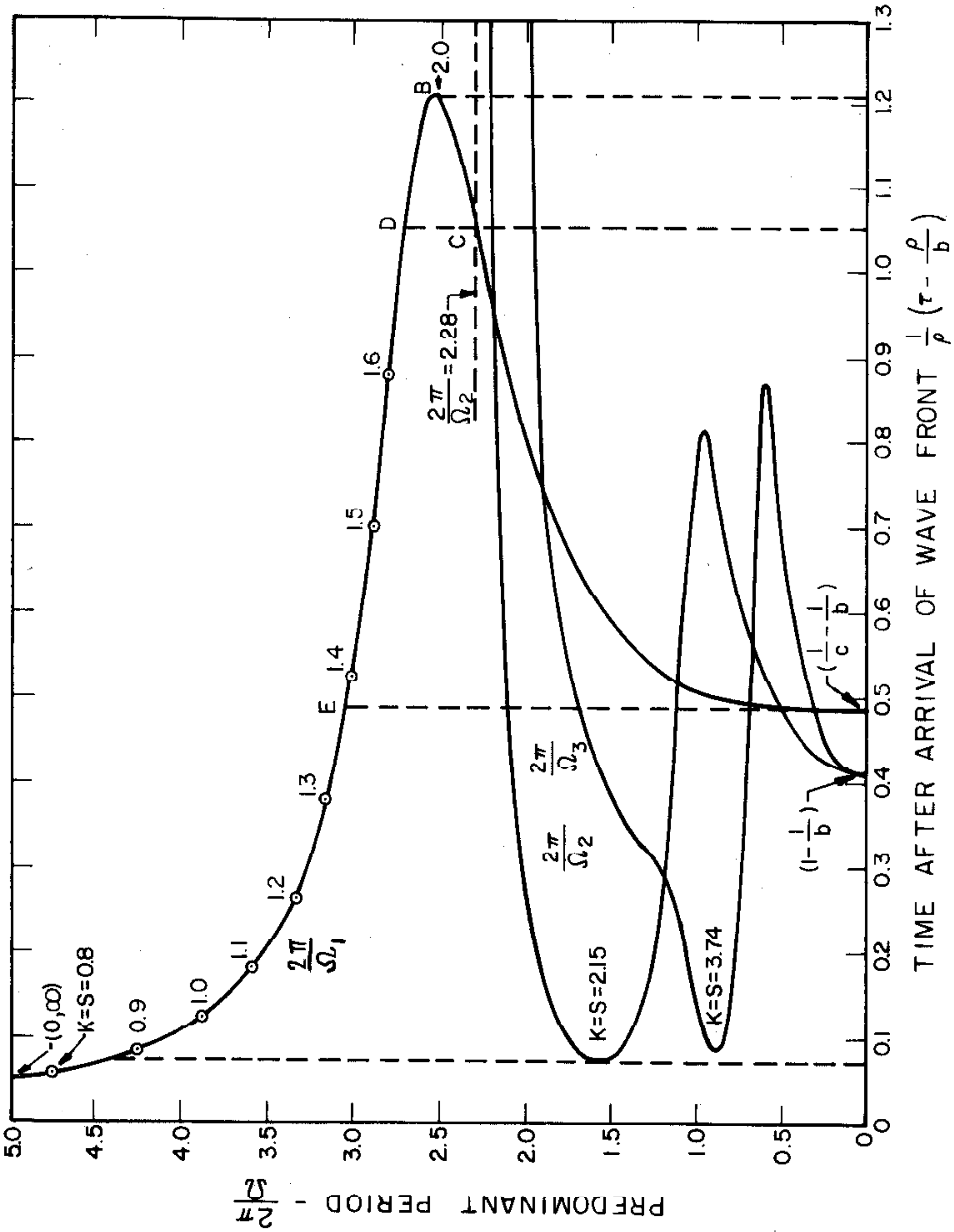

Fig. 4. Predominant period versus time after arrival of wave front ($\sigma = .31$).

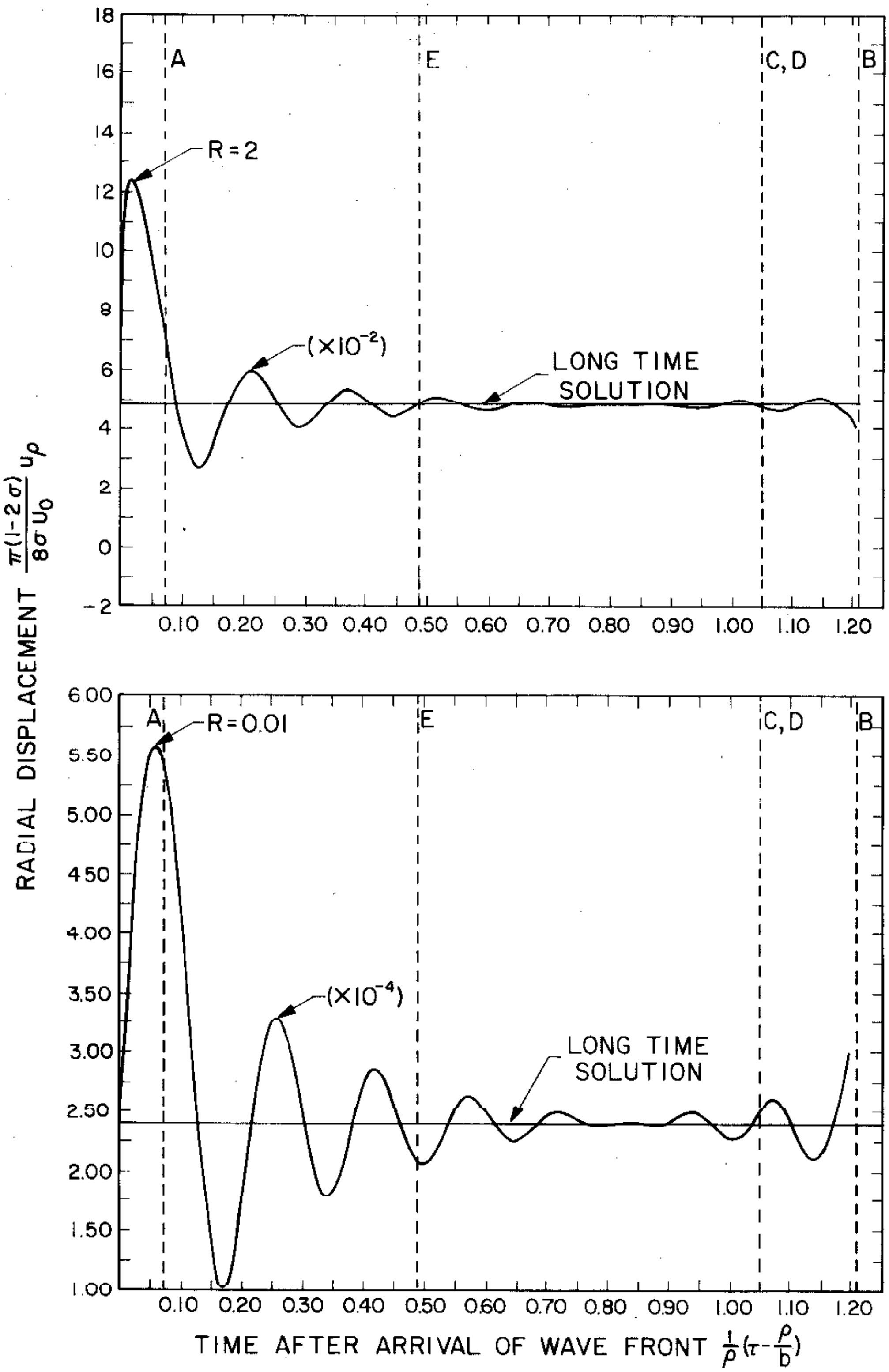

Fig. 5. Radial displacements in plate-cavity problem versus time after arrival of wave front: Dependence on R (ρ = 20, ζ = 1, σ = .31).

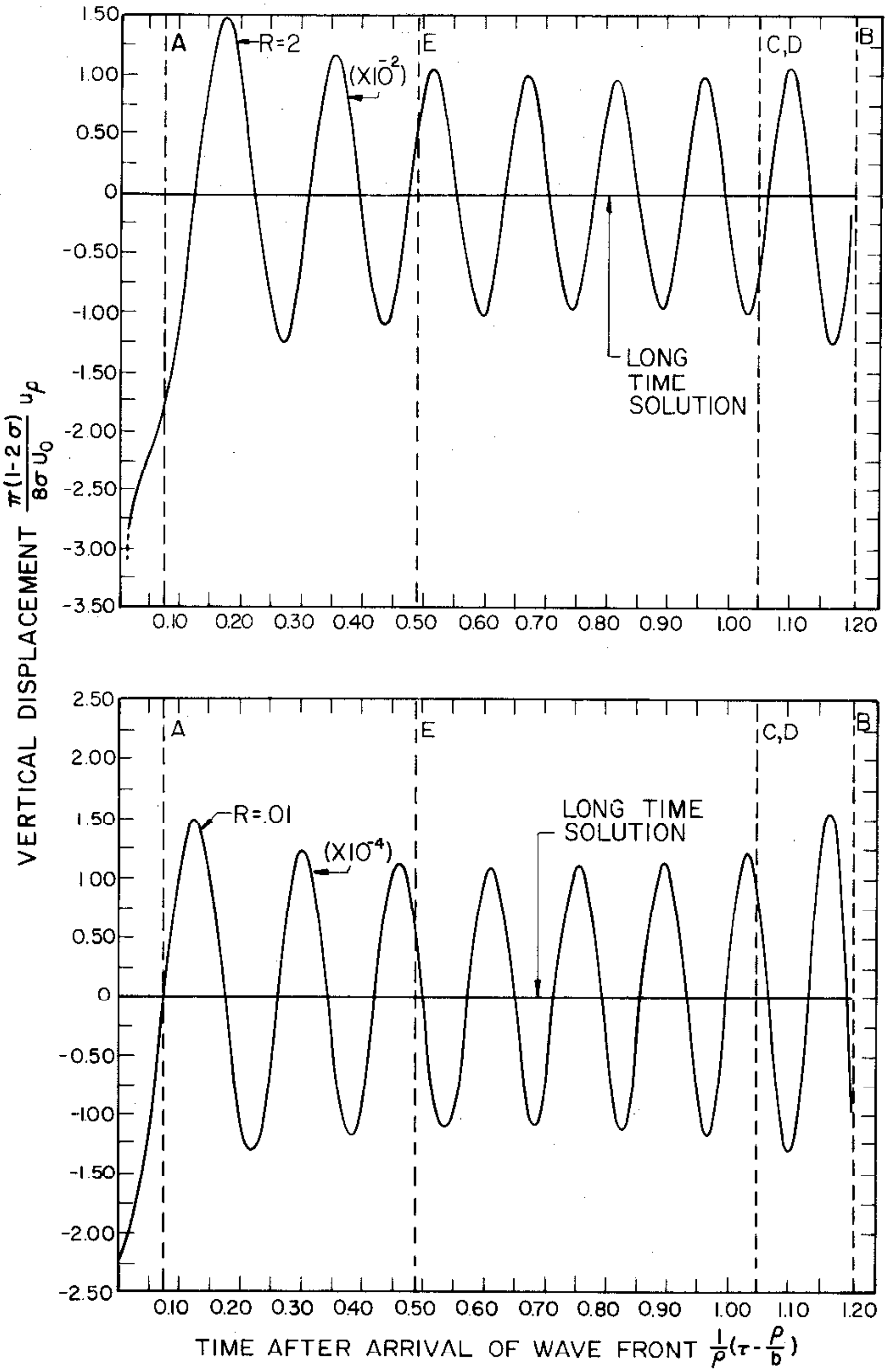

Fig. 6. Vertical displacements in plate-cavity problem versus time after arrival of wave front: Dependence on R (ρ = 20, ζ = 1, σ = .31).

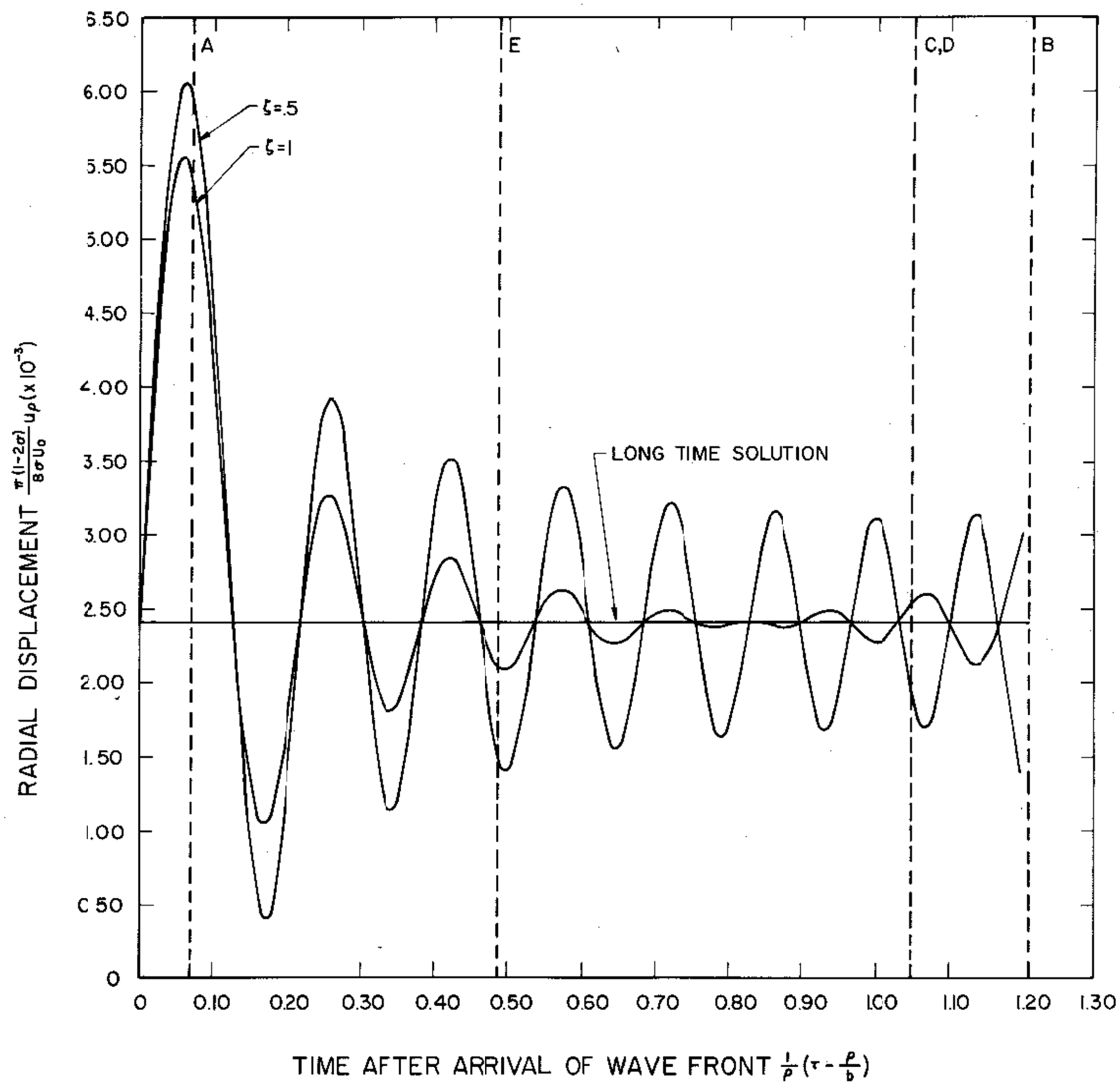

Fig. 7. Radial displacements in plate-cavity problem versus time after arrival of wave front: Dependence on ζ (R = 1, $\rho = 20$, $\sigma = .31$).

