PROPAGATION OF FINITE AMPLITUDE WAVES IN ELASTIC SOLIDS

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ABSTRACT

This thesis is devoted to consideration of finite amplitude waves propagating into an elastic half-space in a direction normal to the boundary. Excitation is by means of strains applied at the boundary as step functions of time.

The solutions obtained are combinations of centered simple waves and shock waves. Longitudinal waves may appear alone but waves with transverse displacement components are always accompanied by longitudinal waves. The foregoing solutions are discussed in general and are illustrated by an example problem involving a special nonlinear, compressible, hyperelastic material. A perturbation method, based on the use of characteristic coordinates, which facilitates approximate solution of the problem for arbitrarily prescribed strain boundary conditions is described.

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CHAPTER I

INTRODUCTION

In this thesis an investigation is made of the simplest boundary and initial value problems associated with the propagation of finite amplitude waves in elastic solids. The direction of the investigation is chosen in accordance with the dictates of the theory and is designed to illuminate the theory. The results obtained are relatively simple, explicit, and easily understood and thus help in visualization of the possibilities for solution of more complicated problems.

As will be seen, the problems here considered are rather like those of wave propagation in perfect gases. From the mathematical point of view both are covered, for example, by Lax's theory of hyperbolic systems of conservation laws.[1, 2] Because of the similarities between plane waves in gases and in elastic solids this study of elastic wave propagation is a natural extension of previous studies of gasdynamics such as those discussed by Courant and Friedrichs.[3] The variety of phenomena which may occur in elastodynamics is greater than that occurring in gasdynamics because of the more general material constitution. In particular, plane elastic waves may involve transverse displacements which do not occur in the plane wave problems of gasdynamics.

A major difficulty of the wave propagation problems considered here is that, due to the generality of the class of admissible materials, a great variety of phenomena are possible. For exhaustive discussion of the problems it is necessary that all these phenomena be identified, and that materials be classified in such a way that it can be predicted which of the various possible phenomena may occur in a given case.

Recent studies of finite amplitude elastic waves include those of Truesdell,[4] W. A. Green,[5] and A. E. Green,[6] which are exact and very general but which are primarily concerned with local behavior at the wavefronts and are therefore of a different character from the present discussion of wave propagation problems in the large. This problem has also been investigated by Fine and Shield[7] by means of a perturbation technique. Studies more like the present one in character are those of Chu[8] and of Bland[9, 10] which deal with shear waves in incompressible materials and purely longitudinal waves in compressible materials, respectively. These two kinds of waves are mathematically analogous and are, in turn, analogous to the problem of propagation of plane waves in a perfect gas. In the next chapter the work of Chu and of Bland will be discussed further and compared with the present work. For references to work done up to 1961 the paper of Truesdell[4] may be consulted.

In Chapter II of this thesis the mathematical theory employed in the discussion of plane finite amplitude waves in elastic solids is described.

Chapters III - V are devoted to the discussion of two special boundary value problems which are fundamental to the subject at hand. Solutions involving shocks and centered simple waves in various combinations are exhibited and the circumstances of their occurrence are described.

In Chapter VI wave propagation in a specific nonlinear, compressible, hyperelastic material is discussed as an illustrative example of the theory of the previous chapters.

An approximate method of solving the problems of the previous chapters is discussed in Chapter VII.

CHAPTER II

FORMULATION OF THE FIELD EQUATIONS AND JUMP CONDITIONS

The governing equations of the theory of propagation of finite amplitude waves in elastic solids express: (1) the mechanical principles of balance of linear momentum, angular momentum, and mass, and, (2) the mechanical constitution of the material. For reasons to be discussed later equations expressing the thermodynamic principles of balance of energy and entropy are not used. The governing equations to be presented are long known. Careful modern derivations are readily available[11] but are briefly included here for completeness and as an aid in clarifying the notation.

Cartesian tensor notation with the associated summation convention is employed in this section. Special tensors which occur are the Kronecker delta and the alternating tensor:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}, \qquad \epsilon_{ijk} = \begin{cases} +1 & \text{if ijk is an even permutation of 123} \\ -1 & \text{if ijk is an odd permutation of 123} \\ 0 & \text{otherwise} \end{cases}.$$

In cartesian coordinates, x_i, Gibbs' vector operations are defined in terms of their tensor counterparts as, for example, $\underline{t} \cdot \underline{n} = \underline{T}$ where $T_i = t_{ij}n_j$, $\underline{a} \cdot \underline{b} = c$ where $c = a_i b_i$, $\underline{a}\,\underline{b} = \underline{c}$ where $c_{ij} = a_i b_j$, $\underline{a} \times \underline{b} = \underline{c}$ where $c_i = \epsilon_{ijk} a_j b_k$. Finally, subscripts following a comma denote partial differentiation with respect to the coordinate bearing that subscript.

The governing equations of the theory will be given in both spatial and material form. The spatial formulation is taken as basic and the material formulation obtained from it.

Spatial (or Eulerian) coordinates are those denoting points of the inertial space in which the body under consideration resides. Material (or Lagrangian) coordinates are those denoting particles of this body. A motion of the body is expressed as a suitable time dependent mapping of the material coordinates into the spatial coordinates, that is, an expression of the place at which each particle of the body resides at each instant of time.

2. 1. Spatial Formulation

Let the material under study reside in an inertial space, the points of which are denoted by coordinates x_i in a cartesian reference frame x. Let the particles of the body be denoted by their place $\underline{X}$ in the x-space when the body is in some given reference configuration.

Equations of balance are written for an arbitrary fixed region $\mathcal{V}$ of the x-space bounded by the fixed surface $\partial\mathcal{V}$. The velocity of the material particle X which resides at the place $\underline{x}$ at the time t is denoted by $\dot{\underline{x}}$ $(\underline{x}, t)$. The mass density of the material is denoted by ρ $(\underline{x}, t)$ and the stress field by the stress tensor $\underline{t}(\underline{x}, t)$. The traction vector on a surface with unit normal $\underline{n}$ is $\underline{t}_{(n)} = \underline{t} \cdot \underline{n}$ or, in component form, $t_{(n)i} = t_{ij}n_j$. The situation described is illustrated in figure 2. 1.

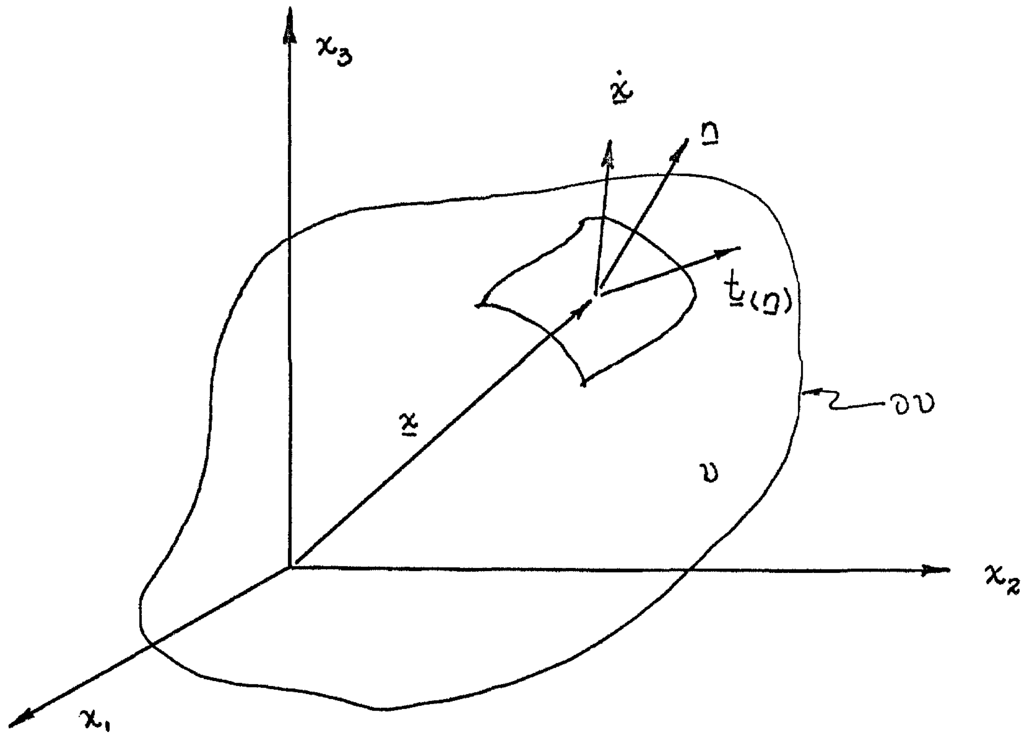


Figure 2. 1

2. 1. 1. Integral Equations of Balance of Mass and Momentum

For an arbitrary fixed region of space $\mathcal{V}$ with boundary $\partial\mathcal{V}$, the equations expressing balance of mass, linear momentum, and angular momentum, respectively, are as follows in the absence of body forces:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, d\underline{x} = -\int_{\partial\mathcal{V}} \rho \, \underline{\dot{x}} \cdot \underline{n} \, ds ,$$

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, \dot{\underline{x}} \, d\underline{x} = -\int_{\partial\mathcal{V}} (\rho \dot{\underline{x}} \, \dot{\underline{x}} - \underline{t}) \cdot \underline{n} \, ds ,$$

$$\frac{d}{dt}\int_{\mathcal{V}} \rho (\dot{\underline{x}} \times \underline{x}) d\underline{x} = -\int_{\partial\mathcal{V}} \left\{ \left[\rho (\underline{x} \times \dot{\underline{x}}) \, \dot{\underline{x}} \right] \cdot \underline{n} - \left[\underline{x} \times (\underline{t} \cdot \underline{n}) \right] \right\} ds .$$

These equations can be written in component form as follows:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, d\underline{x} = -\int_{\partial\mathcal{V}} \rho \, \dot{x}_i n_i \, ds ,$$

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, \dot{x}_i d\underline{x} = -\int_{\partial\mathcal{V}} (\rho \dot{x}_i \dot{x}_j - t_{ij}) n_j ds , \tag{2.1}$$

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, \epsilon_{ijk} x_j \dot{x}_k d\underline{x} = -\int_{\partial\mathcal{V}} \epsilon_{ijk} x_j (\rho \dot{x}_k \dot{x}_\ell - t_{k\ell}) n_\ell ds .$$

2.1.1.1. Differential Equations of Balance of Mass and Momentum

Under the assumptions that $\rho(\underline{x}, t)$ and $\dot{\underline{x}}(\underline{x}, t)$ are continuously differentiable with respect to $\underline{x}$ and to t and that $\underline{t}(\underline{x}, t)$ is continuously differentiable with respect to $\underline{x}$, one may reduce equations (2.1) to the following form by interchange of order of differentiation and integration in the first member and application of the divergence theorem to the second member:

$$\int_{\upsilon}\left[\frac{\partial\rho}{\partial t}+(\rho\dot{x}_i)_{,i}\right]d\underline{x}=0,$$

$$\int_{\upsilon}\left[\frac{\partial}{\partial t}(\rho\dot{x}_i)+(\rho\dot{x}_i\dot{x}_j-t_{ij})_{,j}\right]d\underline{x}=0, \tag{2. 2}$$

$$\int_{\upsilon}\epsilon_{ijk}\left\{\left[\frac{\partial\rho}{\partial t}+(\rho\dot{x}_\ell)_{,\ell}\right]x_j\dot{x}_k-\left[t_{k\ell,\ell}-\rho\left[\frac{\partial\dot{x}_k}{\partial t}+\dot{x}_\ell\dot{x}_{k,\ell}\right]\right]x_j\right.$$

$$\left.+\rho\,\dot{x}_k\left[\frac{\partial x_j}{\partial t}+\dot{x}_\ell\,x_{j,\ell}\right]-t_{kj}\right\}d\underline{x}=0.$$

In accordance with the above regularity conditions the integrands in (2. 2) are continuous. Since υ is an arbitrarily chosen region they must vanish, giving

$$\frac{\partial\rho}{\partial t}+(\rho\dot{x}_i)_{,i}=0\ ,$$

$$t_{ij,j}=\rho\left(\frac{\partial\dot{x}_i}{\partial t}+\dot{x}_j\dot{x}_{i,j}\right), \tag{2. 3}$$

$$t_{ij}=t_{ji}.$$

The case where the regularity conditions used in this section fail to hold is of particular interest in problems of wave propagation. It is discussed in section 2. 1. 1. 2.

This thesis is devoted to the investigation of plane waves propagating into an elastic half-space in a direction normal to the boundary. It is convenient to choose the x_1-axis to be along a normal to this bounding surface and to be directed into the material. Since only plane-polarized transverse displacements are considered it is convenient to choose the x_2-axis parallel to the direction of transverse displacement. Finally, the x_3-axis is chosen normal to the plane of the other two axes, and positive directions are assigned so that the x-coordinates constitute a right-handed system.

A motion of the type described above can be expressed mathematically by the transformation

$$X_1 = x_1 - u_1(x_1, t), \quad X_2 = x_2 - u_2(x_1, t), \quad X_3 = x_3, \tag{2. 4}$$

where $\underline{u} = (u_1, u_2, 0)$ is called the spatial form of the displacement vector. The body is undeformed when $\underline{u} = 0$.

Denote by j the Jacobian of the transformation (2. 4) for any fixed $t \geqslant 0$:

$$j = \det\,(\partial X_i / \partial x_j).$$

As a general mechanical principle it is required that, for each $t \geqslant 0$, the Jacobian satisfy the inequalities

$$0 < j < \infty. \tag{2. 5}$$

For the motion (2. 4),

$$j = 1 - \partial u_1(x_1, t) / \partial x_1. \tag{2. 6}$$

Since equation (2. 1), expressing balance of mass is equivalent to the relation

$$\rho = j \rho_o, \tag{2. 7}$$

where ρ_o is the (constant) density of the material in the undeformed state, the density in the deformed state is given by*

$$\rho = \rho_o(1 - u_{1\,x_1}). \tag{2. 8}$$

* Here and henceforth a variable, independent or dependent, written as a subscript denotes differentiation with respect to that variable.

As a consequence of (2. 5) the transformation (2. 4) is invertable for each time $t \geqslant 0$ and, as is seen from (2. 8), the density of the deformed material is always finite and non-zero.

For the special motion (2. 4) the differential equations of balance reduce to

$$\begin{aligned} &\rho_t + (\rho \dot{x}_1)_{x_1} = 0, \\ &(\rho \dot{x}_1)_t + (\rho \dot{x}_1^2 - t_{11})_{x_1} = 0, \\ &(\rho \dot{x}_2)_t + (\rho \dot{x}_1 \dot{x}_2 - t_{12})_{x_1} = 0, \end{aligned} \tag{2. 9}$$

where the subscripts denote partial differentiation and where, here and henceforth, the tensor $\underline{t}$ is understood to be symmetrical in accordance with $(2.\ 3)_3$.

By (2. 4), the particle velocities are found to be

$$\dot{x}_1 = u_{1_t}/(1-u_{1_{x_1}}), \quad \dot{x}_2 = \left[u_{1_t} u_{2_{x_1}} + u_{2_t}(1-u_{1_{x_1}})\right]/(1-u_{1_{x_1}}). \tag{2. 10}$$

Because of (2. 5) and (2. 6), the denominators of these expressions do not vanish. For convenience the following notation is introduced:

$$p = u_{1_{x_1}}, \quad q = u_{2_{x_1}}, \quad r = u_{1_t}, \quad s = u_{2_t}. \tag{2. 11}$$

With this (2. 10) becomes

$$\dot{x}_1 = r/(1-p), \quad \dot{x}_2 = \left[rq + s(1-p)\right]/(1-p). \tag{2. 12}$$

Assuming that $\underline{u}$ is twice differentiable with respect to x_1 and t the two mixed second derivatives are equal, so that

$$p_t = r_{x_1}, \quad q_t = s_{x_1}. \tag{2.13}$$

By (2.11) equation (2.8) can be written

$$\rho = \rho_o(1-p). \tag{2.14}$$

Substitution of (2.12), (2.13), and (2.14) into $(2.9)_1$, shows the latter to be identically satisfied. Indeed, (2.14) is essentially an integrated form of $(2.9)_1$.

Substitution of (2.11), (2.12) and (2.13) into $(2.9)_{2,3}$ and adjoining (2.13) gives the following equations of motion:

$$\frac{\partial}{\partial t}\left[\rho_o r\right] + \frac{\partial}{\partial x_1}\left[\frac{\rho_o r^2}{1-p} - t_{11}\right] = 0,$$

$$\rho_o \frac{\partial}{\partial t}\left[rq + s(1-p)\right] + \frac{\partial}{\partial x_1}\left[\rho_o r\,\frac{rq + s(1-p)}{1-p} - t_{12}\right] = 0,$$

$$\frac{\partial p}{\partial t} - \frac{\partial r}{\partial x_1} = 0, \tag{2.15}$$

$$\frac{\partial q}{\partial t} - \frac{\partial s}{\partial x_1} = 0.$$

2.1.1.2. Jump Conditions Associated with Balance of Mass and Momentum and with Continuity of Displacement

Equations (2.1) are now investigated for the case where the displacements are everywhere continuous but where the derivatives of the field variables ρ, $\dot{\underline{x}}$, and $\underline{t}$ may fail to be continuous across a propagating singular surface. The field variables are assumed to possess finite limits from either side at the singular surface. This situation is of great practical importance in the theory and its study constitutes a major portion of this thesis.

For simplicity, and because it is the situation relevant to the present investigation, the singular surface is supposed to be a plane normal to the x_1-axis and propagating in the positive direction along this axis. The motion on either side of the singular surface is assumed to consist of plane waves of the type described by (2.4). Let the region $\mathcal{V}$ to which the equations of balance are applied be a rectangular solid having square faces of unit area normal to the x_1-axis and thickness h. A cross-sectional view of $\mathcal{V}$ is shown in figure 2.2.

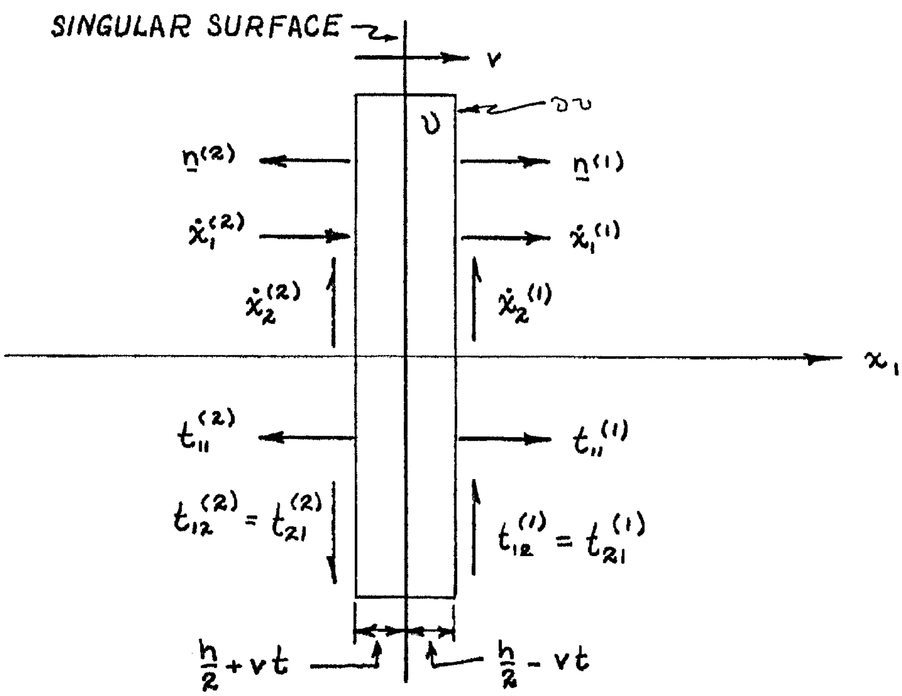


Figure 2.2

Let v be the local propagation speed of the singular surface. Function values for either side of the singular surface are distinguished by superscripts.

Evaluation of (2. 1) for the special geometry of figure 2. 2 and passage to the limit as $h \rightarrow 0$ gives the jump conditions

$$
\begin{aligned}
&[\![\rho]\!] \, v = [\![\rho \dot{x}_1]\!] , \\
&[\![\rho \dot{x}_1]\!] \, v = [\![\rho \dot{x}_1{}^2 - t_{11}]\!] , \\
&[\![\rho \dot{x}_2]\!] \, v = [\![\rho \dot{x}_1 \dot{x}_2 - t_{12}]\!] ,
\end{aligned}
\tag{2. 16}
$$

where

$$[\![A]\!] \quad A^{(2)} - A^{(1)}$$

denotes the jump across the singular surface of an arbitrary field quantity A.

As previously mentioned, the displacement has been assumed to be continuous across the singular surface. By (2. 4) this requirement can be written as

$$[\![\underline{u}]\!] = 0. \tag{2. 17}$$

For the present problem a condition substantially equivalent to (2. 17) but expressed in terms of derivatives of $\underline{u}(\underline{x}, t)$ is much preferable to (2. 17) itself. Such a condition can be obtained by considering the time rate of change of $\underline{u}$ as apparent to an observer moving with the surface of discontinuity (Ref. 11, sec. 180). This rate of change, denoted by $\delta \underline{u} / \delta t$ can be expressed as

$$\delta \underline{u}/\delta t = \underline{u}_{x_1} v + \underline{u}_t .$$

Taking the jump of this equation and using (2.17) gives

$$[\![\underline{u}_{x_1}]\!] v + [\![\underline{u}_t]\!] = 0. \tag{2.18}$$

By (2.11), (2.12), and (2.14) the jump conditions (2.16) and (2.18) can be written in the form

$$\begin{aligned}
&[\![\rho_o r]\!] v = [\![\left(\rho_o r^2/(1-p) \right) - t_{11}]\!] , \\
&[\![\rho_o \left(rq + s(1-p) \right)]\!] v = [\![\left\{ \rho_o r \left(rq + s(1-p) \right) / (1-p) \right\} - t_{12}]\!] , \\
&[\![p]\!] v + [\![r]\!] = 0, \\
&[\![q]\!] v + [\![s]\!] = 0.
\end{aligned} \tag{2.19}$$

Equation $(2.16)_1$ is identically satisfied, as was the case with its counterpart for differentiable fields.

The equations of balance (2.1) have thus been shown to yield the differential equations (2.9) in regions of space-time where continuous derivatives exist, and the jump conditions (2.15) across singular surfaces where the displacements are continuous but where their first derivatives are discontinuous.

<u>2.1.2.</u> <u>Deformation Gradients and Deformation Tensors</u>

Associated with the motion (2.4) are the deformation gradient field,

$$(X_{i,j}) = \begin{Vmatrix} 1-p & 0 & 0 \\ -q & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} , \tag{2.20}$$

the Cauchy deformation tensor,

$$(c_{ij}) \equiv (X_{k,i}\, X_{k,j}) = \begin{Vmatrix} (1-p)^2 + q^2 & -q & 0 \\ -q & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} , \tag{2.21}$$

and the inverse Cauchy deformation tensor,

$$(\overset{-1}{c}_{ij}) = \frac{1}{(1-p)^2} \begin{Vmatrix} 1 & q & 0 \\ q & (1-p)^2+q^2 & 0 \\ 0 & 0 & (1-p)^2 \end{Vmatrix} . \tag{2.22}$$

Since the motion described by (2. 4) is two-dimensional the appropriate two-dimensional versions of (2.18) and (2. 19) are recorded as follows:

$$(c_{\alpha\beta}) = \begin{Vmatrix} (1-p)^2 + q^2 & -q \\ -q & 1 \end{Vmatrix} , \quad (\overset{-1}{c}_{\alpha\beta}) = \frac{1}{(1-p)^2} \begin{Vmatrix} 1 & q \\ q & (1-p)^2 + q^2 \end{Vmatrix} , \tag{2.23}$$

where the lower case greek indices have the range 1, 2. The above deformation tensors measure deformation relative to the undeformed state, $\underline{u} = 0$.

The invariants, under rotations in x-space, of the inverse Cauchy deformation tensor (2.22) are

$$\begin{aligned}
\mathrm{I} &= \overset{-1}{c}_{ii} = \left[1 + q^2 + 2(1-p)^2\right]/(1-p)^2, \\
\mathrm{II} &= \tfrac{1}{2}\left(\mathrm{I}^2 - \mathrm{I}_{\overset{-1}{c}^2}\right) = \left[2 + q^2 + (1-p)^2\right]/(1-p)^2, \qquad (2.24) \\
\mathrm{III} &= \left|\overset{-1}{c}_{ij}\right| = 1/(1-p)^2 ,
\end{aligned}$$

and the invariants, under rotations in the x_1x_2-plane, of $(2.20)_2$ are

$$I_1 = \overset{-1}{c}_{\alpha\alpha} = \left[1 + q^2 + (1-p)^2\right]/(1-p)^2, \quad I_2 = \left|\overset{-1}{c}_{\alpha\beta}\right| = 1/(1-p)^2 . \qquad (2.25)$$

2.1.3. Stresses

The material considered in this thesis is an elastic solid, homogeneous, isotropic, and unstressed in the undeformed state. The stresses in such a material are given in terms of the deformation gradients by the relation

$$t_{ij} = h_{-1}\overset{-1}{c}_{ij} + h_o\delta_{ij} + h_1 c_{ij} \qquad (2.26)$$

where h_{-1}, h_o, and h_1 are functions of the invariants I, II, and III and are characteristic of the material under consideration. An especially important class of elastic materials consists of those called hyperelastic. For these materials a function Σ (I, II, III) exists with the property that

$$h_{-1} = \frac{2}{\sqrt{III}} \; \frac{\partial \Sigma}{\partial I} \, ,$$

$$h_o = \frac{2}{\sqrt{III}} \left(II \frac{\partial \Sigma}{\partial II} + III \frac{\partial \Sigma}{\partial III} \right) , \qquad (2.\ 27)$$

$$h_1 = -2 \sqrt{III} \; \frac{\partial \Sigma}{\partial II} \, .$$

The function Σ represents the strain energy per unit of unstrained volume of the body under consideration.

For the case of plane deformations of the form (2. 4) equation (2. 26) may be simplified to the following:

$$t_{\alpha\beta} = \overline{h}_o \, \delta_{\alpha\beta} + \overline{h}_{-1} \, \overset{-1}{c}_{\alpha\beta} \, , \qquad (2.\ 28)$$

where

$$\overline{h}_\Delta = \overline{h}_\Delta(I_1, I_2), \qquad \Delta = -1, 0 \, . \qquad (2.\ 29)$$

For hyperelastic materials

$$\overline{h}_o = 2\sqrt{I_2} \; \partial \Sigma / \partial I_2, \quad \overline{h}_{-1} = \frac{2}{\sqrt{I_2}} \; \partial \Sigma / \partial I_1. \qquad (2.\ 30)$$

Note that the stresses t_{ij} may be regarded as functions of the coordinates and of time, as in the preceding dynamical calculations, or as functions of some measure of deformation, as in the constitutive equation. Although the same symbol is used to denote both functions these two viewpoints need not cause confusion.

