MINIMUM ENERGY CONTROL OF
ELECTRIC PROPULSION VEHICLES

Thesis by

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ABSTRACT

Minimum-energy control problems for various electric propulsion vehicles are formulated and solved using modern control theory and systems engineering techniques. Analytical results are obtained by making several simplifications and approximations in the dynamical equations of each system whose performance index is related to the minimization of the system energy consumption for a required control action. An attempt is made to implement the resulting control laws using the current engineering practice.

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I. INTRODUCTION

In recent years, following the pioneering works of Pontryagin, Bellman and Kalman, an intense amount of research has been carried out by many workers in the area of modern control theory. The optimization techniques of modern control theory along with digital computers provide a new approach for the design of automatic control systems.

Historically, optimal control theory has been applied to the solution of aerospace problems such as low thrust interplanetary guidance,[15] automatic aircraft landing,[41] re-entry control,[42] soft-landing of a vehicle on the surface of the moon with minimum fuel expenditure,[43] and minimum fuel and minimum time attitude control systems.[44] The literature on the application of optimal control theory to non-aerospace systems is somewhat limited. More recent references deal with optimization of natural gas pipeline systems via dynamic programming,[45] time-optimal positioning control of non-linear hydraulic servomechanisms.[46]

Direct-current and alternating-current electric drive systems utilizing feedback control principles form an important class of systems which play a vital role in modern technology. The application areas are numerous and include many complex industrial processes, portable instruments and devices, and electric propulsion vehicles.

This investigation treats the formulation and solution of minimum-energy control problems associated with electric propulsion vehicles within the context of optimal control theory. The essential elements of a typical electric propulsion vehicle control system are illustrated in Figure 1.

One of the most challenging tasks in the formulation of a practical problem in the context of optimal control theory is the selection of an appropriate performance index. Even though an adequate mathematical description of the plant is available, the quality of the results obtained by the application of optimal control theory is dependent upon the selection of a meaningful performance index. Several contributions are made in this study concerning the selection of meaningful performance indices for the optimization problems considered.

In Chapter II the basic optimization problem for a class of electric propulsion vehicles is formulated and solved under certain assumptions. The performance index is selected to be the control energy consumption which can be positive or negative depending on the type of control action desired. The application of Pontryagin's Maximum Principle together with necessary conditions, in general, indicate a dual-mode control law which may consist of bang-bang and singular control actions during the controlling interval. The engineering realization of such a controller is complex and expensive. Further assumptions convert the basic optimization problem to an approximate optimization problem with a linear plant equation and a quadratic performance index. The solution of this approximate optimization problem yields a feedback control law which has an easily realizable structure. From the digital computer simulations, the salient features of the optimal feedback controller for the approximate optimization problem are determined and compared with those of classical feedback controllers under identical conditions. Also, based on these observations, a suboptimal feedback controller is determined whose performance closely matches that of the optimal feedback controller. Finally, it is shown that the energy consumption and the performance characteristics of the actual system under the action of the optimal and suboptimal control laws of the approximate optimization problem are very nearly the same as those of the actual system under the action of the optimal control law of the basic optimization problem.

In Chapter III a stochastic optimization problem, which is originated from the approximate optimization problem of Chapter II, is formulated and solved. It is assumed that the disturbance torque acting on the vehicle can be modeled by a stationary, Gaussian, exponentially correlated-noise process. The salient features of the stochastic optimization problem are determined using the appropriate digital computer simulations. The behaviour of the exact plant is studied under the action of specific control laws using a Monte-Carlo Simulation Technique. Finally, the effect of observational noise on the system control energy consumption is considered.

In Chapter IV two specific optimization problems are studied under certain simplifying assumptions.

In Chapter V the practical implementation of feedback control laws obtained in Chapters II and IV are considered in the light of current engineering practice.

In Chapter VI the principle results of this investigation are summarized and a list of electric drive systems to which these results may be applicable are given.

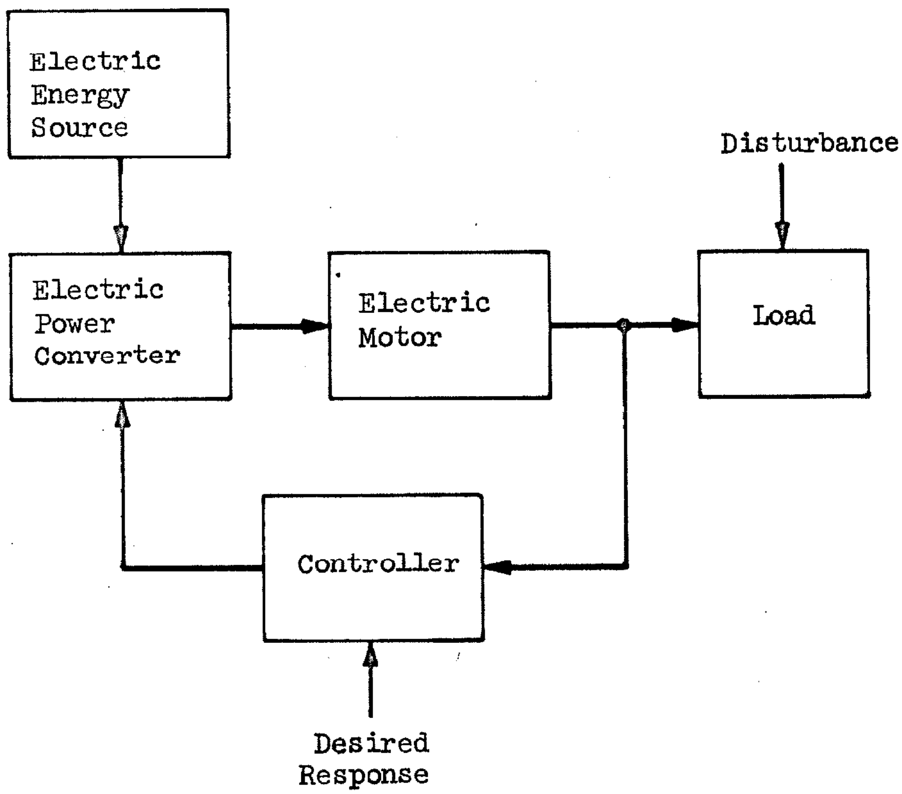


Figure 1. Essential Elements of an Electrical Propulsion Vehicle Control System.

II. THE BASIC OPTIMIZATION PROBLEM

2.1. Introduction.

In this chapter we present the formulation and the solution of the basic optimization problem which will eventually lead us to the determination of the minimum-energy controller for a class of electric propulsion vehicles. The purpose of the controller is to accomplish various control actions for the vehicle while minimizing the net energy flow from a rechargeable battery. The control voltage is applied to the armature circuit of a d-c motor with fixed but reversible field excitation which produces the propulsive force.

A conventional lead-acid battery has a low energy density. It stores, at most, 10 watt-hours of electricity per pound of battery.[1] Therefore, it can not be used as an economical and practical energy source. Fortunately, the current research on new rechargeable battery systems is very encouraging. For example, the energy density of hypothetical metal-air battery is projected to be 5-7 times greater than that of a lead-acid battery.[4] High energy batteries such as sodium-sulphur battery are still in predevelopmental stage. Nuclear energy, solar energy and fuel cells are also possible sources of energy for electric propulsion vehicles. Furthermore, electric energy can be transferred continuously to a moving vehicle by induction, by conduction, or by radiation. However, all of these methods are known to be either uneconomical or impractical.[2,3]

The complete mathematical description of the plant is obtained by applying Lagrange's Energy Method to mechanical and electrical parts of the system, and utilizing basic principles from the theory of electric machines. The performance index is selected to be the electrical energy consumption. The integrand in the performance index represents the electric power which can flow from the battery into the motor circuit or from the motor circuit into the battery over some intervals of time in the controlling process. A number of assumptions are made in the plant equations for mathematical convenience. The control voltage is bounded between zero and the battery voltage.

The exact solution of the optimization problem is obtained by using Pontryagin's Maximum Principle together with necessary conditions from the optimal control theory.[5] For the important case of acceleration it is shown that the singular control[6,7] can not enter into the optimal solution which consists of only the bang-bang solution. The engineering realization of the bang-bang control law or in general dual-mode control law, i.e. optimal solution may consist of bang-bang and singular solutions, is very difficult and expensive. To circumvent this difficulty the following assumptions are made on the basic optimization problem:

1. The control voltage is not bounded
2. The armature inductance is small.

Although the resulting optimization problem is only an approximation of the basic optimization problem; as it will be shown through the use of several digital computer simulations, it yields extremely good results from practical as well as economical points of view. The optimal control law, i.e. feedback solution, is obtained using Bellman's invariant imbedding technique.[10]

2.2. Formulation of the Basic Optimization Problem.

The plant or the process to be controlled is described by the following linear time-invariant differential equations:[13]

$$\dot{x}_1(t) = -\left(\frac{f_e}{J_e}\right)x_1(t) + \left(\frac{k_t}{J_e}\right)x_2(t) - \left(\frac{1}{J_e}\right)v(t) \tag{2-1}$$

$$\dot{x}_2(t) = -\left(\frac{r_a}{\ell_a}\right)x_2(t) + \left(\frac{1}{\ell_a}\right)u(t) - \left(\frac{k_b}{\ell_a}\right)x_1(t) \tag{2-2}$$

where:

$\dot{x}_1(t) \triangleq$ Angular acceleration of the electric motor, [rad/sec^2]

$x_1(t) \triangleq$ Angular velocity of the electric motor, [rad/sec]

$\dot{x}_2(t) \triangleq$ Derivative of armature current, [amps/sec]

$x_2(t) \triangleq$ Armature current, [amps]

$v(t) \triangleq$ Total disturbance torque as referred to motor, [newton-meter]

$u(t) \triangleq$ Armature voltage; control, [volts]

$J_e \triangleq$ Total system inertia as referred to motor, [newton-meter/rad/sec^2]

$f_e \triangleq$ Total system damping coefficient as referred to motor, [newton-meter/rad/sec]

$k_t \triangleq$ Motor torque constant, [newton-meter/amps]

k_t is dependent on the air-gap flux density which is constant for a separately-excited d-c machine.

$k_b \triangleq$ counter emf or back emf constant. [volts/rad/sec]

$\ell_a \triangleq$ Armature inductance, [henry]

$r_a \triangleq$ Armature resistance, [ohms]

In the derivation of Eqs.(2-1) and (2-2) the following assumptions have been made:

1. The electric motor propulsion force is transmitted to vehicle's rear wheels through a speed-reducer and no slipping takes place between the tires and the surface of terrain.

2. The aerodynamic drag force is negligible.

3. The mechanical system parameters; J_e, f_e and the electrical system parameters; ℓ_a, r_a are constants.

4. The motor operation is unsaturated.

The performance index for this optimization problem is selected to be the electric energy consumption of the system and is given by

$$E = \int_0^T u(t)\, x_2(t)\, dt \qquad (2\text{-}3)$$

The integrand in Eq.(2-3) represents the electric power which can flow from the battery into the motor circuit or from the motor circuit into the battery over different intervals of time during the controlling process.

Therefore energy transfer takes place to and from the battery. The nature of electric energy transfer may be best understood by considering the following cases:

1. During speed-control in which it is required to maintain the speed of the vehicle constant while $v(t)$ varies, whenever a positive disturbance torque is applied to the system, i.e. up-hill movement, the control voltage must be increased to keep the vehicle speed constant. Since the control voltage is larger than the back emf voltage, electrical energy is transferred from the battery to the motor circuit.

2. During speed-setting in which it is required to change the speed of the vehicle while $v(t)$ remains constant, whenever the disturbance torque is positive and it is required to increase vehicle speed, the control voltage must be increased in order to obtain the desired speed. Thus, once more the energy is transferred from the battery to the motor circuit.

3. During speed-control of the vehicle, when a negative disturbance torque is applied to the system, i.e. down-hill movement, the control voltage must be reduced in magnitude in order to maintain a constant vehicle speed. Over the interval of time in which back emf voltage is greater than the control voltage plus the voltage drop across the inductance, the energy is transferred from the motor circuit to the battery. The polarity of the armature current is reversed and its magnitude is controlled in such a way that the speed of the vehicle is maintained

constant. In some cases of speed-control, at low speeds with large negative disturbance torques applied to the system, mechanical brakes may be used to supplement the controller effort.

4. During speed-setting of the vehicle under positive disturbance torques, if it is desired to reduce the speed of the vehicle by reducing the control voltage, over the time interval in which the back emf voltage is greater than the control voltage plus the voltage drop in the armature inductance, the energy is transferred from the motor circuit to the battery. This condition exists until the control voltage exceeds the back emf voltage plus the voltage drop in the armature inductance and supplies the necessary motor drive torque corresponding to the new desired speed. The energy is now transferred from the battery to the motor circuit. The above considerations indicate that E, in general, is a measure of net out-flow of energy from the battery in $0 \le t \le T$, i.e. control energy consumption.

The set of boundary conditions to be satisfied by the state variables $x_1(t)$ and $x_2(t)$ for the three cases of control action are given below:

Case 1. Speed-control.

$$x_1(0) = x_{01} = \alpha_1 \quad , \quad x_1(T) = \alpha_1$$

$$x_2(0) = x_{02} \quad , \quad x_2(T) = \alpha_2$$

$$\alpha_2 \text{ is chosen such that } \dot{x}_1(T) = 0 \qquad (2\text{-}4)$$

$$v(0-) \neq v(0) \quad , \quad v(T) = \beta$$

$$v(t) = \beta \quad , \quad 0 \le t \le T$$

Case 2. Speed-setting.

$$x_1(0) = x_{01} \neq \alpha_1 \quad , \quad x_1(T) = \alpha_1$$

$$x_2(0) = x_{02} \quad , \quad x_2(T) = \alpha_2$$

$$\alpha_2 \text{ is chosen such that } \dot{x}_1(T) = 0 \tag{2-5}$$

$$v(0-) = v(0) \quad , \quad v(T) = \beta$$

$$v(t) = \beta \quad , \quad 0 \le t \le T$$

Case 3. Speed-control and speed-setting.

$$x_1(0) = x_{01} \neq \alpha_1 \quad , \quad x_1(T) = \alpha_1$$

$$x_2(0) = x_{02} \quad , \quad x_2(T) = \alpha_2$$

$$\alpha_2 \text{ is chosen such that } \dot{x}_1(T) = 0 \tag{2-6}$$

$$v(0-) \neq v(0) \quad , \quad v(T) = \beta$$

$$v(t) = \beta \quad , \quad 0 \le t \le T$$

The control voltage is constrained by the inequality

$$0 \le u(t) \le U \ . \tag{2-7}$$

The statement of the control problem is as follows: Given the linear time-invariant system, i.e. Eqs.(2-1) and (2-2), the performance index, i.e. Eq.(2-3), a free terminal time T, an inequality constraint on $u(t)$, i.e. Eq.(2-7), determine $u(t)$ which satisfies the boundary conditions on $x_1(t)$ and $x_2(t)$ as described by Eq.(2-4) for speed-control, Eq.(2-5) for speed-setting, Eq.(2-6) for speed-control and speed-setting and minimizes the performance index. It is important to note that for $t \geq T$, it is required to maintain the vehicle speed constant at its terminal value, i.e. $x_1(T) = \alpha_1$, until a new disturbance or a new desired speed-setting is applied to the system. This is accomplished by properly adjusting the value of the armature voltage at $t = T$.

2.3. Solution of the Basic Optimization Problem.

The Hamiltonian function H is defined by:

$$H = \lambda_1(t)\left\{-\left(\frac{f_e}{J_e}\right)x_1(t) + \left(\frac{k_t}{J_e}\right)x_2(t) - \left(\frac{1}{J_e}\right)\beta\right\} + \lambda_2(t)\left\{-\left(\frac{r_a}{\ell_a}\right)x_2(t) - \left(\frac{k_b}{\ell_a}\right)x_1(t)\right\} + u(t)\left\{\left(\frac{\lambda_2(t)}{\ell_a}\right) + x_2(t)\right\} \tag{2-8}$$

where:

$\lambda_1(t)$ and $\lambda_2(t)$ are Lagrange multipliers.

Application of Pontryagin's Maximum Principle yields:

$$u^*(t) = \begin{cases} U & \text{if } \left\{\left(\frac{\lambda_2(t)}{\ell_a}\right) + x_2(t)\right\} < 0 \\ 0 & \text{if } \left\{\left(\frac{\lambda_2(t)}{\ell_a}\right) + x_2(t)\right\} > 0 \end{cases} \tag{2-9}$$

Equation (2-9) denotes a bang-bang type control action in which the control signal takes on its maximum value U or its minimum value 0, depending on the sign of $\left\{\left(\frac{\lambda_2(t)}{\ell_a}\right) + x_2(t)\right\}$.

When $\left\{\left(\frac{\lambda_2(t)}{\ell_a}\right) + x_2(t)\right\} \equiv 0$ over some finite intervals of time in $0 \le t \le T$, singular controls result. If this is the case, Eq.(2-9) yields no information about the desired optimal control function $u^*(t)$.

2.3.1. Determination of the Bang-Bang Control Law.

Assume that singular controls do not exist over any finite time interval in $0 \le t \le T$. Since $v(t)$ is constant in $(0,T)$, from Eqs.(2-1) and (2-2) the following vector differential equation is obtained:

$$\dot{\underline{y}} = A\underline{y} + \underline{v} \tag{2-10}$$

where:

$$\underline{y} \triangleq \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} (x_1(t) - \alpha) \\ \dot{x}_1(t) \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} 0 \\ v_b(t) \end{bmatrix}$$

$$A \triangleq \begin{bmatrix} 0 & 1 \\ -(\mu_1\mu_2) & (\mu_1 + \mu_2) \end{bmatrix}$$

μ_1 and μ_2 are the eigenvalues of A and are defined as:

$$\mu_1 = -\frac{1}{2}\left(\left(\frac{f_e}{J_e}\right) + \left(\frac{r_a}{\ell_a}\right)\right) + \sqrt{\frac{1}{4}\left(\left(\frac{f_e}{J_e}\right) + \left(\frac{r_a}{\ell_a}\right)\right)^2 - \left(\left(\frac{f_e r_a}{J_e \ell_a}\right) + \left(\frac{k_t k_b}{J_e \ell_a}\right)\right)} \tag{2-11}$$

$$\mu_2 = -\frac{1}{2}\left(\left(\frac{f_e}{J_e}\right) + \left(\frac{r_a}{\ell_a}\right)\right) - \sqrt{\frac{1}{4}\left(\left(\frac{f_e}{J_e}\right) + \left(\frac{r_a}{\ell_a}\right)\right)^2 - \left(\left(\frac{f_e r_a}{J_e \ell_a}\right) + \left(\frac{k_t k_b}{J_e \ell_a}\right)\right)} \; . \tag{2-12}$$

In Eqs.(2-11) and (2-12):

$$\frac{1}{4}\left(\left(\frac{f_e}{J_e}\right) + \left(\frac{r_a}{\ell_a}\right)\right)^2 \gg \left(\left(\frac{f_e r_a}{J_e \ell_a}\right) + \left(\frac{k_t k_b}{J_e \ell_a}\right)\right) \tag{2-13}$$

This is, because $\left(\frac{r_a}{\ell_a}\right) \gg 1$ and $\left(\frac{f_e}{J_e}\right)$ is negligible. Equation (2-13) expresses the fact that the system is overdamped which is precisely one of the advantages of using armature controlled d-c motor in electric drive-systems. From Eqs.(2-11), (2-12) and (2-13) it is clear that the eigenvalues μ_1 and μ_2 are real, distinct and negative,

$$v_b(t) \triangleq \left(\frac{k_t u(t) - r_a\beta - (f_e r_a + k_t k_b)\alpha}{J_e \ell_a}\right) \; .$$

Making the following similarity transformation:[8]

$$\underline{y} = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \underline{z} \tag{2-14}$$

where:

$$\underline{z} \triangleq \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

And letting:

$$\underline{z} = \begin{bmatrix} -\left(\frac{1}{\mu_1(\mu_2-\mu_1)}\right) & 0 \\ 0 & \left(\frac{1}{\mu_2(\mu_2-\mu_1)}\right) \end{bmatrix} \underline{\xi}$$

where:

$$\underline{\xi} \triangleq \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}$$

The Eq.(2-10) is transformed into:

$$\dot{\underline{\xi}} = B\underline{\xi} + C\underline{v} \qquad (2\text{-}15)$$

where:

$$B \triangleq \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

$$C \triangleq \begin{bmatrix} -(\mu_1\mu_2) & \mu_1 \\ -(\mu_1\mu_2) & \mu_2 \end{bmatrix}$$

The equation for the switching curve, i.e. closed-loop solution, is determined in the (ξ_1, ξ_2) plane rather than in the (y_1, y_2) plane for the reasons given in reference [8]. From Eq.(2-15):

$$\xi_1(t) = \xi_1(0)\, e^{\mu_1 t} - v_b(t)(1 - e^{\mu_1 t}) \qquad (2\text{-}16)$$

$$\xi_2(t) = \xi_2(0)\, e^{\mu_1 t} - v_b(t)(1 - e^{\mu_2 t}) \qquad (2\text{-}17)$$

Eliminating t in Eqs.(2-16) and (2-17) yields the following equations for the switching curves:

$$S_U = \{(\xi_1, \xi_2) : \xi_2 = a_1 \; ; \; \xi_1 > 0 \, , \; \xi_2 > 0\} \qquad (2\text{-}18)$$

$$S_O = \{(\xi_1, \xi_2) : \xi_2 = a_2 \; ; \; \xi_1 < 0 \, , \; \xi_2 < 0\} \qquad (2\text{-}19)$$

<u>where</u>:

$$a_1 \triangleq - v_U + v_U \left(\frac{\xi_1 + v_U}{v_U} \right)^{\left(\frac{\mu_2}{\mu_1}\right)}$$

$$a_2 \triangleq - v_O + v_O \left(\frac{\xi_1 + v_O}{v_O} \right)^{\left(\frac{\mu_2}{\mu_1}\right)}$$

$$v_U = \left(\frac{k_t U - r_a \beta - (f_e r_a + k_t k_b)\, \alpha}{J_e \ell_a} \right)$$

$$v_0 = \left(\frac{- r_a\beta - (f_e r_a + k_t k_b)\,\alpha}{J_e \ell_a} \right)$$

The corresponding equations for switching regions are given by:

$$R_{UO} = \left\{ \begin{matrix} (\xi_1, \xi_2) \;:\; \xi_2 < a_1 \\ \\ (\xi_1, \xi_2) \;:\; \xi_2 < a_2 \end{matrix} \right\} \tag{2-20}$$

where:

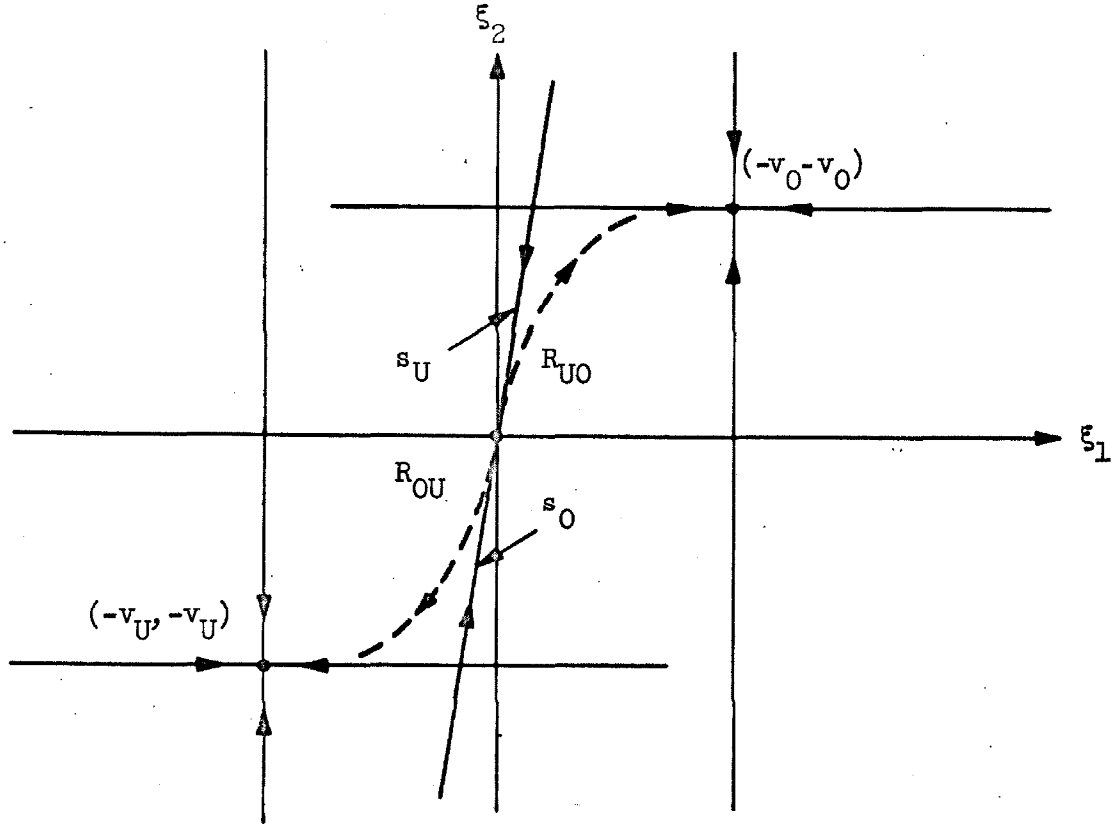
For every state (ξ_1, ξ_2) in the region R_{UO}, the optimal switching sequence is $(U, 0)$, i.e. $u(t) = U$ until the switching boundary given by Eq.(2-19) is reached, then $u(t) = 0$ on this boundary until the origin is reached. Note that $\xi_1 = 0$, $\xi_2 = 0$ in (ξ_1, ξ_2) plane corresponds to $y_1 = 0$, $y_2 = 0$ in (y_1, y_2) plane.

$$R_{OU} = \left\{ \begin{matrix} (\xi_1, \xi_2) \;:\; \xi_2 > a_1 \\ \\ (\xi_1, \xi_2) \;:\; \xi_2 > a_2 \end{matrix} \right\} \tag{2-21}$$

where:

For every state (ξ_1, ξ_2) in the region R_{OU}, the optimal switching sequence is $(0, U)$. Switching curves and regions are shown in Figure 2. Figure 3 displays the configuration of the bang-bang control system.

The following expression for the control energy consumed in $(0, T)$ is obtained by using the appropriate transformations between $x_1(t)$,



R_{UO} = to the right of switching curves s_U and s_O.

R_{OU} = to the left of switching curves s_U and s_O.

Figure 2. Switching Curves and Regions in the (ξ_1, ξ_2) Plane for the Bang-Bang Control.

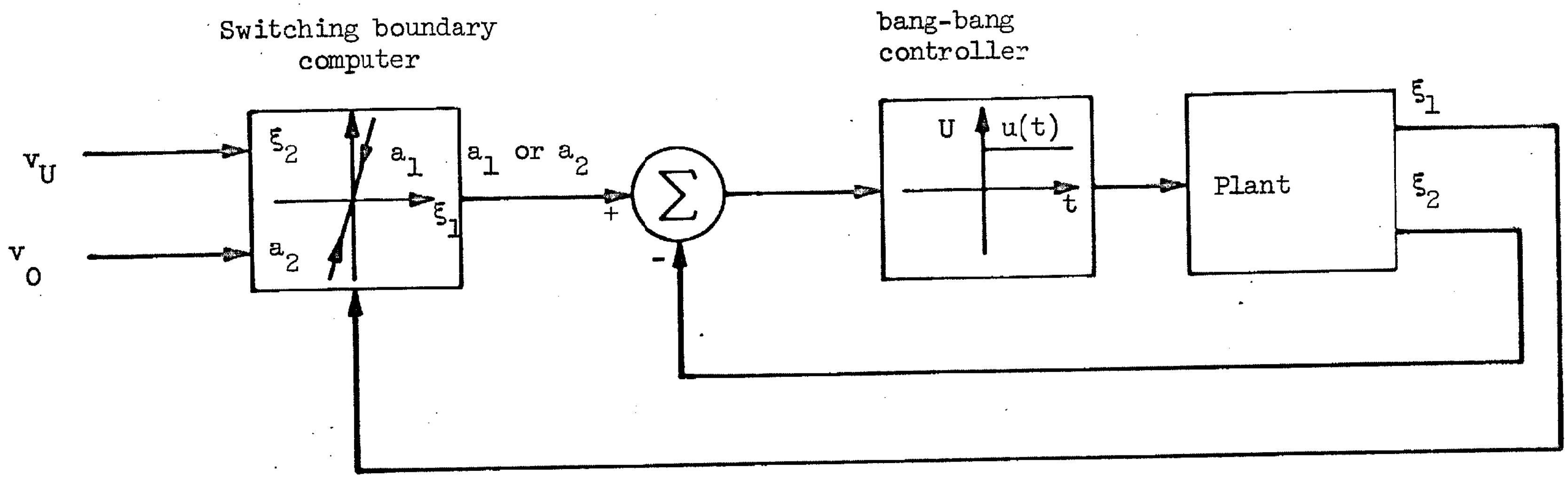


Figure 3. Bang-Bang Control System Block Diagram.

$x_2(t)$ and $\xi_1(t)$, $\xi_2(t)$ variables and by assuming the initial conditions $\xi_1(0)$, $\xi_2(0)$ are in R_{U0}:

$$E = U\left\{\left(\frac{-\left(\frac{f_e}{k_t}\right)-\left(\frac{J_e}{k_t}\right)\mu_1}{(\mu_1(\mu_2-\mu_1))}\right)\left(\left(\frac{\xi_1(0)+v_U}{\mu_1}\right)\left(e^{\mu_1 t_s}-1\right)-v_U t_s\right)\right.$$

$$\left.+\left(\frac{\left(\frac{f_e}{k_t}\right)+\left(\frac{J_e}{k_t}\right)\mu_2}{(\mu_2(\mu_2-\mu_1))}\right)\left(\left(\frac{\xi_2(0)+v_U}{\mu_2}\right)\left(e^{\mu_2 t_s}-1\right)-v_U t_s\right)+\left(\left(\frac{f_e}{k_t}\right)\alpha+\left(\frac{\beta}{k_t}\right)\right)t_s\right\} \tag{2-22}$$

where:

t_s is the time at which the controller switches the battery voltage from $u(t) = U$ to $u(t) = 0$. Equation (2-22) can be evaluated if t_s is known, since all the other terms in (2-22) are known for any control action under consideration. From Eqs.(2-16) and (2-17):

$$t_s = \left(\frac{1}{\mu_1}\right)\log_e\left(\frac{(\xi_1(t_s)+v_U)}{(\xi_1(0)+v_U)}\right) = \left(\frac{1}{\mu_2}\right)\log_e\left(\frac{(\xi_2(t_s)+v_U)}{(\xi_2(0)+v_U)}\right) \tag{2-23}$$

At $t = t_s$ from Eqs.(2-18) and (2-19):

$$(v_0-v_U) + (\xi_2(0)+v_U)\left(\frac{\xi_1(t_s)+v_U}{\xi_1(0)+v_U}\right)^{\left(\frac{\mu_2}{\mu_1}\right)} = v_0\left(\frac{\xi_1(t_s)+v_0}{v_0}\right)^{\left(\frac{\mu_2}{\mu_1}\right)} \tag{2-24}$$

Equation (2-24) is a transcendental equation. For any given $\xi_1(0)$, $\xi_2(0)$; $\xi_1(t_s)$ is determined by trial and error. Substituting this value of $\xi_1(t_s)$ in Eq.(2-23) yields t_s. Figure 4 shows the system trajectories in the (y_1, y_2) plane for the bang-bang control using the following system parameters:

$J_e = 1.42$ newton-meter/rad/sec^2

$f_e = 0.825$ newton-meter/rad/sec

$k_t = 2.0$ newton-meter/amps

$k_b = 2.0$ volts/rad/sec

$r_a = 1.0$ ohm

$\ell_a = 0.010$ henry .

And we have assumed that:

$\alpha = 10.0$ rad/sec

$\beta = 0.0$ newton-meter

$U = 50.0$ volts .