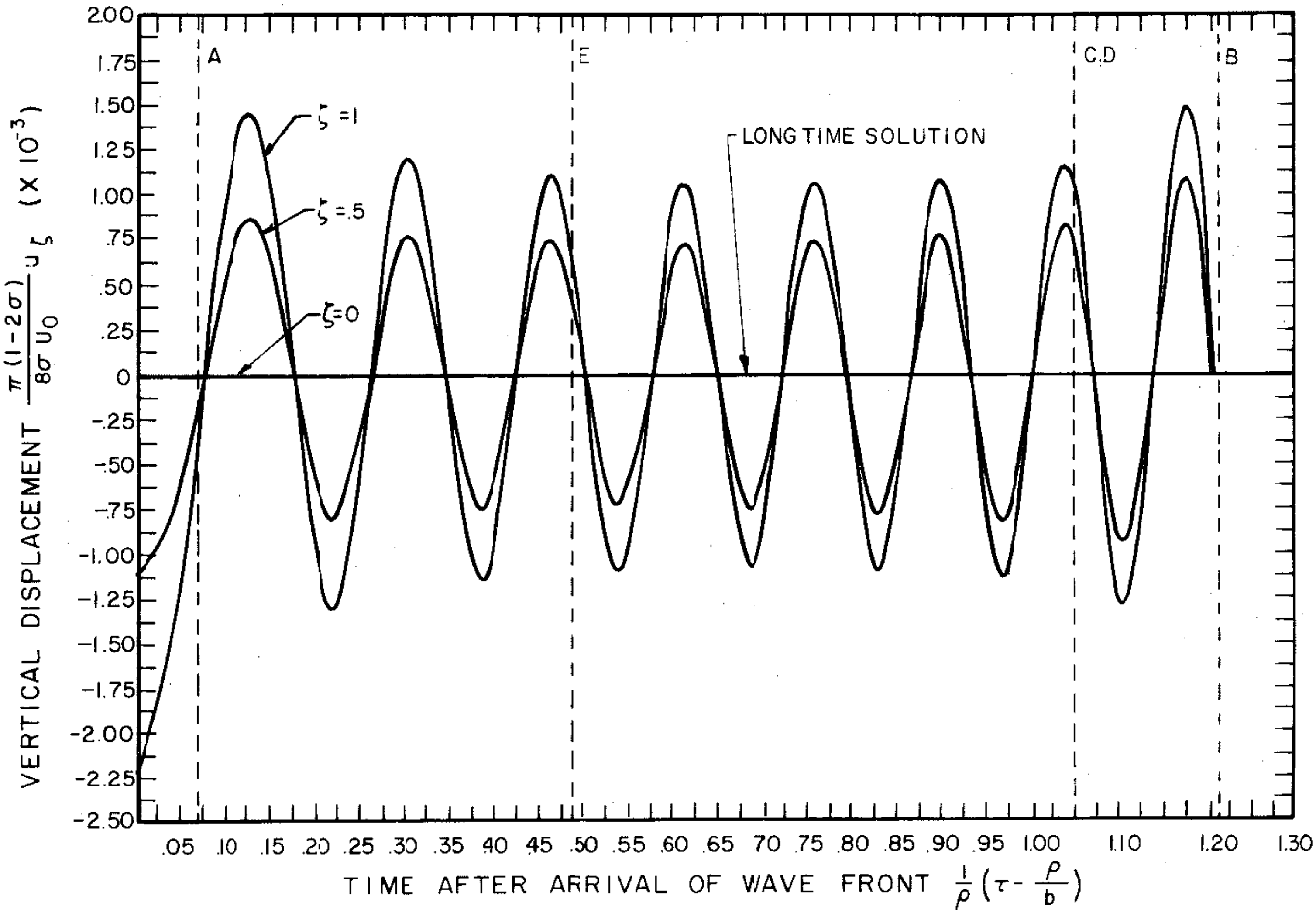

Fig. 8. Vertical displacements in plate-cavity problem versus time after arrival of wave front: Dependence on ζ ($R = 1$, $\rho = 20$, $\sigma = .31$).

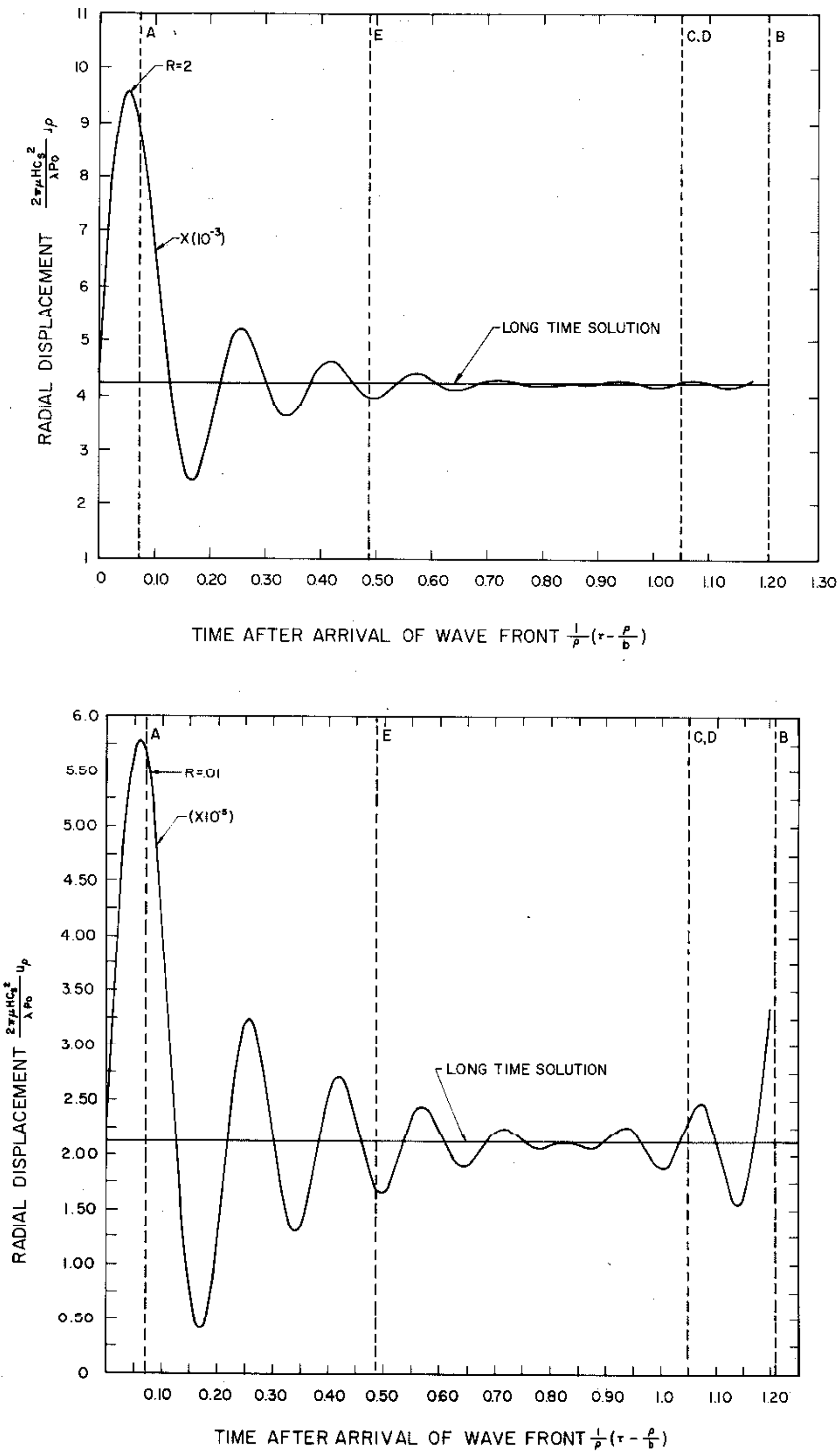

Fig. 9. Radial displacements in body force problem versus time after arrival of wave front: Dependence on R ($\rho = 20$, $\zeta = 1$, $\sigma = .31$).

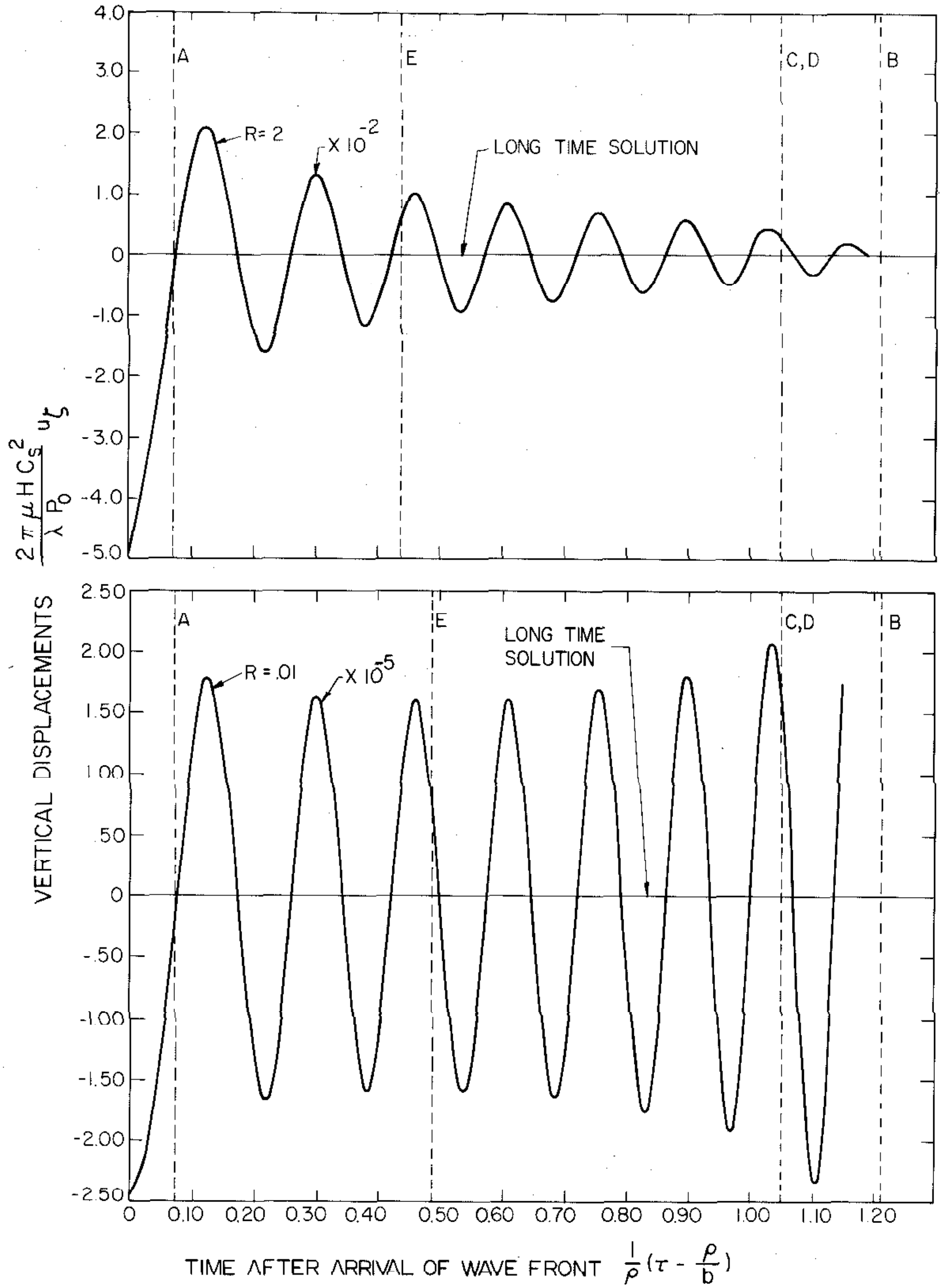

Fig. 10. Vertical displacements in body force problem versus time after arrival of wave front: Dependence on R ($\rho = 20$, $\zeta = 1$, $\sigma = .31$).

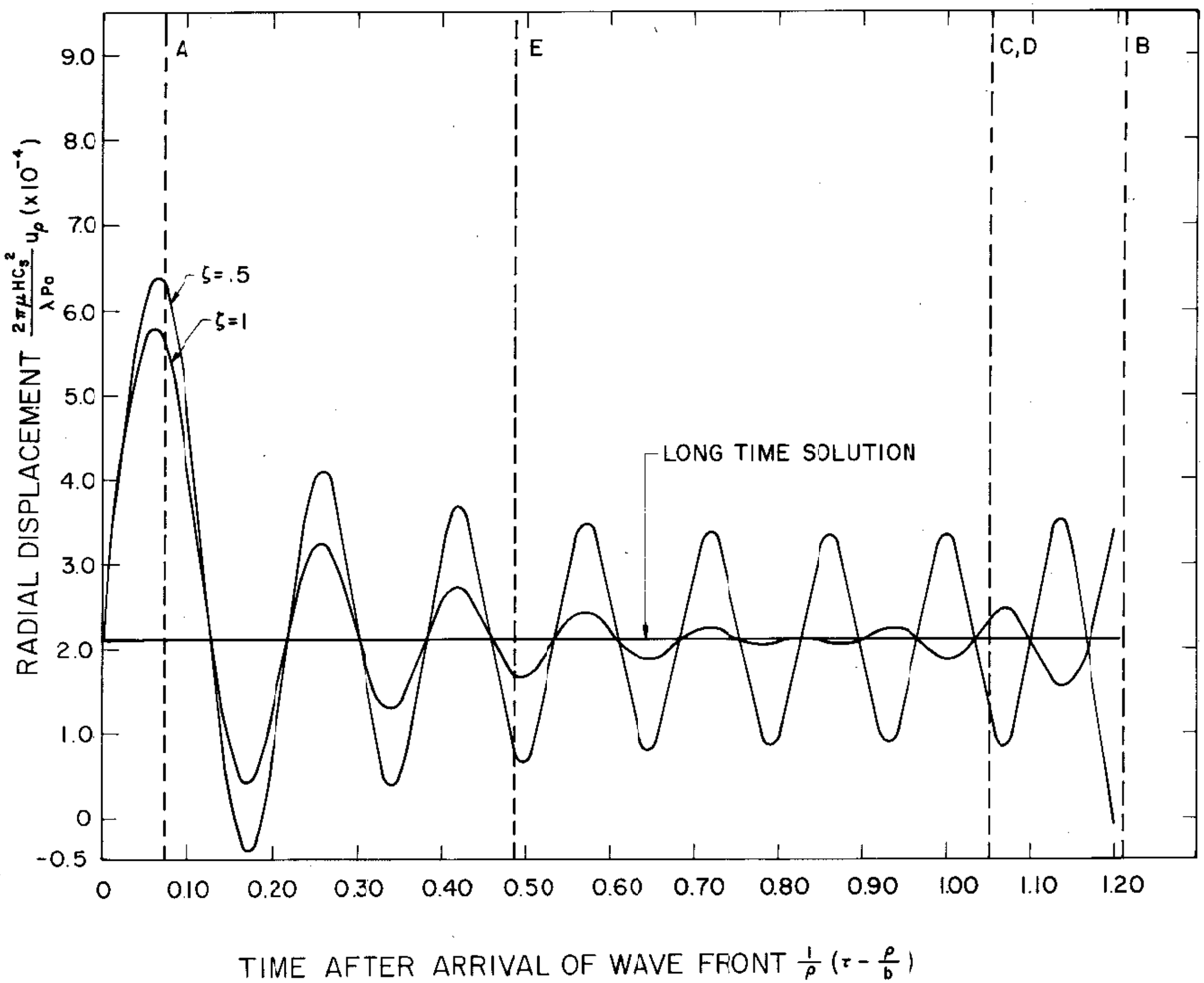

Fig. 11. Radial displacements in body force problem versus time after arrival of wave front: Dependence on ζ ($R = 1$, $\rho = 20$, $\sigma = .31$).