The response functions, or, in the hyperelastic case, the strain energy function, are usually subjected to various restrictions which insure that the material will exhibit "plausible" behavior.[12, 13, 14] These restrictions are many, detailed, and imperfectly understood, and general application of them has not been attempted in this thesis. Instead, restrictions suggested by the present analysis are noted at the places where they become relevant.

2.2. Material Formulation

The material formulation of the field equations and jump conditions is introduced because it contributes to mathematical simplicity in two ways: (1) The field equations and jump conditions themselves take on a simpler form, and, (2) the boundary of the body always has the same material coordinates whatever its motion be.

As mentioned previously, the transformation (2.4) is assumed to be invertible at each time t. Let the inverse be written in the form

$$x_1 = X_1 + U_1(X_1, t), \quad x_2 = X_2 + U_2(X_1, t), \quad x_3 = X_3, \tag{2.31}$$

where $\underline{U}$ is called the material form of the displacement vector. Introduce the notation

$$P = U_{1_{X_1}}, \quad Q = U_{2_{X_1}}, \quad R = U_{1_t}, \quad S = U_{2_t}. \tag{2.32}$$

By (2.4), (2.11), (2.31), and (2.32),

$$p = P/(1+P), \quad q = Q/(1+P), \quad r = R/(1+P), \quad s = [S(1+P)-QR]/(1+P). \tag{2.33}$$

From (2.12), or otherwise, the particle velocities are

$$\dot{x}_1 = R, \quad \dot{x}_2 = S. \tag{2.34}$$

2.2.1. Transformation of Field Equations and Jump Conditions to Material Variables

The field equations (2.15) can be directly transformed into the new variables by means of (2.31). The resulting equations are as follows:

$$
\begin{aligned}
(\rho_o R)_t - (t_{11})_{X_1} &= 0, \\
(\rho_o S)_t - (t_{12})_{X_1} &= 0, \\
P_t - R_{X_1} &= 0, \\
Q_t - S_{X_1} &= 0.
\end{aligned}
\tag{2.35}
$$

Transformation of the jump conditions (2.16) gives

$$
\begin{aligned}
[\![\rho_o R]\!] V + [\![t_{11}]\!] &= 0, \\
[\![\rho_o S]\!] V + [\![t_{12}]\!] &= 0, \\
[\![P]\!] V + [\![R]\!] &= 0, \\
[\![Q]\!] V + [\![S]\!] &= 0,
\end{aligned}
\tag{2.36}
$$

where

$$
V = \frac{v-R}{1+P} \quad . \tag{2.37}
$$

These jump conditions have the same form as those given, for example, by Bland.[9] They are precisely the "Rankine-Hugoniot equations" associated with (2.35) according to the definition of Lax.[1]

In terms of the material variables P and Q, the Cauchy deformation tensors (2.23) become

$$(c_{\alpha\beta}) = \begin{Vmatrix} (1+Q^2)/(1+P)^2 & -Q/(1+P) \\ -Q/(1+P) & 1 \end{Vmatrix}, \quad (c^{-1}_{\alpha\beta}) = \begin{Vmatrix} (1+P)^2 & Q(1+P) \\ Q(1+P) & 1+Q^2 \end{Vmatrix}, \tag{2.38}$$

while the invariants (2.25) take the form

$$I_1 = 1 + Q^2 + (1+P)^2, \qquad I_2 = (1+P)^2. \tag{2.39}$$

The stresses, computed in accordance with (2.38), are now regarded as functions of P and Q:

$$t_{11}(p, q) = \sigma(P, Q), \qquad t_{12}(p, q) = \tau(P, Q), \tag{2.40}$$

with (p, q) and (P, Q) related by $(2.33)_{1,2}$.

Using (2.40) the field equations (2.30) can be placed in the standard form

$$\begin{aligned} \alpha P_X + \beta Q_X &= R_t, \\ \gamma P_X + \delta Q_X &= S_t, \\ R_X &= P_t, \\ S_X &= Q_t, \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} \alpha(P, Q) &= \frac{1}{\rho_o} \sigma_P(P, Q), \quad & \beta(P, Q) &= \frac{1}{\rho_o} \sigma_Q(P, Q), \\ \gamma(P, Q) &= \frac{1}{\rho_o} \tau_P(P, Q), \quad & \delta(P, Q) &= \frac{1}{\rho_o} \tau_Q(P, Q), \end{aligned} \tag{2.42}$$

and where, for brevity, X has been written in place of X_1.

The equations of balance for the problem at hand are now expressed in terms of the differential equations (2.41), valid for those non-negative values of X and t where the indicated derivatives exist, and the jump conditions (2.36) applied across surfaces of discontinuity. The properties of the material, insofar as they are pertinent to the problem at hand, enter through the functions α, β, γ, and δ. The stresses are related to the response functions as follows:

$$\sigma(P,Q) = \bar{h}_o + (1+P)^2\,\bar{h}_{-1}, \quad \tau(P,Q) = Q(1+P)\bar{h}_{-1}. \tag{2.43}$$

By (2.42) and (2.43),

$$\begin{aligned}
\rho_o\alpha(P,Q) &= 2(1+P)\left[\bar{h}_{-1} + \frac{\partial\bar{h}_o}{\partial I_1} + \frac{\partial\bar{h}_o}{\partial I_2} + (1+P)^2\left(\frac{\partial\bar{h}_{-1}}{\partial I_1} + \frac{\partial\bar{h}_{-1}}{\partial I_2}\right)\right], \\
\rho_o\beta(P,Q) &= 2Q\left[\frac{\partial\bar{h}_o}{\partial I_1} + (1+P)^2\,\frac{\partial\bar{h}_{-1}}{\partial I_1}\right], \\
\rho_o\gamma(P,Q) &= Q\left[\bar{h}_{-1} + 2(1+P)^2\left(\frac{\partial\bar{h}_{-1}}{\partial I_1} + \frac{\partial\bar{h}_{-1}}{\partial I_2}\right)\right], \\
\rho_o\delta(P,Q) &= (1+P)\left[\bar{h}_{-1} + 2Q^2\,\frac{\partial\bar{h}_{-1}}{\partial I_1}\right].
\end{aligned} \tag{2.44}$$

These functions may be reduced further in the hyperelastic case by substitution of the forms (2.30) for $\bar{h}_{-1}$ and $\bar{h}_o$. Note that in the hyperelastic case $\gamma = \beta$.

2.2.2. Summary of the Equations of the Material Formulation of the Problem

The material field equations are (2.41):

$$\begin{aligned}
\alpha P_X + \beta Q_X &= R_t, \qquad S_X = Q_t, \\
\gamma P_X + \delta Q_X &= S_t, \\
R_X &= P_t,
\end{aligned} \tag{2.45}$$

where

$$\rho_o \alpha = \sigma_P(P, Q), \quad \rho_o \beta = \sigma_Q(P, Q), \quad \rho_o \gamma = \tau_P(P, Q), \quad \rho_o \delta = \tau_Q(P, Q). \tag{2.46}$$

They are valid for $X, t \geqslant 0$ provided that the indicated derivatives exist and are continuous.

The jump conditions are (2.36):

$$\begin{aligned} [\![\rho_o R]\!] V + [\![\sigma]\!] &= 0, \\ [\![\rho_o S]\!] V + [\![\tau]\!] &= 0, \\ [\![P]\!] V + [\![R]\!] &= 0, \\ [\![Q]\!] V + [\![S]\!] &= 0, \end{aligned} \tag{2.47}$$

and are to be applied at plane surfaces of discontinuity.

The stresses are related to the deformations by means of the equations

$$\sigma(P, Q) = \overline{h}_o + (1+P)^2 \overline{h}_{-1}, \quad \tau(P, Q) = Q(1+P)\overline{h}_{-1}, \tag{2.48}$$

where the functions $\overline{h}_o(I_1, I_2)$ and $\overline{h}_1(I_1, I_2)$ characterize the material under consideration.

In addition to the field equations and jump conditions there are some other conditions to be considered in connection with this subject. Solutions of the jump conditions are admitted as shocks only if they satisfy an admissibility condition, proposed by Lax,[1] which says that their propagation speed be supersonic with respect to the corresponding characteristic wavespeed ahead of the shock, and subsonic with respect to the characteristic wavespeed behind the shock. A detailed statement of this admissibility condition is given later. Whether or not the above condition is satisfied depends strongly on the constitutive equation (2. 26), and these requirements therefore limit the class of materials for which a given solution of the jump conditions is valid.

Finally, the specification of a problem involves specification of suitable boundary and initial conditions. Since equations of the form (2. 45) are the subject of an extensive mathematical literature, the kinds of boundary and initial conditions which may be specified are known in advance.[(2)]

2. 3. Computation of Characteristic Wavespeeds

In order that disturbances of the elastic materials here considered propagate through the material as waves, conditions must be met by the coefficients in the system (2. 45) which guarantee that it be of hyperbolic type.[(2)]

Formal computation of the characteristic wavespeeds gives them as solutions a of

$$a^4 - (\alpha + \delta)\, a^2 + (\alpha\delta - \beta\gamma) = 0. \tag{2. 49}$$

From this a^2 is given by

$$a^2 = \frac{1}{2}\left\{ \alpha + \delta \pm \left[(\alpha - \delta)^2 + 4\beta\gamma\right]^{1/2} \right\}. \tag{2. 50}$$

In order that a^2 be real it is necessary and sufficient that

$$(\alpha - \delta)^2 + 4\beta\gamma \geqslant 0 \tag{2. 51}$$

for the range of P and Q of interest.* This inequality is always satisfied for hyperelastic materials, since $\beta = \gamma$ in this case (see Eqns. (2.44) and (2.30)). The inequality (2.51) will be taken here as a constitutive restriction on all elastic materials. In addition to the requirement that a^2 be real, it is necessary for hyperbolicity of (2.45) that a^2 be positive for either choice of sign in (2.50). The necessary and sufficient condition for this is

$$\alpha + \delta > \left[(\alpha - \delta)^2 + 4\beta\gamma \right]^{1/2} . \qquad (2.52)$$

If (2.51) and (2.52) hold, there are four real wavespeeds:

$$\begin{aligned} a_4 = -a_1 &= \left\{ \frac{1}{2} \left[\alpha + \delta + \left((\alpha - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2} \\ a_3 = -a_2 &= \left\{ \frac{1}{2} \left[\alpha + \delta - \left((\alpha - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2} . \end{aligned} \qquad (2.53)$$

As will be discussed later, these wavespeeds satisfy the relations

$$a_1 < a_2 < 0 < a_3 < a_4 . \qquad (2.54)$$

The following limits are assumed to exist as $P, Q \to 0$:

$$\alpha \to (\lambda + 2\mu)/\rho_0, \quad \beta, \gamma \to 0, \quad \delta \to \mu/\rho_0, \qquad (2.55)$$

where λ and μ are constants. These prescriptions are enforced so that, for small deformations, the nonlinear theory used here becomes consistent with the usual linear theory of elasticity. In particular, the assumption $\beta, \gamma \to 0$ as $P, Q \to 0$ assures that, to

---

* All references to square root (including 1/2 - power) are understood to mean real positive square root.

first approximation, shear and dilitational effects are uncoupled. The constants λ and μ occurring in (2. 55) are the Lamé modulii of the linear theory of elasticity. In the linear theory it is usually assumed (Ref. 11, sec. 301) that $3\lambda + 2\mu > 0$ and $\mu > 0$ in order that the strain energy function be positive definite, and that $\lambda + 2\mu > 0$ and $\mu > 0$ in order that the wavespeeds of that theory be real. Those assumptions are made here and imply

$$\lambda + \mu > 0. \tag{2. 56}$$

Use of these results leads to the limiting values

$$a_3 \rightarrow \sqrt{(\mu / \rho_o)}, \qquad a_4 \rightarrow \sqrt{[(\lambda + 2\mu)/\rho_o]}, \tag{2. 57}$$

as $P, Q \rightarrow 0$, for the wavespeeds (2. 53). In the linear theory $\sqrt{(\mu/\rho_o)}$ is the wavespeed for shear waves and $\sqrt{[(\lambda + 2\mu)/\rho_o]}$ is the wavespeed for longitudinal waves. This terminology is carried over into the nonlinear theory; a_1 and a_4 are called longitudinal wavespeeds and the associated waves longitudinal waves. Similarly, a_2 and a_3 are called shear wavespeeds and the associated waves shear waves. As will be seen in the following chapters, this terminology, particularly in the case of shear waves, is not so meaningful as in the linear theory; nevertheless it is useful in certain respects and will be used in this thesis.

By (2. 56), (2. 57), and the assumed continuity of α, β, γ, and δ, as functions of P and Q,

$$a_3 < a_4 \tag{2. 58}$$

for P and Q sufficiently small. It is assumed in this thesis that (2. 58) is true for all relevant values of P and Q for all materials under consideration.

2.4. Remark on the Neglect of Energy and Entropy Balance

The foregoing theory is based on the mechanical principles of balance of linear and angular momentum, and conservation of mass and on a statement of the mechanical constitution of the material being considered. Two other principles which might be expected to play a role in the formulation of a theory of elastic wave propagation are: (1) balance of energy and, (2) balance of entropy, along with a constitutive equation for heat conduction. These latter principles will, for brevity, be called thermodynamic principles to distinguish them from the mechanical principles and constitutive hypothesis.

The theory resulting from consideration of the mechanical principles and constitutive hypothesis alone is, on the face of it, a complete theory in the sense that it consists of enough variables to describe the motion and enough equations to permit determination of these variables. In so far as smooth solutions are concerned this is true, but when discontinuous (weak) solutions are admitted it is possible to find both smooth and discontinuous solutions satisfying the same boundary and initial conditions and an additional admissibility condition becomes necessary to determine which solution shall be used. In special theories such as that of wave propagation in perfect gases this difficulty is remedied by inclusion of the thermodynamic principles as they are understood in that context. Such considerations have also been attempted in recent theories of elastic wave propagation. For hyperelastic materials the equation of balance of energy is satisfied in regions where the fields are smooth. At discontinuities the jump condition associated with this equation of balance cannot be satisfied, in general, unless a thermodynamical variable such as entropy is introduced and it is not clear even then that an admissible jump in this variable permits satisfaction of the jump condition for energy balance. The equation of entropy balance is satisfied automatically in regions where the fields are smooth if it is assumed that the material is non-heat conducting. No satisfactory statement of this principle seems to have been made for cases of discontinuous fields.

Chu, in his theory of waves in non-heat conducting incompressible hyperelastic solids,[8] gives consideration to energy and entropy balance in smooth waves, but fails to make clear statements of the jump conditions associated with these principles and, although the equations associated with these principles are said to be satisfied by the entropy jump, this entropy jump is not calculated nor is it shown to exist and to be positive. Positivity of entropy jump is not used as an admissibility condition for shocks as is the practice in gas dynamics; rather it is assumed that smooth solutions are proper where they exist and jumps are admitted otherwise.

In the work of Bland[9,10] energy and entropy balance are considered and the entropy change across a weak discontinuity is calculated. Only jumps for which the entropy change of a particle upon passage of the jump is positive are admitted as shocks. This thermodynamical admissibility condition and a condition for mechanical stability of discontinuities are shown to be equivalent for weak shocks.

In the final analysis the theories of Bland and Chu do not make much use of the thermodynamical principles. Both investigators consider isentropic motions where these motions are continuous. At jump Chu abandons thermodynamical considerations altogether; Bland requires entropy increase of a particle upon passage of the jump in order that it be admissible as a shock solution, but carries out the details only for weak shocks.

Because the thermodynamical aspects of the above discussed theories are unsatisfying and because an adequate thermodynamical theory seems beyond reach at the present time, the course taken in this thesis has been to ignore the thermodynamical question entirely. The special and rather artificial admissibility condition discussed has been introduced in place of the thermodynamical principles. With this admissibility condition the theory obtained is, as a practical matter, similar to the above isentropic theories of Chu and of Bland.

Truesdell[4] does not employ thermodynamical principles in his investigations of elastic wave propagation and does not study shocks, hence has no need of admissibility conditions for them.

CHAPTER III

SHOCK WAVE PROPAGATION

The wave propagation problem to be studied has been set in terms of the field equations (2. 45) and the jump conditions (2. 47), along with the admissibility conditions for shocks and suitable boundary conditions yet to be formulated.

Since the jump conditions are algebraic equations, in contrast to the differential field equations, it is natural, in the quest for simple exact solutions for propagating disturbances, to seek solutions corresponding to regions of uniform state (in which the differential equations are trivially satisfied) separated by propagating surfaces of discontinuity. Other forms of solution are considered in later chapters.

Section 3.1 of this chapter is devoted to the consideration of these piecewise constant solutions for the initially deformed half-space $X \geqslant 0$ when the displacement gradients at the boundary are suddenly changed. Later sections of the chapter are devoted to special cases of this problem. In section 3. 1. 1 the problem is first simplified by leaving out the shear disturbances and in sections 3.1.1.1 and 3. 1. 1. 2 further simplified by considering loading and unloading cases separately. In section 3. 1. 2 the problem of application of shear strain to the boundary of an undeformed body is considered. Finally, in section 3. 2, a summary of results of the investigations of this chapter is given.

Conditions on the material constitution are given under which each of the various solutions considered is admissible.

3.1 Sudden Application of Displacement Gradients to the Boundary of an Initially Deformed Half-Space

This section is devoted to consideration of the problem of sudden application of displacement gradients to the boundary of a half-space initially deformed but at rest.

Let the displacement gradients and particle velocities in the initial state have the constant values $P = P_5$, $Q = Q_5$, $R = S = 0$. The stresses in the initial state, $\sigma_5 = \sigma(P_5, Q_5)$ and $\tau_5 = \tau(P_5, Q_5)$, are assumed known in terms of the given displacement gradients. At $t = 0$ let the displacement gradients on the boundary $X = 0$ be changed from P_5 and Q_5 to the new constant values P_o and Q_o, respectively.

The solution is assumed to consist of three regions of constant state separated by two propagating surfaces of discontinuity. In the region farthest from the boundary the state remains unchanged from its initial condition. In the region nearest to the boundary the displacement gradients are those given on the boundary and the stresses are known in terms of these given displacement gradients. The particle velocities in this region have the unknown constant values R_1 and S_1. In the intermediate region the solution corresponds to an unknown constant state, the field quantities of which are denoted by the subscript 3. The discontinuity separating the region adjacent to the boundary from the intermediate region is supposed to propagate at the constant speed V_2 and the other discontinuity is supposed to propagate at the constant speed V_4. This assumed form of solution is shown in figure 3. 1.

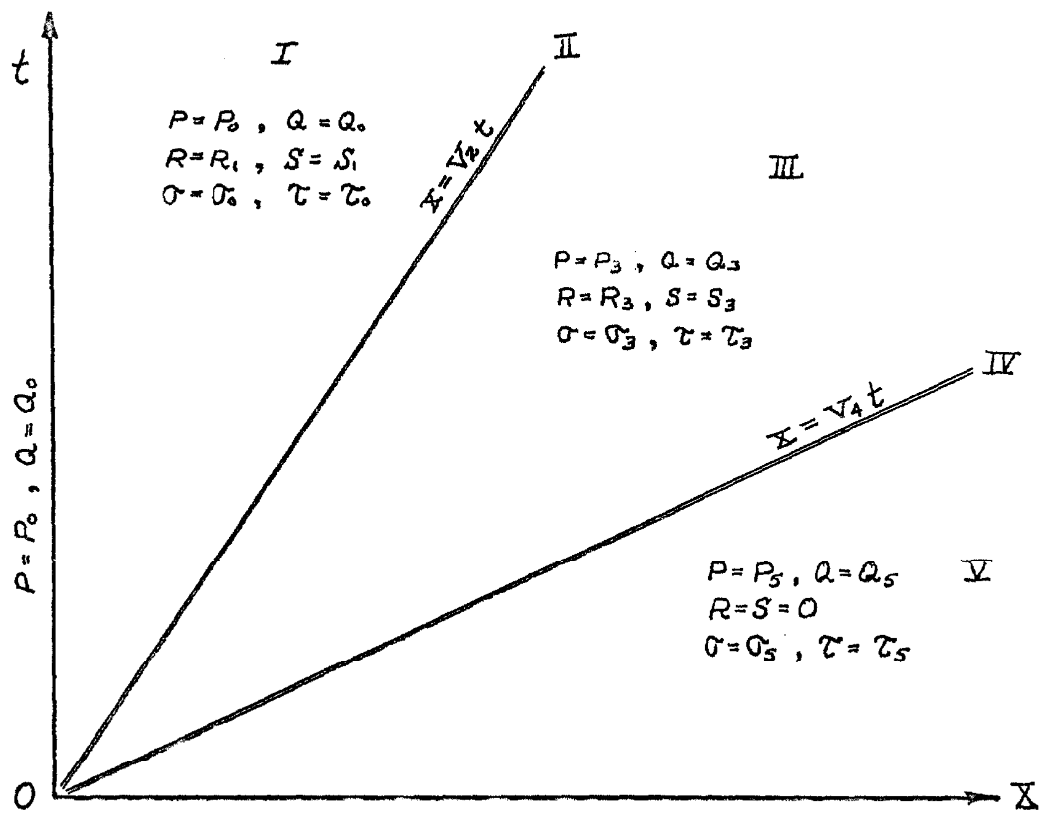


Figure 3. 1

Clearly these fields satisfy the following boundary and initial conditions:

$$P(0,t) = \begin{cases} P_5, & t < 0 \\ P_o, & t > 0 \end{cases}, \quad Q(0,t) = \begin{cases} Q_5, & t < 0 \\ Q_o, & t > 0 \end{cases}, \tag{3.1}$$

$$P(X,0) = P_5, \; Q(X,0) = Q_5, \quad R(X,0) = S(X,0) = 0.$$

Since the assumed fields are constant in the interior of each region the field equations are trivially satisfied there.

Substitution of the assumed fields into the jump conditions (2. 47) and algebraic reduction gives

$$
\begin{aligned}
&(P_o-P_3)(\tau_o-\tau_3)-(Q_o-Q_3)(\sigma_o-\sigma_3) = 0,\\
&(P_3-P_5)(\tau_3-\tau_5)-(Q_3-Q_5)(\sigma_3-\sigma_5) = 0,\\
&V_2 = \sqrt{\frac{\tau_o-\tau_3}{\rho_o(Q_o-Q_3)}}, \qquad V_4 = \sqrt{\frac{\sigma_3-\sigma_5}{\rho_o(P_3-P_5)}},\\
&R_3 = -(P_3-P_5)V_4, \qquad S_3 = -(Q_3-Q_5)V_4,\\
&S_1 = (Q_3-Q_o)V_2 + (Q_5-Q_3)V_4,\\
&R_1 = (P_3-P_o)V_2 + (P_5-P_3)V_4,
\end{aligned}
\tag{3. 2}
$$

where explicit use has been made of the inequalities $V_2 > 0$ and $V_4 > 0$ implicit in the assumed form of the solution shown in figure 3. 1, and where the stresses are related to the displacement gradients through the constitutive equations (2. 48).

Examination of (3. 2) reveals that the question of existence of solutions of the assumed form reduces to the question of existence of solutions P_3 and Q_3 of $(3.1)_{1,2}$, and confirmation of the requirement $V_4 > V_2$.

In order that any solutions of the jump conditions thus found be admissible as shocks they must satisfy conditions given by Lax[1] in the following operational form: At a point on a line of discontinuity in the (X, t)-plane draw, issuing in the positive t-direction, the characteristics with respect to the state ahead of the discontinuity which fall in the region ahead of the discontinuity, and those with respect to the state behind the discontinuity which fall into the region behind the discontinuity. A solution of the jump conditions is admissible as a shock if the number of characteristics drawn in this manner is three.

Lax provides some analytic motivation for the use of such an admissibility condition. In the present context it may be noted that this admissibility condition, as applied to the longitudinal jumps, is equivalent to the condition given by Bland[10] based either on thermodynamic or on mechanical stability considerations.

When jumps are typified as longitudinal or shear in correspondence to the distinction previously made for smooth waves, and when it is assumed, as before, that at each point the characteristic shear wavespeed is less than the characteristic longitudinal wavespeed, then the above admissibility condition may be given as follows: The propagation speed of a jump admissible as a shock is supersonic with respect to the corresponding characteristic wavespeed evaluated ahead of the jump and subsonic with respect to the corresponding characteristic wavespeed evaluated behind the jump. For the jump II this admissibility condition may be expressed analytically as

$$(a_3)_{III} < V_2 < (a_3)_I, \qquad V_2 < (a_4)_{III}, \tag{3. 3}$$

and, for the jump IV, as

$$(a_4)_V < V_4 < (a_4)_{III}, \qquad (a_3)_{III} < V_4, \tag{3. 4}$$

where the subscripts denote the region in which the wavespeeds, as given by (2. 53), are to be evaluated. In addition to the inequalities (3. 3) and (3. 4) it is required that $V_2 < V_4$. This set of inequalities is equivalently, and more symmetrically, given as:

$$\begin{gathered}(a_3)_{III} < V_2 < (a_3)_I \\ V_2 < V_4 \\ (a_4)_V < V_4 < (a_4)_{III}\,. \end{gathered} \tag{3. 5}$$

It is obvious that some restrictions must be placed on the functions $\sigma(P, Q)$ and $\tau(P, Q)$ if solutions of the assumed form are to exist and to satisfy the admissibility conditions. In order that the shock speeds be real it is necessary and sufficient that

$$(\tau_o - \tau_3)(Q_o - Q_3) > 0, \qquad (\sigma_3 - \sigma_5)(P_3 - P_5) > 0, \tag{3. 6}$$

as is seen from $(3.\ 2)_{3,\ 4}$. The admissibility conditions (3. 5) place further restrictions on the material constitution. The question of establishing appropriate constitutive restrictions is the central one of the present problem.

In the remainder of this chapter various special cases of the general problem posed in this section are discussed.

3. 1. 1. Application of Normal Strains to the Boundary of a Normally Strained Half-Space

This section is devoted to consideration of the problem of sudden application of normal strain to the boundary of a normally strained half-space. The problem is essentially that considered previously by Bland,[9, 10] but is included here for completeness and because the results are needed in later sections of this thesis. This problem is seen to be a degenerate case of the problem of the previous section and its solution is obtained from (3. 2) by substituting the further restrictions $Q_o = Q_5 = 0$ of the new problem, along with their consequences $\tau_o = \tau_5 = 0$, and by noting that in this case the jump II must vanish. The resulting situation is as shown in the (X, t)-plane of figure 3.2.

