ASYMPTOTIC L_2 INEQUALITIES OF MARKOFF TYPE

Thesis by

Lawrence Fred Shampine

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1964

ACKNOWLEDGEMENTS

I wish to acknowledge the invaluable guidance and assistance of Professor John Todd throughout my graduate studies.

Thanks are due to an inspiring teacher, Miss Virginia Cromartie, for stimulating my interest in mathematics and to the National Merit Corporation and the National Science Foundation for financially assisting my studies.

I wish to thank Miss Marilyn Clauser for her careful work on a difficult typing job.

ABSTRACT

The Markoff brothers have obtained bounds in the maximum norm for polynomials in terms of bounds for their derivatives. E. Schmidt has obtained asymptotic bounds for the analogous problem for weighted L_2 norms and the first derivative.

This thesis extends Schmidt's work in one case to the second derivative. A new technique is applied which recovers the principal term of Schmidt's asymptotic result and furnishes information about the extremal polynomials. Exact bounds are derived in one case.

INTRODUCTION

In 1890 A. Markoff [6] established a bound on the derivative of a polynomial in terms of a bound on the polynomial itself. He showed

$$\max |f'(x)| \leq n^2 \max |f(x)|$$

where each maximum is over the range $-1 \leq x \leq 1$, where $f(x)$ is any polynomial of degree at most n and where the prime denotes differentiation with respect to x. Equality is obtained for multiples of T_n, the Chebyshev polynomial of degree n.

Repeated application of A. Markoff's inequality gives similar inequalities for higher derivatives which, however, are not sharp. The best possible result is the following, which is due to W. Markoff [7].

$$\max |f^{(k)}(x)| \leq \frac{n^2(n^2-1^2)\cdots[n^2-(k-1)^2]}{1\cdot 3\cdots(2k-1)} \max |f(x)|$$

Again the extremal polynomials are multiples of T_n.

The Markoff's inequalities have suggested a number of related problems. Among such problems are the maxima

$$M_n^p = \max \frac{\int_a^b \{f^{(k)}(x)\}^p w(x)\,dx}{\int_a^b \{f(x)\}^p w(x)\,dx} \tag{0.1}$$

over all non-zero polynomials $f(x)$ of degree at most n for various intervals $[a,b]$, weight functions $w(x)$, powers p and orders of derivative k. In this thesis $p = 2$ will be studied in several situations. Three cases will be distinguished and named after the associated orthogonal polynomials. The Legendre case will be that of interval $[-1,1]$ and $w(x) \equiv 1$. The Laguerre case will be that of interval $(0,\infty)$ and weight function e^{-x}. The Hermite case will be that of interval $(-\infty,\infty)$ and weight function e^{-x^2}.

In 1932 Erhard Schmidt [11] announced an asymptotic estimate of the maximum in the Legendre case for the first derivative:

$$\lim_{n\to\infty} \frac{M_n}{n^2} = \frac{1}{\pi}.$$

Later Hille et al [4] studied the Legendre case for the first derivative for general powers p. They sharpen their results for p = 2 by an argument essentially identical to that used by Schmidt. Bellman [1] obtains a bound for $\frac{M_n}{n^2}$ for this case by elementary means. In 1944 Schmidt [12] obtained several terms in the asymptotic development of M_n for the first derivative in the Legendre and Laguerre cases and the exact result in the Hermite case.

The first section of this thesis deals with the second derivative in the Legendre case. An asymptotic estimate of M_n is derived by an extension of Schmidt's techniques with simplifications introduced by the use of the theory of non-negative matrices. As in W. Markoff's result, repetition of the estimate for the first derivative does not give the best possible result.

The second section applies certain techniques for the numerical estimation of eigenvalues of integral equations to these problems. The first derivative in the Laguerre case is treated as an example and corresponding results are stated for the Legendre case. The Laguerre case has the virtue as an example of permitting an exact solution. The exact results for this case are derived and compared with Schmidt's approximate expansion.

A third section briefly treats the Hermite case for general k.

It would seem possible to extend the method of section 1 to deal with specific cases of higher derivatives yet the method of section 2 seems more attractive. The method of section 2 appears less laborious and, in addition to the maximum being estimated, the extremal polynomial is estimated. Neither method promises to make the general case tractable.

SECTION 1

Let $P_i(x) = \frac{1}{2^i i!} \frac{d^i}{dx^i} (x^2-1)^i$, $i = 1,2,\ldots$ and $P_o(x) = 1$

denote the Legendre polynomials and let $\{\varphi_i(x)\}$ be the corresponding orthonormal system where $\varphi_i(x) = \sqrt{\frac{2i+1}{2}}\ P_i(x)$ for $i = 0,1,2,\ldots.$

Let $f(x) = \sum_{i=1}^{n} a_i \varphi_i(x)$ be an arbitrary polynomial of degree at most n.