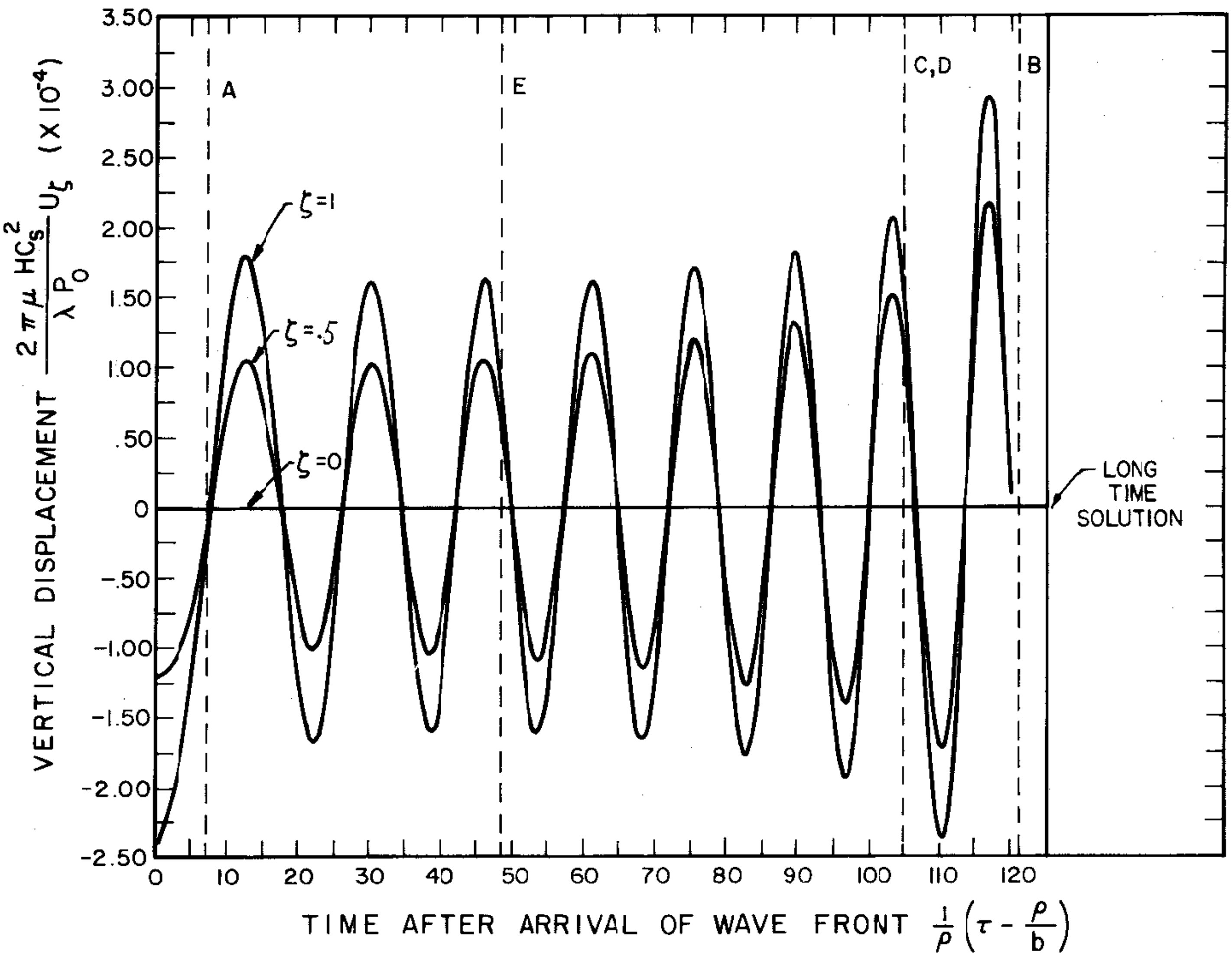

Fig. 12. Vertical displacements in body force problem versus time after arrival of wave front: Dependence on ζ ($R = 1$, $\rho = 20$, $\sigma = .31$).

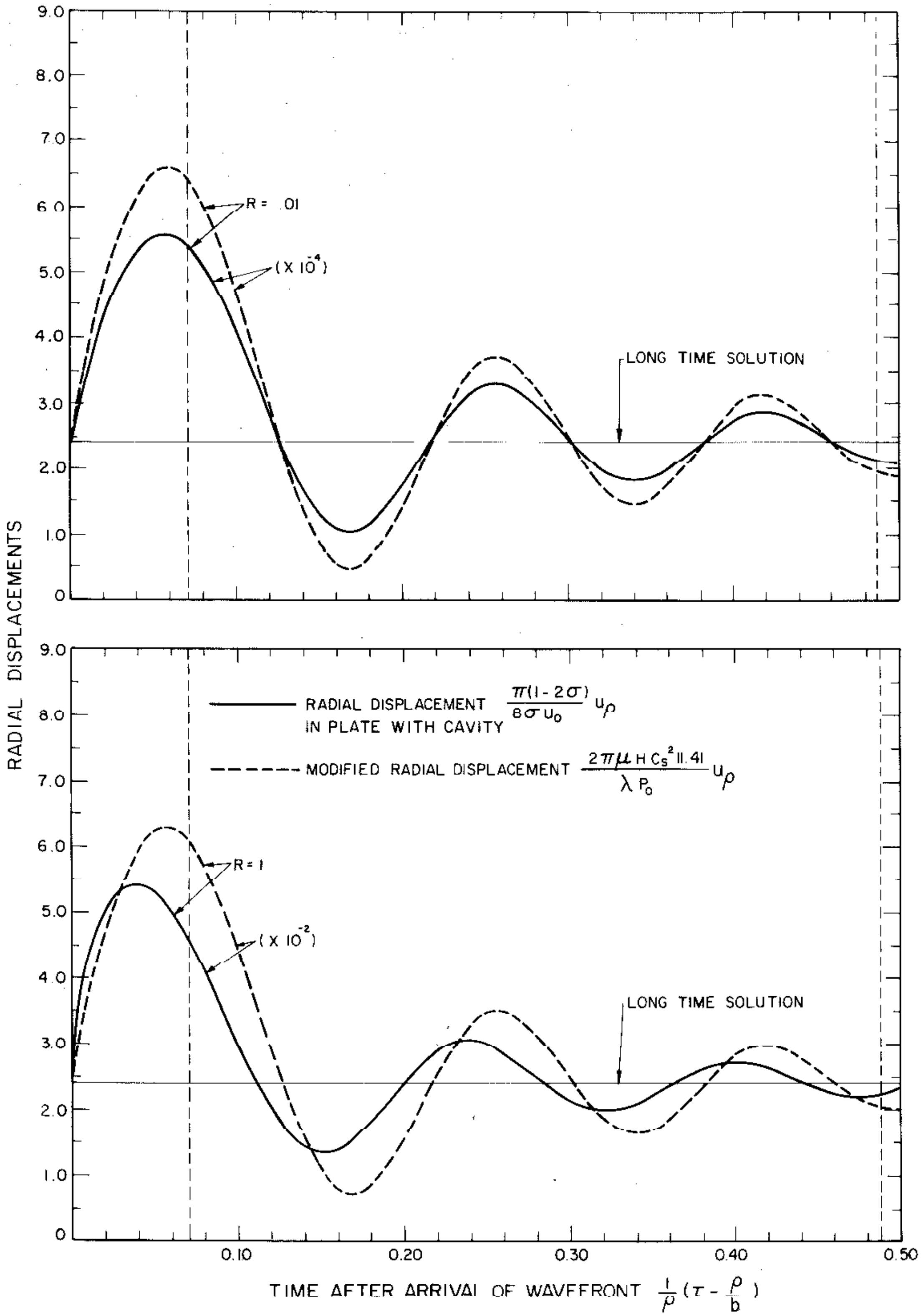

Fig. 13. Radial displacements in plate-cavity and body force problems versus time after arrival of wave front: Comparison ($\rho = 20$, $\zeta = 1$, $\sigma = .31$).

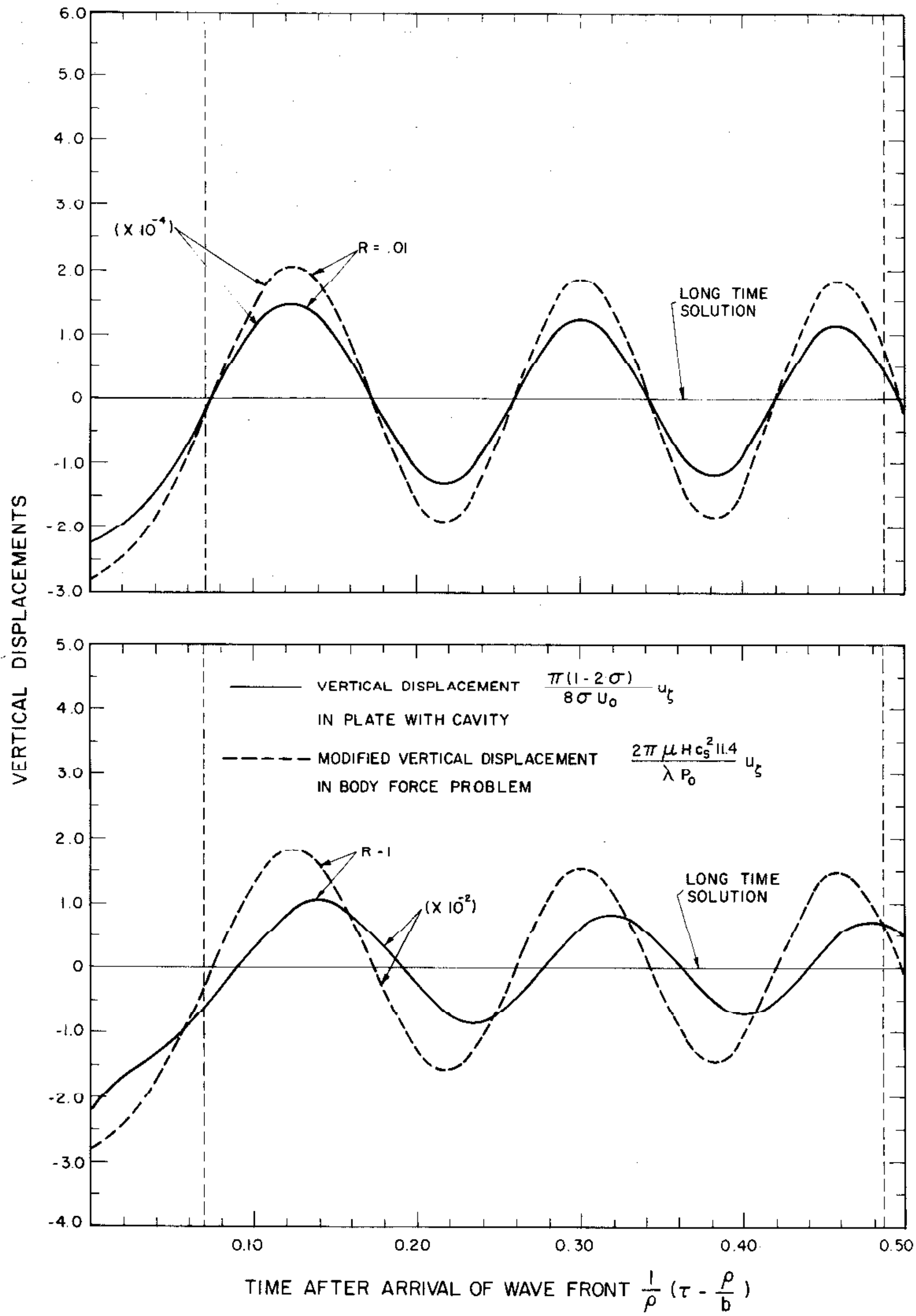

Fig. 14. Vertical displacements in plate-cavity and body force problems versus time after arrival of wave front: Comparison ($\rho = 20$, $\zeta = 1$, $\sigma = .31$).

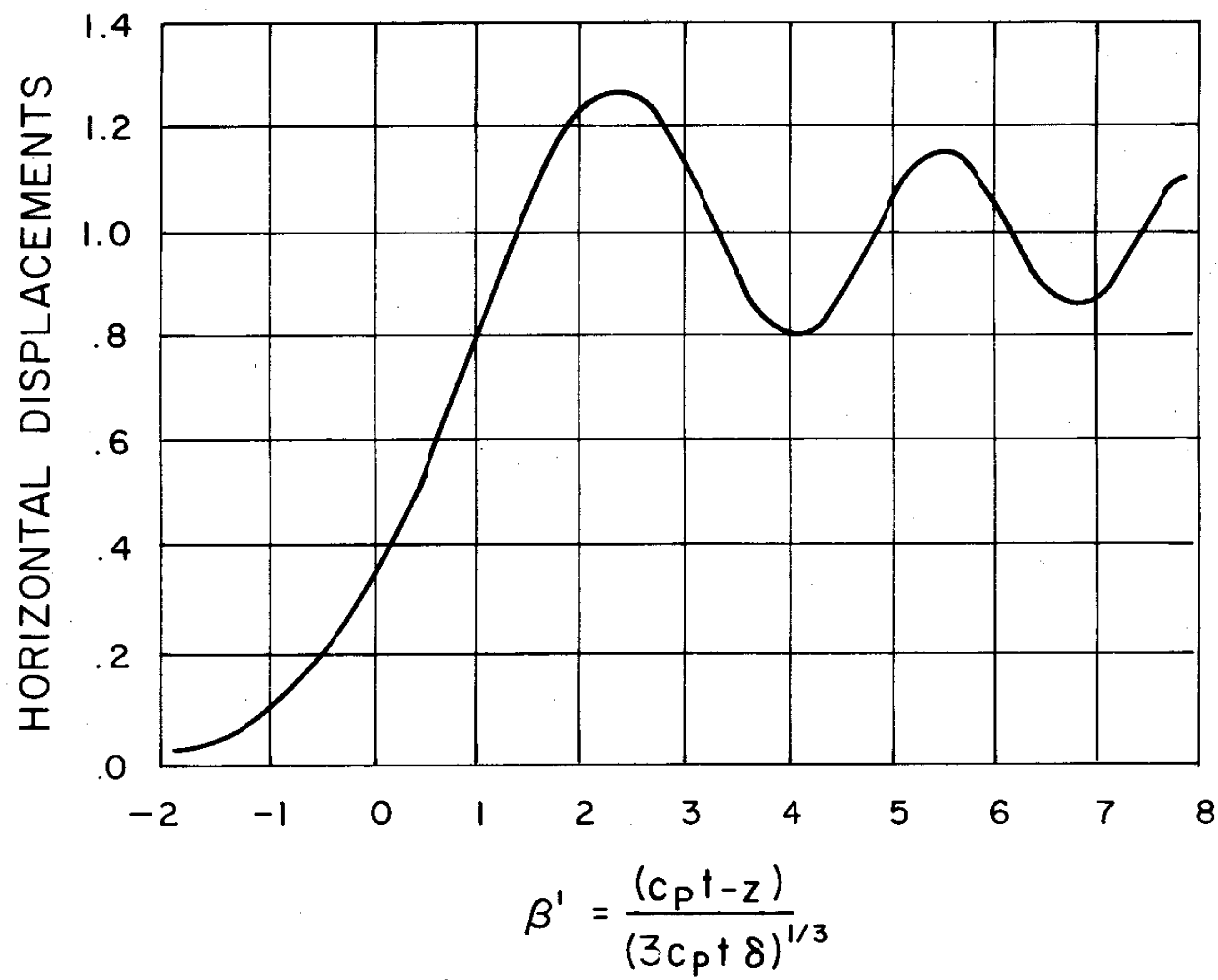

Fig. 15. Horizontal displacements $\frac{\rho' c_P u_z}{P_o}$ (stress input), $\frac{u_z}{U_o}$ (displacement input) in transversely isotropic slab versus time after arrival of wave front (modified).

