S-MATRIX METHODS FOR ELECTROMAGNETIC CORRECTIONS TO STRONG INTERACTIONS WITH AN APPLICATION TO THE PROTON-NEUTRON MASS DIFFERENCE

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ABSTRACT

The first half of this thesis is devoted to the derivation of an S-matrix method for calculating the effect of small perturbations on a partial wave amplitude, and, in particular, on the positions and residues of bound states. The method is applicable to both non-relativistic and relativistic problems. It has, as a particular virtue, rapid convergence of the dispersion integrals. Electromagnetic corrections to strong interactions will be an important application and modifications useful for handling the infrared divergence that occurs in this case are described in detail.

The second half of this work is a detailed, quantitative application to the neutron-proton mass difference. Neutrons and protons are treated as bound state poles in the π-N scattering amplitude and the mass difference is obtained by finding the electromagnetic corrections to their binding energies. The results are in good agreement with experiment. No cutoffs or other purely theoretical parameters are involved. All the long range corrections to the π-N interaction are investigated. Photon exchange turns out to be the most important. Form factors appear as short range modifications of the photon exchange force. The results of the calculation are not sensitive to the detailed behavior of the form factors at large momentum transfer.

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Chapter 1
INTRODUCTION

Traditionally, electromagnetic corrections to strong interactions have been formulated in terms of off-mass-shell propagators and vertex functions. For example, the usual method for calculating the neutron-proton mass difference consists of finding the electromagnetic corrections to the nucleon propagator.

On the other hand, much recent progress in strong interaction dynamics has come from a study of the two-body scattering amplitude on the mass shell. The first half of this thesis is devoted to the derivation of an on-mass-shell, S-matrix formalism for calculating electromagnetic effects, including corrections to masses and coupling constants.

This approach has several advantages: (i) According to the "bootstrap" hypothesis, all strongly interacting particles are bound states or resonances. From this point of view, the mass differences among the members of an isotopic multiplet result from electromagnetic corrections to the interactions which hold the particles together. Now in S-matrix studies, closely related methods apply to both non-relativisitic and relativisitic problems; one can therefore use the understanding of bound states, resonances, and perturbations on the interaction that one has in non-relativistic quantum mechanics as a guide in relativistic problems which, according to the "bootstrap" hypothesis, possess these same features. (ii) The customary approximation scheme in strong interactions emphasizes the long-range parts of the interaction. These are the best understood parts, and there exists ample evidence attesting to their importance. This approximation scheme again appears suitable for estimating corrections due to the electromagnetic interactions, which are of even longer range. It is straightforward to distinguish long range effects in the S-matrix approach, and it seems likely that the dispersion intergrals for the S-matrix are less sensitive to short range or high mass corrections than are the integrals appearing in propagators or vertex functions.

The second half of this work is a detailed, quantitative application to the proton-neutron mass difference. Quantitative calculations of the electromagnetic corrections such as the p-n mass difference are of fundamental importance because they can provide the only real test of the hypothesis that the strong interactions conserve isospin exactly and all deviations from charge independence are of electromagnetic origin.

The traditional propagator or "self energy" approach to the proton-neutron mass difference has been discussed by a number of authors.[1-5] Unfortunately, the dispersion integrals which appear in this formulation of the problem are sensitive to the unknown, high momentum transfer behavior of the nucleon electromagnetic form factors and no one has been able to obtain the observed mass difference without introducing an undetermined "core" parameter.

In the present work neutrons and protons are considered as bound states in the π-N scattering amplitude and the mass difference is estimated by finding the difference in their binding energies. The results are in excellent agreement with experiment. No cutoffs or other purely theoretical parameters are involved. As pointed out above, we expect that, in general, long range effects will play a very important role in the S-matrix. One indeed finds that the dispersion integrals for the p-n mass difference are almost completely dominated by the longest range correction to the πN interaction, namely photon exchange. The electromagnetic form factors of the pions and nucleons will modify the photon exchange force at small distances. This will naturally affect our estimate for the mass difference. However, convergence can be maintained without form factors and the results are not particularly sensitive to the high momentum transfer behavior of the form factors.

There are a number of other possible applications of the present formalism which would be of immediate experimental interest. Among the most interesting would be the mass splittings in the other isospin multiplets. Such calculations, however, depend on parameters which are less well known for most multiplets than for pions and nucleons. For example, one would probably have to know the Σ magnetic moments. Further possible applications include the electromagnetic corrections to the π-N coupling constants and the low-energy π-N phase shifts. Here, our detailed knowledge of pions and nucleons would give the necessary parameters, but the amount of labor involved would be somewhat greater than that required to find mass differences. Similarly, one could obtain estimates for the corrections to low-energy N-N scattering. Actually, our methods are not restricted to electromagnetic problems, and might prove useful in potential theory and other contexts.

Chapter 2 contains a general formalism for calculating electromagnetic corrections to the strong interactions. The methods are first developed within the framework of non-relativistic potential scattering and then carried over to the relativistic domain. A large part of the chapter is devoted to a study of infrared divergence problems. Chapter 3 which deals with the proton-neutron mass difference is fairly self contained. The reader who is mostly interested in this particular application may find it advisable to read only through Equation 6 in Chapter 2 and then proceed to Chapter 3.

Chapter 2

ELECTROMAGNETIC CORRECTIONS IN S-MATRIX THEORY

2.1. Potential Scattering

Our first task is to develop, within the framework of non-relativistic potential scattering, a perturbation theory which can be extended to relativistic problems. For simplicity, we consider only S-wave scattering and define the amplitude

$$A(s) = e^{i\eta} \sin \eta/q \tag{1}$$

where η is the phase shift, q is the momentum, and $s = q^2$. We take the mass of the particle to be 1/2 so that s is the kinetic energy.[6] It is known[7] that, for a superposition of Yukawa potentials, A is an analytic function of s with a right-hand cut required by unitarity and a left-hand cut which comes from the partial wave projections of the Born amplitude and double spectral function. Given the discontinuity across the left cut, one can use the N/D method[8] to obtain the amplitude A.

We will suppose that for some strong potential V, the amplitude has been obtained in the form N/D and derive an expression for the first order change δA in the amplitude when the problem is perturbed by an additional weak potential δV. Let us also assume that the unperturbed problem has a bound state at $s = s_B$ so that $D(s_B) = 0$ and $A = R/s - s_B$ near s_B, and ask for the first order change in the position and residue δs_B and δR of the bound state pole. By definition, we have

$$\delta A = (\delta\eta/q)e^{2i\eta} \tag{2}$$

along the right cut and

$$\delta A = \delta\left(\frac{R}{s-s_B}\right) = \frac{R}{(s-s_B)^2}\,\delta s_B + \frac{\delta R}{s-s_B} \tag{3}$$

near the bound state pole. Since the square of the unperturbed denominator function D^2 has the phase $e^{-2i\eta}$ along the right-hand cut[8] and a double zero at $s = s_B$, the function $D^2\delta A$ has no poles or right-hand cut. The denominator function can be chosen[7] such that D tends to a constant as $s \to \infty$ and a simple application of Cauchy's theorem yields

$$\delta A(s) = \frac{1}{D^2(s)}\,\frac{1}{\pi}\int_{-\infty}^{0} \frac{D^2(s')\,\mathrm{Im}\,\delta A(s')}{s' - s}\,ds' \tag{4}$$

where we have used the fact that D has no left cut. Picking out the coefficients of the poles in Equation 3, we have

$$\delta s_B = \frac{1}{R(D'(s_B))^2}\,\frac{1}{\pi}\int_{-\infty}^{0} \frac{D^2(s')\,\mathrm{Im}\,\delta A(s')}{s' - s_B}\,ds' \tag{5}$$

$$\delta R = \frac{-2RD''(s_B)\delta s_B}{D'(s_B)} + \frac{1}{(D'(s_B))^2}\,\frac{1}{\pi}\int_{-\infty}^{0} \frac{D^2(s')\,\mathrm{Im}\,\delta A(s')}{(s' - s_B)^2}\,ds' \tag{6}$$

Note that the reality of δs_B and δR is guaranteed by the vanishing of $D(s')$ at $s' = s_B$.

