CONTRIBUTIONS TO TENSOR ANALYSIS

Thesis by

William Thomas Guy, Jr.

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1951

It is a pleasure to acknowledge and to express my appreciation for the assistance and encouragement given me by Professor A. D. Michal. His patience and ever-willingness to talk over the various points and possibilities connected with such an undertaking despite other demands on his time and strength are deeply appreciated.

I wish also to acknowledge the help and encouragement given me so freely by my wife, Valaree. Many times her hands were more than filled with the activities of two lively little boys who found it hard to understand why their dad was busy just studying mathematics.

SUMMARY

This thesis treats two separate problems. The first concerns the transverse vibrations of a beam and of a thin rectangular flat plate. These vibrations are associated with a function space which has the properties of a generalized "Riemannian" function space. The geodesics of this space are shown to play a role analogous to that played by the geodesics of the configuration space in the classical treatment of the finite dimensional case. Part I is introductory and treats a few aspects of the vibrations of beams with various end conditions under a change of parameter. Part II develops the integro-differential equation for the thin rectangular flat plate. The associated function space and its geodesics are then studied in some detail. The space is found to be not one of constant Riemannian curvature. An example is worked out to illustrate the ideas, and an extension is suggested. The second problem (Part III) considers the equations of motion of hydrodynamics of viscous flow with moving axes. Use is made of the space of a kinetic metric introduced by McVittie, who considered non-viscous flow only. The Newtonian equations are obtained by taking certain approximations. The equations of motion in terms of the vorticity tensor are developed. Two examples are discussed illustrating the theory, one concerning instability necessary for tropical cyclones.

TABLE OF CONTENTS

PART I

VIBRATION OF BEAMS

1. Notations and conventions. To avoid any misunderstanding we shall explain the notations and conventions used throughout this thesis. Theorems, lemmas, and definitions are numbered consecutively in each paragraph. The integer represents the number of the paragraph, and the number to the right of the decimal point represents the order of the theorem, lemma, or definition. For example, Theorem 4.1 is the first theorem of paragraph 4, while Lemma 5.3 is the third lemma of paragraph 5, etc. The ordered pair of numbers such as (13,417) listed in the text means page 417 of reference number 13 in the list of references given after PART III. The usual Einstein summation convention is used, i.e., repeated lower case indices (Greek or Latin) generate sums. The range of such sums will be given explicitly. Capital indices will be used to forestall summation. As is customary, we let notation indicate some of the restrictions or hypotheses, especially in a series of lemmas. For example, the equation stating Lemma 3.2 certainly presupposes that Definitions 3.3, 3.2, etc., are kept in mind. This should cause no confusion since the required supporting definitions are usually on the same page or on the preceding page. This is done for the obvious reason of keeping the lemmas

from being too bulky. On the other hand, the hypotheses for theorems are listed in the theorem. A function u is said to be of class C^n (briefly, $u \in C^n$) if all of its first n derivatives are continuous.

2. Introduction. The role played by the geodesics of the configuration space of classical mechanics (of generalized coordinates) is well known (cf. (13,31), (15,417), (4,17)). This configuration space is usually a finite dimensional Riemannian space with a fundamental symmetric tensor g_{ij}. The element of arc length is given by $ds^2 = g_{ij}dq^i dq^j$ $(i,j = 1,2,\ldots,n)$ where the q^i are the generalized coordinates, $g_{ij} = g_{ji}$, and the summation convention is used.

We consider here the generalization required for, say, the geodesics of the "configuration space" of a vibrating string, beam, flat plate, etc. Clearly a finite number of independent generalized coordinates is no longer adequate. An introduction of a generalized "Riemannian" function space circumvents this difficulty (9,551), (11,38). This generalization requires the use of arbitrary (except for certain restrictions listed where required) functions in lieu of coordinates, Fréchet differentials, Gâteaux limits, a generalized tensor analysis, and other aspects of modern differential geometry (9,529).

We first consider the "small" transverse vibrations of a straight beam. The effect of the various end conditions will be brought out where appropriate. Vibrations of a flat plate will be considered in the next chapter.

3. <u>Transverse Vibrations of straight beams.</u> The partial differential equation for the "small" transverse vibrations of a straight beam of constant cross section and constructed of material of constant density ρ and constant modulus of elasticity E is (cf. (14,241))

$$\text{(1)} \qquad EI \frac{\partial^4 u(x,t)}{\partial x^4} + \rho \frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

where $u \equiv u(x,t)$ is the displacement of the axis of the beam from its unstrained position as a function of distance x along the length of the beam and of time t. We assume u belongs to class C^4.

$E \equiv$ Young's modulus of elasticity;

$I \equiv$ moment of inertia of the beam cross section about a line through its centroid perpendicular to the plane of the displacement u;

$\rho \equiv$ mass per unit length;

"small" $\equiv \left(\frac{\partial u}{\partial x}\right)^2$ may be neglected in the curvature formula

$$\frac{1}{R} = \frac{\partial^2 u}{\partial x^2} \Big/ \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]^{3/2},$$ which gives the curvature of the axis of the beam.

For ease of reference we tabulate some of the end conditions usually considered in beam problems. The conditions are arranged in columns from which a typical case can be read off by taking one and only one condition from each column. The subscript notation is used for partial derivatives, e.g., $u_{xx}(0,t) \equiv \frac{\partial^2 u(x,t)}{\partial x^2}$ evaluated at $x = 0$.

TABLE I
(or TABLE $\bar{I}$ if $t \to s$, $u(x,t(s)) \to \bar{u}(x,s)$)

	A	B	C	D
(a)	$u_{xx}(0,t)=0$	$u_{xx}(\ell,t)=0$	$u_{xxx}(0,t)=0$	$u_{xxx}(\ell,t)=0$
(b)	$u_x(0,t)=0$	$u_x(\ell,t)=0$	$u(0,t)=0$	$u(\ell,t)=0$

Well known end conditions are

(2)	Aa, Ba, Ca, Da	$\Rightarrow$	free ends
(3)	Aa, Ba, Cb, Db	$\Rightarrow$	hinged (simply supported) ends
(4)	Ab, Bb, Cb, Db	$\Rightarrow$	built-in ends
(5)	Ab, Da, Cb, Da	$\Rightarrow$	built-in, free (cantilever)
(6)	Ab, Ba, Cb, Db	$\Rightarrow$	built-in, hinged

The kinetic energy $T(t)$ and the potential energy $V(t)$ are given by (14,334)

(7)
$$T(t) = \frac{\rho}{2}\int_0^{\ell}\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx$$

(8)
$$V(t) = \frac{EI}{2}\int_0^{\ell}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 dx ,$$
where ℓ is the length of the beam.

<u>Theorem 3.1</u>. If i) $u(x,t) \in C^4$ and satisfies equation (1)

ii) the kinetic and potential energies are given by equations (7) and (8), respectively

iii) $u(x,t)$ satisfies the end condition (2) (free ends)

then $T(t) + V(t) = C$, a constant.

Proof.

$$\frac{d(T+V)}{dt} = \frac{\rho}{2}\frac{d}{dt}\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{EI}{2}\frac{d}{dt}\int_0^\ell \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx$$

$$= \frac{\rho}{2}\int_0^\ell 2u_t u_{tt}\,dx + \frac{EI}{2}\int_0^\ell 2u_{xx}u_{xxt}\,dx$$

$$= \rho\int_0^\ell u_t u_{tt}\,dx + EI\left[u_{xt}u_{xx}\Big|_0^\ell - \int_0^\ell u_{xxx}u_{xt}\,dx\right]$$

$$= \rho\int_0^\ell u_t u_{tt}\,dx - EI\left[u_t u_{xxx}\Big|_0^\ell - \int_0^\ell u_t u_{xxxx}\,dx\right]$$

$$= \rho\int_0^\ell u_t\left(\frac{-EI}{\rho}u_{xxxx}\right)dx + EI\int_0^\ell u_t u_{xxxx}\,dx$$

$$= 0\,, \qquad \text{Q.E.D.}$$

Corollary 3.1. If $u(x,t)$ satisfies any of the end conditions (3), (4), (5), or (6) as well as hypotheses i) and ii) of Theorem 3.1, then $T(t)+V(t)=C$, a constant.

Proof. An analysis of the "u v" terms of the integration by parts in Theorem 3.1 shows that they likewise vanish under any of the end conditions (3), (4), (5), or (6). It is also to be noted that this result also obtains if any set of four conditions are taken from Table I with one and only one from each column.

It is fruitful to introduce another parameter in lieu of t.

Definition 3.1. $$A(\lambda) \equiv 2\left[C - \frac{EI}{2}\int_0^\ell \left(\frac{\partial^2 u(x,\lambda)}{\partial x^2}\right)^2 dx\right]$$

where i) $C = T+V$, and we assume

$$\text{ii)}\quad C - \frac{EI}{2}\int_0^{\ell}\left(\frac{\partial^2 u(x,\lambda)}{\partial x^2}\right)^2 dx > 0.$$

<u>Lemma 3.1.</u> $A(\lambda) = 2T(\lambda)$.

<u>Proof.</u> Clear. Use Definition 3.1 and equation (8).

<u>Definition 3.2.</u> $s \equiv \int_0^t A(\lambda)\,d\lambda$

<u>Definition 3.3.</u> $\bar{A}(s) \equiv A(t(s))$, $\bar{u}(x,s) \equiv u(x,t(s))$

<u>Lemma 3.2.</u>
$$\frac{1}{\bar{A}(s)} = \rho\int_0^{\ell}\left(\frac{\partial \bar{u}(x,s)}{\partial s}\right)^2 dx \tag{9}$$

<u>Proof.</u>

$$\frac{ds}{dt} = A(t) \Rightarrow \frac{dt(s)}{ds} = \frac{1}{A(t(s))} = \frac{1}{\bar{A}(s)}$$ from Definitions 3.2 and 3.3.

$$A(t) = 2T(t) = \rho\int_0^{\ell}\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx$$ from equation (7) and Lemma 3.1.

$$A(t(s)) = \rho\int_0^{\ell}\left(\frac{\partial u(x,t(s))}{\partial s}\right)^2\left(\frac{ds}{dt}\right)^2 dx$$

$$A(t(s)) = \rho\, A^2(t(s))\int_0^{\ell}\left(\frac{\partial u(x,t(s))}{\partial s}\right)^2 dx$$, and using Definition 3.3

$$\frac{1}{\bar{A}(s)} = \rho\int_0^{\ell}\left(\frac{\partial \bar{u}(x,s)}{\partial s}\right)^2 dx \tag{10}$$ Q.E.D.

Theorem 3.1 on the conservation of energy can also be written in terms of the parameter s.

<u>Theorem 3.2.</u> If i) $u(x,t)$ satisfies equation (1),

$\bar{u}(x,s) \equiv u(x,t(s))$, and $u \in C^4$,

ii) $T(t)$ and $V(t)$ are given by equations (7) and (8),

iii) s is given by Definition 3.2,

iv) $\bar{A}(s)$ is given by Definitions 3.1 and 3.3,

then

$$\frac{\rho\,\bar{A}^2(s)}{2}\int_0^\ell\left(\frac{\partial\,\bar{u}(x,s)}{\partial s}\right)^2 dx+\frac{EI}{2}\int_0^\ell\left(\frac{\partial^2\bar{u}(x,s)}{\partial x^2}\right)^2 dx = C\ . \tag{11}$$

<u>Proof</u>. $\dfrac{1}{\bar{A}(s)} = \rho\displaystyle\int_0^\ell\left(\frac{\partial\,\bar{u}(x,s)}{\partial s}\right)^2 dx$ from Lemma 3.2

$-\dfrac{\bar{A}(s)}{2}+\rho\displaystyle\int_0^\ell\left(\frac{\partial\,\bar{u}(x,s)}{\partial s}\right)^2 dx = 0$ using Definition 3.1

$$-C+\frac{EI}{2}\int_0^\ell\left(\frac{\partial^2\bar{u}(x,s)}{\partial x^2}\right)^2 dx+\rho\int_0^\ell\left(\frac{\partial\,\bar{u}(x,s)}{\partial s}\right)^2 dx = 0\ .$$

<u>Lemma 3.3</u>. If i) $u(x,t)$ satisfies equation (1), $\in C^4$,

ii) $u(x,t)$ satisfies the end conditions (2)

iii) $\bar{A}(s)$ is given by Definitions 3.1, 3.2, and 3.3,

then

$$\frac{d\bar{A}(s)}{ds} = -2EI\int_0^\ell \frac{\partial\,\bar{u}(x,s)}{\partial s}\cdot\frac{\partial^4\bar{u}(x,s)}{\partial x^4}\,dx$$

<u>Proof</u>. $A(t) = 2\left[C-\dfrac{EI}{2}\displaystyle\int_0^\ell\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 dx\right]$ from Definition 3.1

$$\frac{dA(t(s))}{ds} = -EI\int_0^\ell 2\,\frac{\partial^2 u(x,t(s))}{\partial x^2}\,\frac{\partial^3 u(x,t(s))}{\partial s\,\partial x^2}\,dx$$

$$= -2EI\left[\frac{\partial^2 u(x,t(s))}{\partial x^2}\,\frac{\partial^2 u(x,t(s))}{\partial s\partial x}\Big|_0^{\ell} - \int_0^{\ell}\frac{\partial^3 u(x,t(s))}{\partial x^3}\,\frac{\partial^2 u(x,t(s))}{\partial s\partial x}dx\right]$$

$$= 2EI\left[\frac{\partial^3 u(x,t(s))}{\partial x^3}\,\frac{\partial u(x,t(s))}{\partial s}\Big|_0^{\ell} - \int_0^{\ell}\frac{\partial^4 u(x,t(s))}{\partial x^4}\,\frac{\partial u(x,t(s))}{\partial s}dx\right]$$

$$\frac{d\bar{A}(s)}{ds} = -2EI\int_0^{\ell}\frac{\partial \bar{u}(x,s)}{\partial s}\,\frac{\partial^4 \bar{u}(x,s)}{\partial x^4}dx \qquad \text{Q.E.D.}$$

Corollary 3.3. Hypothesis ii) of Lemma 3.3 may be extended to encompass any of the end conditions (3), (4), (5), or (6).

Proof. The "u v" terms of the integration by parts vanish for conditions (3), (4), (5), or (6) as well as for condition (2).

Theorem 3.3. If i) $u(x,t)$ satisfies equation (1) with any of the end conditions (2)-(6),

ii) $T(t) + V(t) = C$, a constant

iii) $\bar{A}(s)$ is given by Definitions 3.1, 3.2, and 3.3

iv) $\bar{u}(x,s) \equiv u(x,t(s))$

then $\bar{u}(x,s)$ satisfies the integro-differential equation

$$\frac{\partial^2 \bar{u}(x,s)}{\partial s^2} - \frac{2EI}{\bar{A}(s)}\frac{\partial \bar{u}(x,s)}{\partial s}\int_0^{\ell}\frac{\partial^4 \bar{u}(x,s)}{\partial x^4}\,\frac{\partial \bar{u}(x,s)}{\partial s}dx + \frac{EI}{\bar{A}(s)}\frac{\partial^4 \bar{u}(x,s)}{\partial x^4}\int_0^{\ell}\left(\frac{\partial \bar{u}(x,s)}{\partial s}\right)^2 dx = 0 \tag{12}$$

Proof. $\dfrac{ds}{dt} = A(t)$ from Definition 2.2.

$$\frac{d^2 s}{dt^2} = \frac{dA(t(s))}{ds}\,A(t)\,.$$

Hence $$\frac{\partial u(x,t)}{\partial t} = \frac{\partial \bar{u}}{\partial s} A(t)$$

and $$\frac{\partial^2 u(x,t)}{\partial t^2} \quad \frac{\partial^2 \bar{u}}{\partial s^2} A^2(t) + \frac{\partial \bar{u}}{\partial s} \frac{dA(t)}{ds} A(t) .$$

Substituting these equations into equation (1), we have

$$\rho \frac{\partial^2 \bar{u}}{\partial s^2} + \frac{\partial \bar{u}}{\partial s} \frac{dA\big(t(s)\big)}{ds} \frac{\rho}{A\big(t(s)\big)} + \frac{EI}{A^2\big(t(s)\big)} \frac{\partial^4 u(x,t)}{\partial x^4} = 0 \quad \text{which,}$$

upon using Lemma 3.2 and Corollary 3.3, becomes

$$\rho \frac{\partial^2 \bar{u}}{\partial s^2} + \rho \frac{1}{\bar{A}} \frac{\partial \bar{u}}{\partial s} (-2EI) \int_0^{\ell} \frac{\partial \bar{u}}{\partial s} \frac{\partial^4 \bar{u}}{\partial x^4} dx$$

$$+ \frac{EI}{\bar{A}} \frac{\partial^4 \bar{u}}{\partial x^4} \rho \int_0^{\ell} \left(\frac{\partial \bar{u}}{\partial s} \right)^2 dx = 0 .$$

This proves the theorem upon division by ρ.

We now assume that Table I has been rewritten as Table $\bar{\text{I}}$ in which t is replaced by $t(s)$ and $u(x,t) = u\big(x,t(s)\big) = \bar{u}(x,s)$.

<u>Theorem 3.4.</u> If i) $\bar{u}(x,s) \in C^4$ and satisfies equations (10), (11), and (12)

ii) $\bar{A}(s) > 0$ satisfies Definition 3.1 with s as a parameter

iii) C is a positive constant

iv) $\bar{u}(x,s)$ satisfies the end conditions (2) of Table $\bar{\text{I}}$

v) $t \equiv \int_0^s \frac{d\lambda}{\bar{A}(\lambda)}$

vi) $u(x,t) \equiv \bar{u}\big(x,s(t)\big)$

then a) $u(x,t)$ satisfies the partial differential equation (1)

b) $u(x,t)$ satisfies the end conditions (2) of Table I

c) $u(x,t)$ has the constant total energy level C.

Proof. $t \equiv \int_0^s \frac{d\lambda}{\bar{A}(\lambda)}$ by hypothesis.

$$\frac{dt}{ds} = \frac{1}{\bar{A}(s)}, \quad \therefore \quad \frac{\partial \bar{u}(x,s)}{\partial s} = \frac{\partial \bar{u}(x,s(t))}{\partial t} \frac{1}{\bar{A}(s)}$$

$$\frac{\partial^2 \bar{u}(x,s)}{\partial s^2} = \frac{\partial^2 \bar{u}(x,s(t))}{\partial t^2} \frac{1}{\bar{A}^2(s)} - \frac{\partial \bar{u}(x,s(t))}{\partial t} \frac{d\bar{A}(s(t))}{dt} \frac{1}{\bar{A}^3(s)} .$$

Substituting these into equation (11), we have

$$(13) \quad \frac{\partial^2 u(x,t)}{\partial t^2} \frac{1}{\bar{A}^2(s)} - \frac{\partial u(x,t)}{\partial t} \frac{d\bar{A}(s(t))}{dt} \frac{1}{\bar{A}^3(s)} - \frac{2EI}{\bar{A}(s)} \frac{\partial u}{\partial t} \frac{1}{\bar{A}(s)} \cdot$$

$$\cdot \int_0^{\ell} \frac{\partial^4 \bar{u}(x,s)}{\partial x^4} \frac{\partial \bar{u}(x,s)}{\partial s} dx + \frac{EI}{\bar{A}(s)} \frac{\partial^4 \bar{u}(x,s)}{\partial x^4} \cdot$$

$$\cdot \int_0^{\ell} \left(\frac{\partial \bar{u}(x,s)}{\partial s} \right)^2 dx = 0 .$$

Since $\bar{A}(s) = 2 \left[C - \frac{EI}{2} \int_0^{\ell} \left(\frac{\partial^2 \bar{u}(x,s)}{\partial x^2} \right)^2 dx \right]$

$$\frac{d\bar{A}(s(t))}{dt} = -2EI \int_0^{\ell} \frac{\partial^2 \bar{u}(x,s(t))}{\partial x^2} \frac{\partial^3 \bar{u}(x,s(t))}{\partial t \partial x^2} dx$$

$$= -2EI \int_0^{\ell} \frac{\partial \bar{u}(x,s(t))}{\partial t} \frac{\partial^4 \bar{u}(x,s(t))}{\partial x^4} dx$$

after two integrations by parts like those for the proof of Lemma 3.3. Substituting this into equation (12) we have

$$\frac{\partial^2 u}{\partial t^2} \frac{1}{\bar{A}^2(s)} + \frac{1}{\bar{A}^3(s)} \frac{\partial u}{\partial t} 2EI \int_0^{\ell} \frac{\partial u}{\partial t} \frac{\partial^4 \bar{u}(x,s)}{\partial x^4} dx -$$

$$-2 \frac{EI}{\bar{A}^3(s)} \frac{\partial u}{\partial t} \int_0^{\ell} \frac{\partial^4 \bar{u}(x,s)}{\partial x^4} \frac{\partial \bar{u}(x,s(t))}{\partial t} dx +$$

$$+ \frac{EI}{\bar{A}^3(s)} \frac{\partial^4 \bar{u}(x,s)}{\partial x^4} \int_0^{\ell} \left(\frac{\partial \bar{u}(x,s(t))}{\partial t} \right)^2 dx = 0 \,, \quad \text{or}$$

$$(14) \quad \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 \bar{u}(x,s(t))}{\partial x^4} \left(\frac{ds}{dt} \right)^2 \frac{1}{\bar{A}(s)} \cdot$$

$$\cdot \int_0^{\ell} \left(\frac{\partial \bar{u}(x,s(t))}{\partial t} \right)^2 \left(\frac{dt}{ds} \right)^2 dx = 0 \,.$$

Since $\dfrac{ds}{dt} = \bar{A}(s)$ and $\bar{A}(s) \displaystyle\int_0^{\ell} \left(\frac{\partial \bar{u}(x,s)}{\partial s} \right)^2 dx = \frac{1}{\rho}$,

equation (14) becomes

$\dfrac{\partial^2 u}{\partial t^2} + \dfrac{EI}{\rho} \dfrac{\partial^4 u}{\partial x^4} = 0.$ Hence conclusion a) follows. Conclusion b) follows immediately from the relation of Table I and $\bar{\text{I}}$. Conclusion c) follows from equation (11) of Theorem 3.2 upon noticing that the left side of this equation is just the sum of $T(t)$ and $V(t)$ as given by equations (7) and (8).