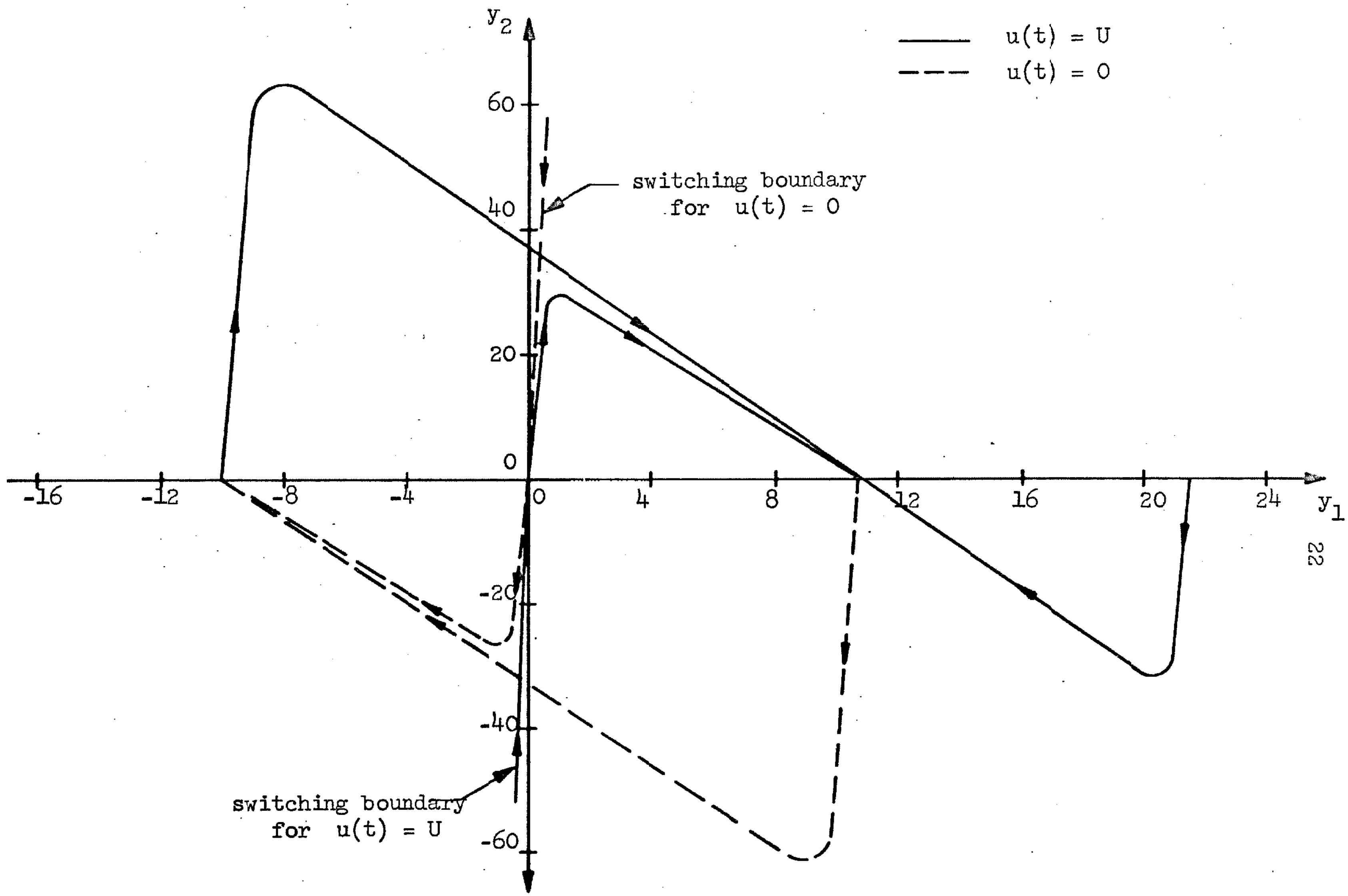


Figure 4. The System Trajectories for the Bang-Bang Control.

2.3.2. Determination of the Singular Control Law.

Let us assume that only the singular controls exist in some time interval. From Eq.(2-8):

$$\dot{x}_1(t) = -\left(\frac{f_e}{J_e}\right) x_1(t) + \left(\frac{k_t}{J_e}\right) x_2(t) - \left(\frac{1}{J_e}\right) \beta \qquad (2\text{-}25)$$

$$\dot{x}_2(t) = -\left(\frac{k_b}{\ell_a}\right) x_1(t) - \left(\frac{r_a}{\ell_a}\right) x_2(t) + \left(\frac{1}{\ell_a}\right) u(t) \qquad (2\text{-}26)$$

$$\dot{\lambda}_1(t) = \left(\frac{f_e}{J_e}\right) \lambda_1(t) + \left(\frac{k_b}{\ell_a}\right) \lambda_2(t) \qquad (2\text{-}27)$$

$$\dot{\lambda}_2(t) = -\left(\frac{k_t}{J_e}\right) \lambda_1(t) + \left(\frac{r_a}{\ell_a}\right) \lambda_2(t) - u(t) \qquad (2\text{-}28)$$

Singular controls exist whenever:

$$\lambda_2(t) = -\ell_a x_2(t) \qquad (2\text{-}29)$$

$$\dot{\lambda}_2(t) = -\ell_a \dot{x}_2(t) \quad . \qquad (2\text{-}30)$$

Furthermore the application of transversality condition yields:

$$H^* \equiv 0 \qquad (2\text{-}31)$$

Using Eqs.(2-29), (2-30) and (2-31) in Eqs.(2-25) through (2-28) yields the following result:

$$u(t) = \left(\frac{\ell_a k_b f_e}{r_a J_e} + k_b\right) x_1(t) + \left(r_a + \ell_a\left(\frac{f_e}{J_e}\right)\right) x_2(t) + \left(\frac{k_b \ell_a}{2.0\; r_a J_e}\right)\beta \tag{2-32}$$

where:

$$x_2(t) = \left\{\left(\frac{f_e}{k_t}\right)x_1(t) + \left(\frac{1}{k_t}\right)\beta\right\} \pm \left\{\left(\left(\frac{f_e}{k_t}\right)x_1(t) + \left(\frac{1}{k_t}\right)\beta\right)^2 + \left(\left(\frac{k_b}{k_t r_a}\right)\beta x_1(t) + \left(\frac{f_e k_b}{r_a k_t}\right) x_1{}^2(t)\right)\right\}^{\frac{1}{2}} .$$

Note that, in the expression for $x_2(t)$:

+ sign denotes motor action

\- sign denotes generator action

and

$$\left\{\left(\left(\frac{f_e}{k_t}\right) x_1(t) + \left(\frac{1}{k_t}\right)\beta\right)^2 + \left(\left(\frac{k_b}{k_t r_a}\right) \beta x_1(t) + \left(\frac{f_e k_b}{r_a k_t}\right)x_1{}^2(t)\right)\right\} > 0 \quad \text{for}$$

any combination between $x_1(t)$ and β.

2.4. An Approximation of the Basic Optimization Problem.

Re-write Eq.(2-2) as

$$\epsilon\, \dot{x}_2(t) + x_2(t) = W(t) \tag{2-33}$$

where:

$$\epsilon \triangleq \left(\frac{\ell_a}{r_a}\right)$$

$$W(t) \triangleq \left(\frac{u(t) - k_b x_1(t)}{r_a}\right)$$

$W(t)$ is the forcing term with $(N+1)$ continuous derivatives in $0 \le t \le T$.

The general solution of Eq.(2-33) is given by:

$$x_2(t) = x_2(0)\, e^{-\left(\frac{1}{\epsilon}\right)t} + \frac{1}{\epsilon}\int_0^t e^{-\left(\frac{1}{\epsilon}\right)(t-\tau)}\, W(\tau)d\tau \qquad (2\text{-}34)$$

After $(N+1)$ times integrating by parts, Eq.(2-34) yields the following result:

$$x_2(t) = \sum_{n=0}^{N} (-1)^n \epsilon^n W^n(t) + R_N(t,\epsilon) \qquad (2\text{-}35)$$ [9]

where:

$$\epsilon^n \triangleq \text{nth power of } \epsilon$$

$$W^n \triangleq \frac{d^n W}{dt^n}$$

and for fixed t, $t > 0$ as $\epsilon \to 0$

$$R_N(t,\epsilon) = O(\epsilon^{(N+1)}) \quad .$$

For most practical systems of interest:

$$O(\epsilon^2) \approx 0$$

Therefore:

$$x_2(t) \cong \left(\frac{u(t)-k_b\dot{x}_1(t)}{r_a}\right) - \epsilon\left(\frac{\dot{u}(t)-k_b\ddot{x}_1(t)}{r_a}\right) \quad . \tag{2-36}$$

For small electric motors, i.e. with ratings below 20hp, Eq.(2-36) can be assumed to be:

$$x_2(t) \cong \left(\frac{u(t) - k_b x_1(t)}{r_a}\right) \tag{2-37}$$

Substituting Eq.(2-37), i.e. zeroth order approximation of $x_2(t)$, into Eqs.(2-1) and (2-3) yields the following simple optimization problem:

Plant:

$$\dot{x}(t) = -\left(\frac{f_e}{J_e} + \frac{k_t k_b}{J_e r_a}\right) x(t) - \left(\frac{1}{J_e}\right) v(t) + \left(\frac{k_t}{J_e r_a}\right) u(t) \tag{2-38}$$

where:

$$x(t) \triangleq x_1(t)$$

Performance index:

$$E = \int_0^T \frac{1}{r_a}\left(u^2(t) - k_b u(t)\, x(t)\right) dt \tag{2-39}$$

Boundary conditions:

Case 1. Speed-control.

$$x(0) = \alpha \quad , \quad x(T) = \alpha$$
$$v(0-) \neq v(0) \quad , \quad v(T) = \beta \qquad (2\text{-}40)$$
$$v(t) = \beta \quad , \quad 0 \leq t \leq T$$

Case 2. Speed-setting.

$$x(0) = x_0 \quad , \quad x(T) = \alpha$$
$$v(0-) = v(0) \quad , \quad v(T) = \beta \qquad (2\text{-}41)$$
$$v(t) = \beta \quad , \quad 0 \leq t \leq T$$

Case 3. Speed-control and Speed-setting.

$$x(0) = x_0 \quad , \quad x(T) = \alpha$$
$$v(0-) \neq v(0) \quad , \quad v(T) = \beta \qquad (2\text{-}42)$$
$$v(t) = \beta \quad , \quad 0 \leq t \leq T$$

The control voltage is not constrained in magnitude. From the observation of Eq.(2-39) it is easy to see that the performance index has the same fundamental characteristic as that of Eq.(2-3), i.e. the integrand of each integral represents the electric power flow which is reversible.

The statement of the control problem is as follows: Given the linear time-invariant system, i.e. Eq.(2-38), the performance index, i.e. (2-39), a terminal time T, and no constraints on the magnitude of control $u(t)$, determine the control $u(t)$ which satisfies the boundary conditions on $x(t)$ as described in Eq.(2-40) for speed-control, Eq.(2-41) for speed-setting, Eq.(2-42) for speed-control and speed-setting and minimizes the performance index. It is important to note that for $t \geq T$, it is required to maintain the vehicle speed constant at its terminal value, i.e. $x(T) = \alpha$, until a new disturbance or a new desired speed-setting is applied to the system.

2.4.1. Solution of the Approximate Optimization Problem.

The Hamiltonian function H for this problem is given by:

$$H = \left(\frac{1}{r_a}\right)u^2(t) - \left(\frac{k_b}{r_a}\right) u(t)x(t) + \left\{-\left(\frac{f_e}{J_e} + \frac{k_t k_b}{J_e r_a}\right) x(t) - \left(\frac{1}{J_e}\right) v(t) + \left(\frac{k_t}{J_e r_a}\right) u(t)\right\} \lambda(t) \qquad (2\text{-}43)$$

Application of Pontriagin's Maximum principle yields the following results:

$$u^*(t) = \left(\frac{k_b}{2}\right) x(t) - \left(\frac{k_t}{2J_e}\right) \lambda(t) \quad , \quad 0 \leq t \leq T \qquad (2\text{-}44)$$

$$H^* = -\left(\frac{k_b^{\,2}}{4r_a}\right) x^2(t) - \left(\frac{f_e}{J_e} + \frac{k_t k_b}{2J_e r_a}\right) x(t)\lambda(t) - \left(\frac{k_t^{\,2}}{4J_e^{\,2} r_a}\right) \lambda^2(t) - \left(\frac{1}{J_e}\right)v(t)\lambda(t) \qquad (2\text{-}45)$$

It is seen from Eq.(2-44) that $u^*(t)$ is a linear function of $x(t)$ and $\lambda(t)$. Since the plant equation is linear, the performance index is quadratic, $u^*(t)$ is unique, and therefore is the optimal solution for this approximate optimization problem.

The canonic equations are then determined to be:

$$\dot{x}(t) = -\, ax(t) - b\lambda(t) - \left(\frac{1}{J_e}\right) v(t) \qquad (2\text{-}46)$$

$$\dot{\lambda}(t) = cx(t) + a\lambda(t) \qquad (2\text{-}47)$$

where:

$$a = \left(\frac{f_e}{J_e} + \frac{k_t k_b}{2J_e r_a}\right) \qquad a > 0$$

$$b = \left(\frac{k_t^{\,2}}{2J_e^{\,2} r_a}\right) \qquad b > 0$$

$$c = \left(\frac{k_b^{\,2}}{2r_a}\right) \qquad c > 0$$

For convenience substitute:

$$\tilde{x}(t) = x(t) - \alpha \qquad (2\text{-}48)$$

$$\beta = v(t) \qquad (2\text{-}49)$$

in Eqs.(2-46) and (2-47) to obtain:

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} -a & -b \\ c & a \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} -\left(\frac{1}{J_e}\right)\beta - a\alpha \\ c\,\alpha \end{bmatrix} . \qquad (2\text{-}50)$$

A. Open-Loop Solution.

The general solution of Eq.(2-50) is given by

$$\begin{bmatrix} \tilde{x}(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} \Phi(t) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + \begin{bmatrix} \underline{p}(t) \end{bmatrix} \qquad (2\text{-}51)$$

where:

Φ is the fundamental matrix solution of Eq.(2-50) which satisfies the matrix differential equation

$$\dot{\Phi}(t) = \begin{bmatrix} -a & -b \\ c & d \end{bmatrix} \Phi(t) \quad , \quad \Phi(0) = I \qquad (2\text{-}52)$$

where:

$I \triangleq$ identity matrix.

$\underline{p}(t)$ is the particular vector solution of Eq.(2-50) which satisfies the vector differential equation:

$$\underline{\dot{p}}(t) = \begin{bmatrix} -a & -b \\ c & a \end{bmatrix} \underline{p}(t) + \begin{bmatrix} -\left(\frac{1}{J_e}\right)\beta - a\alpha \\ c\,\alpha \end{bmatrix}, \quad \underline{p}(0) = \underline{0} \qquad (2\text{-}53)$$

where:

$\underline{0} \triangleq$ Null vector.

The solution of Eqs.(2-52) and (2-53) yields:

$$\Phi(t) = \begin{bmatrix} \left(\text{Cosh}\sqrt{d}\ t - \frac{a}{\sqrt{d}}\ \text{Sinh}\sqrt{d}\ t\right) & \left(-\frac{b}{\sqrt{d}}\ \text{Sinh}\sqrt{d}\ t\right) \\ \left(\frac{c}{\sqrt{d}}\ \text{Sinh}\sqrt{d}\ t\right) & \left(\text{Cosh}\sqrt{d}\ t + \frac{a}{\sqrt{d}}\ \text{Sinh}\sqrt{d}\ t\right) \end{bmatrix} \qquad (2\text{-}54)$$

$$\underline{p}(t) = \begin{bmatrix} \left(-\left(\frac{1}{J_e}\right)\beta - a\alpha\right)\left(\left(\frac{1}{\sqrt{d}}\right)\text{Sinh}\sqrt{d}\ t + \left(\frac{a}{d}\right)(1-\text{Cosh}\sqrt{d}\ t)\right) + (c\alpha)\left(\left(\frac{b}{d}\right)(1-\text{Cosh}\sqrt{d}\ t)\right) \\ \left(-\left(\frac{1}{J_e}\right)\beta - a\alpha\right)\left(\left(\frac{c}{d}\right)(\text{Cosh}\sqrt{d}\ t-1)\right) + (c\alpha)\left(\left(\frac{1}{\sqrt{d}}\right)\text{Sinh}\sqrt{d}\ t + \left(\frac{a}{d}\right)(\text{Cosh}\sqrt{d}\ t-1)\right) \end{bmatrix}$$

(2-55)

where:

$$d = a^2 - bc$$

Substituting Eqs.(2-54) and (2-55) into Eq.(2-51) yields the general solution of Eq.(2-50). The constants k_1 and k_2 are determined in such a way that the two boundary conditions on $\tilde{x}(t)$ are satisfied for a control action under consideration. Consider the most general case:

For speed-control and speed-change

Let:

$$\tilde{x}(0) = \gamma \quad (\text{or} \quad x(0) \neq \alpha)$$

$$\tilde{x}(T) = 0 \quad (\text{or} \quad x(T) = \alpha)$$

Hence,

$$k_1 = \gamma$$

$$k_2 = \gamma\left\{\left(\frac{\sqrt{d}}{b}\right)\left(\frac{\operatorname{Cosh}\sqrt{d}\ T}{\operatorname{Sinh}\sqrt{d}\ T}\right)-\left(\frac{a}{b}\right)\right\}-\left(\left(\frac{1}{J_e}\right)\beta + a\alpha\right)\left\{\left(\frac{1}{b}\right)+\left(\frac{a}{b\sqrt{d}}\right)\left(\frac{1-\operatorname{Cosh}\sqrt{d}\ T}{\operatorname{Sinh}\sqrt{d}\ T}\right)\right\} + \left(a\frac{c}{\sqrt{d}}\right)\left(\frac{1-\operatorname{Cosh}\sqrt{d}\ T}{\operatorname{Sinh}\sqrt{d}\ T}\right)$$

Substituting k_1, k_2 into Eq.(2-51) yields:

$$x(t) = \gamma\left\{\operatorname{Cosh}\sqrt{d}\ t - \left(\frac{\operatorname{Cosh}\sqrt{d}\ T}{\operatorname{Sinh}\sqrt{d}\ T}\right)\operatorname{Sinh}\sqrt{d}\ t\right\} + \alpha\left\{\operatorname{Cosh}\sqrt{d}\ t + \left(\frac{1-\operatorname{Cosh}\sqrt{d}\ T}{\operatorname{Sinh}\sqrt{d}\ T}\right)\operatorname{Sinh}\sqrt{d}\ t\right\} - \beta\left\{\left(\frac{a}{J_e d}\right)(1-\operatorname{Cosh}\sqrt{d}\ t) - \left(\frac{a}{J_e d}\right)\left(\frac{1-\operatorname{Cosh}\sqrt{d}\ T}{\operatorname{Sinh}\sqrt{d}\ T}\right)\operatorname{Sinh}\sqrt{d}\ t\right\} \qquad (2\text{-}56)$$

$$\lambda(t) = \gamma\left\{-\left(\frac{\sqrt{d}}{b}\,\text{Sinh}\sqrt{d}\,t\right)+\left(\frac{\sqrt{d}\,\text{Cosh}\sqrt{d}\,T}{b\,\text{Sinh}\sqrt{d}\,T}\right)\text{Cosh}\sqrt{d}\,t-\left(\frac{a}{b}\right)\text{Cosh}\sqrt{d}\,t\right.$$

$$\left.+\left(\frac{a\,\text{Cosh}\sqrt{d}\,T}{b\,\text{Sinh}\sqrt{d}\,T}\right)\text{Sinh}\sqrt{d}\,t\right\}+\alpha\left\{-\left(\frac{\sqrt{d}}{b}\right)\text{Sinh}\sqrt{d}\,t-\left(\frac{a}{b}\right)\text{Cosh}\sqrt{d}\,t\right.$$

$$\left.-\frac{\sqrt{d}}{b}\left(\frac{1-\text{Cosh}\sqrt{d}\,T}{\text{Sinh}\sqrt{d}\,T}\right)\text{Cosh}\sqrt{d}\,t-\frac{a}{b}\left(\frac{1-\text{Cosh}\sqrt{d}\,T}{\text{Sinh}\sqrt{d}\,T}\right)\text{Sinh}\sqrt{d}\,t\right\}$$

$$-\beta\left\{-\left(\frac{a^2}{J_e bd}\right)(1-\text{Cosh}\sqrt{d}\,t)+\left(\frac{1}{bJ_e}\right)+\left(\frac{a}{bJ_e\sqrt{d}}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\,T}{\text{Sinh}\sqrt{d}\,T}\right)\text{Cosh}\sqrt{d}\,t\right.$$

$$\left.+\left(\frac{a}{bJ_e\sqrt{d}}\right)\text{Sinh}\sqrt{d}\,t+\left(\frac{a^2}{J_e bd}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\,T}{\text{Sinh}\sqrt{d}\,T}\right)\text{Sinh}\sqrt{d}\,t\right\} \qquad (2\text{-}57)$$

Using Eqs.(2-56) and (2-57) in Eq.(2-44) yields the following expression for the optimal control function $u^*(t)$:

$$u^*(t) = \gamma\left\{\left(\frac{k_b}{2}+\frac{k_t a}{2J_e b}\right)\text{Cosh}\sqrt{d}\,t-\left(\frac{k_b}{2}+\frac{k_t a}{2J_e b}\right)\left(\frac{\text{Cosh}\sqrt{d}\,T}{\text{Sinh}\sqrt{d}\,T}\right)\text{Sinh}\sqrt{d}\,t\right.$$

$$\left.+\left(\frac{k_t\sqrt{d}}{2J_e b}\right)\text{Sinh}\sqrt{d}\,t-\left(\frac{k_t\sqrt{d}\,\text{Cosh}\sqrt{d}\,T}{2J_e b\,\text{Sinh}\sqrt{d}\,T}\right)\text{Cosh}\sqrt{d}\,t\right\}+\alpha\left\{\left(\frac{k_b}{2}+\frac{k_t a}{2J_e b}\right)\text{Cosh}\sqrt{d}\,t\right.$$

$$+\left(\frac{k_b}{2}+\frac{k_t a}{2J_e b}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\,T}{\text{Sinh}\sqrt{d}\,T}\right)\text{Sinh}\sqrt{d}\,t+\left(\frac{k_t\sqrt{d}}{2J_e b}\right)\text{Sinh}\sqrt{d}\,t+\left(\frac{k_t\sqrt{d}}{2J_e b}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\,T}{\text{Sinh}\sqrt{d}\,T}\right)$$

$$\left.\text{Cosh}\sqrt{d}\,t\right\}-\beta\left\{\left(\frac{k_b a}{2J_e d}+\frac{k_t a^2}{2J_e^2 bd}\right)(1-\text{Cosh}\sqrt{d}\,t)-\left(\frac{k_b a}{2J_e d}+\frac{k_t a^2}{2J_e^2 bd}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\,T}{\text{Sinh}\sqrt{d}\,T}\right)\right.$$

$$\left.\text{Sinh}\sqrt{d}\,t-\frac{k_t a}{2J_e^2 b\sqrt{d}}\,\text{Sinh}\sqrt{d}\,t-\left(\frac{k_t a}{2J_e^2 b\sqrt{d}}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\,T}{\text{Sinh}\sqrt{d}\,T}\right)\text{Cosh}\sqrt{d}\,t-\frac{k_t}{2J_e^2 b}\right\}$$

$$(2\text{-}58)$$

If $\gamma = 0$ in Eqs.(2-56), (2-57) and (2-58) we obtain the corresponding results for speed-control case. If $\gamma = 0$ and $v(0-) = v(0) = v(T)$, $v(t) = \beta$; $0 \leq t \leq T$, in Eqs.(2-56), (2-57) and (2-58) we obtain the corresponding results for speed-setting case.

B. Closed-Loop Solution.

Since the canonic equations are linear, Bellman's invariant-imbedding technique can be used to determine the feedback solution. Let $r(\eta, \tau)$ be any particular initial condition on $\tilde{x}(t)$ for a process starting at time τ with $\lambda(\tau) = \eta(\tau)$ and satisfying the boundary condition $r(\eta, T) = 0$. Referring to reference [10], $r(\eta, \tau)$ satisfies the following first-order differential equation:

$$\frac{\partial r}{\partial \tau}(\eta, \tau) + \frac{\partial r}{\partial \eta}(\eta, \tau)\, g(\eta, r) = f(\eta, r) \qquad (2\text{-}59)$$

where:

$$f(\eta, r) \triangleq -\, ar(\tau) - b\eta(\tau) - \left(\frac{1}{J_e}\right)\beta - a\alpha$$

$$g(\eta, r) \triangleq cr(\tau) + a\eta(\tau) + c\,\alpha$$

Assume a solution of Eq.(2-59) in the form of

$$r(\eta, \tau) = m(\tau)\,\eta(\tau) + n(\tau) \qquad (2\text{-}60)$$

where:

$$\eta(\tau) \triangleq \lambda(\tau)$$

From Eq.(2-60):

$$\frac{\partial r}{\partial \tau}(\eta, \tau) = \dot{m}(\tau)\ \eta(\tau) + \dot{n}(\tau) \qquad (2\text{-}61)$$

$$\frac{\partial r}{\partial \eta}(\eta, \tau) = m(\tau) \quad . \qquad (2\text{-}62)$$

Substituting Eqs.(2-61), (2-62) into Eq.(2-59) and collecting the coefficients of equal powers of $\eta(\tau)$ and equating them to zero yields:

$$\dot{m}(\tau) + cm^2(\tau) + 2am(\tau) + b = 0 \qquad (2\text{-}63)$$

$$m(T) = 0 \qquad (2\text{-}64)$$

$$\dot{n}(\tau) + (cm(\tau)+a)\ n(\tau) + \alpha cm(\tau) + a\alpha + \left(\frac{1}{J_e}\right)\beta = 0 \qquad (2\text{-}65)$$

$$n(T) = 0 \qquad (2\text{-}66)$$

Equation (2-63) with its boundary condition, i.e. Eq.(2-64), is a first-order nonlinear differential equation of the Riccati type whose solution is given by:

$$m(\tau) = -\ b\left(\frac{\operatorname{Sinh}\sqrt{d}\ (T-\tau)}{a\ \operatorname{Sinh}\sqrt{d}\ (T-\tau) - \sqrt{d}\ \operatorname{Cosh}\sqrt{d}\ (T-\tau)}\right) \qquad (2\text{-}67)$$

Equation (2-65) with its boundary condition, i.e. Eq.(2-66), is a first-order time-varying linear differential equation with a constant forcing term whose solution is given by:

$$n(\tau) = \frac{\alpha\sqrt{d}}{(\sqrt{d}\ \text{Cosh}\sqrt{d}\ (T-\tau) - a\ \text{Sinh}\sqrt{d}\ (T-\tau))}\left(\frac{a}{\sqrt{d}}\ \text{Sinh}\sqrt{d}\ (T-\tau) - \text{Cosh}\sqrt{d}\ (T-\tau) + 1\right) + \frac{\beta}{J_e(\sqrt{d}\ \text{Cosh}\sqrt{d}\ (T-\tau) - a\ \text{Sinh}\sqrt{d}\ (T-\tau)}\left(\text{Sinh}\sqrt{d}\ (T-\tau) - \frac{a}{\sqrt{d}}\ \text{Cosh}\sqrt{d}\ (T-\tau) + \frac{a}{\sqrt{d}}\right) \tag{2-68}$$

Since from Eq.(2-60)

$$\lambda(\tau) = \left(\frac{x(\tau) - \alpha - n(\tau)}{m(\tau)}\right) \tag{2-69}$$

Substituting Eqs.(2-67) and (2-68) into Eq.(2-69) and then using the resulting expression for $\lambda(\tau)$ in Eq.(2-44) yields the following feedback control law for this optimization problem:

$$u^*(t) = \left(\frac{k_b}{2} + \frac{k_t a}{2J_e b}\right)x(t) + \left(\frac{1-\text{Cosh}\sqrt{d}\ (T-t)}{\text{Sinh}\sqrt{d}\ (T-t)}\right)\left(\frac{k_t\sqrt{d}}{2J_e b}\right)x(t) + \left(\frac{k_t}{2J_e^2 b}\right)\beta \left(\frac{1-\text{Cosh}\sqrt{d}\ (T-t)}{\text{Sinh}\sqrt{d}\ (T-t)}\right)\left(\frac{k_t a}{2J_e^2 bd}\right)\beta + \left(\frac{\alpha - x(t)}{\text{Sinh}\sqrt{d}\ (T-t)}\right)\left(\frac{k_t\sqrt{d}}{2J_e b}\right) \ . \tag{2-70}$$

where:

τ is replaced by t without loss of generality. In Eq.(2-70) at $t = T$, using L'Hospital's rule yields the following results:

$$\lim_{t\to T}\left(\frac{1-\text{Cosh}\sqrt{d}\ (T-t)}{\text{Sinh}\sqrt{d}\ (T-t)}\right) = 0 \tag{2-71}$$

$$\lim_{t\to T}\left(\frac{\alpha - x(t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right) = \frac{\dot{x}(T)}{\sqrt{d}} \tag{2-72}$$

$$u^*(T) = \left(\frac{k_b}{2} + \frac{k_t a}{2J_e b}\right) x(T) + \left(\frac{k_t}{2J_e{}^2 b}\right)\beta + \left(\frac{k_t}{2J_e b}\right)\dot{x}(T) \tag{2-73}$$

If, however, $\lim_{t\to T}\left(\frac{\alpha - x(t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right) \equiv 0$ for $t \geq T$ until a new disturbance or a new desired speed-setting is applied to the system, then from Eq.(2-73):

$$u^*(t) = \left(\frac{k_b}{2} + \frac{k_t a}{2J_e b}\right) x(T) + \left(\frac{k_t}{2J_e{}^2 b}\right)\beta \quad . \tag{2-74}$$

Substituting Eq.(2-74) into Eq.(2-38) and noting that $v(t) = \beta$ yields:

$$\dot{x}(t) \equiv 0 \quad \text{for} \quad t \geq T \tag{2-75}$$

If this requirement is met, the controller enters the steady-state phase of its operation. Therefore, it is clear that some means must be incorporated into the control system, to turn the time-varying feedback gains on at $t = 0$ and to turn them off at $t = T$.

When Eq.(2-36), i.e. first order approximation of $x_2(t)$, is substituted into Eqs.(2-1) and (2-3), a better approximation of the basic optimization problem is obtained. In this case, the solution is most conveniently obtained by using the so-called Lagrangian Formulation.[11] However, the performance characteristics of the resulting optimal system are almost identical to those of the optimal system studied above, even though an unrealistically high value of ϵ is assumed in the ensuing computations. For this reason, the performance characteristics obtained from the appropriate digital computer simulation for this optimal system will not be reproduced here. However, it must be mentioned that the optimal system, in this case, requires the derivative of the control $u^*(t)$ to be fed into the plant.

2.4.2. Salient Features of the Minimum-Energy Controller for the Approximate Optimization Problem.

Salient features of the minimum-energy controller are studied by simulating Eqs.(2-56), (2-58) and (2-39) on the digital computer by using the same system parameters as given in Section 2.3.1, except that $\ell_a \equiv 0$. For $t \geq T$ it has been assumed that all time-varying gains of the minimum-energy controller are made zero by means of an auxiliary controller.

2.4.3. Discussion on the Salient Features of the Approximate Optimization Problem.

In reference to Figures 5 through 8, it is seen that the optimal control $u^*(t)$, hence the optimal trajectory $x(t)$ are greatly influenced by the particular choice of T. For the case of speed-control, $\dot{x}(t) = 0$ at $t = \frac{T}{2}$ for all α's, β's, and T's. The proof follows from the use of Rolle's theorem[12] which guarantees that $\dot{x}(t) = 0$ at time t which satisfies the inequality $0 < t < T$. Using Eq.(2-56) with $\gamma = 0$, i.e. speed-control, yields:

$$\dot{x}(t) = \left(\frac{\alpha\sqrt{d} + \dfrac{a}{J_e\sqrt{d}}\beta}{\operatorname{Sinh}\sqrt{d}\,T} \right) \left(\operatorname{Cosh}\sqrt{d}\,t - \operatorname{Cosh}\sqrt{d}\,(T-t) \right) \qquad (2\text{-}76)$$

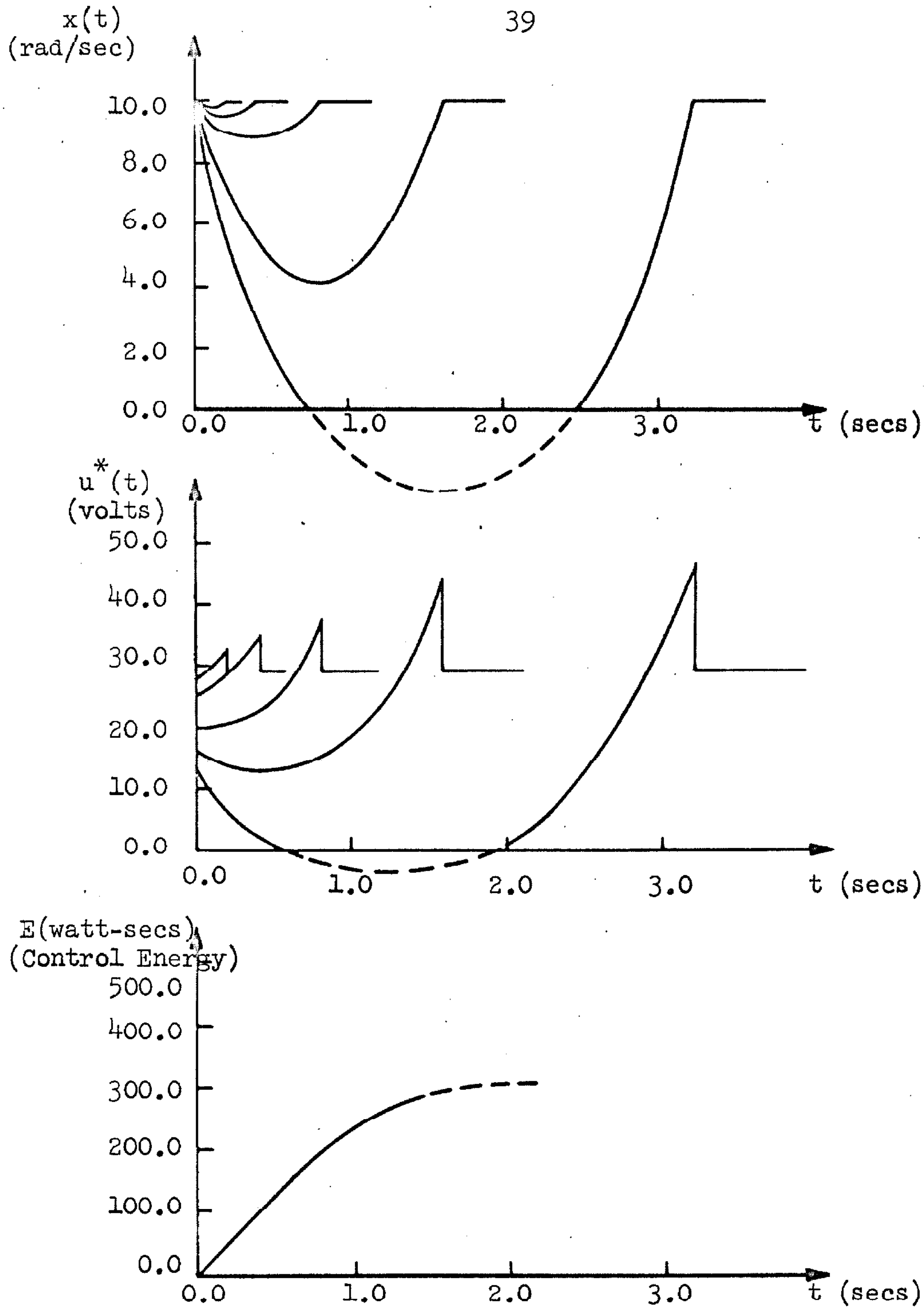


Figure 5. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = 10.0 newton-meter , γ = 0.0 rad/sec (speed-control) , $u^*(0-)$ = 24.2 volts.