If $b_{i,j} = \int_{-1}^{1} \varphi_i'' \varphi_j'' dx$, then, using orthogonality,

$$M_n^2 = \max_{f(x)\not\equiv 0} \frac{\int_{-1}^{1} \{f''(x)\}^2 dx}{\int_{-1}^{1} \{f(x)\}^2 dx} = \max_{\sum_i a_i^2 \neq 0} \frac{\sum_{i,j} a_i a_j b_{i,j}}{\sum_i a_i^2}. \qquad (1.1)$$

Repeated application of integration by parts and the fact that

$P_n^{(k)}(1) = \binom{n+k}{n-k} (2k-1)!!$ [9, p. 252] shows for $i \geq j$

$$b_{i,j} = \sqrt{\frac{2i+1}{2}} \sqrt{\frac{2j+1}{2}}\ \frac{(j-1)(j)(j+1)(j+2)}{48} \{3i(i+1) - j(j+1) + 6\}\{1 + (-1)^{i+j}\}$$

Note that $b_{i,j} \geq 0$ with equality only when $0 \leq i,\ j \leq 1$ or i and j are of opposite parity. The matrix $B = (b_{i,j})$ is symmetric so $Q = \sum_{i,j} a_i a_j b_{i,j}$ is a real quadratic form in $a_o, \ldots, a_n$. The extremal problem (1.1) for the form Q is known to have the solution M_n^2 which is the largest eigenvalue of the matrix B [3, pp. 317-319].

The Legendre polynomials of even (odd) degree consist of even (odd) powers of x. Renumber the P_i so that the ones of even degree come first in ascending order and then those of odd degree in ascending order. For $n \geq 4$ the form of $b_{i,j}$ shows that this operation transforms B into the similar block-diagonal matrix

$$\begin{pmatrix} 0 & & & \\ & E & & \\ & & 0 & \\ & & & U \end{pmatrix} = \begin{pmatrix} b_{0,0} & & & \\ & \begin{matrix} b_{2,2} & b_{2,4} & \cdots \\ b_{4,2} & b_{4,4} & \\ \vdots & & \ddots \end{matrix} & & \\ & & b_{1,1} & \\ & & & \begin{matrix} b_{3,3} & b_{3,5} & \cdots \\ b_{5,3} & b_{5,5} & \\ \vdots & & \ddots \end{matrix} \end{pmatrix},$$

all unlabeled elements are zero. The block matrices E and U are positive matrices. The eigenvalues of B are those of E, those of U and zero as an eigenvalue of multiplicity (at least) two.

The elegant theory of non-negative matrices will be employed now. Some of the main results will be summarized; proofs are found in Varga [13, chap. 2].

Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two mxn matrices. Then $A \geq B$ ($>B$) if $a_{i,j} \geq b_{i,j}$ ($>b_{i,j}$) for all $i = 1,\ldots,m$, $j = 1,\ldots,n$. If 0 is the null matrix and $A \geq 0$ (>0), A is said to be a non-negative (positive) matrix. If $B = (b_{i,j})$ is a complex matrix, then $|B|$ denotes the matrix with entries $|b_{i,j}|$. Let B have eigenvalues λ_i, $i = 1,\ldots,n$. Then $\rho(B) = \max|\lambda_i|$ is the spectral radius of the matrix B.

Let $A > 0$ be an nxn matrix. Then,

1. A has a positive real eigenvalue equal to its spectral radius.

2. To $\rho(A)$ there corresponds a positive eigenvector.

3. $\rho(A)$ increases when any entry of A increases.

4. $\rho(A)$ is a simple eigenvalue of A.

Let $A > 0$ be an nxn matrix and let B be an nxn complex matrix with $A > |B|$. If β is any eigenvalue of B, then $|\beta| < \rho(A)$.

E and U are positive matrices to which this theory may be applied. If n is odd, then E and U are of the same size and $U > E$. Thus $\rho(U) > \rho(E)$ and $\rho(U) = M_n^2$. If n is even, let $\tilde{U}$ be the block-diagonal matrix consisting of a 1x1 zero block and U. The dimensions of E and $\tilde{U}$ are the same and $E > \tilde{U}$, so $\rho(E) > \rho(\tilde{U}) = \rho(U)$ and $\rho(E) = M_n^2$. In terms of polynomials the theory implies $M_n < M_{n+1}$ and if n is even (odd), then the extremal polynomial consists of even (odd) powers of x, excluding x^0, x^1, with positive coefficients.

The case of even $n = 2m$ will be treated. Let

$f(x) = \sum_{i=1}^{m} a_i \varphi_{2i}(x)$ be any polynomial consisting only of even, non-zero, powers of x. The real symmetric matrix E is associated with the real quadratic form $Q = \sum_{i,j=1}^{m} a_i a_j b_{2i,2j}$.

In terms of polynomials $Q(a_1,\dots,a_m) = \int_{-1}^{1} \{f''(x)\}^2 dx \geq 0$.

Equality implies $f''(x) = 0$ hence $a_1 = \dots = a_m = 0$. Thus Q is a positive definite quadratic form.