Equations 2 to 6 are completely adequate for treating short range perturbations. We are developing the non-relativistic formalism in this section, however, primarily for purposes of orientation or introduction to the calculation of electromagnetic corrections to the strong interactions. Electromagnetic corrections are complicated by the infrared divergence associated with the massless nature of the photon, which requires a modification of the formalism presented thus far. To find the appropriate modification, it is convenient to study a non-relativistic perturbation which has essential features of the relativistic problem, including the infrared divergence. Thus, we consider a perturbing potential whose Fourier transform is proportional to $(1/t - \lambda^2)(m^2/t - m^2)^2$ where $t = -2q^2(1 - \cos\theta)$. Potentials of this type are characteristic of the "single photon exchange" potential between two strongly interacting particles whose form factors obey unsubtracted dispersion relations. The constant λ is a fictitious photon mass which must eventually be set equal to zero.

The spatial form of the above potential is

$$\delta V(r) = b\left[\frac{e^{-\lambda r}}{r} - \frac{e^{-mr}}{r} + \left(\frac{\lambda^2 - m^2}{2m}\right)e^{-mr}\right] \tag{7}$$

where we have introduced a strength parameter b which we assume to be small. If the photon mass λ is set equal to zero, Equation 7 becomes a modified Coulomb potential. As is well known, in the limit $\lambda = 0$, logarithmic divergences will appear in the right-hand side of Equation 4 for δA. This is, of course, due to the infinite range of the Coulomb potential. From elementary quantum mechanics, one knows that when the perturbation is summed to all orders the divergent part contributes to the phase shift a term proportional to $\ell n\,(qr)$, or equivalently, $\ell n\,(q/\lambda)$.

The divergent term is common to all partial waves. Thus, above threshold it appears only as a phase factor $\exp\left[-i\frac{b}{q}\,\ell n\,(qr)\right]$ which multiplies the entire S-matrix and has no observable effect. We find it expedient to abandon δA and chose a new amplitude $\delta\hat{A}$ which does not contain this divergent, unobservable phase factor for the following reasons:

(i) The infrared factor introduces an r-dependence into the S-matrix element $e^{2i\eta}$, which would otherwise be independent of r. In a perturbation treatment, this r-dependence appears as a logarithmic divergence in δA.

(ii) Below threshold, q becomes $i|q|$ and the r-dependence takes the form $\exp\,[-\,(b/|q|)\,\ell n\,(|q|\,r)]$. The residue R of a bound state pole will therefore contain a factor which is either zero of infinity, making δR infinite in a perturbation treatment. It is convenient to remove such factors by a redefinition of the amplitude, leading to a redefined residue that depends only on finite quantities.

(iii) Unlike the residue R, the binding energy s_B should not have an infrared divergence. In an approximate evaluation of Equation 5 for δs_B, however, a spurious divergence is likely to appear. Again, this difficulty can be avoided by a redefinition of the amplitude.

The redefinition we shall employ is obtained by removing from the S-matrix the infrared divergent factor $\exp\left[-i(b/q)\,\ell n\,(g(s)/\lambda^2)\right]$ where $g(s)$ is an as yet unspecified function with dimension $(\text{mass})^2$. One readily verifies that $\delta\hat{A}$ is then given by $\delta\hat{A} = \delta A + (b/4q^2)e^{2i\eta}\,\ell n\,(g(s)/\lambda^2)$. For any $g(s)$, the resulting amplitude is well behaved in the limit $\lambda \to 0$. We shall make use of this freedom by choosing $g(s)$ in a way that minimizes the sensitivity of our dispersion relations to distant singularities, which are generally less well known than the nearby ones.

It turns out that the best choice for $g(s)$ corresponds to the following $\delta\hat{A}$:

$$\delta\hat{A} = \delta\hat{\eta}\,\frac{e^{2i\eta}}{q} = (\delta\eta - \delta\eta_{\text{Born}})\,\frac{e^{2i\eta}}{q} \tag{8}$$

$$\delta\eta_{\text{Born}} \equiv -\frac{1}{q}\int_0^{\infty} \sin^2(qr)\,\delta V(r)\,dr \tag{9}$$

Since $\delta\eta_{\text{Born}}$ contains the same $\ell n\,\lambda$ dependence as $\delta\eta$, the infrared phase shift is indeed removed from $\delta\hat{A}$, which is therefore well behaved in the limit $\lambda \to 0$. One will also note that $\delta\hat{A}$ has the correct threshold behavior. It remains to show that, for perturbing potentials of the form of Equation 7, $\delta\hat{A}$ should be quite insensitive to the distant singularities in the dispersion integrals.

The argument about distant singularities runs as follows: For short range V we can, to a good approximation, write

$$\delta\eta = -\frac{1}{q}\int_0^\infty \sin^2(qr+\eta)\,\delta V(r)\,dr \tag{10}$$

Performing a few algebraic manipulations, one finds

$$\hat{\delta\eta} = \delta\eta - \delta\eta_{Born} = -\sin^2\eta\int_0^\infty \frac{\cos 2qr}{q}\,\delta V(r)\,dr$$

$$-\frac{\sin 2\eta}{2}\int_0^\infty \frac{\sin 2qr}{q}\,\delta V(r)\,dr \tag{11}$$

For the perturbing potential Equation 7 with $\lambda = 0$ we find, for $|q| \gg m$,

$$\hat{\delta\eta} = -\frac{bm\sin 2\eta}{8q^2} + O\left(\frac{bm^2}{q^3}\right) \tag{12}$$

which is a more rapid falloff at large q than that of $\delta\eta_{Born}(\sim 1/q)$. Thus, as successively more distant singularities are included in a dispersion-theoretic calculation, $\delta\hat{A}$ is expected to converge more rapidly than would the subtracted term $(\delta\eta_{Born}/q)e^{2i\eta}$.

Equation 12 was obtained from a rather special model, and one naturally wonders which features of the result are general. It is possible, using various simple forms for V and δV, to show that the good convergence of $\delta\hat{A}$ can hold even when V is not restricted to short range. The rapid convergence does depend critically, however, on the "smooth" r-dependence of δV which, as given by Equation 7, is finite at the origin. If more singular behavior appears at $r = 0$, $\hat{\delta\eta}$ falls off more slowly at large q. This is in accord with the physical notion that a rapid spatial variation corresponds to high Fourier components in q. Of course, we do not know the small-distance behavior of electromagnetic corrections to the strong interaction, but at least the apparently rapid convergence of the neutron-proton mass difference calculation in the following chapter is consistent with a smooth, cut off δV at intermediate range. The one contribution to the really short range potential on which detailed information is available is one-photon exchange, and here the recent CEA data[9] on the proton form factor indicates a smooth behavior down to very small distances.

Another possible reason for rapid convergence of $\delta\hat{A}$ is that in potential theory any phase shift tends to its Born approximation at high energy. Thus when potential theory is applicable $(\delta\eta - \delta\eta_{Born})$ approaches zero faster than $\delta\eta_{Born}$ at large s, independently of whether δV is cut off at small distances. This argument is less satisfactory than the arguments of the previous paragraph, however, because: (i) it does not imply that $\delta\eta$ approaches $\delta\eta_{Born}$ any faster than the unperturbed phase shift η approaches its Born value; and (ii) unlike the potentials in ordinary potential theory, the strong interaction depends on energy, with the result that η may approach the Born approximation very slowly if at all. We therefore prefer to rely primarily on the fact that when δV is cut off at small distances, as in Equation 12, $\delta\hat{\eta}$ falls off rapidly at large q no matter what asymptotic behavior the strong interaction phase shift η may have.

The procedure we have adopted of dropping the part of δA which diverges like $\ln(\lambda^2/g(s))$ is roughly equivalent to giving the photon a mass g. Our method differs from handling the infrared divergence by a photon mass, however, because $g(s)$ depends on energy. The energy dependence is needed to make the dispersion relation for the redefined amplitude converge rapidly.