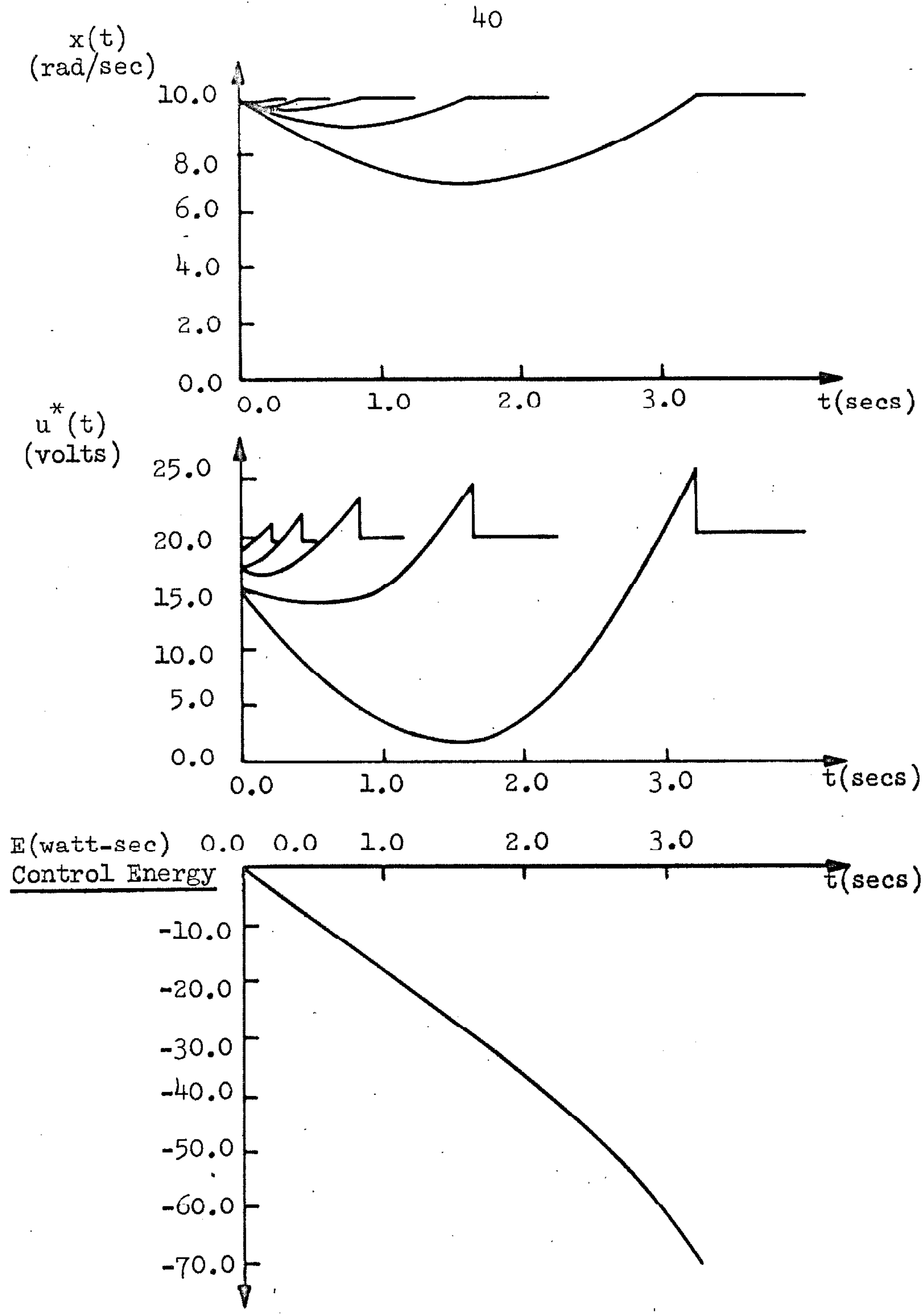


Figure 6. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = - 10.0 newton-meter, γ = 0.0 rad/sec (speed-control) , $u^*(0-)$ = 24.2 volts.

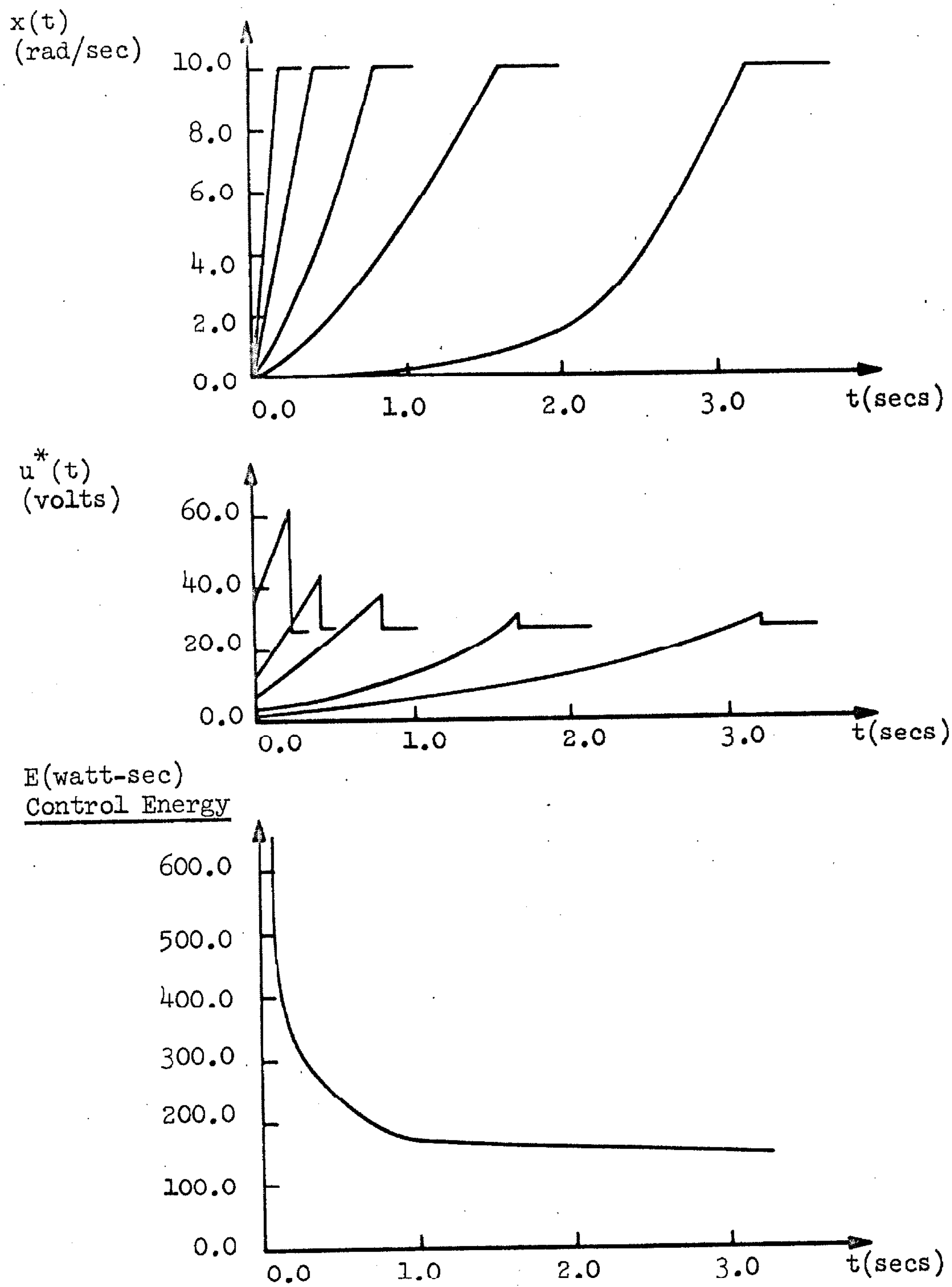


Figure 7. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = 0.0 newton-meter , γ = - 10.0 rad/sec (acceleration) , $u^*(0-)$ = 0.0 volt.

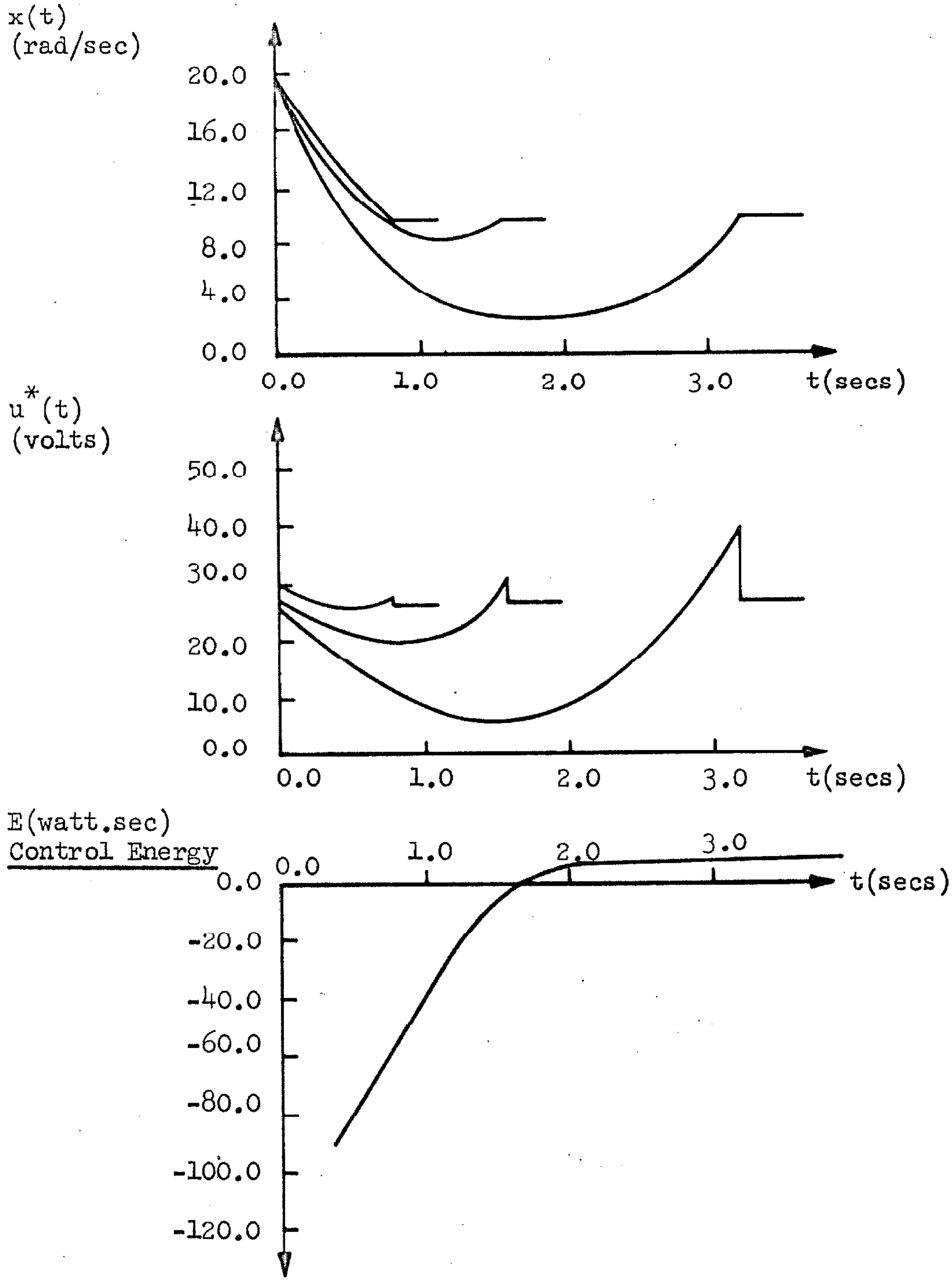


Figure 8. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = 0.0 newton-meter , γ = 10.0 rad/sec (deceleration) , $u^*(0-)$ = 48.4 volts.

From Eq.(2-76) it is clear that $\dot{x}(t) = 0$ in $0 < t < T$ if and only if

$$\mathrm{Cosh}\sqrt{d}\,t - \mathrm{Cosh}\sqrt{d}\,(T-t) = 0 \quad . \tag{2-77}$$

Equation (2-77) is satisfied at $t = \frac{T}{2}$. Also note that at $t = 0+$ whenever $\left(\alpha\sqrt{d} + \frac{a}{J_e\sqrt{d}}\beta\right) > 0$ then $\dot{x}(t)\Big|_{t+0+} < 0$ since

$\left(\mathrm{Cosh}\sqrt{d}\,t - \mathrm{Cosh}\sqrt{d}\,(T-t)\right)\Big|_{t=0+} < 0$ for $T > 0$. If, however,

$\left(\alpha\sqrt{d} + \frac{a}{J_e\sqrt{d}}\beta\right) < 0$, then $\dot{x}(t)\Big|_{t=0+} > 0$ since

$\left(\mathrm{Cosh}\sqrt{d}\,t - \mathrm{Cosh}\sqrt{d}\,(T-t)\right)\Big|_{t=0+} < 0$ for $T > 0$. From Figures 5 and 6 it is seen that, if the speed is not allowed to change considerably by properly choosing T, E vs T relationship becomes linear with almost unity slope. For the case of acceleration, i.e. Figure 7, $u^*(t)$ and $x(t)$ change linearly with time in $0 \le t \le T$ for small T's. Small T means a high acceleration requirement which in turn demands a high control voltage $u(t)$ and from Eq.(2-39) it is clear that the control energy consumption becomes very large. This is because the integrand in Eq.(2-39) is dominated by $u^2(t)$ and the integral of this quantity over $(0,T)$ is extremely large even though T is small. The control energy consumption E becomes independent of T for large T's. This is because, energy saved by keeping the control voltage at a low value for most portion of $(0,T)$, i.e. low acceleration, is diminished by requiring higher acceleration as $t \to T$ for large T's. Therefore, the net gain in control energy consumption for T greater than a certain value is extremely negligible. For the case of deceleration, i.e. Figure 8, considerable amount of energy can be saved for small T's. However for large T's, the energy which can be gained by keeping T small is lost by requiring very high accelerations as $t \to T$ for large T's.

2.4.4. A Suboptimal Feedback Controller for the Approximate Optimization Problem.

Consider Eq.(2-70) which consists of three time varying and two constant gains. Two of the time varying gains are identical and satisfy the following inequality:

$$-1 \le \left(\frac{1-\mathrm{Cosh}\sqrt{d}\,(T-t)}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right) \le 0 \quad \text{in} \quad 0 \le t \le T \quad . \tag{2-78}$$

The third time varying gain becomes infinite* at $t = T$. Therefore; it seems reasonable to assume a suboptimal control law of the following form:

$$u_{so}(t) = \left(\frac{k_b}{2} + \frac{k_t a}{2J_e b}\right) x(t) + \left(\frac{k_t}{2J_e{}^2 b}\right) \beta + \left(\frac{k_t\sqrt{d}}{2J_e b}\right)\left(\frac{1}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right)(\alpha - x(t)) \tag{2-79}$$

Substituting Eq.(2-79) into Eq.(2-38) yields:

$$\dot{x}(t) = \left(\frac{\sqrt{d}}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right)(\alpha - x(t)) \tag{2-80}$$

---

* In practice, it is not possible to have an infinite gain, therefore, more correctly, we should say that the third time varying gain becomes very large at $t = T$.

The solution of Eq.(2-80) is given by:

$$x(t) = x(0)\left(\frac{\text{Cosh}\sqrt{d}\ T+1}{\text{Cosh}\sqrt{d}\ T-1}\right)^{\frac{1}{2}}\left(\frac{\text{Cosh}\sqrt{d}\ (T-t)-1}{\text{Cosh}\sqrt{d}\ (T-t)+1}\right)^{\frac{1}{2}} + \alpha\Bigg\{1 - (\text{Cosh}\sqrt{d}\ (T-t)-1)^{\frac{1}{2}} \cdot \left(\frac{\text{Sinh}\sqrt{d}\ T}{(\text{Cosh}\sqrt{d}\ (T-t)+1)^{\frac{1}{2}}\ (\text{Cosh}\sqrt{d}\ (T-1)}\right)\Bigg\} \tag{2-81}$$

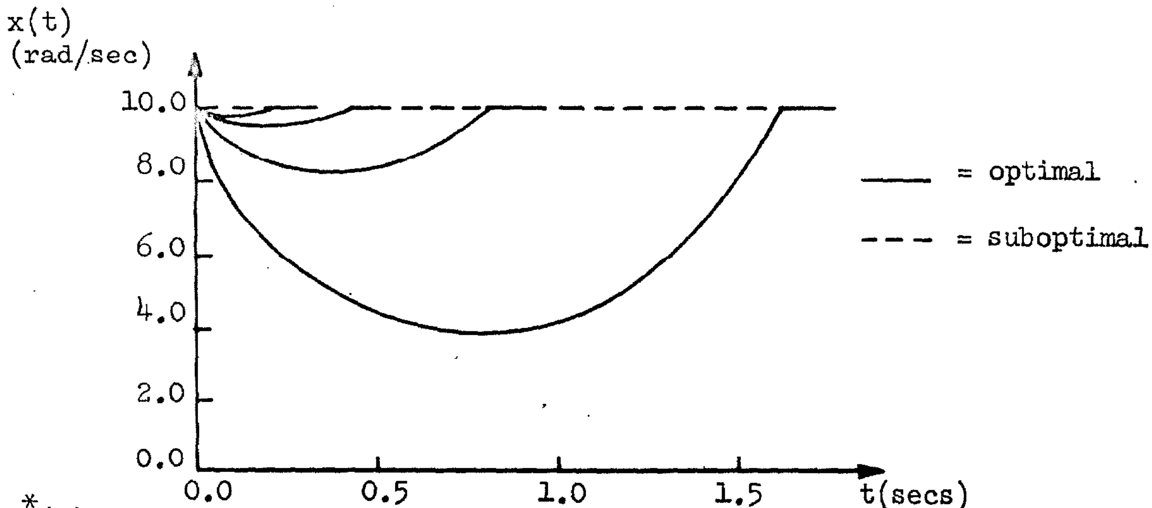
Note that in Eq.(2-81)

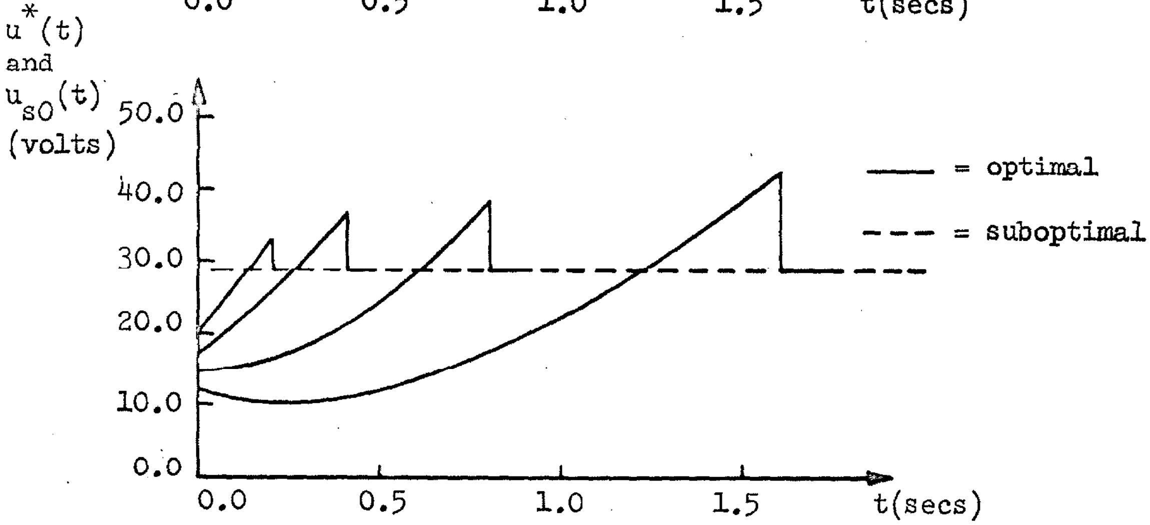
$$x(t)\Big|_{t=0} = x(0)$$

$$x(t)\Big|_{t=T} = \alpha \quad \text{as required in any control action.}$$

2.4.5. Comparison of the Salient Features of the Suboptimal Feedback Controller with those of the Minimum-Energy Controller.

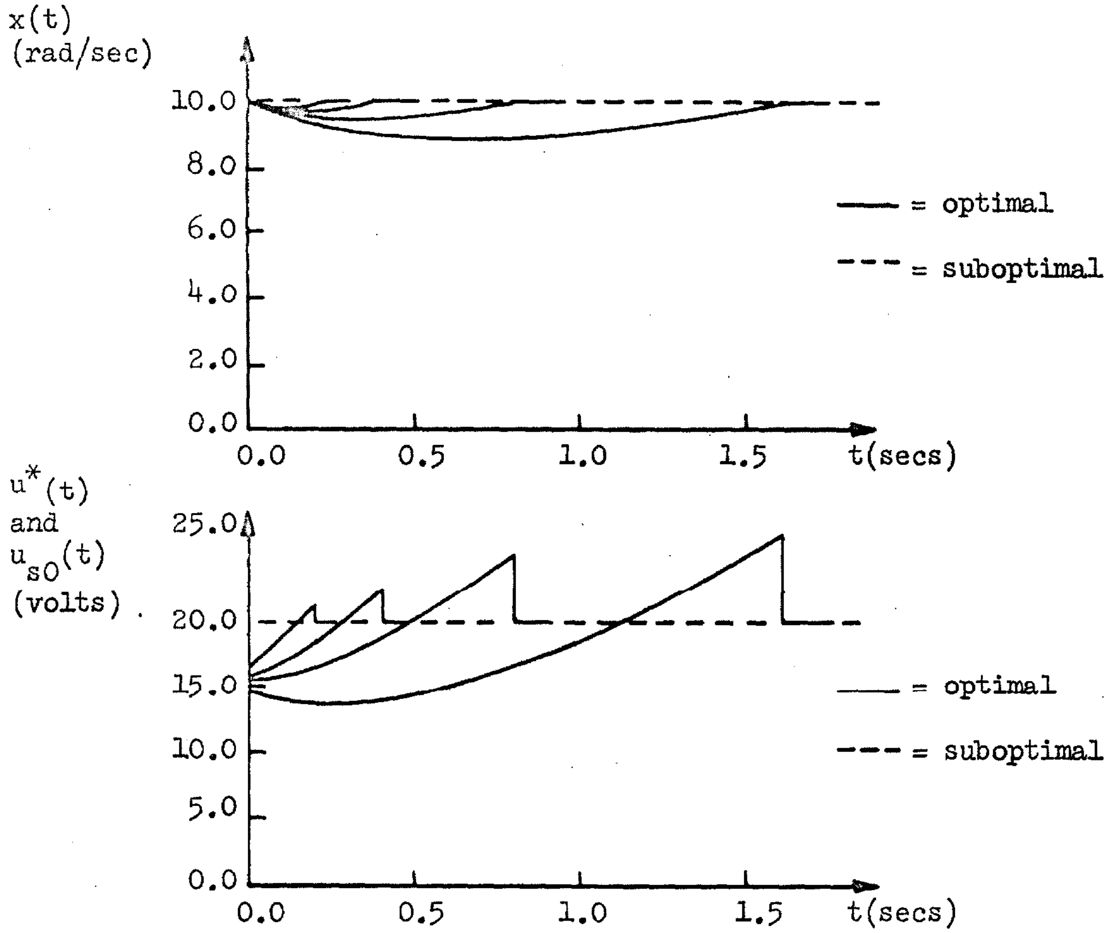
Using Eqs.(2-79), (2-81), (2-56), (2-58) and (2-39), the following results are obtained from the appropriate digital computer simulations.





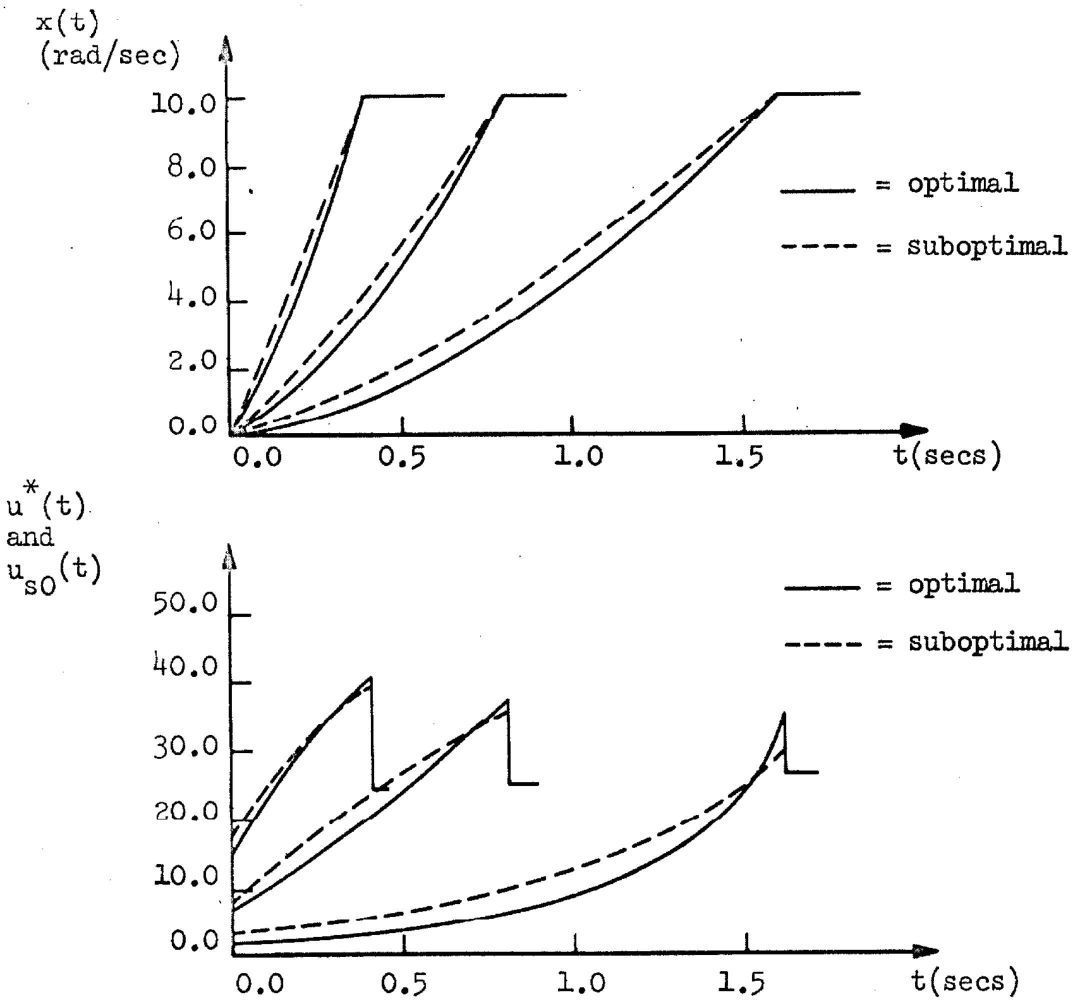
T (sec)	ENERGY CONSUMPTION Optimal (watt-seconds)	Suboptimal (watt-seconds)
0.2	52.59	53.20
0.4	102.62	106.40
0.8	188.82	212.80
1.6	286.38	425.60

Figure 9. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = 10.0 newton-meter , γ = 0.0 rad/sec (speed-control) , $u^*(0-)$ = 24.2 volts.



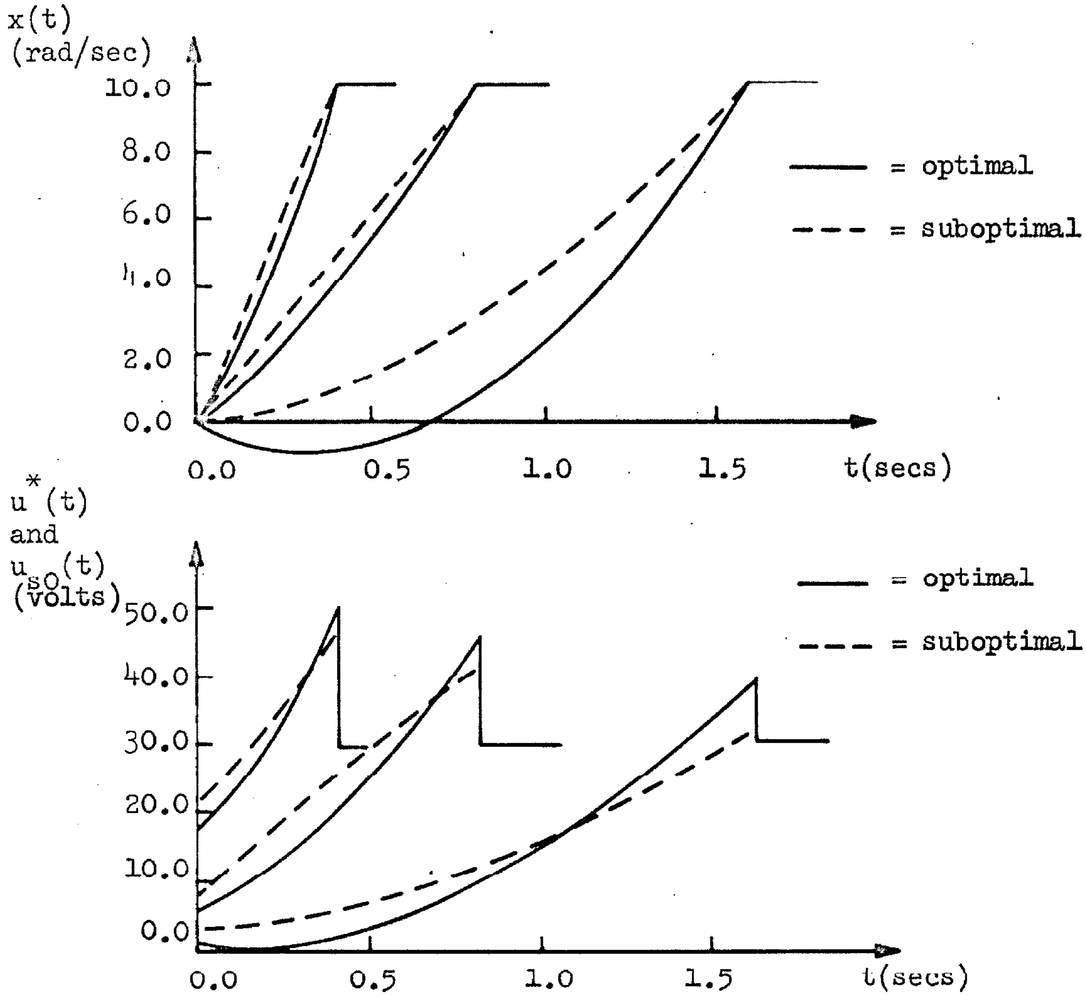
T (sec)	Energy Consumption Optimal (watt-seconds)	Suboptimal (watt-seconds)
0.2	- 3.372	- 3.0
0.4	- 6.8372	- 6.0
0.8	-14.23	- 12.0
1.6	-31.21	- 24.0

Figure 10. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = - 10.0 newton-meter , γ = 0.0 rad/sec (speed-control) , $u^*(0-)$ = 24.2 volts.



T (sec)	Energy Consumption Optimal (watt-secs)	Energy Consumption Suboptimal (Watt-secs)
0.4	237.80	238.0
0.8	185.66	186.84
1.6	168.84	173.43

Figure 11. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = 0.0 newton-meter , γ = - 10.0 rad/sec (acceleration) , $u^*(0-)$ = 0.0 volt.