The real symmetric matrix E is orthogonally similar to a real diagonal matrix D [3, p. 308], i.e., there exists a real orthogonal matrix T such that

$$D = T^{-1}ET \qquad (D = (\lambda_i^{-1}\delta_{i,j}),\ TT^T = I) \qquad (1.2)$$

where $\delta_{i,j}$ is the Kronecker delta and the λ_i^{-1} are the eigenvalues of E. The change of basis carries $\{\varphi_{2i}(x)\}$ into $\{g_i(x)\}$. The new basis is orthonormal and is composed of polynomials in even, non-zero, powers of x. In terms of polynomials, (1.2) is

$$\int_{-1}^{1} g_i g_j \, dx = \delta_{i,j} \quad , \quad \int_{-1}^{1} g_i'' g_j'' \, dx = \lambda_i^{-1} \delta_{i,j} \quad , \quad i,j = 1, \dots, n.$$

1.1 A Differential Equation for $\{g_i(x)\}$.

For $i = 1,\dots,n$ define $h_i(x)$ by $h_i''(x) = g_i(x)$, $h_i'(0) = 0$ and $h_i(1) = 0$. Since $h_i'(x)$ is odd, $h_i'(1) = -h_i'(-1)$ and since $g_i(x)$ is even, $\int_{-1}^{1} g_i dx = 0 = h_i'(1) - h_i'(-1)$. Thus $h_i'(1) = h_i'(-1) = 0$. Because $h_i(x)$ is even, $h_i(1) = h_i(-1) = 0$.

Let $q(x)$ be an arbitrary polynomial of degree at most $n = 2m$ consisting only of even, non-zero powers of x. The basis $\{g_i\}$ gives the representation $q(x) = \sum_{i=1}^{m} c_i g_i(x)$ where $c_i = \int_{-1}^{1} g_i q dx$. The orthogonality of the $\{g_i''\}$ shows $\int_{-1}^{1} g_i'' q'' dx = c_i \lambda_i^{-1}$ and

$$\int_{-1}^{1} \{ g_i'' q'' - \lambda_i^{-1} g_i q \} dx = 0 \qquad i = 1, \dots, m \tag{1.11}$$

Integrating by parts twice and remembering that h_i' and h_i both vanish at ± 1 shows

$$\int_{-1}^{1} g_i q \, dx = \int_{-1}^{1} h_i q'' dx$$

and, using $g_i'' = h_i^{(4)}$, the relation (1.11) can be written in the form

$$\int_{-1}^{1} \{ h_i^{(4)} - \lambda_i^{-1} h_i \} q'' dx = 0 . \tag{1.12}$$

The expression $\{ h_i^{(4)} - \lambda_i^{-1} h_i \}$ is an even polynomial of degree at most n+2 and by (1.12) is orthogonal to any even polynomial q'' of degree at most n-2. The expression is naturally orthogonal to all odd polynomials of degree at most n+1. The orthogonality conditions require

$$h_i^{(4)} - \lambda_i^{-1} h_i = d_i P_{n+2} + k_i P_n . \tag{1.13}$$

Suppose $k_i = 0$ in **(1.13)**. The condition $h_i(1) = 0$ then implies $h_i^{(4)}(1) = d_i$. The polynomial h_i is a solution of

$$h^{(4)} - \lambda_i^{-1} h = h^{(4)}(1)\ P_{n+2} .$$

The homogeneous equation has no polynomial solutions since powers cannot cancel. The inhomogeneous equation has the solution

$$h_i = -h_i^{(4)}(1)\ \lambda_i^{-1} \sum_{\ell=0}^{\infty} \lambda_i^{-\ell}\ P_{n+2}^{(4\ell)} .$$

Remembering the result quoted on p. 4 that $P_{n+2}^{(4\ell)}(1) = 0$ for $n+2 < 4\ell$, $P_{n+2}^{(4\ell)}(1) > 0$ for $n+2 \geq 4\ell$ and $\lambda_i^{-1} > 0$, it is seen that

$$\lambda_i^{-1} \sum_{\ell=0}^{\infty} \lambda_i^{-\ell}\ P_{n+2}^{(4\ell)}(1) > 0 .$$

Thus $h_i(1) = 0$ requires $h_i^{(4)}(1) = 0$ but the polynomial h_i cannot be a solution of the homogeneous equation. This contradiction shows $k_i \neq 0$.

Let $k_i u_i(x) = h_i(x)$. The differential equation (1.13) can be written as follows, when the facts that $h_i(1) = 0$ and $P_{n+2}(1) = P_n(1) = 1$ are used:

$$u^{(4)} - \lambda_i^{-1} u = P_n + [u^{(4)}(1) - 1]\ P_{n+2} .$$

Let $S_i(x) = \sum_{\ell=0}^{\infty} \lambda_i^{\ell}\ P_n^{(4\ell)}(x)$ and $T_i(x) = \sum_{\ell=0}^{\infty} \lambda_i^{\ell}\ P_{n+2}^{(4\ell)}(x)$.