It is simple enough to rephrase the dispersion relations in terms of $\delta\hat{A}$. Using the fact that $D^2\delta\hat{A}$, like $D^2\delta A$, has no poles or right-hand cut, one readily verifies that Equation 4 continues to hold if δA is everywhere replaced by $\delta\hat{A}$. Furthermore, since $\delta\hat{A}$ has the same double pole $R\delta s_B/(s-s_B)^2$ as δA (cf. Equation 3), one can replace $\mathrm{Im}\,\delta A(s')$ by $\mathrm{Im}\,\delta\hat{A}(s')$ in the right-hand side of Equation 5 for δs_B. Finally, if we define $\delta\hat{R}$ as the coefficient of the simple pole in $\delta\hat{A}$ (cf. Equation 3), then $\delta\hat{R}$ is given by Equation 6 with $\mathrm{Im}\,\delta A(s')$ replaced by $\mathrm{Im}\,\delta\hat{A}(s')$.[10]

The following simple example will illustrate a number of the points discussed above. The example involves a comparison between a standard method for calculating δs_B and the S-matrix method of the present paper. The standard expression for δs_B is given by

$$\delta s_B = \int_0^\infty |\psi|^2\, \delta V\, dr \tag{13}$$

where ψ is the unperturbed wave function. We take Equation 7 for δV, and, for simplicity, assume that V is of very short range. For short range V, we can set

$$|\psi|^2 \approx 2|s_B|^{1/2} \exp(-2|s_B|^{1/2} r) \tag{14}$$

and performing the integration in Equation 13 with $\lambda = 0$, we find

$$\delta s_B \approx 2b|s_B|^{1/2}\left[\ell n\left(\frac{m+2|q_B|}{2|q_B|}\right) - \frac{m}{2m+4|q_B|}\right] \tag{15}$$

Next we make an independent calculation of δs_B using the dispersion relation 5 with Im $\delta A(s')$ replaced by Im $\delta\hat{A}(s')$. We will keep only the nearby singularities in the dispersion integral, and then compare the result with Equation 15. Since V is of short range, the only nearby singularities in $\delta\hat{A} = \delta A - (\delta\eta_{Born}/q)e^{2i\eta}$ will come from the Born term in δA and from the singularities in $\delta\eta_{Born}$. Using $\delta A_{Born} \equiv \delta\eta_{Born}/q$, we find

$$\text{Im } \delta\hat{A} = (1-e^{2i\eta})\text{Im}(\delta\eta_{Born}/q) = -2iq\left(\frac{e^{i\eta}\sin\eta}{q}\right)\text{Im}(\delta\eta_{Born}/q) \tag{16}$$

along the nearby part of the left cut. For the perturbing potential Equation 7, it turns out that all of the singularities in $\delta\eta_{Born}/q$ lie in the interval $-m^2/4 \leq s \leq 0$. Using Equation 5 and the relations $e^{i\eta}\sin\eta/q = N/D$ and $q = i|q|$, we can then write

$$\delta s_B = \frac{1}{R(D'(s_B))^2}\,\frac{2}{\pi}\int_{-m^2/4}^{0} |q'|\frac{N(s')D(s')}{s'-s_B}\,\text{Im}(\delta\eta_{Born}(s')/q')\,ds' \tag{17}$$

From the effective range approximation

$$e^{i\eta} \sin \eta/q \approx \frac{1}{iq_B - iq} \tag{18}$$

which for short range V should be good over the range of integration in Equation 17, we extract $N = 1$, $D = iq_B - iq$, $R = -2|q_B|$, and $D'(s_B) = -1/2|q_B|$. The imaginary part of $\delta\eta_{Born}/q$ is known explicitly and it turns out that the integral in Equation 17 can be done analytically. The result is that our S-matrix estimate 17 for δs_B is exactly the same as that given by Equation 15. Notice two important points here: (i) We have obtained this result by keeping only the nearest singularity in $\delta\hat{A}$. (ii) As can be easily verified, if we had tried to estimate δs_B by using Im $\delta A(s')$ instead of Im $\delta\hat{A}(s')$ in Equation 5, a spurious infrared divergence would have appeared in our expression for δs_B. In a more complete calculation, such a divergence would, of course, be cancelled by a divergent term coming from more distant singularities.

2.2. Electromagnetic Corrections to the Strong Interactions

The partial wave scattering amplitudes which appear in relativistic S-matrix theory are believed to obey dispersion relations similar to those which occur in potential scattering theory and, with a few modifications, the results of the previous section will be applicable to relativistic problems. For simplicity, we consider only elastic scattering of two spinless particles of equal mass. The partial wave scattering amplitude then becomes

$$A(s) = \rho(s) e^{i\eta} \sin \eta \tag{19}$$

where s is the total center-of-mass energy squared and ρ is a function which removes the kinematic singularities. (ρ is the analog of the factor $1/q$ in Equation 1. Note that s has been given a new definition.) The phase shift η is in general complex and a measure of the inelasticity is given by

$$I(s) = \rho \left| e^{2i\eta} \right| / 2 \tag{20}$$

We assume that A is an analytic function of s with a right-hand cut controlled by unitarity in the s-channel and a left-hand cut controlled by unitarity in the t- and u-channels. In order to apply the N/D method, we must now be given both $\mathrm{Im}\, A$ along the left cut, and I, or equivalently $\mathrm{Im}\, \eta$, along the right cut as input information.

Again we assume that the strong interaction amplitude A has been obtained in the form N/D and ask for the first order change in A when the electromagnetic corrections are added to the input. We also assume that the unperturbed and perturbed problems have bound state poles at s_B and $s_B + \delta s_B$ with residues R and $R + \delta R$ respectively.

In potential theory, the function $D^2 \delta A$ had no right-hand cut, but it acquires one in strong interaction problems because $\delta\eta$ has an imaginary part, and also because the electromagnetic mass shifts of the scattered particles change the kinematic factor by $\delta\rho$. The analog of Equations 4 to 6 becomes

$$\delta A(s) = \frac{1}{D^2(s)} \frac{1}{\pi}\left[\int_L \frac{D^2(s')\,\mathrm{Im}\,\delta A(s')}{s' - s}\,ds' + \int_R \frac{\mathrm{Im}\left[D^2(s')\,\delta A(s')\right]}{s' - s}\,ds'\right] \tag{21}$$

$$\delta s_B = \frac{1}{R\left[D'(s_B)\right]^2} \frac{1}{\pi}\left[\int_L \frac{D^2(s')\mathrm{Im}\,\delta A(s')}{s' - s_B}\,ds' + \int_R \frac{\mathrm{Im}\left[D^2(s')\,\delta A(s')\right]}{s' - s_B}\,ds'\right] \tag{22}$$

$$\delta R = \frac{-2\,D''(s_B)R\,\delta s_B}{D'(s_B)} + \frac{1}{\left[D'(s_B)\right]^2} \frac{1}{\pi}\left[\int_L \frac{D^2(s')\,\mathrm{Im}\,\delta A(s')}{(s' - s_B)^2}\,ds' + \int_R \frac{\mathrm{Im}\left[D^2(x')\,\delta A(s')\right]}{(s' - s_B)^2}\,ds'\right] \tag{23}$$

where the integrals R and L run over the left and right cuts and we have assumed that $D^2\delta A \rightarrow 0$ for large s. Along the right-hand cut,

$$\mathrm{Im}\,(D^2\delta A) = \mathrm{Im}\left[D^2\left(\frac{\delta I e^{2i\,\mathrm{Re}\,\eta}}{i} - \frac{\delta\rho}{2i}\right)\right] = -\,|D|^2\delta I + \frac{\mathrm{Re}\,(D^2\delta\rho)}{2} \tag{24}$$

Calculation of the new kinematic and inelasticity corrections will be straightforward, with no difficulty of principle.

Whenever the scattered particles are charged, infrared divergences occur and cause us to redefine the amplitude. The redefinition will be carried out in the same spirit as in potential scattering, and we shall mention only the new features.

To begin with, in addition to the "Coulomb divergence" already encountered in potential scattering, new divergences associated with bremsstrahlung appear. The reaction of bremsstrahlung back onto two-body scattering occurs in the form of the diagrams where a photon is emitted by one of the initial particles and absorbed by a particle in the final state. The net effect of all such diagrams[11, 12] is to make the amplitude for scattering without photon emission zero--expressing the fact that charge cannot be accelerated without radiating. When all the soft photon emissions are added in, however, the infrared divergence cancels and the total cross section exhibits only a mild dependence on photon emission. These reasons again call for a redefinition which will allow us to express the amplitude in terms of finite quantities, and to obtain residues, for example, which are only slightly shifted from their strong interaction values. Since "bremsstrahlung diagrams" where a photon is emitted by initial charge line in the s-channel and absorbed by a final charge line can also be interpreted in terms of final state Coulomb interaction in the t- or u-channel, and the Coulomb interaction already required us to redefine the amplitude in potential scattering, it is hardly surprising that the "bremsstrahlung diagrams" require a redefinition of the amplitude.