Corollary 3.4. Hypothesis iv) may be changed to read any one of the end conditions (3), (4), (5), or (6) of Table $\bar{\text{I}}$, and then the conclusion b) will read the respective one of (3), (4), (5), or (6) of Table I.

Proof. Clear. Compare with Corollary 3.3 of Lemma 3.3.

PART II

A FUNCTION SPACE ASSOCIATED WITH A FLAT PLATE

Before proceeding with the geometric ideas and the role of the generalized Riemannian function space, we shall consider the small vibrations of a thin rectangular flat plate.

4. Vibrations of a thin rectangular flat plate. We consider a thin rectangular flat plate of perfectly elastic, homogeneous isotropic material. With the axis selected as shown in Fig. 1, the partial differential equation governing the vibrations in the direction of z, the kinetic energy $T(t)$, and the potential energy $V(t)$ are given respectively by (14,421)

$$(15) \qquad \frac{\partial^2 u(x,y,t)}{\partial t^2} + \frac{aD}{M}\left(\frac{\partial^4 u(x,y,t)}{\partial x^4} + 2\frac{\partial^4 u(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x,y,t)}{\partial y^4}\right) = 0$$

$$(16) \qquad T(t) = \frac{M}{2a}\int_0^c\int_0^b \left(\frac{\partial u}{\partial t}\right)^2 dxdy$$

$$(17) \qquad V(t) = \frac{D}{2}\int_0^c\int_0^b \left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2\right] dxdy$$

where $a =$ area of the plate, $a = bc$

$M =$ mass of the plate

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

E = Young's modulus of elasticity of the material

h = thickness of the plate

ν = Poisson's ratio for the material

$u(x,y,t)$ = deflection of the plate at time t.

$u \in C^4$

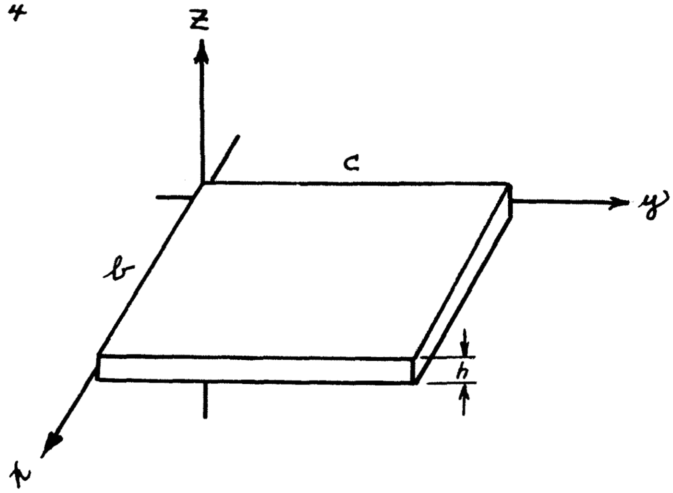

Fig. 1

We shall be concerned with the vibrations $u(x,y,t)$, assumed belonging to C, for which on the boundary $u(x,y,t)$ satisfies

(18) $\quad u(x,y,t) = 0\ ,\quad u_{xx}(x,y,t) = 0 \quad\text{and}\quad u_{yy}(x,y,t) = 0.$

Definition 4.1.

$$\nabla^4 u(x,y,t) \equiv \nabla^2\left(\nabla^2 u(x,y,t)\right) = \frac{\partial^4 u}{\partial x^4} + 2\,\frac{\partial^2 u\,\partial^2 u}{\partial x^2\,\partial y^2} + \frac{\partial^4 u}{\partial y^4}\ .$$

Theorem 4.1. If
i) $u(x,y,t)$ satisfies equation (15) and $\in C^4$
ii) $u(x,y,t)$ satisfies the boundary conditions of equation (18)
iii) the kinetic and potential energies are given by equations (16) and (17), respectively,

then $T(t) + V(t) = C$, a constant.

<u>Proof.</u> $$\frac{d}{dt}\Big(T(t)+V(t)\Big) = \frac{M}{2a}\frac{d}{dt}\int_0^c\int_0^b (u_t)^2\,dxdy \;+$$

$$+\;\frac{D}{2}\frac{d}{dt}\int_0^c\int_0^b \left[u_{xx}^2 + u_{yy}^2 + 2\nu u_{xx}u_{yy} + 2(1-\nu)u_{xy}^2\right]dxdy$$

$$= \frac{M}{a}\int_0^c\int_0^b u_t u_{tt}\,dxdy \;+$$

$$+\;D\int_0^c\int_0^b \Big[u_{xx}u_{xxt} + u_{yy}u_{yyt} + \nu\,(u_{xxt}u_{yy} + u_{xx}u_{yyt}) \;+$$

$$+\;2(1-\nu)u_{xy}u_{xyt}\Big]dxdy \quad \text{and using equation (15)}$$

(19) $$= -\frac{M}{a}\int_0^c\int_0^b u_t\frac{aD}{M}\nabla^4 u\,dxdy \;+$$

$$+\;D\int_0^c\int_0^b \Big[u_{xx}u_{xxt} + u_{yy}u_{yyt} + \nu\,(u_{xxt}u_{yy} + u_{xx}u_{yyt})$$

$$+\;2(1-\nu)u_{xy}u_{xyt}\Big]dxdy\;.$$

In order to simplify the last integral, we integrate it by parts.

We do this a term at a time.

$$D\int_0^c\int_0^b u_{xx}u_{xxt}\,dxdy = D\int_0^c\left[u_{xx}u_{xt}\Big|_0^b - \int_0^b u_{xxx}u_{xt}\,dx\right]dy$$

$$= -D\int_0^c\left[u_{xxx}u_t\Big|_0^b - \int_0^b u_t u_{xxxx}\,dx\right]dy$$

(20) $$= D\int_0^c\int_0^b u_t u_{xxxx}\,dxdy\;. \quad \text{Similarly}$$

(21) $$D\int_0^c\int_0^b u_{yy}u_{yyt}\,dxdy = D\int_0^c\int_0^b u_t u_{yyyy}\,dxdy\;.$$

$$D\int_0^c\int_0^b \nu\, u_{xxt}u_{yy}\,dxdy = D\nu\int_0^c\left[u_{xt}u_{yy}\Big|_0^b - \int_0^b u_{xt}u_{yyx}\,dx\right]dy$$

$$= -D\nu\int_0^c\left[u_t u_{yyx}\Big|_0^b - \int_0^b u_t u_{yyxx}\,dx\right]dy$$

(22) $$= D\int_0^c\int_0^b \nu\, u_t u_{yyxx}\, dxdy.$$ Similarly

(23) $$D\int_0^c\int_0^b \nu\, u_{yyt} u_{xx}\, dxdy = D\int_0^c\int_0^b \nu\, u_t u_{xxyy}\, dxdy.$$

$$D\int_0^c\int_0^b 2(1-\nu) u_{xy} u_{xyt}\, dxdy = 2(1-\nu) D\int_0^c \left[u_{yt} u_{xy} \Big|_{x=0}^{b} - \int_0^b u_{xxy} u_{yt}\, dx \right] dy$$

$$= -2(1-\nu) D\int_0^b \left[u_t u_{xxy} \Big|_{y=0}^{c} - \int_0^c u_t u_{xxyy}\, dy \right] dx$$

(24) $$= D\int_0^c\int_0^b 2(1-\nu) u_t u_{xxyy}\, dxdy .$$

The "u v" term of the integration by parts vanished each time by virtue of the boundary conditions. Using the results of equations (20)-(24), equation (19) becomes

$$\frac{d\big(T(t)+V(t)\big)}{dt} = -D\int_0^c\int_0^b u_t\, \nabla^4 u\, dxdy + D\int_0^c\int_0^b u_t \Big[u_{xxxx} + u_{yyyy} + \nu\, u_{xxyy} + \nu\, u_{xxyy} + 2(1-\nu) u_{xxyy} \Big] dxdy$$

$$= D\int_0^c\int_0^b u_t (-\nabla^4 u + \nabla^4 u)\, dxdy = 0. \qquad \text{Q.E.D.}$$

<u>Definition</u> <u>4.2</u>.

$$A(t) \equiv 2\left[C - \frac{D}{2}\int_0^c\int_0^b \Big\{ u_{xx}^2 + u_{yy}^2 + 2\nu\, u_{xx} u_{yy} + 2(1-\nu) u_{xy}^2 \Big\} dxdy \right]$$

with $A(t) > 0$.

Lemma 4.1. $A(t) = 2T(t)$.

Proof. Clear. Apply Theorem 4.1 and equation (17).

Definition 4.3. $s \equiv \int_0^t A(\lambda)d\lambda$.

Definition 4.4. $\bar{A}(s) \equiv A(t(s))$, $\bar{u}(x,y,s) \equiv u(x,y,t(s))$.

Lemma 4.2. $\frac{d\bar{A}(s)}{ds} = -2D \int_0^c\int_0^b \bar{u}_s \nabla^4 \bar{u}\, dxdy$.

Proof. From Definition 4.2

$$\frac{dA(t(s))}{ds} = -2D \int_0^c\int_0^b \{u_{xx} u_{xxs} + u_{yy}u_{yys} + \nu(u_{xxs} u_{yy} + u_{xx} u_{yys}) + + 2(1-\nu)u_{xy} u_{xys}\} dxdv \quad .$$

Comparing this with the second integral of equation (19) and using equations (20)-(24), we have

$$\frac{d\bar{A}(s)}{ds} = -2D \int_0^c\int_0^b \bar{u}_s \nabla^4 \bar{u}\, dxdy \qquad \text{Q.E.D.}$$

Lemma 4.3. $\frac{1}{\bar{A}(s)} = \frac{M}{a}\int_0^c\int_0^b \bar{u}_s^2\, dxdy$.

Proof. $A(t) = 2T(t) = \frac{M}{a}\int_0^c\int_0^b u_t^2\, dxdy$ from Lemma 4.1

$$A(t(s)) = \frac{M}{a}\int_0^c\int_0^b \bar{u}_s^2 \left(\frac{ds}{dt}\right)^2 dxdy$$

$$\bar{A}(s) = \frac{M}{a}\bar{A}^2(s)\int_0^c\int_0^b \bar{u}_s^2\, dxdy \qquad \text{since}$$

$$\frac{ds}{dt} = A(t(s)) = \bar{A}(s)$$ by Definitions 4.3 and 4.4. Q.E.D.

Theorem 4.2. If i) $u(x,y,t)$ satisfies equations (15) and (18), and $\bar{u}(x,y,s) \equiv u(x,y,t(s))$

ii) $T(t)$ and $V(t)$ are given respectively by equations (16) and (17)

iii) the parameter s is defined by Definition 4.3

iv) $\bar{A}(s)$ is given by Definitions 4.2 and 4.4 ,

then

$$(25) \quad \frac{\bar{A}^2(s)\, M}{2a} \int_0^c\int_0^b \bar{u}_s^2 \, dxdy + \frac{D}{2}\int_0^c\int_0^b \left\{ \bar{u}_{xx}^2 + \bar{u}_{yy}^2 + 2\nu u_{xx}\, u_{yy} + + 2(1-\nu)\bar{u}_{xy}^2 \right\} dxdy = C .$$

Proof. $\frac{\bar{A}(s)}{2} = \frac{\bar{A}^2(s)M}{2a} \int_0^c\int_0^b \bar{u}_s^2 \, dxdy$ from Lemma 4.3.

$$\frac{\bar{A}^2(s)M}{2a} \int_0^c\int_0^b \bar{u}_s^2 \, dxdy = C - \frac{D}{2}\int_0^c\int_0^b \left\{ \bar{u}_{xx}^2 + \bar{u}_{yy}^2 + 2\nu \bar{u}_{xx}\, \bar{u}_{yy} + + 2(1-\nu)\bar{u}_{xy}^2 \right\} dxdy$$

from Definition 4.2. Q.E.D.

Theorem 4.3. If i) $u(x,y,t)$ satisfies equations (15) and (18) and $\in C^4$

ii) $T(t)+V(t) = C$, a constant; where $T(t)$ and $V(t)$ are given by equations (16) and (17)

iii) $\bar{A}(s)$ is given by Definitions 4.2, 4.3, and 4.4

iv) $\bar{u}(x,y,s) \equiv u(x,y,t(s))$

then $\bar{u}(x,y,s)$ satisfies the equation

$$(26) \quad \bar{u}_{ss} + \frac{1}{\bar{A}(s)} \left\{ -2D\bar{u}_s \int_0^c\int_0^b \bar{u}_s \nabla^4 \bar{u} \, dxdy + D \, \nabla^4 \bar{u} \int_0^c\int_0^b \bar{u}_s^2 \, dxdy \right\} = 0 .$$

Proof. $\frac{ds}{dt} = A(t)$, from Definition 4.3

$$\frac{d^2 s}{dt^2} = A(t)\,\frac{dA(t)}{ds} \quad , \qquad u_t = u_s\, A\big(t(s)\big) = u_s \bar{A}(s)$$

$$u_{tt} = u_{ss}\, \bar{A}^2(s) + u_s \frac{d\bar{A}(s)}{ds}\, \bar{A}(s) \ .$$

Substituting these relations as required in equation (15), we have

$\bar{u}_{ss}\, \bar{A}^2 + \bar{u}_s\, \bar{A}\, \dfrac{d\bar{A}}{ds} + \dfrac{aD}{M}\, \nabla^4 \bar{u} = 0$ which, upon using Lemmas 4.2 and 4.3, becomes

$$\bar{u}_{ss} - \frac{2D}{\bar{A}}\, \bar{u}_s \int_0^c\!\!\int_0^b \bar{u}_s\, \nabla^4 \bar{u}\, dxdy + \nabla^4 \bar{u}\, \frac{aD}{M\bar{A}}\, \frac{M}{a} \int_0^c\!\!\int_0^b \bar{u}_s^2\, dxdy = 0 \ .$$

<u>Theorem</u> <u>4.4</u>. If i) $\bar{u}(x,y,s)$ satisfies equations (26) and (25)

ii) $\bar{A}(s) > 0$ and satisfies Definition 4.2 with s a general parameter

iii) C is a positive constant

iv) $t \equiv \displaystyle\int_0^s \frac{d\lambda}{\bar{A}(\lambda)}$,

$u(x,y,t) \equiv \bar{u}\big(x,y,s(t)\big)$

v) $\bar{u}(x,y,s)$ satisfies the boundary conditions of equation (18) with t and u replaced by s and $\bar{u}$, respectively,

then $u(x,y,t)$ satisfies the partial differential equation

$u_{tt} + \dfrac{aD}{M}\, \nabla^4 u = 0$ with boundary conditions of equation (18) and also has the constant energy level C.

<u>Proof</u>. $t \equiv \int_0^s \frac{d\lambda}{\bar{A}(\lambda)}$ by hypothesis. $\frac{dt}{ds} = \frac{1}{\bar{A}(s)}$,

$$\bar{u}_s(x,y,s) = \bar{u}_t(x,y,s(t))\frac{1}{\bar{A}(s)} = u_t\frac{1}{\bar{A}} .$$ Similarly

$$\bar{u}_{ss} = u_{tt}\frac{1}{\bar{A}^2} - \frac{1}{\bar{A}^3}u_t\bar{A}_t .$$

Substituting these into equation (26), we have

$$(27) \qquad u_{tt}\frac{1}{\bar{A}^2} - \frac{1}{\bar{A}^3}u_t\bar{A}_t + \frac{1}{\bar{A}^3}\left\{-2Du_t\int_0^c\int_0^b \bar{u}_t\nabla^4\bar{u}\,dxdy + D\nabla^4 u\int_0^c\int_0^b u_t^2\,dxdy\right\} = 0 .$$

But $\bar{A}(s) = 2\left\{C - \frac{D}{2}\int_0^c\int_0^b\left(\bar{u}_{xx}^2 + \bar{u}_{yy}^2 + 2\nu\bar{u}_{xx}\bar{u}_{yy} + 2(1-\nu)\bar{u}_{xy}^2\right)dxdy\right\}$

$$\frac{d\bar{A}(s(t))}{dt} = -2D\int_0^c\int_0^b\Big(\bar{u}_{xx}\bar{u}_{xxt} + \bar{u}_{yy}\bar{u}_{yyt} + \nu(\bar{u}_{xxt}\bar{u}_{yy} + \bar{u}_{xx}\bar{u}_{yyt}) + 2(1-\nu)\bar{u}_{xy}\bar{u}_{xyt}\Big)dxdy$$

$$(28) \qquad = -2D\int_0^c\int_0^b \bar{u}_t\nabla^4\bar{u}\,dxdy ,$$ where we have repeated the integration by parts of equations (20)-(24) using the given boundary conditions on $\bar{u}$ rather than u. Substituting equation (28) into equation (27), we have

$$(29) \qquad u_{tt} + \frac{2D}{\bar{A}}u_t\int_0^c\int_0^b \bar{u}_t\nabla^4\bar{u}\,dxdy - \frac{2Du_t}{\bar{A}}\int_0^c\int_0^b \bar{u}_t\nabla^4\bar{u}\,dxdy + \frac{D}{\bar{A}}\nabla^4 u\int_0^c\int_0^b u_t^2\,dxdy = 0 ;$$

and since from Lemma 4.3

$$\frac{1}{\bar{A}}\int_0^c\int_0^b u_t^2\,dxdy = \bar{A}(s)\int_0^c\int_0^b \bar{u}_s^2\,dxdy = \frac{a}{M},$$

equation (29) becomes $\quad u_{tt} + \frac{aD}{M}\nabla^4 u = 0\,,\qquad$ Q.E.D.

5. <u>An infinite dimensional generalized Riemannian space</u>. We now consider a function space that can be associated with the vibrations of the beam considered in Part I or the flat plate of paragraph 4. We shall develop it using the results of paragraph 4. The corresponding development for beams can be considered as a special case where $u_y \equiv 0$ and the appropriate boundary conditions are used.

First, we need an expression for arc length in the proposed space.

$$1 = \frac{M}{a}\bar{A}(s)\int_0^c\int_0^b \bar{u}_s^2\,dxdy \qquad \text{from Lemma 4.3,}$$

or $$\left(\frac{ds(\lambda)}{d\lambda}\right)^2 = \frac{M}{a}\bar{A}(s)\int_0^c\int_0^b \bar{u}_\lambda^2\big(x,y,s(\lambda)\big)\,dxdy\,, \qquad \text{or}$$

$$(30)\qquad \left(\frac{ds(\lambda)}{d\lambda}\right)^2 = \frac{2M}{a}\left\{C - \frac{D}{2}\int_0^c\int_0^b\left(\bar{u}_{xx}^2 + \bar{u}_{yy}^2 + 2\nu\bar{u}_{xx}\bar{u}_{yy} + 2(1-\nu)\bar{u}_{xy}^2\right)dxdy\right\}\int_0^c\int_0^b \bar{u}_\lambda^2\big(x,y,s(\lambda)\big)\,dxdy$$

<u>Definition</u> <u>5.1</u>. $v(x,y,\lambda) \equiv u\big(x,y,s(\lambda)\big)$.