Substitution shows that

$$u_i = -\lambda_i S_i(x) - \lambda_i [u_i^{(4)}(1) - 1] T_i(x) . \qquad (1.14)$$

Since $S_i^{(4)}(x) = \lambda_i^{-1}[S_i(x) - P_n(x)]$ and $T_i^{(4)}(x) = \lambda_i^{-1}[T_i(x) - P_{n+2}(x)]$,

differentiating this relation gives

$$u_i^{(4)}(1) = -[S_i(1) - 1] - [u_i^{(4)}(1) - 1][T_i(1) - 1] . \qquad (1.15)$$

The relation (1.15) can be solved for $u_i^{(4)}(1)$ and substituted into (1.14) to find

$$u_i(x) = -\lambda_i S_i(x) + \lambda_i \frac{S_i(1)}{T_i(1)} T_i(x).$$

The requirement $u_i(1) = 0$ is manifestly satisfied. The requirement $u_i'(1) = 0$ gives the condition $S_i'(1)\ T_i(1) = S_i(1)\ T_i'(1)$ which is a polynomial equation for λ_i. In terms of λ the equation is

$$\sum_{\ell=0}^{\infty} \lambda^{\ell} \sum_{i+j=\ell} [P_{n+2}^{(4i+1)}(1)\, P_n^{(4j)}(1) - P_{n+2}^{(4j)}(1)\, P_n^{(4i+1)}(1)] = 0. \qquad (1.16)$$

1.2 Determination of M_n^2

Let $c(i,j)$ denote the expression in square brackets in (1.16). Divide (1.16) by n and let $\eta = \lambda n^8$. The resulting equation can be written as $f_n(\eta) = 0$, where

$$f_n(\eta) = \sum_{\ell=0}^{\infty} \eta^{\ell} \sum_{i+j=\ell} \frac{c(i,j)}{n^{8\ell+1}} . \qquad (1.21)$$

The Hurwitz theorem [10, pp. 156-158] states that if a sequence of functions $\{f_n(\eta)\}_1^{\infty}$ regular in a domain D converges to a non-constant limit function $f(\eta)$ in D, then for every ζ in D such that $f(\zeta) = 0$ and for every disk $|\eta - \zeta| < \delta$, there is a zero of each $f_n(\eta)$ for $n > N(\delta)$ in the disk.

A limit function $f(\eta)$ will be found whose smallest real zero η_0 is the limit of the smallest real zero, $\frac{n^8}{M_n^2}$, of $f_n(\eta)$ for $n = 5,6,\ldots$ which will furnish the desired asymptotic estimate of M_n^2.

Using the representation of $P_m^{(n)}(1)$, $c(i,j)$ is written explicitly as

$$c(i,j) = \binom{n+2+4i+1}{n+2-4i-1}(8i+1)!!\binom{n+4j}{n-4j}(8j-1)!! -$$

$$-\binom{n+2+4j}{n+2-4j}(8j-1)!!\binom{n+4i+1}{n-4i-1}(8i+1)!! .$$

This can be written as

$$\frac{c(i,j)\,(8i+2)!!\,(8j)!!}{n^{8\ell+2}} = \prod_{k=-4i}^{4i+1}\left(1+\frac{2+k}{n}\right)\prod_{k=-4j+1}^{4j}\left(1+\frac{k}{n}\right) -$$

$$-\prod_{k=-4j+1}^{4j}\left(1+\frac{2+k}{n}\right)\prod_{k=-4i}^{4i+1}\left(1+\frac{k}{n}\right)$$

$$= 1+\frac{1}{n}\left[\sum_{k=-4i}^{4i+1}(2+k)+\sum_{k=-4j+1}^{4j}k\right] -$$

$$-1-\frac{1}{n}\left[\sum_{k=-4j+1}^{4j}(2+k)+\sum_{k=-4i}^{4i+1}k\right]+\frac{1}{n^2}d(i,j)$$

$$= \frac{1}{n}(16i-16j+4)+\frac{1}{n^2}d(i,j) \qquad (1.22)$$

In the expansion there are $2\binom{8\ell+2}{m}$ terms in $\frac{1}{n^{2+m}}$. Each term in $\frac{1}{n^{2+m}}$ is of the form $\pm\frac{1}{n^{2+m}}(2+k)^{m-\alpha}k^{\alpha}$ where $|k| \le 4\ell$ hence is dominated in absolute value by $\frac{1}{n^{2+m}}(2+4\ell)^m$. Thus

$$|d(i,j)| < 2\sum_{m=0}^{8\ell+2}\frac{(2+4\ell)^m}{n^m}\binom{8\ell+2}{m}.$$

For $i+j=\ell$ the term $c(i,j)=0$ when $4\ell+1 \geq n+1$ so $\frac{2+4\ell}{n} < 2$ in the non-zero terms. Using the results $\binom{8\ell+2}{m} \leq \binom{8\ell+2}{4\ell+1}$ and $\sum_{m=0}^{8\ell+2} 2^m < 2^{8\ell+3}$, it is seen that

$$|d(i,j)| < 2^{8\ell+4}\binom{8\ell+2}{4\ell+1}. \qquad (1.23)$$

When (1.22) is multiplied by n and substituted in (1.21), $f_n(\eta)$ is given as