A minimum requirement for the redefinition of the amplitude is removal of the $\ln \lambda$ dependence from δA. As in potential scattering, the satisfaction of this requirement leaves arbitrary the coefficient of λ in the logarithm. We could choose the coefficient in the same manner as in potential scattering:

$$\delta \hat{A} = \delta A - \rho \delta \eta_{Born} e^{2i\eta} \tag{25}$$

Here, the Born phase shift is calculated from the electromagnetic correction to the generalized potential defined by Chew and Frautschi.[13] This choice would be expected to give the best convergence of the dispersion relation. On the other hand, there are many terms to keep track of in the relativisitc case and it is tedious to have to compute the contribution of each of them to $\delta\eta_{Born}$ and make the subtraction in Equation 25. The practical compromise which has been adopted in the following chapter is to define the amplitude

$$\delta A' = \delta A - \rho \delta \eta_{infra} e^{2i\eta} \tag{26}$$

where $\delta\eta_{infra}$ contains the infrared terms and the other important high energy contributions from the diagrams in which photons connect external charge lines. The definition adopted for $\delta A'$ is not unique, and $\delta A'$ is not expected to converge as rapidly as $\delta\hat{A}$, but the diagrams in which photons connect external lines do provide the longest range forces and the dominant high energy corrections,[11] so relatively little is lost by using $\delta A'$ in place of $\delta\hat{A}$.

Having redefined the amplitude, one can rephrase the dispersion relations for the residue and mass shift in terms of the new amplitude, or follow a somewhat simplified procedure which will be described in the next section.

2.3. A Simple Method for Subtracting the Infrared Term

The mass shift and the redefined residue shift can be expressed via dispersion relations in which a term such as $\rho\delta\eta_{infra}e^{2i\eta}$ has been subtracted out of δA in the integrand. The subtraction procedure in the integrand can become quite tedious, however, particularly in the relativistic case. We would like to describe a somewhat simpler way to subtract, which would give the same results in an exact calculation and can be shown to give nearly the same result in approximate calculations such as the example at the end of Section 2.1. To see how this goes, let us suppose that the input for Equation 21 has been calculated with a small but finite photon mass λ. The function $\rho\delta\eta_{infra}$ has the form

$$\rho\delta\eta_{infra} = f(s)\,\ell n\,(\lambda^2/g(s)\,) + O(\lambda) \qquad (27)$$

In the limit $\lambda \to 0$, the right-hand side of Equation 21 must therefore be equal to $\delta A' + e^{2i\eta}f(s)\,\ell n\,(\lambda^2/g(s)\,) + O(\lambda)$. Since any function which is logarithmically divergent as $\lambda \to 0$ can be uniquely separated into a part which diverges like $\ell n\,(\lambda^2/g(s)\,)$ and a part which remains finite as $\lambda \to 0$, we can calculate $\delta\hat{A}$ by carrying out the integrations, taking the limit $\lambda \to 0$, and then dropping all terms which diverge like $\ell n(\lambda^2/g(s)\,)$.

If we define $\delta R'$ to be the residue of the simple pole in $\delta A'$ (cf. Equation 3), $\delta R'$ can be extracted from Equation 23 by the same prescription that has been given to find $\delta A'$ from Equation 21.

The calculation of δs_B is on a rather different footing. As we pointed out earlier, δs_B should not diverge in the limit $\lambda \rightarrow 0$. That the formalism is consistent with this property can be seen by observing that, since $\delta\eta_{infra}$ has no pole at $s = s_B$, we have

$$\lim_{s \rightarrow s_B} (s-s_B)^2 \delta A' = \lim_{s \rightarrow s_B} (s-s_B)^2 \delta A = R \delta s_B$$

The limit involving $\delta A'$ is not infrared divergent so δs_B should be finite. The integral in Equation 22 for δs_B, however, is just the infrared divergent integral of Equation 21 for δA, evaluated at $s = s_B$. At the particular point $s = s_B$, the coefficient of the infrared divergent part of this integral must vanish, but the vanishing occurs through cancellations among long and short range contributions which are not enforced in an approximate evaluation of Equation 22. The simplest way to remedy this deficiency of an approximate calculation is to drop the term containing $\ell n\, (\lambda^2/|g(s_B)|)$, since its coefficient should have vanished anyway.[14]

In previous sections we have emphasized the rapid convergence of dispersion relations involving $\delta A'$. The simple subtraction procedure of the present section is essentially equivalent and should also converge rapidly. The short range contributions to Equation 22, then, have little effect on δs_B and act mainly to cancel the spurious infrared divergence produced by integrating only over long range parts. By subtracting the spurious divergence in the above manner, we can avoid the hard work of calculating short range contributions, with little loss of accuracy.

Chapter 3

APPLICATION TO THE PROTON-NEUTRON MASS DIFFERENCE

3.1. Orientation

According to the "bootstrap" hypothesis,[15] nucleons are bound states containing components of π-N, K-Σ, ρ-N and any other system with the proper quantum numbers. In order to apply our formalism to the neutron-proton mass difference, we must concentrate on a specific two-body amplitude. The π-N amplitude has been chosen for the following practical reasons: (i) The π-N system is the least massive two-particle state with the proper quantum numbers; (ii) the nucleon is strongly coupled to it (i.e., the π-N component of the nucleon wave function is certainly large); and (iii) we have a fairly good understanding of the pion-nucleon interaction.

Until now we have been working with the scattering amplitude for two spinless equal mass particles. The fact that nucleons have spin and do not have the same mass as pions does not require any essential changes in the formalism. It does turn out, however, that the partial wave amplitudes have kinematic singularities as a function of s (total cm energy squared). For this reason we will work in the $W = \sqrt{s}$ plane. In what follows the reader will not need a detailed knowledge of the kinematics in π-N scattering.[16,17]

In the following section, the dispersion integral for the mass difference is written down and the general nature of the input singularities is discussed. We then investigate the nature of the infrared divergence in this particular problem. In Section 3.4 we carry out the detailed task of estimating the mass difference from the nearby singularities.

3.2. Formal Considerations

Let us begin by supposing that the electromagnetic interactions have been turned off. Conservation of isospin will then be exact and the nucleons will have a common mass M and the pions a mass μ. We are interested in the nucleon pole which occurs in the analytically continued π-N scattering amplitude. This pole appears in the $J = 1/2$, $L = 1$, $T = 1/2$ channel for which we define the amplitude

$$\begin{aligned} A(W) &= \rho(W) e^{i\eta} \sin\eta \\ \rho(W) &= \frac{W^2}{M^2} \frac{1}{(W-M)^2 - \mu^2} \frac{1}{q} \\ \eta &= \text{phase shift} \\ W &= \text{total cm energy} \\ q &= \text{cm momentum} \end{aligned} \tag{28}$$

and the inelasticity factor

$$I(W) = \frac{\rho(W)\left|e^{2i\eta}\right|}{2} \tag{29}$$

With the above choice for ρ, A has no kinematic singularities in the W-plane(16,17) and for $|W-M| << M$, $A(W) \approx e^{i\eta}\sin\eta/q^3$. The residue of the nucleon pole in the direct channel turns out to be $-3f^2/\mu^2$; $f^2 \approx 0.08$.

We now turn on the electromagnetic interactions and thereby destroy the previous equality of the $T_3 = \pm 1/2$ amplitudes. In particular, the $T_3 = + 1/2$ amplitude now has a pole at the mass of the proton while the $T_3 = - 1/2$ amplitude has a pole at the mass of the neutron. Since the masses of the nucleons and pions are no longer degenerate, the kinematics become more complicated and the kinematic factors ρ will be different for the two charge states. Since we only are interested in the difference between the proton and neutron masses, it is convenient to define

$$\begin{aligned} \delta M &= M_p - M_n \\ \delta A &= A(+1/2) - A(-1/2) \\ \delta I &= I(+1/2) - I(-1/2) \\ \delta\rho &= \rho(+1/2) - \rho(-1/2) \end{aligned} \tag{30}$$

where the indices $\pm 1/2$ refer to states of $T = 1/2$ and $T_3 = \pm 1/2$. (Note that the definitions of δA, δI, and $\delta\rho$ are slightly different from those used in Chapter 2.)