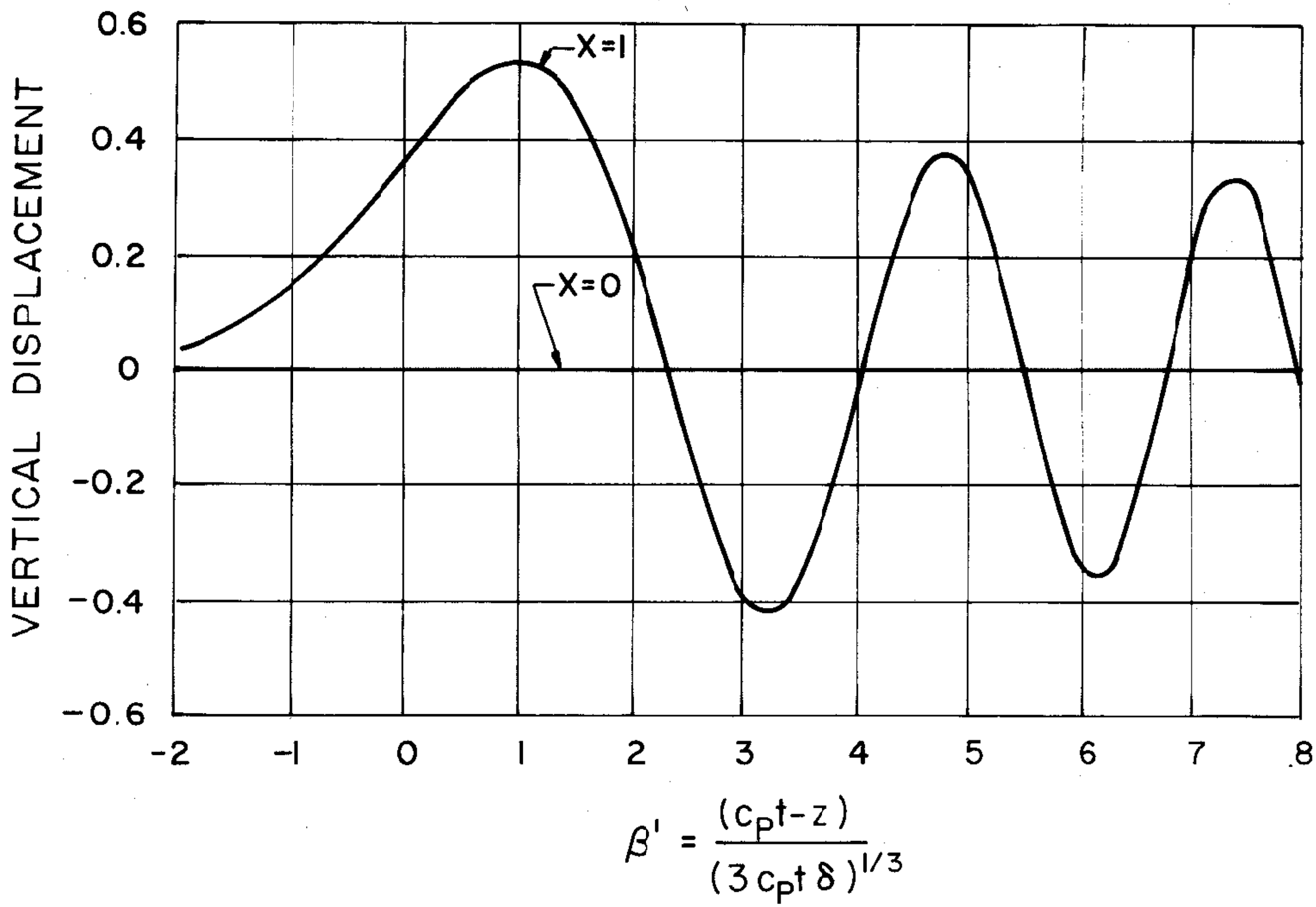

Fig. 16. Modified vertical displacements $\rho' c_{11} c_P (P_o c_{13})^{-1} (3 c_P t \delta)^{1/3} u_x$ (stress input), $c_{11}(c_{13} U_o)^{-1}(3c_P t \delta)^{1/3} u_x$ (displacement input) versus time after arrival of wave front (modified).

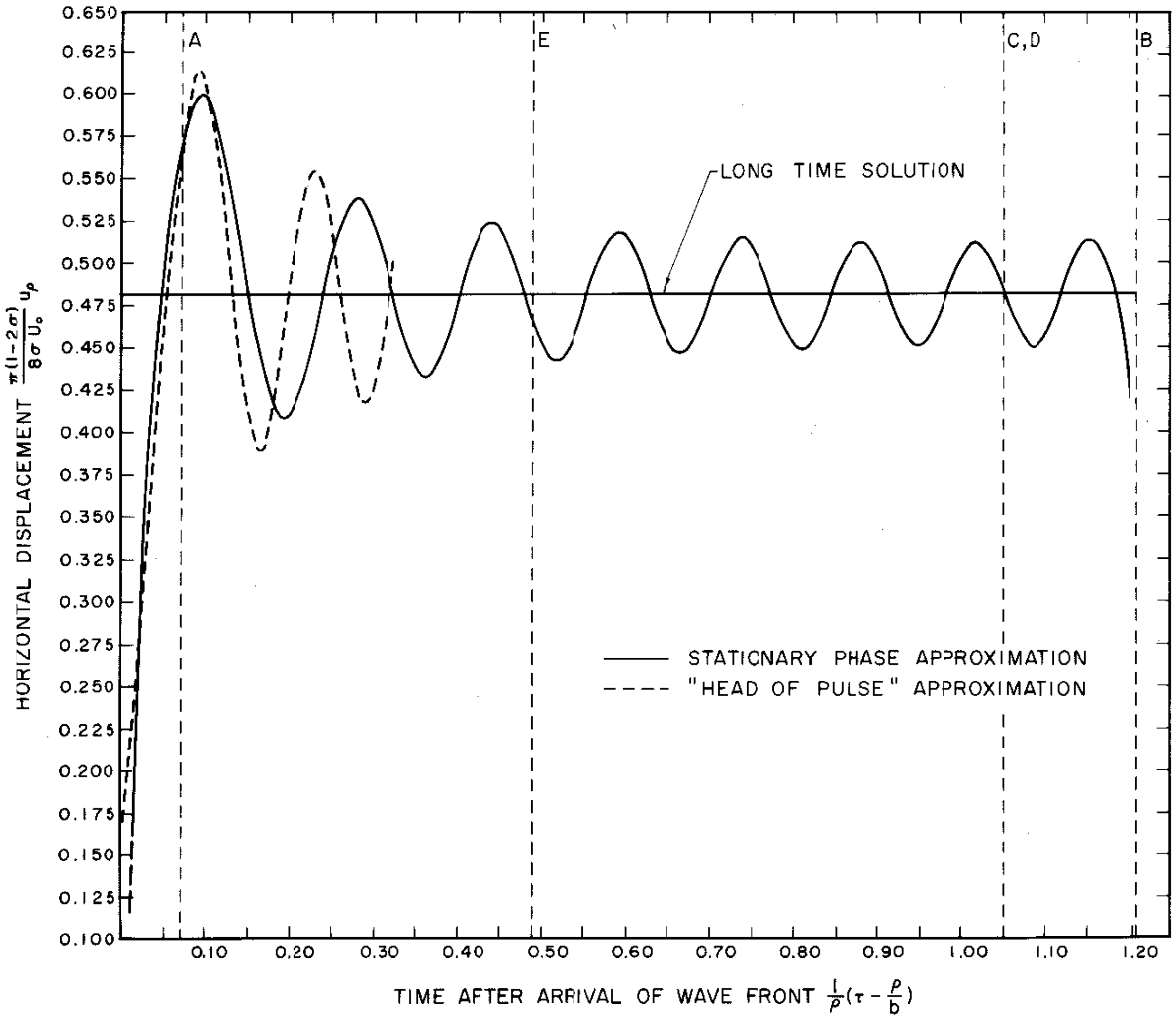

Fig. 17. Horizontal displacement in isotropic slab (displacement input) versus time after arrival of wave front: Comparison between head of the pulse and stationary phase approximations ($\rho = 20$, $\zeta = 1$, $\sigma = .31$).

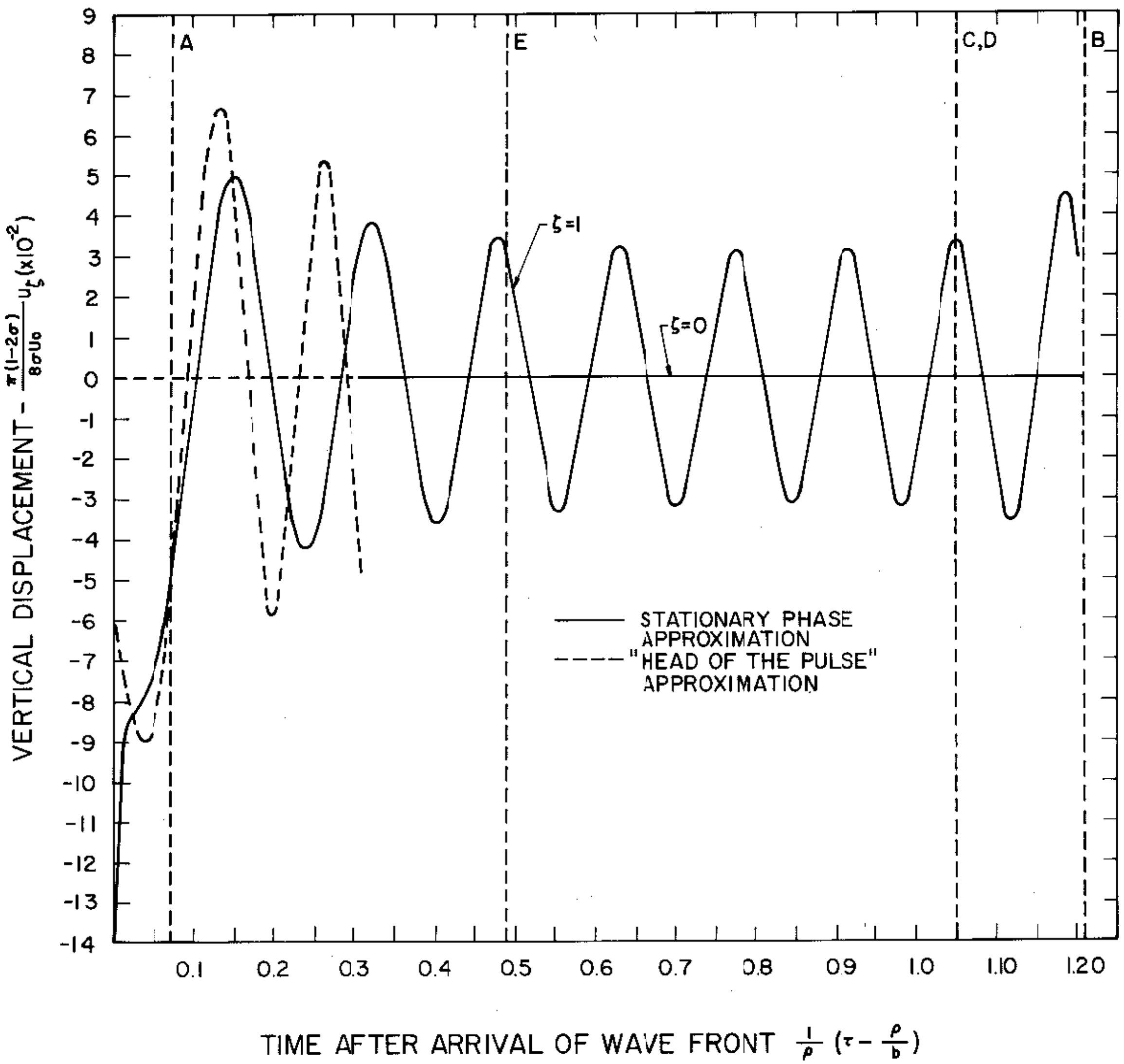

Fig. 18. Vertical displacement in isotropic slab (displacement input) versus time after arrival of wave front: Comparison between head of the pulse and stationary phase approximations ($\rho = 20$, $\zeta = 1$, $\sigma = .31$).

Appendix A. EXTENDED HANKEL TRANSFORMS

INTRODUCTION

As mentioned in the text, Hankel transforms for the interval $(0, \infty)$ are not suitable for application to problems which do not include the origin. In this appendix transform pairs which are adequate for such problems are discussed. They are derived from certain expansion formulas of arbitrary functions, which are discussed in Titchmarsh (61). To the author's knowledge the present work constitutes the first application of such transform pairs to problems in elasticity and for this reason, and for purposes of completeness, a brief account of their derivation is given. Since the proofs given by Titchmarsh are quite complex, alternative formal proofs are given here, along the lines of similar proofs in Sneddon (49). It is hoped that the derivations given here will make the material more readily accessible to the general reader.

A1. FINITE INTERVAL

Consider the differential equation

$$\frac{d}{dr}\left(r\frac{dy}{dr}\right) + rk^2 y = 0 \qquad \text{(A1.1)}$$

in the interval $0 < \alpha \leq r \leq \beta < \infty$, where k is a real parameter. This has the solution (the choice of which will be explained later)

$$y = C_o(k, r, \alpha) \equiv J_o(kr)Y_1(k\alpha) - J_1(k\alpha)Y_o(kr) \qquad \text{(A1.2)}$$

where the J's and Y's are Bessel functions of the first and second kinds, respectively. This solution has the property that

$$\left[\frac{d}{dr} C_o(k, r, \alpha)\right]_{r=\alpha} = 0 \qquad (A1.3)$$

and it is assumed that k is a root of the equation

$$\left[\frac{d}{dr} C_o(k, r, \alpha)\right]_{r=\beta} = 0 \qquad (A1.4)$$

The differential equation A1.1 and conditions A1.3 and A1.4 constitute a Sturm-Liouville system in an interval in which the coefficients of the derivatives in the differential equation are continuous and nonzero. Hence general Sturm-Liouville theory ((62), Chapters 9 and 10) justifies the expansion of an arbitrary function $f(r)$ in terms of the functions $C_o(k_j, r, \alpha)$, where the k_j's are the roots of equation A1.4, provided that the Fourier expansion of the arbitrary function is valid. Using well-known techniques, this expansion can readily be shown to be

$$f(r) = \frac{2}{\beta^2-\alpha^2}\int_\alpha^\beta \xi f(\xi)\, d\xi + \frac{\pi^2}{2}\sum_j \frac{k_j^2 J_1^2(k_j\beta) C_o(k_j, r, \alpha)}{J_1^2(k_j\alpha) - J_1^2(k_j\beta)}$$

$$\times \left\{\int_\alpha^\beta \xi f(\xi) C_o(k_j, \xi, \alpha)\, d\xi\right\}, \quad 0 < \alpha \leq r \leq \beta < \infty \qquad (A1.5)$$

where the k_j's are the positive roots of equation A1.4. This formula was given (without derivation) by Muskat (63) in connection with a problem in fluid mechanics. A transform pair can be obtained from it as follows: If the zero order transform is defined by

$$\tilde{f}^{o}(k_j) = \int_{\alpha}^{\beta} rf(r)C_o(k_j, r, \alpha)\, dr \qquad \text{(Al. 6)}$$

then, from equation Al. 5, the inverse transform is

$$f(r) = \frac{2}{\beta^2-\alpha^2}\int_{\alpha}^{\beta}\xi f(\xi)\, d\xi + \frac{\pi^2}{2}\sum_j \frac{k_j^2 J_1^2(k_j\beta)C_o(k_j, r, \alpha)\tilde{f}^{o}(k_j)}{J_1^2(k_j\alpha) - J_1^2(k_j\beta)} \qquad \text{(Al. 7)}$$

Note the constant term in the last equation, which is analogous to the constant term in a Fourier cosine series.

Similar arguments (and more rigorous ones also; cf. Titchmarsh (61), page 18) can be used to justify the following expansion of an arbitrary function f(r):

$$f(r) = \frac{\pi^2}{2}\sum_j \frac{k_j^2 J_n^2(k_j\beta)C_n(k_j, r, \alpha)}{J_n^2(k_j\alpha) - J_n^2(k_j\beta)}\left\{\int_{\alpha}^{\beta}\xi f(\xi)C_n(k_j, \xi, \alpha)\, d\xi\right\},$$

$$0 < \alpha \leq r \leq \beta < \infty \qquad \text{(Al. 8)}$$

where

$$C_n(k, r, \alpha) \equiv J_n(kr)Y_n(k\alpha) - J_n(k\alpha)Y_n(kr)$$

k_j is a positive root of $C_n(k, \beta, \alpha) = 0$, and n is a positive integer. If the nth order transform is defined by

$$\tilde{f}^{n}(k_j) = \int_{\alpha}^{\beta} rf(r)C_n(k_j, r, \alpha)\, dr \qquad \text{(Al. 9)}$$

then, from equation Al. 8, the inverse transform is

$$f(r) = \frac{\pi^2}{2}\sum_j \frac{k_j^2 J_n^2(k_j\beta)C_n(k_j, r, \alpha)\tilde{f}^{n}(k_j)}{J_n^2(k_j\alpha) - J_n^2(k_j\beta)} \qquad \text{(Al. 10)}$$

Some comments on the choice of the kernel in equation A1.5 should be made. The derivative w.r.t. r of the kernel in equation A1.5 is the kernel in equation A1.8 (with $n = 1$), to within a factor of k. This is a feature of the zero and first order Hankel transforms for the spatial interval $(0, \infty)$ and so the above transforms should be useful for axially symmetric wave problems. A more general expansion of the type A1.5 for arbitrary n appears to be difficult to write down.

A2. INFINITE INTERVAL

For an infinite interval $0 < a \leq r \leq \infty$, k becomes a continuous parameter and integral, instead of series, representations of arbitrary functions are obtained. As mentioned in the introduction, the representation formulas to be given in this section have been derived by Titchmarsh, but here an alternative, more accessible, approach to their derivation is presented. The development used follows closely that given in Sneddon ((49), pp. 56-57) in connection with one of the formulas.