T (sec)	Energy Consumption Optimal (watt-seconds)	Suboptimal (watt-seconds)
0.4	345.02	346.66
0.8	322.11	330.86
1.6	333.62	376.93

Figure 12. Results for: $\alpha = 10.0$ rad/sec , $v(0-) = 0.0$ newton-meter , $\beta = 10.0$ newton-meter , $\gamma = -10.0$ rad/sec (acceleration) , $u^*(0-) = 29.125$ volts.

2.4.6. Comparison of the Performances of the Minimum-Energy Controller and Classical Controllers.[13]

The performance of the minimum-energy controller is compared with the performances of the classical controllers of the following types under the assumption of zero armature inductance:

1. A controller with a high-gain amplifier in the forward path which continuously operates on the error signal in order to produce the required control signal which in turn matches the output speed with the desired speed within a permissible error. This is called a proportional-type speed-controller.
2. A controller with an integrating amplifier in the forward path which continuously integrates the error signal in order to produce the required control signal which in turn matches the output speed with the desired speed exactly. This is called an integral-type speed-controller.

A. Proportional-Type Speed Controller.

The plant equation in terms of small variations is given by:

$$\dot{\omega}(t) + \left(\frac{f_e}{J_e} + \frac{k_t k_b}{J_e r_a}\right) \omega(t) = \left(\frac{k_t}{J_e r_a}\right) u_1(t) - \left(\frac{1}{J_e}\right) v_1(t) \quad (2\text{-}82)$$

where: $\omega(t) \triangleq$ small variation of angular speed, of the motor from its steady-state value

$\dot{\omega}(t) \triangleq$ derivative of $\omega(t)$

$u_1(t) \triangleq$ small variation of the control voltage from its steady-state value

$v_1(t) \triangleq$ small variation of the disturbance torque from its steady-state value

We shall study Eq.(2-82) under the following conditions:

1. The system is operating at steady-state, i.e. $x(0) = \alpha$, $\beta = 0$ and a change in the disturbance torque, i.e. $v_1(t) = \beta_1$, takes place. It is required to maintain the speed near its desired value α.
2. The system is initially at rest, i.e. $x(0) = 0$ and $\beta = 0$. It is required to change the speed of the vehicle from zero to α.

Using elementary tools of the classical control theory, the following results are obtained:

For speed-control.

$$\omega(t) = -\frac{\beta_1}{J_e\tau_{ma}}\left(1-e^{-\tau_{ma}t}\right) \tag{2-83}$$

$$u_1(t) = K\frac{\beta_1}{J_e\tau_{ma}}\left(1-e^{-\tau_{ma}t}\right) \tag{2-84}$$

$$E = u_0\left(\frac{u_0-k_b\alpha}{r_a}\right)T + \frac{u_0}{r_a}\left(\frac{\beta_1}{J_e\tau_{ma}}\right)(K+k_b)\left(T-\frac{1}{\tau_{ma}}\left(1-e^{-\tau_{ma}T}\right)\right)+\left(\frac{u_0-k_b\alpha}{r_a}\right)\frac{K\beta_1}{J_e\tau_{ma}}$$
$$\left(T-\frac{1}{\tau_{ma}}\left(1-e^{-\tau_{ma}T}\right)\right)+\frac{1}{r_a}\left(\frac{K^2\beta_1^{\,2}}{J_e^{\,2}\tau_{ma}^2}+k_bK\left(\frac{\beta_1}{J_e\tau_{ma}}\right)^2\right)\left(T-\frac{2}{\tau_{ma}}\left(1-e^{-\tau_{ma}T}\right)+\right.$$
$$\left.+\frac{1}{2\tau_{ma}}\left(1-e^{-2\tau_{ma}T}\right)\right) \;. \tag{2-85}$$

where:

K = Amplifier gain which is chosen such that the speed of response and the steady-state accuracy of the system are acceptable.

$$\tau_{ma} \triangleq \left(\frac{f_e}{J_e} + \frac{k_t k_b}{J_e r_a} + K \frac{k_t}{J_e r_a} \right)$$

$u_0 \triangleq$ steady-state value of the control voltage at $t = 0$.

<u>For speed-setting</u>.

$$\omega(t) = \frac{\alpha_1 k_{ma}}{\tau_{ma}} \left(1-e^{-\tau_{ma} t} \right) \tag{2-86}$$

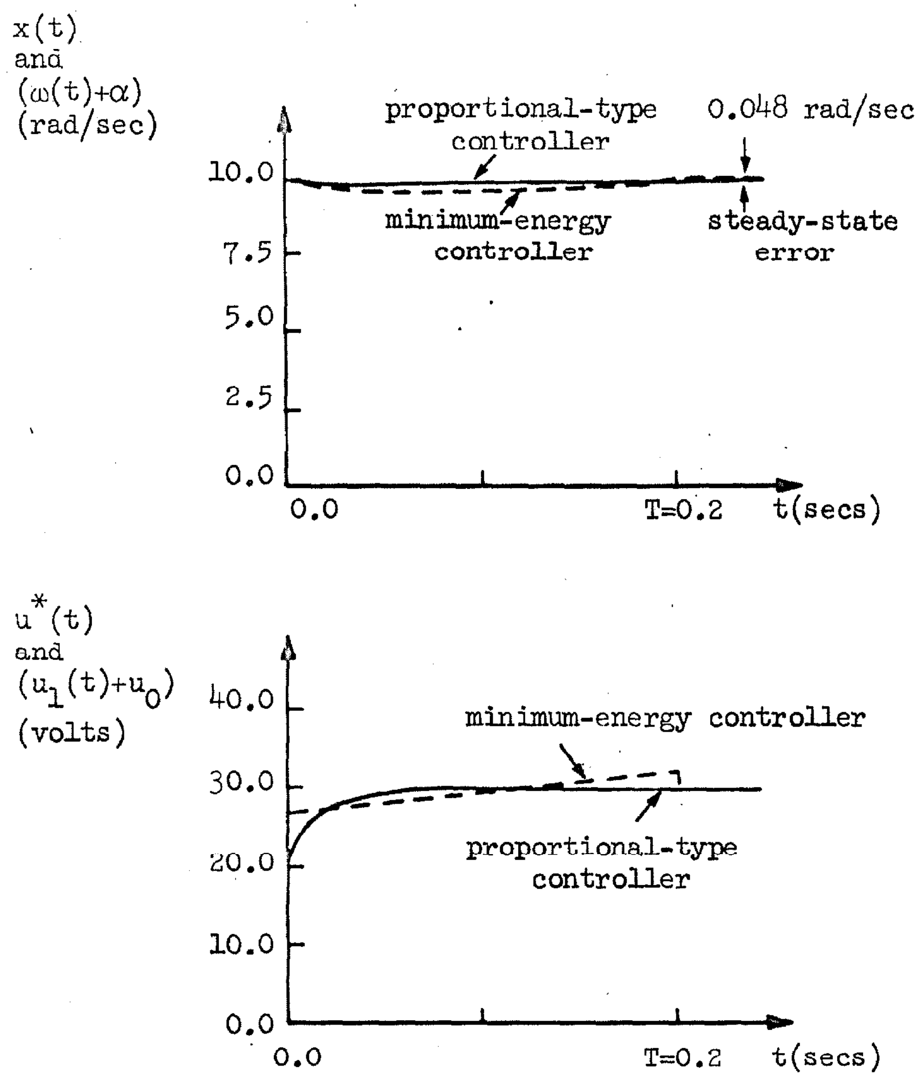
$$u_1(t) = K\alpha \left(1 - \frac{k_{ma}}{\tau_{ma}} + \frac{k_{ma}}{\tau_{ma}} e^{-\tau_m t} \right) \tag{2-87}$$

$$E = \frac{\alpha^2}{r_e} \left(\left[K^2 \left(1 - \frac{k_{ma}}{\tau_{ma}}\right)^2 T - K \frac{k_b k_{ma}}{\tau_{ma}} \left(1 - \frac{k_{ma}}{\tau_{ma}}\right) T \right] + \left[2\left(1 - \frac{k_{ma}}{\tau_{ma}}\right) k_{ma} \frac{K^2}{\tau_{ma}^2} - \frac{2k_{ma}^2 k_b K}{\tau_{ma}^3} \right.\right.$$
$$\left.\left. + K \frac{k_{ma} k_b}{\tau_{ma}^2} \right] \left(1-e^{-\tau_m T}\right) + \left[K^2 \frac{k_{ma}^2}{2\tau_{ma}^3} + \frac{k_{ma}^2 k_b K}{2\tau_{ma}^3} \right] \left(1-e^{-2\tau_m T}\right) \right) . \tag{2-88}$$

where:

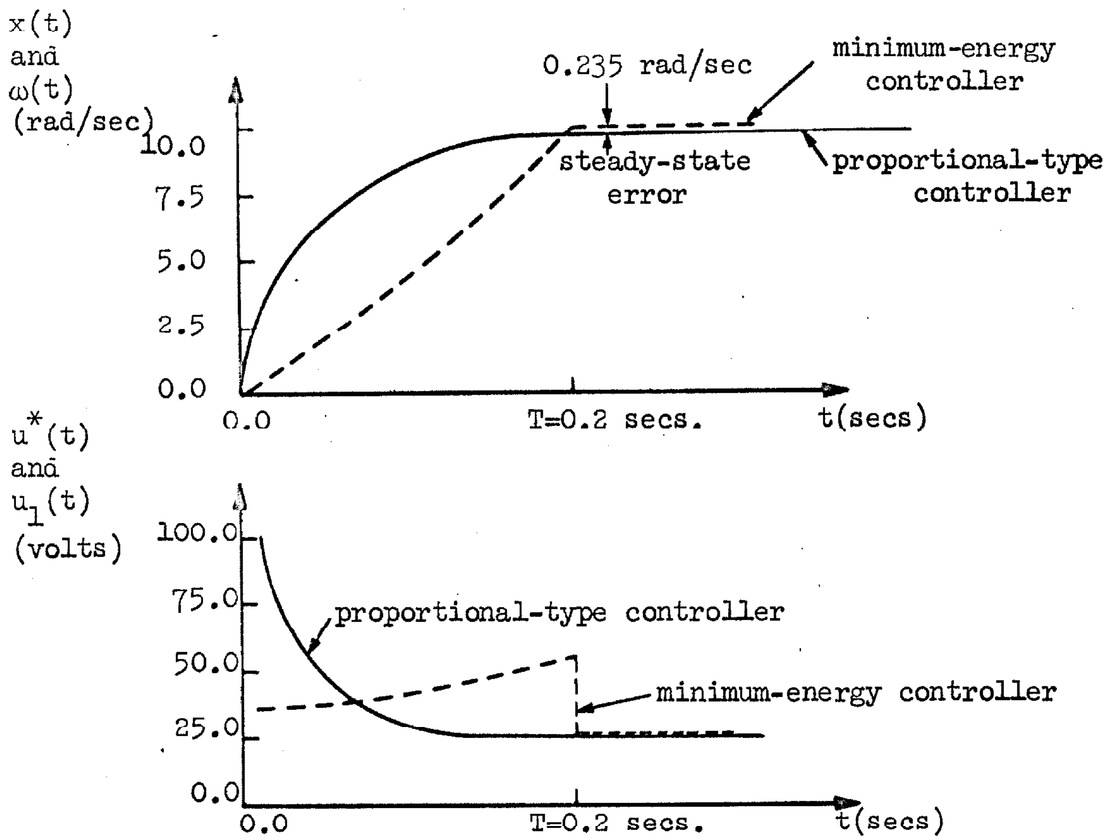
$$k_{ma} \triangleq \left(K \frac{k_t}{J_e r_a} \right)$$

The results of the proposed comparison are summarized in Figures 13 and 14 for the cases of speed-control and speed-setting, respectively.



T (sec)	Energy Consumption (watt-seconds) Classical Controller	Optimal Controller	Suboptimal Controller
0.2	52.80	52.59	53.20

Figure 13. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter, β_1 = 10.0 newton-meter , u_0 = 24.125 volts , K = 100.0.



T (sec)	Energy Consumption (watt-seconds)		
	Classical Controller	Optimal Controller	Suboptimal Controller
0.2	3474.0*	359.00	359.0

Figure 14. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β_1 = 0.0 newton-meter , u_0 = 0.0 volt , K = 100.0.

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* Assuming $u_1(t)$ is not saturated.

B. Integral-Type Speed Controller.

In order to eliminate the steady-state error without using high amplifier gains, an integrator can be placed in front of an amplifier with sufficient gain. Using the elementary tools of classical control theory the following results are obtained:

For speed-control.

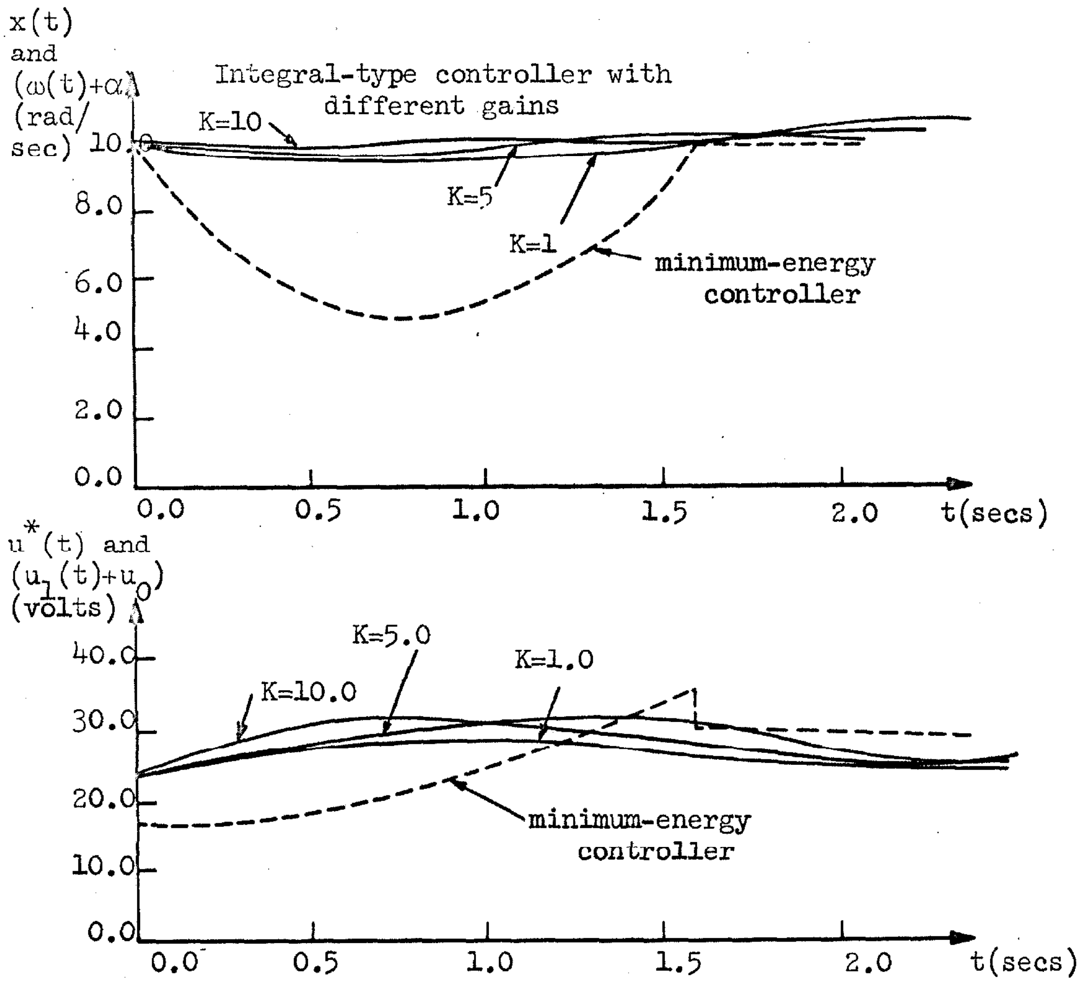
$$\omega(t) = \frac{-(\beta/J_e)}{\sqrt{k_{ma} - \left(\frac{\tau_{ma}}{2}\right)^2}} \; e^{-\left(\frac{\tau_m}{2}\right)t} \; \sin\left(\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}\right)t \qquad (2\text{-}89)$$

$$u_1(t) = \frac{(\beta/J_c)\ K}{\sqrt{k_{ma} - \left(\frac{\tau_{ma}}{2}\right)^2}\ k_{ma}} \left(\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2} - \left(\frac{\tau_m}{2}\right) e^{-\left(\frac{\tau_m}{2}\right)t} \sin\left(\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}\right)t \right.$$
$$\left. - \sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2} \; e^{-\left(\frac{\tau_m}{2}\right)t} \cos\left(\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}\right)t \right) \qquad (2\text{-}90)$$

where:

$$\tau_m \triangleq \left(\frac{f_e}{J_c} + \frac{k_t k_b}{J_c r_a}\right)$$

$$k_{ma} \triangleq \left(K \frac{k_t}{J_e r_a}\right)$$



T (sec)	Energy Consumption (watt-seconds)		
	Classical Controller	Optimal Controller	Suboptimal Controller
1.6	414.36 (K=10) 392.74 (K=5) 368.00 (K=1)	286.38	425.00

Figure 15. Results for: α_1 = 10 rad/sec , β_1 = 10.0 newton-meter , u_0 = 24.125 volts.

For speed-setting.

$$\omega(t) = \alpha\left(1.0 - \frac{\sqrt{k_{ma}}}{\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}}\ e^{-\left(\frac{\tau_m}{2}\right)t}\ \sin\left(\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}\ t + \theta\right)\right) \quad (2\text{-}91)$$

$$u_1(t) = \frac{K\alpha}{\sqrt{k_{ma}\left(k_{ma} - \left(\frac{\tau_m}{2}\right)^2\right)}}\left(\left(\frac{\tau_m}{2}\right)\sin\theta + \left(\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}\right)\cos\theta - \left(\frac{\tau_m}{2}\right)e^{-\left(\frac{\tau_m}{2}\right)t}\right.$$
$$\left.\sin\left(\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}\ t + \theta\right) - \left(\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}\right)e^{-\left(\frac{\tau_m}{2}\right)t}\ \cos\left(\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}\ t + \theta\right)\right)$$
$$(2\text{-}92)$$

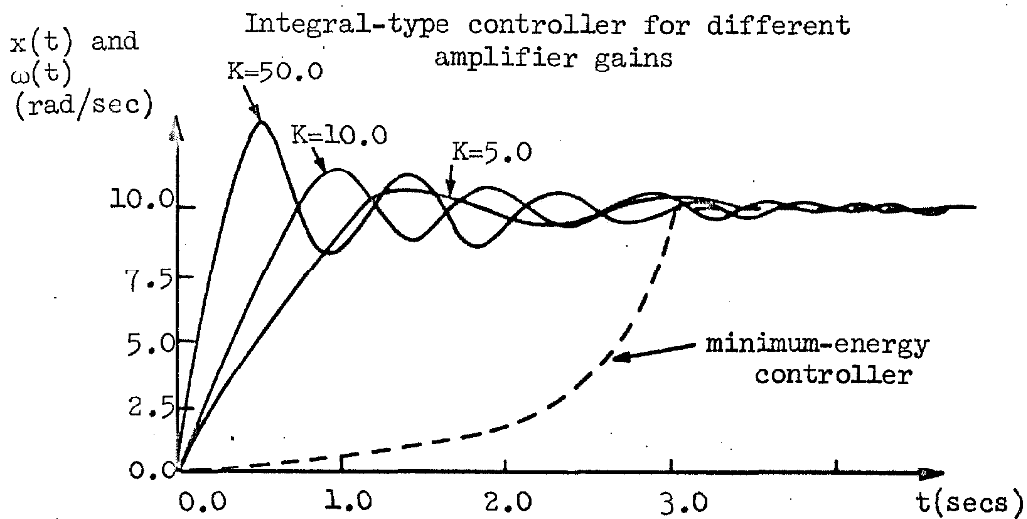
where:

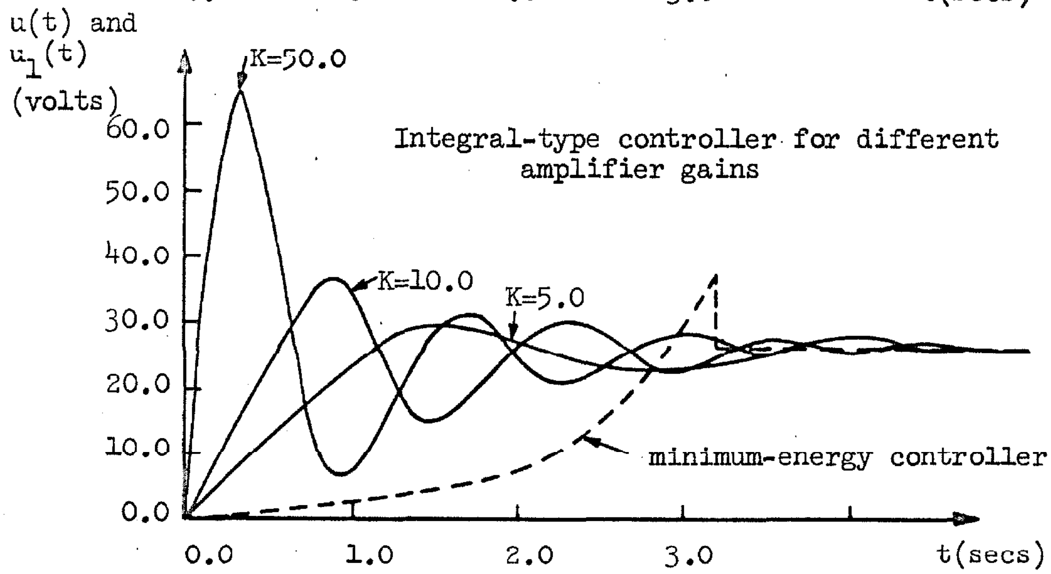
$$\theta \triangleq \mathrm{Tan}^{-1}\left(\frac{\sqrt{k_{ma} - \left(\frac{\tau_m}{2}\right)^2}}{\left(\frac{\tau_m}{2}\right)}\right)$$

The results of the proposed comparison are summarized in Figures 15 and 16 for the cases of speed-control and speed-setting respectively.

2.4.7. Comparisons of the Minimum-Energy and Suboptimal Solutions with the Exact Optimal Solution.

In view of the above results, we conclude that the performances of the minimum-energy and suboptimal controllers show a marked superiority over the performance of so-called classical controllers during the acceleration (and deceleration) of the vehicle. The proposed comparison





T (sec)	Energy Consumption (watt-seconds)		
	Classical Controller	Optimal Controller	Suboptimal Controller
3.2	966.0 (K=50) 501.02 (K=10) 411.37 (K=5)	163.55	167.43

Figure 16. Results for: α_1 = 10 rad/sec , β_1 = 0.0 newton-meter.

is carried out only for the case of acceleration, which is obviously the most important control action, as follows:

Step 1.

Using the results of Sections 2.3.1 and 2.3.2, the system trajectories are determined for the bang-bang and singular controls and shown in Figure 17. It is clear from the observation of Figure 17 that the singular control solution can not enter into the optimal solution. Hence, in this particular control action, the bang-bang solution is the exact optimal solution for the basic optimization problem.

Step 2.

Using the results of Section 2.3.1, the system speed and control trajectories together with the energy-consumption are determined and shown in Figure 18.

Step 3.

The exact plant equations, i.e. Eqs.(2-1) and (2-2), are integrated using the minimum-energy and suboptimal control laws, i.e. Eqs.(2-70) and (2-79) respectively, over the time interval $(0, T)$ which is found in Step 2. The system speed and control trajectories are also shown in Figure 18 for reasons of comparison.

The physical parameters used in the computations are the same as given before and it has been assumed that:

$\alpha = 10.0$ rad/sec

$\beta = 0.0$ n-m

$U = 50.0$ volts.

From the computer results it has been found that the energy-consumption of the bang-bang controller is about 15% higher than that of the minimum-

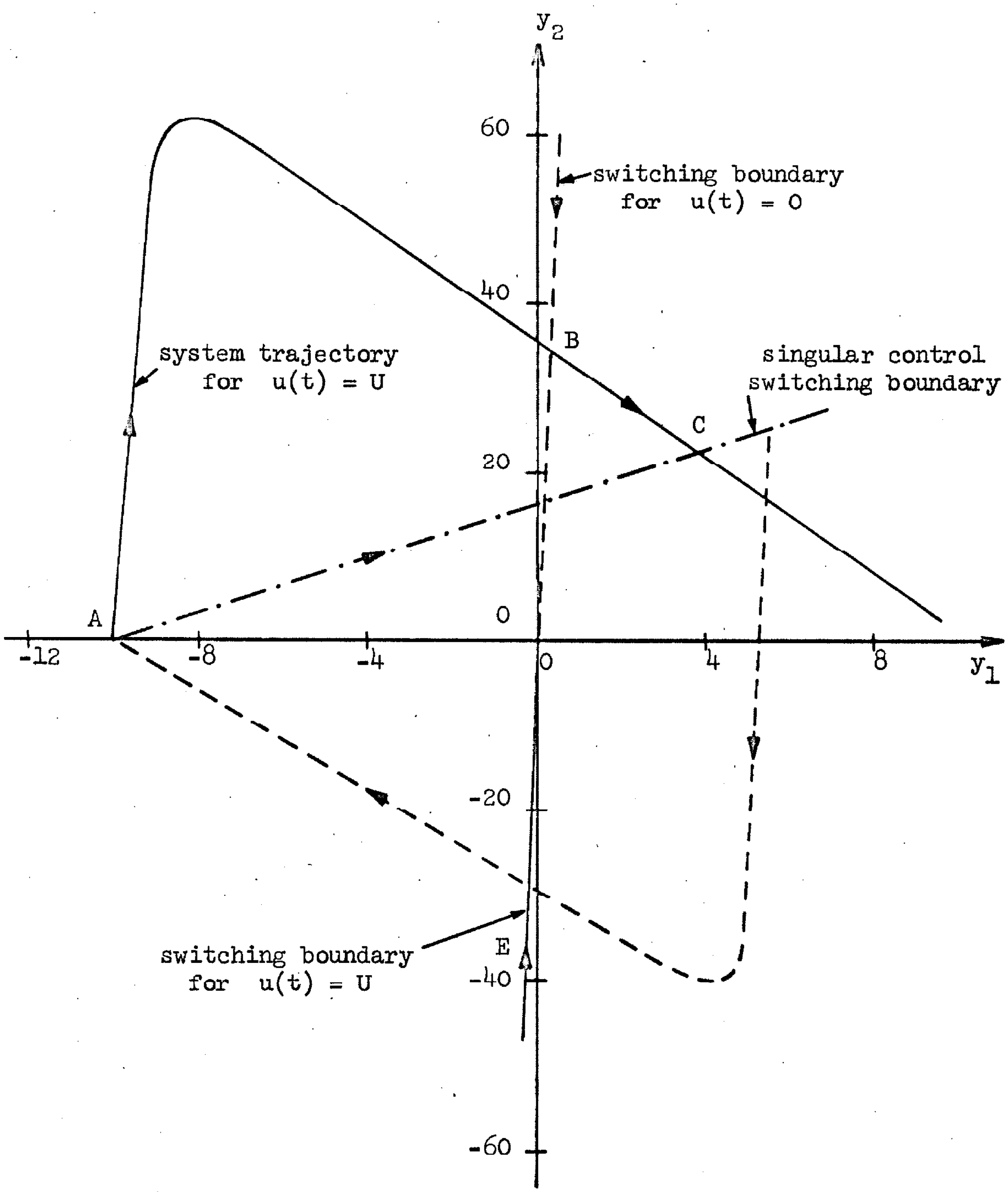
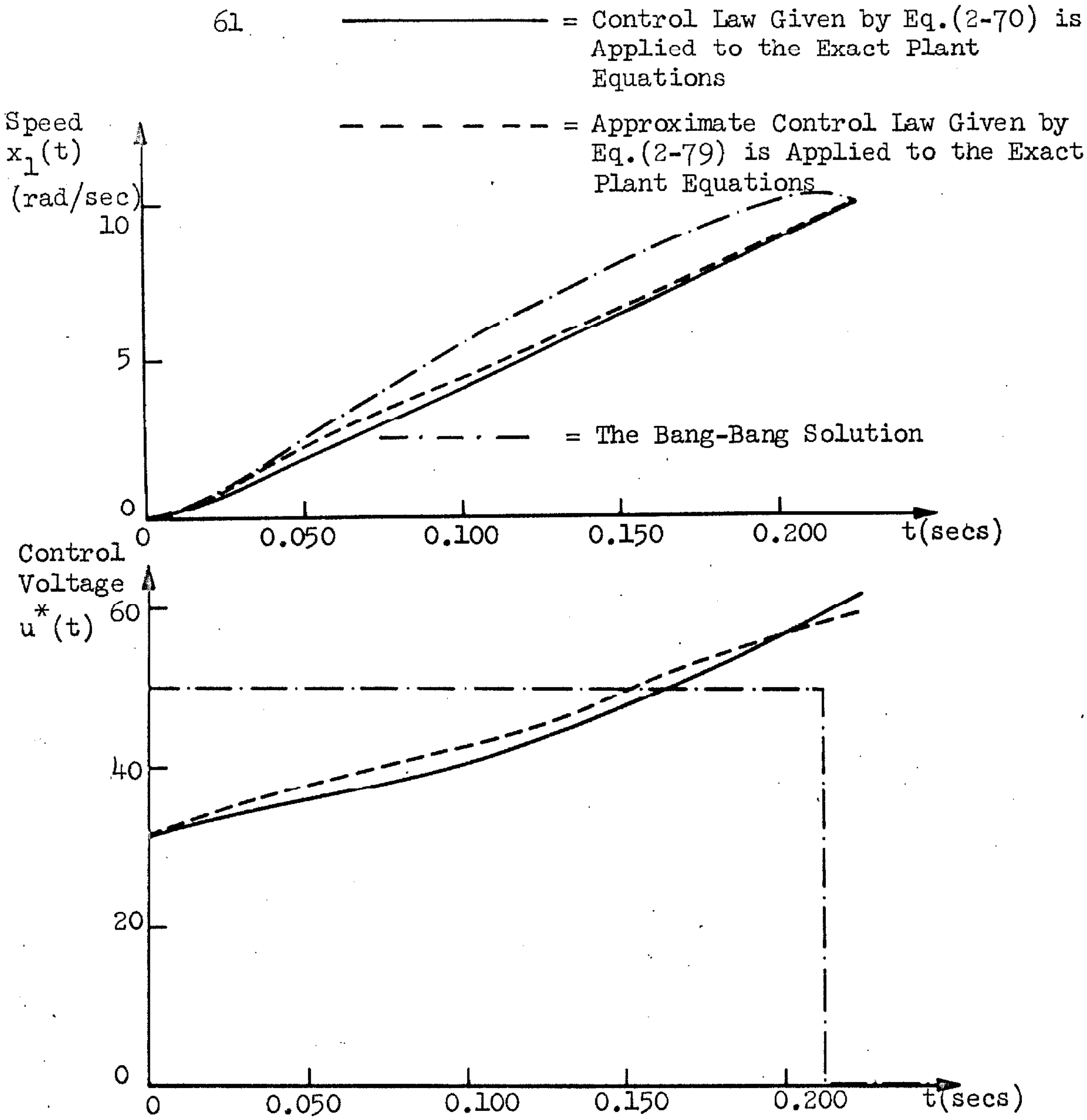


Figure 17. The Determination of the Optimal Solution During the Acceleration.



	Energy Consumption (watt-seconds)		
T (sec)	Bang-Bang Solution	Exact Plant with Minimum Energy Controller	Exact Plant with Suboptimal Controller
0.225	400.0	344.42	343.85

Figure 18. Comparison of Bang-Bang Solution with Minimum-Energy and Suboptimal Solutions.

energy (or suboptimal) controller under the conditions specified above. If, however, $U = 45$ volts, i.e. almost the average value of minimum-energy control law in $(0,T)$, the following results are obtained:

$t_s = 0.220$ second

$T \triangleq 0.225$ second

$E = 334.0$ which is about 3% less than that consumed by the minimum-energy controller.

2.4.8. Remarks on the Approximate Solution.

A careful study of the above results reveals that certain considerations must be taken into account in the selection of T in order to obtain a meaningful solution to this optimization problem. For example, during the forward motion of the vehicle at constant speed, whenever a disturbance comes into the system, the allowable variation of the speed from the desired value α may not be permitted to exceed certain limits. This is, because, the distance moved by the vehicle in $(0,T)$ depends on the variation of speed during the controlling period. Therefore, it may be required to obtain the desired speed as soon as possible. For the case of speed-setting, when T is small, the control energy consumption is large during the acceleration. And there is more electric energy available to be fed back if T is chosen small. The best way of choosing T requires the study of curves of general form as shown in Figure 19, where a possible choice of T denoted by T^* is made which yields a fast response characteristic with good energy consumption. Note that once the selection of T is made, the vehicle's acceleration or deceleration times to any desired speed is fixed. However, as shown in Figure 19 , the energy consumption is different for any possible combinations between α and β. Alternatively we can choose T as the time it takes to make an incremental change in the vehicle speed and the desired speed is reached after several controlling actions each of which lasts T seconds. In either case the performance of the controller is optimal.

In the solution of the approximate optimization problem, we have assumed that $u^*(t)$ is not constrained in magnitude. As a result of this approximation a closed form expression is obtained for $u^*(t)$. However, as seen from the computer results the optimal solution may require $u^*(t)$ to exceed its maximum permissible value or to become negative in any given control action under certain conditions.