The perturbation techniques developed in the previous chapter can now be used to calculate δM to order $\alpha \approx 1/137$. We suppose that the unperturbed amplitude has been obtained in the form N/D with D normalized such that $D'(M) = 1$. The analog of Equation 22 then becomes:

$$\delta M = -\frac{\mu^2}{3f^2}\left[\frac{1}{2\pi i}\int_L \frac{\delta A(W')D^2(W')}{W' - M}\, dW' - \frac{1}{\pi}\int_{M+\mu}^{\infty} \frac{|D(W')|^2\delta I(W') - \frac{1}{2}\mathrm{Re}\,(D^2(W')\delta\rho(W'))}{W' - M}\, dW'\right] \qquad (31)$$

where the contour L encloses all the singularities of δA which lie to the left of $\mathrm{Re}\, W = M + \mu$.

It is convenient to separate the singularities of δA into two classes. The first type of singularity is of purely kinematic origin. The position of a singularity in the W-plane and the kinematic factors which affect its strength are functions of the masses of the scattered particles. When the nucleon and pion mass splittings are taken into account, the original strong interaction singularities in $A(+1/2)$ and $A(-1/2)$ will have slightly different positions and strengths, simply because the kinematics are different in the two channels. The more distant singularities will be affected very little but there will be an imperfect cancellation between the lower mass part of the original singularities in $A(+1/2)$ and $A(-1/2)$. The second type of singularity in δA comes from corrections to the unitarity condition for the T matrix, $\mathrm{Im}\, T_{ab} \propto \sum_c T_{ac}T^*_{bc}$. These will appear either because a new intermediate state has become available (e.g., $N\overline{N} \rightarrow \gamma \rightarrow 2\pi$ in the t-channel), or because of mass shifts in an already existing intermediate state, or because of electromagnetic corrections in a vertex or amplitude leading to one of the original intermediate states (e.g., electromagnetic corrections to π-N scattering in the u-channel).

To order α, any electromagnetic correction must transform under rotations in isospin space like I, T_3, or T_3^2, where I is the unit operator and T_3 is the third component of isotropic spin. For nucleons $T_3^2 = I/4$, and only those corrections which transform like T_3 will contribute to δM. Two important consequences of this observation are: (i) since the pion mass differences transform like T_3^2, they cannot affect δM; (ii) a photon which appears in an intermediate state c of the unitarity condition $\mathrm{Im}\, T_{ab} \propto \sum_c T_{ac} T^*_{cb}$ will not contribute to δM unless it connects one isovector vertex and one isoscalar vertex. The fact that many of the electromagnetic corrections to the π-N interaction do not affect δM makes the calculations of mass difference surprisingly simple. A calculation of the individual nucleon mass shifts would be a more formidable task.

3.3. Treatment of Spurious Infrared Divergences

In the last chapter it was pointed out that, in principle, Equation 31 is convergent in the limit of vanishing photon mass, but in an approximate calculation a spurious infrared divergence will probably appear. The prescription given there for removing a spurious divergence runs as follows. One writes the infrared divergent part of $\delta\eta$ in the form $\delta\eta_{\mathrm{infrared}} = f(W)\, \ell n\, (\lambda/g(W))$, where λ is the photon mass. The factor $f(W)$ is uniquely determined and can be calculated in perturbation theory. The function $g(W)$ has the dimension of mass and is chosen so that $\delta\eta_{\mathrm{infrared}}$ is a good approximation to the phase shift generated by the electromagnetic effects which take place outside of the strong interaction region, i.e., by Coulomb scattering and bremsstrahlung. One then calculates the input for Equation 31 with a finite photon mass λ, does the integration, drops the part which diverges like $\ln(\lambda/|g(M)|)$, and takes the limit $\lambda \to 0$. The calculated value of δM will no longer depend on λ but may depend on the choice of g. This circumstance was discussed at length in Chapter 2. The relevant points were: (i) if one keeps only a few nearby cuts in Equation 31, the best estimate for δM will be obtained by choosing $g(W)$ in the specific manner described above; (ii) if one systematically improves his estimate by keeping more and more distant singularities in Equation 31, the result will become independent of the choice of $g(W)$.

The form of $g(W)$ due to Coulomb scattering alone is obtained as follows. The only $T = 1/2$ state undergoing Coulomb scattering is the $\pi^- p$ component of the $T_3 = -1/2$ state which enters with a Clebsch-Gordon coefficient $(2/3)^{1/2}$. Using $\delta\eta = \eta(+1/2) - \eta(-1/2)$, we find that $\delta\eta_{\text{infrared}}$ equals minus two-thirds the Coulomb phase shift for $\pi^- p$ scattering in the $P_{1/2}$ state. The latter can be obtained from the photon exchange diagram and one finds, for $|W-M| << M$ (here we are interested only in the region around the nucleon pole)

$$\delta\eta_{\text{infrared}} \approx \frac{2}{3}\,\alpha\,\frac{(W-M)}{q}\,\ln\left(\frac{\lambda e}{2q}\right) + O(\lambda) \tag{32}$$

where $e = 2.718\ldots$. From Equation 32 we extract $g(W) \approx 2q/e$. At the nucleon pole, $|q|$ is approximately equal to μ and whenever a $\ln\lambda$ divergence appears in our calculations we will drop the part which diverges like $\ln(\lambda e/2\mu)$. Since the photon exchange diagram is gauge invariant, this method for removing spurious divergences is gauge invariant.

In addition to the Coulomb terms, there are also infrared divergent "bremsstrahlung diagrams" in which a photon connects initial and final charge lines. We now give reasons why the bremsstrahlung terms can be neglected in our particular problem. In a low energy collision, the bremsstrahlung can be computed by finding the energy which would be radiated by the classical currents of the incoming and outgoing particles. According to the last paragraph of Section 3.2, we need only consider the interference terms between the isovector and isoscalar parts of the currents. The isoscalar current comes only from the isoscalar magnetic moment of the nucleon and the isoscalar part of the nucleon recoil current. The isoscalar moment is small and at low energies nucleon recoil is of order q/M. Furthermore, the "bremsstrahlung diagrams" are always of order α/π so that, near the pole, the bremsstrahlung contribution to $\delta\eta_{infrared}$ will be of order $(q/M)(\alpha/\pi)$ and can be safely neglected.

3.4. Approximate Evaluation of the Dispersion Relation

In the previous chapter it was indicated that if the infrared divergent terms are removed as outlined above, there is reason to believe that Equation 31 is dominated by low mass singularities. We will now estimate δM by keeping only those singularities which lie roughly in the region $|W-M| \leq 4\mu$. The calculation is organized as follows. First we estimate the contribution of the kinematic corrections and the nucleon exchange cut and then calculate the contribution of the photon exchange cut. It turns out that these two cuts plus the kinematic corrections yield a value of δM which is in good agreement with experiment. Finally, we estimate the effect of the other nearby singularities and fortunately find that they have little effect on δM.

In order to carry out the above program, we need an expression for the unperturbed denominator function D. To a first approximation, we can set

$$D(W) \approx W - M \tag{33}$$

A somewhat more sophisticated approximation which is better behaved at large W can be obtained by making a one-pole approximation to the cut in D, which gives

$$D(W) \approx (W - M) \frac{(W_o - M)}{(W_o - W)} \tag{34}$$

We can fix W_o by comparing Equation 7 with the denominator function derived by Balazs.[18] Setting $(W_o - M) = 9\mu$ in Equation 34 yields an expression which approximates Balazs' result to within a few per cent throughout the range of interest.[19]

We now proceed with the calculation of δM.

A. Singularities of Kinematic Origin

Since the pion mass differences transform like T_3^2, we need consider only the kinematic effects of the nucleon mass difference. In the energy range under consideration, nucleon recoil can be neglected[20] and in this approximation W and M enter into the kinematics only in the combination (W - M). It is not difficult to convince oneself that this implies that the net effect of the kinematically induced singularities must be to shift the mass of a proton or neutron by an amount equal to the mass shift of its constituent nucleons. Since the proton is two-thirds π^+n and one-third π^0p while the neutron is two-thirds π^-p and one-third π^0n, the total contribution of the singularities of kinematic origin must be $\approx -\delta M/3$. This is clearly a self consistency requirement which arises because we have considered nucleons as bound states of nucleons and pions.