<u>Lemma</u> <u>5.1</u>.

$$(31)\qquad s = \int_{\lambda_0}^{\lambda}\left\{\frac{2M}{a}\left[C - \frac{D}{2}\int_0^c\int_0^b\left(v_{xx}^2 + v_{yy} + 2\nu v_{xx}v_{yy} + \right.\right.\right.$$

$$+ 2(1-\nu)v_{xy}^2\Big)dxdy\Big]\Big\{\int_0^c\int_0^b v_\mu^2(x,y,\mu)dxdy\Big\}^{1/2} d\mu \ .$$

<u>Proof</u>. Clear from equation (30).

<u>Definition 5.2</u>. If i) $v(x,y)$, $\delta v(x,y)$ are arbitrary functions $\in C^4$

ii) v, δv, v_{xx}, v_{yy}, $(\delta v)_{xx}$, and $(\delta v)_{yy}$ all vanish at the boundaries, $x = 0$ or b, $y = 0$ or c

iii) $C - \frac{D}{2}\int_0^c\int_0^b\Big(v_{xx}^2 + v_{yy}^2 + 2\nu v_{xx}v_{yy} + 2(1-\nu)v_{xy}^2\Big)dxdy > 0$,

then the differential of arc length of the infinite dimensional generalized Riemannian space is given by the positive definite functional differential form

$$ds^2 = \frac{2M}{a}\Big\{C - \frac{D}{2}\int_0^c\int_0^b\Big(v_{xx}^2 + v_{yy}^2 + 2\nu v_{xx}v_{yy} + 2(1-\nu)v_{xy}^2\Big)dxdy\Big\}\int_0^c\int_0^b\Big(\delta v(x_1,y_1)\Big)^2 dx_1 dy_1 \tag{32}$$

with function coordinates $v(x,y)$.

We consider the geodesics of this space. They are defined in the usual way, i.e., they are the curves for which s of equation (31) have a stationary value. To facilitate the rather long computations the following notation is used.

<u>Definition 5.3</u>. $$\Big[v_1 , v_2\Big] \equiv \int_0^c\int_0^b v_1(x,y)\, v_2(x,y)dxdy \ . \tag{33}$$

<u>Definition 5.4</u>. $$g(v,\xi) = \frac{2M}{a}\Big\{C - \frac{D}{2}\Big(\Big[v_{x_1x_1} , v_{x_1x_1}\Big] +$$

$$+ \left[v_{y_1 y_1}, v_{y_1 y_1} \right] + 2\nu \left[v_{x_1 x_1}, v_{y_1 y_1} \right] +$$

$$+ 2(1-\nu) \left[v_{x_1 y_1}, v_{x_1 y_1} \right] \Big) \Big\} \ \zeta(x_2, y_2). \qquad (34)$$

Lemma 5.2. $ds^2 = \left[\delta v, g(v, \delta v) \right]$

Proof. Compare Definitions 5.2, 5.3, and equation (32).

The geodesics of this space (9,552-559) are given by the generalized calculus of variations problem

$$(35) \qquad \int_{t_0}^{t_1} \left\{ \left[\frac{dv}{dt}, g\left(v, \frac{dv}{dt}\right) \right] \right\}^{1/2} dt = \min.^{\dagger}$$

and satisfy the generalized Euler-Lagrange equation

$$(36) \qquad \frac{d^2 v}{ds^2} + \Gamma\left(v, \frac{dv}{ds}, \frac{dv}{ds}\right) = 0. \quad \text{Here}$$

Definition 5.5. $s \equiv \int_{t_0}^{t} \left\{ \left[\frac{dv}{dt}, g\left(v, \frac{dv}{dt}\right) \right] \right\}^{1/2} dt.$

Definition 5.6. $G(v, \eta) \equiv$ expression for ζ when $g(v, \zeta) = \eta$ is solved for ζ.

Definition 5.7. $\Gamma(v, \zeta_1, \zeta_2) \equiv G\left(v, \gamma(v, \zeta_1, \zeta_2)\right)$.

Definition 5.8. $\gamma(v, \zeta_1, \zeta_2) \equiv \frac{1}{2} \Big\{ g(v, \zeta_1 ; \zeta_2) +$

$$+ g(v, \zeta_2 ; \zeta_1) - g^{*}_{(3)}(v, \zeta_1 ; \zeta_2) \Big\}.$$

where $g(v, \zeta_1 ; \zeta_2)$ is the Fréchet differential of $g(v, \zeta_1)$ with increment ζ_2.

[†] We use $\frac{dv}{dt}$ $\left(\text{or } \frac{dv}{ds}\right)$ for the derivative with respect to the parameter t (or s) despite the fact that v is also a function of x and y.

<u>Definition</u> <u>5.9</u>. $g^*_{(3)}(v,\zeta;\eta_2)$ is the adjoint of $g(v,\zeta;\eta_1)$ defined by $\left[g(v,\zeta;\eta_1),\eta_2\right] = \left[\eta_1, g^*_{(3)}(v,\zeta;\eta_2)\right]$.

<u>Lemma</u> <u>5.3</u>. $g(v,\zeta;\eta) = \frac{-2MD}{a}\left[\nabla_1^4 v, \eta\right]\zeta(x_2)$

where
$$\nabla_1^4 v \equiv \frac{\partial^4 v}{\partial x_1^4} + 2\frac{\partial^4 v}{\partial x_1^2\partial y_1^2} + \frac{\partial^4 v}{\partial y_1^4}.$$

<u>Proof</u>. We compute $g(v,\zeta;\eta)$ as a Gâteaux limit.

$$g(v,\zeta;\eta) = \lim_{\lambda\to 0}\frac{g(v+\lambda\eta,\zeta) - g(v,\zeta)}{\lambda}$$

$$= \lim_{\lambda\to 0}\frac{1}{\lambda}\Bigg\{\frac{2M}{a}\Bigg(C - \frac{D}{2}\left[v_{x_1x_1}+\lambda\eta_{x_1x_1}, v_{x_1x_1}+\lambda\eta_{x_1x_1}\right]$$
$$-\frac{D}{2}\left[v_{y_1y_1}+\lambda\eta_{y_1y_1}, v_{y_1y_1}+\lambda\eta_{y_1y_1}\right] - D\nu\left[v_{x_1x_1}+\lambda\eta_{x_1x_1}, v_{y_1y_1}+\lambda\eta_{y_1y_1}\right]$$
$$-D(1-\nu)\left[v_{x_1y_1}+\lambda\eta_{x_1y_1}, v_{x_1y_1}+\lambda\eta_{x_1y_1}\right]\Bigg)$$
$$-\frac{2M}{a}\Bigg(C - \frac{D}{2}\left[v_{x_1x_1}, v_{x_1x_1}\right] + \left[v_{y_1y_1}, v_{y_1y_1}\right] - D\nu\left[v_{x_1x_1}, v_{y_1y_1}\right]$$
$$-D(1-\nu)\left[v_{x_1y_1}, v_{x_1y_1}\right]\Bigg)\Bigg\}\zeta(x_2)$$

$$= \lim_{\lambda\to 0}\frac{1}{\lambda}\Bigg\{-\frac{MD}{a}\Bigg(\left[2\lambda v_{x_1x_1}, \eta_{x_1x_1}\right] + \left[2\lambda v_{y_1y_1}, \eta_{y_1y_1}\right] +$$
$$+2\nu\left[\lambda v_{x_1x_1}, \eta_{y_1y_1}\right] + 2\nu\left[\lambda v_{y_1y_1}, \eta_{x_1x_1}\right] +$$
$$+2(1-\nu)\left[2\lambda v_{x_1y_1}, \eta_{x_1y_1}\right]\Bigg)\Bigg\}\zeta(x_2) + O(\lambda)$$

$$g(v,\zeta;\eta) = \frac{-2MD}{a}\Big\{\left[v_{x_1x_1}, \eta_{x_1x_1}\right] + \left[v_{y_1y_1}, \eta_{y_1y_1}\right] + \nu\left[v_{x_1x_1}, \eta_{y_1y_1}\right] +$$
$$+\nu\left[v_{y_1y_1}, \eta_{x_1x_1}\right] + 2(1-\nu)\left[v_{x_1y_1}, \eta_{x_1y_1}\right]\Big\}\zeta(x_2) \qquad (37)$$

Upon checking Definition 5.3 and the boundary conditions of Definition 5.2, we note that the portion between the braces of equation (37) is identical with the second integral of equation (19) if the identifications $u \to v$, $u_t \to \eta$ are made. Also the boundary conditions used for u in equations (20)-(24) are the same as the present ones on v, hence we can use these computations to rewrite equation (37) as

$$g(v,\xi;\eta) = \frac{-2MD}{a}\left[\nabla_1^4 v, \eta\right] \xi(x_2) \qquad \text{Q.E.D.}$$

Lemma 5.4. $g^*_{(3)}(v,\xi;\eta_2) = \frac{-2MD}{a} \nabla_1^4 v(x_1,y_1) \left[\xi, \eta_2\right].$

Proof. $\left[g(v,\xi;\eta_1), \eta_2\right] = \left[\eta_1, g^*_{(3)}(v,\xi;\eta_2)\right]$ from Definition 5.9.

$$g(v,\xi;\eta_1), \eta_2 = \frac{-2MD}{a}\left[\nabla_1^4 v(x_1,y_1), \eta_1(x_1,y_1)\right] \cdot \left[\xi(x_2,y_2), \eta_2(x_2,y_2)\right]$$

$$= \left[\eta_1, \frac{-2MD}{a}\nabla_1^4 v\left[\xi(x_2,y_2), \eta_2(x_2,y_2)\right]\right].$$

Therefore $g^*_{(3)}(v,\xi;\eta_2) = \frac{-2MD}{a}\nabla_1^4 v(x_1,y_1)\left[\xi(x_2,y_2), \eta_2(x_2,y_2)\right]$, Q.E.D.

Lemma 5.5.

$$\gamma(v,\xi_1,\xi_2) = \frac{-MD}{a}\left\{\left[\nabla_1^4 v, \xi_2\right]\xi_1 + \left[\nabla_1^4 v, \xi_1\right]\xi_2 - \left[\xi_1, \xi_2\right]\nabla_1^4 v\right\}.$$

Proof. $\gamma(v,\xi_1,\xi_2) = \frac{1}{2}\left\{g(v,\xi_1;\xi_2) + g(v,\xi_2;\xi_1) - g^*_{(3)}(v,\xi_1;\xi_2)\right\}$ from Definition 5.8. Using Lemmas 5.3 and 5.4 we have

$$\Upsilon(v, \xi_1, \xi_2) = \tfrac{1}{2}\left\{ \frac{-2MD}{a}\left[\nabla_1^{*} v, \xi_2\right]\xi_1 - \frac{2MD}{a}\left[\nabla_1^{*} v, \xi_1\right]\xi_2 + \right.$$

$$\left. + \frac{2MD}{a}\nabla_1^{*} v\left[\xi_1, \xi_2\right]\right\}. \qquad \text{Q.E.D.}$$

<u>Definition</u> <u>5.10</u>.
$$\frac{1}{P[v]} \equiv \frac{2M}{a}\left\{ C - \frac{D}{2}\left(\left[v_{x_1x_1}, v_{x_1x_1}\right] + \left[v_{y_1y_1}, v_{y_1y_1}\right] + \right.\right.$$
$$\left.\left. + 2\nu\left[v_{x_1x_1}, v_{y_1y_1}\right] + 2(1-\nu)\left[v_{x_1y_1}, v_{x_1y_1}\right]\right)\right\}.$$

<u>Lemma</u> <u>5.6</u>. If $\bar{A}(s)$ is defined by Definition 4.2 with u replaced by v, then

$$\frac{a}{M}\,\frac{1}{P[v]} = \bar{A}(s)\ .$$

<u>Proof</u>. Clear, upon checking Definition 4.2.

<u>Lemma</u> <u>5.7</u>.
$$\Gamma(v, \xi, \eta_2) = \frac{-MD}{a} P[v]\left\{\left[\nabla_1^{*} v, \xi_2\right]\xi_1 + \left[\nabla_1^{*} v, \xi_1\right]\xi_2 - \left[\xi_1, \xi_2\right]\nabla_1^{*} v\right\}.$$

Using Definition 5.4 we set

$$\eta = g(v, \xi) = \frac{2M}{a}\left\{ C - \frac{D}{2}\left(\left[v_{x_1x_1}, v_{x_1x_1}\right] + \left[v_{y_1y_1}, v_{y_1y_1}\right] + \right.\right.$$
$$\left.\left. + 2\nu\left[v_{x_1x_1}, v_{y_1y_1}\right] + 2(1-\nu)\left[v_{x_1y_1}, v_{x_1y_1}\right]\right)\right\}\xi(x_2)\ .$$

From Definition 5.6 and Definition 5.10, we have

$$\xi(x_2) = G(v, \eta) = \eta\, P[v]\ .$$

Using Definition 5.7,

$$\Gamma(v, \zeta_1, \eta_2) = G(v, \gamma(v, \zeta_1, \zeta_2))$$
$$= P[v]\,\gamma(v, \zeta_1, \zeta_2) \quad \text{or, from Lemma 5.5}$$

$$\Gamma(v, \zeta_1, \eta_2) = \frac{-MD}{a} P[v] \Big\{ \big[\nabla_1^4 v, \zeta_2\big] \zeta_1 +$$
$$+ \big[\nabla_1^4 v, \zeta_1\big] \zeta_2 - \big[\zeta_1, \zeta_2\big] \nabla_1^4 v \Big\}. \qquad \text{Q.E.D.}$$

Theorem 5.1. The solutions of the generalized geodesic equation $\dfrac{d^2 v(s)}{ds^2} + \Gamma\left(v, \dfrac{dv}{ds}, \dfrac{dv}{ds}\right) = 0$ satisfy the integro-differential equation

(38) $$v_{ss} + \frac{1}{\bar{A}(s)} \Big\{ -2D \big[\nabla_1^4 v, v_s\big] v_s + D \nabla^4 v \big[v_s, v_s\big] \Big\} = 0 ,$$

where $\bar{A}(s)$ is given by Definition 4.2 with u replaced by v and t replaced by s (cf. Theorem 4.3).

Proof. $$0 = v_{ss} + \Gamma(v, v_s, v_s) = v_{ss} - \frac{MD}{a} P[v] \Big\{ \big[\nabla_1^4 v, v_s\big] v_s +$$
$$+ \big[\nabla_1^4 v, v_s\big] v_s - \big[v_s, v_s\big] \nabla_1^4 v \Big\}$$

from Lemma 5.7. But using Lemma 5.6, this becomes

$$v_{ss} + \frac{1}{\bar{A}(s)} \Big\{ -2D \big[\nabla_1^4 v, v_s\big] v_s + D \nabla^4 v \big[v_s, v_s\big] \Big\} = 0. \qquad \text{Q.E.D.}$$

We note that equation (38) is identical with equation (26) of Theorem 4.3 if $\bar{u}$ is replaced by v. Hence we see that the geodesics v of this generalized "Riemannian" space satisfy the integro-differential equation (26) if $\bar{u}(x,y,s) = v(x,y,s)$ where s is the arc length defined by Definition 5.5.

6. Curvature of the generalized Riemannian space. We now desire to compute the generalized Riemannian curvature of our function space. First we compute the generalized Riemann-Christoffel curvature form (9,554) given by

Definition 6.1.
$$
\begin{aligned}
R(v, \xi_1, \xi_2, \xi_3) \equiv \tfrac{1}{2}\Big\{ & g(v, \xi_2 ; \xi_1 ; \xi_3) \\
& + g^{*}_{(3)}(v, \xi_1 ; \xi_3 ; \xi_2) - g^{*}_{(3)}(v, \xi_1 ; \xi_2 ; \xi_3) \\
& - g(v, \xi_3 ; \xi_1 ; \xi_2)\Big\} + \\
& + \gamma^{*}_{(2)}\Big(v,\ G\big(v, \gamma(v, \xi_1, \xi_3)\big), \xi_2\Big) \\
& - \gamma^{*}_{(2)}\Big(v,\ G\ v, \gamma(v, \xi_1, \xi_2)\big), \xi_3\Big) .
\end{aligned}
$$

Definition 6.2. $g(v, \xi_2 ; \xi_1 ; \xi_3)$ is the Fréchet differential of $g(v, \xi_2 ; \xi_1)$ with increment ξ_3.

Definition 6.3. The adjoint $g^{*}_{(3)}(v, \xi ; \zeta ; \delta)$ is defined implicitly by

$$\big[\zeta,\ g(v, \xi ; \eta ; \delta)\big] = \big[\eta,\ g^{*}_{(3)}(v, \xi ; \zeta ; \delta)\big] .$$

Definition 6.4. The adjoint $\gamma^{*}_{(2)}(v, G, \xi_2)$ of $\gamma_{(2)}(v, \zeta, \xi_2)$ is defined implicitly by

$$\big[G,\ \gamma(v, \zeta, \xi_2)\big] = \big[\zeta,\ \gamma^{*}_{(2)}(v, G, \xi_2)\big] .$$

Lemma 6.1.
$$
\begin{aligned}
& \big[\xi_1, g(v, \xi_2)\big]\, g(v, \xi_3) - \big[\xi_1, g(v, \xi_3)\big]\, g(v, \xi_2) \\
& = \frac{1}{P^2[v]} \Big\{ \big[\xi_1, \xi_2\big]\xi_3 - \big[\xi_1, \xi_3\big]\xi_2 \Big\}
\end{aligned}
$$

where $P[v]$ is defined by Definition 5.10.

Proof. Using Definitions 5.4 and 5.10, we have

$$\left[\zeta_1, g(v,\zeta_2)\right] g(v,\zeta_3) - \left[\zeta_1, g(v,\zeta_3)\right] g(v,\zeta_2)$$

$$= \left[\zeta_1, \frac{1}{P[v]} \zeta_2(x_2,y_2)\right] \frac{\zeta_3(x,y)}{P[v]}$$

$$- \left[\zeta_1, \frac{1}{P[v]} \zeta_3(x_3,y_3)\right] \frac{1}{P[v]} \zeta_2(x,y)$$

$$= \frac{1}{P^2[v]} \left\{ \left[\zeta_1, \zeta_2\right]\zeta_3 - \left[\zeta_1, \zeta_3\right]\zeta_2 \right\}, \qquad \text{Q.E.D.}$$

We know that a necessary and sufficient condition that the space is one of constant Riemannian curvature (9,557) is that

$$(39) \qquad R(v,\zeta_1,\zeta_2,\zeta_3) = K \left\{ \left[\zeta_1, g(v,\zeta_2)\right] g(v,\zeta_3) - \left[\zeta_1, g(v,\zeta_3)\right] g(v,\zeta_2) \right\}$$

where K is the Riemannian curvature and is a constant.

<u>Lemma</u> <u>6.2</u>. $g(v,\zeta;\eta;\delta) = \frac{-2MD}{a} \left[\nabla_1^4 \delta, \eta\right] \zeta(x_2)$.

<u>Proof</u>. $g(v,\zeta;\eta;\delta) = \lim\limits_{\lambda\to 0} \frac{g(v+\lambda\delta,\zeta;\eta) - g(v,\zeta;\eta)}{\lambda}$,

or, using Lemma 5.3,

$$g(v,\zeta;\eta;\delta) = \lim_{\lambda\to 0} \frac{-2DM}{a} \frac{\left(\left[\nabla_1^4(v+\lambda\delta),\eta\right] - \left[\nabla_1^4 v,\eta\right]\right)\zeta(x_2)}{\lambda}$$

$$= \lim_{\lambda\to 0} \frac{-2MD}{a} \left(\frac{\left[\nabla_1^4 v,\eta\right] + \lambda\left[\nabla_1^4\delta,\eta\right] - \left[\nabla_1^4 v,\eta\right]}{\lambda}\right) \zeta(x_2)$$

$$= \frac{-2MD}{a} \left[\nabla_1^4\delta,\eta\right] \zeta(x_2) . \qquad \text{Q.E.D.}$$

<u>Lemma</u> <u>6.3</u>. $\underset{(3)}{g}^{*}(v,\zeta;\zeta;\delta) = \frac{-2MD}{a} \left[\zeta,\zeta\right] \nabla_1^4 \delta$.