If the theoretical results show $u^*(t) < 0$ in $0 \le t \le T$, then it is assumed that the desired speed is obtained at a time greater than the prespecified response time T provided that $u^*(t) \equiv 0$ whenever $u^*(t) < 0$ and mechanical brakes are used until $x_1(T) = \alpha$. If the theoretical results show $u^*(t) \ge U$, then it is assumed that the desired speed is obtained at a time which is greater than the prescribed response time T, provided that $u^*(t) = U$ whenever $u^*(t) \ge U$ and the time-varying gain $\left(\frac{1}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right)$ is bounded by $\left(\frac{1}{\mathrm{Sinh}\sqrt{d}\,t_\epsilon}\right)$

where:

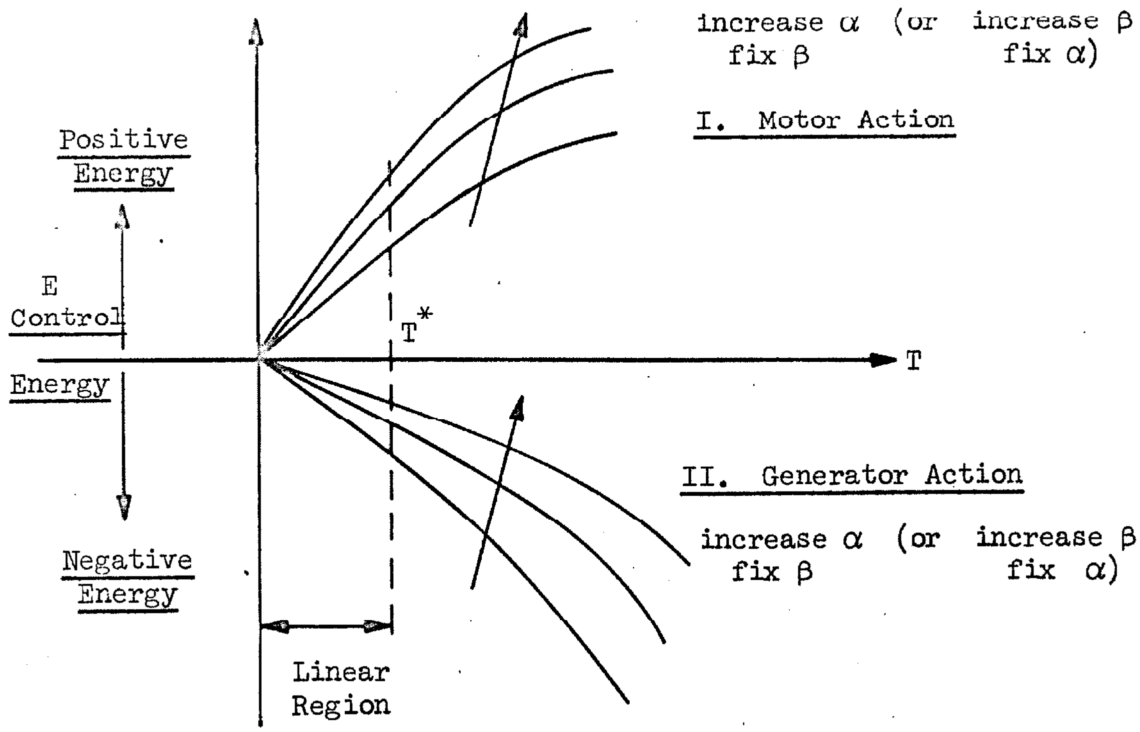
t_ϵ is determined on the basis of following considerations:

1. The physical limitation of the appropriate analog device to generate extremely high gains,
2. The allowable error on the desired speed.

It can be proven easily that even under the above conditions the actual values of the energy consumption of the minimum-energy controller do not deviate appreciably from their calculated values.

From the results of Section 2.4.2 it is known that if T is selected appropriately, then the steady-state value of $u(t)\big|_{t\geq T}$ does not differ considerably from the optimal solution $u^*(t)\big|_{t=T}$. Since, in practice the maximum voltage needed for a control system is determined by using the steady-state form of the dynamical equations under rated speed and load conditions, it can be concluded that the maximum value of the control voltage for a particular system may be best determined from the computer simulation results of the form shown in Section 2.4.2.

Case 1. Speed-control.



Case 2. Speed-setting.

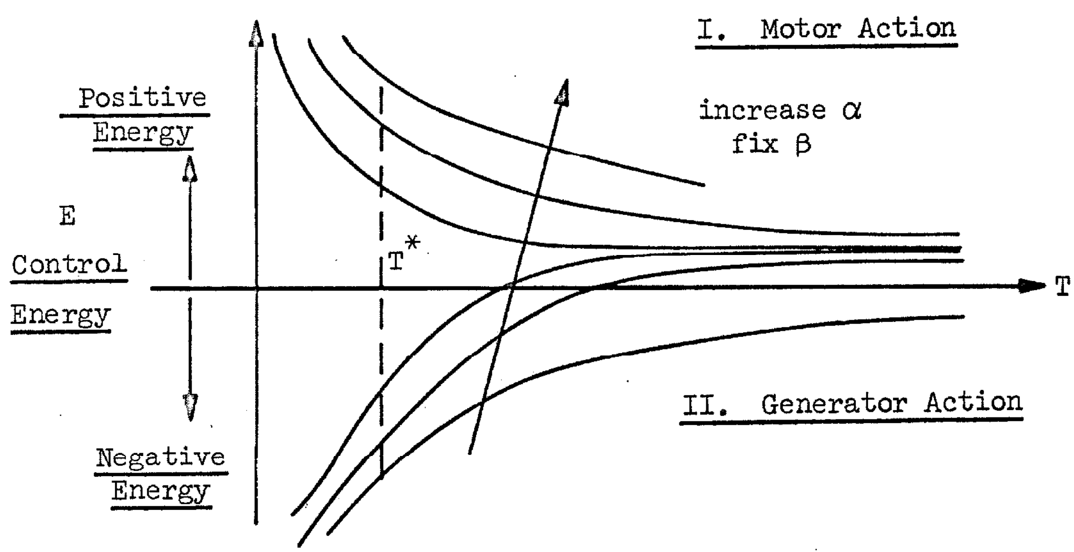


Figure 19. The Variation of Control Energy Consumption with Different Terminal Times.

III. THE STOCHASTIC OPTIMIZATION PROBLEM

3.1. Introduction.

In Chapter II we have assumed that the disturbance torque remains constant during the controlling interval, i.e. (0,T). In reality, the terrain profile encountered by the vehicle varies in a random fashion. Therefore, the disturbance torque is best described by a stochastic process. Other random disturbances acting on the system such as terrain irregularities and wind gusts are assumed to be negligible. In this chapter we shall formulate and solve a stochastic optimization problem originated from the approximate optimization problem studied in Section 2.4. This approach has the following motivation:

In Chapter II we have shown that the solution of the approximate optimization problem yields almost identical performance characteristics* as given by the exact optimal solution under the same conditions. Hence, if the stochastic optimization problem is originated from the approximate optimization problem, it will certainly have the following advantages:

1. The ensuing analysis is mathematically tractable,
2. The desired results are easily obtained and fully justified by using a reasonably small amount of computer time.

---

* The comparison is carried out only for the case of acceleration, since it is by far the most energy consuming operation.

3.2. Formulation of the Stochastic Optimization Problem.

Plant:

$$\dot{x}_1(t) = -\left(\frac{f_e}{J_e} + \frac{k_t k_b}{J_e r_a}\right) x_1(t) - \left(\frac{f_e}{J_e} + \frac{k_t k_b}{J_e r_a}\right)\alpha - \left(\frac{1}{J_e}\right) x_2(t) - \left(\frac{1}{J_e}\right) x_3(t) + \left(\frac{k_t}{J_e r_a}\right) u(t) \qquad (3\text{-}1)$$

$$\dot{x}_2(t) = -(\omega_0)\, x_2(t) + \xi_w(t) \qquad (3\text{-}2)$$

$$\dot{x}_3(t) = 0 \qquad (3\text{-}3)$$

where:

$$x_1(t) \triangleq x(t) - \alpha$$

and $x(t)$ is the speed of the vehicle.

$x_2(t) \triangleq$ A stationary Gaussian, exponentially correlated-noise process having zero mean, variance σ_c^2 and correlation time $\left(\frac{1}{\omega_0}\right)$. It can be shown that $x_2(t)$ process known as Ornstein-Uhlenbeck process[14,15,16] satisfies the Langevin equation given by Eq.(3-2). And $\xi_w(t)$ is a stationary Gaussian, white-noise process having zero mean and spectral density $2\omega_0\,\sigma_c^2$.

$x_3(t) \triangleq$ Known disturbance torque, i.e. $x_3(t) \equiv \beta$ in the deterministic optimization problem studied in Section 2.4.

Performance Index:

$$E = F_w x_1^{\,2}(T) + \int_0^T \left\{ \frac{u^2(t) - k_b u(t)(x_1(t) + \alpha)}{r_a} \right\} dt \qquad (3\text{-}4)$$

where:

T is fixed, $x_1(T)$ is free and $u(t)$ is unconstrained.

$F_w \triangleq$ weighting factor which is artificially introduced here in order to facilitate the use of Bellman's dynamic programming technique.[17]

The set of boundary conditions to be satisfied by the state vector $\underline{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ for the three cases of control action are given below:

Case 1. Speed-control.

$$\begin{aligned} x_1(0) &= 0 \quad , \quad x_1(T) = 0 \\ x_2(0) &= 0 \quad , \quad x_2(T) = \text{Free} \\ x_3(0-) \neq x_3(0) &= \beta \quad , \quad x_3(T) = \beta \end{aligned} \qquad (3\text{-}5)$$

Case 2. Speed-setting.

$$\begin{aligned} x_1(0) &= x(0) - \alpha \quad , \quad x_1(T) = 0 \\ x_2(0) &= 0 \quad , \quad x_2(T) = \text{Free} \\ x_3(0-) = x_3(0) &= \beta \quad , \quad x_3(t) = \beta \end{aligned} \qquad (3\text{-}6)$$

Case 3. Speed-control and Speed-setting.

$$
\begin{aligned}
x_1(0) &= x(0) - \alpha \quad , & x_1(T) &= 0 \\
x_2(0) &= 0 \quad , & x_2(T) &= \text{Free} \\
x_3(0-) \neq x_3(0) &= \beta \quad , & x_3(T) &= \beta
\end{aligned}
\tag{3-7}
$$

3.3. Solution of the Stochastic Optimization Problem.

Define the optimal expected value function for this optimization problem as follows:

$$
S(t,\underline{x}) = \underset{u(t)}{\text{Min}} \ \underset{\xi_w(t)}{\varepsilon} \left(F_w x_1^2(T) + \int_t^T \left\{ \frac{u^2(\tau) - k_b u(\tau)(x_1(\tau)+\alpha)}{r_a} \right\} d\tau \right) \tag{3-8}
$$

where:

$\varepsilon(\cdot) \triangleq$ Expectation operator.

Applying the standard procedure[17,18,19] to Eq.(3-8) yields:

$$
u^*(t) = \left(\frac{k_b}{2}\right)(x_1(t)+\alpha) - \left(\frac{k_t}{2J_e}\right) S_{x_1}(t,\underline{x}) \tag{3-9}
$$

where:

$$
S_{x_1}(t,\underline{x}) = \frac{\partial}{\partial x_1} S(t,\underline{x})
$$

and $S_{x_1}(t,\underline{x})$ is obtained by solving the following Bellman-Hamilton-

Jacobi equation:

$$S_t(t,\underline{x}) - \left(\frac{c}{2}\right)(x_1(t)+\alpha)^2 - a(x_1(t)+\alpha)\, S_{x_1}(t,\underline{x}) - \left(\frac{b}{2}\right) S^2_{x_1}(t,\underline{x}) - \left(\frac{1}{J_e}\right) x_2(t) S_{x_1}(t,\underline{x}) -$$

$$\left(\frac{1}{J_e}\right) x_3(t)\, S_{x_1}(t,\underline{x}) - (\omega_0) x_2(t) S_{x_2}(t,\underline{x}) + (\omega_0 \sigma_c{}^2)\, S_{x_2 x_2}(t,\underline{x}) = 0 \qquad (3\text{-}10)$$

with the boundary condition:

$$S(T,\underline{x}) = F_w x_1{}^2(T) \qquad (3\text{-}11)$$

where:

a, b, c are defined as before and

$$S_t(t,\underline{x}) = \frac{\partial}{\partial t} S(t,\underline{x})$$

$$S_{x_2}(t,\underline{x}) = \frac{\partial}{\partial x_2} S(t,\underline{x})$$

$$S_{x_2 x_2}(t,\underline{x}) = \frac{\partial}{\partial x_2 \partial x_2} S(t,\underline{x}) \quad .$$

The solution of Eq.(3-10) is obtained as follows:

Let:

$$S(t,\underline{x}) = \frac{1}{2} \langle \underline{x}, P(t)\, \underline{x} \rangle + \underline{q}\, \underline{x} + r(t) \qquad (3\text{-}12)$$

where:

$P(t) \triangleq$ A 3 x 3 symmetric matrix

$\langle\, ,\, \rangle \triangleq$ Inner product of two vectors.

$\underline{q} \triangleq$ A raw vector with two components

$r(t) \triangleq$ A scalar function.

Determining $S_t(t,\underline{x})$, $S_{x_1}(t,\underline{x})$, $S_{x_2}(t,\underline{x})$, $S_{x_2x_2}(t,\underline{x})$ from Eq.(3-12) and substituting them into Eq.(3-10), collecting the coefficients of equal powers of x's and equating them to zero yields the following equations:

$$\dot{P}_{11}(t) - bP_{11}^2(t) - 2aP_{11}(t) - c = 0 \tag{3-13}$$

$$P_{11}(T) = 2F_W$$

$$\dot{P}_{12}(t)-(a+bP_{11}(t)+ \omega_0)\ P_{12}(t) -\left(\frac{1}{J_e}\right)P_{11}(t) = 0 \tag{3-14}$$

$$P_{12}(T) = 0$$

$$\dot{P}_{13}(t)-(a+bP_{11}(t))P_{13}(t)-\left(\frac{1}{J_e}\right)P_{11}(t) = 0 \tag{3-15}$$

$$P_{13}(T) = 0$$

$$\dot{P}_{22}(t)-(2\omega_0)\ P_{22}(t)-bP_{12}^2(t)-\left(\frac{2}{J_e}\right)P_{12}(t) = 0 \tag{3-16}$$

$$P_{22}(T) = 0$$

$$\dot{P}_{23}(t)-\omega_0\ P_{23}(t)- bP_{12}(t)\ P_{13}(t)-\left(\frac{1}{J_e}\right)(P_{12}(t)+ P_{13}(t)) = 0 \tag{3-17}$$

$$P_{23}(T) = 0$$

$$\dot{P}_{33}(t) - bP_{13}^2(t) - \left(\frac{2}{J_e}\right)P_{13}(t) = 0 \tag{3-18}$$

$$P_{13}(T) = 0$$

$$\dot{q}_1(t) - (a + bP_{11}(t))\, q_1(t) - (a\alpha)\, P_{11}(t) - \alpha c = 0 \tag{3-19}$$

$$q_1(T) = 0$$

$$\dot{q}_2(t) - \omega_0 q_2(t) - \left(\frac{1}{J_e} + bP_{12}(t)\right) q_1(t) - (a\alpha)\, P_{12}(t) = 0 \tag{3-20}$$

$$q_2(T) = 0$$

$$\dot{q}_3(t) - a\alpha\, P_{13}(t) - bP_{13}(t)\, q_1(t) - \left(\frac{1}{J_e}\right)q_1(t) = 0 \tag{3-21}$$

$$q_3(T) = 0$$

$$\dot{r}(t) - \left(\frac{b}{2}\right)q_1^{\,2}(t) - (a\alpha)\, q_1\,(t) - \left(\frac{c}{2}\right)\alpha^2 + (\omega_0 \sigma_c^{\,2})P_{22}(t) = 0 \tag{3-22}$$

$$r(T) = 0$$

Note that:

$$S_{x_1}(t,\underline{x}) = P_{11}(t)\, x_1(t) + P_{12}(t)\, x_2(t) + P_{13}(t)\, x_3(t) + q_1(t) \tag{3-23}$$

where:

$P_{11}(t)$, $P_{12}(t)$, $P_{13}(t)$ and $q_1(t)$ can be determined in this case by solving analytically the Eqs.(3-13), (3-14), (3-15) and (3-19). The results are:

$$P_{11}(t) = -\left\{\frac{a}{b} - \left(\frac{\sqrt{d}}{b}\right)\left(\frac{\operatorname{Cosh}\sqrt{d}\ (T-t)}{\operatorname{Sinh}\sqrt{d}\ (T-t)}\right)\right\} \tag{3-24}$$

$$P_{12}(t) = \left(\frac{1}{J_e b(\omega_0{}^2 - d)}\right)\left\{a\omega_0 + d + \sqrt{d}(a+\omega_0)\left(\frac{e^{-\omega_0(T-t)} - \operatorname{Cosh}\sqrt{d}\ (T-t)}{\operatorname{Sinh}\sqrt{d}\ (T-t)}\right)\right\} \tag{3-25}$$

Note that $\omega_0{}^2 = d$ does not represent a singularity in Eq.(3-25) since $\lim_{\omega_0{}^2 \to d} P_{12}(t)$ exists.

$$P_{13}(t) = -\left(\frac{1}{J_e b \operatorname{Sinh}\sqrt{d}\ (T-t)}\right)\left\{\operatorname{Sinh}\sqrt{d}\ (T-t) - \frac{a}{\sqrt{d}}(\operatorname{Cosh}\sqrt{d}\ (T-t) - 1)\right\} \tag{3-26}$$

$$q_1(t) = \left(\frac{-\alpha\sqrt{d}}{b \operatorname{Sinh}\sqrt{d}\ (T-t)}\right)\left\{\frac{a}{\sqrt{d}} \operatorname{Sinh}\sqrt{d}\ (T-t) - \operatorname{Cosh}\sqrt{d}\ (T-t) + 1\right\} \tag{3-27}$$

where:

$$d \triangleq a^2 - bc$$

and the operation $\lim \frac{F}{W} \to \infty$ has been performed on the expressions obtained from the solution of Eqs.(3-13), (3-14), (3-15) and (3-19). Substituting Eq.(3-23) into Eq.(3-9) yields:

$$u^*(t) = \left(\frac{k_b}{2} + \frac{k_t a}{2J_e b}\right)x(t) + \left(\frac{k_t\sqrt{d}}{2J_e b}\right)\left(\frac{1-\mathrm{Cosh}\sqrt{d}\ (T-t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right)x(t) + \left(\frac{k_t}{2J_e{}^2 b}\right)x_3(t) +$$

$$\left(\frac{k_t\ a}{2J_e{}^2 b\sqrt{d}}\right)\left(\frac{1-\mathrm{Cosh}\sqrt{d}\ (T-t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right)x_3(t) - \left(\frac{k_t}{2J_e{}^2 b(\omega_0{}^2-d)}\right)\Bigg\{(a\omega_0+d) + \sqrt{d}\ (a+\omega_0)$$

$$\left(\frac{e^{-\omega_0(T-t)} - \mathrm{Cosh}\sqrt{d}\ (T-t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right)\Bigg\}x_2(t) + \left(\frac{k_t\sqrt{d}}{2J_e b}\right)\left(\frac{\alpha-x(t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right) \qquad (3\text{-}28)$$

where:

$$x(t) = x_1(t) + \alpha \quad .$$

The optimal expected value of the control energy is obtained by solving Eqs.(3-13) through (3-22) in the computer and evaluating the following expression:

$$S(t,\underline{x})\Big|_{t=0} = \{\tfrac{1}{2}\ \langle\underline{x}, P(t)\underline{x}\rangle + \underline{q}\ \underline{x} + r(t)\}\ \Big|_{t=0} \quad . \qquad (3\text{-}29)$$

Note, however, that in actual practice the state variables $x_2(t)$ and $x_3(t)$ can not be measured separately.* Hence the implementation of the control law, i.e. Eq.(3-28), can not be accomplished unless a Kalman filter[20] is used to estimate the state vector $\underline{x}$ from the measurable states. This follows from the concept of separation

---

* On the other hand it is possible to measure the disturbance torque $v(t)$ which is the sum of state variables $x_2(t)$ and $x_3(t)$.

principle[21,22] which allows the estimation of the state vector $\underline{x}$ and the computation of the control law to be performed independently.[19] We shall study this so-called estimation and control problem in more detail later in this chapter.

3.4. Salient Features of the Stochastic Optimization Problem.

Let:

$$x(t) = x_d(t) + \tilde{x}(t) \tag{3-30}$$

where:

$x_d(t) \triangleq$ Deterministic speed or $\varepsilon(x(t))$.

$x(t) \triangleq x_1(t) + \alpha$ = Stochastic speed

$\tilde{x}(t)$ = Speed dispersion. Note that $\varepsilon(\tilde{x}(t)) = 0$.

Substituting Eq.(3-30) into Eqs.(3-1), (3-2) and (3-3) yields:

$$\dot{\tilde{x}}(t) = -\sqrt{d}\left(\frac{\text{Cosh}\sqrt{d}\ (T-t)}{\text{Sinh}\sqrt{d}\ (T-t)}\right)\tilde{x}(t) - \left(\frac{1}{J_e} + \frac{1}{J_e(\omega_0^2-d)}\left\{a\omega_0 + d + \sqrt{d}\ (a+\omega_0)\right.\right.$$

$$\left.\left.\left(\frac{e^{-\omega_0(T-t)} - \text{Cosh}\sqrt{d}\ (T-t)}{\text{Sinh}\sqrt{d}\ (T-t)}\right)\right\}\right)\tilde{x}_2(t) \tag{3-31}$$

$$\dot{\tilde{x}}_2(t) = -\omega_0\,\tilde{x}_2(t) + \xi_w(t) \tag{3-32}$$

or in matrix form:

$$\dot{\underline{z}}(t) = A\,\underline{z}(t) + \underline{y}(t) \tag{3-33}$$

where:

$$\underline{z}(t) \triangleq \begin{bmatrix} \tilde{x}(t) \\ \tilde{x}_2(t) \end{bmatrix}$$

$$\underline{y}(t) \triangleq \begin{bmatrix} 0 \\ \xi_w(t) \end{bmatrix}$$

and A is the coefficient matrix of Eqs.(3-31) and (3-32).

Let:

$$Q(t) = \varepsilon(\underline{z}(t)\underline{z}^T(t)) \triangleq \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{12}(t) & Q_{22}(t) \end{bmatrix} \tag{3-34}$$

Then it is known that the variance matrix Q satisfies the following differential equation:

$$\dot{Q}(t) = AQ(t) + Q(t)\,A^T + \underline{y}(t)\,\underline{y}^T(t) \tag{3-35}$$

where:

superscript T denotes a matrix transpose. In this case:

$$Q(0) = \text{A } 2 \times 2 \text{ zero matrix.} \tag{3-36}$$

The solution of Eq.(3-35) yields the following equations:

$$\dot{Q}_{11}(t) = -2\sqrt{d}\left(\frac{\mathrm{Cosh}\sqrt{d}\ (T-t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right)Q_{11}(t) - \frac{2}{J_e}\left(1 + \frac{1}{(\omega_0{}^2-d)}\left\{(a\omega_0+d)+\sqrt{d}(a+\omega_0)\right.\right.$$

$$\left.\left.\left(\frac{e^{-\omega_0(T-t)} - \mathrm{Cosh}\sqrt{d}\ (T-t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right)\right\}\right)Q_{12}(t) \tag{3-37}$$

$$Q_{11}(0) = 0$$

$$\dot{Q}_{12}(t) = -\left(\omega_0 + \sqrt{d}\left(\frac{\mathrm{Cosh}\sqrt{d}\ (T-t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right)\right)Q_{12}(t) - \frac{1}{J_e}\left(1 + \frac{1}{(\omega_0{}^2-d)}\left\{(a\omega_0+d)+\sqrt{d}(a+\omega_0)\right.\right.$$

$$\left.\left.\left(\frac{e^{-\omega_0(T-t)} - \mathrm{Cosh}\sqrt{d}\ (T-t)}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right)\right\}\right)Q_{22}(t) \tag{3-38}$$

$$Q_{12}(0) = 0$$

$$\dot{Q}_{22}(t) = -2\ \omega_0 Q_{22}(t) + 2\omega_0\sigma_c{}^2 \tag{3-39}$$

Let:

$$R(t) \triangleq \varepsilon(\tilde{u}^2(t)) \tag{3-40}$$

where:

$$\tilde{u}(t) = u^*(t) - u_d^*(t) \tag{3-41}$$

$u^*(t)$ is given by Eq.(3-28) and $u_d^*(t)$ is the solution of the deterministic optimization problem, i.e. Eq.(2-70).

Hence:

$$R(t) = \left(\frac{k_b}{2} + \frac{k_t a}{2J_e b} - \left(\frac{k_t\sqrt{d}}{2J_e b}\right)\left(\frac{\operatorname{Cosh}\sqrt{d}\ (T-t)}{\operatorname{Sinh}\sqrt{d}\ (T-t)}\right)\right)^2 Q_{11}(t) - \left(\frac{k_b}{2} + \frac{k_t a}{2J_e b} - \left(\frac{k_t\sqrt{d}}{2J_e b}\right)\right.$$

$$\left.\left(\frac{\operatorname{Cosh}\sqrt{d}\ (T-t)}{\operatorname{Sinh}\sqrt{d}\ (T-t)}\right)\right)\left(\frac{k_t}{2J_e^{\,2} b(\omega_0^{\,2}-d)}\left((a\omega_0+d)+\sqrt{d}\ (a+\omega_0)\left\{\frac{e^{-\omega_0(T-t)} - \operatorname{Cosh}\sqrt{d}\ (T-t)}{\operatorname{Sinh}\sqrt{d}\ (T-t)}\right\}\right)\right)$$

$$Q_{12}(t) + \left(\frac{k_t\sigma_c}{2J_e^{\,2} b(\omega_0^{\,2}-d)}\left((a\omega_0+d)+\sqrt{d}\ (a+\omega_0)\left\{\frac{e^{-\omega_0(T-t)} - \operatorname{Cosh}\sqrt{d}\ (T-t)}{\operatorname{Sinh}\sqrt{d}\ (T-t)}\right\}\right)\right)^2$$

$$\left(1-e^{-2\omega_0 t}\right) \tag{3-42}$$

Figures 20 and 21 show the $Q_{11}(t)$ vs time, $R(t)$ vs time relationships for any control action under the following conditions:

1. $\left(\frac{1}{\omega_0}\right) = 1.0$ second,

   $\sigma_c = 1.0$ newton-meter,

2. $\left(\frac{1}{\omega_0}\right) = 1.0$ second,

   $\sigma_c = 10.0$ newton-meter.

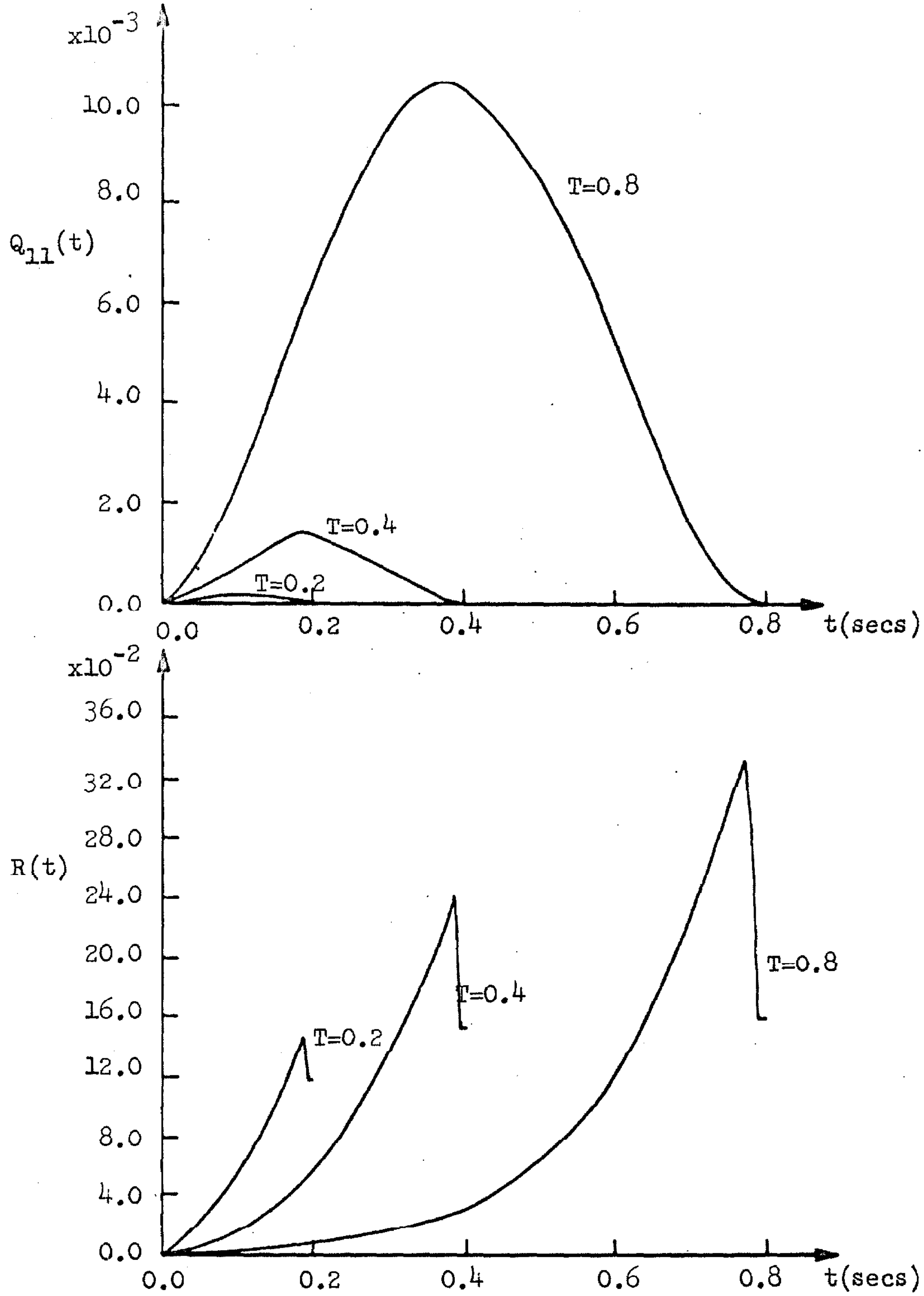


Figure 20. Results for:

$\left(\frac{1}{\omega_0}\right) = 1.0$ second

$\sigma_c = 1.0$ newton-meter

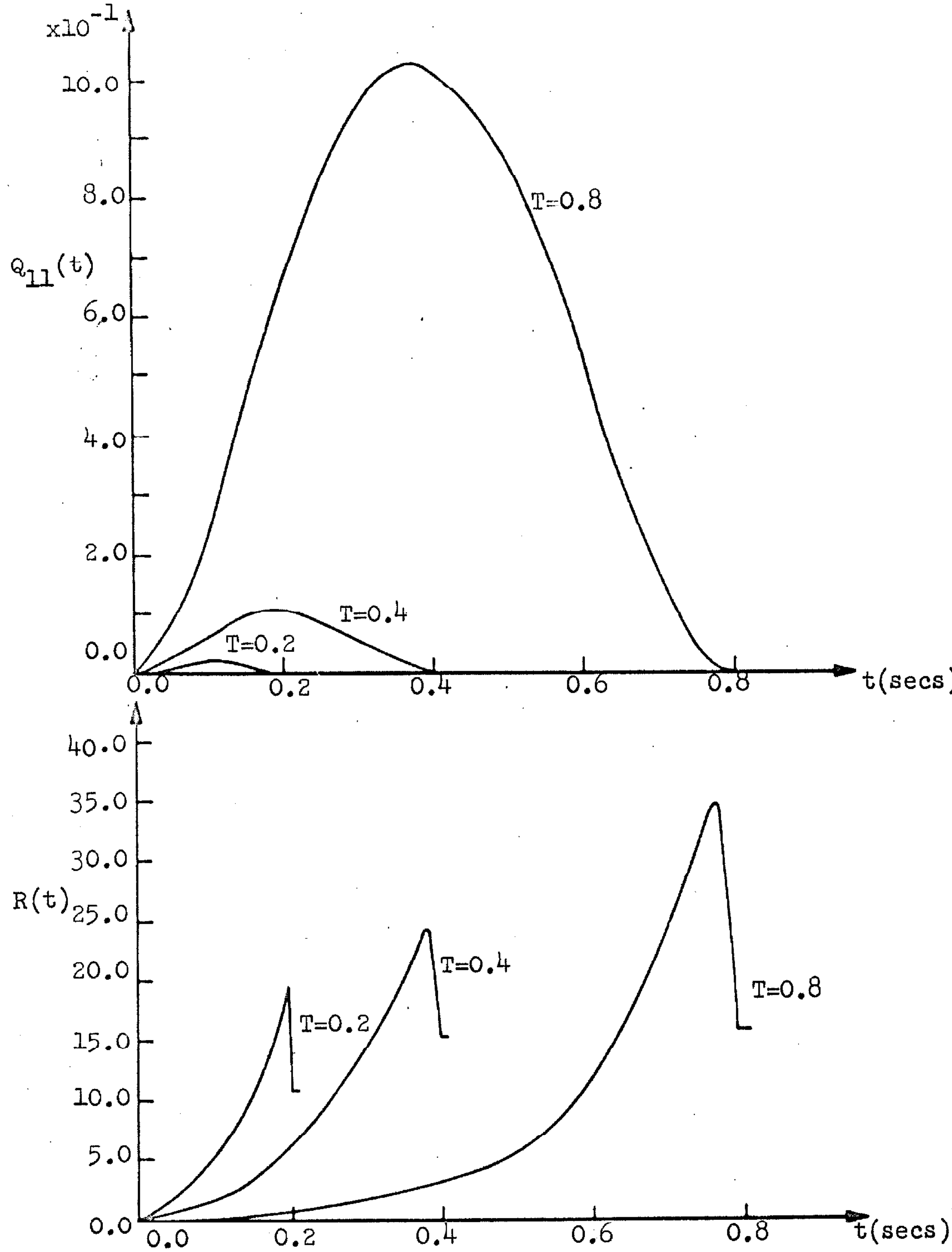


Figure 21. Results for:

$\left(\frac{1}{\omega_0}\right) = 1.0$ second

$\sigma_c = 10.0$ newton-meter

When σ_c is kept constant and $\left(\frac{1}{\omega_0}\right)$ is decreased the ordinates of $Q_{11}(t)$ vs time, $R(t)$ vs time curves decrease for the same times but their general forms remain the same as shown in Figures 20 and 21. The two most interesting characteristics from the observation of Figures 20 and 21 are:

1. The speed variance $Q_{11}(t)$ becomes very small in the interval $t_\epsilon \le t \le T$. Where:

$$t_\epsilon \text{ is the time at which } \lim_{t \to T} \left(\frac{1}{\operatorname{Sinh}\sqrt{d}\,(T-t)} \right) = \frac{1}{\operatorname{Sinh}\sqrt{d}\, t_\epsilon} \quad .$$

And $Q_{11}(T) = 0$ as expected.