B. Corrections to Nucleon Exchange

Nucleon exchange in the u-channel gives rise to a short-cut which, for practical purposes, can be considered as a pole at $W = M$. Electromagnetic corrections to nucleon exchange will come from changes in the π-N coupling constants and the masses of the exchanged nucleons. Since $D(M) = 0$, it follows from Equation 31 that a change in the residue of the pole will not affect δM. The changes in the position of the pole due to the mass shifts of the exchanged nucleons can be easily calculated. Near the pole $D \approx W-M$ and using Equation 31, one finds that electromagnetic corrections to the crossed nucleon pole contribute $+ 5\,\delta M/27$ to the mass difference. Again we have a self-consistency requirement. The net contribution of the kinematic effects and crossed nucleon pole is $- 4\,\delta M/27$. There are no other low-lying singularities which are proportional to δM.

One should note that we are not trying to make the mass difference "bootstrap" itself. In order to obtain a non-vanishing mass difference we must introduce a driving force, i.e., electromagnetism. The reaction of the mass difference back on itself is only about a 15% effect.

C. Photon Exchange

Photon exchange gives rise to the most important singularity. This cut is contained in the photon exchange amplitude

$$\delta A_\gamma = \frac{\alpha}{6M^2}\left\{\frac{(W+M)^2-\mu^2}{(W-M)^2-\mu^2}(W-M)I_1+(W+M)I_2\right\}$$

$$I_1 = \int_{-1}^{1}\frac{x}{t-\lambda^2}F_\pi(t)F_{1s}(t)\,dx \tag{35}$$

$$I_2 = \int_{-1}^{1}\frac{1}{t-\lambda^2}F_\pi(t)F_{1s}(t)\,dx$$

$$t = -2q^2(1-x)$$

where F_π is the pion form factor, F_{1s} is the nucleon isoscalar charge form factor, and we have neglected the small isoscalar anomalous magnetic moment of the nucleon. Since the form factors have been included in its definition, δA_γ includes not only the singularity due to the photon intermediate state but also a number of other t-channel processes. For example, we have included the singularity arising from $N\bar{N} \to \omega \to 2\pi$ with the approximation $\omega \to \gamma \to 2\pi$ for the $\omega \to 2\pi$ amplitude.

With the choice $m_\rho^2/(m_\rho^2 - t)$, $m_\rho \approx 750$ MeV, for F_π, one can verify that Equation 4 will be sensitive only to the low t-behavior of F_{1s} and a sufficiently general expression for the latter is $F_{1s} = 1 - c + cm_s^2/(m_s^2 - t)$, where m_s is some effective resonance mass. The best fit to the low t-behavior[21] of F_{1s} is obtained with $c \approx 1$ and $m_s^2 \approx 20\ \mu^2$.

Upon inserting the above form factors into Equation 35, one finds that all the important singularities of the functions I_1 and I_2 lie in the region[14] $|W - M| \leq m_\rho/2$. In this region one of our approximate expressions (33-34) for D should be adequate. Using the straight line approximation 33 for D and substituting Equation 35 into 31, one finds that the photon exchange contribution to δM is approximately[22, 23]

$$- \frac{5}{9} \frac{\alpha}{f^2} \frac{\mu^2}{M} \left[\ell n\ (m_\rho/\lambda) - c \frac{m_\rho^2}{m_\rho^2 - m_s^2} \ell n(m_\rho/m_s) \right] \tag{36}$$

According to our earlier discussion, the spurious infrared divergence is to be removed by dropping the part proportional to $\ell n\ (e\lambda/2\mu)$ which yields

$$- \frac{5}{9} \frac{\alpha}{f^2} \frac{\mu^2}{M} \left[\ell n\ (\frac{em_\rho}{2\mu}) - c \frac{m_\rho^2}{m_\rho^2 - m_s^2} \ell n\ (m_\rho/m_s) \right] \tag{37}$$

For $m_s^2 = 20\,\mu^2$ and $c = 1$, Equation 37 is numerically equal to - 1.4 MeV.

The cuts in δA_γ are arranged such that to zeroth order in $1/M$, the photon exchange contribution vanishes when D is approximated by a straight line. (Note that Equation 37 contains a factor

μ/M.) Because of this circumstance, the corrections to Equation 37 due to the curvature of D are not entirely negligible. Taking the one-pole approximation (34) for D, one finds a photon exchange contribution of[22, 23]

$$-\frac{\mu^2}{3f^2}\,(W_o - M)^2\,\frac{d}{dW}\big(\,(W-M)\delta A_\gamma(W)\,\big)_{W=W_o} \tag{38}$$

After removing the spurious infrared divergence and setting $W_o = 9\mu$, $c = 1$, and $m_s^2 = 20\mu^2$, one finds that Equation 38 has the numerical value -1.6 MeV. Reasonable upper and lower bounds[19] on W_o would be $5\mu \le W_o - M \le 15\mu$. Variation of W_o over this region cannot change the estimated photon exchange contribution by more than about ± 0.1 MeV.

The singularities of kinematic origin contribute $-\delta M/3$, the crossed nucleon pole $+5\,\delta M/27$ and photon exchange -1.6 MeV. Summing these contributions, we find

$$(31/27)\ \delta M \approx -1.6\ \text{MeV}$$

or

$$\delta M \approx -1.4\ \text{MeV} \tag{39}$$

which is in remarkable agreement with the experimental value of -1.3 MeV.

In Subsection D, we will show that the remaining low mass singularities are very weak so that Equation 39 emerges as our final estimate for δM. One should note that: (i) since the photon exchange amplitude is gauge invariant, Equation 39 is gauge invariant. (ii) The photon exchange contribution given by Equation 37 diverges if both

m_s and m_ρ tend to infinity. This is due to the bad asymptotic behavior of the straight line denominator function 33. With a better behaved D such as 34, the photon exchange contribution is finite without form factors. For this reason, Equation 38 and 39 are not particularly sensitive to the detailed behavior of the form factors. The form factors should, of course, be included and are necessary to obtain the observed value for δM.

D. Other Low Mass Singularities

We will now make a survey of the remaining nearby singularities and will find that they are very weak.

(a) t-Channel Cuts

In this channel we have the process $N\overline{N} \rightarrow 2\pi$. We will first consider electromagnetic corrections to ρ exchange. Since the ρ mass differences transform like T_3^2 they will not affect δM. Furthermore, one can convince himself that electromagnetic corrections to the ρ-π coupling constants must also transform like I or T_3^2 and need not be considered. This leaves only corrections to the ρ-N coupling constants. Since the ρ and ω masses are nearly equal, the process $\rho^o \rightarrow \gamma \rightarrow \omega \rightarrow N\overline{N}$ could produce unusually large electromagnetic effects at the $\rho N\overline{N}$ vertex. However, by including the form factors in the photon exchange amplitude, we have already taken these particular corrections into account. The effect of further corrections to the ρN couplings can be estimated as follows. It can be shown that, among the possible splittings of the ρN coupling constants,[24] only a difference in the magnitudes of $f_{\rho^o nn}$ and $f_{\rho^o pp}$ can affect δM. We define $\Delta = |(|f_{\rho^o pp}| - |f_{\rho^o nn}|)/f_{\rho^o pp}|$ and assume that Δ is about one per cent. One can then insert the ρ exchange amplitude into Equation 31 and estimate the effect on δM. Keeping the nearby part of the ρ cut would, for $\Delta \approx 1\%$, change the calculated value of δM by less than $\pm$ 0.05 MeV.

We must also consider the ϕ and ω intermediate states. A number of authors[25-27] have pointed out that mechanisms like $\omega \rightarrow \gamma \rightarrow 2\pi$ and $\phi \rightarrow \gamma \rightarrow \pi\pi$ could lead to anomalously large amplitudes for the electromagnetic transitions $\omega \rightarrow 2\pi$ and $\phi \rightarrow 2\pi$. One will recall, however, that by including the form factors in the photon exchange amplitude we have already taken the $\omega(\phi) \rightarrow \gamma \rightarrow 2\pi$ mechanism into account. It is not possible to estimate the remaining part of the $\omega(\phi) \rightarrow 2\pi$ amplitude but there does not appear to be any reason to believe that it is particularly large. An additional $\omega\pi\pi$ coupling[24] of order $f^2_{\omega\pi\pi}/4\pi \approx \alpha$ would change our estimate for δM by only a few per cent.