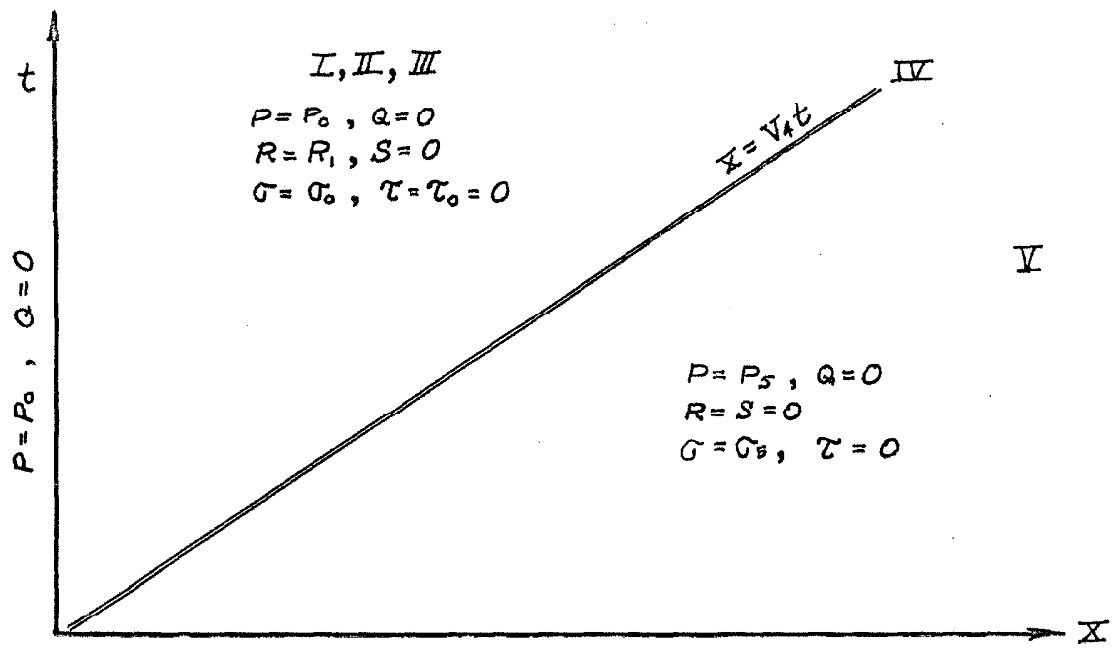


Figure 3. 2

The field quantities are given in terms of the boundary and initial conditions as

$$V_4 = \sqrt{\frac{\sigma_o - \sigma_5}{\rho_o(P_o - P_5)}}, \quad R_1 = -(P_o - P_5)V_4. \tag{3. 7}$$

The displacement can be computed from (3. 7) if desired; the result is

$$U_1 = \begin{cases} P_o X - (P_o - P_5) V_4 t, & X - V_4 t < 0 \\ P_5 X, & X - V_4 t > 0. \end{cases}$$

For the purposes of the present discussion the positivity of the radicand in $(3.7)_1$ is taken as a constitutive requirement; the necessary inequality,

$$(\sigma_o - \sigma_5)(P_o - P_5) > 0, \tag{3.8}$$

is precisely the T-E inequality discussed in terms of "static plausibility" by Truesdell and Toupin.[12] The characteristic longitudinal wavespeed is

$$a_4 = \sqrt{\left[\sigma_P(P, O) / \rho_o\right]}. \tag{3.9}$$

In order that the jump here considered be a shock it is necessary that the admissibility condition

$$\sqrt{\sigma_P(P_5, O)} < \sqrt{\frac{\sigma(P_o, O) - \sigma(P_5, O)}{P_o - P_5}} < \sqrt{\sigma_P(P_o, O)} \tag{3.10}$$

be satisfied.

This solution illustrates the fact that purely longitudinal waves can propagate unaccompanied by any waves involving transverse displacement.

It may be observed here that if the stress-strain function $\sigma = \sigma(P, O)$ is invertable the solutions here presented are immediately interpretable in terms of a boundary value problem in which the normal stress σ, rather than the displacement gradient P, is prescribed at the boundary $X = 0$.

3.1.1.1. Application of Normal Strain to the Boundary of an Undeformed Half-Space

It is of interest to further simplify the problem of the above paragraph to the case $P_5=0$, so that in advance of the disturbance the material is undeformed. In this case σ_5 is also zero and the solution (3.7) becomes

$$V_4 = \sqrt{\frac{\sigma(P_o, O)}{\rho_o P_o}} \quad , \quad R_1 = -P_o V_4 . \tag{3.11}$$

The admissibility condition (3.10) becomes

$$\sqrt{(\lambda+2\mu)} < \sqrt{[\sigma(P_o, O)/P_o]} < \sqrt{\sigma_P(P_o, O)}, \tag{3.12}$$

where the left member is taken in accordance with the limit given by (2.57). The requirement (3.12) can be conveniently described as one of convexity: $|\sigma(P, O)|$ must be convex for $P > 0$ or for $P < 0$ as the case may be. A necessary and sufficient condition that (3.12) be satisfied for all P_o of interest is that $\sigma_P(P,O)$ be monotone increasing for $P > 0$ and monotone decreasing for $-1 < P < 0$. Materials satisfying (3.12) for all $P_o > -1$ can be called monotonically hardening in uniaxial extension. If (3.12) is satisfied for $P_o > 0$ (for $-1 < P_o < 0$) then the material is called monotonically hardening in tension (compression). An example stress-strain curve for a material monotonically hardening in uniaxial extension is given as figure 3.3.

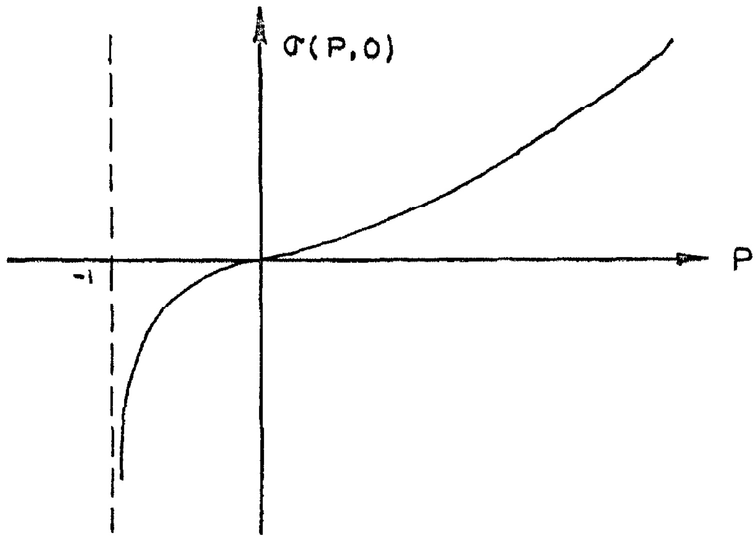


Figure 3. 3

This problem is clarified by considering, by way of example, the second order theory outlined in the appendix. In this case

$$\sigma(P, O) = \rho_o a_{oo} P + \frac{1}{2} \rho_o a_{10} P^2 + \ldots \tag{3.13}$$

where $a_{oo} > 0$. The exact results (3.11) become

$$V_4 = \sqrt{a_{oo}} \left(1 + \frac{a_{10}}{4a_{oo}} P_o + \ldots \right), \quad R_1 = -P_o V_4, \tag{3.14}$$

and, to first approximation, the admissibility condition (3.12) becomes

$$a_{10} P_o > 0 \quad . \tag{3.15}$$

The tensile jump is an admissible solution if the material is such that $a_{10} > 0$; otherwise the compression jump is admissible. It is seen that, for example, compression jumps are not admissible in materials for which $a_{10} < 0$. Since there is no obvious reason why a compressive strain could not be applied to the boundary of such a material it is presumed that solutions other than those of the presently assumed form exist. This is indeed the case, and in a later section it will be shown that for $\alpha_{10}P_o < 0$ a smooth wave solution exists.

3.1.1.2. Removal of Strain from the Boundary of a Normally Strained Half-Space

The case in which the half-space is initially deformed so that $P=P_5$ throughout is now considered. If at t=0 the strain is removed from the boundary, the solution (3.7) becomes

$$V_4 = \sqrt{[\sigma(P_5, O)/\rho_o P_5]}, \quad R_1 = P_5 V_4, \tag{3.16}$$

and the admissibility condition (3.10) takes the form

$$\sqrt{\sigma_P(P_5, O)} < \sqrt{[\sigma(P_5, O)/P_5]} < \sqrt{(\lambda+2\mu)}. \tag{3.17}$$

Materials having the property (3.17) for all $P_5 > -1$ are said to be monotonically softening in uniaxial extension.

In second order approximation (3.16) becomes

$$V_4 = \sqrt{\alpha_{oo}}\left(1 + \frac{\alpha_{10}}{4\alpha_{oo}} P_5 + \ldots\right), \quad R_1 = P_5 V_4, \tag{3.18}$$

and, for satisfaction of the admissibility condition (3.17) to first approximation it is necessary and sufficient that

$$\alpha_{10} P_5 < 0. \tag{3.19}$$

This condition is opposite to (3.15); it implies that unloading of an elongated body takes place by means of a shock if $\alpha_{10} < 0$. If $\alpha_{10} > 0$ a compressed body can be unloaded by a shock. It will be seen in subsequent chapters that the remaining cases involve propagation of smooth waves.

From the preceeding examples one sees that, assuming the material is admissible in the sense that the wavespeed is real ($a_{oo} > 0$), the sign of the second coefficient, that is, the curvature of the stress-strain curve, is the crucial feature so far as the nature of admissible waves is concerned. This situation will be seen to prevail in more complicated situations where, however, the resulting inequalities are not so easily visualized in terms of a simple experiment. The fact that the admissibility conditions (3. 15) for the loading and (3. 19) for the unloading problems are satisfied by materials of opposite character is true of the exact conditions (3. 12) and (3. 17) as well and, furthermore, carries over to problems involving shear waves. For this reason only loading problems are considered subsequently.

3. 1. 2. Application of Shear Strain to the Boundary of an Undeformed Half-Space

The complexity of problems involving shear wave propagation is such that the treatment is best restricted to consideration of loading of an initially undeformed half-space or unloading of an initially sheared half-space, since any more general problem is not significantly simpler than the case of section 3. 1.

In the previous sections the essential difference between the loading and the unloading problem was seen to be in the reversal of comparisons in the admissibility conditions for shocks. This situation prevails in shear wave problems as well, and consequently only loading waves are explicitly considered in the following work.

The problem of application of shear strain to the boundary of an undeformed half-space is covered by equations (3. 2) and figure 3. 1 with the specializations

$$P_5 = Q_5 = 0, \quad P_o = 0 \tag{3. 20}$$

to account, respectively, for the facts that the body is initially undeformed and that no normal strains are applied at the boundary. Consistent with the assumption that longitudinal disturbances propagate faster than shear disturbances and the observation that they propagate into an unsheared body unaccompanied by shear disturbances it is conjectured that

$$Q_3 = 0. \tag{3. 21}$$

By (3. 20) and (3. 21), (3. 2) becomes

$$\begin{aligned}
&P_3 \tau_o + Q_o(\sigma_o - \sigma_3) = 0 \\
&V_2 = \sqrt{(\tau_o / \rho_o Q_o)}, \qquad V_4 = \sqrt{(\sigma_3 / \rho_o P_3)}, \\
&R_3 = -P_3 V_4, \qquad S_3 = 0, \\
&R_1 = -P_3(V_4 - V_2), \qquad S_1 = -Q_o V_2.
\end{aligned} \tag{3. 22}$$

Formally, the solution of the problem is now obtained when $(3. 22)_1$ is solved for P_3 in terms of Q_o; the remaining equations in (3. 22) serve to determine the remaining quantities directly. The question of the existence of a solution $P_3 = P_3(Q_o) > -1$ of $(3. 22)_1$, and of the admissibility of such a solution as a shock remains to be settled.

It is not to be expected that solutions of $(3. 22)_1$ exist for all functions $\sigma(P, Q)$ and $\tau(P, Q)$ nor that all solutions which may exist will be admissible as shocks. Indeed, if smooth wave solutions are to exist it must be expected that there be some cases where either solutions $P_3 = P_3(Q_o)$ of $(3.22)_1$ fail to exist, or exist but are not admissible as shocks.

Conditions (3.5) that the jumps be admissible as shocks are that

$$\sqrt{\tau_Q(P_3, 0)} < \sqrt{[\tau(0, Q_o)/Q_o]} < \sqrt{\left\{\tfrac{1}{2}\left[\sigma_P + \tau_Q - \left((\sigma_P - \tau_Q)^2 + 4\sigma_Q\tau_P\right)^{1/2}\right]\right\}_{P=0,\, Q=Q_o}},$$

$$\sqrt{[\tau(0, Q_o)/Q_o]} < \sqrt{[\sigma(P_3, 0)/P_3]}\,, \qquad (3.23)$$

$$\sqrt{(\lambda + 2\mu)} < \sqrt{[\sigma(P_3, 0)/P_3]} < \sqrt{\sigma_P(P_3, 0)}.$$

Since P_3 and Q_o appearing in (3.23) are related by $(3.22)_1$ these inequalities cannot immediately be regarded solely as constitutive restrictions. However, $(3.23)_3$ is certainly satisfied for a material monotonically hardening in extension. Similarly, since, by (2.42) and (2.53),

$$(a_3)_{P=0,\, Q=Q_o} < \sqrt{\tau_Q(0, Q_o)},$$

it is seen from the right hand inequality of $(3.23)_1$ that a material monotonically hardening in shear is necessary in order that a solution of the form considered may be admissible as a shock. Since a solution of $(3.22)_1$ which fails to satisfy (3.23) is of no use in the present connection, conditions of this sort can be placed upon

$\sigma(P, Q)$, and $\tau(P, Q)$ as hypotheses of an existence theorem for a solution of $(3.22)_1$.

Equation $(3.22)_1$ is now considered with a view toward establishing some simple sufficient conditions for the existence of a solution. With

$$\begin{aligned} A(P;Q_o) &= \sigma(O, Q_o)/P \\ B(P;Q_o) &= \sigma(P, O)/P - \tau(O, Q_o)/Q_o \end{aligned} \tag{3.24}$$

$(3.22)_1$ takes the form

$$A(P_3; Q_o) = B(P_3;Q_o). \tag{3.25}$$

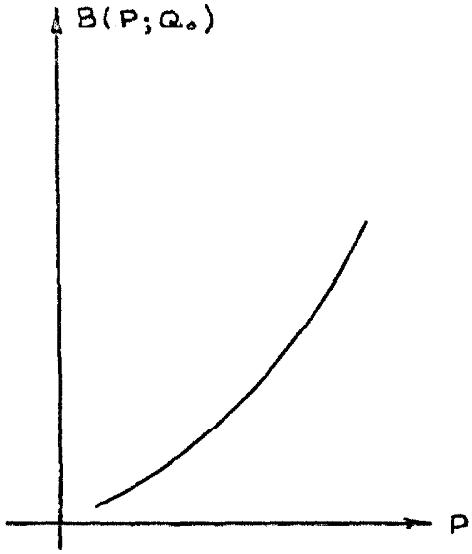
It is assumed, as a constitutive requirement, that

$$\sigma(P, O)/P > 0, \quad \tau(O, Q_o)/Q_o > 0. \tag{3.26}$$

The limiting values of these ratios as P, $Q_o \to 0$ exist by assumption and are given by

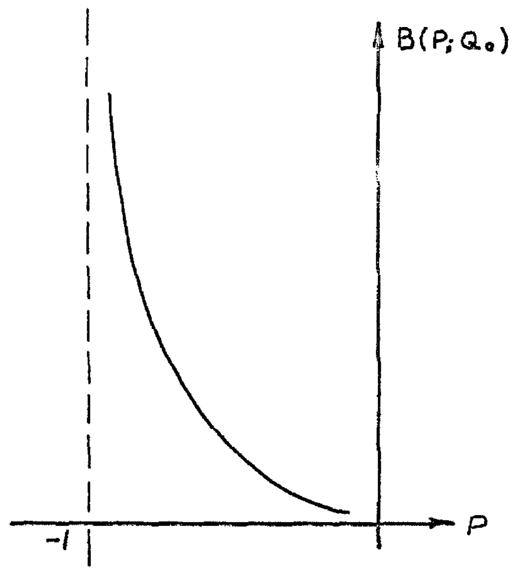
$$\sigma(P, O)/P \to \lambda + 2\mu, \quad \tau(O, Q_o)/Q_o \to \mu, \tag{3.27}$$

where λ and μ are the Lame modulii of the linear theory. Based on the results obtained in section 3.1.1.1 convexity of $|\sigma(P, O)|$ is assumed since it is necessary for the admissibility of any solution which may exist. Also, as observed in this section, attention may be restricted to materials monotonically hardening in shear. Whether $B(O;Q_o)$ is positive or negative is immaterial in the present case, but the hardening requirement implies that for sufficiently large P, and fixed Q_o, $B(P;Q_o) > 0$. In this case $B(P;Q_o)$ has the form shown in figure 3.4A and/or 3.4B. The character of the



HARDENING IN TENSION

Figure 3. 4A



HARDENING IN COMPRESSION

Figure 3. 4B

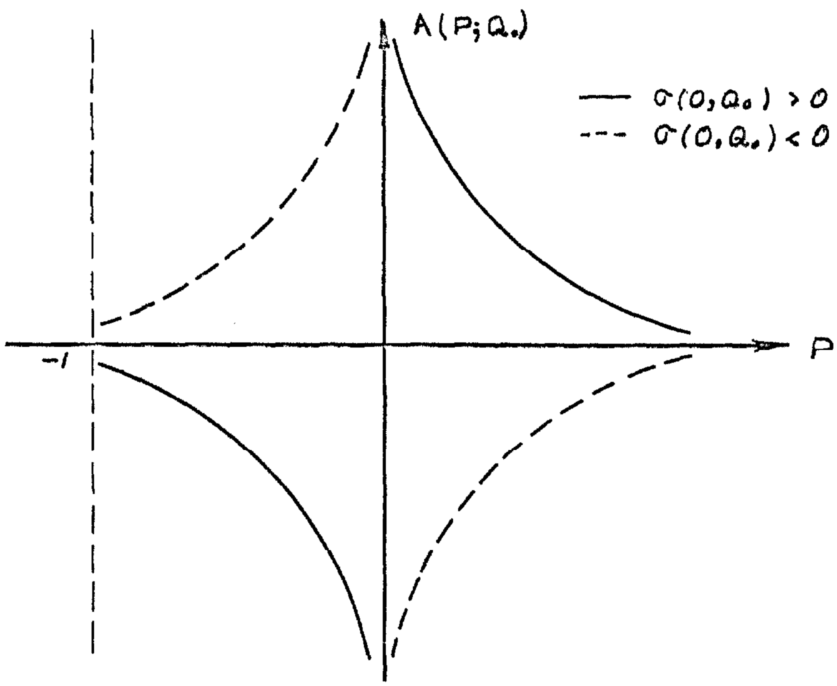


Figure 3. 5

function $A(P;Q_o)$ varies with the sign of $\sigma(O, Q_o)$ as shown in figure 3. 5. Superimposing figure 3. 5 on each part of figure 3. 4 reveals that:

1) If the material is hardening in both tension and compression there exists a solution P_3 of $(3.22)_1$ whatever sign $\sigma(O, Q_o)$ may have.
2) If the material is hardening in tension there exists a solution $P_3 > 0$ of $(3.22)_1$ provided $\sigma(O, Q_o) > 0$.
3) If the material is hardening in compression there exists a solution $P_3(-1 < P_3 < 0)$ of $(3.22)_1$ provided $\sigma(O, Q_o) < 0$.

There may be solutions in some other special circumstances as well. For all three cases above the left hand parts of the pairs of inequalities $(3.23)_3$ and $(3.23)_2$ are satisfied. If the left hand part of $(3.23)_1$ is satisfied then $(3.22)_2$ plus the condition that the material be hardening in shear implies the right hand part of $(3.23)_3$.

To illustrate this loading shear wave problem it is convenient to consider the third order theory outlined in the appendix. According to (A. 9) the stresses are given by

$$\frac{1}{\rho_o}\sigma(P, Q) = \alpha_{oo}P + \frac{1}{2}\alpha_{10}P^2 + \frac{1}{2}\beta_{01}Q^2 + \frac{1}{2}\beta_{11}PQ^2 + \frac{1}{3}\alpha_{20}P^3 + \dots$$
$$\frac{1}{\rho_o}\tau(P, Q) = \delta_{oo}Q + \delta_{10}PQ + \delta_{20}P^2Q + \frac{1}{3}\delta_{02}Q^3 + \dots \quad . \tag{3. 28}$$

In terms of these stress-strain relations $(3.22)_1$ has the solution

$$P_2 = \frac{\beta_{01} Q_o^{\,2}}{2(\alpha_{oo} - \delta_{oo})} + \dots , \tag{3. 29}$$

and, with this, $(3.\ 22)_{2,\ 3}$ become

$$\begin{aligned} V_2 &= \sqrt{\delta_{oo}} \left(1 - \frac{\beta_{01}\, \gamma_{01}}{4\delta_{oo}(\alpha_{oo} - \delta_{oo})} Q_o^{\,2} + \dots \right) \\ V_4 &= \sqrt{\alpha_{oo}} \left(1 + \frac{\alpha_{10}\, \beta_{01}}{\alpha_{oo}(\alpha_{oo} - \delta_{oo})} Q_o^{\,2} + \dots \right) . \end{aligned} \tag{3. 30}$$

It is required here, as in (2. 56), that $\alpha_{oo} > \delta_{oo}$, hence for sufficiently small $Q_o^{\,2}$ (3. 30) implies $V_2 < V_4$. Similarly, assuming $\alpha_{oo} > \delta_{oo}$ and monotonicity, the admissibility conditions are found to be

$$\alpha_{10}\beta_{01} > 0, \quad 0 < \delta_{02} - \frac{\delta_{10}\beta_{01}}{\alpha_{oo} - \delta_{oo}} . \tag{3. 31}$$

3. 2 Summary of Shock Solutions

In section 3. 1 the shock waves arising as result of a change of strain at the boundary of a deformed half-space were considered. The problem was reduced to the solution of two algebraic equations but these equations were not solved explicitly. In section 3. 1. 1 this problem was simplified by eliminating shear effects from the initial state of deformation and from the boundary conditions. This resulted in an explicitly solvable problem which was considered in its generality and also in the special cases of loading of an undeformed body and unloading of a deformed body. Conditions for admissibility of shocks as solutions of these problems were seen to be that, in the loading case, the material be of hardening type and, in the unloading case, that the material be of the monotonically softening type. These conditions are familiar to workers in other areas of nonlinear wave propagation.

In section 3. 1. 2 the shear loading problem was discussed. It was shown that the shear disturbances there considered could not propagate alone, but were always accompanied by longitudinal waves. Conditions of admissibility were given which, as in the normal loading case, could be described as requirements that the appropriate stress-strain curves be of hardening character. Since two jumps were involved in the solution of this problem, separate admissibility conditions had to be met on each one. For the longitudinal jump (IV) the admissibility condition was, as before, that the material be monotonically hardening in uniaxial extension. For the shear jump (II) it was seen to be necessary that the material be monotonically hardening in shear. The exact statement of the necessary and sufficient condition for this jump is given in $(3.23)_1$.

Various of the above problems were illustrated by second-order solutions which, presumably accurate for moderate deformation gradients, have the virtue of displaying very clearly the effects of the nonlinearities involved.

CHAPTER IV

PROPAGATION OF CENTERED SIMPLE WAVES

In the previous chapter various problems of sudden application of strain to the boundary of a half-space were found to be solved in terms of shocks. Other problems, seemingly well set, could not be solved in this manner because the admissibility conditions were not satisfied. It is reasonable to assume that in these cases the disturbance will propagate as a smooth wave. For the present purpose, waves are considered to be smooth if the field variables P, Q, R, S are continuously differentiable with respect to X and to t except perhaps at certain propagating singular surfaces across which it is required that the field variables themselves be continuous. Such waves must be solutions of the field equations (2. 45):

$$\begin{aligned} \alpha P_X + \beta Q_X &= R_t, \\ \gamma P_X + \delta Q_X &= S_t, \\ R_X &= P_t, \\ S_X &= Q_t. \end{aligned} \tag{4. 1}$$

As in the previous chapter, the problems to be considered involve boundary conditions in which the displacement gradients P and Q are prescribed step functions of time. In accordance with Lax's results,[(1)] the fact that the boundary conditions are step functions implies that smooth solutions are made up of centered simple waves and regions of uniform state. The centered simple waves are familiar to workers in other areas of nonlinear wave propagation, for example gasdynamics,[3] and are discussed in later sections of this chapter.

In section 4.1 the simplified problem involving only longitudinal disturbances is discussed. In sections 4.1.1 and 4.1.2 this problem is further specialized to cases of loading waves and unloading waves, respectively. In section 4.2 the problem of application of shear strain to the boundary of an unloaded half-space is treated. The problems of this chapter have exact counterparts in the previous and the following chapters; the difference lies in the class of materials to which the solutions are applicable.

4.1. Application of Normal Strain to the Boundary of a Normally Strained Half-Space

This section is devoted to consideration of the problem of wave propagation occurring in a normally strained half-space when the normal strain applied at the boundary is suddenly changed. This problem will be seen to be analogous to the shock tube problem of gasdynamics[3] and to the shear wave problem in an incompressible elastic half-space.[8] It has been rather completely discussed by Bland[10] but is briefly included here for completeness and to permit ready comparison with the results of other parts of this thesis. Only centered simple waves are considered.

For the present problem symmetry considerations indicate that a solution be sought for which Q=0 and S=0 everywhere in the domain. Under this hypothesis the field equations become

$$a(P)P_X = R_t, \qquad R_X = P_t, \tag{4.2}$$

where, from (2.46),

$$a(P) \equiv a(P,0) = \sqrt{\left[\sigma_P(P,0)/\rho_0\right]}.$$

Centered simple wave solutions are those of the form

$$P = P(Z), \qquad R = R(Z), \tag{4.3}$$

where

$$Z = X/t. \tag{4.4}$$

In this case (4.2) becomes

$$\begin{aligned} a(P)P_Z + ZR_Z &= 0, \\ ZP_Z + R_Z &= 0, \end{aligned} \tag{4.5}$$

and elimination of R gives

$$\left[a(P) - Z^2\right]P_Z = 0. \tag{4.6}$$

For a non-trivial solution of (4.6) it is necessary that

$$Z^2 = a(P). \tag{4.7}$$

From this equation P can be obtained in the form

$$P = P(Z), \qquad V_{34} < Z < V_{45}, \tag{4.8}$$

provided that $a(P)$ is a monotone function increasing as P varies from P_0 to P_5.