Proof. $\left[\zeta, g(v,\xi;\eta;\delta)\right] = \left[\eta, g^{*}_{(3)}(v,\xi;\zeta;\delta)\right]$
from Definition 6.3.

$$\left[\zeta, g(v,\xi;\eta;\delta)\right] = \frac{-2MD}{a}\left[\nabla_1^{4}\delta, \eta\right]\left[\zeta, \xi\right]$$

$$\left[\eta, \frac{-2MD}{a}\left[\zeta, \xi\right]\nabla_1^{4}\delta\right]$$

$$\therefore \quad g^{*}_{(3)}(v,\xi;\zeta;\delta) = \frac{-2MD}{a}\left[\zeta, \xi\right]\nabla_1^{4}\delta \ . \qquad \text{Q.E.D.}$$

Lemma 6.4.

$$\gamma^{*}_{(2)}(v,G,\xi_2) = \frac{-DM}{a}\left\{\left[\nabla^{4}v, \xi_2\right]G + \left[G,\xi_2\right]\nabla_1^{4}v - \left[G,\nabla_1^{4}v\right]\xi_2\right\}.$$

Proof. $\left[G, \gamma(v,\zeta,\xi_2)\right] = \left[\zeta, \gamma^{*}_{(2)}(v,G,\xi_2)\right]$

by Definition 6.4. Hence on using Lemma 5.5 we have

$$\left[G, \gamma(v,\zeta,\xi_2)\right] = \frac{-DM}{a}\left\{\left[\nabla_1^{4}v,\xi_2\right]\left[G,\zeta\right] + \left[\nabla_1^{4}v,\zeta\right]\left[G,\xi_2\right] - \left[\zeta,\xi_2\right]\left[G,\nabla_1^{4}v\right]\right\}$$

$$= \left[\zeta, \frac{-DM}{a}\left(\left[\nabla_1^{4}v,\xi_2\right]G + \left[G,\xi_2\right]\nabla_1^{4}v - \left[G,\nabla_1^{4}v\right]\xi_2\right)\right] \ .$$

Hence,

$$\gamma^{*}_{(2)}(v,G,\xi_2) = \frac{-DM}{a}\left\{\left[\nabla_1^{4}v,\xi_2\right]G + \left[G,\xi_2\right]\nabla_1^{4}v - \left[G,\nabla_1^{4}v\right]\xi_2\right\}.$$

Lemma 6.5.

(40) $\quad \gamma^{*}_{(2)}\Big(v, G\big(v,\gamma(v,\xi_1,\xi_3)\big), \xi_2\Big) - \gamma^{*}_{(2)}\Big(v, G\big(v,\gamma(v,\xi_1,\xi_2)\big), \xi_3\Big)$

$$= P[v]\,\frac{D^2M^2}{a^2}\Big\{\, 3\left[\nabla_1^{*}v,\xi_1\right]\left[\nabla_1^{*}v,\xi_2\right]\xi_3$$

$$-3\left[\nabla_1^{*}v,\xi_2\right]\left[\xi_1,\xi_3\right]\nabla_1^{*}v+$$

$$+\left[\xi_1,\xi_2\right]\left[\nabla_1^{*}v,\xi_3\right]\nabla_1^{*}v$$

$$-3\left[\nabla_1^{*}v,\xi_1\right]\left[\nabla_1^{*}v,\xi_3\right]\xi_2+\left[\nabla_1^{*}v,\nabla_1^{*}v\right]\cdot$$

$$\cdot\left[\xi_1,\xi_3\right]\xi_2-\left[\nabla_1^{*}v,\nabla_1^{*}v\right]\left[\xi_1,\xi_2\right]\xi_3\Big\}.$$

<u>Proof.</u> Using Lemma 6.4, Definition 5.7, and Lemma 5.7, we have

$$(41)\qquad \gamma^{*}_{(2)}\Big(v,G\big(v,\gamma(v,\xi_1,\xi_3)\big),\xi_2\Big)=\frac{-DM}{a}\left(\frac{-MD}{a}\right)P[v]\,\Big\{\left[\nabla_1^{*}v,\xi_2\right]\cdot$$

$$\cdot\Big(\left[\nabla_1^{*}v,\xi_3\right]\xi_1+\left[\nabla_1^{*}v,\xi_1\right]\xi_3-\left[\xi_1,\xi_3\right]\nabla_1^{*}v\Big)$$

$$+\nabla_1^{*}v\Big(\left[\nabla_1 v,\xi_3\right]\left[\xi_2,\xi_1\right]+\left[\nabla_1^{*}v,\xi_1\right]\cdot$$

$$\cdot\left[\xi_2,\xi_3\right]-\left[\xi_1,\xi_3\right]\left[\xi_2,\nabla_1^{*}v\right]\Big)$$

$$-\xi_2\Big(\left[\nabla_1^{*}v,\xi_3\right]\left[\nabla_1^{*}v,\xi_1\right]+\left[\nabla_1^{*}v,\xi_1\right]\cdot$$

$$\left[\nabla_1^{*}v,\xi_3\right]-\left[\xi_1,\xi_3\right]\left[\nabla_1^{*}v,\nabla_1^{*}v\right]\Big\}\quad .$$

Similarly,

$$(42)\qquad -\gamma^{*}_{(2)}\Big(v,G\big(v,\gamma(v,\xi_1,\xi_2)\big),\xi_3\Big)=\left(\frac{DM}{a}\right)^2P[v]\,\Big\{-\left[\nabla_1^{*}v,\xi_3\right]\cdot$$

$$\cdot\Big(\left[\nabla_1^{*}v,\xi_2\right]\xi_1+\left[\nabla_1^{*}v,\xi_1\right]\xi_2+\left[\xi_1,\xi_2\right]\nabla_1^{*}v\Big)$$

$$-\nabla_1^{*}v\Big(\left[\nabla_1^{*}v,\xi_2\right]\left[\xi_3,\xi_1\right]+\left[\nabla_1^{*}v,\xi_1\right]\cdot$$

$$\cdot\left[\xi_3,\xi_2\right]-\left[\xi_1,\xi_2\right]\left[\xi_3,\nabla_1^{*}v\right]\Big)$$

$$+\xi_3\Big(\left[\nabla_1^{*}v,\xi_2\right]\left[\nabla_1^{*}v,\xi_1\right]+\left[\nabla_1^{*}v,\xi_1\right]\cdot$$

$$\cdot\left[\xi_3,\xi_2\right]-\left[\xi_1,\xi_2\right]\left[\xi_3,\nabla_1^{*}v\right]\Big)\Big\}\,.$$

Adding equations (41) and (42) we get the required result.

Lemma 6.6.

$$(43) \quad R(v, \xi_1, \xi_2, \xi_3) = \frac{-MD}{a} \Big\{ \left[\nabla_1^* \xi_3, \xi_1\right] \xi_2 + \left[\xi_3, \xi_1\right] \nabla_1^* \xi_2 + \left[\xi_2, \xi_1\right] \nabla^* \xi_3 + \left[\nabla_1^* \xi_2, \xi_1\right] \xi_3 \Big\}$$
$$+ \left(\frac{DM}{a}\right)^2 P[v] \Big\{ 3\left[\nabla_1^* v, \xi_1\right] \left[\nabla_1^* v, \xi_2\right] \xi_3 - 3\left[\nabla_1^* v, \xi_2\right] \left[\xi_1, \xi_3\right] \nabla_1^* v - 3\left[\nabla_1^* v, \xi_1\right] \left[\nabla_1^* v, \xi_3\right] \xi_2$$
$$+ \left[\xi_1, \xi_2\right] \left[\nabla_1^* v, \xi_3\right] \nabla_1^* v + \left[\xi_1, \xi_3\right] \left[\nabla_1^* v, \nabla_1^* v\right] \xi_2 - \left[\nabla_1^* v, \nabla_1^* v\right] \left[\xi_1, \xi_2\right] \xi_3 \Big\}.$$

Proof. First we consider the first four terms of $R(v, \xi_1, \xi_2, \xi_3)$ as given by Definition 6.1. Using Lemmas 6.2 and 6.3, they become

$$(44) \quad \tfrac{1}{2} \Big\{ g(v, \xi_2 ; \xi_1 ; \xi_3) + g^*_{(3)}(v, \xi_1 ; \xi_3 ; \xi_2) - g^*_{(3)}(v, \xi_1 ; \xi_2 ; \xi_3) - g(v, \xi_3 ; \xi_1 ; \xi_2) \Big\}$$
$$= \frac{-MD}{a} \left[\nabla_1^* \xi_3, \xi_1\right] \xi_2 - \frac{MD}{a} \left[\xi_3, \xi_1\right] \nabla_1^* \xi_2 +$$
$$+ \frac{MD}{a} \left[\xi_2, \xi_1\right] \nabla_1^* \xi_3 + \frac{MD}{a} \left[\nabla_1^* \xi_2, \xi_1\right] \xi_3 .$$

Since the right hand side of equation (40) is the last two terms of $R(v, \xi_1, \xi_2, \xi_3)$ as given by Definition 6.1, we get the required result by adding equations (40) and (44). Q.E.D.

Theorem 6.1. The infinite dimensional function space determined by ds^2 of Lemma 5.2 (or equation (32)) is not one of constant Riemannian curvature.

Proof. Comparing Lemmas 6.1 and 6.6 we see that

$$R(v, \xi_1, \xi_2, \xi_3) \neq K\left\{ \left[\xi_1, g(v, \xi_2)\right] g(v, \xi_3) - \left[\xi_1, g(v, \xi_3)\right] g(v, \xi_2) \right\}$$

where K is a constant, as required by equation (39) for a space of constant Riemannian curvature. Q.E.D.

7. An example. For an example we consider the geodesics for the function space of the flat rectangular plate of paragraph 4 with $u(x,y,t)$ assumed in the form (for fixed m,n)

(45) $$u_{mn} = B_{mn} \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c} \sin pt \,.$$ This form of u clearly satisfies the boundary conditions of equation (18). Equation (15) becomes

$$u_{tt} + \frac{aD}{M}\left(u_{xxxx} + 2u_{xxyy} + u_{yyyy}\right)$$

$$= - B_{mn} p^2 \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c} \sin pt \; +$$

$$+ \frac{AD}{M} \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c} \sin pt \; B_{mn} \, \cdot$$

$$\cdot \left\{ \left(\frac{m\pi}{b}\right)^4 + 2\left(\frac{m\pi}{b}\right)^2 \left(\frac{n\pi}{c}\right)^2 + \left(\frac{n\pi}{c}\right) \right\} = 0 \,.$$

Therefore $$p^2 = \frac{AD}{M}\left[\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2\right]$$ or

$$p = \pi^2\left(\frac{aD}{M}\right)^{1/2}\left(\frac{m^2}{b^2}+\frac{n^2}{c^2}\right) . \tag{46}$$

We desire to compute the constant $C = C_{mn} = T_{mn}(t) + V_{mn}(t)$ which represents the total energy level for a fixed m and n. From equation (16),

$$T_{mn}(t) = \frac{M}{2a}\int_0^c\int_0^b u_t^2\,dxdy \tag{47}$$

$$= \frac{B_{mn}^2 M}{2a}\int_0^c\int_0^b p^2\cos^2 pt\,\sin^2\frac{m\pi x}{b}\sin^2\frac{n\pi y}{c}\,dxdy$$

$$= B_{mn}^2\frac{Mp^2}{2a}\cos^2 pt\,\frac{bc}{4} = B_{mn}^2\frac{Mp^2 bc}{8a}\cos^2 pt .$$

From equation (17),

$$V_{mn}(t) = \frac{B_{mn}^2 D}{2}\int_0^c\int_0^b\left[u_{xx}^2 + u_{yy}^2 + 2\nu u_{xx}u_{yy} + 2(1-\nu)u_{xy}^2\right]dxdy$$

$$= \frac{B_{mn}^2 D}{2}\int_0^c\int_0^b\Big[\frac{m^4\pi^4}{b^4}\sin^2\frac{m\pi x}{b}\sin^2\frac{n\pi y}{c}\sin^2 pt +$$

$$+\frac{n^4\pi^4}{c^4}\sin^2\frac{m\pi x}{b}\sin^2\frac{n\pi y}{c}\sin^2 pt +$$

$$+ 2\nu\frac{m^2\pi^2}{b^2}\frac{n^2\pi^2}{c^2}\sin^2\frac{m\pi x}{b}\sin^2\frac{n\pi y}{c}\sin^2 pt$$

$$+ 2(1-\nu)\frac{m^2\pi^2}{b^2}\frac{n^2\pi^2}{c^2}\cos^2\frac{m\pi x}{b}\cos^2\frac{n\pi y}{c}\cdot$$

$$\cdot\sin^2 pt\Big]\,dxdy$$

$$= \frac{B_{mn}^2\pi^4 D}{2}\left(\frac{m^4}{b^4}+\frac{n^4}{c^4}\right)\sin^2 pt\int_0^c\int_0^b\sin^2\frac{m\pi x}{b}\cdot$$

$$\cdot\sin^2\frac{n\pi y}{c}\,dxdy + B_{mn}^2\pi^4 D\frac{m^2n^2}{b^2c^2}\sin^2 pt\cdot$$

$$\cdot \int_0^c \int_0^b \cos^2 \frac{m\pi x}{b} \cos^2 \frac{n\pi y}{c}\, dxdy$$

$$V_{mn} = B_{mn}^2 \frac{D\pi^4}{2} (\sin^2 pt) \frac{bc}{4} \left(\frac{m^4}{b^4} + \frac{n^4}{c^4} + \frac{2m^2n^2}{b^2c^2} \right)$$

$$(48) \quad = \frac{D\pi^4 bc \sin^2 pt}{8} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^2 B_{mn}^2 \ .$$

<u>Lemma 7.1.</u> $T_{mn}(t) + V_{mn}(t) = C_{mn} = B_{mn}^2 \frac{D\pi^4 bc}{8} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^2$.

<u>Proof.</u> Adding equations (47) and (48), and after substituting for p^2 from equation (46), we have

$$T_{mn}(t) + V_{mn}(t) = B_{mn}^2 M \frac{\left[\pi^4 \frac{aD}{M} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^2 \right]}{8a} bc \cos^2 pt$$

$$+ B_{mn}^2 \frac{D\pi^4 bc}{8} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^2 \sin^2 pt$$

$$= B_{mn}^2 \frac{D\pi^4 bc}{8} \left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^2 = C_{mn}$$

which is a constant for fixed m and n. Q.E.D.

We note that if B_{mn}^2 is chosen as $\left(\frac{m^2}{b^2} + \frac{n^2}{c^2} \right)^{-2}$ we have the

$C_{mn} = \frac{D\pi^4 bc}{8} = C$, which is independent of m and n and hence the energy level is the same for all modes of vibration.

<u>Lemma 7.2.</u> For a fixed m and n, the arc length in our generalized Riemannian space is given by

$$s_{mn} = B_{mn}^2 \frac{Mp^2\ bc}{8a}\left(t + \frac{1}{2p}\sin 2pt\right).$$

Proof. Using Lemma 4.1, Definition 4.3, and equation (47), we have

$$s_{mn} = \int_0^t B_{mn}^2 \frac{Mp^2\ bc}{4a}\cos^2 pt,\ dt,$$

$$= \frac{B_{mn}^2\ Mp^2\ bc}{8a}\left(t + \frac{1}{2p}\sin 2pt\right). \qquad \text{Q.E.D.}$$

Theorem 7.1. If i) $C = \dfrac{D\pi^4 bc}{8}$ (where D, b, c are defined after equation (17))

ii) $B_{mn} = \left(\dfrac{m^2}{b^2} + \dfrac{n^2}{c^2}\right)^{-1}$

iii) $p^2 = \dfrac{aD}{M}\pi^4\left(\dfrac{m^2}{b^2} + \dfrac{n^2}{c^2}\right)^2$

then

$$v_{mn} = u_{mn} = B_{mn}\left[\sin \frac{m\pi x}{b} \quad \sin \frac{n\pi y}{c} \quad \sin pt\right]$$

is a geodesic of the infinite dimensional "Riemannian" space defined by equation (32) of Definition 5.2 with total energy level C for each harmonic.

Proof. The ds of Definition 5.2 is the differential of arc length as given by Lemma 5.1, which in turn is equivalent to s as given in Definition 4.3. Definition 4.3 gives the form of s actually used in the computation of s of Lemma 7.2. That each harmonic has a total energy level C follows from the remark just before Lemma 7.2.

Lemma 7.3. If $v_{mn}(x,y) = u_{mn}(x,y,t)$ for a fixed value of t, then the geodesics $v_{mn}(x,y)$ are closed geodesics.

Proof. From equation (45) we observe that we return to the same function v_{mn} if t is increased by 2π.

Theorem 7.2. The functional elements v_{11}, v_{12}, v_{21}, v_{13}, v_{22}, v_{31}, ... of the double infinity family of geodesics are orthogonal at the origin.

Proof. The origin is given by $v = 0$, i.e., by $t = 0$ in $u_{mn}(x,y,t)$. The tangent vector to v_{mn} at the origin is given by

$$\left.\frac{\partial u_{mn}(x,y,t)}{\partial t}\right|_{t=0} = B_{mn}\, p \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c} \quad .$$

By orthogonality of two functions ξ_1 and ξ_2 we mean that $\left[\xi_1, g(v,\xi_2)\right] = 0$ where $\left[\xi_1, g(v,\xi_2)\right]$ is given by Definitions 5.3 and 5.4. From Definitions 5.4 and 5.10 we have

$$g(v,\xi_2) = \frac{1}{P[v]}\,\xi_2(x_2,y_2) \quad .$$

Hence

$$\left[B_{mn}\, p \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c},\ g\left(v,\ B_{ij}\, p \sin\frac{i\pi x}{b} \sin\frac{j\pi y}{c}\right)\right]$$

$$= \frac{1}{P[v]} B_{mn} B_{ij}\, p^2 \int_0^c\int_0^b \sin\frac{m\pi x}{b} \sin\frac{i\pi x}{b} \sin\frac{n\pi y}{c} \sin\frac{j\pi y}{c}\, dxdy$$

$$= \frac{1}{P[v]} B_{mn} B_{ij}\, p^2\, \delta_{ij}\, \delta_{mn} \frac{bc}{4}$$

where the repeated indices do not generate sums and δ_{ij} is the well known Kronecker delta symbol. Hence v_{mn} is orthogonal to v_{ij}, unless $m = n$ and $i = j$. Q.E.D.

8. Remarks and suggested problem. The function space associated with the thin rectangular flat plate of paragraph 4 was developed in detail in paragraphs 5 through 7. The function space for the vibrating beam of Part I with various end conditions can be obtained similarly. In this latter case, however, a different function space is obtained for different end conditions in spite of the fact that many of the leading equations involved are not affected by the end conditions. That this is true can be readily seen by rewriting Definition 5.2 for the case of the vibrating beam. The new hypothesis ii) certainly varies with the end conditions, thus changing the elements of the function space.

An essential part of the development was the fact that the sum of the kinetic and potential energy was a constant. This was obtained by assuming that the modulus of elasticity E, the density ρ, the moment of inertia I, etc. were constants. This condition can be relaxed so that these quantities vary with position and time. In this event, the requirement that $\frac{d\left(T(t)+V(t)\right)}{dt}=0$ means that an integral involving E_t, I_t, ρ_t, etc. must be 0. This places an additional restriction on the functions v of the associated function space. The study of such a space has interesting possibilities, one of them being the possibility of its having a constant curvature.

PART III

MOVING AXES IN HYDRODYNAMICS OF VISCOUS FLUIDS

9. Introduction. If the equations of hydrodynamics are desired in terms of a curvilinear system of coordinates, they can be derived from the known equations expressed in a "fixed" rectangular coordinate system. Then using a transformation for the space variables only, we can arrive at the desired equations via the methods of tensor analysis. On the other hand, these operations fail if the desired curvilinear system is in motion with respect to the fixed rectangular coordinate system.

We shall now show how the required equations for viscous flow may be obtained by the introduction of a "kinetic metric."[†] From this, using some of the techniques of the theory of special relativity, we can obtain the Newtonian equations by successive approximations in terms of $\frac{1}{c^2}$, where c is the velocity of light.