Some algebraic simplifications occur when the second linearly independent solution of Bessel's equation is taken to be G_n ((64), page 23) instead of Y_n. The relation between G, J, and Y, is given by

$$G_n(z) = \frac{i\pi}{2}\left[J_n(z) + iY_n(z)\right] \tag{A2.1}$$

The functions

$$T_o(k, r, a) \equiv J_o(kr)G_1(ka) - J_1(ka)G_o(kr)$$

$$T_o(\zeta, r, a) \equiv J_o(\zeta r)G_1(\zeta a) - J_1(\zeta a)G_o(\zeta r)$$

where k and ζ are positive real parameters, a is a constant, and r a variable, are solutions of Bessel's equation of order zero. Inserting these into the Lommel integral formula ((64), page 69), and noting that

$$\left[\frac{d}{dr} T_o(\zeta, r, a)\right]_{r=a} = \left[\frac{d}{dr} T_o(k, r, a)\right]_{r=a} = 0$$

one obtains:

$$(k^2 - \zeta^2)\int_a^h rT_o(k, r, a)T_o(\zeta, r, a)\, dr$$

$$= [\zeta rT_o(k, r, a)\frac{d}{dr} T_o(k, r, a) - krT_o(\zeta, r, a)\frac{d}{dr} T_o(k, r, a)]_{r=h}, \quad \text{(A2. 2)}$$

where h is a constant. The J functions on the right side of the last equation are now replaced by (Watson (48))

$$J_n(z) = \frac{1}{\pi i}[G_n(z) - e^{in\pi}G_n(ze^{i\pi})]$$

Then, assuming h is large, the G functions involving h on the right side of the resulting equation are replaced by the leading terms of their large-argument asymptotic expansions. Multiplying through the resulting equation by $\zeta\varphi(\zeta)/B_1(\zeta a)$, where $B_1(\zeta a) = e^{i\pi}G_1(\zeta a)G_1(\zeta a e^{i\pi})$ and $\varphi(\zeta)$ is an arbitrary real-valued function, integrating w.r.t. ζ between 0 and ∞, and letting $h \to \infty$, there results:

$$\lim_{h\to\infty}\int_0^\infty \frac{\zeta\varphi(\zeta)}{B_1(\zeta a)}\,d\zeta \int_a^h r\,T_0(k,r,a)\,T_0(\zeta,r,a)\,dr =$$

$$-\lim_{h\to\infty}\int_0^\infty \left\{\frac{\sqrt{\zeta}\,[G_1(ka)G_1(\zeta a)+G_1(kae^{i\pi})G_1(\zeta ae^{i\pi})]}{2\pi\sqrt{k}\,B_1(\zeta a)(\zeta+k)}\right\}\varphi(\zeta)\cos(\zeta+k)h\,d\zeta$$

$$-\,i\lim_{h\to\infty}\int_0^\infty \left\{\frac{\sqrt{\zeta}\,[G_1(kae^{i\pi})G_1(\zeta ae^{i\pi})-G_1(ka)G_1(\zeta a)]}{2\pi\sqrt{k}\,B_1(\zeta a)(\zeta+k)}\right\}\varphi(\zeta)\sin(\zeta+k)h\,d\zeta$$

$$-\,i\lim_{h\to\infty}\int_0^\infty \left\{\frac{\sqrt{\zeta}\,[G_1(kae^{i\pi})G_1(\zeta a)-G_1(ka)G_1(\zeta ae^{i\pi})]}{2\pi\sqrt{k}\,B_1(\zeta a)(\zeta-k)}\right\}\varphi(\zeta)\cos(\zeta-k)h\,d\zeta$$

$$-\,i\lim_{h\to\infty}\int_0^\infty \left\{\frac{\sqrt{\zeta}\,[G_1(kae^{i\pi})G_1(\zeta a)+G_1(ka)G_1(\zeta ae^{i\pi})]}{2\pi\sqrt{k}\,B_1(\zeta a)}\right\}\varphi(\zeta)\frac{\sin(\zeta-k)h}{(\zeta-k)}\,d\zeta$$

$$+\lim_{h\to\infty}\frac{1}{h}\int_0^\infty \frac{\zeta P\varphi(\zeta)}{B_1(\zeta a)(\zeta+k)(\zeta-k)}\,d\zeta \qquad \text{(A2. 3)}$$

where P represents the contribution from the higher order terms in the asymptotic expansions of the G functions.

On separating the first and second terms of the right side of equation A2. 3 into their real and imaginary parts, it can be seen from the Riemann-Lebesque lemma that the resulting integrals are zero, provided the conditions of the lemma are met. The restrictions imposed on φ by these conditions are not too severe, as can be seen on observing that the {} bracket terms in the integrals are bounded (noting that $B_1(\zeta a)$ and $(\zeta + k)$ have no zeros in the interval in question). The last term on the right side vanishes if the integral multiplying $1/h$ exists.

Now $(\zeta - k)$ is a factor of P, since it is a factor of equation A2.2, and so no difficulty arises due to the presence of this term in the denominator. It is henceforth assumed that φ is such that this integral exists. The third term on the right side is of the form

$$\lim_{h\to\infty} \int_0^\infty \frac{F(\zeta,k)}{\zeta - k}\,\varphi(\zeta)\cos(\zeta - k)h\,d\zeta$$

where $F(\zeta,k)\Big|_{\zeta=k} = 0$, $\partial F/\partial\zeta$ is non-singular, and $F(\zeta,k)$ is bounded in the interval. Hence it is zero, by the Riemann-Lebesque lemma, as can be seen on expanding $F(\zeta,k)$ in a Taylor series about $\zeta = k$. Thus the only nonvanishing term on the right side is the fourth term, which can readily be evaluated by means of the theory of Dirichet integrals ((69), pp. 9-15). There results, assuming φ is continuous in $0 \le k \le \infty$:

$$\int_0^\infty \frac{\zeta\varphi(\zeta)}{B_1(\zeta a)}\,d\zeta \int_a^\infty r T_0(k,r,a)\,T_0(\zeta,r,a)\,dr = \varphi(k), \quad 0 \le k \le \infty \tag{A2.4}$$

The argument up to now has followed closely that in Sneddon, except for the choice of limits and the location of the factor $B_1(\zeta a)$, which here has been chosen to allow inclusion of the origin in the range of ζ (without it singularities would arise). Interchanging the order of integration, equation A2.4 may be written:

$$\int_a^\infty r T_0(k,r,a)\,dr \int_0^\infty \frac{\zeta\varphi(\zeta)\,T_0(\zeta,r,a)}{B_1(\zeta a)}\,d\zeta = \varphi(k) \tag{A2.5}$$

Letting

$$\psi(r) = \int_o^\infty \frac{\zeta\varphi(\zeta)T_o(\zeta, r, a)}{B_1(\zeta a)} d\zeta \qquad (A2.6)$$

equation A2. 5 may be written

$$\varphi(k) = \int_a^\infty rT_o(k, r, a)\psi(r)\, dr$$

Substituting this value of φ into equation A2. 6 gives, on changing the dummy variables of integration:

$$\psi(r) = \int_o^\infty \frac{\zeta T_o(\zeta, r, a)}{B_1(\zeta a)} d\zeta \int_a^\infty \xi T_o(\zeta, \xi, a)\psi(\xi)\, d\xi$$

or, using equation A1. 2, A2. 1, and the definition of B_1:

$$\psi(r) = \int_o^\infty \frac{\zeta C_o(\zeta, r, a)\, d\zeta}{J_1^2(\zeta a) + Y_1^2(\zeta a)} d\zeta \int_a^\infty \xi\psi(\xi)C_o(\zeta, \xi, a)\, d\xi \qquad (A2.7)$$

If the zero order transform is defined by

$$\tilde{\psi}^o(k) = \int_a^\infty r\psi(r)C_o(k, r, a)\, dr \qquad (A2.8)$$

then it follows from equation A2. 7 that the inverse transform is

$$\psi(r) = \int_o^\infty \frac{k\tilde{\psi}^o(k)C_o(k, r, a)}{J_1^2(ka) + Y_1^2(ka)} dk \qquad (A2.9)$$

Using a different technique, Tichmarsh has derived the expansion formula given by equation A2. 7 ((61), set $\nu = 0$ in last example in §4.10). However as mentioned before, the above formal approach appears to be simpler.

Using arguments similar to the above, the following expansion can be derived:

$$\psi(r) = \int_0^{\infty} \frac{\zeta C_n(\zeta, r, a)}{J_n^2(\zeta a) + Y_n^2(\zeta a)} \, d\zeta \int_a^{\infty} \xi\psi(\xi) C_n(\zeta, \xi, a) \, d\xi \qquad (A2.10)$$

If the nth order transform is defined by

$$\tilde{\psi}^n(k) = \int_a^{\infty} r\psi(r) C_n(k, r, a) \, dr \qquad (A2.11)$$

then it follows from equation A2.10 that the inverse transform is

$$\psi(r) = \int_0^{\infty} \frac{k\tilde{\psi}^n(k) C_n(k, r, a)}{J_n^2(\zeta a) + J_n^2(\zeta a)} \, dk \qquad (A2.12)$$

Equation A2.10 has been derived by Titchmarsh ((61), page 87) and is termed the Weber integral formula.

A3. TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

Here the transforms of derivatives, and various combinations of derivatives, are formally derived. The transforms to be considered are those given by equations A1.6, A1.9 (with n = 1), A2.8, and A2.11 (with n = 1). Since the question of whether k is continuous or discrete becomes important only when inverse transforms are being considered, these transforms can be treated simultaneously, with the aid of the following notation. The first and zero transforms of a function $\varphi(r)$ are defined to be

$$\tilde{\varphi}^1(k) = \int_\alpha^\beta r\varphi(r)C_1(k, r, \alpha)\, dr \qquad (A3.1)$$

$$\tilde{\varphi}^o(k) = \int_\alpha^\beta r\varphi(r)C_o(k, r, \alpha)\, dr \qquad (A3.2)$$

respectively, where it is to be understood that if β is finite then k is to be replaced by k_j (in A3.1 a root of $C_1(k, \beta, \alpha) = 0$, in A3.2 a root of A1.4).

Integration by parts gives:

$$\int_\alpha^\beta r\frac{\partial\varphi}{\partial r} C_1(k, r, \alpha)\, dr$$

$$= \Big[r\varphi C_1(k, r, \alpha)\Big]_\alpha^\beta - k\int_\alpha^\beta r\varphi C_o(k, r, \alpha)\, dr$$

i.e.,

$$\int_\alpha^\beta r\frac{\partial\varphi}{\partial r} C_1(k, r, \alpha)\, dr$$

$$= \Big[r\varphi C_1(k, r, \alpha)\Big]_\alpha^\beta - k\tilde{\varphi}^o(k) \qquad (A3.3)$$

Similarly, the following results can be established on integrating by parts:

$$\int_\alpha^\beta r\left[\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} - \frac{1}{r^2}\varphi\right]C_1(k, r, \alpha)\, dr$$

$$= \left\{\left[r\frac{\partial\varphi}{\partial r} + \varphi\right]C_1(k, r, \alpha) - kr\varphi C_o(k, r, \alpha)\right\}_\alpha^\beta - k^2\tilde{\varphi}^1(k)$$

(A3.4)

$$\int_a^\beta r\left[\frac{\partial\varphi}{\partial r} + \frac{1}{r}\varphi\right]C_o(k, r, a)\,dr = \Big[r\varphi C_o(k, r, a)\Big]_a^\beta + k\tilde{\varphi}^1(k) \qquad (A3.5)$$

$$\int_a^\beta r\left[\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}\right]C_o(k, r)\,dr$$

$$= \left[r\frac{\partial\varphi}{\partial r}C_o(k, r, a) + kr\varphi C_1(k, r, a)\right]_a^\beta - k^2\tilde{\varphi}^o(k) \qquad (A3.6)$$

The choice of the above groups of derivatives has been dictated by their occurrence in the equations in the text. Note the great similarity between the above results and similar results for the zero and first order Hankel transforms for the interval $(0, \infty)$, as given by Sneddon ((49), pp. 60-62).

Appendix B. NATURE OF THE ZEROS OF CERTAIN TRANSCENDENTAL FUNCTIONS

INTRODUCTION

The roots of certain transcendental equations (frequency equations) are examined analytically in this appendix. The main technique used is a generalization of one used for locating the roots of certain Bessel functions (64) and employed by Tranter (65) in connection with a problem on the vibrations of an elastic cylinder. In Tranter's case the system is governed by one differential equation, whereas in the present work systems governed by several differential equations are treated. The discussions are mainly confined to problems arising in the text, but some extensions to cases of interest in their own right are also included.

B1.1. ISOTROPIC PLATE; COMPLEX ROOTS

The first cast to be examined is that of the isotropic plate treated in Section I. Detailed description will be limited to the plate of infinite radial extent, since, as will emerge subsequently, the question of whether the outer radius of the plate is finite or infinite is not critical. Substituting equations 1.81 and 1.82 into equations 1.25, 1.26, 1.50, and 1.51, and using equations 1.27, 1.28, 1.29, 1.30, 1.63 and 1.71, one obtains:

$$\frac{d^2F_{1C}^{(n)}}{dz^2} - \frac{k^2c_d^2 - p_n^2}{c_s^2} F_{1C}^{(n)} - k\left(\frac{c_d^2}{c_s^2} - 1\right)\frac{dF_{2C}^{(n)}}{dz} = 0 \qquad \text{(B1.1)}$$

$$\frac{d^2F_{2C}^{(n)}}{dz^2} - \frac{k^2c_s^2 + p_n^2}{c_d^2} F_{2C}^{(n)} - k\left(\frac{c_s^2}{c_d^2} - 1\right)\frac{dF_{1C}^{(n)}}{dz} = 0 \tag{Bl. 2}$$

$$\frac{dF_{1C}^{(n)}}{dz} - kF_{2C}^{(n)} = 0, \quad z = \pm H \tag{Bl. 3}$$

$$(\lambda + 2\mu)\frac{dF_{2C}^{(n)}}{dz} + \lambda k F_{1C}^{(n)} = -\frac{\mu}{k} D_C(k, p_n), \quad z = \pm H^* \tag{Bl. 4}$$

where $F_{1C}^{(n)}$ and $F_{2C}^{(n)}$ are the values of the functions corresponding to the value p_n of the parameter p. Let p_1 and p_2 be arbitrary values of p. Equation Bl. 1 with $n = 2$ is now multiplied by $F_{2C}^{(1)}$ and the result is subtracted from the product of $F_{2C}^{(2)}$ and equation Bl. 1 with $n = 1$. Then equation Bl. 2 with $n = 2$ is multiplied by $F_{1C}^{(1)}$ and the result is subtracted from the product of $F_{1C}^{(2)}$ and equation Bl. 2 with $n = 1$. Adding the results of these operations and integrating (w. r. t. z) between - H and H gives, after some rearranging:

$$\begin{aligned}(p_2^2 - p_1^2)\int_{-H}^{H}\left[F_{1C}^{(1)}F_{1C}^{(2)} - F_{2C}^{(1)}F_{2C}^{(2)}\right]dz \\ = -\Bigg\{c_s^2F_{1C}^{(2)}\left[\frac{dF_{1C}^{(1)}}{dz} - kF_{2C}^{(1)}\right] - c_s^2F_{1C}^{(1)}\left[\frac{dF_{1C}^{(2)}}{dz} - kF_{2C}^{(2)}\right] \\ + F_{2C}^{(2)}\left[c_d^2\frac{dF_{2C}^{(1)}}{dz} + k(c_d^2 - 2c_s^2)F_{1C}^{(1)}\right] \\ - F_{2C}^{(1)}\left[c_d^2\frac{dF_{2C}^{(2)}}{dz} + k(c_d^2 - 2c_s^2)F_{1C}^{(2)}\right]\Bigg\}_{-H}^{H}\end{aligned} \tag{Bl. 5}$$

---

*These equations also follow directly from the definition of the functions (cf. Appendix C).

for $\mathrm{Re}\ \eta_j < \infty$.* This last condition is necessary to insure convergence of the integrals. Using equations B1.3 and B1.4, and the even and odd nature of the functions $F_{1C}^{(n)}$ and $F_{2C}^{(n)}$, respectively, equation B1.5 may be written:

$$(p_2^2 - p_1^2)\int_0^H \left[F_{1C}^{(1)} F_{1C}^{(2)} + F_{2C}^{(1)} F_{2C}^{(2)}\right] dz$$

$$= \frac{2\mu}{k}\left[F_{2C}^{(2)} D_C(k, p_1) - F_{2C}^{(1)} D_C(k, p_2)\right]_H \qquad \text{(B1.6)}$$