The situation is as shown in the (X, t)-plane of figure 4.1.

According to (4.7),

$$V_{34} = \sqrt{a(P_0)} = \sqrt{\left[\sigma_P(P_0,0)/\rho_0\right]}, \quad V_{45} = \sqrt{a(P_5)} = \sqrt{\left[\sigma_P(P_5,0)/\rho_0\right]}. \tag{4.9}$$

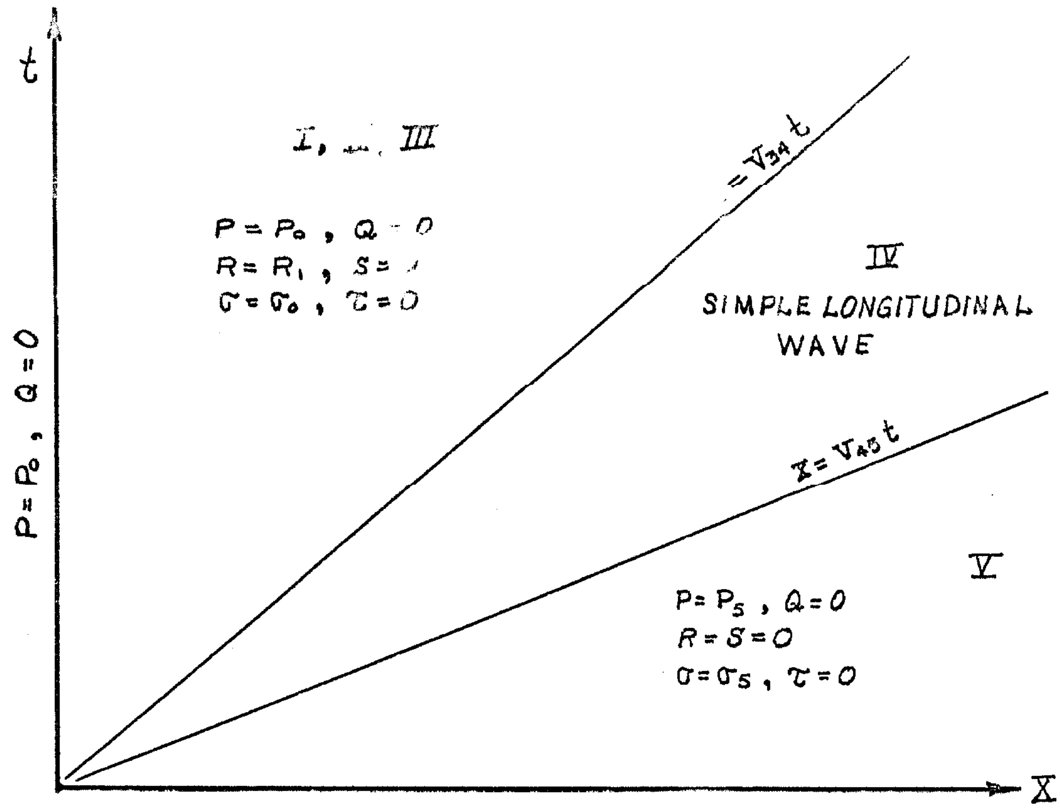


Figure 4. 1

Substitution of (4. 7) into (4. 5) and integration gives

$$R(Z) = 2\int_{Z}^{V_{45}} \frac{a(P(Z'))}{a_P(P(Z'))}\, dZ', \tag{4. 10}$$

for $V_{34} \leqslant Z \leqslant V_{45}$. R_1 is obtained from (4. 10) as $R(V_{34})$, and this completes the solution of the problem.

As is seen from figure 4. 1, Eq. (4. 7), and the monotonicity assumption on $a(P)$, it is necessary that $V_{34} < \sqrt{a(P)} < V_{45}$ for all P between P_0 and P_5:

$$\sqrt{\sigma_P(P_0, 0)} < \sqrt{\sigma_P(P, 0)} < \sqrt{\sigma_P(P_5, 0)}. \qquad (4.11)$$

Comparison of the extreme members of (4. 11) with those of (3. 10) shows that situations where the shock solution is admissible and where a simple wave solution exists are complementary. In this sense the admissibility condition here employed for shocks can be replaced by the principle of using smooth wave solutions where they exist and otherwise admitting shocks.

4. 1. 1. Application of Normal Strain to the Boundary of an Unstrained Half-Space

In the case of longitudinal waves propagating into a body unstrained and at rest the solutions of the previous section hold with

$$P_5 = 0. \qquad (4.12)$$

With this simplification (4. 9) and (4. 11) reduce to

$$V_{34} = \sqrt{[\sigma_P(P_0, 0)/\rho_0]}, \quad V_{45} = \sqrt{[\sigma_P(0, 0)/\rho_0]} = \sqrt{[(\lambda+2\mu)/\rho_0]}, \qquad (4.13)$$

and

$$\sqrt{\sigma_P(P_0, 0)} < \sqrt{\sigma_P(P, 0)} < \sqrt{(\lambda+2\mu)} \qquad (4.14)$$

for all P between P_0 and zero. Observe that (4. 14) is characteristic of softening materials. From $(4.13)_2$ one sees that a smooth longitudinal wave, however strong, advances into the undeformed body at the linear dilitation wavespeed. This is to be expected, since the continuity properties of such waves indicate that $\sqrt{[\sigma_P(P, 0)/\rho_0]}$ must be near to its limiting value for P=0 at the wave front.

For illustration of the results of this section an example is given using the second order theory outlined in the appendix. By (A. 8 $)_1$,

$$a(P) = a_{00} + a_{10}P + O(P^2), \qquad \text{as } P \to 0 . \tag{4. 15}$$

Substitution of this equation into (4. 7), the exact solution in the simple wave region, gives

$$Z^2 = a_{00} + a_{10}P + O(P^2), \tag{4. 16}$$

and the wave speeds of interest are

$$V_{34} = \sqrt{a_{00}}\left(1 + \frac{a_{10}}{2a_{00}}P_0 + \ldots\right), \quad V_{45} = \sqrt{a_{00}} \, , \tag{4. 17}$$

with $\rho_0 \, a_{00} = \lambda + 2\mu$. Condition (4. 14) for the existence of a smooth solution becomes

$$a_{10}P_0 < 0,$$

which is complementary to (3. 15), the admissibility condition for a shock.

4. 1. 2. Unloading of a Normally Strained Half-Space

The unloading problem for a normally strained half-space is included in the case of section 4. 1. Those results are reduced to the present situation by letting

$$P_0 = 0. \tag{4. 19}$$

With this simplification (4. 9) becomes

$$V_{34} = \sqrt{[(\lambda+2\mu)/\rho_0]}, \quad V_{45} = \sqrt{[G_P(P_5, 0)/\rho_0]}, \tag{4. 20}$$

and (4.11) becomes

$$\sqrt{(\lambda+2\mu)} < \sqrt{G_P(P,0)} < \sqrt{\sigma_P(P_5,0)}, \tag{4.21}$$

for all P between zero and P_5, an inequality which characterizes monotonically hardening materials. Comparison of the extreme members of (4.21) with those of (3.17) shows, again, that the cases where smooth waves exist are complementary to those where shocks are admissible.

Calculation of a second order example can be carried out just as in the previous section.

4.2. Application of Shear Strain to the Boundary of an Unloaded Half-Space

As in the case of shocks (section 3.2.2), smooth shear waves cannot propagate unaccompanied by dilitation effects. This section is devoted to investigation of smooth waves analogous to the shocks of section 3.2.2.1.

Consider an elastic half-space $X \geqslant 0$ initially at rest and unstrained so that

$$P(X,0) = Q(X,0) = 0, \qquad R(X,0) = S(X,0) = 0. \tag{4.22}$$

Suppose that the displacement gradients

$$P(0,t) = 0, \qquad Q(0,t) = \begin{cases} 0, & t < 0 \\ Q_0, & t > 0 \end{cases}, \tag{4.23}$$

are applied. Since all waves propagate with finite speed there is a rest zone ahead of the disturbance, and since the boundary conditions do not change after a certain time there is a uniform state behind the disturbance. In accordance with Lax's[(1)] results the regions adjacent to regions of uniform state are simple waves, and because the boundary conditions are step functions of time these simple waves are centered. A solution of this form is assumed for the present problem and is depicted in the (X, t)-plane of figure 4. 2.

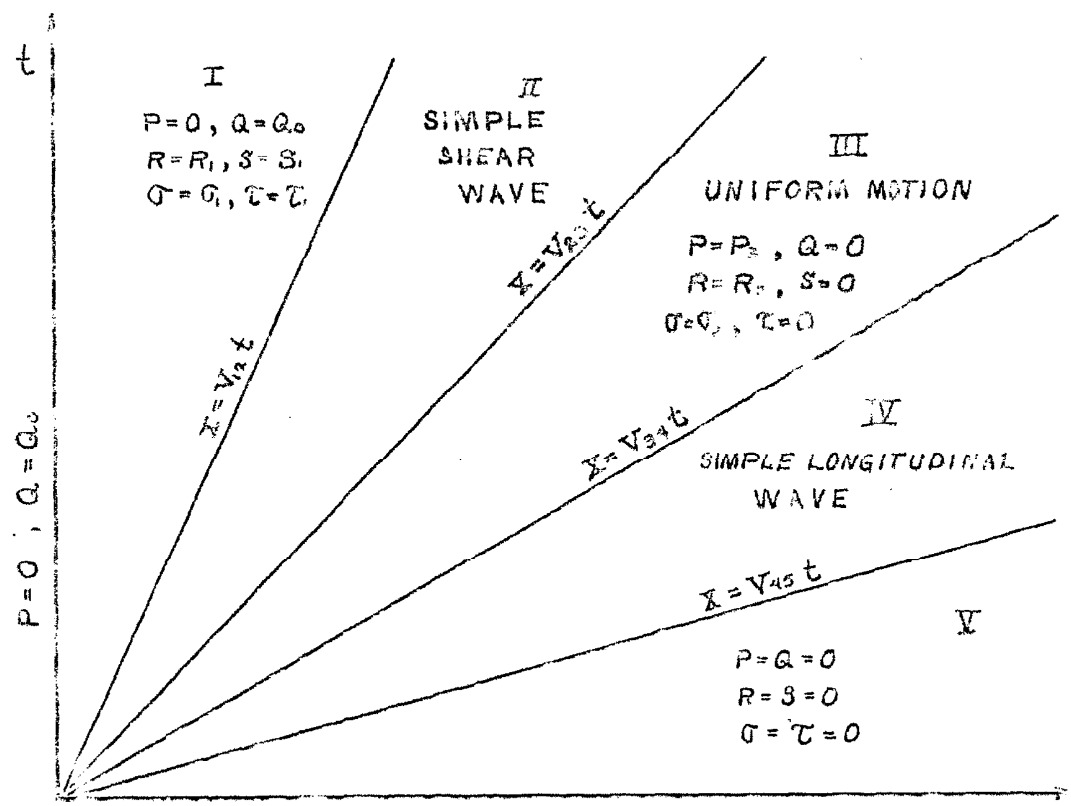


Figure 4. 2

This assumed solution involves a longitudinal wave in region IV, followed by a region of uniform motion, region III, also not involving shear effects, propagating ahead of the shear disturbance into the rest zone. This is consistent with the assumption that longitudinal waves propagate faster than shear waves. Region II is occupied by the centered simple shear wave which is the main feature of the solution. The region between the trailing edge of the shear wave and the boundary is, again, one of uniform motion.

The process of constructing in detail the solution described above may be broken down into several parts. The first step is to obtain a centered simple wave solution in region IV, about which everything is known except the normal displacement gradient P_3 on the boundary between regions III and IV. Next the centered simple shear wave of region II is constructed so that it matches the uniform state in region I. It remains to fix P_3 and the boundaries of region III so that the field variables are continuous across these boundaries, and, finally, to write down the conditions under which all the above steps can be carried out.

Solution in Region IV

Under the assumption $Q=0$ in region IV the field equations there become

$$\alpha(P)P_X = R_t, \qquad R_X = P_t, \tag{4.23}$$

with the conditions on the boundary of the region that $P=0$ when $X/t = V_{45}$ and $P=P_3$ when $X/t = V_{34}$. This problem is seen to reduce to one of the type considered in section 4.1.1, except that here P_3 is not yet determined. The results of that section give

$$V_{45} = \sqrt{[(\lambda + 2\mu)/\rho_0]}, \quad V_{34} = \sqrt{[\sigma_P(P_3, 0)/\rho_0]}. \qquad (4.25)$$

As in section 4.1.1 the solution in the longitudinal wave region is obtained from

$$(X/t)^2 = a(P),$$
$$R(X/t) = 2 \int_{X/t}^{\sqrt{[(\lambda+2\mu)/\rho_0]}} \frac{a(P(Z))}{a_P(P(Z))} \, dZ, \qquad (4.26)$$

by inversion of $(4.26)_1$, to obtain $P = P(X/t)$, and substitution of this result into $(4.26)_2$. From figure 4.2 it is seen that $V_{34} < V_{45}$ is a necessary condition for existence of this longitudinal wave. When this condition is coupled with the invertibility requirement placed on $(4.26)_1$ the condition

$$\sqrt{[\sigma_P(P_3, 0)/\rho_0]} < \sqrt{a(P)} < \sqrt{[(\lambda + 2\mu)/\rho_0]} \qquad (4.27)$$

for all P between P_3 and zero results.

Solution in Region II

Consideration of the simple wave occupying region II is more troublesome because the full set of field equations (4.1) is involved. Since the disturbance in this region is a centered simple wave, solutions are sought in the form

$$P = P(Z), \quad Q = Q(Z), \quad R = R(Z), \quad S = S(Z), \qquad (4.28)$$

where $Z = X/t$. Substitution of (4.28) into (4.1) gives

$$\begin{aligned} \alpha P_Z + \beta Q_Z + Z R_Z &= 0, \\ \gamma P_Z + \delta Q_Z + Z S_Z &= 0, \\ R_Z + Z P_Z &= 0, \\ S_Z + Z Q_Z &= 0. \end{aligned} \tag{4. 29}$$

Since α, β, γ, and δ are functions of P and Q alone, R and S can be eliminated from (4. 29) to give the second order quasi-linear system,

$$\begin{aligned} (\alpha - Z^2) P_Z + \beta Q_Z &= 0, \\ \gamma P_Z + (\delta - Z^2) Q_Z &= 0, \end{aligned} \tag{4. 30}$$

which describes the solution in region II. The initial conditions for (4. 30) are

$$P(V_{12}) = 0, \qquad Q(V_{12}) = Q_0 , \tag{4. 31}$$

as is seen from figure 4. 2. A necessary condition for a non-trivial solution of (4. 30) is that

$$(\alpha - Z^2)(\delta - Z^2) - \beta\gamma = 0,$$

or,

$$Z^4 - (\alpha - \delta) Z^2 + (\alpha\delta - \beta\gamma) = 0. \tag{4. 32}$$

With this (4. 30) gives

$$\frac{dP}{dQ} = - \frac{\beta}{\alpha - Z^2} . \tag{4. 33}$$

The solutions of (4. 32) are

$$Z = \pm \left\{ \frac{1}{2} \left[\alpha + \delta \pm \left((\alpha - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2} \tag{4. 34}$$

For the case at hand only solutions $Z > 0$ are appropriate and, in addition, $Z(0, Q_0)$ must be the wavespeed for a characteristic associated with shear wave propagation. On this basis the signs in (4. 34) are chosen such that

$$Z = \left\{ \frac{1}{2} \left[\alpha + \delta - \left((\alpha - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2} . \tag{4. 35}$$

From this,

$$V_{12} = \left\{ \frac{1}{2} \left[\alpha + \delta - \left((\alpha - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2}_{P=0,\ Q=Q_0} . \tag{4. 36}$$

Substitution of (4. 35) into (4. 33) gives

$$\frac{dP}{dQ} = \frac{-2\beta}{\alpha - \delta + \left[(\alpha - \delta)^2 + 4\beta\gamma \right]^{1/2}} , \tag{4. 37}$$

which is a first order ordinary differential equation for $P=P(Q)$. The initial condition is

$$P(Q_0) = 0, \tag{4. 38}$$

and solution is to be obtained on the interval between Q_0 and zero. In accordance with figure 4. 2, P_3 is given as

$$P_3 = P(0). \tag{4. 39}$$

The shear wavefront propagates at the speed

$$V_{23} = \left\{ \frac{1}{2} \left[\alpha + \delta - \left((\alpha - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2}_{P=P_3,\ Q=0}$$

which becomes, since $\beta(P, 0) = 0$,

$$V_{23} = \sqrt{\sigma(P_3, 0)} = \sqrt{\left[\tau_Q(P_3, 0)/\rho_0 \right]} . \tag{4. 40}$$

Substitution of the solution $P=P(Q)$ of (4. 37) into (4. 30) gives

$$\left\{ \left[\alpha(P(Q), Q) - Z^2 \right] P'(Q) + \beta[P(Q), Q] \right\} Q_Z = 0 . \tag{4. 41}$$

For non-trivial solution $Q(Z)$ of (4. 41) it is necessary that

$$Z^2 = \alpha[P(Q), Q] + \beta[P(Q), Q]/P'(Q) \tag{4. 42}$$

and, providing that the right hand member of (4. 42) is a monotone function of Q increasing as Q varies from Q_0 to zero, it is invertable to give a solution

$$Q = Q(Z). \tag{4. 43}$$

In accordance with (4. 35) and $(2.53)_2$, the right hand member of (4. 42) is just the square of the shear wavespeed in region II, as is to be expected, and the condition that it be monotonic can be rephrased as a condition that the wavespeed be greater at the leading edge of the wave and that it decrease through the wave to a minimum at the trailing edge. This condition for the propagation of smooth waves is familiar to workers in other areas of nonlinear wave propagation. Substitution of (4. 43) into the solution of (4. 37) gives P as a function of Z in region II:

$$P = P(Z). \tag{4. 44}$$

With this and (4. 43) equations $(4.29)_{3,4}$ can be integrated to give $R = R(Z)$ and $S = S(Z)$ in region II.

Admissibility

In order that the computed wavespeeds not contradict the implications of figure 4. 2 it is necessary that

$$0 < V_{12} < V_{23} < V_{34} < V_{45} \tag{4. 45}$$

where, as has been shown,

$$V_{12}=\left\{\frac{1}{2\rho_0}\left[\sigma_P+\tau_Q-\left((\sigma_P-\tau_Q)^2+4\sigma_Q\tau_P\right)^{1/2}\right]\right\}^{1/2}_{P=0,\ Q=Q_0},$$

$$V_{23}=\sqrt{[\tau_Q(P_3,0)/\rho_0]},\quad V_{34}=\sqrt{[\sigma_P(P_3,0)/\rho_0]}, \tag{4.46}$$

$$V_{45}=\sqrt{[(\lambda+2\mu)/\rho_0]}.$$

Equations (4. 45) and (4. 46) can be rephrased as

$$(a_3)_I < (a_3)_{III},\quad (a_3)_{III} < (a_4)_{III},\quad (a_4)_{III} < (a_4)_V . \tag{4.47}$$

If account be taken of the invertability requirement on $(4.26)_1$ and (4. 42), this can be extended to

$$\begin{aligned} &(a_3)_I < Z_{II} < (a_3)_{III},\\ &(a_3)_{III} < (a_4)_{III},\\ &(a_4)_{III} < Z_{IV} < (a_4)_V, \end{aligned} \tag{4.48}$$

where

$$\begin{aligned} Z_{II} &= \left\{\frac{1}{2}\left[\alpha+\delta-\left((\alpha-\delta)^2+4\beta\gamma\right)^{1/2}\right]\right\}^{1/2},\\ Z_{IV} &= \sqrt{\alpha}, \end{aligned} \tag{4.49}$$

with the right hand members of (4. 49) permitted to take on all values in the appropriate wave regions. The inequality $(4.48)_2$ reflects the assumption that longitudinal waves propagate faster than shear waves. The inequalities $(4.48)_{1,3}$ express the requirement that the wavespeed must decrease monotonically with passage of a smooth wave. Note that the extreme members of $(4.48)_{1,3}$ form a comparison opposite to that of the extreme members of $(3.5)_{1,3}$.

Summary of shear loading problem

The exact solution of the shear loading problem in terms of smooth waves has been obtained. The (X, t)-diagram of figure 4.2 has been shown to be appropriate to this case. The solution in the longitudinal wave region is given by (4.25) and (4.26). In the shear wave region the solution is given by (4.36), (4.40), (4.43), and (4.44). In order that the assumed solution be valid the inequalities (4.48) requiring that the material be softening in character must be satisfied.

Example calculation

As an illustration of the results of this section an approximate solution is given.

The first step to be taken is approximate solution of (4.37):

$$\frac{dP}{dQ} = \frac{-2\beta}{\alpha - \delta + \left[(\alpha - \delta)^2 + 4\beta\gamma\right]^{1/2}} \, . \qquad (4.50)$$

Examination of this equation discloses that, in region II, $P = O(Q_0^{\ 2})$ as $Q_0^{\ 2} \to 0$. This means that a change of variable is convenient to facilitate accurate ordering of terms.

Introduce $\mathcal{P}$, $\mathcal{Q}$ such that

$$\begin{aligned} \mathcal{P} &= -2P/Q_0^{\ 2} & \qquad P &= -Q_0^{\ 2}\mathcal{P}/2 \\ \mathcal{Q} &= Q^2/Q_0^{\ 2} & \qquad Q^2 &= Q_0^{\ 2}\mathcal{Q} \, . \end{aligned} \qquad (4.51)$$

Substitution of this into (A. 13) and neglect of terms of order higher than $Q_0^{\,2}$ gives

$$\begin{aligned} a &= a_{00} + \left(- \frac{a_{10}}{2} \mathcal{P} + a_{02}\mathcal{Q} \right) Q_0^{\,2} + \dots , \\ \beta &= \beta_{01}\, Q_0 \sqrt{\mathcal{Q}} + \dots , \\ \gamma &= \delta_{10}\, Q_0 \sqrt{\mathcal{Q}} + \dots , \\ \delta &= \delta_{00} + \left(- \frac{\delta_{10}}{2} \mathcal{P} + \delta_{02}\mathcal{Q} \right) Q_0^{\,2} + \dots . \end{aligned} \tag{4. 52}$$

With this, (4. 50) becomes

$$\begin{aligned} \frac{d\mathcal{P}}{d\mathcal{Q}} = \frac{\beta_{01}}{a_{00} - \delta_{00}} \Bigg\{ 1 - \frac{1}{2(a_{00} - \delta_{00})} \Bigg[(\delta_{10} - a_{10}) \mathcal{P} \\ + 2 \left(a_{02} - \delta_{02} + \frac{\beta_{01}\, \delta_{10}}{a_{00} - \delta_{00}} \right) \mathcal{Q} \Bigg] Q_0^{\,2} + \dots \Bigg\} . \end{aligned} \tag{4. 53}$$

Suppose $\mathcal{P}(\mathcal{Q})$ is representable in the form

$$\mathcal{P} = \mathcal{P}_0 + Q_0^{\,2} \mathcal{P}_1 + \dots \quad . \tag{4. 54}$$

Substitution of (4. 54) into (4. 53) and equation of terms of the same degree in $Q_0^{\,2}$ gives

$$\begin{aligned} \frac{d\mathcal{P}_0}{d\mathcal{Q}} &= \frac{\beta_{01}}{a_{00} - \delta_{00}} \\ \frac{d\mathcal{P}_1}{d\mathcal{Q}} &= - \frac{\beta_{01}}{2(a_{00} - \delta_{00})^2} \Bigg[(\delta_{10} - a_{10}) \mathcal{P}_0 \\ &\quad + 2 \left(a_{02} - \delta_{02} + \frac{\beta_{01}\, \delta_{10}}{a_{00} - \delta_{00}} \right) \mathcal{Q} \Bigg] \\ &\ \ \vdots \end{aligned} \tag{4. 55}$$

The initial condition on (4. 53) is $\mathcal{P}(1) = 0$, hence the conditions for (4. 55) are

$$\mathcal{P}_0(1) = 0, \quad \mathcal{P}_1(1) = 0, \ \ldots \quad . \tag{4. 56}$$

With this the solution of $(4.\ 55)_1$ is

$$\mathcal{P}_0 = \frac{\beta_{01}}{\alpha_{00} - \delta_{00}} \ (\mathcal{Q} -1), \tag{4. 57}$$

and the solution of $(4.\ 55)_2$ is

$$\mathcal{P}_1 = - \frac{\beta_{01}}{4(\alpha_{00} - \delta_{00})^2} \left[\frac{\beta_{01}(\delta_{10} - \alpha_{10})}{\alpha_{00} - \delta_{00}} \ (\mathcal{Q} -1)^2 \right.$$
$$\left. + \ 2\left(\alpha_{02} - \delta_{02} + \frac{\beta_{01}\,\delta_{10}}{\alpha_{00} - \delta_{00}}\right)(\mathcal{Q}^2 -1) \right]. \tag{4. 58}$$

With these results (4. 54) becomes

$$\mathcal{P} = \frac{\beta_{01}}{\alpha_{00} - \delta_{00}} \ (\mathcal{Q} -1) - \frac{\beta_{01}}{4(\alpha_{00} - \delta_{00})^2} \left[\frac{\beta_{01}(\delta_{10} - \alpha_{10})}{\alpha_{00} - \delta_{00}} \ (\mathcal{Q} -1)^2 \right.$$
$$\left. + \ 2\left(\alpha_{02} - \delta_{02} + \frac{\beta_{01}\,\delta_{10}}{\alpha_{00} - \delta_{00}}\right)(\mathcal{Q}^2 -1) \right] Q_0{}^2 + \ldots \quad . \tag{4. 59}$$

Substitution of (4. 51) into (4. 30) gives

$$\begin{aligned} &Q_0(\alpha - Z^2)\sqrt{\mathcal{Q}}\ \mathcal{P}_Z - \beta\, \mathcal{Q}_Z = 0, \\ &Q_0\,\gamma\sqrt{\mathcal{Q}}\ \mathcal{P}_Z - (\delta - Z^2)\mathcal{Q}_Z = 0. \end{aligned} \tag{4. 60}$$

Since $\mathcal{P} = \mathcal{P}(\mathcal{Q})$, $(4.\ 60)_1$ can be rewritten

$$\left[Q_0(\alpha - Z^2)\sqrt{\mathcal{Q}}\ \mathcal{P}_{\mathcal{Q}} \ - \beta\right] \mathcal{Q}_Z = 0.$$

For non-trivial solution of this equation it is necessary that

$$Q_0(\alpha - Z^2)\sqrt{\mathcal{Q}}\,\mathcal{P}_{\mathcal{Q}} - \beta = 0$$

or

$$Z^2 = \alpha - \frac{\beta}{Q_0\sqrt{\mathcal{Q}}\,\mathcal{P}_{\mathcal{Q}}} \quad . \tag{4.61}$$

To $0(Q_0{}^2)$ this is

$$Z^2 = \alpha_{00} + \left(-\frac{\alpha_{10}}{2}\,\mathcal{P}_0 + \alpha_{02}\mathcal{Q}\right) Q_0{}^2 + \dots$$

$$= \left(\delta_{00} + \frac{\beta_{01}\,\delta_{10}}{2(\alpha_{00} - \delta_{00})}\,Q_0{}^2 + \dots\right) +$$

$$+ \left(\delta_{02} - \frac{3\beta_{01}\,\delta_{10}}{2(\alpha_{00} - \delta_{00})}\right)\mathcal{Q}\,Q_0{}^2 + \dots \quad . \tag{4.62}$$

From this

$$V_{12} = \sqrt{\delta_{00}}\left[1 + \frac{1}{2\,\delta_{00}}\left(\delta_{02} - \frac{\beta_{01}\,\delta_{10}}{\alpha_{00} - \delta_{00}}\right) Q_0{}^2 + \dots\right] ,$$

$$V_{23} = \sqrt{\delta_{00}}\left[1 + \frac{\beta_{01}\,\delta_{10}}{4\,\delta_{00}(\alpha_{00} - \delta_{00})}\,Q_0{}^2 + \dots\right] . \tag{4.63}$$

The necessary condition, $V_{12} < V_{23}$, for the existence of a solution of the form considered becomes

$$\delta_{02} < \frac{\beta_{01}\,\delta_{10}}{\alpha_{00} - \delta_{00}} \quad , \tag{4.64}$$

which is the complement of $(3.31)_2$. From (4.59),

$$\mathcal{P}(0) = -\frac{\beta_{01}}{\alpha_{00} - \delta_{00}}\left\{1 - \frac{1}{4(\alpha_{00} - \delta_{00})}\left[2(\alpha_{02} - \delta_{02}) + \frac{\beta_{01}(\alpha_{10} + \delta_{01})}{\alpha_{00} - \delta_{00}}\right] Q_0{}^2 + \dots\right\} . \tag{4.65}$$

This gives

$$P_3 = -\frac{Q_0^{\;2}}{2}\,\mathcal{P}(0) = \frac{\beta_{01}}{2(\alpha_{00}-\delta_{00})}\,Q_0^{\;2} + \ldots \quad . \tag{4.66}$$

The solution in the longitudinal wave region, region IV, is much more easily obtained. From (4.25),

$$V_{34} = \sqrt{\alpha_{00}}\left(1 + \frac{\alpha_{10}\,\beta_{01}}{4\alpha_{00}(\alpha_{00}-\delta_{00})}\,Q_0^{\;2} + \ldots\right)\ , \quad V_{45} = \sqrt{\left[(\lambda+2\mu)/\rho_0\right]}\,. \tag{4.67}$$

The requirement $V_{34} < V_{45}$ becomes

$$\alpha_{10}\,\beta_{01} < 0,$$

the condition complementary to the admissibility condition for a shock, $(3.31)_1$. Estimates of the field quantities in this region are obtained from (4.26).