10. Space of the kinetic metric. We first require the square of the differential element of arc length for the proposed space. This is obtained by considering a fixed coordinate system with an assumed absolute time and transforming to a moving coordinate system. We let X^1, X^2, X^3 be the space coordinates in the fixed

[†]The non-viscous case was discussed by G. C. McVittie (6,285). We follow his notation rather closely.

coordinate system and X^4 be the assumed absolute time coordinate. Also let x^1, x^2, x^3 and x^4 be the space and time coordinates in the moving coordinate system. Let the transformation from fixed to moving coordinates be given by

<u>Definition</u> <u>10.1</u>.
$$X^r = h^r(x^1, x^2, x^3, x^4) \quad , \qquad r = 1,2,3$$
$$X^4 = x^4$$
where $h^r \in C^2$.

Clearly the kinetic energy T of a unit mass of the fluid in the X-system is given by

$$T = \tfrac{1}{2}\ \delta_{rs} \frac{dX^r}{dX^4} \frac{dX^s}{dX^4} \qquad\qquad (r,s = 1,2,3)$$

where δ_{rs} is the well known 2-index Kronecker delta symbol.

<u>Definition</u> <u>10.2</u>.
$$ds^2 \equiv (dX^4)^2 - \frac{1}{c^2}\ \delta_{rs}\ dX^r dX^s \quad .$$

We now introduce the convention that Latin lower case indices have the range 1 to 3, and Greek lower case indices have the range 1 to 4, unless otherwise stated.

<u>Definition</u> <u>10.3</u>.
$$\gamma_{pq} \equiv \delta_{rs} \frac{\partial h^r}{\partial x^p} \frac{\partial h^s}{\partial x^q}$$
$$\gamma_{4p} \equiv \delta_{rs} \frac{\partial h^r}{\partial x^4} \frac{\partial h^s}{\partial x^p} \equiv \gamma_{p4}$$
$$\gamma_{44} \equiv \delta_{rs} \frac{\partial h^r}{\partial x^4} \frac{\partial h^s}{\partial x^4}$$

<u>Definition 10.4.</u> $$g_{pq} \equiv \frac{-\gamma_{pq}}{c^2} \equiv g_{qp}$$

$$g_{p4} \equiv \frac{-\gamma_{p4}}{c^2} \equiv g_{4p}$$

$$g_{44} \equiv 1 - \frac{\gamma_{44}}{c^2} \quad .$$

<u>Lemma 10.1.</u> $ds^2 = g_{\sigma\tau}\, dx^\sigma dx^\tau$.

<u>Proof.</u> From Definition 10.1, $dX^r = \dfrac{\partial h^r}{\partial x^\sigma} dx^\sigma$ and $dX^4 = dx^4$.

Substituting these into Definition 10.2, we have

$$ds^2 = (dx^4)^2 - \frac{1}{c^2}\, \delta_{rs} \frac{\partial h^r}{\partial x^\sigma} dx^\sigma \frac{\partial h^s}{\partial x^\tau} dx^\tau$$

$$= \left(1 - \frac{1}{c^2}\, \delta_{rs} \frac{\partial h^r}{\partial x^4} \frac{\partial h^s}{\partial x^4}\right)(dx^4)^2 - \frac{1}{c^2}\, 2\, \delta_{rs} \frac{\partial h^r}{\partial x^p} \frac{\partial h^s}{\partial x^4} dx^p dx^4$$

$$- \frac{1}{c^2}\, \delta_{rs} \frac{\partial h^r}{\partial x^p} \frac{\partial h^s}{\partial x^q} dx^p dx^q \ .$$

Upon using Definitions 10.3 and 10.4, this becomes

(1) $$ds^2 = \left(1 - \frac{1}{c^2}\, \gamma_{44}\right)(dx^4)^2 - \frac{2}{c^2}\, \gamma_{p4}\, dx^p dx^4 - \frac{1}{c^2}\, \gamma_{pq}\, dx^p dx^q$$

$$= g_{44}(dx^4)^2 + 2\, g_{p4}\, dx^p dx^4 + g_{pq}\, dx^p dx^q$$

$$= g_{\sigma\tau}\, dx^\sigma dx^\tau \qquad \text{Q.E.D.}$$

This ds is the desired differential of arc length of the space associated with the unit mass kinetic energy T.

<u>Definition 10.5.</u> $g \equiv |g_{\sigma\tau}|$, $\gamma \equiv |\gamma_{\sigma\tau}|$, $\Delta \equiv |\gamma_{pq}|$,

$g^{\tau\sigma} \equiv$ (cofactor of $g_{\sigma\tau}$ in $g)g^{-1}$,

$\gamma^{qp} \equiv$ (cofactor of γ_{pq} in Δ) Δ^{-1} .

Here the vertical bars are a short notation for the determinant of the elements enclosed between them.

Lemma 10.2. $g = -c^{-6}\Delta + c^{-8}\gamma'$.

Proof. Observe $g = \begin{vmatrix} -\frac{\gamma'_{11}}{c^2} & -\frac{\gamma'_{12}}{c^2} & -\frac{\gamma'_{13}}{c^2} & 0-\frac{\gamma'_{14}}{c^2} \\ -\frac{\gamma'_{21}}{c^2} & -\frac{\gamma'_{22}}{c^2} & -\frac{\gamma'_{23}}{c^2} & 0-\frac{\gamma'_{24}}{c^2} \\ -\frac{\gamma'_{31}}{c^2} & -\frac{\gamma'_{32}}{c^2} & -\frac{\gamma'_{33}}{c^2} & 0-\frac{\gamma'_{34}}{c^2} \\ -\frac{\gamma'_{41}}{c^2} & -\frac{\gamma'_{42}}{c^2} & -\frac{\gamma'_{43}}{c^2} & 1-\frac{\gamma'_{44}}{c^2} \end{vmatrix}$

by Definition 10.4. Rewrite g as the sum of two determinants by splitting the summands of the fourth column and apply Definition 10.5.

Lemma 10.3. $(-g)^{1/2} = \frac{\Delta^{1/2}}{c^3}\left(1 - \frac{\gamma'}{2c^2\Delta}\right)$ to terms of order $\frac{1}{c^5}$.

Proof. Clear.

Lemma 10.4. For the fixed rectangular cartesian coordinate system X

$$g_{44} = 1, \quad g_{\rho\rho} = -\frac{1}{c^2}, \quad (-g)^{1/2} = \frac{1}{c^3}, \quad g^{44} = 1, \quad g^{\rho\rho} = -c^2$$

with remaining $g^{\sigma\tau}$ and $g_{\sigma\tau}$ equal 0.

Proof. Clear. Use Definitions 10.2 and 10.5.

11. <u>Equations of motion and of continuity</u>. For purposes of reference and completeness we shall give a brief sketch of the steps required to write the equations of motion and of continuity (Navier-Stokes differential equations) of a fluid in the form used in the special theory of relativity.

<u>Definition 11.1.</u> $\rho \equiv$ density of the fluid

$p \equiv$ pressure of the fluid

$\mu \equiv$ coefficient of viscosity of the fluid

$(X,Y,Z) \equiv$ external forces per unit volume in the X^1, X^2, X^3 (fixed system) directions, respectively.

$(U^1,U^2,U^3,1) \equiv \left(\frac{dX^1}{dX^4}, \frac{dX^2}{dX^4}, \frac{dX^3}{dX^4}, \frac{dX^4}{dX^4}\right)$ are the velocity components in the X^1, X^2, X^3, X^4 directions, respectively.

$$\Theta \equiv \frac{\partial U^1}{\partial X^1} + \frac{\partial U^2}{\partial X^2} + \frac{\partial U^3}{\partial X^3} \quad \begin{cases} = 0 \text{ for incompressible fluids} \\ \neq 0 \text{ for compressible fluids} \end{cases}$$

The well known Navier-Stokes differential equations (3,577) are

$$\rho\left(\frac{\partial U^1}{\partial X^4} + U^1\frac{\partial U^1}{\partial X^1} + U^2\frac{\partial U^1}{\partial X^2} + U^3\frac{\partial U^1}{\partial X^3}\right)$$

$$= \rho X - \frac{\partial p}{\partial X^1} + \frac{1}{3}\mu\frac{\partial \Theta}{\partial X^1} + \mu\nabla^2 U^1$$

(1) $$\rho\left(\frac{\partial U^2}{\partial X^4} + U^1\frac{\partial U^2}{\partial X^1} + U^2\frac{\partial U^2}{\partial X^2} + U^3\frac{\partial U^2}{\partial X^3}\right)$$

$$= \rho Y - \frac{\partial p}{\partial X^2} + \frac{1}{3}\mu\frac{\partial \Theta}{\partial X^2} + \mu\nabla^2 U^2$$

$$\rho\left(\frac{\partial U^3}{\partial X^4}+U^1\frac{\partial U^3}{\partial X^1}+U^2\frac{\partial U^3}{\partial X^2}+U^3\frac{\partial U^3}{\partial X^3}\right)$$

$$=\rho\, Z-\frac{\partial p}{\partial X^3}+\frac{1}{3}\mu\frac{\partial \Theta}{\partial X}+\mu\nabla^2 U^3$$

(2) $$\frac{\partial \rho}{\partial X^4}+\frac{\partial \rho U^1}{\partial X^1}+\frac{\partial \rho U^2}{\partial X^2}+\frac{\partial \rho U^3}{\partial X^3}=0 \quad .$$

Using the equation of continuity (2), equation (1) may be rewritten as

$$\frac{\partial(\rho U^1)}{\partial X^4}+\frac{\partial}{\partial X^1}\left(\rho(U^1)^2+p\right)+\frac{\partial}{\partial X^2}(\rho U^1U^2)+\frac{\partial}{\partial X^3}(\rho U^1U^3)$$

$$=\rho X+\frac{1}{3}\mu\frac{\partial \Theta}{\partial X^1}+\mu\nabla^2 U^1$$

$$\frac{\partial(\rho U^2)}{\partial X^4}+\frac{\partial(\rho U^1U^2)}{\partial X^1}+\frac{\partial\left(\rho(U^2)^2+p\right)}{\partial X^2}+\frac{\partial \rho U^2U^3}{\partial X^3}$$

(3)

$$=\rho Y+\frac{1}{3}\mu\frac{\partial \Theta}{\partial X^2}+\mu\nabla^2 U^2$$

$$\frac{\partial(\rho U^3)}{\partial X^4}+\frac{\partial(\rho U^1U^3)}{\partial X^1}+\frac{\partial(\rho U^2U^3)}{\partial X^2}+\frac{\partial}{\partial X^3}\left(\rho(U^3)^2+p\right)$$

$$=\rho Z+\frac{1}{3}\mu\frac{d\Theta}{dX^3}+\mu\nabla^2 U^3 \quad .$$

<u>Definition 11.2.</u> $T^{\sigma\tau}\equiv\rho U^\sigma U^\tau-\frac{g^{\sigma\tau}}{c^2}p$.

<u>Lemma 11.1.</u> If terms of order $\frac{1}{c^2}$ are neglected, then $\frac{\partial T^{\sigma\tau}}{\partial X^\tau}$ is the left side of equation (3) for $\sigma = 1, 2,$ or 3 and the left side of equation (2) for $\sigma = 4$.

Proof. Using Lemma 10.4, we have

$$\frac{\partial T^{\kappa\tau}}{\partial X^{\tau}} = \frac{\partial(\rho U^{\kappa} - \frac{g^{\kappa 4}}{c^2} p)}{\partial X^4} + \frac{\partial(\rho U^{\kappa} U^{\lambda} - \frac{g^{\kappa\lambda}}{c^2} p)}{\partial X^{\lambda}}$$

$$= \frac{\partial(\rho U^{\kappa})}{\partial X^4} + \frac{\partial(\rho U^{\kappa} U^{\lambda} + \delta^{\kappa\lambda} p)}{\partial X^{\lambda}}$$

which is clearly the left side of equation (3).

$$\frac{\partial T^{4\tau}}{\partial X^{\tau}} = \frac{\partial(\rho U^4 U^4 - \frac{g^{44}}{c^2} p)}{\partial X^4} + \frac{\partial(\rho U^4 U^{\kappa} - \frac{g^{4\kappa}}{c^2} p)}{\partial X^{\kappa}}$$

$$= \frac{\partial \rho}{\partial X^4} + \frac{\partial \rho U^{\kappa}}{\partial X^{\kappa}}$$

which is the left side of equation (2). Q.E.D.

Definition 11.3. $(f^1, f^2, f^3, f^4) \equiv (X, Y, Z, 0)$

Lemma 11.2. If terms of order $\frac{1}{c^2}$ are neglected, then

$$-\frac{1}{3}\mu \frac{g^{\sigma\tau}}{c^2} \frac{\partial U^{\nu}{}_{,\nu}}{\partial X^{\tau}} = \frac{1}{3}\mu\, \delta^{\sigma\lambda} \frac{\partial \Theta}{\partial X^{\lambda}}$$

where the comma denotes partial differentiation.

Proof. Since $U^{\nu}{}_{,\nu} = \frac{\partial U^{\nu}}{\partial X^{\nu}} = \Theta$, we have

$$-\frac{1}{3}\mu \frac{g^{\sigma\tau}}{c^2} \frac{\partial U^{\nu}{}_{,\nu}}{\partial X^{\tau}} = -\frac{1}{3}\mu \frac{g^{\sigma 4}}{c^2} \frac{\partial \Theta}{\partial X^4} - \frac{1}{3}\mu \frac{g^{\sigma\kappa}}{c^2} \frac{\partial \Theta}{\partial X^{\kappa}} .$$

Hence, upon using Lemma 10.4,

$$-\frac{1}{3}\mu \frac{g^{\lambda\tau}}{c^2} \frac{\partial U^{\nu}{}_{,\nu}}{\partial X^{\tau}} = --\frac{1}{3}\mu \frac{g^{\lambda 4}}{c^2} \frac{\partial \Theta}{\partial X^4} - -\frac{1}{3}\mu \frac{g^{\lambda\kappa}}{c^2} \frac{\partial \Theta}{\partial X^{\kappa}} = \frac{1}{3}\mu\, \delta^{\lambda\kappa} \frac{\partial \Theta}{\partial X^{\kappa}}$$

and

$$-\frac{1}{3}\mu\frac{g^{4\tau}}{c^2}\frac{\partial U^{\nu}{}_{,\nu}}{\partial X^{\tau}}=-\frac{1}{3}\mu\frac{g^{44}}{c^2}\frac{\partial \Theta}{\partial X^4}-\frac{1}{3}\mu\frac{g^{4\gamma}}{c^c}\frac{\partial \Theta}{\partial X^{\gamma}}=0\ ,$$

if we neglect terms of order $\frac{1}{c^2}$.

Therefore $$-\frac{1}{3}\mu\frac{g^{\sigma\tau}}{c^2}\frac{\partial U^{\nu}{}_{,\nu}}{\partial X^{\tau}}=\frac{1}{3}\mu\,\delta^{\sigma\gamma}\frac{\partial\Theta}{\partial X^{\gamma}}\ .\qquad \text{Q.E.D.}$$

<u>Lemma</u> <u>11.3</u>. If terms of order $\frac{1}{c^2}$ are neglected, then $-\frac{1}{c^2}\mu\, g^{\sigma\tau}\, U^{\nu}{}_{,\sigma,\tau}=\mu\,\delta^{\nu}_{k}\nabla^2 U^{k}$ where ∇^2 is the three dimensional Laplacian operator.

<u>Proof</u>. $-\frac{1}{c^2}\mu\, g^{\sigma\tau}\, U^{4}{}_{,\sigma,\tau}=0=\mu\,\delta^{4}_{k}\nabla^2 U^{k}$

$$-\frac{1}{c^2}\mu\, g^{\sigma\tau}\, U^{\gamma}{}_{,\sigma,\tau}=\frac{-1}{c^2}\mu\, g^{44}\, U^{\gamma}{}_{,4,4}-\frac{1}{c^2}\mu\, g^{RR}\frac{\partial^2 U^{\gamma}}{(\partial X^{r})^2}$$

$$=\mu\,\nabla^2 U^{\gamma},\qquad \text{neglecting}$$

terms of order c^{-2}. Therefore,

$$-c^{-2}\mu\, g^{\sigma\tau}\, U^{\nu}{}_{,\sigma,\tau}=\mu\,\delta^{\nu}_{k}\nabla^2 U^{k}\ .\qquad \text{Q.E.D.}$$

<u>Theorem</u> <u>11.1</u>. If terms of order $\frac{1}{c^2}$ are neglected, then equations (3) and (2) become

$$(4)\qquad \frac{\partial T^{\sigma\tau}}{\partial X^{\tau}}=\rho f^{\sigma}-\frac{\mu}{3c}g^{\sigma\tau}\frac{\partial U^{\nu}_{,\nu}}{\partial X^{\tau}}-c^{-2}\mu\, g^{\sigma\tau}\, U^{\nu}_{,\sigma,\tau}\ .$$

<u>Proof</u>. By Lemma 11.1, $\frac{\partial T^{\sigma\tau}}{\partial X^{\tau}}$ gives the left side of equations (3) and (2). Comparing Lemmas 11.2 and 11.3 with the right side of equations (3) and (2) and remembering Definition 11.3, the desired results follow.

<u>Definition</u> <u>11.4</u>. If x^{σ} and s are the coordinates and

arc length, respectively, of Lemma 10.1, then we define

$$v^{\sigma} = \frac{dx^{\sigma}}{ds} \quad .$$

Definition 11.5. Regarding ρ and p as invariants, we define $T^{\sigma\tau}$ in the x-coordinate system by (cf. Definition 11.2)

$$T^{\sigma\tau} \equiv \rho v^{\sigma} v^{\tau} - \frac{g^{\sigma\tau}}{c^2} p \; .$$

Lemma 11.4. $T^{\sigma\tau}_{,\tau} = \rho f^{\sigma} - \frac{1}{3c^2} \mu \, g^{\sigma\tau} \frac{\partial v^{\nu}_{,\nu}}{\partial x^{\tau}} - \frac{1}{c^2} \mu \, g^{\nu\sigma} v^{\tau}_{,\nu,\tau}$.

Proof. In changing from rectangular coordinates X to the general coordinates x of the space of Lemma 10.1, we replace the partial derivative of $T^{\sigma\tau}$ by its covariant derivative. Hence rewriting equation (4) of Theorem 11.1, we get the desired result.

The equations of motion and continuity of Lemma 11.4 may also be written in terms of the vorticity tensor.

Definition 11.6. If $v_{\sigma} \equiv g_{\sigma\tau} v^{\tau}$, then the vorticity tensor, $\xi_{\sigma\tau}$, is defined by (cf. expression for vortex vector of (5,277))

$$\xi_{\sigma\tau} \equiv c^{-1} (-g)^{-\frac{1}{2}} \; (v_{\sigma,\tau} - v_{\tau,\sigma}) \; .$$

Lemma 11.5. If terms of order c^{-2} are neglected and the fixed rectangular coordinate system X is used, then

$$\xi_{23} = \frac{\partial U^3}{\partial X^2} - \frac{\partial U^2}{\partial X^3} , \quad \xi_{31} = \frac{\partial U^1}{\partial X^3} - \frac{\partial U^3}{\partial X^1} , \quad \xi_{12} = \frac{\partial U^2}{\partial X^1} - \frac{\partial U^1}{\partial X^2}$$

$$\xi_{41} = \xi_{42} = \xi_{43} = \xi_{11} = \xi_{22} = \xi_{33} = \xi_{44} = 0$$

<u>Proof</u>. $$\zeta_{23} = \frac{1}{c\sqrt{-g}}\left(U_{2,3} - U_{3,2}\right) = \frac{1}{cc^{-3}}\left(\frac{\partial U_2}{\partial X^3} - \frac{\partial U_3}{\partial X^2}\right)$$

$$= c^2\left(g_{2\sigma}\frac{\partial U^\sigma}{\partial X^3} - g_{3\sigma}\frac{\partial U^\sigma}{\partial X^2}\right) = c^2\left(g_{2\gamma}\frac{\partial U^\gamma}{\partial X^3} - g_{3\gamma}\frac{\partial U^\gamma}{\partial X^2}\right)$$

since we are neglecting terms of order c^{-2}. Using Lemma 10.4, we have $\zeta_{23} = -\frac{\partial U^2}{\partial X^3} + \frac{\partial U^3}{\partial X^2}$. The other relations follow similarly.