So far nothing specific has been said about p_1 and p_2. It is now assumed that they are zeros of $D(k, p)$ and hence equation B1.6 may be written:

$$p_2^2 - p_1^2)\int_0^H \left[F_{1C}^{(1)} F_{1C}^{(2)} + F_{2C}^{(1)} F_{2C}^{(2)}\right] dz = 0 \qquad \text{(B1.7)}$$

Before proceeding further with the main argument, some discussion of the functions $F_{1C}^{(n)}$, $F_{2C}^{(n)}$ and $D(k, p_n)$ is necessary. Since these functions are odd functions of η_1, they have branch points at the zeros of η_1 and so suitable branch cuts in the p-plane must be introduced. These cuts are so chosen that it is the principle branches of the functions which arise, i.e., the cuts run from the branch points to $-\infty$, parallel to the real p-axis. From the location of the cuts and the form of the functions it follows that they are analytic functions of p in a domain which is symmetric w.r.t. the real p-axis and are real when p is real, provided k is real. Hence, by Schwarz' principle of

*For finite k, this is equivalent to $\mathrm{Re}\ p < \infty$.

reflection (66),

$$F_{1C}(p^*) = F_{1C}^*(p)$$
$$F_{2C}(p^*) = F_{2C}^*(p) \qquad \text{(Bl. 8)}$$

$$D_C(k, p^*) = D_C^*(k, p) \qquad \text{(Bl. 9)}$$

where p^* denotes the complex conjugate of p, etc. From equation Bl. 9 it follows that the zeros of $D_C(k, p)$ occur in complex conjugate pairs. It is now supposed that $p_1 = \alpha + i\beta$, and p_2 are complex conjugates. Then, on using equations Bl. 8, equation Bl. 7 gives:

$$\alpha\beta\int_o^H \left[|F_{1C}(k, \alpha+i\beta, z)|^2 + |F_{2C}(k, \alpha+i\beta, z)|^2 \right] dz = 0 \qquad \text{(Bl.10)}$$

But the integrand in this last equation is always positive and so a contradiction is reached. Hence there are no complex conjugate, and hence no complex, zeros, except possibly those for which α or β vanish, i.e., real or pure imaginary zeros. Note also the previous restriction on the points for which Re $\eta_j = \infty$. These points receive special discussion in the text.

Some general comments on the method and its scope are appropriate here, particularly in connection with axially symmetric problems. The work of Tranter (65) on the cylindrical shell, of Miklowitz (12) on the sudden punching of a hole in an infinite elastic plate, and of Selberg (67) on a cylindrical cavity in an infinite medium, is typical of such problems. In Selberg's and Miklowitz's solutions the expression corresponding to $D_C(k, p)$ contains functions of the type $K_m(pa/c_d)$, where K_m is a modified Bessel function of the second kind, and a is the radius of the cavity or hole. The limits of integration in the equation analogous to equation B1.10 are from a to ∞. Hence the integrals converge only if $\mathrm{Re}\, p > 0$, as may be seen from the asymptotic form of $K_m(pr/c_d)$, and so the above method works in the right half p-plane only. It is interesting to note that Selberg, using the principle of the argument (68), located a root of his "frequency equation" in the second quadrant of the p-plane. Thus a portion of his solution decays exponentially with time. That this portion stems from the form of the solution rather than any physical damping can be seen from the work of Miklowitz, in which, by suitable choice of the modified Bromwich contour, this term was replaced by certain integrals which do not have this exponential time character. However the occurrence of the term indicates that physical arguments regarding roots in the half-plane $\mathrm{Re}\, p < 0$ must be treated with some caution. The issue does not arise in Tranter's work, since the limits of integration there are finite and hence the above argument is valid for the whole p-plane, except $p = \infty$.

It should also be noted that the above proof holds only for those cuts which give domains which are symmetric w. r. t. the real p-axis. For other branches complex roots may exist.

The essential features of the proof did not critically hinge on the particular form of F_{1C}, F_{2C} and D_C. In fact the only use made of their explicit form was in assessing them as analytic functions in the cut p-plane. On the introduction of suitable branch cuts the functions

F_{1F}, F_{2F} and D_F, given by equations 1.59, 1.90 and 1.91, respectively, are also analytic functions in the cut p-plane. Hence the details of the proof showing $D_F(k, p)$ has no complex zeros for real k except possibly real or pure imaginary ones, go through exactly as above and will not be discussed any further here.

B1.2. ISOTROPIC PLATE; REAL ROOTS

The zeros of equation 1.56 satisfy the equation

$$\frac{\tanh \eta_1 H}{\tanh \eta_2 H} = \frac{4k^2 \eta_1 \eta_2}{(\eta_1^2 + k^2)^2} \qquad \text{(B1.11)}$$

where

$$\eta_1 = \left(k^2 + \frac{p^2}{c_s^2}\right)^{1/2} \qquad \text{(B1.12)}$$

$$\eta_2 = \left(k^2 + \frac{p^2}{c_d^2}\right)^{1/2} \qquad \text{(B1.13)}$$

Assuming k is real and p is real and nonvanishing, and noting that $c_d > c_s$, it is seen that $\eta_1 > \eta_2$. Hence $\tanh \eta_1 H > \tanh \eta_2 H$, since $\tanh x$ is a monotonic increasing function of x. Hence the left hand side of equation B1.11 is greater than one. Now, since $\eta_1 > \eta_2$, it follows that

$$\begin{aligned}(\eta_1^2 + k^2)^2 - 4k^2\eta_1\eta_2 &> (\eta_1^2 + k^2)^2 - 4k^2\eta_1^2 \\ &= (\eta_1^2 - k^2)^2 \\ &> 0\end{aligned}$$

Therefore

$$\frac{4k^2\eta_1\eta_2}{(\eta_1^2 + k^2)^2} < 1$$

i.e., the right hand side of equation B1.11 is less than one, which is a contradiction. Hence there are no roots of equation B1.11 for which p and k are real, except possibly $p = 0$ (this point receives special attention in the text).

Using similar arguments it can readily be shown that $D_F(k, p)$ also has no real roots for which p and k are real, except possibly $p = 0$.

B1.3. ISOTROPIC PLATE; REPEATED ROOTS

As in the previous cases the detailed discussion will be confined to the problem for $D_C(k, p)$. On letting $p_2 = p_1 + \epsilon$, where $\epsilon << 1$, and expanding the various functions containing p_2 in Taylor series, equation B1.5 in the limit gives, after some rearranging:

$$-2p_1\int_{-H}^{H}\left[F_{1C}^2(k, p_1, z) + F_{2C}^2(k, p_1, z)\right]dz$$

$$-\left\{c_s^2\left[\frac{\partial F_{1C}}{\partial z}\frac{\partial F_{1C}}{\partial p_1} - F_{1C}\frac{\partial^2 F_{1C}}{\partial z\,\partial p_1}\right] + c_d^2\left[\frac{\partial F_{2C}}{\partial z}\frac{\partial F_{2C}}{\partial p_1} - F_{2C}\frac{\partial^2 F_{2C}}{\partial z\,\partial p_1}\right]\right.$$

$$\left. + k(c_d^2 - c_s^2)\left[F_{1C}\frac{\partial F_{2C}}{\partial p_1} - F_{2C}\frac{\partial F_{1C}}{\partial p_1}\right]\right\}_{-H}^{H} \qquad \text{(B1.14)}$$

where $\partial/\partial p_1$ denotes the derivative w.r.t. p evaluated at $p = p_1$.

Differentiating equations Bl. 3 and Bl. 4 w. r. t. p and evaluating the results at $p = p_1$, gives:

$$\frac{\partial^2 F_{1C}}{\partial z\, \partial p_1} - k \frac{\partial F_{2C}}{\partial p_1} = 0, \quad z = \pm H \tag{Bl.15}$$

$$(\lambda + 2\mu) \frac{\partial^2 F_{2C}}{\partial z\, \partial p_1} + \lambda k \frac{\partial F_{1C}}{\partial p_1} = - \frac{\mu}{k} \frac{\partial D_C}{\partial p_1}, \quad z = \pm H$$

If p_1 is a double zero of D_C, this last equation may be written:

$$(\lambda + 2\mu) \frac{\partial^2 F_{2C}}{\partial z\, \partial p_1} + \lambda k \frac{\partial F_{1C}}{\partial p_1} = 0, \quad z = \pm H \tag{Bl.16}$$

Substituting equations Bl.15 and Bl.16 into equation Bl.14, and using equations Bl. 3 and Bl. 4, one obtains:

$$2p_1 \int_o^H \left[F_{1C}^2(k, p_1, z) + F_{2C}^2(k, p_1, z) \right] dz = 0 \tag{Bl.17}$$

It has been shown above that the only possible zeros of D_C are complex conjugate, pure imaginary, given by $p = \pm i\omega_{nC}(k)$, say. Substituting these values of p into equations Bl.12 and Bl.13 it is seen that, for real k, three cases of equation Bl.17 need to be considered, viz.:

(i) η_1 and η_2 both real, corresponding to $k^2 c_s^2 > \omega_{nC}^2(k)$

(ii) η_1 pure imaginary, η_2 real, corresponding to

$$(c_d^2)^{-1} \omega_{nC}^2(k) < k^2 < (c_s^2)^{-1} \omega_{nC}^2(k)$$

(iii) η_1 and η_2 both pure imaginary, corresponding to $c_d^2 k^2 < \omega_{nC}^2(k)$.

Other possible cases corresponding to $p_1 = 0$ or to the vanishing of either η_1 or η_2 are excluded from the discussion, since these points receive special attention in the text.

Case (i). It is seen from equations 1.84 and 1.85 that F_{1C} and F_{2C} are both real when η_1 and η_2 are real and hence the integrand in equation B1.17 is always positive. Thus the equation leads to a contradiction and hence there are no repeated zeros of D_C for which $k^2 c_s^2 > \omega_{nC}^2(k)$.

Case (ii). On substituting $\eta_1 = \pm i\xi_1$, where ξ_1 is real, into equations 1.84 and 1.85, expressions of the type if_1 and if_2, where f_1 and f_2 are real-valued functions, are obtained for F_1 and F_2. Hence the integrand in equation B1.17 is always negative and so a contradiction is again reached. Hence there are no repeated zeros of D_C for which

$$(c_d^2)^{-1}\omega_{nC}^2(k) < k^2 < (c_s^2)^{-1}\omega_{nC}^2(k)$$

Case (iii). On substituting $\eta_1 = \pm i\xi$, $\eta_2 = \pm i\xi_2$, where ξ_1 and ξ_2 are real, into equations 1.84 and 1.85, expressions of the type ig_1 and ig_2, where g_1 and g_2 are real-valued functions, are obtained for F_1 and F_2. Hence the integrand in equation B1.17 is always negative and so a contradiction is reached once more. Hence there are no repeated zeros of D_C for which $c_d^2 k^2 < \omega_{nC}^2(k)$ and so, in summary, there are no repeated zeros of D_C for real k.

A similar proof showing that D_F has no repeated zeros for real k follows readily.

B1.4. TRANSVERSELY ISOTROPIC PLATE

The next case to be considered is that of the transversely isotropic plate treated in Section III. The notation in that section is adopted here also. In particular note the difference in the choice of z here and in the previous sections of this appendix.

For the case of complex roots the argument is very similar to that given for the isotropic plate, the only difference lying in algebraic details. Either of the two problems treated in Section III can be focussed upon, since the same frequency equation arises in both cases. Choosing the longitudinal impact type problem and going through the procedure given above (§B1.1), equation B1.10 is obtained again, where now F_1 and F_2 are given by equations 3.19 and 3.20, respectively, and p_1 and p_2 are roots of equation 3.21 set equal to zero. The branch points of the functions in this case do not necessarily lie on the imaginary p-axis, but this makes no basic difference in the argument, since the branch cuts can still be so chosen that the principle of reflection applies. The remainder of the proof now goes through as above, and it can be concluded that $D(k, p)$, as given by equation 3.21, has no complex roots for real k, except possibly pure imaginary or real roots. The points $p = 0, \infty$ and the zeros of η_1 and η_2 are not covered by the above discussion, but receive special attention in the text.

Proofs of the absence of real and repeated roots are not readily forthcoming in the present example, not only because of greater algebraic complexity, but also because the relative magnitudes of the various elastic constants are unknown. For the problem at hand the absence of such roots must be deduced from the physical arguments given previously (§1.2), i.e., negative real roots are excluded on the basis that the basic physical model has no dissipative mechanism and positive real and repeated roots are excluded on the basis that they lead to solutions which diverge with increasing time.

B1.5. PLATE CONSISTING OF ISOTROPIC PARALLEL LAYERS; COMPLEX ROOTS

Some of the scope and limitations of the present technique for determining the nature of the roots of equations have been given in the preceding sections of this appendix. Since the method appears to be of general use in multi-integral transform techniques, some further illustrations of its applicability will now be given. The problem to be considered here is that of an infinite plate consisting of $(2n - 1)$ isotropic layers in welded contact, with the symmetries indicated in Figure 19, subjected to surface loadings which generate either symmetric or antisymmetric waves. It should be noted that a detailed solution of the problem is not aimed at here, but instead general information pertinent to the above techniques is sought.

The notation used is shown in Figure 19. Assuming zero initial and "radiation" conditions, zero body forces, and axial symmetry, application of the Laplace and Hankel transforms (for the interval $(0, \infty)$) to equations 1.3, 1.4, 1.10, and 1.11, gives the following transformed equations for the jth layer:

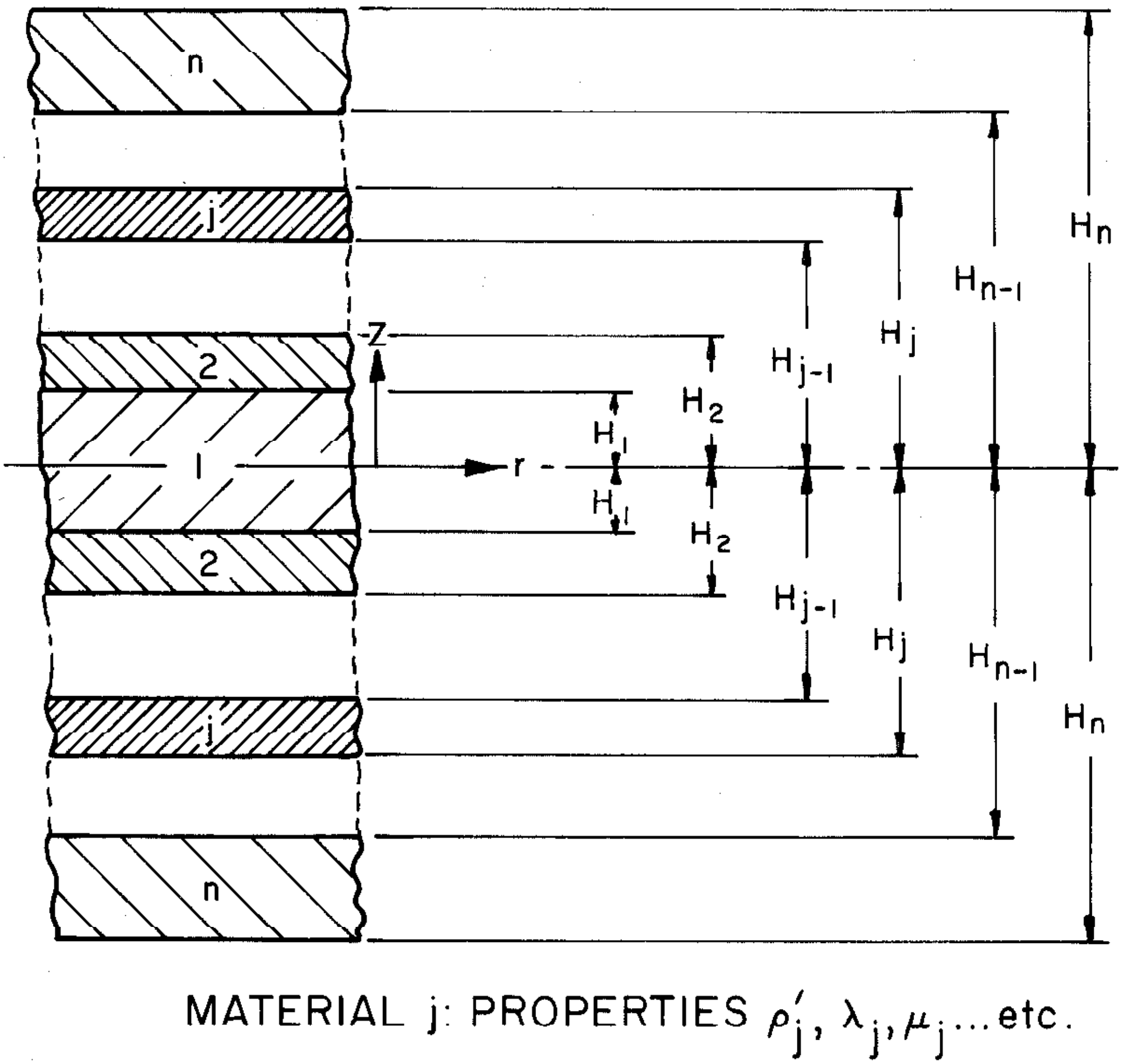

Fig. 19. Geometry of layered plate and coordinates used.