CHAPTER V

SHEAR LOADING PROBLEM - SOLUTIONS COMBINING CENTERED SIMPLE WAVES AND SHOCKS

The solutions previously given for the shear loading problem presuppose that the shear and dilitation effects both propagate as smooth waves or both propagate as shocks. In this chapter cases where one of the waves is smooth and the other is a shock are considered.

In section 5.1 the solution of the problem of sudden application of a shear displacement gradient to the boundary of an undeformed elastic half-space is solved under the assumption that the disturbance propagates as a smooth shear wave preceeded by a normal shock. The situation is relevant in connection with the special hyperelastic material discussed in the following chapter. In section 5.2 the same boundary and initial value problem as above is considered, but this time subject to the assumption that the disturbance propagates as a shear shock preceeded by a smooth longitudinal wave. Section 5.3 is devoted to a summary of the results obtained for the shear loading problem in Chapters III, IV, and V. Finally, in section 5.4 an example problem is discussed which illustrates an effect which may occur when a monotonicity condition is violated.

5.1. Solution Combining a Normal Shock and a Centered Shear Wave

In this section the shear loading problem is again considered. A solution of the form depicted in figure 5.1 is assumed. This figure shows a disturbance consisting of a normal shock followed in turn by a region of uniform motion and a centered simple shear wave propagating into the undeformed half-space.

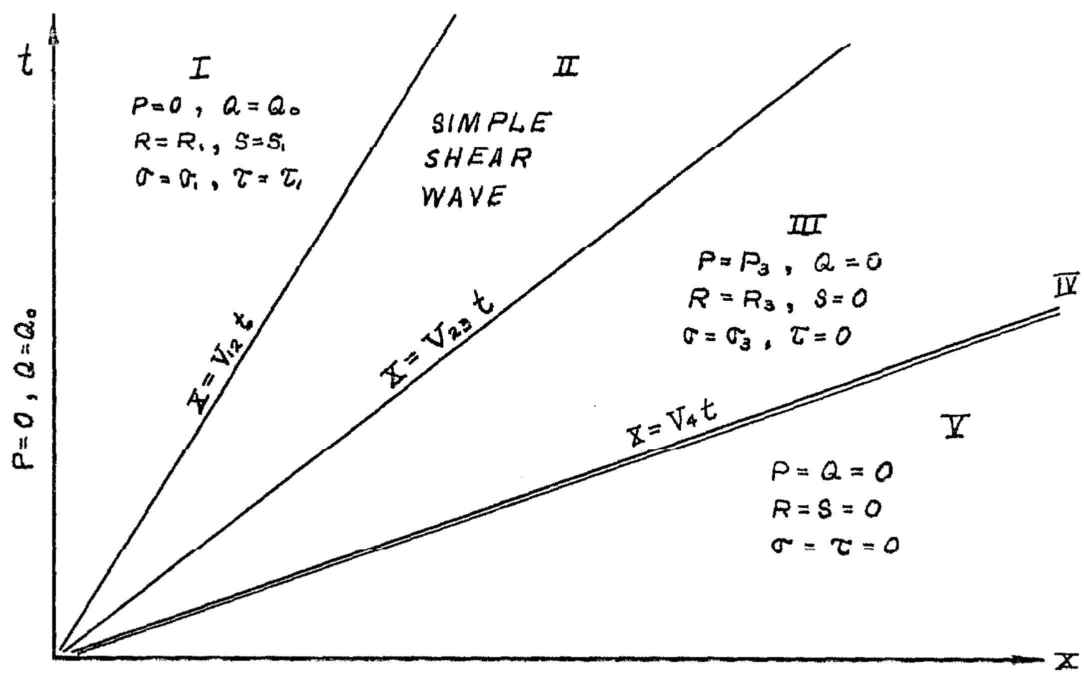


Figure 5.1

Clearly the initial conditions

$$P(X,0) = Q(X,0) = 0, \quad R(X,0) = S(X,0) = 0, \tag{5.1}$$

and the boundary conditions

$$P(0, t) = 0, \qquad Q(0, t) = \begin{cases} 0, & t < 0 \\ Q_0, & t > 0 \end{cases} , \tag{5.2}$$

are satisfied by the fields shown in figure 5.1. Application of the jump conditions (2.47) to the jump IV gives the shock speed V_4 and the particle velocity R_3 as

$$V_4 = \sqrt{[\sigma(P_3, 0)/\rho_0 P_3]}, \quad R_3 = P_3{}^2 V_4 , \tag{5.3}$$

where P_3 remains to be determined. The admissibility condition on the jump is

$$\sqrt{(\lambda+2\mu)} < \sqrt{[\sigma(P_3, 0)/P_3]} < \sqrt{\sigma_P(P_3, 0)} . \tag{5.4}$$

In region II a centered simple wave solution is sought. The analysis is the same as that carried out in the previous chapter with the result that P and Q are related by (4.37):

$$\frac{dP}{dQ} = \frac{-2\beta}{(\alpha - \delta) + \left[(\alpha - \delta)^2 + 4\beta\gamma\right]^{1/2}} , \tag{5.5}$$

subject to the initial condition

$$P(Q_0) = 0 . \tag{5.6}$$

P_3 is given, in terms of the solution $P(Q)$ of (5.5), as

$$P_3 = P(0). \tag{5.7}$$

As in section 4.2, the leading and trailing edges of the shear wave have the speeds

$$V_{12} = \left\{ \frac{1}{2} \left[\alpha + \delta - \left((\alpha - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2}_{P=0,\ Q=Q_0} ,$$

$$V_{23} = \sqrt{\delta}\,(P_3, 0), \tag{5.8}$$

with the inequalities

$$V_{12} < V_{23} < V_4 \tag{5.9}$$

implied in figure 5.1. The inequalities (5.9) and (5.4) combined with the monotonicity condition on the shear wavespeed in the simple wave can be written

$$(a_3)_I < Z_{II} < (a_3)_{III} ,$$

$$(a_3)_{III} < V_4 , \tag{5.10}$$

$$(a_4)_V < V_4 < (a_4)_{III} ,$$

where

$$Z_{II} = \left\{ \frac{1}{2} \left[\alpha + \delta - \left((\alpha - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2} , \tag{5.11}$$

with $Q=Q(P)$ in accordance with (5.5), and P taking on all its values in the wave region.

A third order example can be calculated as has been done in the previous sections. Among the results of such a calculation are

$$P_3 = \frac{\beta_{01}}{2(\alpha_{00} - \delta_{00})} Q_0^{\ 2} + \ldots ,$$

$$V_{12} = \sqrt{\delta_{00}} \left[1 + \frac{1}{2\delta_{00}} \left(\delta_{02} - \frac{\beta_{01}\,\delta_{10}}{2(\alpha_{00} - \delta_{00})} \right) Q_0^{\ 2} + \ldots \right] , \tag{5.12}$$

$$V_{23} = \sqrt{\delta_{00}}\left[1 + \frac{\beta_{01}\,\delta_{10}}{4\,\delta_{00}(\alpha_{00} - \delta_{00})}\,Q_0^{\,2} + \dots\right],$$

$$V_4 = \sqrt{\alpha_{00}}\left[1 + \frac{\alpha_{10}\,\beta_{01}}{8\alpha_{00}(\alpha_{00} - \delta_{00})}\,Q_0^{\,2} + \dots\right],$$

where it is recalled that $\alpha_{00}=(\lambda+2\mu)/\rho_0$, and $\delta_{00}=\mu/\rho_0$, but that the higher order elastic constants have no counterparts in the linear theory and in the nonlinear theory have magnitude and sign depending on the particular material under consideration. For this case the conditions (5.10) become

$$0 < \alpha_{10}\beta_{01}, \quad \delta_{02} < \beta_{01}\,\delta_{10}/(\alpha_{00} - \delta_{00}). \tag{5.13}$$

5.2. Solution Combining Centered Longitudinal Wave ans Shear Shock

This section is again devoted to the shear loading problem, but this time a solution of the form depicted in figure 5.2 is considered. This figure shows a disturbance, consisting of a centered simple longitudinal wave followed, in turn, by a region of uniform state and a shear shock, propagating into the undeformed half-space.

The fields shown are again seen to satisfy the boundary and initial conditions (5.1) and (5.2), and the field equations in the interior of regions I, III, and V. Application of the jump conditions to II gives

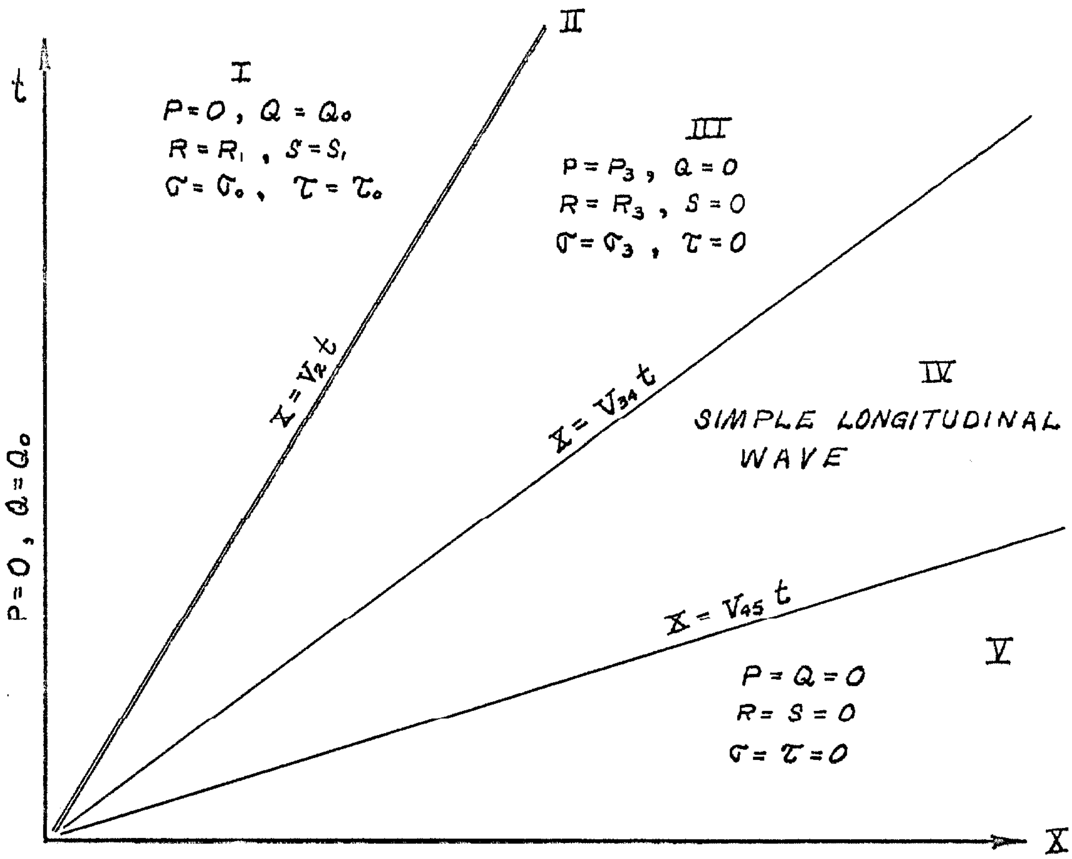


Figure 5. 2

$$\begin{aligned} & P_3\,\tau_0 + Q_0(\sigma_0 - \sigma_3) = 0, \\ & V_2 = \sqrt{[\tau(0, Q_0)/\rho_0 Q_0]}, \\ & S_1 = -Q_0 V_2, \quad R_1 - R_3 = -P_3 V_2, \end{aligned} \tag{5.14}$$

and the burden of the analysis falls on the solution of $(5.14)_1$. This equation has been discussed in section 3. 2. 2. 1.

In region IV the field equations become

$$a(P)P_X = R_t, \quad R_X = P_Z\ , \tag{5.15}$$

and, as discussed previously, the solutions are

$$(X/t)^2 = a(P), \qquad R(X/t) = 2\int_{X/t}^{V_{45}} \frac{a(P(Z))}{a_P(P(Z))}\, dZ\ , \tag{5.16}$$

with

$$V_{34} = \sqrt{[\sigma_P(P_3, 0)/\rho_0]}, \qquad V_{45} = \sqrt{[(\lambda+2\mu)/\rho_0]}. \tag{5.17}$$

R_3 is given by $(5.16)_2$ as $R(V_{34})$. As is seen from figure 5.2, the wavespeeds must satisfy the inequalities

$$V_2 < V_{34} < V_{45}\ ,$$

that is,

$$\sqrt{[\tau(0, Q_0)/Q_0]} < \sqrt{\sigma_P(P_3, 0)} < \sqrt{(\lambda+2\mu)}\ . \tag{5.18}$$

In addition to this, the admissibility conditions for the shock must be satisfied. All these inequalities, together with the invertibility condition on $(5.16)_1$ give the inequalities

$$\begin{aligned} (a_3)_{III} &< V_2 < (a_3)_I\ ,\\ V_2 &< (a_4)_{III}\ ,\\ (a_4)_{III} &< Z_{IV} < (a_4)_V\ , \end{aligned} \tag{5.19}$$

with

$$Z_{IV} = \sqrt{a(P)}\ , \tag{5.20}$$

P taking all its values in region IV.

Computation of a third order example gives the results

$$P_3 = \frac{\beta_{01}}{2(\alpha_{00} - \delta_{00})} Q_0^{\ 2} + \ldots ,$$

$$V_2 = \sqrt{\delta_{00}} \left(1 - \frac{\beta_{01} \delta_{01} Q_0^{\ 2}}{4\, \delta_{00}(\alpha_{00} - \delta_{00})} + \ldots \right) ,$$

$$V_{34} = \sqrt{\alpha_{00}} \left(1 + \frac{\alpha_{10}\beta_{01}}{4\alpha_{00}(\alpha_{00} - \delta_{00})} Q_0^{\ 2} + \ldots \right) , \tag{5.21}$$

$$V_{45} = \sqrt{\alpha_{00}} \ .$$

In terms of (5. 21) the inequalities (5. 19) become

$$\alpha_{10}\beta_{01} < 0 \ , \quad \frac{\beta_{01}\ \delta_{10}}{\alpha_{00} - \delta_{00}} < \delta_{02} \ , \tag{5.22}$$

provided the monotonicity condition is satisfied. If $\delta_{02} = 0$ then $(5.22)_2$ becomes $\beta_{01}\ \delta_{10} < 0$ which is also sufficient for monotonicity.

5. 3. Shear Loading Problem - Summary

Review of sections 3. 1. 2, 4. 2, 5. 1 and 5. 2 reveals that the same shear loading problem has been studied in each of these sections and a different solution obtained in each case. This does not give rise to contradiction or ambiguity because the inequalities controlling admissibility are different. The situation is most easily understood by considering the extreme members of the first and third lines of the sets of inequalities (3. 23), (4. 48), (5. 10), and (5. 19). These are shown in table 5. 1 where, it will be recalled, a_3 and a_4 stand for the shear and longitudinal wavespeeds, respectively, and where the subscripts on these quantities denote the region in which they are to be evaluated.

Thesis Section	Eqn. No.	Condition on and Character of Shear Disturbance	Condition on and Character of Longitudinal Wave
3. 1. 2	3. 23	$(a_3)_{III} < (a_3)_I$ Shock	$(a_4)_V < (a_4)_{III}$ Shock
4. 2	4. 48	$(a_3)_I < (a_3)_{III}$ Smooth	$(a_4)_{III} < (a_4)_V$ Smooth
5. 1	5. 10	$(a_3)_I < (a_3)_{III}$ Smooth	$(a_4)_V < (a_4)_{III}$ Shock
5. 2	5. 19	$(a_3)_{III} < (a_3)_I$ Shock	$(a_4)_{III} < (a_4)_V$ Smooth

TABLE 5. 1

It is seen from the table that shocks prevail when the wavespeed increases upon passage of the disturbance; smooth waves prevail when the wavespeed decreases upon passage of the disturbance. Each of the four cases considered is different; together they exhaust the possibilities.

The inequalities shown in the table are incomplete in the sense that the middle part has been omitted. This middle part is the condition that the appropriate stress-strain curve be monotonic. If this condition fails to be met then even though the extreme members are in the proper relation, the solution, as given, is inadmissible. An example of the additional complexity introduced in this situation is given in the next section.

The inequalities given in the table incompletely describe the situation in another sense: the second inequality of each set has been omitted. In each case this omitted inequality reflects the assumption that the longitudinal disturbance propagates faster than the shear disturbance. As discussed in section 2.3.1, this is necessarily true for sufficiently small disturbances. It seems probable that it is true more generally but this is not proved. If there should be a material for which a shear disturbance can propagate faster than a dilitational disturbance then solutions for this case could be computed as above but with the order of the waves reversed in the (X, t)-diagram.

5.4. Special Example: Smooth Longitudinal Wave — Normal Shock Combination

The example of this section is given to illustrate the difficulties attendant upon violation of the motonicity requirements. For this purpose a special material (not necessarily representative of any real solid) having an inflection point in the stress-strain curve for uniaxial extension is considered. This example also illustrates the larger problem that for exhaustive discussion of plane waves a great variety of phenomena must be identified and materials classified in such a way that it can be predicted which of the various possible phenomena can occur in a given case. This is particularly important since the method of solution is semi-inverse in the sense that one must begin by making a conjecture about the appropriate configuration in the (X, t)-plane.

The stress-strain curve in uniaxial extension for the material of this example is

$$\sigma(P) = \rho_0 \left(C_1^{\,2} P + \tfrac{1}{3} C_3 P^3\right), \qquad C_1 > 0, \quad C_3 > 0 . \tag{5. 23}$$

Consideration is restricted to longitudinal waves, and in this case the field equations are

$$a(P)P_X = R_t, \qquad R_X = P_t , \tag{5. 24}$$

where, using (5. 23),

$$a(P) = C_1^{\,2} + C_3 P^2 . \tag{5. 25}$$

The jump conditions are

$$[\![P]\!]\, V + [\![R]\!] = 0, \qquad V = \sqrt{\{[\![\sigma]\!] / \rho_0 [\![P]\!]\}} , \tag{5. 26}$$

and the admissibility condition for a normal shock is

$$\sqrt{[\sigma_P(P_3^{\,+})/\rho_0]} < V_2 < \sqrt{[\sigma_P(P_0)/\rho_0]} , \tag{5. 27}$$

where $P_3^{\,+}$ and P_0 are displacement gradients ahead of and behind the shock, respectively.

The boundary and initial conditions considered are

$$P(X,0) = P_4 \ (-1 < P_4 < 0), \qquad R(X,0) = 0, \qquad P(0,t) = \begin{cases} 0, & t < 0 \\ P_0, & t > 0 \end{cases}, \ (P_0 > 0) . \tag{5. 28}$$

It is conjectured that the (X, t)-diagram for this problem is as depicted in figure 5. 3. This figure shows the undisturbed material at rest in a compressed state. Expansion to the extended final state takes place in part through the simple wave of region III and in part through the normal shock II.

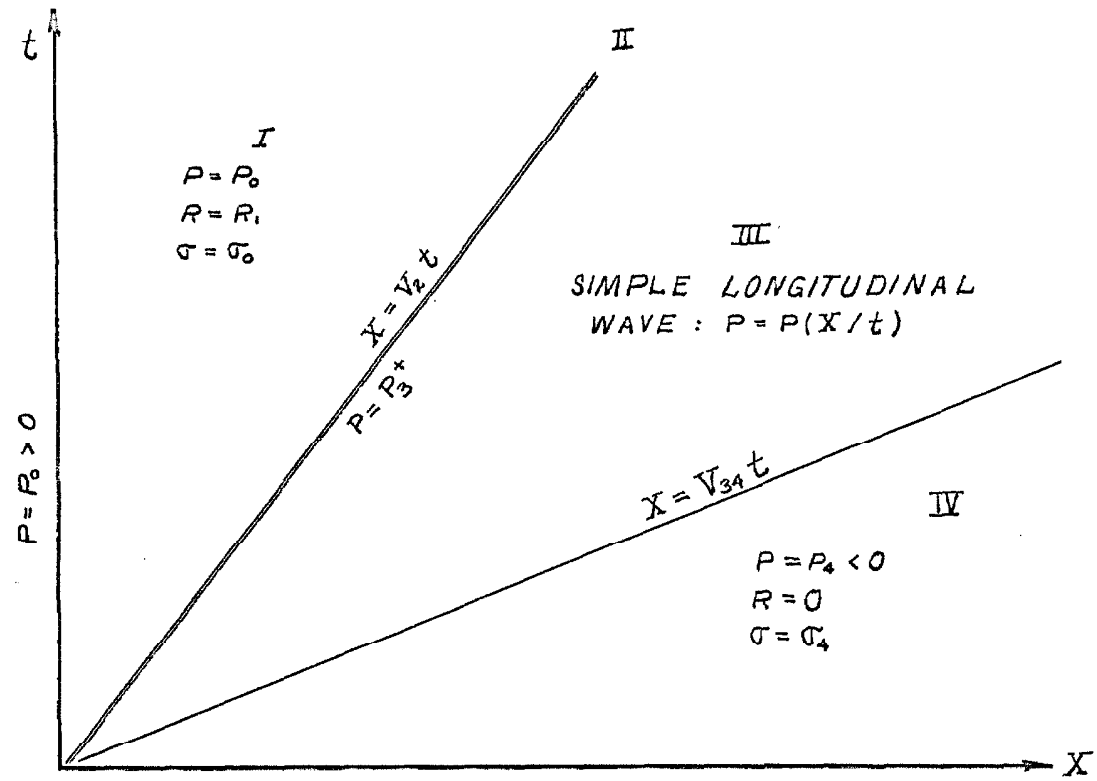


Figure 5. 3

In the simple wave region

$$Z^2 \equiv (X/t)^2 = a(P) \tag{5. 29}$$

or, for the material at hand,

$$Z^2 = C_1{}^2 + C_3 P_3{}^2, \tag{5. 30}$$

the subscript being adjoined to P to denote validity in region III only. From (5. 30),

$$V_{34} = C_1 \sqrt{\left[1 + \frac{C_3}{C_1^2} P_4^2\right]}, \quad V_2 = C_1 \sqrt{\left[1 + \frac{C_3}{C_1^2} (P_3^+)^2\right]}. \tag{5. 31}$$

The jump condition gives

$$V_2 = C_1 \sqrt{\left[1 + \frac{C_3}{3C_1^2}\left(\frac{(P_3^+)^3 - P_0^3}{P_3^+ - P_0}\right)\right]}, \tag{5. 32}$$

and, for agreement between $(5.\ 31)_2$ and (5. 32), it is necessary that

$$P_3^+ = -P_0/2\ . \tag{5. 33}$$

In this case

$$V_2 = C_1 \sqrt{\left[1 + \frac{C_3}{4C_1^2} P_0^2\right]}.$$

A check of the right hand part of (5. 27) shows that the membe are in the relation indicated. On the left equality prevails, as may be seen from figure 5..3. This is a condition mathematically like the contact discontinuity of shock tube theory and of Lax's theory(1, 2) although, because of the coordinates used here, II is not a contact surface in the physical sense.

The situation here presented can be described as a shock with a smooth precursor wave. The case $C_3 < 0$ leads to a similar situation except that the shock preceeds the smooth wave.