<u>Lemma</u> <u>11.6</u>. The equations of motion can be written

$$(5) \quad v_\tau(\rho v^\sigma)_{,\sigma} + c\sqrt{-g}\;\zeta_{\tau\sigma}\,\rho v^\sigma + \tfrac{1}{2}\rho\frac{\partial(v^\sigma v_\sigma)}{\partial x^\tau} - \frac{1}{c^2}\frac{\partial p}{\partial x^\tau}$$

$$= \rho f_\tau - \frac{1}{3c^2}\mu\frac{\partial v^\sigma_{,\sigma}}{\partial x^\tau} - \frac{1}{c^2}\mu\, g_{\mu\tau}\, g^{\sigma\nu}\, v^\mu_{,\sigma,\nu} \quad .$$

<u>Proof</u>. Taking the covariant derivative of $T^{\sigma\tau}$ as given by Definition 11.5, we have

$$T^{\sigma\tau}_{\cdot\cdot,\tau} = v^\sigma(\rho v^\tau)_{,\tau} + \rho v^\tau v^\sigma_{,\tau} - \frac{g^{\sigma\tau}}{c^2}\,p_{,\tau}$$

$$= v^\sigma(\rho v^\tau)_{,\tau} + \rho v^\tau g^{\sigma\mu} v_{\mu,\tau} - \frac{g^{\sigma\tau}}{c^2}\frac{\partial p}{\partial x^\tau}$$

Now substituting for $v_{\mu,\tau}$ from Definition 11.6, we get

$$T^{\sigma\tau}_{\cdot\cdot,\tau} = v^\sigma(\rho v^\tau)_{,\tau} + \rho\, v^\tau g^{\sigma\mu}\left(c\sqrt{-g}\;\zeta_{\mu\tau} + v_{\tau,\mu}\right) - \frac{g^{\sigma\tau}}{c^2}\frac{\partial p}{\partial x^\tau}$$

$$= v^\sigma(\rho v^\tau)_{,\tau} + c\sqrt{-g}\,\rho v^\tau g^{\sigma\mu}\zeta_{\mu\tau} + \rho\, g^{\sigma\mu} v^\tau v_{\tau,\mu} - \frac{g^{\sigma\tau}}{c^2}\frac{\partial p}{\partial x^\tau} \quad .$$

Setting this last expression equal to the right side of Lemma 11.4, we get

$$T^{\sigma\tau}_{\cdot\cdot,\tau} = v^{\sigma}(\rho\ v^{\tau})_{,\tau} + c\sqrt{-g}\,\rho\, v^{\tau} g^{\sigma\mu}\,\zeta_{\mu\tau} + \rho\ g^{\sigma\mu}\ v^{\tau} v_{\tau,\mu} - \frac{g^{\sigma\tau}}{c^2}\frac{\partial p}{\partial x^{\tau}}$$

$$= \rho\ f^{\sigma} - \frac{1}{3c^2}\mu\ g^{\sigma\tau}\frac{\partial v^{\nu}_{,\nu}}{\partial x^{\tau}} - \frac{1}{c^2}\mu\ g^{\nu\sigma}\ v^{\tau}_{,\nu,\tau}\ .$$

"Multiplying" by $g_{\sigma\delta}$ and summing on σ we have

$$v_{\delta}(\rho v^{\tau})_{,\tau} + c\sqrt{-g}\,\rho\, v^{\tau}\ \zeta_{\delta\tau} + \rho\ g_{\sigma\delta}\ g^{\sigma\mu}\ v^{\tau}\ v_{\tau,\mu} - \frac{1}{c^2}\ \frac{\partial p}{\partial x^{\delta}}$$

$$= \rho\ f_{\delta} - \frac{1}{3c^2}\mu\frac{\partial v^{\nu}_{,\nu}}{\partial x^{\delta}} - \frac{1}{c^2}\mu g_{\sigma\delta}\ g^{\nu\sigma}\ v^{\tau}_{,\nu,\tau}\ .$$

Hence if we observe that $v^{\tau}v_{\tau,\delta} = \frac{1}{2}\,\dfrac{\partial(v^{\tau}v_{\tau})}{\partial x^{\delta}}$, and if we write

τ for δ and σ for τ , we get the desired result.

<u>Lemma 11.7</u>. If terms of order $\frac{1}{c^2}$ are neglected, then for $\tau = 4$, equation (5) becomes

$$(\rho v^{\sigma})_{,\sigma} = 0\ .$$

<u>Proof</u>. Since $v_{\tau} = g_{\tau\sigma}\, v^{\sigma}$, then

$$v_4 = g_{4\sigma}\ v^{\sigma} = g_{44} + g_{4g}\ v^{g}\ .$$

Or by using Definition 10.4 and $v^4 = \dfrac{dx^4}{ds} = 1$, we have

(6) $\qquad v_4 = 1 - \gamma_{44}\ c^{-2} - \gamma_{4g}\ c^{-2}\ v^{g}$. Similarly,

$$v_g = g_{g\sigma}\ v^{\sigma} = g_{g4} + g_{gp}\ v^{p}$$

(7) $\qquad = -\gamma_{g4}\ c^{-2} - \gamma_{gp}\ c^{-2}\ v^{p}$. By Definition 10.4

$$v^{\sigma}v_{\sigma} = g_{\sigma\tau}\ v^{\sigma}v^{\tau} = g_{44} + 2g_{4g}\ v^{g} + g_{pg}\ v^{p}v^{g}$$

(8) $\qquad = 1 - \gamma_{44}\ c^{-2} - 2\gamma_{g4}\ c^{-2}\ v^{g} - \gamma_{pg}\ c^{-2}\ v^{p}v^{g}$.

Using Lemma 10.3,

(9) $\qquad c\sqrt{-g}\ \zeta_{\tau\sigma} = c^{-2}\sqrt{\Delta}\left(1 - \dfrac{\gamma}{2c^2\Delta}\right)\zeta_{\tau\sigma}$. Since $f^4 = 0$,

(10) $$f_4 = g_{4\sigma} f^\sigma = g_{4q} f^q = -\gamma_{4q} c^{-2} f^q .$$

Substituting these results into equation (5), we have for the case $\tau = 4(v^4 = 1)$,

$$\left(1 - \frac{\gamma_{44} + \gamma_{4q} v^q}{c^2}\right)(\rho v^\sigma)_{,\sigma} + \frac{\sqrt{\Delta}}{c^2}\left(1 - \frac{\gamma}{2c^2\Delta}\right)\xi_{4\sigma} v^\sigma$$
$$+ \frac{\rho}{2}\frac{\partial}{\partial x^4}\left(\frac{-\gamma_{44}}{c^2} - \frac{2\gamma_{q4} v^q}{c^2} - \frac{\gamma_{pq}}{c^2} v^p v^q\right) - c^{-2}\frac{\partial p}{\partial x^4}$$
$$= \rho f_4 - \frac{\mu}{3c^2}\frac{\partial v^\sigma_{,\sigma}}{\partial x^4} - c^{-2}\mu g_{\mu 4} g^{\sigma\nu} v^\mu_{,\sigma,\nu} .$$

Neglecting terms of order c^{-2} this becomes

(11) $$(\rho v^\sigma)_{,\sigma} = 0 .$$

<u>Theorem 11.2</u>. If terms of order c^{-2} are neglected, then for $\tau = q = 1,2$, or 3, the equations of motion (5) become

(12) $$\sqrt{\Delta}\rho(\xi_{q4} + \xi_{qp} v^p) - \tfrac{1}{2}\rho\frac{\partial}{\partial x^q}\left(\gamma_{44} + 2\gamma_{p4} v^p + \gamma_{pr} v^p v^r\right) - \frac{\partial p}{\partial x^q}$$
$$= -\rho\gamma_{pq} f^p - \frac{\mu}{3}\frac{\partial v^p_{,p}}{\partial x^q} - \mu\gamma_{pq}\gamma^{rs} v^p_{,r,s} .$$

<u>Proof</u>. $$v_q = v^\sigma g_{\sigma q} = g_{4q} + v^p g_{pq} = -\gamma_{4q} c^{-2} - \gamma_{pq} c^{-2} v^p$$
$$f_q = g_{\tau q} f^\tau = g_{pq} f^p = -\gamma_{pq} c^{-2} f^p$$
$$g_{\mu q} g^{\sigma\nu} v^\mu_{,\sigma,\nu} = g_{pq}\left(g^{44} v^p_{,4,4} + g^{4s} v^p_{,4,s} + g^{s4} v^p_{,s,4} + g^{rs} v^p_{,r,s}\right)$$
$$= -c^{-2}\gamma_{pq}\left(v^p_{,4,4} + g^{4s} v^p_{,4,s} + g^{s4} v^p_{,s,4} - c^2\gamma^{rs} v^p_{,r,s}\right)$$
$$= \gamma_{pq}\gamma^{rs} v^p_{,r,s} \text{ neglecting terms of order } c^{-2} .$$

Setting $\tau = q = 1,2$, or 3 in equation (5) of Lemma 11.6 and using the results of equations (6)-(11) as well as those just above, we have

$$0+\frac{\sqrt{\Delta}}{c}\left(1-\frac{\gamma'}{2c^2\Delta}\right)\left(\zeta_{q4}\,\rho+\zeta_{qp}\,\rho\, v^p\right)+$$

$$+\tfrac{1}{2}\rho\,\frac{\partial}{\partial x^q}\left(1-\gamma_{44}\,c^{-2}-2\gamma_{q4}\,c^{-2}v^q-\gamma_{pr}\,v^p v^r c^{-2}\right)-c^{-2}\,\frac{\partial p}{\partial x^q}$$

$$=-\rho\,\frac{\gamma_{pq}}{c^2}\,f^p-\frac{1}{3c^2}\,\mu\,\frac{\partial v^{\sigma}_{,\sigma}}{\partial x^q}-\frac{1}{c^2}\,\mu\,g_{\mu q}\,g^{\sigma\nu}\,v^{\mu}_{,\sigma,\nu}\;.$$

Multiplying by c^2 and neglecting terms of order c^{-2}, we have

$$\sqrt{\Delta}\,\rho\left(\zeta_{q4}+\zeta_{qp}\,v^p\right)-\tfrac{1}{2}\rho\,\frac{\partial}{\partial x^q}\left(\gamma_{44}+2\gamma_{p4}v^p+\gamma_{pr}\,v^p v^r\right)-\frac{\partial p}{\partial x^q}$$

$$=-\rho\,\gamma_{pq}\,f^p-\frac{\mu}{3}\,\frac{\partial v^p_{,p}}{\partial x^q}-\mu\,\gamma_{pq}\,\gamma^{rs}\,v^p_{,r,s}\;.$$

<u>Corollary 11.1</u>. If the coordinate system x is at rest relative to the "fixed" coordinate system X, then equation (12) becomes

$$(13)\qquad \sqrt{\Delta}\,\rho\left(\zeta_{q4}+\zeta_{qp}\,v^p\right)-\tfrac{1}{2}\rho\,\frac{\partial}{\partial x^q}\left(\gamma_{pr}\,v^p v^r\right)-\frac{\partial p}{\partial x^q}$$

$$=-\rho\,\gamma_{pq}\,f^p-\frac{\mu}{3}\,\frac{\partial v^p_{,p}}{\partial x^q}-\mu\,\gamma_{pq}\,\gamma^{rs}\,v^p_{,r,s}\;.$$

<u>Proof</u>. From Definition 10.4 we see that $\gamma_{44}=0=\gamma_{p4}$ if the transformation function h^r is independent of x^4, and hence equation (12) simplifies to equation (13).

<u>Corollary 11.2</u>. If the coordinate system x is at rest relative to the coordinate system X and is such that $\gamma_{pq}=0$ unless $p=q$ (that is, the directions of x^1, x^2, and x^3 are orthogonal) then the equations of motion (12) become

$$(14) \qquad \sqrt{\Delta}\,\rho\left(\zeta_{q4} + \zeta_{qp} v^p\right) - \tfrac{1}{2}\rho\frac{\partial}{\partial x^q}\left(\gamma_{tp}\, v^t v^p\right) - \frac{\partial p}{\partial x^q}$$
$$= -\rho\,\gamma_{Qq}\, f^Q - \frac{\mu}{3}\frac{\partial v^p_{,p}}{\partial x^q} - \mu\,\gamma_{Qq}\,\gamma^{tt}\, v^Q_{,t,t} \; .$$

Proof. Clear, when we remember that $\gamma_{pq} = \delta_{pq}\,\gamma_{QQ}$ and that $\gamma^{pq} = \delta_{pq}\,(\gamma_{QQ})^{-1}$, and use Corollary 11.1. The capital indices are used, as mentioned in paragraph 1, to forestall summation.

12. Rate of change of the vorticity tensor. In this paragraph we construct an expression involving the x^4 derivative of the vorticity tensor. First we write the equations of motion as given by equation (5) of Lemma 11.6 as

$$(15) \qquad c\sqrt{-g}\;\zeta_{\tau\sigma}\,\rho\, v^\sigma + \tfrac{1}{2}\rho\frac{\partial (v^\sigma v_\sigma)}{\partial x^\tau} = -v_\tau(\rho v^\sigma)_{,\sigma} + \frac{1}{c^2}\frac{\partial p}{\partial x^\tau} +$$
$$+ \rho f_\tau - \frac{c^{-2}\mu}{3}\frac{\partial v^\sigma_{,\sigma}}{\partial x^\tau} - c^{-2}\mu\, g_{\mu\tau}\, g^{\sigma\nu} v^\mu_{,\sigma,\nu} \; .$$

Definition 12.1. $\rho\, g_\tau \equiv$ right side of equation (15). We see that g_τ is a covariant vector since it is a sum of covariant vectors.

Lemma 12.1.
$$g_{\tau,\sigma} - g_{\sigma,\tau} = \frac{\partial g_\tau}{\partial x^\sigma} - \frac{\partial g_\sigma}{\partial x^\tau} \; .$$
$$v_{\tau,\sigma} - v_{\sigma,\tau} = \frac{\partial v_\tau}{\partial x^\sigma} - \frac{\partial v_\sigma}{\partial x^\tau} \; .$$

Proof. Clear.

Theorem 12.1.

$$(16)\qquad v^{\nu}\frac{\partial}{\partial x^{\nu}}\left(c\sqrt{-g}\,\zeta_{\tau\sigma}\right) = -\,c\sqrt{-g}\,\frac{\partial v^{\nu}}{\partial x^{\sigma}}\left(\zeta_{\tau\nu} - \frac{\partial v^{\nu}}{\partial x^{\tau}}\,\zeta_{\sigma\nu}\right) + \frac{\partial g_{\tau}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma}}{\partial x^{\tau}}\,.$$

Proof. From Definition 12.1 we have

$$g_{\tau} = c\sqrt{-g}\,\zeta_{\tau\mu}\,v^{\mu} + \tfrac{1}{2}\frac{\partial\,(v^{\mu}v_{\mu})}{\partial x^{\tau}}\,.$$

Therefore, using Lemma 12.1,

$$(17)\qquad g_{\tau,\sigma} - g_{\sigma,\tau} = c\sqrt{-g}\left(\frac{\partial v^{\mu}}{\partial x^{\sigma}}\,\zeta_{\tau\mu} + v^{\mu}\frac{\partial \zeta_{\tau\mu}}{\partial x^{\sigma}} - \frac{\partial v^{\mu}}{\partial x^{\tau}}\,\zeta_{\sigma\mu} - \frac{\partial \zeta_{\sigma\mu}v^{\mu}}{\partial x^{\tau}}\right) + v^{\mu}\left\{\frac{\partial}{\partial x^{\sigma}}\left[\frac{\partial v_{\tau}}{\partial x^{\mu}} - \frac{\partial v_{\mu}}{\partial x^{\tau}}\right] - \frac{\partial}{\partial x^{\tau}}\left[\frac{\partial v_{\sigma}}{\partial x^{\mu}} - \frac{\partial v_{\mu}}{\partial x^{\sigma}}\right]\right\}.$$

But from Definition 11.6 and Lemma 12.1 we have

$c\sqrt{-g}\,\zeta_{\tau\mu} = \dfrac{\partial v_{\tau}}{\partial x^{\mu}} - \dfrac{\partial v_{\mu}}{\partial x^{\tau}}$, and hence equation (17) can be written as

$$\frac{\partial g_{\tau}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma}}{\partial x^{\tau}} = c\sqrt{-g}\,\frac{\partial v^{\mu}}{\partial x^{\sigma}}\,\zeta_{\tau\mu} - \frac{\partial v^{\mu}}{\partial x^{\tau}}\,\zeta_{\sigma\mu} + {}$$

$$+\, v^{\mu}\,\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial v_{\tau}}{\partial x^{\sigma}} - \frac{\partial v_{\sigma}}{\partial x^{\tau}}\right)$$

$$= c\sqrt{-g}\left[\frac{\partial v^{\mu}}{\partial x^{\sigma}}\,\zeta_{\tau\mu} - \frac{\partial v^{\mu}}{\partial x^{\tau}}\,\zeta_{\sigma\mu}\right] + v^{\mu}\frac{\partial\,(c\sqrt{-g}\,\zeta_{\tau\sigma})}{\partial x^{\mu}}$$

The desired result is obtained after transferring the first term following the equal sign to the left side.

Theorem 12.1.

$$(16)\qquad v^{\nu}\frac{\partial}{\partial x^{\nu}}\left(c\sqrt{-g}\;\zeta_{\tau\sigma}\right) = -c\sqrt{-g}\;\frac{\partial v^{\nu}}{\partial x^{\sigma}}\left(\zeta_{\tau\nu} - \frac{\partial v^{\nu}}{\partial x^{\tau}}\zeta_{\sigma\nu}\right) + \frac{\partial g_{\tau}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma}}{\partial x^{\tau}}\,.$$

Proof. From Definition 12.1 we have

$$g_{\tau} = c\sqrt{-g}\;\zeta_{\tau\mu}\;v^{\mu} + \tfrac{1}{2}\frac{\partial (v^{\mu}v_{\mu})}{\partial x^{\tau}}\,.$$

Therefore, using Lemma 12.1,

$$(17)\qquad g_{\tau,\sigma} - g_{\sigma,\tau} = c\sqrt{-g}\left(\frac{\partial v^{\mu}}{\partial x^{\sigma}}\zeta_{\tau\mu} + v^{\mu}\frac{\partial \zeta_{\tau\mu}}{\partial x^{\sigma}} - \frac{\partial v^{\mu}}{\partial x^{\tau}}\zeta_{\sigma\mu} - \frac{\partial \zeta_{\sigma\mu}v^{\mu}}{\partial x^{\tau}}\right) + v^{\mu}\left\{\frac{\partial}{\partial x^{\sigma}}\left[\frac{\partial v_{\tau}}{\partial x^{\mu}} - \frac{\partial v_{\mu}}{\partial x^{\tau}}\right] - \frac{\partial}{\partial x^{\tau}}\left[\frac{\partial v_{\sigma}}{\partial x^{\mu}} - \frac{\partial v_{\mu}}{\partial x^{\sigma}}\right]\right\}.$$

But from Definition 11.6 and Lemma 12.1 we have

$c\sqrt{-g}\;\zeta_{\tau\mu} = \dfrac{\partial v_{\tau}}{\partial x^{\mu}} - \dfrac{\partial v_{\mu}}{\partial x^{\tau}}$, and hence equation (17) can be

written as

$$\frac{\partial g_{\tau}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma}}{\partial x^{\tau}} = c\sqrt{-g}\;\frac{\partial v^{\mu}}{\partial x^{\sigma}}\zeta_{\tau\mu} - \frac{\partial v^{\mu}}{\partial x^{\tau}}\zeta_{\sigma\mu} + $$

$$+\, v^{\mu}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial v_{\tau}}{\partial x^{\sigma}} - \frac{\partial v_{\sigma}}{\partial x^{\tau}}\right)$$

$$= c\sqrt{-g}\left[\frac{\partial v^{\mu}}{\partial x^{\sigma}}\zeta_{\tau\mu} - \frac{\partial v^{\mu}}{\partial x^{\tau}}\zeta_{\sigma\mu}\right] + v^{\mu}\frac{\partial\;(c\sqrt{-g}\;\zeta_{\tau\sigma})}{\partial x^{\mu}}$$

The desired result is obtained after transferring the first term following the equal sign to the left side.