$$\frac{d^2\tilde{\tilde{u}}^1_{rj}}{dz^2} - (c^2_{sj})^{-1}(p^2 + k^2 c^2_{dj})\tilde{\tilde{u}}^1_{rj} - k(c^2_{sj})^{-1}(c^2_{dj} - c^2_{sj})\frac{d\tilde{\tilde{u}}^o_{zj}}{dz} = 0,$$

$$j = 1, 2, \ldots, n \qquad \text{(B1.18)}$$

$$\frac{d^2\tilde{\tilde{u}}^o_{zj}}{dz^2} - (c^2_{dj})^{-1}(p^2 + k^2 c^2_{sj})\tilde{\tilde{u}}^o_{zj} + k(c^2_{dj})^{-1}(c^2_{dj} - c^2_{sj})\frac{d\tilde{\tilde{u}}^1_{rj}}{dz} = 0,$$

$$j = 1, 2, \ldots, n \qquad \text{(B1.19)}$$

$$\tilde{\tilde{\sigma}}^1_{rzj} = \mu_j\left(\frac{d\tilde{\tilde{u}}^1_{rj}}{dz} - k\tilde{\tilde{u}}^o_{zj}\right), \quad j = 1, 2, \ldots, n \qquad \text{(B1.20)}$$

$$\tilde{\tilde{\sigma}}^o_{zzj} = (\lambda_j + 2\mu_j)\frac{d\tilde{\tilde{u}}^o_{zj}}{dz} + \lambda_j k\tilde{\tilde{u}}^1_{rj}, \quad j = 1, 2, \ldots, n \qquad \text{(B1.21)}$$

The boundary conditions are taken to be:

$$\sigma_{rzn} = 0, \quad z = \pm H_n$$

$$\sigma_{zzn} = \pm Vh(r)H(t), \quad z = \pm H_n$$

where $H(t)$ is the Heaviside step function, $h(r)$ is an arbitrary function, V is a constant, and an appropriate choice of the plus and minus signs leads to the generation of either symmetric or antisymmetric waves. Transforming these conditions gives:

$$\tilde{\tilde{\sigma}}^1_{rzn} = 0, \quad z = \pm H_n$$

$$\tilde{\tilde{\sigma}}^o_{zzn} = \pm \frac{V}{p}\tilde{h}^o(k), \quad z = \pm H_n \qquad \text{(B1.22)}$$

The transformed continuity conditions at the interfaces may be written:

$$\tilde{\tilde{u}}^1_{rj} = \tilde{\tilde{u}}^1_{r(j+1)}, \quad z = \pm H_j, \quad j = 1, 2, \ldots, (n-1) \tag{Bl. 23}$$

$$\tilde{\tilde{u}}^o_{zj} = \tilde{\tilde{u}}^o_{z(j+1)}, \quad z = \pm H_j, \quad j = 1, 2, \ldots, (n-1) \tag{Bl. 24}$$

$$\mu_j \left[\frac{d\tilde{\tilde{u}}^1_{rj}}{dz} - k\tilde{\tilde{u}}^o_{zj} \right] = \mu_{j+1} \left[\frac{d\tilde{\tilde{u}}^1_{r(j+1)}}{dz} - k\tilde{\tilde{u}}^o_{z(j+1)} \right], \quad z = \pm H, \quad j = 1, 2, \ldots, (n-1) \tag{Bl. 25}$$

$$(\lambda_j + 2\mu_j) \frac{d\tilde{\tilde{u}}^o_{zj}}{dz} + \lambda_j k\tilde{\tilde{u}}^1_{rj} = (\lambda_{j+1} + 2\mu_{j+1}) \frac{d\tilde{\tilde{u}}^o_{z(j+1)}}{dz} + \lambda_{j+1} k\tilde{\tilde{u}}^1_{r(j+1)},$$

$$z = \pm H, \quad j = 1, 2, \ldots, (n-1) \tag{Bl. 26}$$

An outline of the process whereby the solution may be determined will now be given. The displacements are written as linear combinations of appropriate hyperbolic functions, the exact form depending on whether symmetric or antisymmetric waves are generated. In general, having $(2n - 2)$ interfaces, with 4 conditions at each, and 4 surface conditions, there are $(8n - 4)$ conditions to be satisfied. In the case of either symmetric or antisymmetric waves there are only $(4n - 2)$ independent conditions. In such cases the displacements in each layer, except the layer of material one, are written containing 8 arbitrary constants. That the layer of material one differs from the other layers is due to the fact that the displacements there are either symmetric or antisymmetric w. r. t. the plane $z = 0$, whereas the displacements in all other layers contain both symmetric and antisymmetric components w. r. t. the <u>middle plane of the layer</u>, even though they are either symmetric or antisymmetric w. r. t. the plane $z = 0$. The satisfaction of the transformed displacement equations of motion imposes 4 relations between the 8 constants in each layer and so, in effect, the displacements in each layer other than layer one, contain 4 arbitrary constants. The number of such layers is $(n - 1)$ so that, on taking into account the 2 constants coming from layer one, there are $(4n - 2)$ constants to be determined from the $(4n - 2)$ conditions.

These conditions lead to a set of $(4n - 2)$ linear algebraic nonhomogeneous equations for the $(4n - 2)$ arbitrary constants A_q, $q = 1, 2, \ldots, (4n-2)$. The nonhomogeneous terms in these equations correspond to the transformed boundary conditions given by equations Bl. 22 and so may be written as a $(4n - 2)$ dimensional column vector, whose only nonvanishing element is the term $(p)^{-1}V\tilde{h}^o(k)$. Thus, using Cramer's rule, the solutions to the set of equations may be written:

$$A_q = \frac{\det B_q}{\det C}, \quad q = 1, 2, \ldots, (4n-2)$$

where C is the matrix of the coefficients of the A_q's and B_q is the matrix C with its qth column replaced by the column vector corresponding to the nonhomogeneous terms. Suppose the only nonvanishing term of this column vector occurs in the ith row. Then

$$A_q = \frac{V\tilde{h}^o(k)C_{iq}}{p \det C}$$

where C_{iq} is the cofactor of the element c_{iq} of $\det C$. Hence the solutions for the displacements in the jth layer are of the form:

$$\tilde{\bar{u}}^1_{rj} = \frac{V\tilde{h}^o(k)F_j(k, p, z)}{pD(k, p)}, \quad j = 1, 2, \ldots, n \qquad \text{(Bl. 27)}$$

$$\tilde{\tilde{u}}^{o}_{zj} = \frac{V\tilde{h}^{o}(k)G_j(k,p,z)}{pD(k,p)}, \quad j = 1, 2, \ldots, n \tag{B1.28}$$

where

$$F_j(k,p,z) = \sum_{y=1}^{4n-2} C_{iy}\beta_{yj}(k,p,z), \quad j = 1, 2, \ldots, n$$

$$G_j(k,p,z) = \sum_{y=1}^{4n-2} C_{iy}\,\alpha_{yj}(k,p,z), \quad j = 1, 2, \ldots, n$$

and the α_{yq}'s and the β_{yq}'s denote the coefficients of the arbitrary constants in the expressions for the transformed displacements. Since each of these displacements does not involve all the constants, many of the α's and β's will be zero.*

Substituting equations B1.27 and B1.28 into equations B1.18, B1.19, B1.22, B1.23, B1.24, B1.25 and B1.26 there results:

$$\frac{d^2F_j^{(\gamma)}}{dz^2} - (c_{sj}^2)^{-1}(p_\gamma^2 + k^2c_{dj}^2)F_j^{(\gamma)} - k\left(\frac{c_{dj}^2}{c_{sj}^2} - 1\right)\frac{dG_j^{(\gamma)}}{dz} = 0, \quad j = 1, 2, \ldots, n \tag{B1.29}$$

$$\frac{d^2G_j^{(\gamma)}}{dz^2} - (c_{dj}^2)^{-1}(p_\gamma^2 + k^2c_{sj}^2)G_j^{(\gamma)} + k\left(1 - \frac{c_{sj}^2}{c_{dj}^2}\right)\frac{dF_j^{(\gamma)}}{dz} = 0, \quad j = 1, 2, \ldots, n \tag{B1.30}$$

$$\frac{dF_n^{(\gamma)}}{dz} - kG_n^{(\gamma)} = 0, \quad z = \pm H_n \tag{B1.31}$$

---

*Illustrations of the above statements and techniques can be found, for example, in the work of Saito and Sato (69).

$$(\lambda_n + 2\mu_n)\frac{dG_n^{(\gamma)}}{dz} + \lambda_n k F_n^{(\gamma)} = D(k, p_n), \quad z = \pm H_n \qquad \text{(Bl. 32)}$$

$$F_j^{(\gamma)} = F_{j+1}^{(\gamma)}, \quad z = \pm H_j, \quad j = 1, 2, \ldots, (n-1) \qquad \text{(Bl. 33)}$$

$$G_j^{(\gamma)} = G_{j+1}^{(\gamma)}, \quad z = \pm H_j, \quad j = 1, 2, \ldots, (n-1) \qquad \text{(Bl. 34)}$$

$$\mu_j\left(\frac{dF_j^{(\gamma)}}{dz} - kG_j^{(\gamma)}\right) = \mu_{j+1}\left(\frac{dF_{j+1}^{(\gamma)}}{dz} - kG_{j+1}^{(\gamma)}\right), \quad z = \pm H_j, \quad j = 1, 2, \ldots, (n-1) \qquad \text{(Bl. 35)}$$

$$(\lambda_j + 2\mu_j)\frac{dG_j^{(\gamma)}}{dz} + \lambda_j k F_j^{(\gamma)} = (\lambda_{j+1} + 2\mu_{j+1})\frac{dG_{j+1}^{(\gamma)}}{dz} + \lambda_{j+1} k F_{j+1}^{(\gamma)},$$

$$z = \pm H_j, \quad j = 1, 2, \ldots, (n-1) \qquad \text{(Bl. 36)}$$

where $F_j^{(\gamma)}$ and $G_j^{(\gamma)}$ are the values of $F_j(k, p, z)$ and $G_j(k, p, z)$ corresponding to $p = p_\gamma$. Equation Bl. 29 with $\gamma = \ell$ is now multiplied by $F_j(m)$ and the result is subtracted from the product of $F_j(\ell)$ with equation Bl. 29 with $\gamma = m$. Then equation Bl. 30 with $\gamma = \ell$ is multiplied by $G_j^{(m)}$ and the result is subtracted from the product of $G_j^{(\ell)}$ with equation Bl. 30 with $\gamma = m$. Adding the results of these operations and integrating between Q and T, where Q and T are arbitrary, gives:

$$(p_\ell^2 - p_m^2)\int_Q^T I_j(z)\, dz = \left[\varphi_j(z)\right]_Q^T, \quad j = 1, 2, \ldots, n \qquad \text{(Bl. 37)}$$

where

$$I_j(z) = F_j^{(m)} F_j^{(\ell)} + G_j^{(m)} G_j^{(\ell)}, \quad j = 1, 2, \ldots, n \tag{B1.38}$$

$$\varphi_j(z) = (\rho_j')^{-1} \Big\{ G_j^{(m)} \Big[(\lambda_j + 2\mu_j) \frac{dG_j^{(\ell)}}{dz} + k\lambda_j F_j^{(\ell)} \Big] - G_j^{(\ell)} \Big[(\lambda_j + 2\mu_j) \frac{dG_j^{(m)}}{dz} + k\lambda_j F_j^{(m)} \Big] + \mu_j F_j^{(m)} \Big[\frac{dF_j^{(\ell)}}{dz} - kG_j^{(\ell)} \Big] \mu_j F_j^{(\ell)} \Big[\frac{dF_j^{(m)}}{dz} - kG_j^{(m)} \Big] \Big\}, \quad j = 1, 2, \ldots, n \tag{B1.39}$$

and it is to be understood that $p = \infty$ is excluded from the discussion.

Taking $j = 1$, $Q = -H_1$, $T = H_1$, $j = 2$, $Q = H_1$, $T = H_2$, $j = 2$, $Q = -H_1$, $T = -H_2$, $j = j$, $Q = H_{j-1}$, $T = H_j$, $j = j$, $Q = -H_{j-1}$, $T = -H_j$, etc., successively in equation B1.39 and adding the results, gives:

$$(p_\ell^2 - p_m^2) \Big\{ N_1 \int_{-H_1}^{H_1} I_1(z)\,dz + \sum_{t=2}^{n} N_t \Big[\int_{H_{t-1}}^{H_t} I_t(z)\,dz - \int_{-H_{t-1}}^{-H_t} I_t(z)\,dz \Big] \Big\}$$

$$= \sum_{q=1}^{n-1} [N_q \varphi_q(H_q) - N_q \varphi_q(-H_q) - N_{q+1} \varphi_{q+1}(H_q) + N_{q+1} \varphi_{q+1}(-H_q)] + [N_n \varphi_n(H_n) - N_n \varphi_n(-H_n)] \tag{B1.40}$$

where the N's are constants to be determined.

From equations B1.27 and B1.28 it is seen that for the symmetric case F_j and G_j, $j = 1, 2, \ldots, n$, are even and odd functions of z, respectively. Hence equations B1.38 and B1.39 give that $I_j(z)$ and $\varphi_j(z)$, $j = 1, 2, \ldots, n$, are even and odd functions of z, respectively. It can readily be seen that the same is true in the antisymmetric case.

Hence in either case equation Bl. 40 may be written:

$$(p_\ell^2 - p_m^2)\left\{N_1\int_{-H_1}^{H_1} I_1(z)\,dz + 2\sum_{t=2}^{n} N_t\int_{H_{t-1}}^{H_t} I_t(z)\,dz\right.$$

$$= 2\sum_{q=1}^{n-1}\left[N_q\varphi_q(H_q) - N_{q+1}\varphi_{q+1}(H_q)\right] + 2N_n\varphi_n(H_n) \qquad \text{(Bl. 41)}$$

Using equations Bl. 31 through Bl. 36, it can be shown that, if $N_q = \rho_q'$, $q = 1, 2, \ldots, n$, then

$$N_{q+1}\varphi_{q+1}(H_q) - N_q\varphi_q(H_q) = 0, \quad q = 1, 2, \ldots, (n-1)$$

$$N_n\varphi_n(H_n) = G_n^{(m)}(H_n)D(k, p_\ell) - G_n^{(\ell)}(H_n)D(k, p_m)$$

Hence, with this choice of the N's, equation Bl. 41 gives:

$$(p_\ell^2 - p_m^2)\sum_{t=1}^{n} \rho_t'\int_{H_{t-1}}^{H_t} I_t(z)\,dz = G_n^{(m)}(H_n)D(k, p_\ell) - G_n^{(\ell)}(H_n)D(k, p_m)$$

where $H_o \equiv 0$. On assuming p_ℓ and p_m are zeros of $D(k, p)$, this last equation may be written:

$$(p_\ell^2 - p_m^2)\sum_{t=1}^{n} \rho_t'\int_{H_{t-1}}^{H_t} I_t(z)\,dz = 0 \qquad \text{(Bl. 42)}$$

The remainder of the proof now proceeds as above (§Bl. 1) and it can be concluded that the frequency equation, i.e., $D(k, p) = 0$, has no complex roots. The problems of real or pure imaginary zeros, and repeated zeros, must be examined separately for each individual case.