To round out the survey of low mass singularities in the t-channel, we consider the $\pi + \gamma$ intermediate state. The effect of this cut can be estimated as follows. To contribute to the mass difference, the photon must connect one isoscalar and one isovector vertex. We expect that $\pi + \gamma \rightarrow 2\pi$ is dominated by $\pi + \gamma \rightarrow \rho \rightarrow 2\pi$ which requires an isoscalar photon. On the other side of the diagram, the amplitude for $N\overline{N} \rightarrow \pi$ + (isovector γ) is probably dominated by

$N\bar{N} \rightarrow \omega \rightarrow \pi + \gamma$ (the amplitude for $\phi \rightarrow \pi + \gamma$ is expected to be very small[27]). Putting the diagram together, we have $N\bar{N} \rightarrow \omega \rightarrow \pi + \gamma \rightarrow \rho \rightarrow 2\pi$ which resembles the one photon exchange diagram with the photon replaced by $\pi + \gamma$. The relative importance of these two diagrams can be deduced by comparing their contributions to the imaginary part of the off-mass-shell amplitude for $\omega \rightarrow \rho$. With the usual definition for the coupling constants,[26, 27] one finds that the single photon intermediate state produces an imaginary part given by

$$\mathrm{Im}_{\gamma} T(\rho \rightarrow \omega;x) = \pi\delta(x)\gamma_{\rho\gamma}\gamma_{\omega\gamma}/\mu^2 \qquad (40)$$

while the $\pi + \gamma$ intermediate state yields an imaginary part of

$$\mathrm{Im}_{\pi+\gamma} T(\rho \rightarrow \omega;x) = f_{\omega\pi\gamma} f_{\rho\pi\gamma}/4\pi)(\mu^4/24)(x-1)^3/x \qquad (41)$$

where we have introduced the dimensionless mass variable $x = t/\mu^2$. The product $f_{\omega\pi\gamma} f_{\rho\pi\gamma}/4\pi$ is expected[27] to be on the order of $(0.5)(\gamma_{\rho\gamma}\gamma_{\omega\gamma})/(m_\rho^2 m_\omega^2 \mu^2)^{-1}$ which means that Equation 41 is roughly equal to

$$\mathrm{Im}_{\pi+\gamma} T(\rho \rightarrow \omega;x) \approx (3 \times 10^{-5})(\gamma_{\rho\gamma}\gamma_{\omega\gamma}/\mu^2)(x-1)^3/x \qquad (42)$$

Comparing Equations 40 and 42, it is easy to see that the nearby part of the $\pi + \gamma$ cut will have a negligible effect on δM.

(b) <u>s-Channel Cuts</u>

Here we have to consider the inelastic processes $N+\pi \rightarrow N+\gamma \rightarrow N+\pi$ and $N+\pi \rightarrow N+\pi+\gamma \rightarrow N+\pi$. It follows from the last paragraph of Section 3.3 that the $N + \pi + \gamma$ (Bremsstrahlung) cut will be negligible. The following physical argument will show that the $N + \gamma$ cut is also weak. At low energies the π-N system can radiate a real photon through the spin flip current of the nucleon, the recoil current of the pion, or the formation and radiative decay of the (3-3) resonance. The latter two photon vertices are by far the most important but both are pure isovector. The only available isoscalar vertex comes from the nucleon spin flip current and has a strength $\sqrt{\alpha}\,(\mu_p + \mu_n)k/2M$, where the μ_p and μ_n are the nucleon magnetic moments and k is the photon momentum. To estimate the strength of the $N + \gamma$ cut, we multiply $\sqrt{\alpha}\,(\mu_p + \mu_n)k/2M$ by the strength of the isovector vertices $\approx \sqrt{\alpha}$ and the strength squared of the πN interaction in the $J = 1/2^+$, $T = 1/2$ state. At low energies, the latter is on the order of $f^2 \approx 0.08$. For $k = \mu$, the discontinuity across the cut will then be on the order of $\alpha f^2(\mu_p + \mu_n)\mu/2M \approx 0.005\ \alpha$ which is down by a factor of 1/200 as compared to the discontinuity across the photon exchange cut. A straightforward calculation based on the photoproduction amplitudes of CGLN[28] confirms the above estimate. The nearby part of the $N + \gamma$ cut could have at most a 2% effect on δM.

(c) u-Channel Cuts

This channel also involves the process $N + \pi \rightarrow N + \pi$ and one can use the static crossing relations[20] to estimate the nearby part of the u-channel cuts; one finds, in the usual notation,[20]

$$\mathrm{Im}\ \delta A(M-W) = \frac{1}{9}\left[\mathrm{Im}\ \delta\eta_{11}\frac{\cos 2\eta_{11}}{q^3} - 4\ \mathrm{Im}\ \delta\eta_{31}\frac{\cos 2\eta_{31}}{q^3} - 4\ \mathrm{Im}\ \delta\eta_{13}\frac{\cos 2\eta_{13}}{q^3} + 16\ \mathrm{Im}\ \delta\eta_{33}\frac{\cos 2\eta_{33}}{q^3}\right] + \frac{1}{9}\left[\mathrm{Re}\ \delta\eta_{11}\frac{\sin 2\eta_{11}}{q^3} - 4\ \mathrm{Re}\ \delta\eta_{31}\frac{\sin 2\eta_{31}}{q^3} - 4\ \mathrm{Re}\ \delta\eta_{13}\frac{\sin 2\eta_{13}}{q^3} + 16\ \mathrm{Re}\ \delta\eta_{33}\frac{\sin 2\eta_{33}}{q^3}\right] \tag{43}$$

The first term in square brackets comes from the inelastic intermediate states $N + \pi + \gamma$ and $N + \gamma$. The $N + \pi + \gamma$ cut will be weak for the same reason as in the s-channel. Turning to the $N + \gamma$ cut, we note that the sequence $N + \pi \to N + \gamma \to N + \pi$ with both the initial and final pions and nucleons in $T = 3/2$ states requires two isovector photon vertices and cannot affect the mass difference. The $N + \gamma$ intermediate state therefore contributes only to $\mathrm{Im}\ \delta\eta_{11}$ and $\mathrm{Im}\ \delta\eta_{31}$. Since the strength of the low energy π-N interaction in the (3, 1) state is of the same order as in the (1, 1) state, our estimate for the s-channel $N + \gamma$ cut also holds for its u-channel counterpart and we see the latter will have little effect on δM.

The second term in square brackets on the right-hand side of Equation 43 comes from electromagnetic corrections to elastic πN scattering. The largest part of the Re $\delta\eta$'s should come from Coulomb scattering. One can replace the Re $\delta\eta$'s in Equation 43 by the Born approximation to the Coulomb phase shift, remove the infrared divergence by dropping the part which diverges like

$\ell n\,(e\lambda/2\mu)$ and use the observed strong interaction η's to estimate the effect of this cut. Since $\sin 2\eta_{33}$ changes sign at resonance, the (3, 3) resonance term which usually dominates the left cut in pion-nucleon processes has very little effect on δM. The remaining phase shifts η_{11}, η_{31}, and η_{13} are very small and one finds that the crossed πN cut probably has less than a 4 or 5% effect on the mass difference.

To summarize this section: photon exchange and corrections to nucleon exchange and the kinematics lead to an estimate of - 1.4 MeV for δM. The net contribution of all the other cuts which lie in the region $|W-M| \leq 4\mu$ is almost certainly less than $\pm$ 0.1 MeV.