$$f_n(\eta) = f(\eta) + \frac{1}{n}\sum_{\ell=0}^{\infty}\eta^\ell \sum_{i+j=\ell}\frac{d(i,j)}{(8\ell+2)!!(8j)!!}$$

where $f(\eta) = \sum_{\ell=0}^{\infty}\eta^\ell \sum_{i+j=\ell}\frac{(16i-16j+4)}{(8i+2)!!(8j)!!}$. By the bound (1.23) and $(8i+2)!!(8j)!! \geq (4\ell)!!$ for $i+j=\ell$,

$$|f_n(\eta) - f(\eta)| < \frac{1}{n}\left|\sum_{\ell=0}^{\infty}\eta^\ell \frac{\ell\, 2^{8\ell+4}}{(4\ell)!!}\binom{8\ell+2}{4\ell+1}\right|. \qquad (1.24)$$

The series appearing in (1.24) is an entire function hence is bounded on compact subsets of the complex plane. Because of the factor $\frac{1}{n}$, $f_n(\eta)$ converges uniformly to $f(\eta)$ on compact subsets of the plane.

A more convenient expression for $f(\eta)$ will be derived now.

For $\ell \geq 1$, $\sum_{i+j=\ell}\frac{(16i-16j+4)}{(8i+2)!!(8j)!!} = \frac{4}{2^{4\ell+1}}\sum_{i+j=\ell}\frac{(4i+1)-4j}{(4i+1)!(4j)!}$

$$= \frac{2}{2^{4\ell}(4\ell)!}\left[\sum_{i=0}^{\ell}\binom{4\ell}{4i} - \sum_{i=1}^{\ell}\binom{4\ell}{4i-1}\right].$$

The identities, where $i^2 = -1$,

$$\frac{1}{4}\left[(1+i)^{4\ell} + (1-i)^{4\ell} + (1+1)^{4\ell} + (1-1)^{4\ell}\right] = \sum_{k=0}^{\ell} \binom{4\ell}{4k} = 2^{4\ell-2} + (-1)^{\ell} 2^{2\ell-1}$$

$$\frac{1}{4}\left[(1+1)^{4\ell} - (1-1)^{4\ell} + i(1+i)^{4\ell} - i(1-i)^{4\ell}\right] = \sum_{k=1}^{\ell} \binom{4\ell}{4k-1} = 2^{4\ell-2}$$

show that $f(\eta) = 2 + \sum_{\ell=1}^{\infty} \eta^{\ell} \frac{(-1)^{\ell}}{4^{\ell}(4\ell)!}$. Since

$\cos \omega \cosh \omega = \sum_{\ell=0}^{\infty} \frac{\omega^{4\ell} 4^{\ell} (-1)^{\ell}}{(4\ell)!}$, the new variable y defined

by $16y^4 = \eta$ for real $\eta \geq 0$ can be introduced to write f as

$$f(y) = 1 + \cos y \cosh y \quad \text{for } y \geq 0. \tag{1.25}$$

The smallest real zero of $f(\eta)$ has been designated η_0.

Let y_0 be the smallest real zero of $f(y)$. For each n, $M_n^{-2} = \min_i \lambda_i$

so $n^8 M_n^{-2}$ is the smallest real root of $f_n(\eta)$. The Hurwitz theorem asserts that

$$\lim_{n \to \infty} n^8 M_n^{-2} = \eta_0 = 16 y_0^4 .$$

These results are summarized in

Theorem 1. Let $f(x)$ be any polynomial of degree at most n. Let M_n^2 be defined by

$$M_n^2 = \max_{f(x) \not\equiv 0} \frac{\int_{-1}^{1} \{f''(x)\}^2 dx}{\int_{-1}^{1} \{f(x)\}^2 dx} .$$

Then, $\lim_{n \to \infty} \frac{M_n^2}{n^8} = \frac{1}{16 y_0^4}$ where y_0 is the smallest positive zero of

$$1 + \cos y \cosh y = 0.$$

Plummer [8] has determined y_0 to be 1.8751041 correct to 10^{-7}.

Repetition of Schmidt's result for the first derivative gives the bound π^{-4} whereas the best possible value is approximately $(3.7502082)^{-4}$.

Section 2

In this section the first derivative Laguerre case will be handled in two ways. The exact results will be derived by elementary methods. Approximate results will be derived as an illustration of a general technique which relates the matrix problems arising in various cases to integral equations.

2.1 Exact Treatment of First Derivative Laguerre Case.

As in section 1

$$M_n^2 = \max \frac{\int_0^\infty \{f'(x)\}^2 e^{-x} dx}{\int_0^\infty \{f(x)\}^2 e^{-x} dx}, \qquad (2.11)$$

where the maximum is over all non-zero polynomials $f(x)$, is the dominant eigenvalue of the matrix $B = (b_{i,j})$ where $b_{i,j} = \int_0^\infty L_i' L_j' e^{-x} dx$, $0 \leq i,j \leq n$, and L_i is the ith Laguerre polynomial. Integration by parts shows $b_{i,j} = -L_i'(0) L_j(0)$ for $i \leq j$. The relations

$$L_i(x) = \sum_{k=0}^{i} (-1)^k \frac{1}{k!} \binom{i}{k} x^k \quad \text{and} \quad L_{i+1}' = L_i' - L_i \qquad [9, \text{pp. } 296\text{-}297]$$

show $b_{i,j} = i$ for $0 \leq i \leq j \leq n$.