CHAPTER VI

EXAMPLE PROBLEM: WAVE PROPAGATION IN A SPECIAL HYPERELASTIC SOLID

As an example of the results presented in the foregoing chapters the problem of propagation of loading waves in a special material is solved. This material, chosen for this example solely as a matter of convenience, was proposed by Ko,[15] on the basis of experimental observations, as a model for static deformation of a foam rubber. The stress-strain relation is

$$t_{ij} = \mu\left(\delta_{ij} - \frac{c_{ij}}{\sqrt{III_{c^{-1}}}}\right), \qquad \mu > 0. \tag{6.1}$$

By (2.38), (6.1) implies

$$\sigma(P, Q) = \mu\left(1 - \frac{1+Q^2}{(1+P)^3}\right), \quad \tau(P, Q) = \mu\,\frac{Q}{(1+P)^2}, \tag{6.2}$$

and, from this

$$\begin{aligned} \sigma_P &= 3\mu\,\frac{1+Q^2}{(1+P)^4}, \qquad \sigma_Q = -2\mu\,\frac{Q}{(1+P)^3}, \\ \tau_P &= -2\mu\,\frac{Q}{(1+P)^3}, \qquad \tau_Q = \frac{\mu}{(1+P)^2}. \end{aligned} \tag{6.3}$$

By (6.3), (2.46), and (2.52) it is seen that the inequality

$$Q^2 < 3 \tag{6.4}$$

must be satisfied in order that the field equations (2.45) be of hyperbolic type. For uniaxial extension $(6.2)_1$ becomes

$$\sigma(P, 0) = \mu\left(1 - \frac{1}{(1+P)^3}\right), \tag{6.5}$$

which function is plotted as figure 6.1. Materials having such a stress-strain curve are describable as hardening in compression and softening in tension. Expansion of (6.5) about P=0 discloses that

$$\lambda = \mu \tag{6.6}$$

for this material.

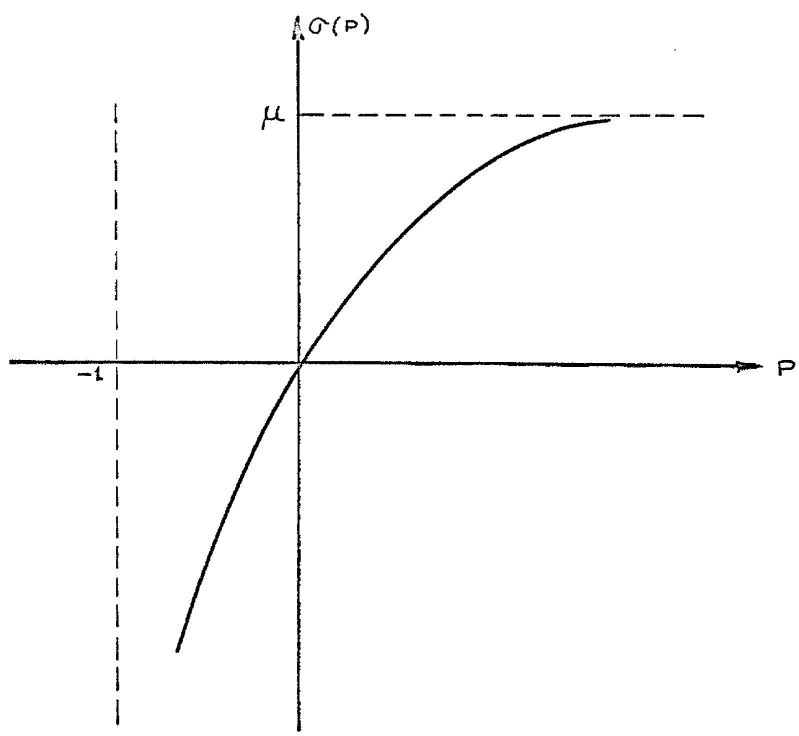


Figure 6.1

In accordance with (2. 42) and (2. 53), the characteristic wavespeeds associated with this material are

$$a_3=-a_2=\left\{\frac{\mu}{2\rho_0}\left[\frac{3(1+Q^2)}{(1+P)^4}+\frac{1}{(1+P)^2}-\left\langle\left(\frac{3(1+Q^2)}{(1+P)^4}-\frac{1}{(1+P)^2}\right)^2+\frac{16Q^2}{(1+P)^6}\right\rangle^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}},$$

$$a_4=-a_1=\left\{\frac{\mu}{2\rho_0}\left[\frac{3(1+Q^2)}{(1+P)^4}+\frac{1}{(1+P)^2}+\left\langle\left(\frac{3(1+Q^2)}{(1+P)^4}-\frac{1}{(1+P)^2}\right)^2+\frac{16Q^2}{(1+P)^6}\right\rangle^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}. \tag{6. 7}$$

Whether longitudinal loading disturbances propagate as shocks or as smooth waves has been seen to depend on whether the inequality

$$\sqrt{(\lambda+2\mu)} < \sqrt{[\sigma\ (P,0)/P]} < \sqrt{\sigma_P(P,0)} \tag{6. 8}$$

or one with reversed comparison holds. Substitution of (6. 2), (6. 3), and (6. 6) into (6. 8) shows that loading disturbances propagate as:

$$\begin{aligned} &1)\ \text{Shocks if } P < 0 \\ &2)\ \text{Smooth waves if } P > 0. \end{aligned} \tag{6. 9}$$

6. 1. Application of Normal Strain to the Boundary of an Unloaded Half-Space

In this section the propagation of a normal loading disturbance in the material (6. 1) is considered. In accordance with (6. 9) this disturbance propagates as a shock for $P < 0$ and as a smooth wave for $P > 0$.

The shock situation ($P < 0$) shown in figure 6. 2 is considered first.

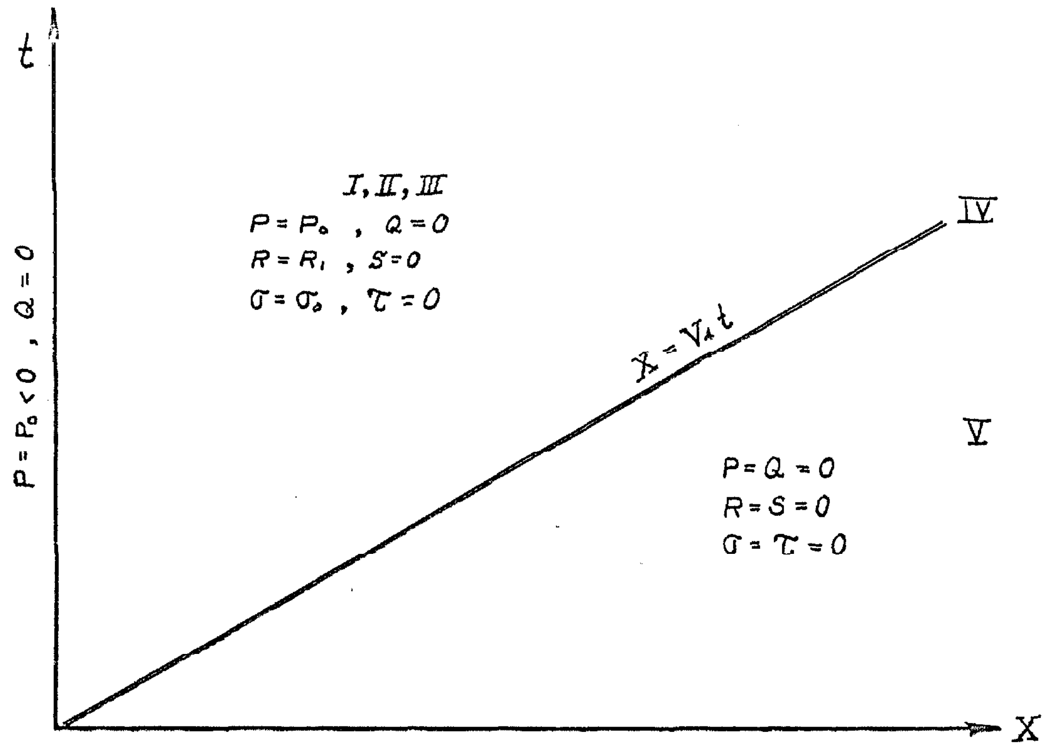


Figure 6. 2

By (3. 11) the shock speed and the particle velocity in region I are given by

$$V_4 = \sqrt{[\sigma_0/\rho_0 P_0]}, \qquad R_1 = -P_0 V_4, \tag{6.10}$$

so that, using (6. 4),

$$V_4 = \sqrt{\left[\frac{\mu}{\rho_0 P_0}\left(1 - \frac{1}{(1+P_0)^3}\right)\right]}, \qquad R_1 = -P_0 V_4. \tag{6.11}$$

To complement this solution, centered smooth wave propagation corresponding to a loading for which $P > 0$ is also considered. This situation is as shown in figure 6. 3, and corresponds to a centered simple wave advancing into the undeformed medium, followed by a uniform state of tension.

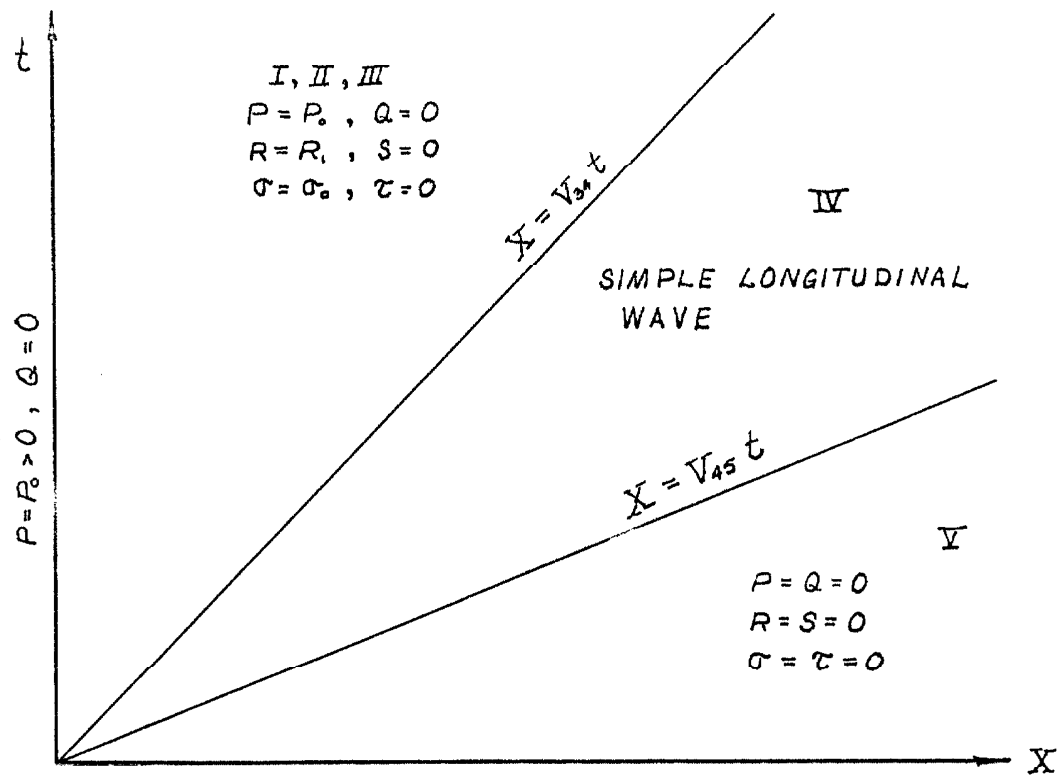


Figure 6. 3

The solution of this problem is as given in section 4. 1. 1. The speeds of the leading and trailing edges of the simple wave are given by

$$V_{34} = \sqrt{\left[\sigma_P(P_0, 0)/\rho_0\right]}, \quad V_{45} = \sqrt{\left[(\lambda+2\mu)/\rho_0\right]}, \tag{6.12}$$

respectively. In the simple wave region

$$(X/t)^2 = \sigma_P(P, 0)/\rho_0, \quad R(X/t) = 2\int_{X/t}^{V_{45}} \frac{a(P(Z))}{a_P(P(Z))}\, dZ . \tag{6.13}$$

By (6. 4)

$$a(P) = \sigma_P(P, 0)/\rho_0 = \frac{3\mu}{\rho_0}\frac{1}{(1+P)^4}$$

so

$$V_{34} = \sqrt{\left[3\mu/\rho_0(1+P)^4\right]}, \quad V_{45} = \sqrt{(3\mu/\rho_0)} , \tag{6.14}$$

and

$$(X/t)^2 = 3\mu/\rho_0(1+P)^4, \qquad V_{34} \leqslant X/t \leqslant V_{45} , \tag{6.15}$$

$$R(X/t) = -(3\mu/\rho_0)^{1/4}\left[\sqrt{V_{45}} - \sqrt{(X/t)}\right] . \tag{6.16}$$

In summary, the solution of the smooth wave problem is described explicitly by

$$P = \begin{cases} 0 , & \text{Region V} \\ (3\mu/\rho_0)^{1/4}\sqrt{(t/X)} - 1 , & \text{Region IV} \\ P_0, & \text{Region I} \end{cases} , \tag{6.17}$$

$$R = \begin{cases} 0, & \text{Region V} \\ -(3\mu/\rho_0)^{1/4}\left[(3\mu/\rho_0)^{1/4} - (X/t)^{1/2}\right], & \text{Region IV} , \\ -(3\mu/\rho_0)^{1/4}\left[(3\mu/\rho_0)^{1/4} - \left\langle 3\mu/\rho_0(1+P_0)^4\right\rangle^{1/4}\right. , & \text{Region I} \end{cases}$$

where the various regions of interest are

$$\begin{aligned}
&\text{Region I:} && 0 \leqslant X/t \leqslant \sqrt{\left[3\mu/\rho_0(1+P_0)^4\right]} \\
&\text{Region IV:} && \sqrt{\left[3\mu/\rho_0(1+P_0)^4\right]} < X/t < \sqrt{(3\mu/\rho_0)} \qquad (6.18) \\
&\text{Region V:} && \sqrt{(3\mu/\rho_0)} \leqslant X/t\,.
\end{aligned}$$

The displacement can be obtained from (6.18) by integrating the equations $\partial U/\partial X = P$, and $\partial U/\partial t = R$.

6.2. Application of Shear Strain to the Boundary of an Unstrained Elastic Half-Space

This section is devoted to investigation of the shear loading problem for the material characterized by (6.1). According to $(3.23)_1$ a shear disturbance can propagate as a shock only if

$$\sqrt{\left[\tau(0,Q_0)/Q_0\right]} < \sqrt{\left\{\frac{1}{2}\left[\sigma_P + \tau_Q - \left((\sigma_P - \tau_Q)^2 + 4\sigma_Q\tau_P\right)^{1/2}\right]\right\}}_{\substack{P=0,\\ Q=Q_0}}.$$

For the material at hand this inequality becomes

$$1 < \left\{\frac{1}{2}\left[(4+3Q_0^{\,2}) - \sqrt{(4+28Q_0^{\,2}+9Q_0^{\,4})}\right]\right\}\,,$$

an inequality which is not satisfied for any choice of Q_0. It is concluded that shear loading disturbances will always be propagated as smooth waves for the material presently considered.

It has been shown in general that shear effects propagate in company with longitudinal effects. Until the shear wave is investigated it cannot be determined whether the accompanying longitudinal disturbance will propagate as a smooth wave or as a shock.

The initial and boundary conditions are again taken in the following form:

$$P(X,0) = Q(X,0) = 0, \quad R(X,0) = S(X,0) = 0$$

$$P(0,t) = 0, \quad Q(0,t) = \begin{cases} 0, & t < 0 \\ Q_0, & t > 0 \end{cases} . \tag{6.19}$$

It has been shown that in the centered shear wave the ordinary differential equation (4.37) must be satisfied. For the present material this can be written

$$\frac{d(Q^2)}{dP} = \frac{3(1+Q^2)-(1+P)^2}{2(1+P)} + \left\{\left[\frac{3(1+Q^2)-(1+P)^2}{2(1+P)}\right]^2 + 4Q^2\right\}^{1/2} . \tag{6.20}$$

The initial condition on this equation is

$$Q^2(0) = Q_0^{\ 2} \tag{6.21}$$

where it must be recalled that, in accordance with (6.4), $Q_0^{\ 2} < 3$.

Integration of (6.20) has not been accomplished but, for the present purpose, it is sufficient to investigate some properties of the solution. In particular, the existence of a number $P_2 > -1$ such that $Q^2(P_2) = 0$ is to be proved. The right hand member of (6.20) is seen to be positive if

$$3(1+Q^2) - (1+P)^2 > 0, \tag{6.22}$$

a relation satisfied in the shaded region of figure 6.4. In this region solutions $Q^2 = Q^2(P)$ of (6.20) are monotonically increasing functions of P. By (6.20) and (6.19),

$$\left.\frac{d(Q^2)}{dP}\right|_{P=0,\,Q=Q_0} = \frac{2+3Q_0^{\,2}}{2} + \left[\left(\frac{2+3Q_0^{\,2}}{2}\right)^2 + 4Q_0^{\,2}\right]^{1/2} > 0. \tag{6.23}$$

In accordance with this result, the solution curve has positive slope at $P=0$. It is appropriate to try to prove that a solution through $(0, Q_0^{\,2})$, continued into the region $P < 0$, will cross the line $Q^2=0$ for some P_2 in the interval $(-1, 0)$.

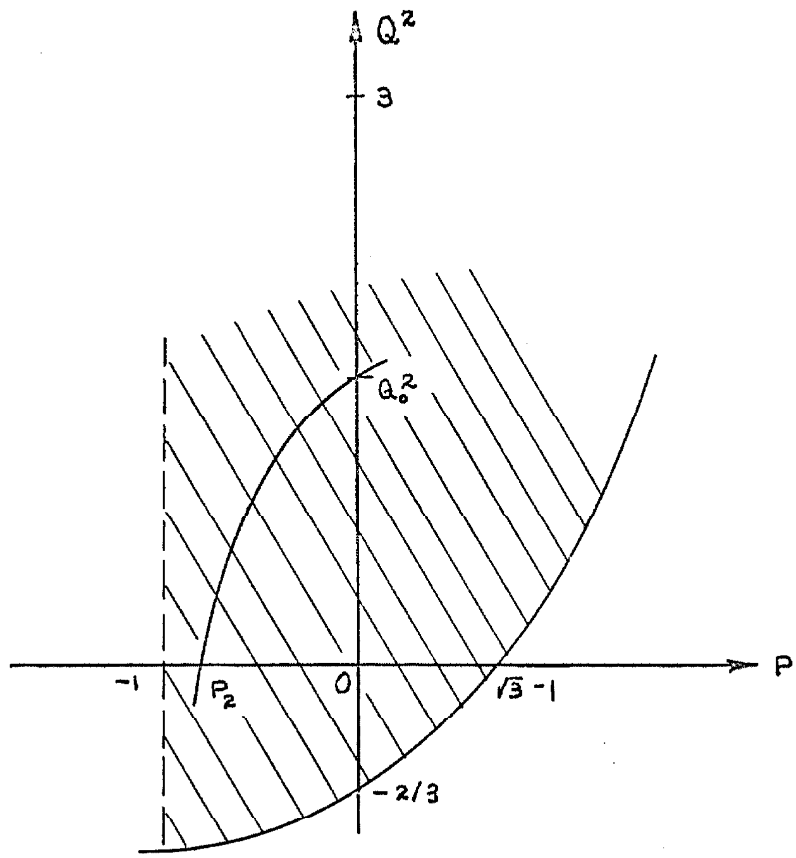


Figure 6.4

Within the shaded region of figure 6.4

$$\frac{dQ^2}{dP} > \frac{3(1+Q^2) - (1+P)^2}{2(1+P)} > 0 . \tag{6.24}$$

Since $Q^2 > 0$, (6.23) can be replaced by

$$\frac{dQ^2}{dP} > \frac{3-(1+P)^2}{2(1+P)} > 0, \tag{6.25}$$

for P in the range $(-1, \sqrt{3}-1)$. By (6.24),

$$Q^2 < {Q_0}^2 + \frac{3}{2}\log(1+P) + \frac{1}{4} - \frac{1}{4}(1+P)^2 \tag{6.26}$$

for $-1 < P < 0$. This bound guarantees a zero, P_2, of $Q^2(P)$ in the interval $(-1, 0)$. The speeds of the leading and trailing edges of the shear wave are, in accordance with $(6.6)_1$,

$$\begin{aligned} V_{23} &= \sqrt{\frac{\mu}{\rho_0 (1+P_2)^2}} \ , \\ V_{12} &= \sqrt{\left\{ \frac{\mu}{2\rho_0}\left[4 + 3{Q_0}^2 - \left((2 + 3{Q_0}^2) + 16{Q_0}^2\right)^{1/2}\right]\right\}} . \end{aligned} \tag{6.27}$$

That $V_{12} \lessgtr V_{23}$ is seen by first noting that $V_{12} = V_{23} \Rightarrow P_2 = {Q_0}^2 = 0$, a trivial case. For ${Q_0}^2$ small and using the fact that $-1 < P_2 < 0$ it follows that $V_{12} < V_{23}$. Since these two wavespeeds depend continuously on ${Q_0}^2$ and P_2 the inequality must be true for all non-vanishing P_2, ${Q_0}^2$.

Since $P_2 < 0$ it follows from (6.8) that the longitudinal disturbance associated with this problem will be a shock, that is, the solution of section 5.1 is applicable. The appropriate (X, t)-diagram is that of figure 5.1. Adaptation of the results (6.11) to the present notation gives

$$V_4 = \left[\frac{\mu}{\rho_0}\,\frac{1}{P_2}\left(1 - \frac{1}{(1+P_2)^3}\right)\right]^{1/2}, \qquad R_2 = -P_2 V_4. \tag{6. 28}$$

It is readily seen that $V_{23} < V_4$ for whatever value in $(-1, 0)$ P_2 may have.

Substitution of $P(Q)$, as obtained by inverting the solution of (6. 20), into (4. 42) and proceeding as indicated in the text following that equation completes the solution of the problem.

6. 3. Wave Propagation in Harmonic Materials

The "harmonic material" of Fritz John[16] is of some interest in connection with the present wave propagation problems. This is a hyperelastic material with the following stress-strain equations.

$$\sigma = 2\,\mu\left[(2+P)\,\frac{F'(W)}{W} - 1\right], \qquad \tau = 2\,\mu\,Q\,F'(W)/W, \tag{6. 29}$$

where

$$W = \sqrt{\left[(2+P)^2 + Q^2\right]} \tag{6. 30}$$

with $F(W)$ a function characterizing the various materials in this class. The function F is supposed here to have the following properties:

$$\begin{aligned} &F'(W) > 0, \quad F''(W) > 0, \quad \text{for all } W, \\ &F'(2) = 1, \quad F''(2) = (\lambda + 2\mu)/2\mu, \\ &F'(W)/W : \text{ monotonically increasing fn. of } W. \end{aligned} \tag{6. 31}$$

Consideration of longitudinal waves in this material poses no special problems. It is found that either smooth longitudinal waves or normal shocks propagate into an undeformed body according as

$F''(W)$ is a monotonically increasing or a monotonically decreasing function of W.

The shear loading problem is, of course, more complicated. For the harmonic material the equation (4. 37) is explicitly integrable and gives the result

$$(P+2)^2 + Q^2 = 4 + Q_0^{\,2}, \tag{6. 32}$$

which implies that W is constant in the shear wave region. This, in turn, means that the shear wavespeed is a constant in this region, hence that the wave region degenerates to a single ray. The jump conditions are found to be satisfied along this ray but the inequalities to be satisfied for admissibility do not hold; equalities hold in their place. This gives a very special situation similar to that discussed in section 5. 4. The solution is neither a smooth wave nor a shock. If this special disturbance is admitted then the solution to the boundary value problem can be completed by using the kind of longitudinal wave appropriate to the character of $F''(W)$, as discussed above.

CHAPTER VII

PERTURBATION SOLUTION OF THE FIELD EQUATIONS

It is clear from the examples included in the foregoing parts of this thesis that a method of obtaining an approximate solution of the wave propagation problem which possesses the qualitative features of the exact solution would be very useful.

An obvious choice for such a method would be some kind of perturbation analysis. One such method has been proposed by Fine and Shield(7) for general three-dimensional elastodynamic problems, but their method is deficient for the present purpose in that it gives its best results only for a rather short time interval, and then in a form which somewhat obscures the qualitative aspects of the solution. Their approximation is obtained by subjecting the field equations and auxiliary conditions, expressed in the physical variables $(\underline{X}, t)$ or $(\underline{x}, t)$, to a straightforward perturbation analysis. This method fails to provide improved estimates of the wavespeed at each stage of the computation, and thus generates secular terms in the solution. A similar difficulty frequently occurs in investigations of ordinary differential equations having oscillatory solutions but is avoided there by applying, for example, Lindsteds method of correcting the frequency of oscillation.

A perturbation analysis is described in this chapter which, while of less general applicability than that of Fine and Shield, corrects the wavespeed at each stage of the analysis thus avoiding secular terms in the solution and giving results which are felt to be better and more easily interpretable. This method is essentially that proposed by Phyllis A. Fox[17] in a gasdynamical context. Slight modification has been necessary to include shear wave problems.

7.1. Application of Normal Strain to the Boundary of an Undeformed Half-Space

In the case Q=0 the field equations (2.45) reduce to

$$\alpha(P)\, P_X = R_t, \qquad R_X = P_t, \tag{7.1}$$

with

$$\alpha(P) = \sigma_P(P, 0)/\rho_0 \, . \tag{7.2}$$

This problem has been treated fully and exactly in the literature and, indeed, has been discussed in this thesis but is included here as an example problem to illustrate the solution of wave propagation problems by perturbation methods.

Following Fox,[17] characteristic variables $s_1 = s_1(X, t)$ and $s_2 = s_2(X, t)$ are introduced into the (X, t)-plane in such a way that

$$X_{s_1} + \sqrt{\alpha}\, t_{s_1} = 0, \qquad X_{s_2} - \sqrt{\alpha}\, t_{s_2} = 0. \tag{7.3}$$

The s_i=constant curves constitute an admissible set of coordinates so long as the wavespeed $\sqrt{\alpha}$ is finite and non-zero. In these coordinates the problem (7.1) becomes one of solving the system of equations

$$
\begin{aligned}
\sqrt{a}\,P_{s_1} + R_{s_1} &= 0, \\
\sqrt{a}\,P_{s_2} - R_{s_2} &= 0, \\
\sqrt{a}\,t_{s_1} + X_{s_1} &= 0, \\
\sqrt{a}\,t_{s_2} - X_{s_2} &= 0,
\end{aligned}
\tag{7.4}
$$

subject to suitable boundary and initial conditions. If consideration is limited to disturbances propagating into an undeformed body at rest the boundary and initial conditions are

$$
\begin{aligned}
&P(X, 0) = 0\ , \qquad R(X, 0) = 0 \\
&P(0, t) = \epsilon\,\Phi(t), \qquad \Phi(0) = 0,
\end{aligned}
\tag{7.5}
$$

where $\Phi(t)$ and ϵ are given and ϵ is small. In addition to this, initial conditions must be given which fix the parameterization of the characteristic curves s_i=constant. Take

$$
X(x_1, s_2)\Big|_{s_1=s_2=s} = 0, \quad \sqrt{a_{00}}\; t(s_1, s_2)\Big|_{s_1=s_2=s} = s \tag{7.6}
$$

where, as before,

$$
a_{00} = a(0, 0) = (\lambda + 2\mu)/\rho_0\ . \tag{7.7}
$$

Using (7.6) the boundary condition $(7.5)_3$ can be rewritten in terms of the new variables as

$$
P(s, s) = \epsilon\,\Phi(s/a_{00})\ . \tag{7.8}
$$

The initial conditions $(7.5)_{1,2}$ are not conveniently expressed in the characteristic coordinates, but since there is a rest zone ahead of the wave, an initial condition may be applied on the wavefront.