3.5. Discussion

In Chapter 2 it was argued that the high mass singularities, which we have neglected, will contribute mostly to an infrared divergent term which cancels the spurious divergence encountered above. During our treatment of potential scattering, we worked out an example in which the "strong" potential was short ranged and the "electromagnetic" potential was cut off by form factors at small distances. There, we were able to remove the infrared terms from the dispersion integral and show explicitly that the remaining integral was completely dominated by the nearby singularities. Because of the similarity between the potential theory example and our present calculation of δM and because the physics which underlies the infrared divergence is the same in both cases, one can be fairly confident that the above conjecture is correct. Assuming this to be true, none of the neglected singularities in δA is likely to have a large effect on δM, and the agreement between the theoretical and experimental values of the mass difference is not accidental.[29]

In conclusion, we discuss what physical interpretation may be placed on the calculation. We have found that the dispersion integrals are dominated by the photon exchange term connecting the isovector $\pi\pi\gamma$ vertex to the isoscalar $NN\gamma$ vertex. The isoscalar anomalous magnetic moment is small, so the $NN\gamma$ vertex essentially reduces to the isoscalar Dirac term. For the purposes of the present argument, it is convenient to add in the photon exchange connecting the isovector $\pi\pi\gamma$ vertex and the isovector $NN\gamma$ Dirac vertex which, we recall, shifts both proton and neutron masses in the same way and thus does not affect their splitting. We now have the full Dirac vertex with charge $+e$ for the proton and 0 for the neutron. In terms of this vertex, the only component of n or p exhibiting an important one-photon exchange is $\pi^- p$, which makes up two-thirds of the neutron.

Evidently the Coulomb part of the interaction is attractive for $\pi^- p$. Thus one might expect the neutron to become lighter than the proton--exactly opposite to our result, not to mention experiment!

To see what is going on here, it is sufficient to take Equation 37 as the contribution of the photon exchange cut. This formula, which was obtained with the approximation $D = (W - M)$, is much simpler than Equation 38 and contains the essential physics of the situation. One will note that Equation 37 contains a factor $1/M$ and it is not difficult to convince oneself that the only part of the photon exchange force between a π^- and p which can make a contribution of order $1/M$ is the interaction of the Dirac $(e/2M)$ magnetic moment of the proton with the magnetic field produced by the moving pion. (The ordinary electrostatic attraction does not vanish as $M \to \infty$ and nucleon recoil effects will be of order $1/M^2$.) So the main effect of the $\pi^- p$ interaction is magnetic and the Coulomb term, which was expected to make the neutron lighter, does not in fact affect the neutron mass at all in the approximation $D = (W - M)$!

A heuristic explanation of this result is as follows. Loosely speaking, we have considered the neutron to be made up of a π^- bound to a fixed proton. By virtue of our assumption that D is practically a straight line, we have also assumed that the forces which bind the π^- have a range which is short compared to the inverse binding energy $1/\mu$. This means that most of the time the pion will be found outside of the region in which the binding forces operate. In this outside region the pion, with binding energy μ, has zero total energy. Now the standard expressions for the charge and current densities of a spin zero (Klein-Gordan) particle in a potential-free region are

$$\rho = \frac{i}{2\mu} \left(\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t} \phi\right) \tag{44}$$

$$\vec{j} = \frac{1}{2\mu i} \left(\phi^* \nabla \phi - \phi \nabla \phi^*\right) \tag{45}$$

Zero total energy implies $\partial\phi/\partial t = 0$, so the charge density vanishes and it is not surprising that the Coulomb term vanishes. On the other hand, the pion momentum does not vanish so there will be a current which can interact with the magnetic moment of the nucleon to produce the mass difference.

REFERENCES

1. R. Feynman and G. Speisman, Phys. Rev. 94, 500 (1954).

2. M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters 2, 7 (1959).

3. S. Sunakawa and K. Tanaka, Phys. Rev. 115, 754 (1959).

4. H. Katsumori and M. Shimada, Phys. Rev. 124, 1203 (1961).

5. A. Solomon, Nuovo Cimento 27, 748 (1963).

6. We take $\hbar = c = 1$, in addition to setting the kinetic energy equal to q^2.

7. R. Blankenbecler, M. Goldberger, N. Khuri, S. Trieman, Ann. Phys. 10, 62 (1960).

8. For the purposes at hand it is sufficient to know that, if the potential has a single bound state at $s = s_B$, A can be written in the form $A = N/D$ with

$$D(s) = \frac{s - s_B}{s_o - s_B} \exp \frac{s - s_o}{\pi} \int_0^{\infty} \frac{\eta(s')}{(s' - s_o)(s' - s)} ds'$$

where s_o is an arbitrary normalization point. A discussion of the integral equations satisfied by N and D can be found, for example, in S. C. Frautschi, Regge Poles and S-Matrix Theory, W. A. Benjamin, Inc., New York.

9. K. W. Chen, A. A. Cone, J. R. Dunning, Jr., S. G. F. Frank, N. F. Ransey, J. K. Walker, and Richard Wilson, Phys. Rev. Letters 11, 561 (1963).

10. The favorable convergence of dispersion relations involving the redefined amplitude has nothing particular to do with the fact that we started with an infrared-divergent amplitude. This suggests the following treatment for other perturbation problems which have no infrared divergence: (i) calculate $\delta\eta_{Born}$. (ii) Define $\delta\hat{A}$ as in Equation 8. Calculate the dispersion relations for $\delta\hat{A}$, δs_B, and $\delta\hat{R}$, which converge more rapidly than the dispersion relations involving the usual amplitude. (iii) The mass shift is correctly obtained in this way. Naturally, we do not want a redefined amplitude or residue as the final answer, but to obtain the final expressions for δA and δR, one simply adds the known $\delta\eta_{Born}$ term back in.

11. D. Yennie, S. Frautschi, and H. Suura, Ann. Phys. 13, 379 (1961).

12. J. Jauch and F. Rohrlich, The Theory of Photons and Electrons (Addison-Wesley, Cambridge, 1955).

13. G. F. Chew and S. C. Frautschi, Phys. Rev. 124, 264 (1961).

14. We take the absolute value of g in the logarithm because Equation 22 does guarantee that δs_B is real, so only the real part of $\ln(\lambda^2/g(s_B))$ can appear in the spurious term.

15. C. Chew and S. Frautschi, Phys. Rev. Letters 8, 41 (1962). For a treatment of the nucleon as a bound state in the πN system, see E. Abers and C. Zemach, Phys. Rev. 131, 1205 (1963).

16. W. Fraser and J. Fulco, Phys. Rev. 119, 1420 (1960).

17. S. Frautschi and J. Walecka, Phys. Rev. 120, 1486 (1960).

18. L. Balazs, Phys. Rev. 128, 1935 (1962).

19. Physically, $(W_0 - M)^{-1}$ should correspond roughly to the range of the forces which bind the nucleon. Since N^* exchange is generally believed to be the most important of these forces, setting $W_0 - M = 9\mu \approx M_{N^*}$ would seem to be very reasonable. It might be objected that N^* exchange also provides a longer range force at low energies; however, the short range part is more important in determing the behavior of D.

20. G. Chew, M. Goldberger, F. Low and Y. Nambu, Phys. Rev. 106, 1337 (1957).

21. C. de Vries, R. Herman, and R. Hofstadter, Phys. Rev. Letters 9, 381 (1962).

22. In addition to P-wave cuts near $W = M$, δA_γ has S-wave cuts[16,17] near $W = -M$. The latter are outside of our region of interest and will be neglected. One can verify that setting $W + M = 2M$ and $q^2 = (W-M)^2 - \mu^2$ in the part of δA_γ containing I_2 will have little effect on the contribution to δM of the P-wave cuts and will suffice to make the S-wave cuts negligible.

23. The integrations leading to Equations 36 and 38 can be carried out as follows. By explicit calculation, one finds that the cuts in δA_γ extend a finite distance into the left half plane. The contour L in Equation 31 will therefore be a closed loop around these cuts. With the approximation 33 for D, the integrand has no further singularities in the W plane. Expanding the contour to infinity yields Equation 36. If one uses Equation 34 for D, the integrand goes to $1/W^2$ at infinity but has a pole at $W = W_o$. A simple contour integration gives Equation 38.

24. The vector meson coupling constants $f_{\rho NN}$ and $f_{\omega\pi\pi}$ are defined in analogy with the electric charge e. We take $f^2_{\rho NN}/4\pi \approx 1/2$.

25. Y. Nambu and J. J. Sakurai, Phys. Rev. Letters 8, 79 (1962).

26. M. Gell-Mann, D. Sharp, and W. Wagner, Phys. Rev. Letters 8, 261 (1962).

27. R. Dashen and D. Sharp, Phys. Rev. 133, 1585 (1964).

28. G. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. 106, 1345 (1957).

29. The reader who objects to our treatment of the infrared divergences should note that, according to Equation 36, if we had given the photon a finite mass on the order of the inverse "radius" of a nucleon $\approx 2\mu$, our estimate for δM would still be of the correct sign and order of magnitude.