The cases $n = 0,1,2$ are trivially solved directly. The eigenvalues of B for $n \geq 3$ will be obtained as the reciprocals of the eigenvalues of B^{-1}. Define A as the symmetric, tridiagonal matrix for which

$$\left.\begin{aligned} a_{i,i+1} &= -1 \\ a_{i,i} &= 2 \end{aligned}\right\} \quad 1 \leq i \leq n$$

$$a_{n,n} = 1 \quad .$$

Let the second difference operator Δ_j^2 operating on $r = \{b_1, \dots, b_n\}$ be defined by $\Delta_j^2 r = b_{j-1} - 2b_j + b_{j+1}$. Let r_i be the ith row of B and C_j the jth column of A. Then $E = BA = (e_{i,j})$ is defined by $e_{i,j} = (r_i, C_j)$. Since

$$e_{i,j} = (r_i, c_j) = -\Delta_j^2 r_i = \delta_{i,j} \qquad j \neq 1, n$$

$$e_{1,1} = 1 \quad , \quad e_{i,1} = 0 \qquad i \neq 1$$

$$e_{n,n} = 1 \quad , \quad e_{i,n} = 0 \qquad i \neq n$$

it is seen that E = I and $A = B^{-1}$.

Let $x = (x_j)$, $1 \leq j \leq n$, be an eigenvector of B^{-1}. The requirement that x be an eigenvector is equivalent to the satisfaction of the difference equation

$$-x_{j-1} + 2x_j - x_{j+1} = \lambda x_j \qquad j = 2, \dots, n-1$$

with boundary conditions

$$2x_1 - x_2 = \lambda x_1$$

$$-x_{n-1} + x_n = \lambda x_n .$$

This system is conveniently written as

$$(2-\lambda)\, x_j = x_{j-1} + x_{j+1} \qquad j = 2, \dots, n-1 \qquad (2.12)$$

$$(2-\lambda)\, x_1 = x_2 \qquad (2.13)$$

$$(2-\lambda)\, x_n = x_{n-1} + x_n \qquad (2.14)$$

This system has the solution $x_j = \sin \alpha j$ where $2-\lambda = 2 \cos \alpha$ and $\alpha = \frac{2k-1}{2n+1}\pi$, $k = 1, \dots, n$, since (2.12) becomes the identity

$$2 \cos \alpha \sin \alpha j = \sin \alpha (j-1) + \sin \alpha (j+1),$$

(2.13) becomes

$$2 \cos \alpha \sin \alpha = \sin 2\alpha$$

and (2.14) is

$$2 \cos \alpha \sin \alpha n = \sin \alpha (n-1) + \sin \alpha n$$

which is equivalent to

$$2 \sin \frac{\alpha}{2} \cos \frac{2n+1}{2} \alpha = 2 \sin \frac{2k-1}{2(2n+1)} \pi \cos \frac{\pi}{2}(2k-1) = 0 .$$

The smallest eigenvalue of B^{-1} is $M_n^{-2} = 2 - 2\cos\frac{\pi}{2n+1} =$

$= 4\sin^2 \frac{\pi}{2(2n+1)}$. These results are summarized in

Theorem 2. Let $f(x)$ be any polynomial of degree at most n. Then, for $n \geq 3$

$$\int_0^\infty \{f'(x)\}^2 e^{-x} dx \leq \frac{1}{4} \sin^{-2} \frac{\pi}{2(2n+1)} \int_0^\infty \{f(x)\}^2 e^{-x} dx$$

where equality is obtained for multiples of $\sum_{j=1}^{n} L_j(x) \sin \frac{\pi j}{2n+1}$.

Schmidt [12, p. 167] obtained the asymptotic result that for $n \geq 2$

$$\frac{2n+1}{\pi M_n} = 1 - \frac{\pi^2}{24(2n+1)^2} + \frac{R}{(2n+1)^4} \qquad -\tfrac{1}{3} < R < \tfrac{4}{3}$$

which is seen to be the first terms of the series expansion of the exact result:

$$\frac{2n+1}{\pi M_n} = 1 - \frac{\pi^2}{24(2n+1)^2} + \cdots + \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2(2n+1)}\right)^{2k} + \cdots .$$

2.2 The Use of Integral Equations

Wielandt [14] has studied the approximation of the eigenvalues of the integral equation $\int_0^1 K(x,y)\phi(y)dy = \lambda \phi(x)$ by the eigenvalues of the n-dimensional matrix formed by replacing the integral by an n-point quadrature and the integral equation by a system of n linear equations. The relevant results follow. The integral equation is considered for symmetric kernels K continuous on $[0,1] \times [0,1]$. The quadrature rule R_n approximates $\int_0^1 f(y)dy$ by $\sum_{i=1}^{n} \frac{1}{n} f(\frac{i}{n})$. The matrix $K^R = (\frac{1}{n} K(\frac{i}{n},\frac{j}{n}))$ has eigenvalues λ^R. The λ^R and λ are real and will be ordered so that λ_p^R (λ_p) is the pth positive eigenvalue of $K^R(K)$, counting multiplicities. If a pth positive eigenvalue does not exist, define $\lambda_p^R = 0$. Similarly define $\lambda_{-p}^R(\lambda_{-p})$.