<u>Corollary 12.1</u>. If terms of order $\frac{1}{c^2}$ are neglected, then equation (16) can be written as

$$(18)\quad \left(\frac{\partial}{\partial x^4}+v^g\frac{\partial}{\partial x^g}\right)\sqrt{\Delta}\,\zeta_{\tau\sigma}=-\sqrt{\Delta}\left(\frac{\partial v^g}{\partial x^\sigma}\zeta_{\tau g}-\frac{\partial v^g}{\partial x^\tau}\zeta_{\sigma g}\right)$$

$$+\frac{\partial}{\partial x^\sigma}\left(\frac{1}{\rho}\frac{\partial p}{\partial x^\tau}-\gamma_{\tau g}f^g-\frac{1}{3\rho}\mu\frac{\partial v^g_{,g}}{\partial x^g}-\frac{\mu}{\rho}\gamma_{g\tau}\gamma^{rs}v^g_{,r,s}\right)$$

$$-\frac{\partial}{\partial x^\tau}\left(\frac{1}{\rho}\frac{\partial p}{\partial x^\sigma}-\gamma_{\sigma g}f^g-\frac{1}{3\rho}\mu\frac{\partial v^g_{,g}}{\partial x^\sigma}-\frac{\mu}{\rho}\gamma_{g\sigma}\gamma^{rs}v^g_{,r,s}\right).$$

<u>Proof</u>. Using the approximate formulas of Lemma 10.3,

$$(19)\quad v^\mu\frac{\partial\,(c\sqrt{-g}\,\zeta_{\tau\sigma})}{\partial x^\mu}=\frac{1}{c^2}\,\frac{\partial\,(\sqrt{\Delta}\,\zeta_{\tau\sigma})}{\partial x^4}+v^g\frac{\partial(\sqrt{\Delta}\,\zeta_{\tau\sigma})}{\partial x^g}\quad .$$

Similarly,

$$(20)\quad -c\sqrt{-g}\left(\frac{\partial v^\nu}{\partial x^\sigma}\zeta_{\tau\nu}-\frac{\partial v^\nu}{\partial x^\tau}\zeta_{\sigma\nu}\right)=\frac{-\sqrt{\Delta}}{c^2}\left(\frac{\partial v^g}{\partial x^\sigma}\zeta_{\tau g}-\frac{\partial v^g}{\partial x^\tau}\zeta_{\sigma g}\right)\quad .$$

From Definition 12.1 and Lemma 11.7

$$(21)\quad g_\tau=\frac{1}{\rho c^2}\frac{\partial p}{\partial x^\tau}+f_\tau-\frac{1}{3\rho c^2}\mu\frac{\partial v^\sigma_{,\sigma}}{\partial x^\tau}-\frac{1}{\rho c^2}\mu\, g_{\mu\tau}\, g^{\sigma\nu}v^\mu_{,\sigma,\nu}\,.$$

But $f_\tau=g_{\tau\sigma}f^\sigma=g_{\tau 4}\,f^4+g_{\tau g}f^g=-c^{-2}\gamma_{\tau g}\,f^g$ from Definitions 10.5 and 11.3. From Definition 11.4,

$$\frac{\partial v^\sigma_{,\sigma}}{\partial x^\tau}=\frac{\partial v^g_{,g}}{\partial x^\tau}\quad .$$

Also $g_{\mu\tau}\,g^{\sigma\nu}v^\mu_{,\sigma,\nu}=g_{g\tau}\,g^{\sigma\nu}\,v^g_{,\sigma,\nu}$

$$=-c^2\gamma_{g\tau}\left(g^{44}v^g_{,4,4}+g^{4r}v^g_{,4,r}+g^{r4}v^g_{,r,4}+g^{rs}v^g_{,r,s}\right)$$

$$=-c^2\gamma_{g\tau}\,g^{rs}v^g_{,r,s}=-c^2\gamma_{g\tau}\left(-c^2\gamma^{rs}v^g_{,r,s}\right)$$

$$=\gamma_{g\tau}\,\gamma^{rs}\,v^g_{,r,s}\quad .$$

Substituting the expressions for f_τ , $\dfrac{\partial\, v^{\sigma}_{,\sigma}}{\partial\, x^{\tau}}$, and $g_{\mu\tau}\, g^{\sigma\nu}\, v^{\mu}_{,\sigma,\nu}$ into equation (21), we have

$$(22) \qquad g_\tau = c^{-2}\ \frac{1}{\rho}\frac{\partial p}{\partial x^{\tau}} - \gamma_{\tau\vartheta}\ f^{\vartheta} - \frac{1}{3\rho}\mu\,\frac{\partial\, v^{\vartheta}_{,\vartheta}}{\partial x^{\tau}} - \frac{\mu}{\rho}\,\gamma_{\vartheta\tau}\,\gamma^{rs}\,v^{\vartheta}_{,r,s}$$

Now using equations (19), (20), and (22), we see that equation (16) becomes

$$\frac{1}{c^2}\left(\frac{\partial}{\partial x^4} + v^{\vartheta}\frac{\partial}{\partial x^{\vartheta}}\right)\sqrt{\Delta}\ \zeta_{\tau\sigma} = \frac{-\sqrt{\Delta}}{c^2}\left(\frac{\partial v^{\vartheta}}{\partial x^{\sigma}}\,\zeta_{\tau\vartheta} - \frac{\partial v^{\vartheta}}{\partial x^{\tau}}\,\zeta_{\sigma\vartheta}\right) +$$

$$+\ c^{-2}\,\frac{\partial}{\partial x^{\sigma}}\left(\frac{1}{\rho}\frac{\partial p}{\partial x^{\tau}} - \gamma_{\tau\vartheta}\ f^{\vartheta} - \frac{1}{3\rho}\mu\frac{\partial v^{\vartheta}_{,\vartheta}}{\partial x^{\tau}} - \frac{\mu}{\rho}\,\gamma_{\vartheta\tau}\,\gamma^{rs}\,v^{\vartheta}_{,r,s}\right)$$

$$-\ c^{-2}\,\frac{\partial}{\partial x^{\tau}}\left(\frac{1}{\rho}\frac{\partial p}{\partial x^{\sigma}} - \gamma_{\sigma\vartheta}\ f^{\vartheta} - \frac{1}{3\rho}\mu\frac{\partial v^{\vartheta}_{,\vartheta}}{\partial x^{\sigma}} - \frac{\mu}{\rho}\,\gamma_{\vartheta\sigma}\,\gamma^{rs}\,v^{\vartheta}_{,r,s}\right).$$

Multiplying by c^2 we see that this equation coincides with equation (18). Q.E.D.

<u>Lemma 12.2</u>. If terms of order c^{-2} are neglected, then

$$(23) \qquad \sqrt{\Delta}\ \zeta_{\tau\sigma} = \frac{\partial\, \gamma_{\sigma 4}}{\partial\, x^{\tau}} - \frac{\partial\, \gamma_{\tau 4}}{\partial x^{\sigma}} + \frac{\partial\,(\gamma_{\sigma\vartheta}\ v^{\vartheta})}{\partial\, x^{\tau}} - \frac{\partial}{\partial x^{\sigma}}\,(\gamma_{\tau\vartheta}\ v^{\vartheta}) \quad .$$

<u>Proof</u>. From Definition 11.6,

$$c\sqrt{-g}\ \zeta_{\tau\sigma} = v_{\tau,\sigma} - v_{\sigma,\tau} = \frac{\partial\, v_{\tau}}{\partial\, x^{\sigma}} - \frac{\partial\, v_{\sigma}}{\partial\, x^{\tau}} \ .$$

Or, $\qquad c\sqrt{-g}\ \zeta_{\tau\sigma} = \dfrac{\partial}{\partial x^{\sigma}}\,(g_{\tau\nu}\ v^{\nu}) - \dfrac{\partial}{\partial x^{\tau}}\,(g_{\sigma\nu}\ v^{\nu})$. Using

Definition 10.4 and Lemma 10.3, this becomes

$$c^{-2}\sqrt{\Delta}\ \zeta_{\tau\sigma} = \frac{\partial}{\partial x^{\sigma}}(g_{\tau 4}v^4 + g_{\tau g}v^g) - \frac{\partial}{\partial x^{\tau}}(g_{\sigma 4}v^4 + g_{\sigma g}v^g)$$

$$= \frac{\partial}{\partial x^{\sigma}}(-c^{-2}\gamma_{\tau 4} - c^{-2}\gamma_{\tau g}v^g) - \frac{\partial}{\partial x^{\tau}}(-c^{-2}\gamma_{\sigma 4} - c^{-2}\gamma_{\sigma g}v^g).$$

Hence, $\sqrt{\Delta}\ \zeta_{\tau\sigma} = \dfrac{\partial \gamma_{\sigma 4}}{\partial x^{\tau}} - \dfrac{\partial \gamma_{\tau 4}}{\partial x^{\sigma}} + \dfrac{\partial(\gamma_{\sigma g}v^g)}{\partial x^{\tau}} - \dfrac{\partial(\gamma_{\tau g}v^g)}{\partial x^{\sigma}}$. Q.E.D.

It is convenient to write $\zeta_{\sigma\tau}$ as the sum of two terms. We make the

<u>Definition 12.2</u>. $\omega_{\tau\sigma} = \Delta^{-\frac{1}{2}}\left(\dfrac{\partial \gamma_{\sigma 4}}{\partial x^{\tau}} - \dfrac{\partial \gamma_{\tau 4}}{\partial x^{\sigma}}\right)$

$$\Omega_{\tau\sigma} = \Delta^{-\frac{1}{2}}\left(\frac{\partial\, \gamma_{\sigma g}v^g}{\partial x^{\tau}} - \frac{\partial\, \gamma_{\tau g}v^g}{\partial x^{\sigma}}\right).$$

Hence

(24) $$\zeta_{\tau\sigma} = \omega_{\tau\sigma} + \Omega_{\tau\sigma}\ .$$

We shall now write equation (18) in a more convenient form for use in the examples to be considered in the next paragraph. In these examples we will need the equations for ζ_{23} , ζ_{31} , and ζ_{12} . We simplify the writing of this equation by using the following definitions (lmn is a cyclic permutation of 123).

<u>Definition 12.3</u>. $\dfrac{d}{dx} \equiv \dfrac{\partial}{\partial x^4} + v^g\dfrac{\partial}{\partial x^g}$.

<u>Definition 12.4</u>. $\omega_l = \omega_{mn}$, $\Omega_l = \Omega_{mn}$ where $\omega_{\mu\nu}$ and $\Omega_{\mu\nu}$ are given by Definition 12.2.

<u>Definition 12.5</u>. $\gamma_{mg}f^g = F_m$, $X_l = \dfrac{\partial F_m}{\partial x^n} - \dfrac{\partial F_n}{\partial x^m}$,

$$Y_{\ell} = \frac{\partial \rho^{-1}}{\partial x^m} \frac{\partial p}{\partial x^n} - \frac{\partial \rho^{-1}}{\partial x^n} \frac{\partial p}{\partial x^m} .$$

<u>Definition 12.6.</u> $$A_{\ell} = \frac{\mu}{3}\left\{\frac{\partial}{\partial x^m}\left(\rho^{-1}\frac{\partial v^g_{,g}}{\partial x^n}\right) - \frac{\partial}{\partial x^n}\left(\rho^{-1}\frac{\partial v^g_{,g}}{\partial x^m}\right)\right\}$$

$$= \frac{\mu}{3}\left\{\frac{\partial \rho^{-1}}{\partial x^m} \frac{\partial v^g_{,g}}{\partial x^n} - \frac{\partial \rho^{-1}}{\partial x^n} \frac{\partial v^g_{,g}}{\partial x^m}\right\}$$

$$B_{\ell} = \mu\left\{\frac{\partial}{\partial x^m}\left(\rho^{-1} \gamma_{gn} \gamma^{ts} v^g_{,t,s}\right) - \frac{\partial}{\partial x^n}\left(\rho^{-1} \gamma_{gm} \gamma^{ts} v^g_{,t,s}\right)\right\}$$

First we rewrite the left side of equation (18) by using Definitions 12.3, 12.2, and 12.4 and setting $(\tau, \sigma) = (m,n)$.

$$\left(\frac{\partial}{\partial x^4} + v^g \frac{\partial}{\partial x^g}\right)\sqrt{\Delta}\, \zeta_{mn} = \sqrt{\Delta}\, \frac{d\, \zeta_{mn}}{dx} + \zeta_{mn} \frac{d \sqrt{\Delta}}{dx}$$

(25) $$\frac{d}{dx^4}\left(\sqrt{\Delta}\, \zeta_{mn}\right) = \sqrt{\Delta}\, \frac{d\,(\omega_{\ell} + \Omega_{\ell})}{dx^4} + (\omega_{\ell} + \Omega_{\ell}) \frac{d \sqrt{\Delta}}{dx^4} .$$

Now using equation (25) and the notation of Definitions 12.5 and 12.6, we have for equation (18),

$$\sqrt{\Delta}\, \frac{d\, \Omega_{\ell}}{dx^4} = -\sqrt{\Delta}\, \frac{d\, \omega_{\ell}}{dx^4} - (\omega_{\ell} + \Omega_{\ell}) \frac{d \sqrt{\Delta}}{dx^4}$$

$$- \sqrt{\Delta}\left\{\frac{\partial v^g}{\partial x^n} \zeta_{mg} - \frac{\partial v^g}{\partial x^m} \zeta_{ng}\right\} +$$

$$+ \frac{\partial}{\partial x^n}\left(\rho^{-1}\frac{\partial p}{\partial x^m}\right) - \frac{\partial}{\partial x^m}\left(\rho^{-1}\frac{\partial p}{\partial x^n}\right) +$$

$$+ \frac{\partial}{\partial x^m}(\gamma_{ng} f^g) - \frac{\partial}{\partial x^n}(\gamma_{mg} f^g)$$

$$+ \frac{\mu}{3}\left\{\frac{\partial}{\partial x^m}\left(\rho^{-1}\frac{\partial v^g_{,g}}{\partial x^n}\right) - \frac{\partial}{\partial x^n}\left(\rho^{-1}\frac{\partial v^g_{,g}}{\partial x^m}\right)\right\} +$$

$$+\mu\left\{\frac{\partial}{\partial x^m}\left(\rho^{-1}\gamma_{gm}\gamma^{rs}v^g_{,r,s}\right)-\frac{\partial}{\partial x^r}\left(\rho^{-1}\gamma_{gm}\gamma^{rs}v^g_{,r,s}\right)\right\}.$$

Or,

$$(26)\quad \frac{d\Omega_\ell}{dx^4} = -\frac{d\omega_\ell}{dx} - \Theta\,(\omega_\ell+\Omega_\ell) + \sum_{p=1}^{3}(\omega_p+\Omega_p)\frac{\partial v^\ell}{\partial x^p} +$$

$$+\left(-Y_\ell - X_\ell + A_\ell + B_\ell\right)\Delta^{-\frac{1}{2}},$$

$$\text{where}\quad \Theta = \frac{1}{\sqrt{\Delta}}\left\{\frac{\partial\sqrt{\Delta}}{\partial x^4} + \frac{\partial(\sqrt{\Delta}\,v^g)}{\partial x^g}\right\}.$$

Equation (26) is the required equation giving the rate of change of the vorticity tensor in a form used in the examples.

13. <u>Examples</u>. In this paragraph we consider two examples, one with local rectangular cartesian coordinates and the other with local cylindrical coordinates. We consider the motion of the air within a distance of 200 km. of a point 0 on the surface of the earth at North latitude ϕ.

First select the x and y directions so that positive x is toward the East, Ox and Oy lie in the tangent plane to the surface at O, and the coordinate system is a right handed system in the usual order. The kinetic metric is given by (6,296)

$$(27)\quad ds^2 = \left(1+\frac{\omega^2K_2}{c^2}\right)dt^2 - \frac{1}{c^2}\left\{dx^2+dy^2+dz^2+\left(k(z\ a)-\ell\, y\right)dxdt\right.$$

$$\left. +\ell\, xdydt - kxdzdt\right\}$$

$$\text{where}\quad K_2 = -\tfrac{1}{2}\left\{\left((z+a)\cos\phi - y\sin\phi\right)^2 + x^2\right\}$$

ω = angular speed of the earth about its polar axis

a = radius of the earth (considered as a sphere)

$$k = 2\,\omega\,\cos\phi$$
$$\ell = 2\,\omega\,\sin\phi$$

Comparing equation (27) with that for $(ds)^2$ in equation (1) of the proof of Lemma 10.1, neglecting terms in ω^2, and using the notation $(x^1, x^2, x^3, x^4) = (x, y, z, t)$ we see that

(28) $$\gamma_{44}=0, \quad \gamma_{14} = \gamma_{41} = \tfrac{1}{2}\{k(z+a)-\ell y\}, \quad \gamma_{24} = \gamma_{42} = \tfrac{1}{2}x\ell$$
$$\gamma_{34} = \gamma_{43} = -\tfrac{1}{2}kx, \quad \gamma_{pq} = \delta_{pq}, \quad \sqrt{\Delta} = 1, \quad \gamma^{pq} = \delta_{pq} \quad .$$

Following the classical notation (3,31-32) we use

$$v^1 = u, \quad v^2 = v, \quad v^3 = w, \quad \Omega_1 = \xi, \quad \Omega_2 = \eta, \quad \Omega_3 = \zeta,$$

and hence

(29) $$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
$$\omega_1 = 0, \quad \omega_2 = k, \quad \omega_3 = \ell$$

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

For the case $q = 1$, equation (14) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial}{\partial t}\{k(z+a)-\ell y\} - v\ell + wk$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + f' + \frac{\mu}{3\rho}\frac{\partial \Theta}{\partial x} + \frac{\mu}{\rho}\nabla^2 u \quad .$$

Similar equations are obtained for $q = 2$ or 3.

We now consider the equations given by the rate of change of the vorticity tensor for this example. We first consider the case where $\ell = 1$.

By Definition 12.5

(30) $$Y_1 = \frac{\partial \rho^{-1}}{\partial y}\frac{\partial p}{\partial z} - \frac{\partial \rho^{-1}}{\partial z}\frac{\partial p}{\partial y}$$

$$(31) \qquad X_1 = \frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}$$

$$A_1 = \frac{\mu}{3}\left(\frac{\partial \rho^{-1}}{\partial y} \; \frac{\partial v^{q}_{,q}}{\partial z} - \frac{\partial \rho^{-1}}{\partial z} \; \frac{\partial v^{q}_{,q}}{\partial y} \right)$$

$$(32) \qquad = \frac{\mu}{3}\left(\frac{\partial \rho^{-1}}{\partial y} \; \frac{\partial \Theta}{\partial z} - \frac{\partial \rho^{-1}}{\partial z} \; \frac{\partial \Theta}{\partial y} \right)$$

$$B_1 = \mu \left\{ \frac{\partial}{\partial y}\left(\rho^{-1} \gamma_{q3} \gamma^{rs} v^{q}_{,r,s} \right) - \frac{\partial}{\partial z}\left(\rho^{-1} \gamma_{q2} \gamma^{rs} v^{q}_{,r,s} \right) \right\}$$

$$= \mu \left\{ \frac{\partial}{\partial y}\left(\rho^{-1} \gamma_{33} \gamma^{rr} v^{3}_{,r,r} \right) - \frac{\partial}{\partial z}\left(\rho^{-1} \gamma_{22} \gamma^{rr} v^{2}_{,r,r} \right) \right\}$$

$$(33) \qquad B_1 = \mu \left\{ \frac{\partial}{\partial y}\left(\rho^{-1} \nabla^2 w \right) - \frac{\partial}{\partial z}\left(\rho^{-1} \nabla^2 v \right) \right\} .$$

We now write equation (26) for the case $\ell = 1$.

$$(34) \qquad \frac{d\xi}{dt} = -\Theta \xi + \xi \frac{\partial u}{\partial x} + (k+\eta)\frac{\partial u}{\partial y} + (\ell+\zeta)\frac{\partial u}{\partial z}$$

$$- \frac{\partial F_2}{\partial z} + \frac{\partial F_3}{\partial y} - \frac{\partial \rho^{-1}}{\partial y} \; \frac{\partial p}{\partial z} + \frac{\partial \rho^{-1}}{\partial z} \; \frac{\partial p}{\partial y} +$$

$$+ \frac{\mu}{3}\left(\frac{\partial \rho^{-1}}{\partial y} \; \frac{\partial \Theta}{\partial z} - \frac{\partial \rho^{-1}}{\partial z} \; \frac{\partial \Theta}{\partial v} \right) +$$

$$+ \mu \left(\frac{\partial}{\partial y}\left(\rho^{-1} \nabla^2 w \right) - \frac{\partial}{\partial z}\left(\rho^{-1} \nabla^2 v \right) \right) .$$

Similarly, for $\ell = 2$ and 3,

$$(35) \qquad \frac{d\eta}{dt} = -\Theta(k+\eta) + \xi \frac{\partial v}{\partial x} + (k+\eta)\frac{\partial v}{\partial y} + (\ell+\zeta)\frac{\partial v}{\partial z} +$$

$$- \frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z} - \frac{\partial \rho^{-1}}{\partial z} \; \frac{\partial p}{\partial x} + \frac{\partial \rho^{-1}}{\partial x} \; \frac{\partial p}{\partial z} +$$

$$+ \frac{\mu}{3}\left(\frac{\partial \rho^{-1}}{\partial z}\frac{\partial \Theta}{\partial x} - \frac{\partial \rho^{-1}}{\partial x}\frac{\partial \Theta}{\partial z}\right) +$$

$$+ \mu\left(\frac{\partial}{\partial z}\left(\rho^{-1}\nabla^2 u\right) - \frac{\partial}{\partial x}\left(\rho^{-1}\nabla^2 w\right)\right).$$

(36) $$\frac{d\zeta}{dt} = -\Theta\,(\ell+\zeta) + \xi\,\frac{\partial w}{\partial x} + (k+\eta)\frac{\partial w}{\partial y} + (\ell+\zeta)\frac{\partial w}{\partial z}$$

$$-\frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x} - \frac{\partial \rho^{-1}}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial \rho^{-1}}{\partial y}\frac{\partial p}{\partial x} +$$

$$+ \frac{\mu}{3}\left\{\frac{\partial \rho^{-1}}{\partial x}\frac{\partial \Theta}{\partial x} - \frac{\partial \rho^{-1}}{\partial y}\frac{\partial \Theta}{\partial x}\right\} + \mu\left\{\frac{\partial}{\partial x}\left(\rho^{-1}\nabla^2 v\right) - \frac{\partial}{\partial y}\left(\rho^{-1}\nabla^2 u\right)\right\}.$$

We now consider the same problem using cylindrical coordinates.