Successful applications of the method to other geometries, e.g., infinite rod consisting of concentric layers, can readily be made.

Appendix C. ANALYSIS OF FORMAL SOLUTIONS

A number of formal steps have been taken in arriving at equations 1.134, 1.135, 2.30 and 2.31, but, appealing to the elastodynamic uniqueness theorem, if it can be shown that these equations satisfy the differential equations, boundary and initial conditions, then they are the solutions. One method of approach to this problem is to differentiate through the series and integrals, and manipulate the terms as convenient, assuming that the necessary convergence properties are satisfied. Unfortunately a rigorous demonstration of these convergence properties is impractical, in view of the complicated algebraic form of the equations. [For some measure of the tasks involved in such a demonstration, cf. Cagniard (15).] In order to have some confidence in the solutions, equations 1.134, 1.135, 2.30 and 2.31, will now be examined formally, though the final criterion as to their validity would have to be experimental.

Inspection of equations 1.134 and 1.135 shows that possible sources of trouble are (i) zeros of $J_1^2(ka) + Y_1^2(ka)$ (ii) zeros of $N_C[k, i\omega_{nC}(k)]$ (iii) zeros of $[k^2c_d^2 - \omega_{nC}^2(k)]$ (iv) $k = 0$ (v) $k = \infty$.

(i) Zeros of $J_1^2(ka) + Y_1^2(ka)$

Since $J_1^2(ka) + Y_1^2(ka) \geq 0$, for positive real k, the only values of k for which the expression vanishes are those for which $J_1(ka)$ and $Y_1(ka)$ vanish simultaneously. But the zeros of $J_1(ka)$ and $Y_1(ka)$ are distinct and so $J_1^2(ka) + Y_1^2(ka)$ is never zero, for positive real k.

(ii) Zeros of $N_C[k, i\omega_{nC}(k)]$

The argument given here is a somewhat expanded version of a similar argument given by Skalak for a related rod problem (2). It is valid also for $N_F[k, i\omega_{nF}(k)]$ and both cases can be considered together by deleting the suffixes C and F.

Equation 1.121 (or 1.122) gives that

$$N[k, i\omega_n(k)] = \frac{1}{i\omega_n(k)} \frac{\partial D}{\partial p}[k, i\omega_n(k)]$$

and so the zeros of N are the zeros of $\partial D/\partial p$, since $\omega_n(k) \neq \infty$ for $k \neq \infty$ (the case $k = \infty$ will be considered later). Consider the kp-plane, k, p real, and let $\vec{e}_p$, $\vec{e}_k$, $\vec{e}_S$ and $\vec{e}_T$ be unit vectors in this plane, in the p, k, S and T directions, respectively, where T is the tangent direction of an $\omega_n(k)$ versus k curve and S is an arbitrary direction. Along an $\omega_n(k)$ versus k curve $D(k, p) = 0$ and hence

$$\frac{\partial D}{\partial T} = 0$$

i.e.,

$$\frac{\partial D}{\partial T} = \vec{e}_T \cdot \nabla D = \frac{\partial D}{\partial k}(\vec{e}_k \cdot \vec{e}_T) + \frac{\partial D}{\partial p}(\vec{e}_p \cdot \vec{e}_T) = 0 \qquad \text{(C1.1)}$$

Hence, if at the point on the curve in question $\partial D/\partial p = 0$, or $\vec{e}_p \cdot \vec{e}_T = 0$, then equation C1.1 gives that either $\partial D/\partial k = 0$, or $\vec{e}_k \cdot \vec{e}_T = 0$ (discounted, since inspection of the modes shows that no vertical tangents occur). Hence

$$\frac{\partial D}{\partial S} = \vec{e}_S \cdot \nabla D = \frac{\partial D}{\partial k}(\vec{e}_k \cdot \vec{e}_S) + \frac{\partial D}{\partial p}(\vec{e}_p \cdot \vec{e}_S) = 0$$

showing that D is either a maximum, minimum, or is stationary, at the point in question. However it cannot be a maximum or minimum, since $D(k, p) = 0$ along the curve. If D is stationary at the point, then two $\omega_n(k)$ versus k curves intersect there. Inspection of the modes, noting that Figures such as 2 give only one quadrant (the other quadrants are obtained on noting that the modes are even functions of k and ω), shows that no such intersections occur, except at $k = 0$, where the lowest mode and its reflection in the k-axis touch. Thus this point is not covered by the above argument, a restriction which Skalak failed to mention. In fact it can readily be shown by an expansion that $N = 0$ at $k = 0$.

(iii) Zeros of $[k^2 c_d^2 - \omega_{nC}^2(k)]$

The $\omega_{nC}(k)$ versus k curves are the zeros of $D_C(k, p)$. Substituting $p = \pm ikc_d$ into equation 1.56 it is found that $D_C(k, \pm ikc_d) = 0$, for $kH[(c_d^2/c_s^2) - 1]^{1/2} = m\pi$, $m = 0, 1, 2, 3, \ldots$, or $k = k_m$, say, so that zeros of $[k^2 c_d^2 - \omega_{nC}^2(k)]$ can indeed occur. These isolated points are scattered throughout the entire mode spectrum and some modes may not contain any of them. For instance, there are no points on the lowest mode for which $\omega_{1C}(k) = kc_d$. However $[k^2 c_d^2 - \omega_{1C}^2(k)]$ does vanish for $k = 0$, but this point receives separate discussion later. Taylor expansions about k_m and use of the fact that

$$F_{1C}[k, \pm ikc_d, z] = F_{2C}[k, \pm ikc_d, z] = 0$$

shows that

$$\lim_{k \to k_m} \frac{F_{1C}[k, \pm ikc_d, z]}{[k^2c_d^2 - \omega_{nC}^2(k)]} = \frac{\text{constant}}{\frac{\partial \omega_{nC}(k_m)}{\partial k}}$$

which is finite, since $\partial\omega_{nC}(k_m)/\partial k \neq 0$ for $k_m \neq 0$ (however horizontal tangents do occur for other k values), and

$$\lim_{k \to k_m} \frac{F_{2C}[k, \pm ikc_d, z]}{[k^2c_d^2 - \omega_{nC}^2(k)]} = 0$$

Hence the points k_m, except possibly $k_m = 0$, are not singularities of the integrands.

(iv) $k = 0$

The cut-off frequencies, i.e., the limiting values of $\omega_{nC}(k)$ as $k \to 0$, of the symmetric Rayleigh-Lamb frequency equation are (16):

$$n = 1, \quad \lim_{k \to 0} \omega_{1C}(k) = \lim_{k \to 0} kc_P$$

$$n > 1, \quad \omega_{nC}(k) = \frac{n\pi c_n}{2H} \quad , \quad n = 2, 3, 4, \ldots$$

where

$$c_n = c_d, \quad n = 3, 5, 7, \ldots$$

$$= c_s, \quad n = 2, 4, 6, \ldots$$

Using these results, Taylor expansions about $k = 0$ show that the integrands in equations 1.134 and 1.135 go to zero as $k \to 0$ (for finite n).

(v) $k = \infty$

It is known (16) that $\omega_{\ell C}(k) \to kc_\ell$, $k \to \infty$, where $c_\ell = c_R$ (Rayleigh wave speed), $\ell = 1$, $c_\ell = c_s$, $\ell > 1$. Using these results, expressions for large k can be obtained for F_{1C}, F_{2C} and N_C. For $n > 1$ some caution is necessary, since the value of $[k^2 - (c_s^2)^{-1}\omega_{nC}^2(k)]$, which arises in the terms involving η_1, is uncertain. Assuming that, for $n > 1$,

$$\lim_{k\to\infty} \omega_{nC}(k) = \lim_{k\to\infty} [kc_s \pm \epsilon(k)]$$

where $\epsilon(k) = \partial/k^m$, ∂ and m being positive constants, and replacing the Bessel functions by the leading terms of their large-argument asymptotic expansions, it can be shown that the integrands are well behaved as $k \to \infty$.

The above arguments showing that the integrands are well behaved over the entire range of integration, not only facilitate the determination of the stationary phase solutions given in §1.3, but also make more plausible the following formal verification of the solutions, in which the various expressions are obtained by differentiating through the series and integrals.

Substituting equations 1.134 and 1.135 into equations 1.10, 1.11 (with $F_r = F_z = 0$), 1.3, and 1.4, gives:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{c_s^2}{c_d^2}\frac{\partial^2}{\partial z^2} - \frac{1}{c_d^2}\frac{\partial^2}{\partial t^2}\right) {}_C u_r + \left(1 - \frac{c_s^2}{c_d^2}\right)\frac{\partial^2}{\partial z \partial r}\, {}_C u_z$$

$$= \frac{4\lambda U_o c_s^2}{\pi\mu c_d^2} \sum_{n=1}^{\infty} \int_o^{\infty} kC_1(k, r, a)\, W_n^{-1}(k) \left\{ \frac{\partial^2}{\partial z^2} F_{1C} - \left(\frac{c_d^2}{c_s^2} - 1\right) k \frac{\partial}{\partial z} F_{2C} - \frac{c_d^2}{c_s^2}\eta_2^2 F_{1C} \right\}$$

$$\times \cos \omega_{nC}(k)t \, dk \qquad \text{(C1. 2)}$$

$$\left(1 - \frac{c_s^2}{c_d^2}\right)\left(\frac{\partial^2}{\partial z \partial r} + \frac{1}{r}\frac{\partial}{\partial z}\right) {}_C u_r + \left[\frac{\partial^2}{\partial z^2} + \frac{c_s^2}{c_d^2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\right] - \frac{1}{c_d^2}\frac{\partial^2}{\partial t^2} \quad {}_C u_z$$

$$= \frac{4\lambda U_o}{\pi\mu} \sum_{n=1}^{\infty} \int_o^{\infty} kC_o(k, r, a) W_n^{-1}(k) \left\{ \frac{\partial^2}{\partial z^2} F_{2C} - \frac{c_s^2}{c_d^2} \eta_1^2 F_{2C} \right.$$

$$\left. + \left(1 - \frac{c_s^2}{c_d^2}\right) k \frac{\partial}{\partial z} F_{1C} \right\} \cos \omega_{nC}(k)t \; dk \qquad \text{(C1. 3)}$$

$${}_C\sigma_{rz} = \frac{4\lambda U_o}{\pi} \sum_{n=1}^{\infty} \int_o^{\infty} kC_1(k, r, a) W_n^{-1}(k) \left\{ \frac{\partial}{\partial z} F_{1C} - kF_{2C} \right\} \cos \omega_{nC}(k)t \; dk$$

(C1. 4)

$${}_C\sigma_{zz} = \frac{4\lambda U_o}{\pi} \sum_{n=1}^{\infty} \int_o^{\infty} kC_o(k, r, a) W_n^{-1}(k) \left\{ (\lambda + 2\mu) \frac{\partial}{\partial z} F_{2C} \right.$$

$$\left. + \lambda k F_{1C} \right\} \cos \omega_{nC}(k)t \; dk \qquad \text{(C1. 5)}$$

Direct substitution of equations 1. 84 and 1. 85 shows that the terms in the { } brackets in equations C1. 2 and C1. 3 are zero, so that equations 1. 134 and 1. 135 satisfy the differential equations. Again direct substitution of equations 1. 84 and 1. 85, and use of equation 1. 109, shows that the terms in the {} brackets in equations C1. 4 and C1. 5 are zero at $z = \pm H$, so that equations 1. 134 and 1. 35 satisfy the boundary conditions on the lateral surfaces of the plate. Setting $r = a$ in equations 1. 134 and C1. 3 gives

$${}_C u_r \Big|_{r=a} = U_o, \quad {}_C\sigma_{rz} \Big|_{r=a} = 0$$

so that the boundary conditions at the cavity are satisfied. The time dependence of the u_r condition follows from the fact that u_r (and u_z) are zero for times less than the arrival time of the wave front (which stems from the order condition of the Laplace transform, cf. §1. 2). This property also insures the satisfaction of the initial conditions given by equations 1. 21. The "radiation" conditions given by equations 1. 71 follow on taking the limits inside the series and integral signs.

Another feature which strongly supports the validity of the solutions is the correctness of the long time, or static, solutions, viz.,

$$_C u_r = U_o \frac{a}{r}$$

$$_C u_z = 0$$

It can easily be shown that these are in fact the solutions to the equivalent static boundary value problem.

Some comments should be made here regarding the solutions to finite plate problems. It can be formally shown that the equations analogous to 1.134 and 1.135 satisfy the differential equations, initial conditions, and the boundary conditions on the lateral surfaces of the plate. However, as pointed out by Muskat (63), difficulties arise when one attempts to verify the boundary conditions on the cylindrical surfaces, since the eigenfunctions in the series are identically zero at both ends of the interval. Hence the series must be summed before verification of these boundary conditions can be attempted.

Most of the preceding discussions are relevent to the body force problem also and so the discussion of equations 2. 30 and 2. 31 will be correspondingly brief.

Using arguments similar to those given above, it can be shown that the integrands in equations 2.30 and 2.31 are well behaved over the entire range of integration. However in this case it is quite difficult to show even formally that the equations 2.30 and 2.31 satisfy the differential equations, because of the presence of the nonhomogeneous body force terms. To establish this aspect of the equations, suitable limiting processes would have to be undergone and much greater care taken in obtaining derivatives. This difficulty however does not seriously detract from the plausibility of the results, since it is a feature of many solutions to nonhomogeneous equations, established, for example, by Green's function methods, and these solutions are known to be of value.

The satisfaction of the boundary conditions can be shown as follows: Substituting equations 2.30 and 2.31 into equations 1.3 and 1.4 gives

$$
{}_C\sigma_{rz} = \mu\left(\frac{\partial}{\partial z} R_{or} + \frac{\partial}{\partial r} R_{oz}\right) - \frac{\lambda P_o}{2\pi H}\sum_{n=1}^{\infty}\int_o^{\infty}\frac{k^3 J_1(kr)J_1(ka)\cos\omega_{nC}(k)t}{[k^2c_d^2-\omega_{nC}^2(k)]\,\omega_{nC}^2(k)N_C[k, i\omega_{nC}(k)]}\left\{\frac{\partial}{\partial z}F_{1C} - kF_{2C}\right\}dk \tag{C1.6}
$$

$$
\begin{aligned}
{}_C\sigma_{zz} = {} & (\lambda + 2\mu)\left[\frac{\partial}{\partial z} R_{oz} + \left(1 - 2\frac{c_s^2}{c_d^2}\right)\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)R_{or}\right] \\
& - \frac{\lambda P_o}{2\pi\mu H}\sum_{n=1}^{\infty}\int_o^{\infty}\frac{k^3 J_o(kr)J_1(ka)\cos\omega_{nC}(k)t}{[k^2c_d^2-\omega_{nC}^2(k)]\,\omega_{nC}^2(k)N_C[k, i\omega_{nC}(k)]} \\
& \times\left\{(\lambda+2\mu)\frac{\partial}{\partial z}F_{2C} + \lambda kF_{1C}\right\}dk + \begin{cases} 0, & r > a \\ \dfrac{\lambda P_o}{4\pi H c_d^2 a}, & 0 < r < a \end{cases}
\end{aligned} \tag{C1.7}
$$

Direct substitution of equations 2.32 and 2.33 shows that

$$\frac{\partial}{\partial z} R_{or} + \frac{\partial}{\partial r} R_{oz} = 0, \quad z = \pm H \tag{Cl. 8}$$

and

$$(\lambda + 2\mu)\left[\frac{\partial}{\partial z} R_{oz} + \left(1 - 2\frac{c_s^2}{c_d^2}\right)\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)R_{or}\right]$$

$$= -\frac{\lambda P_o}{4\pi H c_d^2}\int_o^{\infty} J_o(kr)J_1(ka)\, dk, \quad z = \pm H \tag{Cl. 9}$$

Now ((36), page 92)

$$\int_o^{\infty} J_o(kr)J_1(ka)\, dk = \begin{cases} 0, & r > a \\ \frac{1}{a}, & r < a \end{cases}$$

and using this, and equations Cl. 8 and Cl. 9, and the fact that the terms in the { } brackets in equations Cl. 6 and Cl. 7 are zero at $z = \pm H$, it follows that the boundary conditions on the lateral surfaces of the plate are satisfied.

The satisfaction of the initial conditions follows from the wave front property and the radiation conditions follow on taking the limits inside the series and integrals.

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