2. The control variance $R(t)$ is bounded at $t = T$. This is because in Eq.(3-28), the error term $(\alpha - x(t))$ which is a stochastic process decreases to zero fast enough as $\lim_{t \to T} \left(\frac{1}{\operatorname{Sinh}\sqrt{d}\,(T-t)} \right) = \infty$ so that $u^*(t)$ is bounded at $t = T$. It can be shown easily that if $x_2(t)$ is assumed to be a stationary Gaussian white-noise with zero mean, the corresponding $u^*(t)$ and hence $R(t)$ become unbounded at $t = T$.

The study of statistical properties of the nature shown in Figures 20 and 21 may prove to be invaluable assets in predicting the behaviour of the system in a practical situation. For example it may be important to be able to answer the questions of the following types:

1. How does the variance of the speed vary in $0 \le t \le T$ for different terrain models and what values does it take in $t_\epsilon \le t \le T$?

2. What is the best value of t_ϵ so that a compromise between $Q(t)$ and the overall energy consumption is reached?

From Eq.(3-28):

$$u_{SO}(t) = \left(\frac{k_b}{2} + \frac{k_t a}{2J_e b}\right)x(t) + \left(\frac{k_t}{2J_e{}^2 b}\right)\left(x_2(t)+ x_3(t)\right) + \left(\frac{k_t\sqrt{d}}{2J_e b}\right)\left(\frac{\alpha - x(t)}{\text{Sinh}\sqrt{d}\ (T-t)}\right) \tag{3-43}$$

Equation (3-43) is obtained from Eq.(3-28) by assuming that:

$$\left(\frac{1-\text{Cosh}\sqrt{d}\ (T-t)}{\text{Sinh}\sqrt{d}\ (T-t)}\right) \equiv \lim_{t\to T} (\cdot) = 0$$

$$\left\{a\omega_0+d+\sqrt{d}\ (a+\omega_0)\left(\frac{e^{-\omega_0(T-t)} - \text{Cosh}\sqrt{d}\ (T-t)}{\text{Sinh}\sqrt{d}\ (T-t)}\right)\right\} \equiv \lim_{t\to T} \{\cdot\} = -\ (\omega_0{}^2-d) \quad .$$

The validity of the above assumptions will be justified in the next section.

3.5. A Monte Carlo Simulation.

Re-write the exact plant equations as follows:

$$\dot{x}_1(t) = -\left(\frac{f_e}{J_e}\right)x_1(t) - \left(\frac{f_e}{J_e}\right)\alpha + \left(\frac{k_t}{J_e}\right)x_4(t) - \left(\frac{1}{J_e}\right)\left(x_2(t) + x_3(t)\right) \tag{3-44}$$

$$\dot{x}_2(t) = -\ (\omega_0)\ x_2(t) + \xi_w(t) \tag{3-45}$$

$$\dot{x}_3(t) = 0 \tag{3-46}$$

$$\dot{x}_4(t) = -\left(\frac{k_b}{\ell_a}\right)\left(x_1(t) + \alpha\right) - \left(\frac{r_a}{\ell_a}\right)x_4(t) + \left(\frac{1}{\ell_a}\right)u(t) \tag{3-47}$$

where:

$x_1(t)$, $x_2(t)$, $x_3(t)$ are as defined in Section 3.2.

$x_4(t) \triangleq$ Armature current.

The Eqs.(3-44) through (3-47) which characterize the actual system behaviour in response to a specific control law $u^*(t)$ are studied by using the following Monte Carlo Simulation Algorithm:

(i) Assume the values of ω_0 and σ_c.

(ii) Generate the Ornstein-Uhlenbeck (O-U) process using the equation given in reference [16]:

$$x_2(t_{i+1}) = N_{i+1}\sigma_c\left(1-e^{-2\omega_0(t_{i+1}-t_i)}\right)^{\frac{1}{2}} + x_2(t_i)e^{-\omega_0(t_{i+1}-t_i)} \tag{3-48}$$

where:

N_{i+1} = output of a Gaussian random number generator with zero mean and standard deviation with a value one. A typical sample function of the O-U process and its corresponding terrain profile is shown in Figure 22.

(iii) Use a specific control law to integrate Eqs.(3-44), (3-46), (3-47) over $(0,T)$ to determine $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $u(t)$.

(iv) Using the expression:

$$E = \int_0^T u(t)\, x_4(t)\, dt \tag{3-49}$$

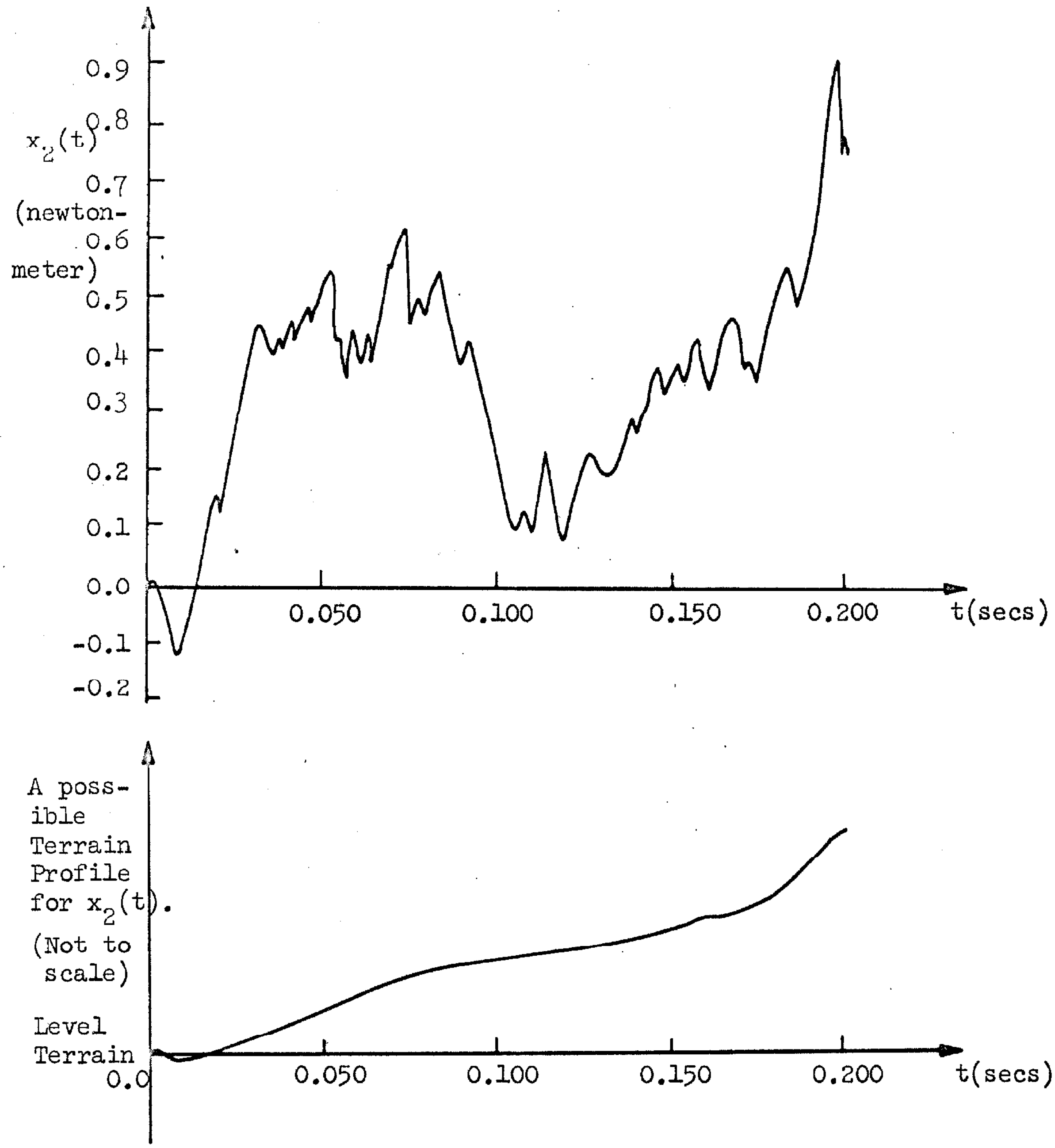


Figure 22. A sample function of the Ornstein-Uhlenbeck process and its corresponding terrain profile for:

$$\left(\frac{1}{\omega_0}\right) = 1.0 \text{ second}$$

$$\sigma_c = 1.0 \text{ newton-meter}$$

determine the energy consumption.

(v) Repeat the steps (i) through (iv) over $(0,T)$ for the next sample function of the O-u process of the same statistical properties.

The following control laws are used:

Case 1.

Equation (3-28).

Case 2.

Equation (3-43).

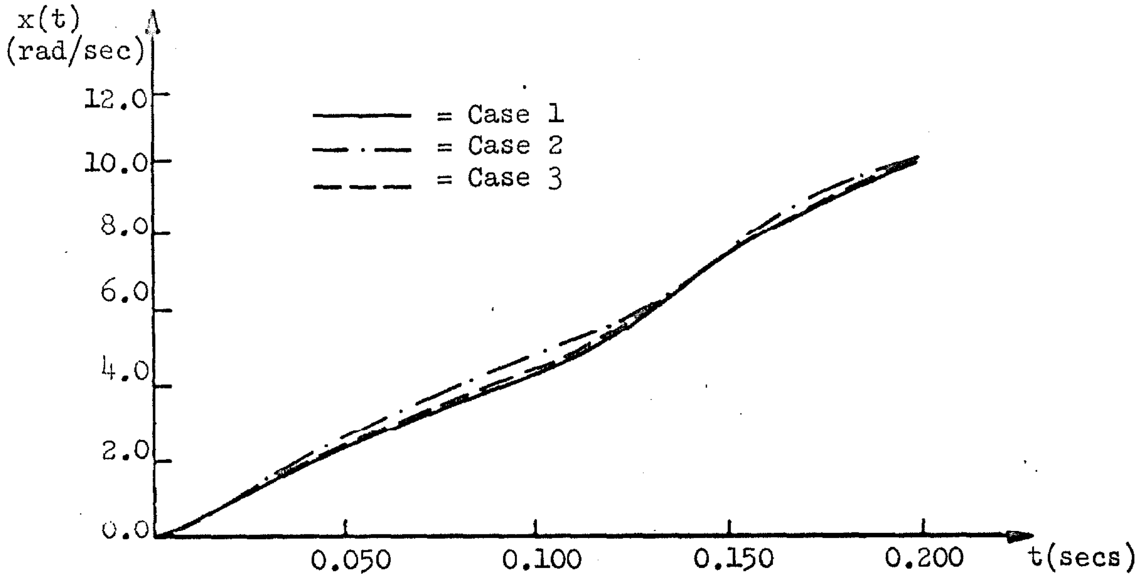
Case 3.

Equation (2-70).

Where:

β is replaced by a stochastic process, i.e. $v(t) = x_2(t) + x_3(t)$. Note that the suboptimal control law for Case 3 is identical to Eq. (3-43). Figures 23 through 26 display the results for the first sample function. The results for other sample functions differ slightly since $x_2(t)$ is a random process.

The abrupt variation of $u^*(t)$ as $t \to T$ in some cases is due to effect of $\left(\frac{\alpha - x(t)}{\text{Sinh}\sqrt{d}\,(T-t)}\right)$ term which appears in the expression for $u^*(t)$. Also note that the performance characteristics in Figures 23 through 26 indicate that, in the absence of observation noise, the control law of Case 3 yields results which are extremely close to the results yielded by the optimal solution for any sample function of the (O-U) process in $(0,T)$. This is important because in practice we can easily implement the control law given for Case 3 as opposed to that given for Case 1. The control law given for Case 2 is the suboptimal



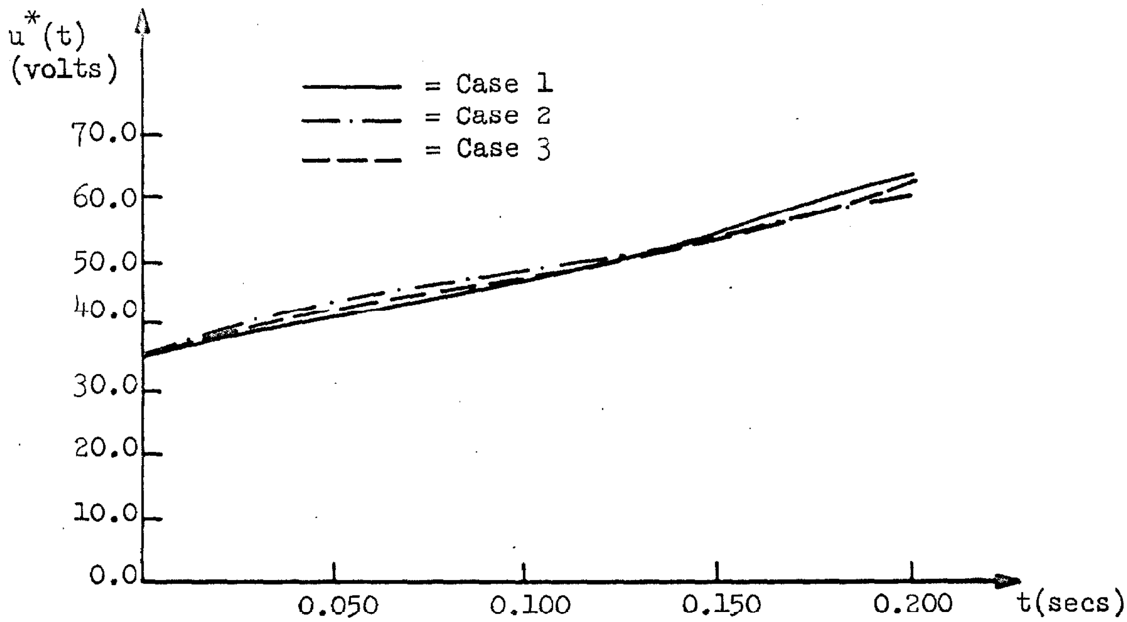


Figure 23. Results for:

$x(0) = 0.0$ rad/sec, $x(T) = 10.0$ rad/sec

$\left(\frac{1}{\omega_0}\right) = 1.0$ second

$\sigma_c = 1.0$ newton-meter

$x_3(t) \equiv 0.0$ newton-meter

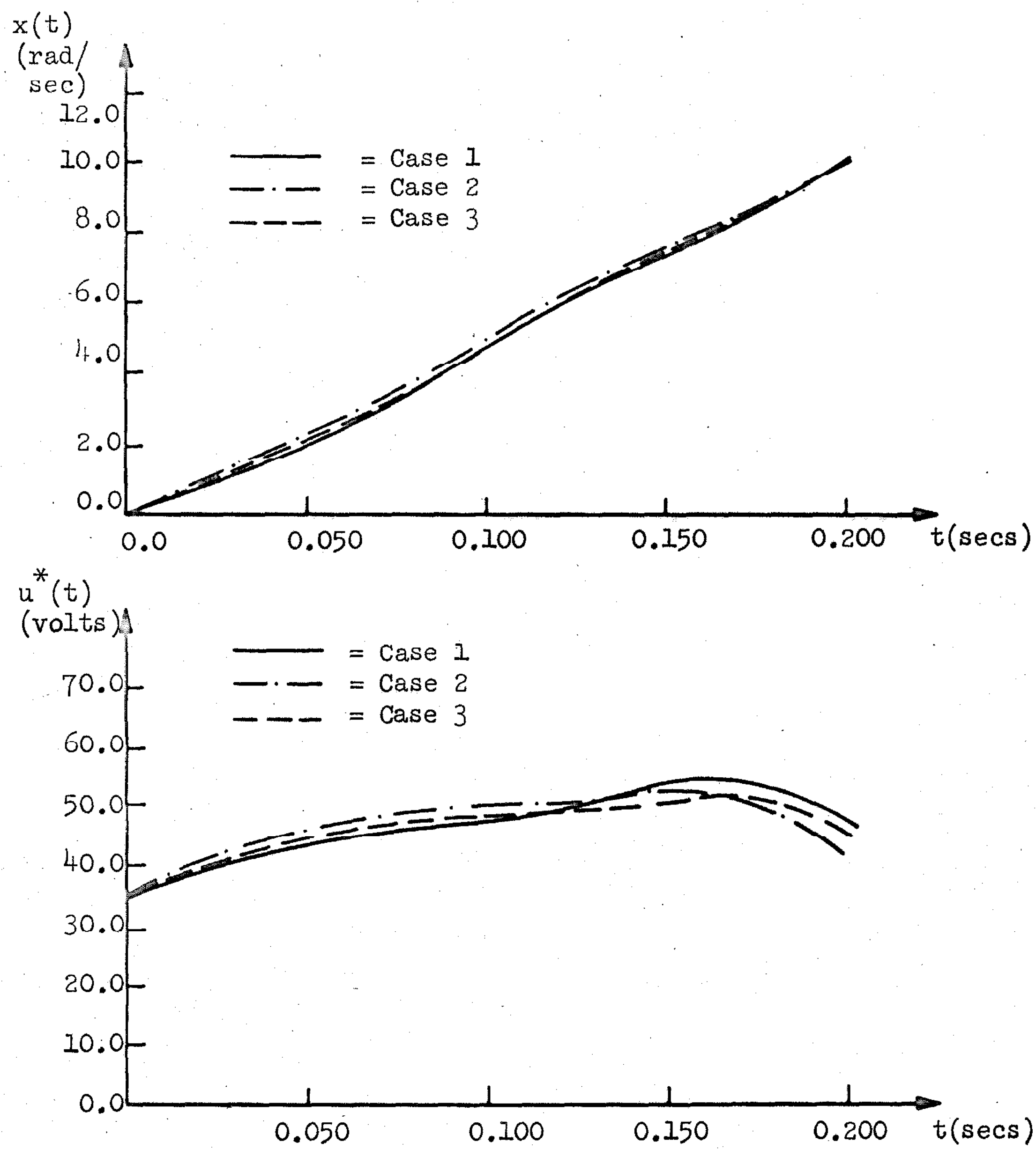


Figure 24. Results for:

$x(0) = 0.0$ rad/sec, $x(T) = 10.0$ rad/sec

$\left(\frac{1}{\omega_0}\right) = 1.0$ second

$\sigma_c = 10.0$ newton-meter

$x_3(t) \equiv 0.0$ newton-meter

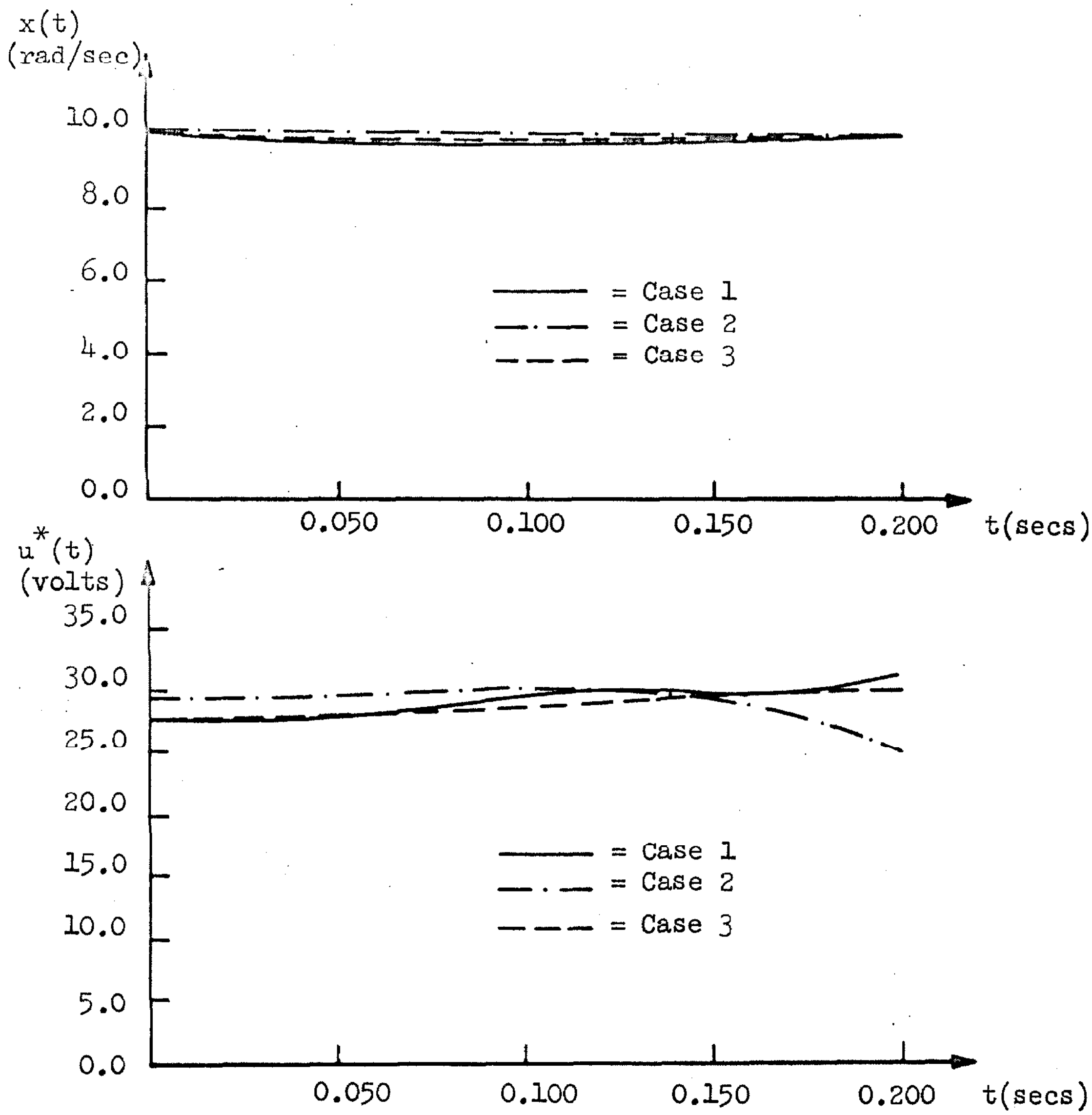


Figure 25. Results for:

$x(0) = 10.0$ rad/sec, $x(T) = 10.0$ rad/sec

$\left(\frac{1}{\omega_0}\right) = 1.0$ second

$\sigma_c = 1.0$ newton-meter

$x_3(t) \equiv 10.0$ newton-meter

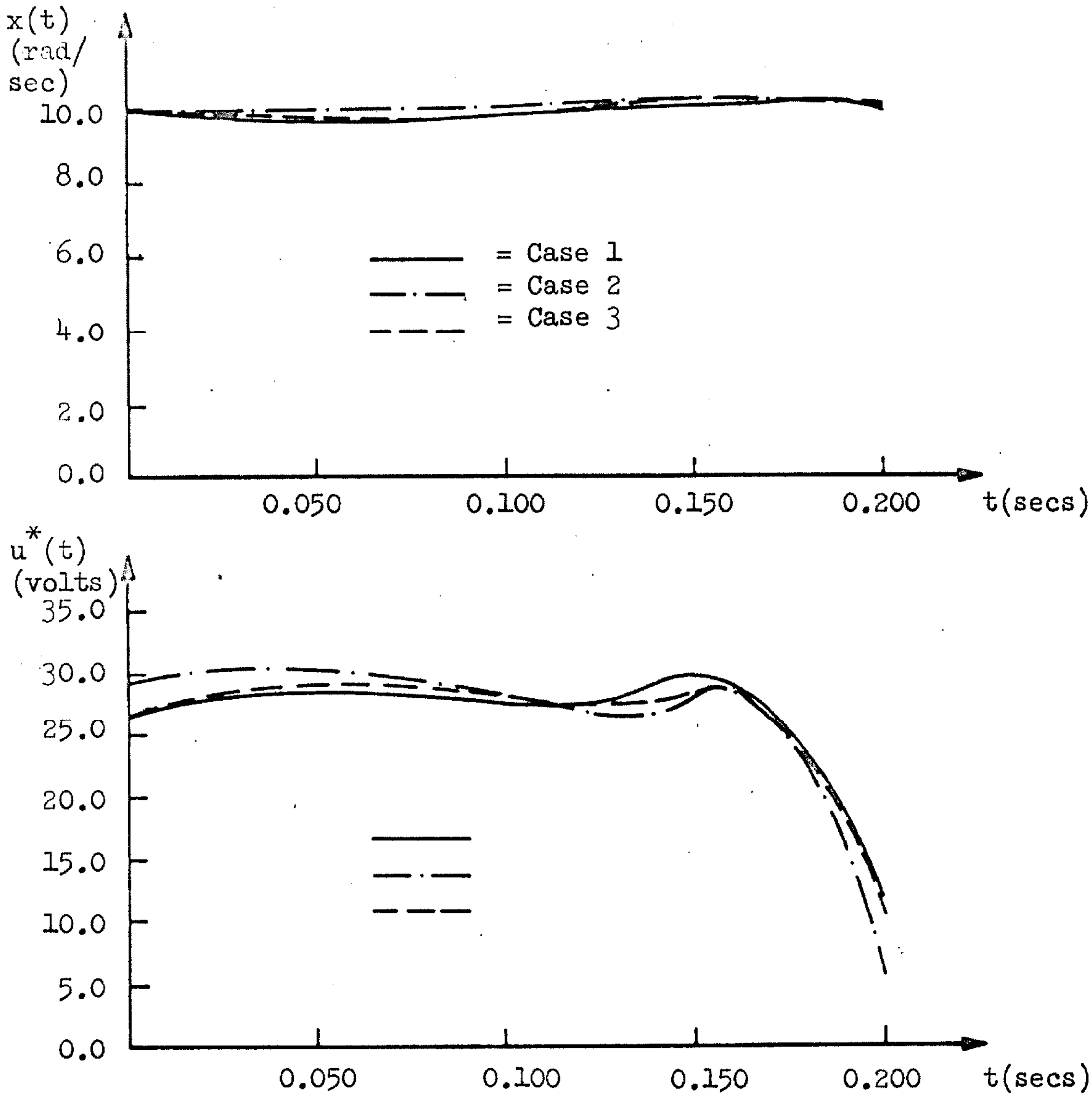


Figure 26. Results for:

$x(0) = 10.0$ rad/sec, $x(T) = 10.0$ rad/sec

$\left(\frac{1}{\omega_0}\right) = 1.0$ second

$\sigma_c = 10.0$ newton-meter

$x_3(t) \equiv 10.0$ newton-meter

TABLE 1

Results of the Monte Carlo Simulation for Speed-setting:

$x(0) = 0$ rad/sec , $x(T) = 10.0$ rad/sec , $x_3(t) \equiv 0.0$ newton-meter.

Sample Number	Energy Consumption for Case 1		Energy Consumption for Case 2		Energy Consumption for Case 3	
	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=1.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=10.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=1.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=10.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=1.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=10.0$
1	374.77	375.09	374.40	374.84	374.76	375.05
2	374.64	375.05	374.26	375.09	374.62	375.50
3	374.77	374.63	374.39	374.42	374.77	374.78
4	374.81	375.14	374.44	374.77	374.81	375.17
5	374.94	376.61	374.59	376.49	374.95	376.72
6	374.86	375.59	374.51	375.37	374.87	375.66
7	374.83	376.29	374.47	376.12	374.82	376.26
8	374.84	375.36	374.47	374.94	374.84	375.37
9	374.89	376.86	374.53	376.82	374.90	377.05

TABLE 2

Results of the Monte Carlo Simulation for Speed-control:

$x(0) = x(T) = 10.0$ rad/sec , $x_3(t) \equiv 10.0$ newton-meter.

Sample Number	Energy Consumption for Case 1		Energy Consumption for Case 2		Energy Consumption for Case 3	
	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=1.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=10.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=1.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=10.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=1.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=10.0$
1	53.70	54.56	53.64	55.56	53.697	54.591
2	53.69	55.751	53.709	57.841	53.687	55.874
3	53.68	54.093	53.602	54.222	53.684	54.110
4	53.66	53.850	53.514	53.717	53.668	53.876
5	53.724	54.572	53.500	53.818	53.727	54.644
6	53.667	53.797	53.470	53.283	53.668	53.820
7	53.777	55.840	53.710	56.762	53.772	55.855
8	53.665	53.797	53.50	53.600	53.665	53.780
9	53.640	54.530	53.39	53.670	53.644	54.644

control law for both Case 1 and Case 3 and is the simplest and most economical to implement. Tables 1 and 2 show that the control energy consumption in a given control action for each case does not differ much from one sample function to another which has the same statistical properties. Hence the expected control energy consumption may be determined by computing the arithmetic average of the energy consumptions of all sample functions used in a particular experiment.

3.6. Estimation and Control.

In practice it is never possible to measure the state vector $\underline{x}$ exactly. Let us assume that $\underline{x}$ can be measured to within an additive error which can be modeled as a Gaussian white noise with zero mean:

$$\underline{y} = h\,\underline{x} + \underline{\eta} \tag{3-50}$$

From Eqs.(3-1), (3-2), (3-50) and assumption of separating estimation and control we determine the following Kalman-Filter equations:[18]

$$\dot{\underline{\hat{x}}}(t) = A\,\underline{\hat{x}}(t) + B\,\underline{u}(t) + K(t)\,h^T N_0^{-1}[\underline{y}-h\underline{\hat{x}}] \tag{3-51}$$

$$\dot{K}(t) = A\,K(t) + K(t)\,A^T + \Gamma M_0 \Gamma^T - [K(t)h^T]\,N_0^{-1}[hK(t)] \tag{3-52}$$

where:

$$A \triangleq \begin{bmatrix} -\left(\frac{f_e}{J_e} + \frac{k_t k_b}{J_e r_a}\right) & -\left(\frac{1}{J_e}\right) & -\left(\frac{1}{J_e}\right) \\ 0 & -(\omega_0) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B \triangleq \begin{bmatrix} \left(\frac{k_t}{J_e r_a}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & \xi_w(t)\xi_w(\tau) & 0 \\ 0 & 0 & 0 \end{bmatrix} \triangleq M_0 \; \delta(t-\tau)$$

$$\varepsilon[\underline{\eta}(t)\underline{\eta}^T(\tau)] \triangleq N_0 \; \delta(t-\tau)$$

$K(t) = K^T(t) = $ A 3 x 3 matrix.

$h \triangleq$ A rectangular matrix.

There are two cases of interest:

Case 1.

$$\underline{y} = y_1(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \eta_1(t) \qquad (3\text{-}53)$$

where:

$$x(t) = x_1(t) + \alpha$$

$\eta_1(t)$ and $\xi_w(t)$ are uncorrelated since $\eta_1(t)$ evolves completely independent of $\xi_w(t)$.

Case 2.

$$\underline{y} = \begin{bmatrix} y_1(t) \\ \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \eta_1(t) \\ \\ \eta_2(t) \end{bmatrix} \qquad (3\text{-}54)$$

where:

$\eta_1(t)$, $\eta_2(t)$ and $\xi_w(t)$ are uncorrelated, since $x(t)$ and $v(t)$ are measured by two independent devices and $\eta_1(t)$ and $\eta_2(t)$ evolve completely independent of each other and $\xi_w(t)$.