A kernel G is said to allow the rule R_n if G and the matrix G^R derived by using R_n have the same eigenvalues. A set of functions $g_1(x),\ldots,g_m(x)$ is said to admit the quadrature formula R_n if each $g_i(x)$ is square-integrable over $[0,1]$ and

$$\int_0^1 g_i(y) g_j(y) dy = \sum_{k=1}^{n} \frac{1}{n} g_i(\tfrac{k}{n}) g_j(\tfrac{k}{n}), \quad 1 \le i,j \le m.$$

If $g_1(x),\ldots,g_m(x)$ admit R_n and if $c_{ij} = c_{ji}$ are real constants, then

$$G = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}\, g_i(x) g_j(y)$$

is a symmetric kernel allowing R_n. Let $\|K\| = \max |\lambda|$ where λ runs over all the eigenvalues of the kernel or square matrix K. If $K(x,y)$ is a symmetric kernel and $G(x,y)$ is a symmetric kernel allowing R_n such that

$$G(\tfrac{i}{n}, \tfrac{j}{n}) = K(\tfrac{i}{n}, \tfrac{j}{n}), \quad 1 \le i,j \le n,$$

then

$$|\lambda_p - \lambda_p^R| \le \|K - G\|, \quad p = \pm 1, \pm 2, \ldots.$$

A bound for $\|K\|$ is given by

$$\|K\|^2 \leq \int_0^1 \int_0^1 |K(x,y)|^2 dx\, dy .$$

If $|K(x,y)| \leq G(x,y)$, then $\|K\| \leq \|G\|$.

The n piecewise constant functions

$$g_i(x) = \begin{cases} 1 & \frac{i-1}{n} < x \leq \frac{i}{n} \qquad i = 1, \ldots, n \\ 1 & i = 1 \ , \ x = 0 \\ 0 & \text{otherwise} \end{cases}$$

obviously admit R_n. Let $K(x,y) = \min (x,y)$. The kernel

$$G(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} K\left(\frac{i}{n}, \frac{j}{n}\right) g_i(x) g_j(y)$$

allows the rule R_n and $G^R = \frac{1}{n^2} B$ so $\|G\| = \frac{M_n^2}{n^2}$. The equation for the eigenfunctions is

$$\lambda \varphi(x) = \int_0^x y\, \varphi(y)\, dy + x \int_x^1 \varphi(y)\, dy .$$

By induction $\varphi(x)$ is infinitely differentiable and $\varphi''(x) = -\lambda^{-1}\varphi(x)$ with $\varphi(0) = \varphi'(1) = 0$. The solutions are

$$\lambda_k = (k + \tfrac{1}{2})^{-2} \pi^{-2}, \quad \varphi_k = \sin (k + \tfrac{1}{2}) \pi x \ , \quad k = 0, 1, \ldots .$$

Thus $|\lambda_1 - \lambda_1^R| = \left| \frac{M_n^2}{n^2} - \frac{4}{\pi^2} \right| \leq \|K - G\|$.

A bound for $\|K - G\|$ is obtained by

$$\|K - G\|^2 \leq \int_0^1 \int_0^1 |G(x,y) - K(x,y)|^2 dx\, dy <$$

$$< \int_0^1 \frac{x}{n} \int_0^{\frac{1}{n}} \left(\frac{1}{n} - x\right)^2 dx\, dy = \frac{1}{6n^4} .$$

Since $G(x,y) > K(x,y)$, $\|G\| > \|K\|$ and

$$M_n^2 = \frac{4n^2}{\pi^2} + R \qquad \text{where} \qquad 0 < R < \frac{1}{\sqrt{6}} .$$

From the theory of positive matrices and its extension to positive kernels by Jentzsch [5], it is known that λ_1 is simple as is λ_1^R for each n. The kernel K is the uniform limit of the kernels G, so the eigenfunction belonging to λ_1 is the uniform limit of those belonging to λ_1^R [2, p. 151].