On this basis $(7.5)_{1,2}$ are replaced by the conditions

$$P(0, s_2) = 0, \quad R(0, s_2) = 0, \quad s_2 > 0. \tag{7.9}$$

The (X, t)-diagram of figure 7.1 illustrates the coordinates and auxiliary conditions.

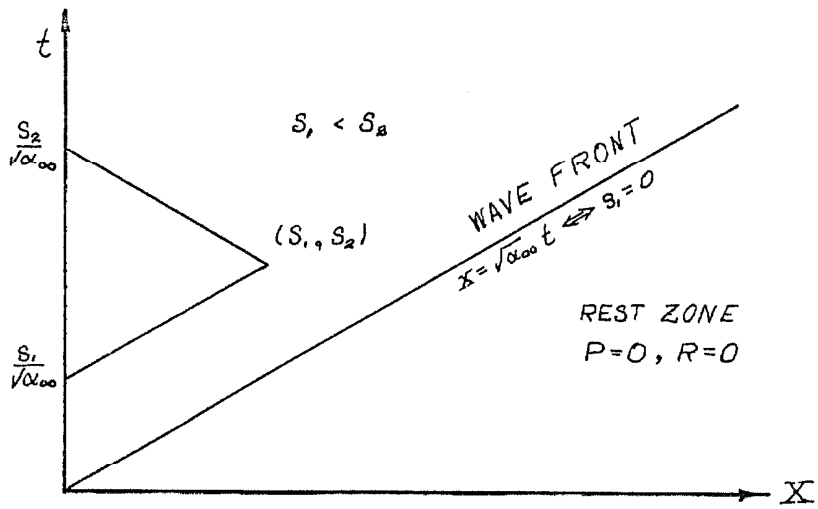


Figure 7.1

It is assumed that $\alpha(P)$ is expandable near P=0 as in $(A.8)_1$,

$$\alpha(P) = \alpha_{00} + \alpha_{10}P + \alpha_{20}P^2 + \dots , \tag{7.10}$$

and that P, R, X, and t are representable by the perturbation series

$$\begin{aligned} P &= \epsilon P_0 + \epsilon^2 P_1 + \dots , \\ R &= \epsilon R_0 + \epsilon^2 R_1 + \dots , \\ X &= X_0 + \epsilon X_1 + \dots , \\ t &= t_0 + \epsilon t_1 + \dots , \end{aligned} \tag{7.11}$$

where the functions depend on the arguments s_1 and s_2. Substitution of $(7.10)_1$ into (7. 9) gives

$$\alpha(P) = \alpha_{00} + \epsilon\,\alpha_{10}\,P_0 + \ldots, \tag{7.12}$$

and, from this,

$$\sqrt{\alpha(P)} = \sqrt{\alpha_{00}}\left(1 + \frac{\epsilon\,\alpha_{10}}{2\alpha_{00}}\,P_0 + \ldots\right). \tag{7.13}$$

Substitution of (7. 10) and (7. 12) into (7. 4) and equation to zero of the coefficient of each power of ϵ in each equation gives

$$\epsilon^0:\qquad \sqrt{\alpha_{00}}\,t_{0_{s_1}} + X_{0_{s_1}} = 0,\quad \sqrt{\alpha_{00}}\,t_{0_{s_2}} - X_{0_{s_2}} = 0, \tag{7.14}$$

$$\begin{aligned}
\epsilon^1:\qquad & \sqrt{\alpha_{00}}\,P_{0_{s_1}} + R_{0_{s_1}} = 0,\quad \sqrt{\alpha_{00}}\,P_{0_{s_2}} - R_{0_{s_2}} = 0,\\
& \sqrt{\alpha_{00}}\,t_{1_{s_1}} + X_{1_{s_1}} = -\frac{\alpha_{10}}{2\sqrt{\alpha_{00}}}\,P_0\,t_{0_{s_1}},\\
& \sqrt{\alpha_{00}}\,t_{1_{s_2}} - X_{1_{s_2}} = -\frac{\alpha_{10}}{2\sqrt{\alpha_{00}}}\,P_0\,t_{0_{s_2}},
\end{aligned} \tag{7.15}$$

$$\begin{aligned}
\epsilon^2:\qquad & \sqrt{\alpha_{00}}\,P_{1_{s_1}} + R_{1_{s_1}} = -\frac{\alpha_{10}}{2\sqrt{\alpha_{00}}}\,P_0\,P_{0_{s_1}},\\
& \sqrt{\alpha_{00}}\,P_{1_{s_2}} - R_{1_{s_2}} = -\frac{\alpha_{10}}{2\sqrt{\alpha_{00}}}\,P_0\,P_{0_{s_2}}.
\end{aligned} \tag{7.16}$$

Substitution of (7. 10) into the boundary and initial conditions (7. 6) - (7. 9) gives

$$
\begin{aligned}
&X_i(s, s) = 0, \quad i=0,1,\ldots, \\
&\sqrt{\alpha_{00}}\, t_0(s, s)=s, \quad t_i(s, s)=0, \; i=1, 2, \ldots, \\
&P_0(s, s)= \Phi(s/\sqrt{\alpha_{00}}), \quad P_i(s, s)=0, \quad i=1, 2, \ldots, \\
&P_i(0, s_2)=0, \quad R_i(0, s_2)=0, \quad i=0, 1, \ldots .
\end{aligned} \tag{7.17}
$$

The solution of (7.14) consistent with (7.16) is

$$
t_0 = \frac{1}{2\sqrt{\alpha_{00}}}(s_1 + s_2), \quad X_0 = \frac{1}{2}(s_1 - s_2). \tag{7.18}
$$

Solution of $(7.15)_{1,2}$, with due regard to (7.17), gives

$$
P_0 = \Phi(s_1/\sqrt{\alpha_{00}}), \quad R_0 = -\sqrt{\alpha_{00}}\, \Phi(s_1/\sqrt{\alpha_{00}}), \tag{7.19}
$$

as is readily verified. Substitution of (7.19) into $(7.15)_{3,4}$ gives

$$
\begin{aligned}
(\sqrt{\alpha_{00}}\, t_1 + X_1)_{s_1} &= -\frac{\alpha_{10}}{4\alpha_{00}} \Phi(s_1/\sqrt{\alpha_{00}}), \\
(\sqrt{\alpha_{00}}\, t_1 - X_1)_{s_2} &= -\frac{\alpha_{10}}{4\alpha_{00}} \Phi(s_1/\sqrt{\alpha_{00}}),
\end{aligned} \tag{7.20}
$$

and solutions of these equations satisfying (7.17) are

$$
\begin{aligned}
\sqrt{\alpha_{00}}\, t_1 &= \frac{\alpha_{10}}{8\alpha_{00}}(s_1 - s_2)\, \Phi(s_1/\sqrt{\alpha_{00}}) + \frac{\alpha_{10}}{8\sqrt{\alpha_{00}}} \int_{s_1/\sqrt{\alpha_{00}}}^{s_2/\sqrt{\alpha_{00}}} \Phi(z)\,dz \\
X_1 &= -\frac{\alpha_{10}}{8\alpha_{00}}(s_1 - s_2)\, \Phi(s_1/\sqrt{\alpha_{00}}) + \frac{\alpha_{10}}{8\sqrt{\alpha_{00}}} \int_{s_1/\sqrt{\alpha_{00}}}^{s_2/\sqrt{\alpha_{00}}} \Phi(z)\,dz.
\end{aligned} \tag{7.21}
$$

Connections to P and R are computed from (7.16) which now take the form

$$(\sqrt{a_{00}}\, P_1 + R_1)_{s_1} = -\frac{a_{10}}{4\sqrt{a_{00}}} \left[\Phi^2(s_1/\sqrt{a_{00}})\right]_{s_1} = -\frac{a_{10}}{4a_{00}} \left[\Phi^2(s_1/\sqrt{a_{00}})\right]',$$

$$(\sqrt{a_{00}}\, P_1 - R_1)_{s_2} = 0 . \tag{7. 22}$$

The solution of (7. 22) compatible with (7. 17) is

$$P_1 = 0, \quad R_1 = -\frac{a_{10}}{4\sqrt{a_{00}}}\, \Phi^2(s_1/\sqrt{a_{00}}). \tag{7. 23}$$

The collected results are as follows:

$$P = \epsilon\, \Phi(s_1/\sqrt{a_{00}}) + O(\epsilon^3),$$

$$R = -\epsilon \sqrt{a_{00}}\, \Phi(s_1/\sqrt{a_{00}}) - \frac{\epsilon^2 a_{10}}{4\sqrt{a_{00}}}\, \Phi^2(s_1/\sqrt{a_{00}}) + O(\epsilon^3),$$

$$\sqrt{a_{00}}\, t = \frac{1}{2}(s_1+s_2) + \frac{\epsilon\, a_{10}}{8a_{00}}(s_1-s_2)\, \Phi(s_1/\sqrt{a_{00}}) + \frac{\epsilon\, a_{10}}{8\sqrt{a_{00}}} \int_{s_1/\sqrt{a_{00}}}^{s_2/\sqrt{a_{00}}} \Phi(z)\, dz + O(\epsilon^2), \tag{7. 24}$$

$$X = -\frac{1}{2}(s_1-s_2) - \frac{\epsilon\, a_{10}}{8\sqrt{a_{00}}} \int_{s_1/\sqrt{a_{00}}}^{s_2/\sqrt{a_{00}}} \Phi(z)\, dz + O(\epsilon^2).$$

As an example problem (7. 24) is evaluated for the case

$$\epsilon = P_0, \quad \Phi(t) = \frac{t}{\tau} H(t) - \left(\frac{t}{\tau} - 1\right) H(t-\tau), \tag{7. 25}$$

where τ is a given positive constant. The function Φ is displayed as figure 7.2.

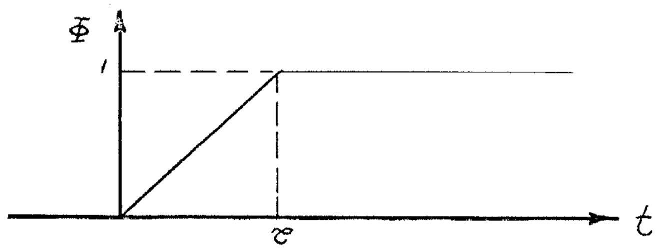


Figure 7.2

Substitution of (7.25) into (7.24) gives

$$P = \frac{P_0}{\sqrt{a_{00}}\,\tau}\left\{ s_1 H(s_1/\sqrt{a_{00}}) - (s_1-\sqrt{a_{00}}\,\tau)\, H(s_1-\sqrt{a_{00}}\,\tau)\right\} + O(P_0^{\,3})$$

$$R = -\frac{P_0}{\tau}\left\{ s_1 H(s_1) - (s_1-\sqrt{a_{00}}\,\tau)\, H(s_1-\sqrt{a_{00}}\,\tau)\right\}$$

$$-\frac{P_0^{\,2} a_{10}}{4a_{00}\sqrt{a_{00}}\,\tau^2}\left\{ s_1^{\,2} H(s_1) - (s_1^{\,2}-a_{00}\tau^2) H(s_1-\sqrt{a_{00}}\,\tau)\right\} + O(P_0^{\,3})$$

$$\sqrt{a_{00}}\,t = \frac{1}{2}(s_1+s_2) + \frac{P_0 a_{10}}{8a_{00}\sqrt{a_{00}}\,\tau}(s_1-s_2)\left\{ s_1 H(s_1) - (s_1-\sqrt{a_{00}}\,\tau) H(s_1-\sqrt{a_{00}}\,\tau)\right\}$$

$$+\frac{P_0 a_{10}}{16a_{00}\sqrt{a_{00}}\,\tau}\left\{ s_2^{\,2} - s_1^{\,2} + (s_1-\sqrt{a_{00}}\,\tau)^2\, H(s_1-\sqrt{a_{00}}\,\tau)\right.$$

$$\left. -(s_2-\sqrt{a_{00}}\,\tau)\, H(s_2-\sqrt{a_{00}}\,\tau)\right\} + O(P_0^{\,2})$$

$$X = -\frac{1}{2}(s_1-s_2) - \frac{P_0 a_{10}}{8a_{00}\sqrt{a_{00}}\,\tau}(s_1-s_2)\left\{s_1 H(s_1) - (s_1-\sqrt{a_{00}}\,\tau)H(s_1-\sqrt{a_{00}}\,\tau)\right\}$$

$$+\frac{P_0 a_{10}}{16a_{00}\sqrt{a_{00}}\,\tau}\left\{(s_2^{\,2}-s_1^{\,2}) + (s_1-\sqrt{a_{00}}\,\tau)^2\,H(s_1-\sqrt{a_{00}}\,\tau)\right.$$

$$\left.-(s_2-\sqrt{a_{00}}\,\tau)^2\,H(s_2-\sqrt{a_{00}}\,\tau)\right\} + O(P_0^{\,2}) \qquad (7.\,27)$$

In the region $\sqrt{a_{00}}\,\tau \leqslant s_1 < s_2$ (7. 27) becomes

$$P = P_0 + O(P_0^{\,3}), \quad R = -P_0\sqrt{a_{00}} - \frac{P_0^{\,2} a_{10}}{4\sqrt{a_{00}}},$$

$$\sqrt{a_{00}}\,t = \frac{1}{2}(s_1+s_2) + O(P_0^{\,3}), \qquad (7.\,28)$$

$$X = -\frac{1}{2}\left(1+\frac{P_0 a_{10}}{2a_{00}}\right)(s_1-s_2) + O(P_0^{\,3}).$$

From $(7.\,38)_{3,4}$

$$s_1 = \sqrt{a_{00}}\,t - \frac{1}{\left(1+\frac{P_0 a_{10}}{2a_{00}}+\ldots\right)}X, \quad s_2 = \sqrt{a_{00}}\,t + \frac{1}{\left(1+\frac{P_0 a_{10}}{2a_{00}}+\ldots\right)}X\,. \qquad (7.\,29)$$

Along s_1=constant

$$\frac{dX}{dt} = \sqrt{a_{00}} + \frac{P_0 a_{10}}{2\sqrt{a_{00}}} + O(P_0^{\,2}) \qquad (7.\,30)$$

and similarly for s_2=constant. This shows how the present method provides corrections to the wavespeeds beyond their values according to the linearized theory. In the region $0 \leqslant s_1 < \sqrt{a_{00}}\,\tau < s_2$ (7. 27) becomes

$$P = P_0 \frac{s_1}{\sqrt{a_{00}}\,\tau} + O(P_0{}^3), \quad R = -\frac{P_0 s_1}{\tau} - \frac{P_0{}^2 a_{10}}{4a_{00}\sqrt{a_{00}}\,\tau^2}\, s_1{}^2 + O(P_0{}^3)$$

$$\sqrt{a_{00}}\; t = \frac{1}{2}(s_1+s_2) + \frac{P_0 a_{10}}{16 a_{00}\sqrt{a_{00}}\,\tau}(s_1-s_2)^2 + O(P_0{}^2) \qquad (7.\ 31)$$

$$X = -\frac{1}{2}(s_1-s_2) - \frac{P_0 a_{10}}{16 a_{00}\sqrt{a_{00}}\,\tau}\left\{2s_1(s_1-s_2)-(s_2{}^2-s_1{}^2)\right\} + O(P_0{}^2)\ .$$

a rough figure displaying the solution (7. 27) is given as figure 7. 3. The drawing is for $P_0 a_{10} < 0$, the case in which shock formation does not occur.

The perturbation method here expounded is seen both in its generality (7. 24) and by example (7. 27), figure 7. 3, to provide solutions having the qualitative features of the exact solution. Observe in equation (7. 27) and in figure 7. 3 the straight advancing characteristics in the simple wave region, the proper fit and smoothness across the wavefront, and, particularly the absence of the secular terms obtained by Fine and Shield.

As discussed by Fox, this method can also be used to predict shock formation.

The results shown on figure 7. 3 agree, in the limit $\tau \rightarrow 0$, with the perturbation results of section 4. 1. 1.

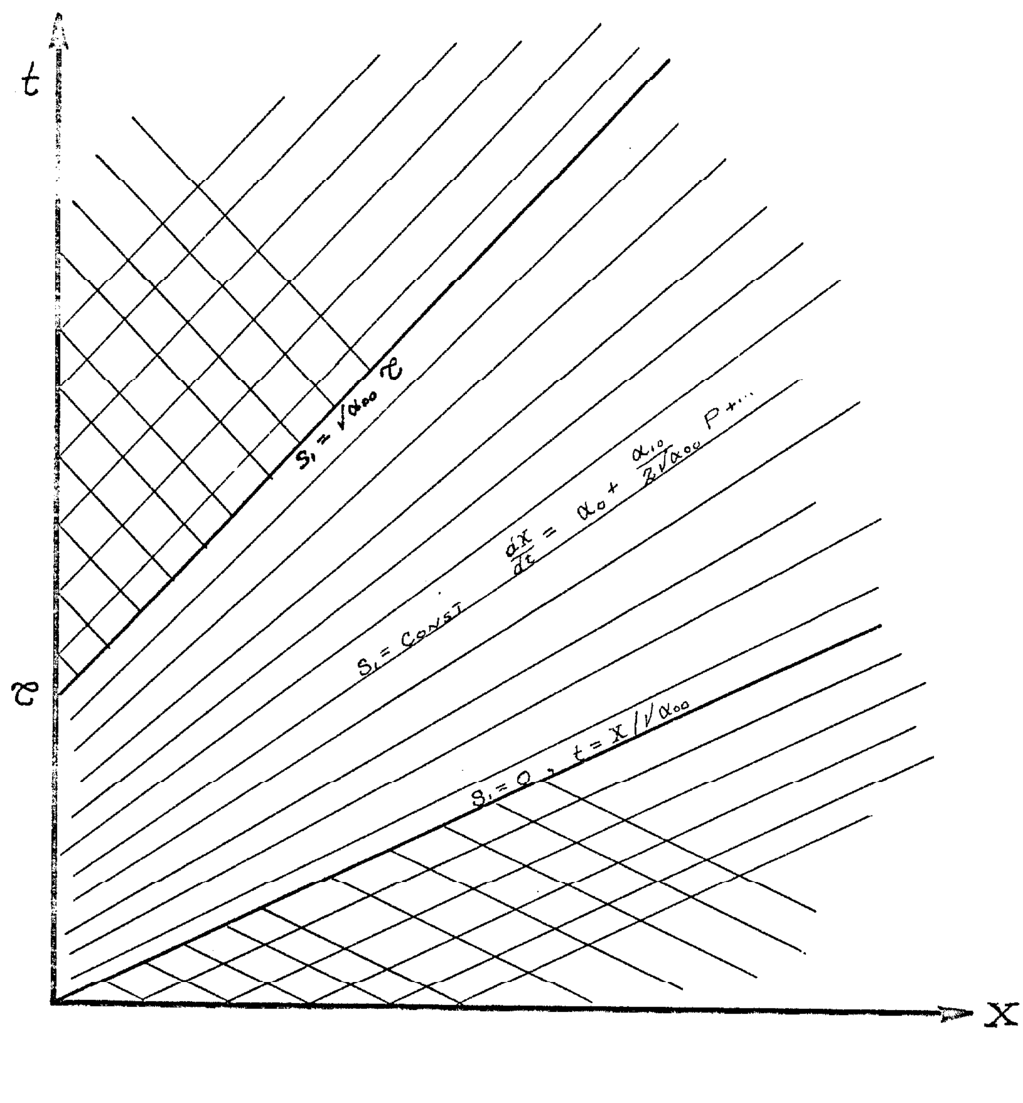

t
τ
X
$S_1 = \sqrt{\alpha_{00}}\,\tau$
$\frac{dX}{dt} = \alpha_0 + \frac{\alpha_{10}}{2\sqrt{\alpha_{00}}} P + \cdots$
$S_1 =$ CONST
$S_1 = 0,\ t = X/\sqrt{\alpha_{00}}$

Figure 7.3

7.2. Shear Strain Applied to the Boundary of an Undeformed Half-Space

In this section perturbation solution of the field equations (2.45) is considered for cases in which $Q \neq 0$. These equations are

$$\begin{aligned} \alpha P_X + \beta Q_X &= R_t, \\ \gamma P_X + \delta Q_X &= S_t, \\ R_X &= P_t, \\ S_X &= Q_t . \end{aligned} \tag{7.32}$$

New independent variables s_1 and s_2 related to the physical independent variables by the equations

$$X_{s_1} - a_4 t_{s_1} = 0, \quad X_{s_2} - a_3 t_{s_2} = 0, \tag{7.33}$$

where

$$a_3 = \left\{ \frac{1}{2} \left[(\alpha + \delta) - \left((\alpha - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2},$$

$$a_4 = \left\{ \frac{1}{2} \left[(a + \delta) + \left((a - \delta)^2 + 4\beta\gamma \right)^{1/2} \right] \right\}^{1/2},$$

are the two positive characteristic wavespeeds are introduced. The new variables constitute an admissible coordinate system so long as the wavespeeds a_3 and a_4 are distinct, finite, and non-zero. The advancing characteristics have been chosen as coordinates because it is expected that the field variables associated with advancing waves will be most easily expressed as functions of these quantities. Differentiation is accomplished by means of the formulae

$$\frac{\partial}{\partial s_1} = X_{s_1} \frac{\partial}{\partial X} + t_{s_1} \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial s_2} = X_{s_2} \frac{\partial}{\partial X} + t_{s_2} \frac{\partial}{\partial t}, \tag{7.35}$$

or, equivalently,

$$\frac{\partial}{\partial X} = \frac{1}{\Delta}\left[t_{s_2}\frac{\partial}{\partial s_1} - t_{s_1}\frac{\partial}{\partial s_2}\right], \quad \frac{\partial}{\partial t} = \frac{1}{\Delta}\left[X_{s_1}\frac{\partial}{\partial s_2} - X_{s_2}\frac{\partial}{\partial s_1}\right] \tag{7.36}$$

where

$$\Delta \equiv X_{s_1}t_{s_2} - X_{s_2}t_{s_1} \neq 0. \tag{7.37}$$

Transforming (7.32) into (s_1, s_2)-coordinates and adjoining (7.33) gives:

$$\begin{aligned}
&t_{s_2}(\alpha P_{s_1} + \beta Q_{s_1} + a_3 R_{s_1}) - t_{s_1}(\alpha P_{s_2} + \beta Q_{s_2} + a_4 R_{s_2}) = 0,\\
&t_{s_2}(\gamma P_{s_1} + \delta Q_{s_1} + a_3 S_{s_1}) - t_{s_1}(\gamma P_{s_2} + \delta Q_{s_2} + a_4 S_{s_2}) = 0,\\
&t_{s_2}(R_{s_1} + a_3 P_{s_1}) - t_{s_1}(R_{s_2} + a_4 P_{s_2}) = 0,\\
&t_{s_2}(S_{s_1} + a_3 Q_{s_1}) - t_{s_1}(S_{s_2} + a_4 Q_{s_2}) = 0,\\
&X_{s_1} - a_4 t_{s_1} = 0, \quad X_{s_2} - a_3 t_{s_2} = 0.
\end{aligned} \tag{7.38}$$

The problem of application of strains to the boundary of a half-space initially undeformed and at rest is set in terms of the field equations (7.38) and the following initial and boundary conditions.

$$\begin{aligned}
&P(X,0) = Q(X,0) = 0, \quad R(X,0) = S(X,0) = 0,\\
&P(0,t) = 0, \quad Q(0,t) = \epsilon\mathscr{F}(t),
\end{aligned} \tag{7.39}$$

where ϵ and $\mathscr{F}$ are given and

$$\mathscr{F}(0) = 0. \tag{7.40}$$

In addition to (7. 39) initial conditions must be placed on X and t. These conditions are taken as

$$X(s, s) = 0, \qquad a_0 t(s, s) = s, \tag{7. 41}$$

where a_0 is some constant having the dimensions of velocity. In consideration of (7. 41) equations $(7.\,39)_{5,6}$ can be rewritten in terms of the (s_1, s_2) coordinates as

$$P(s, s) = 0, \qquad Q(s, s) = \epsilon \Psi(s/a_0). \tag{7. 42}$$

Since the longitudinal wavefront is $s_2=0$ and the shear wavefront is $s_1=0$ equations $(7.\,39)_{1-4}$ can be replaced by

$$P(s_1, 0) = Q(0, s_2) = 0, \quad R(s_1, 0) = S(0, s_2) = 0. \tag{7. 43}$$

It is assumed that the dependent variables are representable by the following perturbation series:

$$\begin{aligned} P &= \epsilon^2 P_2 + O(\epsilon^4), \\ Q &= \epsilon Q_1 + O(\epsilon^3), \\ R &= \epsilon^2 R_2 + O(\epsilon^4), \\ S &= \epsilon S_1 + O(\epsilon^3), \\ t &= t_0 + \epsilon^2 t_2 + O(\epsilon^4), \\ X &= X_0 + \epsilon^2 X_2 + O(\epsilon^4), \end{aligned} \tag{7. 44}$$

where the perturbation quantities are regarded as functions of s_1, s_2.