It is known that the optimal expected value of the control energy is given by:

$$S(t,\hat{\underline{x}})\Big|_{t=0} = \{\tfrac{1}{2} \langle \hat{\underline{x}}, P(t)\, \hat{\underline{x}} \rangle + \underline{q}\, \hat{\underline{x}} + r(t)\}\Big|_{t=0} \qquad (3\text{-}55)$$

P(t) and $\underline{q}$ are known from Eqs.(3-13) through (3-21). Only the equation for r(t) changes:

For Case 1.

$$\dot{r}(t) - \left(\frac{c}{2}\right)\alpha^2 - (a\alpha)q_1(t) - \left(\frac{b}{2}\right)q_1^2(t) + \frac{1}{2n_0}\Big\{(k_{11}^2(t))P_{11}(t) + 2(k_{11}(t)k_{12}(t))$$

$$P_{12}(t) + 2(k_{11}(t)k_{13}(t))P_{13}(t) + 2(k_{12}(t)k_{13}(t))P_{23}(t) + (k_{12}^2(t))$$

$$P_{22}(t) + (k_{13}^2(t))\ P_{33}(t)\Big\} = 0 \qquad (3\text{-}56)$$

$$r(T) = 0$$

where:

k's are the elements of K matrix.

For Case 2.

$$\dot{r}(t) - \left(\frac{c}{2}\right)\alpha^2 - (a\alpha)q_1(t) - \frac{b}{2}\,q_1^2(t) + 0.5\Bigg\{\left(\left(\frac{1}{n_{01}}\right)(k_{11}^2(t)) + \left(\frac{1}{n_{02}}\right)(k_{12}(t)+k_{13}(t))^2\right)$$

$$P_{11}(t) + 2.0\left(\left(\frac{1}{n_{01}}\right)(k_{11}(t)k_{12}(t)) + \left(\frac{1}{n_{02}}\right)(k_{12}(t)+k_{13}(t))(k_{22}(t)+k_{23}(t))\right)$$

$$P_{12}(t) + 2.0\left(\left(\frac{1}{n_{01}}\right)(k_{11}(t)k_{13}(t)) + \left(\frac{1}{n_{02}}\right)(k_{12}(t)+k_{13}(t))(k_{23}(t)+k_{33}(t))\right)$$

$$P_{13}(t) + 2.0\left(\left(\frac{1}{n_{01}}\right)(k_{12}(t)k_{13}(t)) + \left(\frac{1}{n_{02}}\right)(k_{22}(t)+k_{23}(t))(k_{23}(t)+k_{33}(t))\right)$$

$$P_{23}(t) + \left(\left(\frac{1}{n_{01}}\right)(k_{12}^2(t)) + \left(\frac{1}{n_{02}}\right)(k_{22}(t)+k_{23}(t))^2\right)P_{22}(t) + \left(\left(\frac{1}{n_{01}}\right)k_{13}^2(t) +\right.$$

$$\left.\left(\frac{1}{n_{02}}\right)(k_{23}(t)+k_{33}(t))^2\right)P_{33}(t)\Bigg\} = 0 \qquad (3\text{-}57)$$

$$r(T) = 0.$$

Let:

$$K(0) = \begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 1 & 10 \end{bmatrix}$$

<u>For Case 1</u>.

$$\varepsilon[\underline{\eta}(t)\underline{\eta}^T(\tau)] = n_0 \, \delta(t-\tau)$$

$$\text{and} \quad n_0 = 1.0$$

<u>For Case 2</u>.

$$\varepsilon[\underline{\eta}(t)\underline{\eta}^T(\tau)] = \begin{bmatrix} n_{01} & 0 \\ 0 & n_{02} \end{bmatrix} \delta(t-\tau)$$

and

$$n_{01} = 1.0$$

$$n_{02} = 1.0$$

Equation (3-55) is then evaluated in the usual manner and the results are summarized in Table 3 where the results obtained from the exact solution are also shown for the sake of comparison.

Table 3.

Results for: T = 0.2 second.

Control Action	Energy Consumption for Exact Measurements		Energy Consumption for Case 1		Energy Consumption for Case 2	
	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=1.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=10.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=1.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=10.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=1.0$	$\left(\frac{1}{\omega_0}\right)=1.0$ $\sigma_c=10.0$
$x(0)=0$ $x(T)=10.0$ $x_3(t)\equiv 0.0$	358.95	359.79	359.0	360.16	359.0	360.53
$x(0)=x(T)=10.0$ $x_3(t)\equiv 10.0$	52.793	53.638	52.842	54.01	52.84	54.37

The results for large T's or for different measurement noise levels have the same general nature as shown in Table 3. In conclusion, from this study, it is clear that it is totally unnecessary to carry out a Monte Carlo Simulation to investigate the behaviour of the exact plant equations and to determine the energy consumption under the action of $u^*(t)$ which operates on the estimated states.

IV. EXTENSIONS TO SPECIFIC OPTIMIZATION PROBLEMS

4.1. Introduction.

In this chapter we shall formulate and solve the following optimization problems by using some of the principal results of Chapter II:

1. The minimum-energy control of electric propulsion vehicles powered by a-c induction motors,

2. The minimum-energy control of electric propulsion vehicles powered by d-c traction motors.

We shall not consider the stochastic control aspects of these problems. This is because the results of Chapter III show clearly that the minimum-energy control law obtained from the study of a deterministic linear model will perform almost in an optimal fashion under the actual environmental conditions.

4.2. Formulation of Problem 1.

Plant:

$$\dot{x}(t) = -\left(\frac{f_e}{J_e}\right)x(t) - \left(\frac{1}{J_e}\right)v(t) + \left(\frac{k_t}{J_e}\right)u(t) \qquad (4\text{-}1)$$

where:

$\dot{x}(t)$, $x(t)$, $v(t)$, f_e, J_e are as defined in Section 2.4

$k_t \triangleq$ Motor torque constant, [newton-meter/rad/sec]

$u(t) \triangleq$ Slip-frequency; control, [rad/sec].

In the derivation of Eq.(4-1) it has been assumed that the torque of a 3-phase squirrel-cage induction motor is controlled by controlling its slip frequency while keeping the ratio of the stator voltage to stator frequency constant.*[23,24,25,26,27,28,29,30,31,32] Under these conditions the induction motor behaves like a separately excited and armature-controlled d-c motor.[47] Note that, without this assumption the solution of a well-formulated minimum-energy control problem for an a-c drive system is a very difficult task. This is because the mechanical and electrical parts of such a control system are described by nonlinear differential equations and also the corresponding minimum-energy control problem, in general, is associated with many optimal trajectories depending on the number of possible combinations between the desired speed and disturbance torque entering into the system. At present there exists no satisfactory theoretical approach in the determination of control law for such a control problem.

Performance Index:

Performance index is selected to be:

$$E = \int_0^T u^2(t)\, dt \quad . \tag{4-2}$$

The integrand of Eq.(4-2) represents the rotor copper losses since the rotor rms current is linearly proportional to the slip-frequency over a small region about zero slip-frequency. Note that zero to maximum positive or negative torque can be obtained by varying the slip-frequency between zero and some small positive value or between zero and some

---

* Except at very low stator frequencies.

small negative value respectively. The minimization of $\int_0^T u^2(t)\,dt$ means a reduction in the rotor copper energy losses while satisfying a given load condition. In other words, most of the energy supplied to the machine in the motoring mode is converted into the desired mechanical energy with very little rotor losses or most of the mechanical energy available at the motor shaft is converted into electric energy in the generating mode and transferred back to the battery with very little rotor losses.

The set of boundary conditions to be satisfied by the state variable $x(t)$ are the same as given in Section 2.4.

In practice $u(t)$ is forced to satisfy the following inequality constraint:

$$-U \le u(t) \le U \quad . \tag{4-3}$$

This is because whenever $|u(t)| > U$ the motor torque is no longer linearly proportional to $u(t)$ and the motor actually generates less torque than its maximum rated torque which is obtained when $|u(t)| = U$. For mathematical tractability and for reasons of simplicity of the implementation of the resulting optimal solution let us assume that $u(t)$ is unconstrained.

4.2.1. Solution of Problem 1.

Using the same approach as outlined in Section 2.4.1 under the assumption of no constraints on $u(t)$, the following results are obtained:

A. Open-Loop Solution.

The optimal state trajectory $x(t)$ is determined to be:

$$x(t) = \gamma\left\{e^{-at} - \left(\frac{e^{-aT}}{\text{Sinh } aT}\right)\text{Sinh } at\right\} + \alpha\left\{e^{-at} + \left(\frac{1-e^{-aT}}{\text{Sinh } aT}\right)\text{Sinh } at\right\}$$
$$- \beta\left\{\frac{1}{aJ_e}\left(1-e^{at}\right) - \frac{1}{aJ_e}\left(\frac{1-e^{-aT}}{\text{Sinh } aT}\right)\text{Sinh } at\right\} \tag{4-4}$$

where:

$$a \triangleq \left(\frac{f_e}{J_e}\right)$$

Note that in Eq.(4-4)

$$\dot{x}(t) = 0 \quad \text{at} \quad t = \frac{T}{2} \quad . \tag{4-5}$$

The lagrange multiplier function $\lambda(t)$ is determined to be:

$$\lambda(t) = e^{at}\left\{\gamma\left(\frac{ae^{-aT}}{b \text{ Sinh } aT}\right) - \frac{1}{b}\left(\frac{1-e^{-aT}}{\text{Sinh } aT}\right)\left(a\alpha + \frac{\beta}{J_e}\right)\right\} \tag{4-6}$$

where:

$$b \triangleq \frac{1}{2}\left(\frac{k_t}{J_e}\right)^2$$

And the optimal control function $u^*(t)$ is determined to be:

$$u^*(t) = -\gamma\left\{\left(\frac{ak_t}{2bJ_e}\right)\left(\frac{e^{a(t-T)}}{\text{Sinh } aT}\right)\right\} + \left\{\left(\frac{ak_t}{2bJ_e}\right)\left(\frac{e^{at}-e^{a(t-T)}}{\text{Sinh } aT}\right)\right\}\alpha + \left\{\left(\frac{k_t}{2J_e^2 b}\right)\left(\frac{e^{at}-e^{a(t-T)}}{\text{Sinh } aT}\right)\right\}\beta \quad (4\text{-}7)$$

Equations (4-5), (4-6) and (4-7) hold for the general control action of speed-control and speed setting. However, in the above equations; if $v(0-) \neq v(0) = v(T) = \beta$, $\gamma = 0$, speed-control action results and if $v(0-) = v(0) = v(T) = \beta$, $\gamma \neq 0$, speed-setting action results.

B. Closed-Loop Solution.

The feedback control-law is determined to be:

$$u^*(t) = \left(\frac{2f_e}{k_t}\right)\left\{\left(\alpha + \frac{1}{J_e a}\beta\right)\left(\frac{e^{a(T-t)}-1}{e^{2a(T-t)}-1}\right) + (\alpha - x(t))\left(\frac{1}{e^{2a(T-t)}-1}\right)\right\} \quad (4\text{-}8)$$

C. A Suboptimal Control Law.

For all control actions of interest:

$$0 < \left\{\frac{e^{a(T-t)}-1}{e^{2a(T-t)}-1}\right\} \leq 0.5 \quad .$$

For T sufficiently small we can assume that:

$$\left\{\frac{e^{a(T-t)}-1}{e^{2a(T-t)}-1}\right\} = 0.5 \quad (4\text{-}9)$$

Substituting Eq.(4-9) into Eq.(4-8) yields:

$$u_{s0}(t) = \left(\frac{f_e}{k_t}\right)\alpha + \left(\frac{1}{k_t}\right)\beta + \left\{\frac{2\left(\frac{f_e}{k_t}\right)}{(e^{2a(T-t)}-1)}\quad(\alpha - x(t))\right\} \quad (4\text{-}10)$$

It can be easily shown that $u_{s0}(t)$ yields $x(T) = \alpha$ for any control action of interest. When the time-varying gain in Eq.(4-10) is made zero for $t \geq T$, then, $\dot{x}(t) \equiv 0$.

4.2.2. Salient Features of the Solution of Problem 1.

Salient features of the optimal and suboptimal solutions are obtained by simulating the appropriate equations on the digital computer. The results are shown in Figures 27 through 30. For $t \geq T$, it has been assumed that all time varying gains are made zero by means of an auxiliary controller. The following system parameters are used:

J_e = 1.42 newton-meter/rad/sec^2

f_e = 0.825 newton-meter/rad/sec

k_t = 3.25 newton-meter/rad/sec.

The considerations necessary for the best choice of T are exactly the same as given in Section 2.4.8.

4.3. Formulation of Problem 2.

Plant:

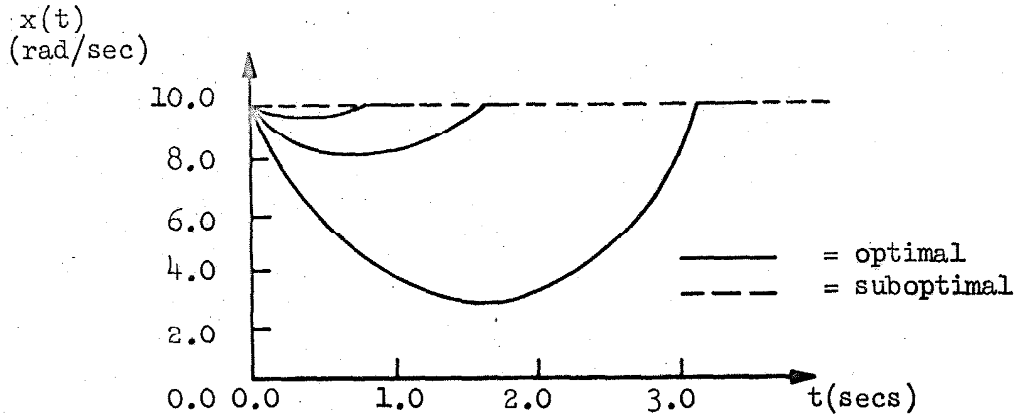
$$\dot{x}_1(t) = -\left(\frac{f_e}{J_e}\right)x_1(t) + \left(\frac{m_{af}}{J_e}\right)x_2^2(t) - \left(\frac{1}{J_e}\right)v(t) \tag{4-11}$$

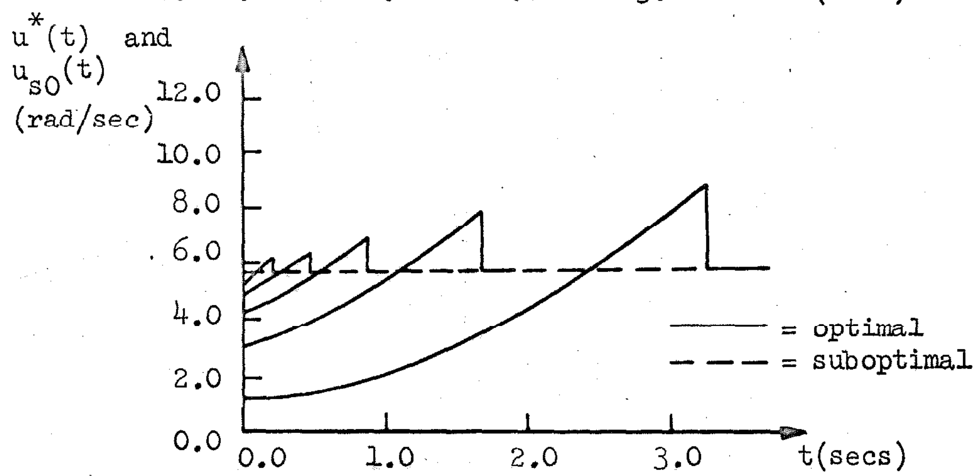
$$\dot{x}_2(t) = -\left(\frac{r_e}{\ell_e}\right)x_2(t) - \left(\frac{m_{af}}{\ell_e}\right)x_2(t)\, x_1(t) + \left(\frac{1}{\ell_e}\right)u(t) \tag{4-12}$$

where:

$m_{af} \triangleq$ Mutual inductance between the field and armature circuits, [henry]

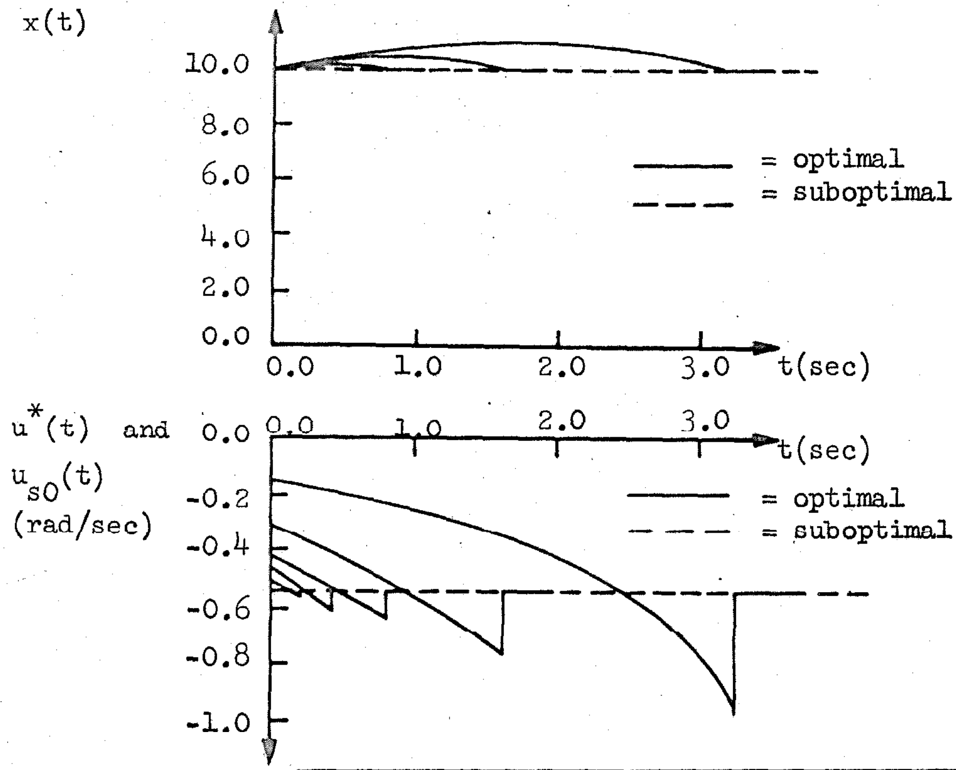
$\ell_e \triangleq$ Sum of the field and armature inductances, [henry]





T (sec)	Performance Index	
	Optimal	Suboptimal
0.2	6.2921	6.32
0.4	12.52	12.64
0.8	24.66	25.28
1.6	46.67	50.56
3.2	77.81	101.12

Figure 27. Results for:
α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = 10.0 newton-meter ,
γ = 0.0 rad/sec (speed-control) , $u^*(0-)$ = 2.54 rad/sec.

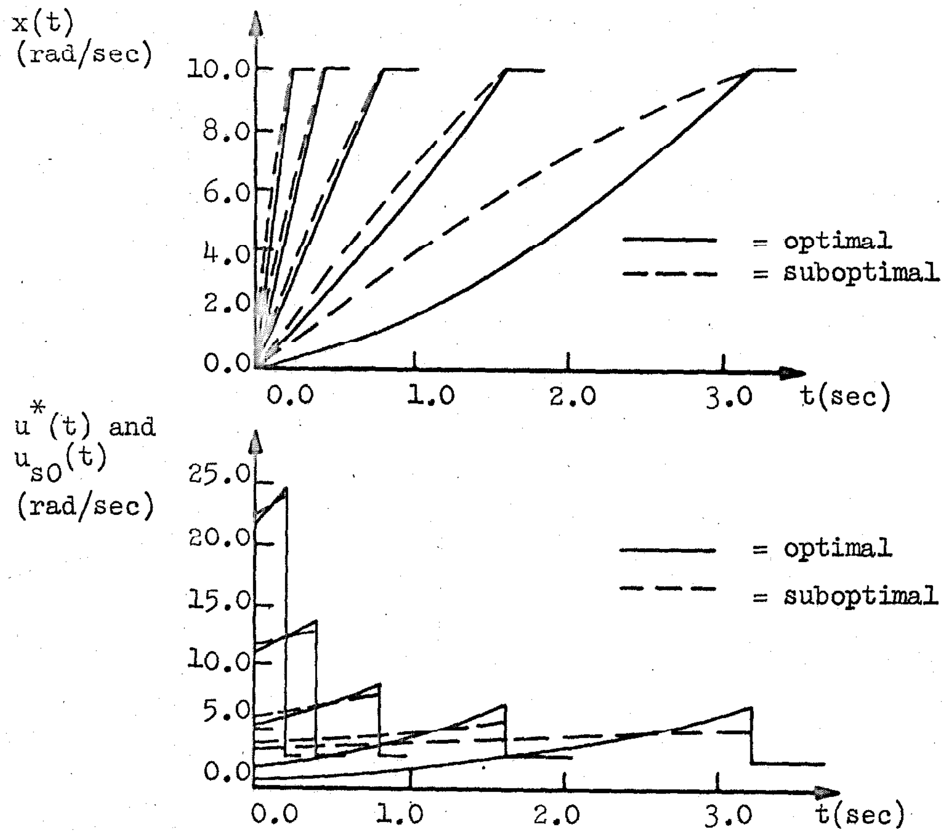


T (sec)	Performance Index	
	optimal	suboptimal
0.2	0.0578	0.0600
0.4	0.11519	0.1200
0.8	0.2268	0.2400
1.6	0.42914	0.480
3.2	0.71554	0.960

Figure 28. Results for:

α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = -10.0 newton-meter ,

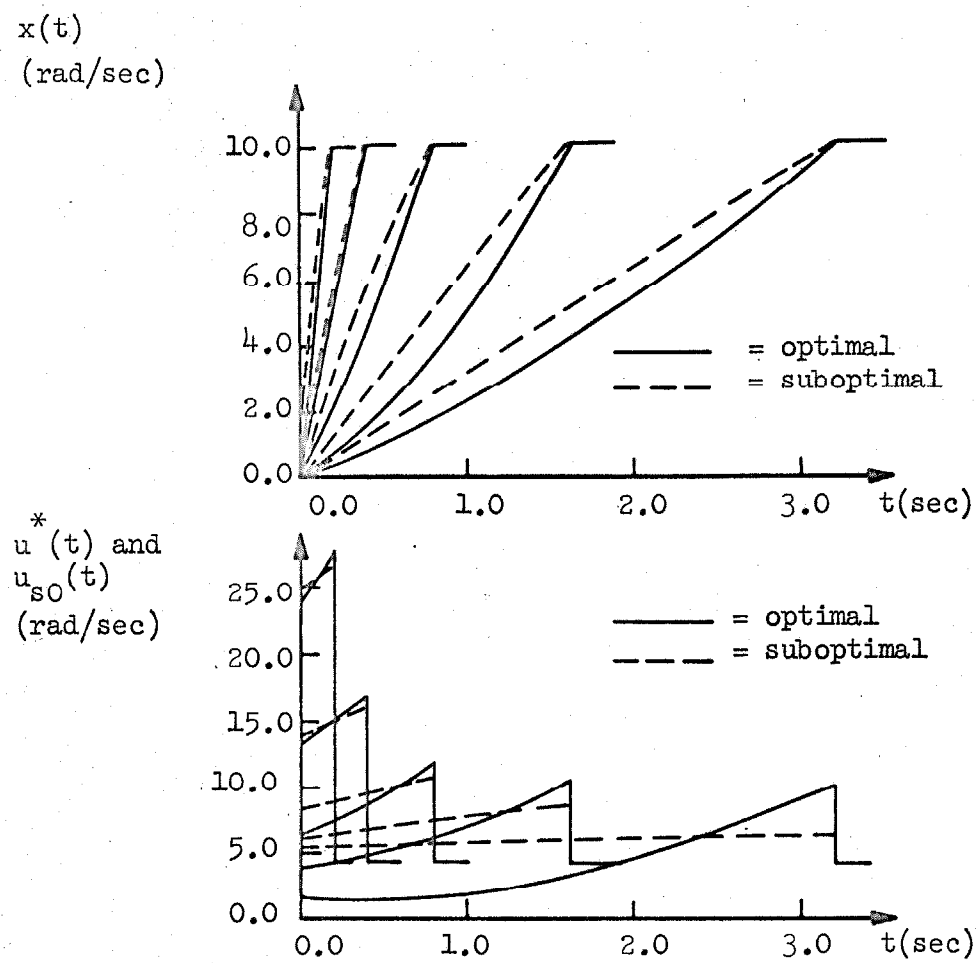
γ = 0.0 rad/sec (speed-control) , $u^*(0-)$ = 2.54 rad/sec.



T (sec)	Performance Index	
	Optimal	Suboptimal
0.2	106.85	106.85
0.4	59.534	59.547
0.8	36.478	36.576
1.6	26.033	26.755
3.2	22.314	26.925

Figure 29. Results for:

α = 10.0 rad/sec , v(0-) = 0.0 rad/sec , β = 0.0 newton-meter ,
γ = -10.0 rad/sec (acceleration) , u^*(0-) = 0.0 rad/sec.



T (sec)	Performance Index	
	Optimal	Suboptimal
0.2	137.15	137.18
0.4	93.22	93.326
0.8	76.757	77.324
1.6	78.244	81.97
3.2	91.345	114.23

Figure 30. Results for: α = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = 10.0 newton-meter , γ = -10.0 rad/sec (acceleration) , $u^*(0-)$ = 0.0 rad/sec.

$r_e \triangleq$ Sum of the field and armature resistances, [ohms].

and all other variables and parameters are the same as defined in Section 2.2.

Equations (4-11) and (4-12)[13] describe the dynamical behaviour of an electric propulsion vehicle powered by a d-c traction (series) motor. Note that in practice the lowest value of $u(t)$ is zero. Hence from Eq.(4-12) it is clear that $x_2(t)$ can not be made negative and therefore re-generative braking action can not be obtained with a series motor. The performance index for this optimization problem is selected to be:

$$E_a = \int_0^T \left(\frac{u^2(t) - m_{af} x_2(t) x_1(t) u(t)}{r_e} \right) dt \quad . \tag{4-13}$$

Note that from Eqs.(2-3) and (4-12):

$$E = \int_0^T \left(\frac{u^2(t) - m_{af} x_2(t) x_1(t) u(t)}{r_e} \right) dt - \left(\frac{\ell_e}{r_e} \right) \int_0^T u(t) \dot{x}_2(t) dt \tag{4-14}$$

For a practical system, $\dot{x}_2(t)$ is always finite. Let:

$$|u(t)\dot{x}_2(t)| \leq M \tag{4-15}$$

Furthermore, let us assume that the major contribution in $\int_0^T u(t)\dot{x}_2(t)dt$ comes in a time interval; $0 \leq t \leq \frac{\ell_e}{r_e}$. Hence,

$$E \cong \int_0^T \left(\frac{u^2(t) - m_{af} x_2(t) x_1(t) u(t)}{r_e} \right) dt - \left(\frac{\ell_e}{r_e} \right) \int_0^{\left(\frac{\ell_e}{r_e}\right)} u(t) \dot{x}_2(t) dt \tag{4-16}$$

Let:

$$R_E = \left(\frac{\ell_e}{r_e}\right) \int_0^{\left(\frac{\ell_e}{r_e}\right)} u(t)\dot{x}_2(t)dt \tag{4-17}$$

Then:

$$|R_E| \le \left(\frac{\ell_e}{r_e}\right) \int_0^{\left(\frac{\ell_e}{r_e}\right)} |u(t)\dot{x}_2(t)|dt = \left|\left(\frac{\ell_e}{r_e}\right)^2\right| M \tag{4-18}$$

Equation (4-18) means:

$$R_E = O(\epsilon^2) \tag{4-19}$$

where: $\epsilon \triangleq \left(\frac{\ell_e}{r_e}\right)$

Thus:

$$E \cong \int_0^T \left(\frac{u^2(t)-m_{af}x_2(t)x_1u(t)}{r_e}\right) dt + O(\epsilon^2) \tag{4-20}$$

But all practical systems of interest:

$$O(\epsilon^2) \cong 0.0 \tag{4-21}$$

Therefore minimization of Eq.(4-13) is approximately equivalent to the minimization of system energy consumption. The integrand of Eq. (4-13) represents the electric power flow which can flow only from the battery into the motor circuit.

The set of boundary conditions to be satisfied by the state variables $x_1(t)$ and $x_2(t)$ for three cases of control action are the same as given in Section 2.2.

Assume for all cases of interest the optimal solution $u^*(t)$ satisfies the following inequality constraint:

$$0 \le u^*(t) \le U. \tag{4-22}$$

But for reasons of simplicity let us assume that u(t) is unconstrained.

4.3.1. Solution of Problem 2.

Define the Hamiltonian function H as:

$$H = \left(\frac{u^2(t)}{r_e}\right) - \frac{m_{af}}{r_e} x_2(t)\, x_1(t)u(t) + \lambda_1(t)\left\{-\left(\frac{f_e}{J_e}\right)x_1(t) + \left(\frac{m_{af}}{J_e}\right)x_2^2(t) - \left(\frac{1}{J_e}\right)\beta\right\}$$
$$+ \lambda_2(t)\left\{-\left(\frac{r_e}{\ell_e}\right)x_2(t) - \left(\frac{m_{af}}{\ell_e}\right)x_2(t)x_1(t) + \left(\frac{1}{\ell_e}\right)u(t)\right\} \tag{4-23}$$

where:

$\lambda_1(t),\ \lambda_2(t) \triangleq$ Lagrange multipliers.

Using Pontryagin's Maximum Principle yields:

$$u^*(t) = 0.5\, m_{af}x_2(t)x_1(t) - 0.5\left(\frac{r_e}{\ell_e}\right)x_4(t) \tag{4-24}$$

$$H^* = -\left(\frac{m^2_{af}}{4r_e}\right)x_2^2(t)x_1^2(t) + x_3(t)\left(-\left(\frac{f_e}{J_e}\right)x_1(t) + \left(\frac{m_{af}}{J_e}\right)x_2^2(t) - \left(\frac{1}{J_e}\right)\beta\right) +$$
$$x_4(t)\left(-\left(\frac{r_e}{\ell_e}\right)x_2(t) - \left(\frac{m_{af}}{2\ell_e}\right)x_2(t)x_1(t)\right) - \left(\frac{r_e}{4\ell_e^2}\right)x_4^2(t) \tag{4-25}$$

where:

$$x_3(t) \triangleq \lambda_1(t)$$

$$x_4(t) \triangleq \lambda_2(t) \quad .$$

A. Open-Loop Solution.

From Eq.(4-25) the following canonic equations are obtained:

$$\dot{x}_1(t) = -\left(\frac{f_e}{J_e}\right)x_1(t) + \left(\frac{m_{af}}{J_e}\right)x_2^2(t) - \left(\frac{1}{J_e}\right)\beta \tag{4-26}$$

$$\dot{x}_2(t) = -\left(\frac{r_e}{\ell_e}\right)x_2(t) - \left(\frac{r_e}{2\ell_e^2}\right)x_4(t) - \left(\frac{m_{af}}{2\ell_e}\right)x_2(t)x_1(t) \tag{4-27}$$

$$\dot{x}_3(t) = \left(\frac{f_e}{J_e}\right)x_3(t) + \left(\frac{m_{af}^2}{2r_c}\right)x_1(t)x_2^2(t) + \left(\frac{m_{af}}{2\ell_e}\right)x_2(t)x_4(t) \tag{4-28}$$

$$\dot{x}_4(t) = \left(\frac{r_e}{\ell_e}\right)x_4(t) + \left(\frac{m_{af}^2}{2r_e}\right)x_2(t)x_1^2(t) - \left(\frac{2m_{af}}{J_e}\right)x_2(t)x_3(t) + \left(\frac{m_{af}}{2\ell_e}\right)x_1(t)x_4(t) \tag{4-29}$$

Let us assume that a unique solution of Eqs.(4-26) through (4-29) satisfying certain boundary conditions exists. In (k+1)st stage of quasilinearization method[34]:

$$\underline{\dot{x}}^{(k+1)} = A^{(k)}(t)\ \underline{x}^{(k+1)} + \underline{d}^{(k)} \tag{4-30}$$

where:

$$\underline{x} \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$A(t) \triangleq$ coefficient matrix of the system of Eqs.(4-26) through (4-29)

$\underline{d} \triangleq$ Forcing vector of the system of Eqs.(4-26) through (4-29).

The general solution of Eq.(4-30) is given by

$$\underline{x}^{(k+1)} = \Phi^{(k+1)}(t)\ \underline{c}^{(k+1)} + \underline{p}^{(k+1)} \tag{4-31}$$

where:

$\Phi^{k+1}(t)$ satisfies the matrix differential equation:

$$\dot{\Phi}^{k+1}(t) = A^{k}(t)\ \Phi^{k+1}(t) \tag{4-32}$$

with $\Phi^{k+1}(0) = I =$ Identity matrix.

$\underline{p}^{(k+1)}$ satisfies the vector differential equation:

$$\dot{\underline{p}}^{(k+1)} = A^{k}(t)\ \underline{p}^{(k+1)} + \underline{d}^{k} \tag{4-33}$$

with $\underline{p}^{k+1}\Big|_{t=0} = \underline{0} =$ Null vector.

$\underline{c} \triangleq$ constant vector.