<u>Lemma 13.1</u>. $$ds^2 = dt^2 - \frac{1}{c^2}\Big(dr^2 + r^2 d\theta^2 + dz^2 + k(z+a)\sin\theta\; dr\,dt$$
$$+ \left(k(z+a)r\cos\theta + \ell\, r^2\right)d\theta\, dt - kr\sin\theta\; dz\,dt\Big).$$

<u>Proof</u>. Make the transformations $x = r\sin\theta$, $y = -r\cos\theta$, $z = z$ in the expression for ds^2 of equation (27).

<u>Lemma 13.2</u>. If $(r, \theta, z, t) = (x^1, x^2, x^3, x^4)$ then

(37) $$\gamma_{44} = 0, \quad \gamma_{14} = \gamma_{41} = \tfrac{1}{2}k(z+a)\sin\theta,$$
$$\gamma_{24} = \gamma_{42} = \tfrac{1}{2}\left(k(z+a)\, r\cos\theta + \ell\, r^2\right), \quad \gamma_{34} = \gamma_{43} = \frac{-kr}{2}\sin\theta,$$
$$\gamma_{11} = 1, \quad \gamma_{22} = r^2, \quad \gamma_{33} = 1, \quad \gamma_{pq} = 0 \text{ for } p \neq q, \quad \sqrt{\gamma} = r,$$
$$\gamma^{11} = 1, \quad \gamma^{22} = r^{-2}, \quad \gamma^{33} = 1, \quad \gamma^{pq} = 0 \text{ for } p \neq q.$$

<u>Proof</u>. Clear.

<u>Lemma 13.3</u>. If $(U, V, W) = (v^1, rv^2, v^3)$ then divergence

(38) $$\circledH = \frac{U}{r} + \frac{\partial U}{\partial r} + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial z} .$$

Proof. Clear.

Lemma 13.4. $$\zeta_1 = -k \cos\theta + r^{-1}\left(\frac{\partial W}{\partial \theta} - r\frac{\partial V}{\partial z}\right)$$

(39) $$\zeta_2 = r^{-1} k \sin\theta + r^{-1}\left(\frac{\partial U}{\partial z} - \frac{\partial W}{\partial r}\right)$$

$$\zeta_3 = \ell + r^{-1}\left(\frac{\partial (rV)}{\partial r} - \frac{\partial U}{\partial \theta}\right) .$$

Proof. Using Definitions 12.2 and 12.4, we have

$$\zeta_1 = \omega_1 + \Omega_1 = r^{-1}\left(\frac{\partial \gamma_{34}}{\partial x^2} - \frac{\partial \gamma_{24}}{\partial x^3}\right) + r^{-1}\left(\frac{\partial \gamma_{3g} v^g}{\partial x^2} - \frac{\partial \gamma_{2g} v^g}{\partial x^3}\right)$$

$$= r^{-1}(-kr \cos\theta) + r^{-1}\left(\frac{\partial W}{\partial \theta} - r\frac{\partial V}{\partial z}\right) .$$

ζ_2 and ζ_3 are computed similarly.

McVittie (6,299) discusses Sawyer's theory of the development of tropical cyclones and suggests a more reasonable criterion for instability than that given by Sawyer. We give a criterion applicable to viscous flow under certain simplifying assumptions.

Sawyer considers small perturbations of a symmetrical circular motion whose basic flow is defined by

(40) $$U = 0, \quad V = v_0(r,z), \quad W = 0, \quad p_0 = p_0(r,z),$$

$$\rho_0 = \rho_0(r,z), \quad \ell v_0 + \frac{v_0^2}{r} = \frac{1}{\rho_0}\frac{\partial p_0}{\partial r} .$$

He finds that the flow is unstable if

(41) $$B = \left(\ell + \frac{2v_o}{r}\right)\left(\ell + \frac{\partial v_o}{\partial r} + \frac{v_o}{r}\right)$$ is negative.

Lemma 13.5. Assuming U, V, and W are given by equation (40), then

$$\zeta_1 = -k\cos\theta - \frac{\partial v_o}{\partial z}$$

$$\zeta_2 = r^{-1} k \sin\theta$$

$$\zeta_3 = \ell + \frac{\partial v_o}{\partial r} + \frac{v_o}{r}$$

$$\Theta = 0 .$$

Proof. Clear. Use equations (38) to (40).

Lemma 13.6. $Y_1 = Y_3 = 0$, $Y_2 = \frac{\partial \rho_o^{-1}}{\partial z}\frac{\partial p_o}{\partial r} - \frac{\partial \rho_o^{-1}}{\partial r}\frac{\partial p_o}{\partial z}$,

$$X_1 = X_2 = X_3 = 0 .$$

Proof. Use Definition 12.5.

Lemma 13.7. $A_1 = 0$, $A_2 = 0$, $A_3 = 0$.

Proof. Using Definition 12.6,

$$A_1 = \frac{\mu}{3}\left(\frac{\partial \rho_o^{-1}}{\partial\theta}\frac{\partial \Theta}{\partial z} - \frac{\partial \rho_o^{-1}}{\partial z}\frac{\partial \Theta}{\partial\theta}\right) = 0 .$$

Similarly $A_2 = A_3 = 0$.

Lemma 13.8. $\nabla^2 v' = 0$.

$$\nabla^2 v^2 = \nabla^2(r^{-1} v_o) = r^{-3} v_o + r^{-1}\frac{\partial^2 v_o}{\partial r^2} + r^{-2}\frac{\partial v_o}{\partial r} + r^{-1}\frac{\partial^2 v_o}{\partial z^2} .$$

$$\nabla^2 v^3 = 0.$$

Proof. Using equation (40), we have

$$\nabla^2 v' = \nabla^2 U = 0.$$

$$\nabla^2 v^2 = \nabla^2 (r^{-1} V) = \nabla^2 \left(r^{-1} v_o (r,z)\right)$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)(r^{-1} v_o)$$

$$= 2r^{-3} v_o + r^{-1}\frac{\partial^2 v_o}{\partial r^2} - v_o r^{-3} + r^{-2}\frac{\partial v_o}{\partial r} + r^{-1}\frac{\partial^2 v_o}{\partial z^2} \quad .$$

$$\nabla^2 v^3 = \nabla^2 W = 0 \ .$$

Lemma 13.9. $B_1 = -\mu \ r^2 \frac{\partial}{\partial z}\left(\rho_o^{-1} \nabla^2 (r^{-1} v_o)\right)$

$B_2 = 0$

$B_3 = \mu \frac{\partial}{\partial r}\left(\rho_o^{-1} r^2 \nabla^2 (r^{-1} v_o)\right) \quad .$

Proof. Using Definition 12.6 and Lemma 13.8,

$$B_1 = \mu\left\{\frac{\partial}{\partial x^2}\left(\rho_o^{-1} \gamma_{q3} \gamma^{rs} v^q_{,r,s}\right) - \frac{\partial}{\partial x^3}\left(\rho_o^{-1} \gamma_{q2} \gamma^{rs} v^q_{,r,s}\right)\right\}$$

$$= \mu\left\{\rho_o^{-1}\frac{\partial}{\partial \theta}\left(\gamma^{rs} v^3_{,r,s}\right) - \frac{\partial}{\partial z}\left(\rho_o^{-1} r^2 \gamma^{rs} v^2_{,r,s}\right)\right\}$$

$$= \mu\left\{\rho_o^{-1}\frac{\partial}{\partial \theta}(\nabla^2 v^3) - \frac{\partial}{\partial z}(\rho_o^{-1} r^2 \nabla^2 v^2)\right\}$$

$$B_1 = -\mu \ r^2 \frac{\partial}{\partial z}(\rho_o^{-1} \nabla^2 v^2) \ .$$ Similarly,

$$B_2 = \mu\left\{\frac{\partial}{\partial x^3}(\rho_o^{-1} \nabla^2 v') - \frac{\partial}{\partial x'}(\rho_o^{-1} \nabla^2 v^3)\right\} = 0 \ ,$$

$$B_3 = \mu\left\{\frac{\partial}{\partial x'}(\rho_o^{-1} r^2 \nabla^2 v^2) - \frac{\partial}{\partial x^2}(\rho_o^{-1} \nabla^2 v')\right\}$$

$$= \mu\frac{\partial}{\partial r}(\rho_o^{-1} r^2 \nabla^2 v^2) \ .$$

Lemma 13.10.
$$\frac{d\zeta_1}{dx^4} = -\mu\, r\frac{\partial}{\partial z}\left(\rho_0^{-1}\,\nabla^2(r^{-1}\, v_0)\right)$$
$$\frac{d\zeta_2}{dx^4} = \zeta_1\frac{\partial(r^{-1}\, v_0)}{\partial r} + \zeta_3\frac{\partial(r^{-1}\, v_0)}{\partial z} - \frac{Y_2}{r}$$
$$\frac{d\zeta_3}{dx^4} = \mu\, r^{-1}\frac{\partial}{\partial r}\left(\rho_0^{-1}\, r^2\,\nabla^2(r^{-1}\, v_0)\right).$$

Proof. From equation (26)
$$\frac{d\zeta_1}{dx^4} = \frac{d(\Omega_1 + \omega_1\quad)}{dx^4} = -\Theta\,(\omega_1 + \Omega_1\,) + \sum_{\mu=1}^{3}(\omega_\mu + \Omega_\mu)\frac{\partial v'}{\partial x^\mu}$$
$$+ (\Delta)^{-\frac{1}{2}}(-Y_1 - X_1 + A_1 + B_1) = r^{-1}\, B_1$$
$$= -\mu\, r\frac{\partial}{\partial z}\left(\rho_0^{-1}\,\nabla^2(r^{-1}\, v_0)\right).$$ The other two expressions are found similarly.

Lemma 13.11. Using Lemma 13.5,
$$\frac{d\zeta_1}{dx^4} = k\, r^{-1}\, v_0\, \sin\theta\ ,$$
$$\frac{d\zeta_2}{dx^4} = k\, r^{-2}\, v_0\, \cos\theta\ ,$$
$$\frac{d\zeta_3}{dx^4} = 0\ .$$

Proof.
$$\frac{d\zeta_1}{dx^4} = \left(\frac{\partial}{\partial t} + v^\delta\frac{\partial}{\partial x^\delta}\right)\left(-k\cos\theta - \frac{\partial v_0}{\partial z}\right)$$
$$= v^2\frac{\partial}{\partial x^2}(-k\cos\theta) = r^{-1}\, v_0\, k\sin\theta\ .$$

The other two are found similarly.

Equating the two expressions for $\frac{d\xi_\ell}{dx^4}$ in Lemmas 13.10 and 13.11, we have

$$k\ r^{-1} v_o \sin\theta = -\mu\ r \frac{\partial}{\partial z}\left(\rho_o^{-1} \nabla^2 (r^{-1} v_o)\right)$$

(42) $$k\ r^{-2} v_o \cos\theta = -(k \cos\theta + \frac{\partial v_o}{\partial z}\ \frac{\partial (r^{-1} v_o)}{\partial r} +$$

$$+\left(\ell + \frac{\partial v_o}{\partial r} + \frac{v_o}{r}\right) r^{-1} \frac{\partial v_o}{\partial z} - \frac{Y_2}{r}$$

$$0 = \frac{\mu}{r} \frac{\partial}{\partial r}\left(\rho_o^{-1} r^2 \nabla^2 (r^{-1}\ v_o)\right)$$

where $\nabla^2 (r^{-1} v_o)$ is given in Lemma 13.8.

The second of these equations can be rewritten as

$$k\ r^{-1} \cos\theta \frac{\partial v_o}{\partial r} = \frac{1}{r}\frac{\partial v_o}{\partial z}\left(\ell + \frac{2v_o}{r}\right) - \frac{Y_2}{r}\ .$$

<u>Lemma 13.12</u>. If i) terms involving $r^{-1}\frac{\partial v_o}{\partial r}$, $r^{-1}\rho_o^{-1}$, and $r^{-1} v_o$ (except when multiplied by $\frac{\partial v_o}{\partial z}$) are neglected

ii) v_o satisfies the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) v_o = 0\ ,$$

then equation (42) can be written as

$$0 = 0$$

(43) $$0 = \frac{\partial v_o}{\partial z}\left(\ell + \frac{2v_o}{r}\right) - Y_2$$

$$0 = 0\ .$$

<u>Proof</u>. The first equation written out is

$$k r^{-1} v_o \sin\theta = -\mu\, r\left\{\frac{\partial \rho_o^{-1}}{\partial z}\left(r^{-3} v + r^{-1}\frac{\partial^2 v_o}{\partial r^2} + r^{-2}\frac{\partial v_o}{\partial r} + r^{-1}\frac{\partial^2 v_o}{\partial z^2}\right)\right\}$$

$$-\mu\, r \rho_o^{-1}\left\{r^{-3}\frac{\partial v_o}{\partial z} + r^{-1}\frac{\partial^3 v_o}{\partial z \partial r^2} + r^{-2}\frac{\partial^2 v_o}{\partial z \partial r} + r^{-1}\frac{\partial^3 v_o}{\partial z^3}\right\}$$

This clearly reduces to $0 = 0$ under the hypothesis upon noting that $\frac{\partial}{\partial z}\left(\frac{\partial^2 v_o}{\partial r^2} + \frac{\partial^2 v_o}{\partial z^2}\right) = 0$.

The second equation (as rewritten below equation (42)) gives

$$0 = \frac{\partial v_o}{\partial z}\left(\ell + \frac{2v}{r}\right) - Y_2 .$$

The third equation becomes

$$0 = \frac{\mu}{r}\frac{\partial}{\partial r}\left\{\rho_o^{-1}\left(r^{-1} v_o + r\frac{\partial^2 v_o}{\partial r^2} + \frac{\partial v_o}{\partial r} + r\frac{\partial^2 v_o}{\partial z^2}\right)\right\}$$

$$= \mu\frac{\partial \rho_o^{-1}}{\partial r}\left\{-r^{-2} v_o + r^{-1}\frac{\partial v_o}{\partial r}\right\}$$

$$+\frac{\mu}{r}\rho_o^{-1}\left\{r^{-2} v_o + r^{-1}\frac{\partial v_o}{\partial r} + \frac{\partial^2 v_o}{\partial r^2}\right\}$$. The approximations then reduce it to $0 = 0$.

<u>Theorem 13.1</u>. If i) the basic flow of a symmetrical circular motion is given by $U = 0$, $V = v_o(r,z)$, $W = 0$, $p_o = p_o(r,z)$, $\rho_o = \rho_o(r,z)$,

$$\ell v_o + \frac{v_o^2}{r} = \frac{1}{\rho_o}\frac{\partial p_o}{\partial r}$$

ii) terms involving $r^{-1}\frac{\partial v_o}{\partial r}$, $r^{-1}\rho_o^{-1}$,

$\dfrac{\partial \rho_o^{-1}}{\partial r}$, and $r^{-1} v_o$ (except when multi-

plied by $\dfrac{\partial v_o}{\partial z}$) are neglected

iii) v_o satisfies the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) v_o = 0$$

iv) $\dfrac{\partial \rho_o}{\partial z} < 0$,

then the flow is unstable (by the criterion of equation (41)) if

(a) $\zeta_3 = \ell + \dfrac{\partial v_o}{\partial r} + \dfrac{v_o}{r} > 0$ and $\dfrac{\partial \rho_o}{\partial r} \dfrac{\partial v_o}{\partial z} < 0$

or (b) $\zeta_3 < 0$ and $\dfrac{\partial \rho_o}{\partial r} \dfrac{\partial v_o}{\partial z}$ is positive.

<u>Proof</u>. From Lemma 13.6 and equation (43) of Lemma 13.12, we have $\dfrac{\partial \rho_o^{-1}}{\partial z} \dfrac{\partial p_o}{\partial r} - \dfrac{\partial \rho_o^{-1}}{\partial r} \dfrac{\partial p_o}{\partial z} = \dfrac{\partial v_o}{\partial z}\left(\ell + \dfrac{2v_o}{r}\right)$. Now neglecting $\dfrac{\partial \rho_o^{-1}}{\partial r}$ and substituting the resulting expression for $\ell + \dfrac{2v_o}{r}$ into equation (41), we obtain $\dfrac{\partial v_o}{\partial z} B = \dfrac{\partial \rho_o^{-1}}{\partial z} \dfrac{\partial p_o}{\partial r} \zeta_3$. Here ζ_3 is given by Lemma 13.5. Hence $B = -\rho_o^{-2} \dfrac{\partial \rho_o}{\partial z} \dfrac{\partial p_o}{\partial r} \left(\dfrac{\partial v_o}{\partial z}\right)^{-1} \zeta_3$. Since $\rho_o > 0$, $\dfrac{\partial \rho_o}{\partial z} > 0$, we see that B is negative, and hence the flow is unstable in the cases listed in (a) and (b).

We note that in the proof of Lemma 13.12 it was not necessary to assume that the viscosity was zero. Hence the results of Theorem 13.1 apply in the viscous as well as the non-viscous case previously discussed by McVittie.

REFERENCES

1. Courant, R., and Hilbert, D., Methoden der mathematischen Physik, Vol. I and II. Berlin: Springer, 1931 and 1937.

2. Kármán, T. v., and Biot, M. A., Mathematical Methods in Engineering. New York: McGraw-Hill, 1940.

3. Lamb, H., Hydrodynamics. New York: Dover, 1945.

4. Lanczos, C., The Variational Principles of Mechanics. Toronto: University of Toronto Press, 1949.

5. McConnell, A. J., Applications of the Absolute Differential Calculus. London: Blackie, 1931.

6. McVittie, G. C., "A Systematic Treatment of Moving Axes in Hydrodynamics," Proc., Royal Society of London, Series A, Vol. 196 (1949) pp. 285-300.

7. Michal, A. D., Matrix and Tensor Calculus with Applications to Mechanics, Elasticity, and Aeronautics. New York: Wiley, 1947.

8. -------- "General Tensor Analysis," Bull., Amer. Math. Soc., Vol. 43 (1937) pp. 394-401.

9. -------- "General Differential Geometries and Related Topics," Bull., Amer. Math. Soc., (Aug., 1939) pp. 529-563.

10. -------- "The Vibration of Elastic Strings as Studies in Geodesics," Actas, Academia Nacional de Ciencias Exactas, Fisicas y Naturales de Lima, Vol. 9, Fasciculo I (1946) pp. 3-27.

11. -------- "Studies on Geodesics in Vibrations of Elastic Beams," Proc., National Academy of Sciences, Vol. 31, No. 1 (Jan., 1945) pp. 38-43.

12. Rayleigh, Lord, The Theory of Sound, Vol. I and II. New York: Dover, 1945.

13. Synge, J. L., "On the Geometry of Dynamics," Philos. Trans., Series A, 226 (1927) p. 31.

14. Timoshenko, S., Vibration Problems in Engineering. New York: Van Nostrand, 1937.

15. Whittaker, E. T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. New York: Dover, 1944.