The above results are summarized in

Theorem 3. Let f(x) be any polynomial of degree at most n. Let M_n^2 be defined by

$$M_n^2 = \max_{f(x)\not\equiv 0} \frac{\int_0^\infty \{f'(x)\}^2 e^{-x}\,dx}{\int_0^\infty \{f(x)\}^2 e^{-x}\,dx} .$$

Then $M_n^2 = \frac{4n^2}{\pi^2} + R$ where $0 < R < \frac{1}{\sqrt{6}}$. The maximum is attained for multiples of $L_n(x) + \sum_{i=1}^{n-1} a_i L_i(x)$. Given $\epsilon > 0$, there exists $N(\epsilon)$ such that for $n > N(\epsilon)$

$$|a_i - \sin \frac{\pi i}{2n}| < \epsilon \qquad i = 1, \ldots, n-1.$$

Application of this technique to the first derivative Legendre case leads to the kernel $K(x,y) = \frac{1}{2} x^{5/2} y^{1/2}$, $x \leq y$ and

Theorem 4. Let f(x) be any polynomial of degree at most n. Let M_n^2 be defined by

$$M_n^2 = \max_{f(x)\not\equiv 0} \frac{\int_{-1}^1 \{f'(x)\}^2 dx}{\int_{-1}^1 \{f(x)\}^2 dx} .$$

Then $M_n^2 = \frac{n^4}{\pi^2} + R$ where $0 < R < 4n + 7$. The maximum is attained for multiples of $\varphi_n(x) + \sum_{r=1}^{2[\frac{n-1}{2}]} a_{n-2r}\varphi_{n-2r}(x)$. Given $\epsilon > 0$, there exists $N(\epsilon)$ such that for $n > N(\epsilon)$

$$| a_{n-2r} - \sqrt{\frac{n-2r}{n}} \sin \frac{\pi (n-2r)^2}{2n^2} | < \epsilon \qquad r = 1, \ldots, 2\left[\frac{n-1}{2}\right]$$

Here the $\{\varphi_j\}$ are the orthonormalized Legendre polynomials.

Section 3

In this section the Hermite case of (0.1) will be treated for general k.

An arbitrary polynomial f(x) of degree at most n can be expressed in terms of the Hermite polynomials defined by

$$H_0(x) = 1\ , \quad H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m}\left(e^{-x^2}\right), \quad m = 1, 2, \cdots .$$

These polynomials have the properties that

$$H_m^{(k)}(x) = \frac{2^k m!}{(m-k)!} H_{m-k}(x) \qquad m \geq k,$$

$$\int_{-\infty}^{\infty} H_m^2(x)\, e^{-x^2} dx = 2^m m! \sqrt{\pi} \qquad \text{[2, pp. 91-93].}$$

If $f(x) = \sum_{m=0}^{n} a_m H_m(x)$, then

$$\int_{-\infty}^{\infty} \{f(x)\}^2 e^{-x^2} dx = \sum_{m=0}^{n} a_m^2\, 2^m m! \sqrt{\pi},$$

$$\int_{-\infty}^{\infty} \{f^{(k)}(x)\}^2 e^{-x^2} dx = \sum_{m=k}^{n} a_m^2\, 2^m m! \sqrt{\pi} \left[\frac{2^k m!}{(m-k)!}\right].$$

From this formulation it is easily seen that

Theorem 5. Let f(x) be any polynomial of degree at most n. Then,

$$\int_{-\infty}^{\infty} \{f^{(k)}(x)\}^2 e^{-x^2} dx \leq \frac{2^k n!}{(n-k)!} \int_{-\infty}^{\infty} \{f(x)\}^2 e^{-x^2} dx.$$

Equality is obtained for multiples of $H_n(x)$.

REFERENCES

1. R. Bellman, A Note on an Inequality of E. Schmidt, Bull. Amer. Math. Soc. 50 (1944), 734-736.
2. R. Courant and D. Hilbert, Methods of Mathematical Physics, 1st ed., vol. 1, Interscience Publishers, New York, 1953.
3. F. Gantmacher, The Theory of Matrices, vol. 1, Chelsea, New York, 1959.
4. E. Hille, G. Szego and J. D. Tamarkin, On Some Generalizations of a Theorem of A. Markoff, Duke Math. J. 3 (1937), 729-739.
5. R. Jentzsch, Über Integralgleichungen mit positivem Kern, J. fur Math. 141 (1912), 235-244.
6. A. Markoff, Sur une question posée par Mendeleieff, Bull. Aci. Sci. St. Petersburg 62 (1889), 1-24.
7. W. Markoff, Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen, Math. Ann. 77(1916), 213-258.
8. H. Plummer, The Numerical Solution of a Type of Equation, Phil. Mag. 32 (1941), 505-512.
9. G. Sansone, Orthogonal Functions, Interscience Publishers, New York, 1959.
10. G. Sansone and J. Gerretsen, Lectures on the Theory of Functions of a Complex Variable, P. Noordhoff, Groningen, 1960.
11. E. Schmidt, Die asymptotische Bestimmung des Maximums des Integrals über das quadrat der Ableitung eines normierten Polynoms, dessen Grad ins Unendliche wächst, Sitzungsber. d. Preuss. Akad. d. Wiss. (1932), 287.

12. E. Schmidt, Über die nebst ihren Ableitungen orthogonalen Polynomensysteme und das zugehörige Extremum, Math Ann. 119 (1944), 165-204.

13. R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

14. H. Wielandt, Error Bounds for Eigenvalues of Symmetric Integral Equations, Proceedings of Symposia in Applied Mathematics, vol. VI, McGraw-Hill, New York, 1956.