The following digital computer algorithm is used in the determination of the open-loop trajectories:

1. Assume $x_3(0)$, $x_4(0)$ and obtain $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ by integrating Eqs.(4-26) through (4-29).
2. Determine $A(t)$ and $\underline{d}$ using $\underline{x}$ obtained in 1.
3. Solve Eqs.(4-32) and (4-33).
4. From Eq.(4-31) determine constant $\underline{c}$ in such a way that the boundary conditions on $x_1(t)$ and $x_2(t)$ are satisfied, i.e.:

$$x_1(0) = x_{01} \quad , \quad x_1(T) = \alpha_1$$

$$x_2(0) = x_{02} \quad , \quad x_2(T) = \alpha_2 \quad .$$

5. From Eq.(4-31) obtain the solution of $\underline{x}$ for the first iteration.
6. From Eq.(4-24) determine the value of $u^*(t)$ for the first iteration.
7. Use the value of $\underline{x}$ obtained in 5, in 2 and repeat the same procedure as above to determine $\underline{x}$ and $u^*(t)$ for the second iteration and so on until the following set of iteration stopping conditions are satisfied.

(i) Stop the iteration if the number of iterations are greater than ten.

(ii) Stop the iteration if $\int_0^T |u^*(t)^{k+1} - u^*(t)^k| dt \le \epsilon_s$ where ϵ_s is a suitably selected small number.

B. A Suboptimal Closed-Loop Solution.

Since the plant is described by two nonlinear differential equations and the performance index contains a nonlinear term in its integrand and also there are infinite number of trajectories associated with this optimization problem the solution of the Bellman-Hamilton-Jacobi equation for this optimization problem is a very difficult task. Therefore, let us assume a suboptimal feedback solution of the following form:

$$u_{s0}(t) = m_{af} x_2(t) x_1(t) + r_e x_2(t) + \ell_e \sqrt{d} \left(\frac{\alpha_2 - x_2(t)}{\operatorname{Sinh} \sqrt{d}\ (t_f - t)} \right) \qquad (4\text{-}34)$$

where:

d is defined as in Chapter II and for the d-c traction motor it is evaluated at the rated current, i.e. $m_{af}x_2(t)\big|_{rated} = k_t = k_b$,

t_f is the time at which $x_2(t_f) = \alpha_2$ and remains at that value until $x_1(T) \cong \alpha_1$.

The motivation for the choice of Eq.(4-34) is as follows:

$$\lim_{t\to t_f} (u_{s0}(t)) = m_{af}\alpha_2 x_1(t) + r_e\alpha_2 + \ell_e\sqrt{d}\lim_{t\to t_f}\left(\frac{\alpha_2 - x_2(t)}{\mathrm{Sinh}\sqrt{d}\ (t_f-t)}\right) \quad (4\text{-}35)$$

If the time-varying gain is made zero at $t = t_f$, from Eq.(4-35):

$$u_{s0}(t) = m_{af}\alpha_2 x_1(t) + r_e\alpha_2 \quad ; \quad t_f \le t \le T \ . \quad (4\text{-}36)$$

Substitution of Eq.(4-36) into Eq.(4-12) yields:

$\dot{x}_2(t) \equiv 0$ for $t \ge t_f$ until a new disturbance comes into the system.

Note that when Eq.(4-34) is substituted into Eq.(4-12) the following equation is obtained:

$$\dot{x}_2(t) = -\left(\frac{\sqrt{d}}{\mathrm{Sinh}\sqrt{d}\ (t_f-t)}\right) x_2(t) + \left(\frac{\sqrt{d}}{\mathrm{Sinh}\sqrt{d}\ (t_f-t)}\right)\alpha_2 \quad (4\text{-}37)$$

The general solution of Eq.(4-37) is given by

$$x_2(t) = x_2(0)\left(\frac{\mathrm{Cosh}\sqrt{d}\ (t_f-t)-1}{\mathrm{Cosh}\sqrt{d}\ (t_f-t)+1}\right)^{1/2}\left(\frac{\mathrm{Cosh}\sqrt{d}\ t_f+1}{\mathrm{Cosh}\sqrt{d}\ t_f-1}\right)^{1/2}$$
$$+ \alpha_2\left(1.0-\left(\mathrm{Cosh}\sqrt{d}(t_f-t)-1.0\right)^{1/2}\frac{\mathrm{Sinh}\sqrt{d}\ t_f}{(\mathrm{Cosh}\sqrt{d}\ (t_f-t)+1)^{1/2}(\mathrm{Cosh}\sqrt{d}\ t_f-1)}\right) \quad (4.38)$$

where:

$$x_2(t)\Big|_{t=0} = x_{02} \quad \text{and} \quad x_2(t)\Big|_{t=t_f} = \alpha_2 \quad \text{as expected.}$$

Thus when $u_{s0}(t)$ given by Eq.(4-34) is applied into the plant and t_f is selected appropriately, then $x_2(t_f) = \alpha_2$ in $t_f \le t \le T$. In general, $T-t_f \gg t_f$ since the mechanical time constant of the vehicle is much greater than its electrical time constant. The suboptimal feedback control law $u_{s0}(t)$ depends on $x(t)$ in $t_f \le t \le T$. Substituting $x_2(t) = \alpha_2$ in Eq.(4-11) yields:

$$\dot{x}_1(t) + \left(\frac{f_e}{J_e}\right)x_1(t) = \left(\frac{f_e}{J_e}\right)\alpha_1 \qquad t_f \le t \le T \tag{4-39}$$

The general solution of (4-39) is determined to be:

$$x_1(t) = x_1(t_f)e^{-\left(\frac{f_e}{J_e}\right)(t-t_f)} + \alpha_1\left(1-e^{-\left(\frac{f_e}{J_e}\right)(t-t_f)}\right) \tag{4-40}$$

Hence as $t \gg t_f \quad x_1(t) \to \alpha_1$.

Note that for $t_f \le t \le T$, Eq.(4-12) becomes an algebraic equation and Eq.(4-11) becomes a linear equation under the action of $u_{s0}(t)$. The suboptimal control law $u_{s0}(t)$, as shown above satisfies the boundary conditions $x_1(T) \hat{=} \alpha_1$ and $x_2(T) = x_2(t_f) = \alpha_2$ if the time varying gain in Eq.(4-34) is turned on at $t = 0$ and is turned off at $t = t_f$ until a new disturbance comes into the system.

4.3.2. Salient Features of the Solution of Problem 2.

Salient features of the optimal and suboptimal solutions are determined by simulating the appropriate equations on the digital computer. The results are shown in Figures 31 and 32. For $t \geq t_f$, it has been assumed that the time varying gain in Eq.(4-34) is turned off at $t = t_f$ by means of an auxiliary controller. The following system parameters are used:

$$J_e = 1.42 \text{ newton-meter/rad/sec}^2$$

$$f_e = 0.825 \text{ newton-meter/rad/sec}$$

$$\ell_e = 0.010 \text{ henry}$$

$$r_e = 1.0 \text{ ohm}$$

$$m_f = 0.040 \text{ henry}$$

$$t_f = \frac{\ell_e}{r_e} = 0.010 \text{ second.}$$

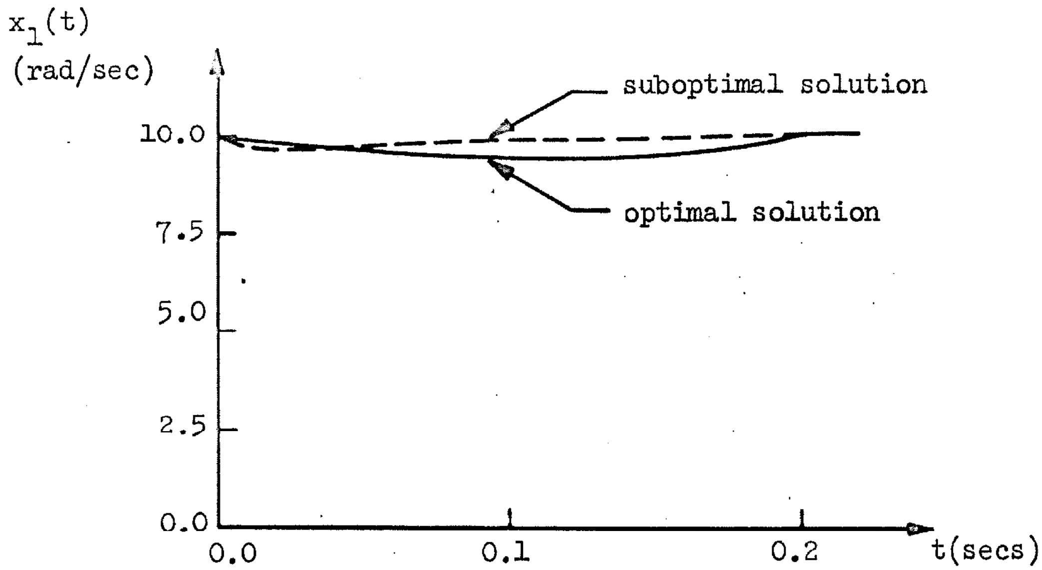
In Figure 31; the energy consumed by the suboptimal controller is 129.263 watt-seconds which is slightly higher than that consumed by the optimal controller. The evaluation of Eq.(4-14) yields:

$$E = 123.22 \text{ watt-seconds.}$$

Hence the Eq.(4-13) is a very closed approximation of Eq.(4-20). For Figure 32; the following corresponding results are obtained: Energy consumed by the suboptimal controller is 57.87 watt-seconds, and

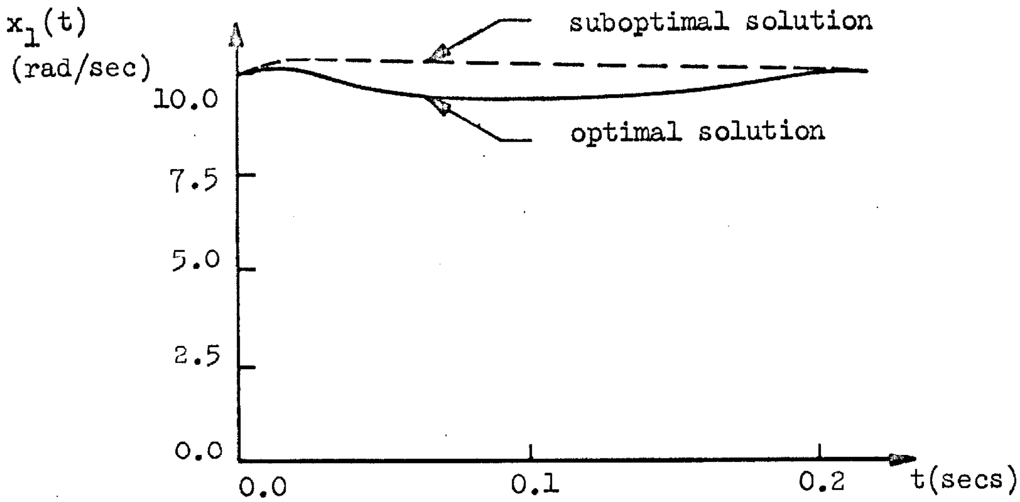
$$E = 55.81 \text{ watt-seconds}$$

Therefore, all the assumptions we made above are justified completely for the two cases of control action considered.



Iteration Number	Performance Index	$\int_0^T \lvert u^*(t)^{(k+1)} - u^*(t)^{(k)} \rvert dt$
1	725.11	5547.2
2	288.04	3174.0
3	213.90	2398.7
4	143.17	741.92
5	128.52	312.71
6	126.79	24.44
7	~ 126.79	0.3854

Figure 31. Results for: α_1 = 10.0 rad/sec , v(0-) = 0.0 newton-meter , β = 10.0 newton-meter , α_2 = 21.40 amperes.



Iteration Number	Performance Index	$\int_0^T \lvert u^*(t)^{(k+1)} - u^*(t)^{(k)} \rvert dt$
1	62.63	705.19
2	55.19	130.96
3	54.82	6.13
4	54.82	0.05568

Figure 32. Results for: α_1 = 10.0 rad/sec , v(0-) = 10.0 newton-meter , β = 0.0 newton-meter , α_2 = 14.30 amperes.

V. IMPLEMENTATION

5.1. Introduction.

In this chapter we shall comment briefly on some design techniques which may be useful in the practical implementation of the suboptimal control laws for the problems studied in Chapters II and IV which exhibit the best overall characteristics with respect to other control laws.

5.2. The Practical Implementation of the Suboptimal Control Law for the Separately-Excited and Armature-Controlled d-c Motor Drive System.

Re-write Eq.(2-79) as follows:

$$u_{s0}(t) = \left(\frac{k_b}{2} + \frac{k_t a}{2J_e b}\right)x(t) + \left(\frac{k_t}{2J_e{}^2 b}\right)v(t) + \left(\frac{k_t\sqrt{d}}{2J_e b}\right)\left(\frac{1}{\mathrm{Sinh}\sqrt{d}\ (T-t)}\right)(\alpha - x(t)) \tag{5-1}$$

where:

$v(t) \triangleq$ Disturbance torque, [newton-meter].

The current speed $x(t)$ can be measured by a tachometer which produces a voltage proportional to the speed and the disturbance torque $v(t)$ can be measured by a pendulum-type device[33] which measures the terrain slope with respect to horizontal. A detailed analysis shows that the following functional relationship between the terrain slope $\phi(t)$ and the disturbance torque $v(t)$ for a wheeled-vehicle with rubber tires:

$$v(t) = \left(w_v \mathrm{Sin}\ \phi(t) + f_1\right)\left(\frac{r_w c_F}{k_G}\right) \tag{5-2}$$

where:

w_v = Total weight of the vehicle, [lbs]

$\phi(t)$ = Slope angle of the terrain with respect to horizontal, [degrees]

f_1 = Tire resistance at zero speed [lbs]

f_1 depends on the rubber material from which tires are made and also on the degree of inflation of tires,

r_w = effective radius of the wheel-tire assembly, [ft]

k_G = Gear ratio

c_F = Conversion factor from ft.-lb. to newton-meter

The time-varying gain $\left(\frac{1}{\text{Sinh}\sqrt{d}\,(T-t)}\right)$ can be realized by a simple electronic system whose block diagram is shown in Figure 33. A brief description of the function of each block in Figure 33 follows: Input circuit furnishes a voltage v_D; which is proportional to the desired response time setting, for the electronic delay circuit whose output voltage v_z varies as a linear function of time in $(0, T)$. This voltage v_z is then applied to a reverse biased zener diode whose current I_z increases slightly as v_z increases and becomes very large when v_z becomes slightly greater than the rated zener voltage. The interval of time over which $v_z >$ rated zener voltage is extremely small and corresponds to the interval of time as $t \to T$ and the sudden increase in I_z corresponds to $\lim_{t \to T}\left(\frac{1}{\text{Sinh}\sqrt{d}\,(T-t)}\right) \to \infty$. Since $\left.\left(\frac{1}{\text{Sinh}\sqrt{d}\,(T-t)}\right)\right|_{t=0}$ is larger for smaller T's it is necessary to modify the bias setting in the preamplifier whose output is applied to a high

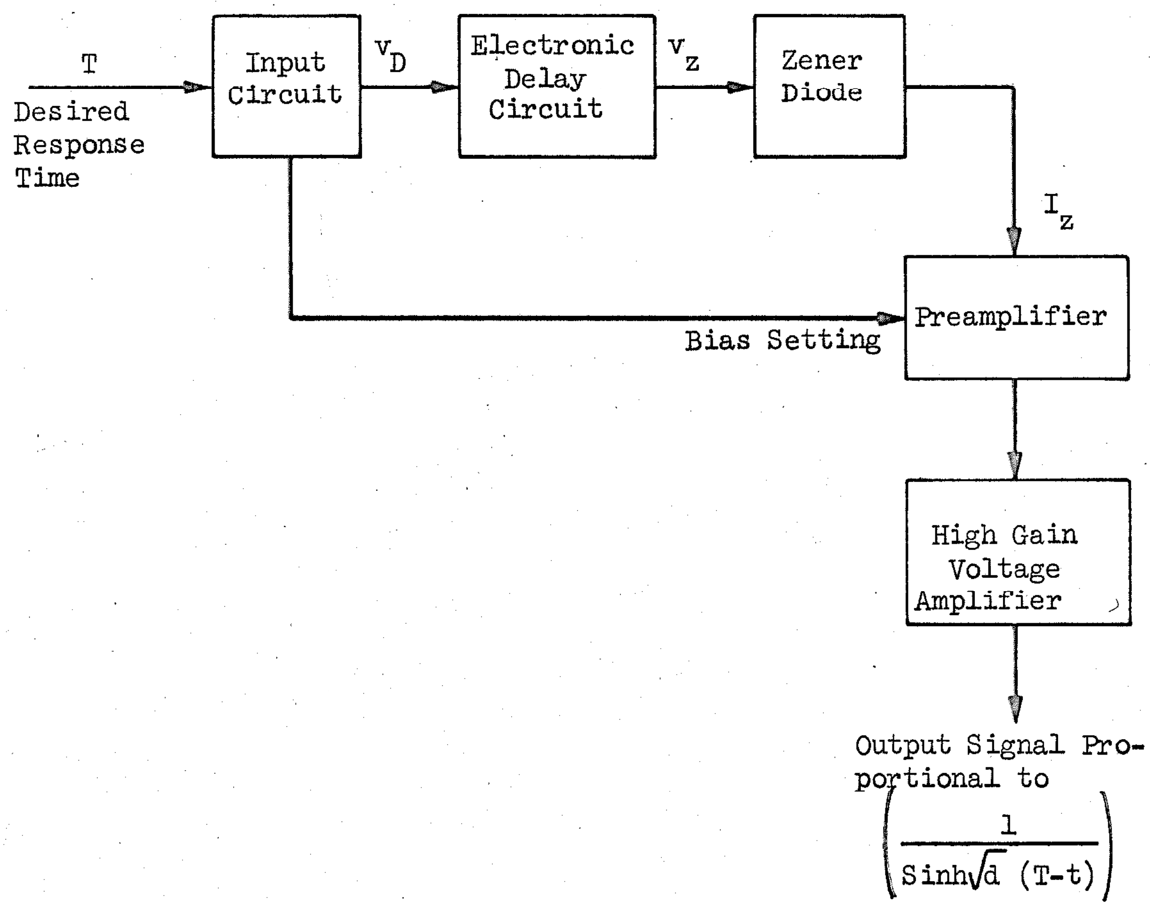


Figure 33. Engineering Realization of the Time-Varying Gain Function of the Control Law.

gain voltage amplifier which in turn yields a voltage at its output which is proportional to $\left(\frac{1}{\text{Sinh}\sqrt{d}\,(T-t)}\right)$.

Desired speed is normally fed into the system by positioning a throttle mechanism. Each throttle position corresponds to a particular desired speed which in turn is proportional to a particular voltage. Let:

$$k_1 \triangleq \left(\frac{\text{Current speed}}{\text{Voltage proportional to the current speed}}\right), \qquad (5\text{-}3)$$

$$k_2 \triangleq \left(\frac{\text{Desired speed}}{\text{Voltage proportional to the desired speed}}\right) \qquad (5\text{-}4)$$

where:

k_1 and k_2 are constants and $k_1 \equiv k_2$,

$$k_3 \triangleq \left(\frac{\text{Disturbance torque}}{\text{Voltage proportional to disturbance torque}}\right) \qquad (5\text{-}5)$$

Note that from Eq.(5-2) it is clear that $\phi(t)$ and $v(t)$ are related to each other by a nonlinear relationship. Here it is necessary to design a circuit which will make measurements on $\phi(t)$ and will yield a signal proportional to $v(t)$ in such a way that k_3 is constant,

$$k_4 = \left(\frac{\text{Actual value of the time-varying gain}}{\text{Voltage proportional to the actual value of the time-varying gain}}\right) \qquad (5\text{-}6)$$

where k_4 is constant.

Equation (5-1) represents the value of the control voltage, at each instant of time in $(0,T)$, to be applied into the armature circuit by modulating the fixed battery voltage. The problem is how do we obtain the armature voltage according to Eq.(5-1) by using the measurements on $x(t)$, $v(t)$, $\dot{\alpha}$ and generating the time-varying gain $\left(\frac{1}{\sinh\sqrt{d}\,(T-t)}\right)$ by means of analog signals? One way of accomplishing this task is as follows:

1. Use an appropriate pulse-width modulation technique[35,36] to control the fixed battery voltage* through a d-c to d-c silicon controlled rectifier (SCR) converter. In this way the armature voltage may be controlled from a very low value; which may be a few percent of the battery voltage, to a high value; which may be equal to the battery voltage. In order to obtain the armature voltage described by Eq.(5-1) it is necessary to supply the firing signals to SCR gates in the d-c to d-c converter in such a way that the converter acts like a voltage amplifier with a voltage gain of K_A.

2. The bias signal into the firing circuits of the d-c to d-c converter can be made equal to the sum of the following voltages:

$$\left\{\left(\frac{k_b}{2} + \frac{k_t a}{2J_e b}\right)\left(\frac{k_1}{K_A}\right)\right\} \text{ x voltage output of the tachogenerator,} \tag{5-7}$$

$$\left\{\left(\frac{k_t\sqrt{d}}{2J_e b}\right)\left(\frac{k_1 k_4}{K_A}\right)\right\} \text{ x voltage output of the device generating the time-varying gain x error voltage,} \tag{5-8}$$

---

* Assume that the battery voltage reduction is negligible over a time interval during which many control actions are performed.

$$\left\{\left(\frac{k_t}{2J_e{}^2 b}\right)\left(\frac{k_3}{K_A}\right)\right\} \times \text{voltage output of the device measuring the disturbance torque.} \qquad (5\text{-}9)$$

If the firing circuits are designed in such a way that any change in the amplitude of the bias signal yields a change at the output of the d-c to d-c converter which is K_A times larger than the bias voltage we can claim that our theoretical results may form the basis for a practical control system that possesses many unique characteristics. We must emphasize the fact that the output voltage of the d-c to d-c controller is only an approximation of the desired armature voltage as given by Eq.(5-1).

However, it is a very good approximation since in general the pulse period, i.e. the sum of the on and off times, is much smaller than T.

The d-c to d-c converter must also have the ability to transfer the electrical energy from the motor circuit to the battery during the generating mode of the operation. Therefore, while one portion of the controller provides the control voltage $u_{s0}(t)$ to the armature circuit at all times, the other portion of the controller must be capable of stepping up the control voltage $u_{s0}(t)$ by several times in order to provide a voltage output to charge the battery during the generating mode. This can also be accomplished by using a suitable pulse-modulation technique. Note, however, that when $u_{s0}(t)$ becomes small during the regeneration mode, the amplification factor of the step-up portion of the converter must be increased by an auxiliary controller to such a level from which the battery can be charged. Conversely, if $u_{s0}(t)$ is high, the amplification gain must be decreased to limit the battery charging voltage. Figure 34 shows the complete block diagram of the

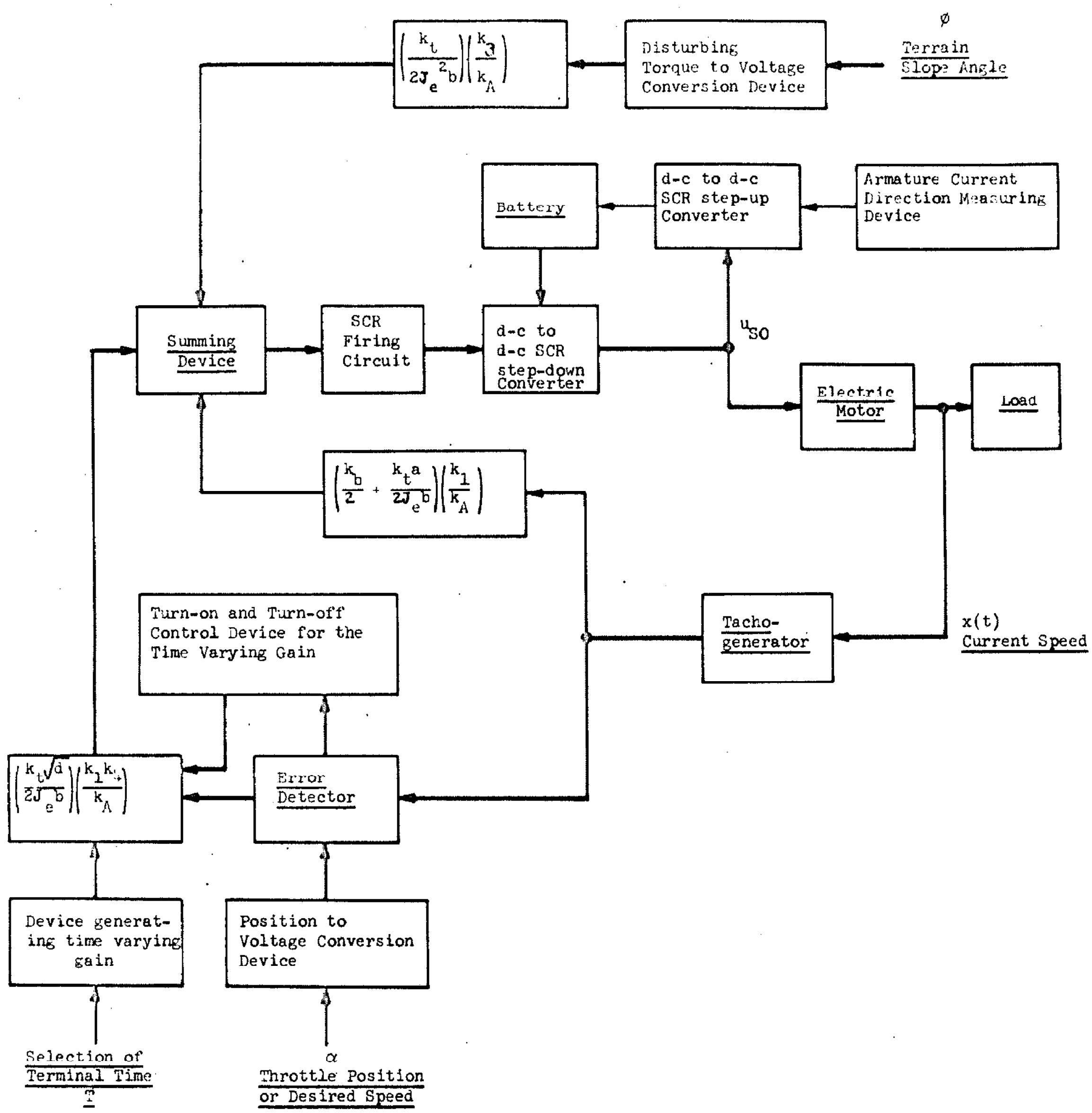


Figure 34. The Practical Implementation of the Control System.

control system with its auxiliary controllers.

5.2.1. Remarks on System Parameters.

All the system parameters appearing in Eq.(5-1) must be specified with reasonable accuracy for the electric propulsion vehicle of interest. The total inertia of wheels, tires, brake drums, axles and motor armature may be computed from dimensions. The torque constant, hence the back emf constant, i.e. they are equivalent if m.k.s. system of units is used, may be obtained from locked rotor test.[37] Gear ratio is easily determined by studying the required range of vehicle speeds with the available range of motor speeds.

Tire rolling resistance, viscous friction may be computed by performing an experiment on the vehicle under consideration and using a parameter estimation technique.[38]

5.3. The Practical Implementation of the Suboptimal Control Law for the Slip-frequency and Constant (volt/cycle) Controlled 3-Phase Induction Motor Drive System.

The practical implementation of Eq.(4-10) requires the following considerations:

1. $u_{s0}(t)$ is generated according to Eq.(4-10) where β is replaced by $v(t)$. Most of the ideas presented in Section (5-2) are also applicable and therefore will not be repeated here.
2. $u_{s0}(t)$ is then added to the current angular frequency of the rotor to determine the stator frequency which is then applied to a d-c to a-c SCR converter, i.e. an inverter, to yield the desired inverter frequency.

3. The stator voltage is obtained according to a pre-specified constant ratio of stator voltage to stator frequency to yield a constant air gap flux. This can be accomplished by modulating the battery voltage via a d-c to d-c converter with pulse-modulation technique in such a way that the ratio of the output voltage of the d-c to d-c converter to stator frequency of the motor is constant. The output voltage is then applied to an inverter circuit which in turn produces the motor voltage at the desired magnitude and frequency.
4. During the generating mode a reverse path must be provided from the output of the inverter circuit back to the battery. This may be accomplished by designing an appropriate step-up a-c to d-c converter and placing it between the battery and inverter output circuit.
5. Auxiliary circuits to turn the time varying gains on and off at $t = 0$ and $t = T$ respectively and to identify the mode of operation by measuring the sense of slip-frequency must be incorporated in the overall control system.

Note that practical reasons require the use of 3-phase squirrel-cage induction motors. Hence when the command slip frequency is obtained, it must be applied into the appropriate locations in the firing circuits in such a way that we obtain 3-phase a-c voltages at the output of the inverter.

5.4. The Practical Implementation of the Suboptimal Control Law for the Series-Excited and Armature-Controlled d-c Motor Drive System.

The practical implementation of Eq.(4-34) requires the following considerations:

1. $u_{s0}(t)$ is generated according to Eq.(4-34) where it can be assumed that:

$$\alpha_2 \hat{=} \left(\frac{r_e\alpha_1 + \beta}{m_{af}}\right)^{1/2} \tag{5-10}$$

   where β represents the mean value of $v(t)$ in $0 \leq t \leq T$ which is measured by suitable instrumentation.

2. There is no provision for the regeneration in the control system.

3. The controller measures the current speed, the current value of the armature current, the disturbance torque $v(t)$. The desired speed α_1 is fed into the system in the same manner as described in Section 5.2 and the desired value α_2 is obtained according to Eq.(5-10).

The same design techniques discussed in Section 5-2 are also applicable here; therefore, they will not be repeated here.

VI. CONCLUSIONS

6.1. Introduction.

The primary aim of this investigation has been to demonstrate the application of theoretical concepts of modern control theory into a class of engineering problems. Any application of an electric motor as a prime mover with electricity as the basic energy source in variable speed drive systems creates a need for a controller linking the energy source to the motor for speed setting and speed control actions. Such systems span the applications spectrum from large industrial complexes to small portable devices and recently an intensive effort initiated by several engineering institutions to extend them to electric propulsion vehicles such as electric cars, electric trains, electric earth-moving vehicles, electric lift trucks, submerged vehicles, etc. However, as is typical of such problems solved by classical techniques, the existing designs are motivated largely by experience and in no sense are the resulting systems optimal. The recent advances in optimal control theory and digital computers make it feasible to replace all the existing empirical design techniques by a scientifically motivated and mathematically sound design procedure.

6.2. Techniques for Developing Mathematical Models of Electric Propulsion Systems.

Developing a reasonably accurate mathematical model for a specific electric propulsion vehicle is a necessity before the theoretical concepts of modern control theory can be applied to synthesize a feedback controller. The following procedure has been used in the determination of the plant equations:

1. Obtain the expressions for the total kinetic, potential and dissipation energies of the system which consists of several electrical and mechanical interacting components.

2. Select the suitable generalized coordinates for the expressions obtained in 1 and use the auxiliary expressions for the generalized forces to write down the plant equations according to Lagrange's energy method.[39,40]

6.3. The Selection of the System Performance Index.

The system energy consumption has been considered as the most important performance requirement and it has been shown that by considering response time T as a fixed but arbitrary quantity other important performance requirements can also be satisfied. This particular approach shows a marked departure from the usual method of specifying the performance index which in general may consist of several performance requirements weighed according to their importance.

6.4. The Synthesis Techniques.

In general, closed-form solutions for the Bellman-Hamilton-Jacobi equation in nonlinear problems can not be obtained except in very special cases. Therefore, the synthesis problem, by necessity, must be treated as an approximation problem.[41] Fortunately, the approximate optimization problem of Chapter II and problem 1 of Chapter IV can be described by linear plant equations, and their performance indices are of quadratic nature, to a high degree of approximation. This in turn allows us to obtain several useful analytical results by using Bellman's dynamic programming approach. The digital computer simulations of the results have yielded the following response characteristics for all the problems considered in this investigation:

1. The response of the system is very stable in $0 \le t \le T$ for any required control action.
2. The desired speed is obtained with a high degree of accuracy in a pre-specified response time T.

6.5. Future Efforts and Applications.

The final justification of the assumptions and approximations made in this investigation require the prototype development of minimum-energy controllers. If the test results on a vehicle under actual environmental conditions verify the theoretical predictions then we can claim that we have developed a novel and powerful technique in designing the electronic controllers for electric propulsion vehicles that minimizes the system energy consumption while satisfying the performance requirements in any required control action. Note that the design procedure developed in this study is also applicable to the following electric drive systems:

a.c. source → rectifier → d.c. motor (shunt, series)

a.c. source → rectifier-inverter → a.c. motor (induction)
or cycloconverter

The specific results obtained so far apply to electric drive systems with (1) constant inertia, (2) constant frictional coefficients, e.g. air drag, viscous friction, tire rolling friction, and variable but measurable external loads during the controlling process.

For applications in which some or all of the constraints are relaxed, additional analytical investigations are required, but these do not appear too difficult to perform. It is the belief of the author that the contents of this investigation may be very useful in the practical development of feedback controllers for the following electric drive systems:

Electric Propulsion Vehicles: Electric cars, Electrically powered Tractors and Road Building Machinery, Rapid Transit Systems, Electric Trains, Submerged Vehicles, Electric Lift Trucks, Lunar Roving Vehicles, etc.

Industrial Process Control: Paper Mills, Steel Mills, etc.

Miscellaneous: Antenna Drives, Cargo Winch Drives, Capstan Drives, Welding Positioner Drives, Conveyor Drives, Drill Press Drives, Packaging Machine Drives, Printing Press Drives, Pump Drives, etc.

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