Substitution of (7. 44) into (A. 8) and simplification gives

$$\begin{aligned} \alpha &= \alpha_{00} + \epsilon^2(\alpha_{10} P_2 + \alpha_{02} Q_1^{\,2}) + \dots, \\ \beta &= \epsilon \beta_{01} Q_1 + \dots, \\ \gamma &= \epsilon \delta_{10} Q_1 + \dots, \\ \delta &= \delta_{00} + \epsilon^2(\delta_{10} P_2 + \delta_{02} Q_1^{\,2}) + \dots \quad . \end{aligned} \tag{7. 45}$$

The characteristic wavespeeds are found to be

$$a_3 = \sqrt{\delta_{00}} + \epsilon^2(a_{31}P_2 + a_{32}Q_1^2) + \dots,$$
$$a_4 = \sqrt{\alpha_{00}} + \epsilon^2(a_{41}P_2 + a_{42}Q_1^2) + \dots, \tag{7.46}$$

where

$$a_{31} = \frac{\delta_{10}}{2\sqrt{\delta_{00}}}, \qquad a_{32} = \frac{1}{2\sqrt{\delta_{00}}}\left(\delta_{02} - \frac{\beta_{01}\,\delta_{10}}{\alpha_{00} - \delta_{00}}\right),$$
$$a_{41} = \frac{\alpha_{10}}{2\sqrt{\alpha_{00}}}, \qquad a_{42} = \frac{1}{2\sqrt{\alpha_{00}}}\left(\alpha_{02} + \frac{\beta_{01}\,\delta_{10}}{\alpha_{00} - \delta_{00}}\right). \tag{7.47}$$

Substitution of (7. 45) - (7. 47) into (7. 38) and equation of terms of the same degree in ϵ to zero in each equation gives

$$\epsilon^0: \; (X_0 - \sqrt{\delta_{00}}\,t_0)_{s_2} = 0, \quad (X_0 - \sqrt{\alpha_{00}}\,t_0)_{s_1} = 0, \tag{7.48}$$

$$\epsilon^1: \; t_{0_{s_2}}(S_1 + \sqrt{\delta_{00}}\,Q_1)_{s_1} - t_{0_{s_1}}(S_1 + \sqrt{\alpha_{00}}\,Q_1)_{s_2} = 0,$$
$$t_{0_{s_2}}(\sqrt{\delta_{00}}\,S_1 + \delta_{00}Q_1)_{s_1} - t_{0_{s_1}}(\sqrt{\alpha_{00}}\,S_1 + \delta_{00}Q_1)_{s_2} = 0, \tag{7.49}$$

$$\epsilon^2: \; t_{0_{s_2}}(R_2 + \sqrt{\delta_{00}}\,P_2)_{s_1} - t_{0_{s_1}}(R_2 + \sqrt{\alpha_{00}}\,P_2)_{s_2} = 0,$$
$$t_{0_{s_2}}(\sqrt{\delta_{00}}\,R_2 + \alpha_{00}P_2)_{s_1} - t_{0_{s_1}}(\sqrt{\alpha_{00}}\,R_2 + \alpha_{00}P_2)_{s_2}$$
$$= -\frac{1}{2}\beta_{01}\left[t_{0_{s_2}}(Q_1^2)_{s_1} - t_{0_{s_1}}(Q_1^2)_{s_2}\right],$$

$$(X_2 - \sqrt{\alpha_{00}}\, t_2)_{s_1} = (a_{41}P_2 + a_{42}Q_1^{\,2})t_{0_{s_1}}, \qquad (7.50)$$

$$(X_2 - \sqrt{\delta_{00}}\, t_2)_{s_2} = (a_{31}P_2 + a_{32}Q_1^{\,2})t_{0_{s_2}}.$$

Substitution of (7. 44) into (7. 41) - (7. 43) gives the boundary and initial conditions

$$\begin{aligned}
&X_i(s, s) = 0, \quad i=0, 1, \ldots, \quad a_0t_0(s, s) = s,\\
&a_0t_i(s, s) = 0, \quad i=1, 2, \ldots,\\
&P_i(s, s) = 0, \quad i=1, 2, \ldots, \quad Q_1(s, s) = \epsilon\, \Psi(s/a_0),\\
&Q_i(s, s) = 0, \quad i=2, 3, \ldots,\\
&P_i(s_1, 0) = Q_i(0, s_2) = 0, \quad R_i(s_1, 0) = S_i(0, s_2) = 0, \quad i=1, 2, \ldots .
\end{aligned} \qquad (7.51)$$

The solution of (7. 48) satisfying (7. 51) is

$$X_0 = \frac{\sqrt{\alpha_{00}\,\delta_{00}}}{a_0(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})}(s_2 - s_1), \qquad a_0t_0 = \frac{\sqrt{\alpha_{00}}\,s_2 - \sqrt{\delta_{00}}\,s_1}{\sqrt{\alpha_{00}} - \sqrt{\delta_{00}}} \quad . \qquad (7.52)$$

With (7. 52), (7. 49) becomes

$$\begin{aligned}
&\sqrt{\alpha_{00}}(\sqrt{\delta_{00}}S_1 + \delta_{00}Q_1)_{s_1} + \sqrt{\delta_{00}}(\sqrt{\delta_{00}}S_1 + \sqrt{\alpha_{00}\delta_{00}}Q_1)_{s_2} = 0,\\
&\sqrt{\alpha_{00}}(\sqrt{\delta_{00}}S_1 + \delta_{00}Q_1)_{s_1} + \sqrt{\delta_{00}}(\sqrt{\alpha_{00}}S_1 + \delta_{00}Q_1)_{s_2} = 0.
\end{aligned} \qquad (7.53)$$

The solution of these equations satisfying (7. 51) is seen to be

$$\begin{aligned}
&Q_1 = \Psi(s_1/a_0), \qquad S_1 = -\sqrt{\delta_{00}}\,\Psi(s_1/a_0),\\
&\qquad \Psi(z) = 0, \qquad z \leq 0.
\end{aligned} \qquad (7.54)$$

With this and (7. 42), (7. 50)$_{1,2}$ become

$$\sqrt{\alpha_{00}}(R_2 + \sqrt{\delta_{00}}P_2)_{s_1} + \sqrt{\delta_{00}}(R_2 + \sqrt{\alpha_{00}}P_2)_{s_2} = 0,$$

$$\sqrt{\alpha_{00}}(\sqrt{\delta_{00}}R_2 + \alpha_{00}P_2)_{s_1} + \sqrt{\delta_{00}}(\sqrt{\alpha_{00}}R_2 + \alpha_{00}P_2)_{s_2} = \qquad (7.\ 55)$$

$$- \frac{\beta_{01}\sqrt{\alpha_{00}}}{2}\left[\Psi^2(s_1/a_0)\right]_{s_1}.$$

The solution of (7. 55) consistent with (7. 51) is

$$P_2 = -\frac{\beta_{01}}{2(\alpha_{00} - \delta_{00})}\left[\Psi^2(s_1/a_0) - \Psi^2(s_2/a_0)\right]$$

$$(7.\ 56)$$

$$R_2 = \frac{\beta_{01}}{2(\alpha_{00} - \delta_{00})}\left[\sqrt{\delta_{00}}\ \Psi^2(s_1/a_0) - \sqrt{\alpha_{00}}\ \Psi^2(s_2/a_0)\right].$$

It remains to consider (7. 50)$_{3,4}$. Using the previous results the solutions of these equations consistent with (7. 51) are

$$t_2 = \frac{(s_2 - s_1)}{a_0(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})^2}\left[\sqrt{\alpha_{00}}\left(a_{32} - \frac{\beta_{01}a_{31}}{2(\alpha_{00} - \delta_{00})}\right)\Psi^2(s_1/a_0)\right.$$

$$\left. - \frac{\beta_{01}a_{41}\sqrt{\delta_{00}}}{2(\alpha_{00} - \delta_{00})}\Psi^2(s_2/a_0)\right] + \frac{1}{(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})^2}\left[\frac{\beta_{01}a_{31}\sqrt{\alpha_{00}}}{2(\alpha_{00} - \delta_{00})}\right.$$

$$\left. - \sqrt{\delta_{00}}\left(a_{42} - \frac{\beta_{01}a_{41}}{2(\alpha_{00} - \delta_{00})}\right)\right]\int_{s_1/a_0}^{s_2/a_0}\Psi^2(z)dz \qquad (7.\ 57)_1$$

$$X_2 = \sqrt{\delta_{00}}\,t_2 + \frac{\sqrt{\alpha_{00}}}{a_0(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})}\left(a_{32} - \frac{\beta_{01} a_{31}}{2(\alpha_{00} - \delta_{00})}\right)(s_2 - s_1)\,\Psi^2(s_1/a_0)$$

$$+ \frac{\beta_{01}\, a_{31} \sqrt{\alpha_{00}}}{2(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})(\alpha_{00} - \delta_{00})} \int_{s_1/a_0}^{s_2/a_0} \Psi^2(z)\,dz \,. \qquad (7.57)_2$$

As an illustration of the foregoing perturbation solution an explicit calculation is given for the case

$$\epsilon = Q_0, \quad \Psi(t) = \frac{t}{\tau} H(t) - \left(\frac{t}{\tau} - 1\right) H(t - \tau) \,. \qquad (7.58)$$

where τ is a given constant. Associated with this problem is the (X, t)-diagram of figure 7.4. Evaluation of (7.54), (7.56), and (7.57) in region IV of figure 7.4 gives

$$Q_1 = 0, \quad S_1 = 0, \quad P_2 = \frac{\beta_{01}\, s_2^{\,2}}{2a_0^{\,2}\, \tau^2 (\alpha_{00} - \delta_{00})} \,, \quad R_2 = -\sqrt{\alpha_{00}}\, P_2$$

$$t_2 = \frac{1}{(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})^2}\left\{ - \frac{\beta_{01} a_{41} \sqrt{\delta_{00}}}{2(\alpha_{00} - \delta_{00})}\, \frac{(s_2 - s_1)}{a_0} \right. \qquad (7.59)$$

$$\left. + \left[\frac{\beta_{01} a_{31} \sqrt{\alpha_{00}}}{2(\alpha_{00} - \delta_{00})} - \sqrt{\delta_{00}} \left(a_{42} - \frac{\beta_{01} a_{41}}{2(\alpha_{00} - \delta_{00})}\right)\right] \frac{s_2}{3a_0} \right\} \frac{s_2^{\,2}}{a_0^{\,2}\, \tau^2}$$

$$X_2 = \sqrt{\delta_{00}}\, t_2 + \frac{\beta_{01} a_{31} \sqrt{\alpha_{00}}}{2(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})(\alpha_{00} - \delta_{00})} \quad \frac{s_2^{\,3}}{2a_0^{\,3}\, \tau^2} \,.$$

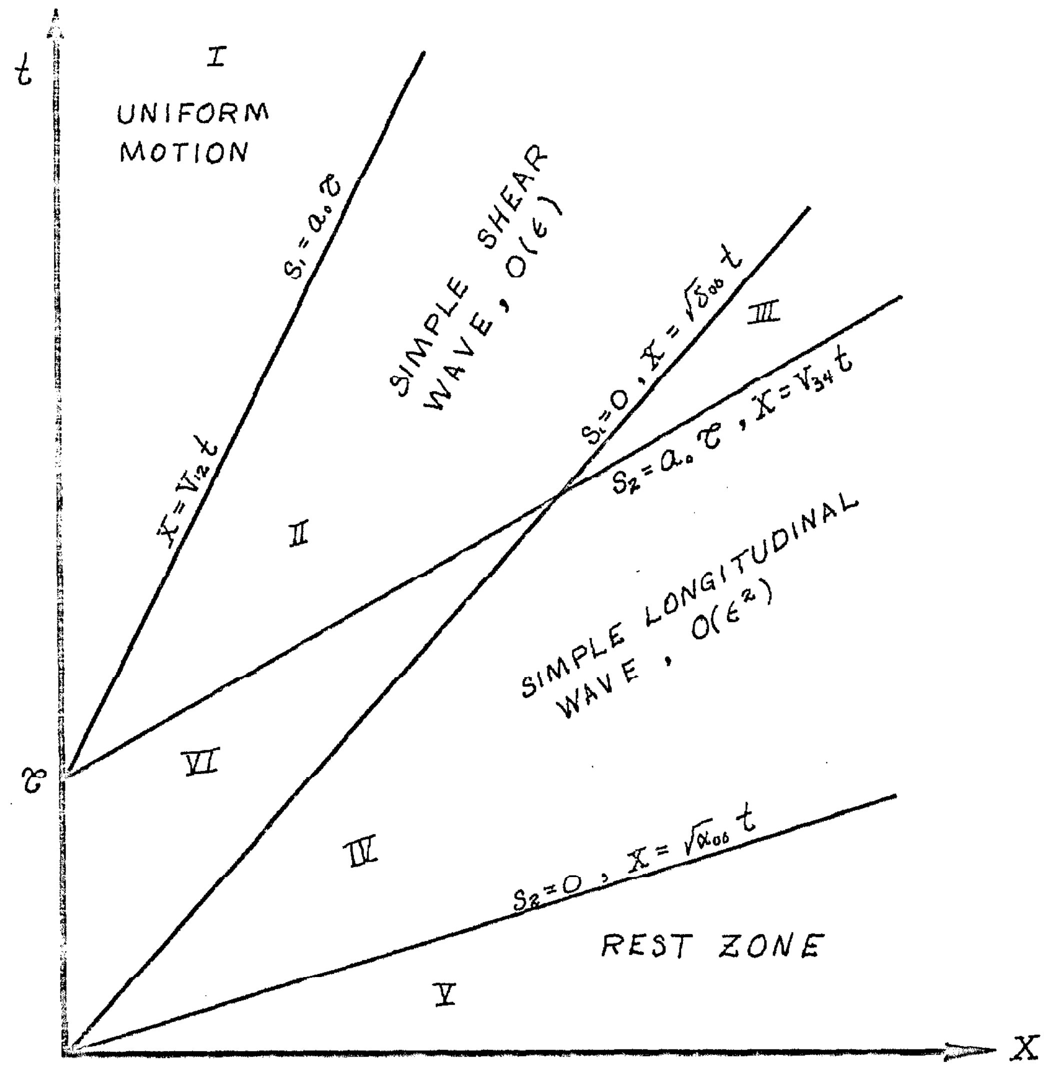

t
I
UNIFORM MOTION
$s_1 = a_0\tau$
SIMPLE SHEAR WAVE, $O(\epsilon)$
$X = V_{12}t$
II
$s_1 = 0$, $X = \sqrt{\delta_{00}}\,t$
III
$s_2 = a_0\tau$, $X = V_{34}t$
SIMPLE LONGITUDINAL WAVE, $O(\epsilon^2)$
τ
VI
IV
$s_2 = 0$, $X = \sqrt{\alpha_{00}}\,t$
REST ZONE
V
X

Figure 7.4

From this and (7. 52)

$$V_{34} = \sqrt{\alpha_{00}}\left(1 + \frac{\alpha_{10}\,\beta_{01}}{4\alpha_{00}(\alpha_{00}-\delta_{00})} Q_0^{\,2} + \dots\right) , \tag{7. 60}$$

a result which agrees with $(4.\ 67)_1$. Evaluation of (7. 54), (7. 56), and (7. 57) in region II of the figure gives

$$Q_1 = \frac{s_1}{a_0\tau} \quad , \qquad S_1 = -\sqrt{\delta_{00}}\,\frac{s_1}{a_0\tau}\,, \quad P_2 = -\frac{\beta_{01}}{2(\alpha_{00}-\delta_{00})}\left[\frac{s_1^{\,2}}{(a_0\tau)^2} - 1\right]$$

$$R_2 = \frac{\beta_{01}}{2(\alpha_{00}-\delta_{00})}\left[\sqrt{\delta_{00}}\,\frac{s_1^{\,2}}{(a_0\tau)^2} - \sqrt{\alpha_{00}}\right]$$

$$\begin{aligned} t_2 = {} & \frac{s_2 - s_1}{a_0(\sqrt{\alpha_{00}} - \sqrt{\delta_{00}})^2}\left[\sqrt{\alpha_{00}}\left(a_{32} - \frac{\beta_{01}a_{31}}{2(\alpha_{00}-\delta_{00})}\right)\frac{s_1^{\,2}}{(a_0\tau)^2}\right. \\ & \left. - \frac{\beta_{01}a_{41}\sqrt{\delta_{00}}}{2(\alpha_{00}-\delta_{00})}\right] + \frac{1}{(\sqrt{\alpha_{00}}-\sqrt{\delta_{00}})^2}\left[\frac{\beta_{01}a_{31}\sqrt{\alpha_{00}}}{2(\alpha_{00}-\delta_{00})}\right. \\ & \left. - \sqrt{\delta_{00}}\left(a_{42} - \frac{\beta_{01}a_{41}}{2(\alpha_{00}-\delta_{00})}\right)\right]\left[-\frac{s_1^{\,3}}{3\tau^2 a_0^{\,3}} + \frac{s_2}{a_0} - \frac{2\tau}{3}\right], \\ X_2 = {} & \sqrt{\delta_{00}}\,t_2 + \frac{\sqrt{\alpha_{00}}}{a_0(\sqrt{\alpha_{00}}-\sqrt{\delta_{00}})}\left(a_{32} - \frac{\beta_{01}a_{31}}{2(\alpha_{00}-\delta_{00})}\right)\frac{s_1^{\,2}(s_2 - s_1)}{(a_0\tau)^2} \\ & + \frac{\beta_{01}a_{31}\sqrt{\alpha_{00}}}{2(\sqrt{\alpha_{00}}-\sqrt{\delta_{00}})(\alpha_{00}-\delta_{00})}\left[-\frac{s_1^{\,3}}{3\tau^2 a_0^{\,3}} + \frac{s_2}{a_0} - \frac{2\tau}{3}\right] . \end{aligned} \tag{7. 61}$$

From (7. 61) and (7. 52),

$$V_{12} = \sqrt{\delta_{00}}\left[1 + \frac{1}{2\delta_{00}}\left(\delta_{02} - \frac{\beta_{01}\,\delta_{10}}{\alpha_{00} - \delta_{00}}\right)Q_0^{\,2} + \dots\right]$$

$$V_{23} = \sqrt{\delta_{00}}\left[1 + \frac{\beta_{01}\,\delta_{10}}{4\delta_{00}(\alpha_{00} - \delta_{00})}\,Q_0^{\,2} + \dots\right], \qquad (7.62)$$

results which agree with (4.63).

APPENDIX

Third Order Elastic Materials

The stress-strain equations of the theory of plane strain here considered are (2. 28):

$$t_{\alpha\beta} = \overline{h}_0\, \delta_{\alpha\beta} + \overline{h}_{-1}\, c_{\alpha\beta}, \quad \alpha, \beta = 1, 2 \tag{A. 1}$$

where $\overline{h}_0(I_1, I_2)$ and $\overline{h}_{-1}(I_1, I_2)$ are response functions characteristic of the material. The invariants of the inverse Cauchy deformation tensor associated with the motion (2. 4) are (2. 39):

$$I_1 = 1 + Q^2 + (1+P)^2, \quad I_2 = (1+P)^2. \tag{A. 2}$$

Assume that $\overline{h}_0$ and $\overline{h}_{-1}$ admit the Taylor expansions

$$\begin{aligned}\overline{h}_0 = {} & c_{00} + c_{01}(I_1-2) + c_{02}(I_2-1) + c_{03}(I_1-2)^2 \\ & + c_{04}(I_1-2)(I_2-1) + c_{05}(I_2-1)^2 + c_{06}(I_1-2)^3 \\ & + c_{07}(I_1-2)^2(I_2-1) + c_{08}(I_1-2)(I_2-1)^2 + c_{09}(I_2-1)^3 + \dots\end{aligned} \tag{A. 3}$$

$$\begin{aligned}\overline{h}_{-1} = {} & c_{10} + c_{11}(I_1-2) + c_{12}(I_2-1) + c_{13}(I_1-2)^2 \\ & + c_{14}(I_1-2)(I_2-1) + c_{15}(I_2-1)^2 + c_{16}(I_1-2)^3 \\ & + c_{17}(I_1-2)^2(I_2-1) + c_{18}(I_1-2)(I_2-1)^2 + c_{19}(I_2-1)^3 + \dots\ .\end{aligned}$$

In accordance with (2. 43), the stresses are given by

$$\sigma(P, Q) = \overline{h}_0 + (1+P)^2\,\overline{h}_{-1}, \quad \tau(P, Q) = Q(1+P)\overline{h}_{-1}\,. \tag{A. 4}$$

By (A. 3) and (A. 4)

$$\begin{aligned} \sigma(P, Q) &= a_1 P + a_3 P^2 + a_5 Q^2 + a_7 P Q^2 + a_9 P^3 + \dots \\ \tau(P, Q) &= a_2 Q + a_4 P Q + a_6 P^2 Q + a_8 Q^3 + \dots \end{aligned} \tag{A. 6}$$

where

$$c_{00} + c_{10} = 0,$$

in order that $\sigma(0, 0) = 0$, and where

$$a_1 = 2(c_{01} + c_{02} + c_{10} + c_{11} + c_{12})$$

$$a_2 = c_{10}$$

$$a_3 = c_{01} + c_{02} + c_{10} + c_{11} + c_{12} + 4(c_{03} + c_{04} + c_{05} + c_{11} + c_{12} + c_{13} + c_{14} + c_{15})$$

$$a_4 = c_{10} + 2(c_{11} + c_{12})$$

$$a_5 = c_{01} + c_{11}$$

$$a_6 = 3(c_{11} + c_{12}) + 4(c_{13} + c_{14} + c_{15})$$

$$a_7 = 4(c_{03} + c_{13}) + 2(c_{11} + c_{14} + c_{04})$$

$$a_8 = c_{11}$$

$$\begin{aligned} a_9 = {} & 4(c_{03} + c_{04} + c_{05} + c_{11} + c_{12} + c_{13} + c_{14} + c_{15}) \\ & + 8(c_{06} + c_{07} + c_{08} + c_{09} + c_{13} + c_{14} + c_{15} \\ & + c_{16} + c_{17} + c_{18} + c_{19}). \end{aligned}$$

By (A. 6)

$$\begin{aligned} \sigma_P &= a_1 + 2a_3P + a_7Q^2 + 3a_9P^2 + \ldots \\ \sigma_Q &= 2a_5Q + 2a_7PQ + \ldots \\ \tau_P &= a_4Q + 2a_6PQ + \ldots \\ \tau_Q &= a_2 + a_4P + a_6P^2 + 3a_8Q^2 + \ldots \quad . \end{aligned} \tag{A. 7}$$

Writing

$$a_1 = \rho_0\alpha_{00}, \quad 2a_3 = \rho_0\alpha_{10}, \quad a_7 = \rho_0\alpha_{02}, \quad 3a_9 = \rho_0\alpha_{20},$$

$$2a_5 = \rho_0\beta_{01}, \quad 2a_7 = \rho_0\beta_{11}, \quad a_4 = \rho_0\gamma_{01}, \quad 2a_6 = \rho_0\gamma_{11},$$

$$3a_8 = \rho_0\gamma_{02}, \quad a_2 = \rho_0\delta_{00}, \quad a_4 = \rho_0\gamma_{01} = \rho_0\delta_{10},$$

$$2a_6 = 2\rho_0\delta_{20} = \rho_0\gamma_{11}, \quad 3a_8 = \rho_0\delta_{02} = \rho_0\gamma_{02},$$

(A. 7) becomes

$$\begin{aligned} \alpha &= \alpha_{00} + \alpha_{10}P + \alpha_{02}Q^2 + \alpha_{20}P^2 + \ldots \\ \beta &= \beta_{01}Q + 2\alpha_{02}PQ + \ldots \\ \gamma &= \delta_{10}Q + 2\,\delta_{20}PQ + \ldots \\ \delta &= \delta_{00} + \delta_{10}P + \delta_{20}P^2 + \delta_{02}Q^2 + \ldots, \end{aligned} \tag{A. 8}$$

and (A. 6) becomes

$$\begin{aligned} \sigma(P,Q) &= \rho_0\left[\alpha_{00}P + \tfrac{1}{2}\alpha_{10}P^2 + \tfrac{1}{2}\beta_{01}Q^2 + \alpha_{02}PQ^2 + \tfrac{1}{3}\alpha_{20}P^3 + \ldots\right] \\ \tau(P,Q) &= \rho_0\left[\delta_{00}Q + \delta_{10}PQ + \delta_{20}P^2Q + \tfrac{1}{3}\delta_{02}Q^3 + \ldots\right] . \end{aligned} \tag{A. 9}$$

The coefficients α_{00} and δ_{00} may be expressed in terms of the Lamé modulii of the linear theory of elasticity:

$$\alpha_{00} = (\lambda + 2\mu)/\rho_0, \quad \delta_{00} = \mu/\rho_0 . \tag{A. 10}$$

REFERENCES

1. P. D. Lax, "Hyperbolic Systems of Conservation Laws II", Communications on Pure and Applied Mathematics, Vol. X (1957), pp. 537-566.

2. A. Jeffrey and T. Taniuti, Non-Linear Wave Propagation (New York: Academic Press, 1964).

3. R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves (New York: Interscience Publishers, Inc., 1948).

4. C. Truesdell, "General and Exact Theory of Waves in Finite Elastic Strain", Archive for Rational Mechanics and Analysis, Vol. 8 (1961), pp. 263-296.

5. W. A. Green, "The Growth of Plane Discontinuities Propagating into a Homogeneously Deformed Elastic Material", Archive for Rational Mechanics and Analysis, Vol. 16 (1964), pp. 79-88.

6. A. E. Green, "A Note on Wave Propagation in Initially Deformed Bodies", Journal of the Mechanics and Physics of Solids, Vol. 11 (1963), pp. 119-126.

7. A. D. Fine and R. T. Shield, Second Order Effects in the Propagation of Elastic Waves (a report published by: Division of Applied Mathematics, Brown University, Providence, R.I., December, 1964).

8. Boa-Teh Chu, "Finite Amplitude Waves in Incompressible Perfectly Elastic Materials", Journal of the Mechanics and Physics of Solids, Vol. 12 (1964), pp. 45-57.

9. D. R. Bland, "On Shock Waves in Hyperelastic Media", Proceedings of the International Symposium on Second Order Effects in Elasticity, Plasticity, and Fluid Dynamics, (Oxford: Pergammon Press, 1964) pp. 93-108.

10. D. R. Bland, "Dilitational Waves and Shocks in Large Displacement Isentropic Dynamic Elasticity", Journal of the Mechanics and Physics of Solids, Vol. 12 (1964), pp. 245-267.

11. C. Truesdell and R. A. Toupin, "The Classical Field Theories", Handbuch der Physik, Ed. S. Flugge, Vol. 3/1, (1960), pp. 226-858.

12. C. Truesdell and R. Toupin, "Static Grounds for Inequalities in Finite Strain of Elastic Materials", Archive of Rational

Mechanics and Analysis, Vol. 12 (1963), pp. 1-33.

13. Bernard D. Coleman, "Mechanical and Thermodynamical Admissibility of Stress-Strain Functions", Archive for Rational Mechanics and Analysis, Vol. 9 (1962), pp. 172-186.

14. Lincoln E. Bragg, "Monotonicity on Curves in Lieu of the C-N Inequalitics for Finite Elasticity", Archive for Rational Mechanics and Analysis, Vol. 17 (1964), pp. 327-338.

15. William L. Ko, "Application of Finite Elastic Theory to the Behavior of Rubber-Like Materials", (Ph. D. Thesis, California Institute of Technology), p. 22.

16. Fritz John, "Plane Strain Problems for a Perfectly Elastic Material of Harmonic Type", Communications on Pure and Applied Mathematics, Vol. 13 (1960), pp. 239-296.

17. Phyllis A. Fox, "Perturbation Theory of Wave Propagation Based on the Method of Characteristics", Journal of Mathematics and Physics, Vol. 34 (1955), pp. 133-151.