STRUCTURE THEOREMS FOR LOCAL NOETHER LATTICES

Thesis by

Kenneth Paul Bogart

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1968

(Submitted April 1, 1968)

ACKNOWLEDGMENTS

Professor Robert Dilworth has been the ideal thesis advisor for me. His incisive questions have often helped me to clarify my ideas. His encouragement has pulled me through several periods of stress. His patience in reading this thesis in many stages of its development and his comments about it have made it far more readable than it would have been otherwise.

My wife has put up with being alone more nights and weekends than I care to remember. Her help in proofreading the manuscript has been invaluable.

I want to thank the National Science Foundation for their support in the form of a Graduate Fellowship during my tenure as a graduate student. I also want to thank the California Institute of Technology for supplementing this fellowship with a Graduate Teaching Assistantship during the 1966-67 and 1967-68 school years.

ABSTRACT

A local Noether lattice of dimension n is _regular_ if and only if its maximal element is a join of n principal elements. A set of n principal elements whose join is the maximal element is called a _regular system of parameters_. An element of a regular system of parameters is called a _regular parameter_.

The main results of this thesis describe the structure of distributive regular local Noether lattices, and relate the structure of certain broad classes of local Noether lattices to the structure of distributive regular local Noether lattices.

A distributive regular local Noether lattice of dimension n is isomorphic to RL_n, the sublattice of the lattice of ideals of $F[x_1, \ldots, x_n]$ (F a field) consisting of all joins of products of powers of the principal ideals $(x_1), \ldots, (x_n)$. A local Noether lattice L of dimension n is regular if and only if there exists a sublattice L' of L isomorphic to RL_n, and prime, primary, and principal elements in L' are prime, primary, and principal, respectively, in L.

A lattice L with a unique proper maximal element which has a minimal representation as a join of n principal elements is a distributive local Noether lattice if and only if it is isomorphic to RL_n/θ where θ is an equivalence relation which is first defined on the principal elements of RL_n and is then extended to all of RL_n by preserving join, and where, in addition, θ preserves multiplication and preserves the cancellation of principal elements in nonzero products.

A final result which shows the strength of the condition that a local Noether lattice be regular is an abstract characterization of RL_3. If L is a regular local Noether lattice with precisely three minimal primes, and if each minimal prime of L is a regular parameter, then L is isomorphic to RL_3.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

This chapter introduces the basic ideas of abstract commutative ideal theory and describes the main results of this thesis. The proofs of these results and some additional supporting results are given in later chapters of the thesis.

The basic concept of abstract commutative ideal theory is the concept of a Noether lattice which was introduced by R. P. Dilworth [2] as an abstraction of the concept of the lattice of ideals of a Noetherian ring. A Noether lattice is a modular multiplicative lattice satisfying the ascending chain condition in which every element is a join of elements called principal elements. The principal elements are characterized by a pair of identities that are satisfied by the principal ideals of a ring, and they play the same role in the abstract theory that principal ideals play in the ideal theory of Noetherian rings.

The multiplication, meet, and join in a Noether lattice are supposed to mirror the multiplication, intersection, and sum of ideals. Because of this, a <u>multiplicative lattice</u> is defined to be a complete lattice L containing a unit element I and a null element O, and provided with a commutative, associative, join-distributive multiplication for which I is an identity element. We will use $\wedge$ and $\vee$ to denote meet and join, respectively, and $\leq$ to denote lattice partial orderings, with $<$ reserved for strict inequality. For each A, B in L, A:B, the residual of A by B, is the join of all X in L such that $XB \leq A$. An element E in L is principal if

(1. 1) $(A \wedge B{:}E)E = AE \wedge B$ (all $A, B \in L$) and

(1. 2) $(A \vee BE){:}E = A{:}E \vee B$ (all $A, B \in L$) .

(An element E in L is meet principal if it satisfies (1. 1); join principal if satisfies (1. 2).) We will reserve the letters E, F, H, K, and N for principal elements. Since all elements in the lattice of ideals of a multiplication ring (for example, a Dedekind domain) are principal [3], the concept of a principal element is broader than that of a principal ideal.

An element P of a Noether lattice L is called prime if for all A, B in L, $P \geq AB$ implies $P \geq A$ or $P \geq B$. An element Q in L is primary if for all A, B in L, $Q \geq AB$ implies $Q \geq A$ or $Q \geq B^k$ for some integer k. If Q is primary, the join P_Q of all X such that $X^k \leq Q$ for some integer k is a prime containing Q and is called the associated prime of Q. The usual theorems about the existence and uniqueness of primary decompositions [4, pp. 14-15, p. 21], [2, pp. 483-86] hold for Noether lattices.

Let P be a prime element of a Noether lattice L. P has rank r if r is the maximum of the lengths of chains of distinct primes less than P. P has dimension d if d is the maximum of the lengths of chains of distinct proper primes greater than P. Let $A \in L$. Then A has rank r if r is the minimum of the ranks of its associated primes; A has dimension d if d is the maximum of the dimensions of its associated primes. A Noether lattice is local if it has precisely one proper maximal element. If L is local then, as for rings [4, p. 63], dim(0) is finite and is called the dimension of L.

Two very important theorems of commutative ideal theory which Dilworth has generalized to Noether lattices are Krull's intersection theorem and Krull's principal ideal theorem. Restricted to a local Noether lattice with maximal element M, the intersection theorem states that $\wedge_k M^k = 0$. The principal element theorem states that the rank of a minimal prime containing a principal element is at most one.

If L is a local Noether lattice of dimension n, a set of n principal elements whose join is primary with respect to the maximal element of L is a system of parameters. A set of n principal elements (called regular parameters) whose join is the maximal element is a regular system of parameters. If L has a regular system of parameters, L is a regular local Noether lattice.

The structure of arbitrary regular local Noether lattices is closely related to a special class $\{RL_n\}$ of Noether lattices. The elements of RL_n are those ideals of $F[x_1, \ldots, x_n]$ (F a field) which are joins of products of the principal ideals $(x_1), (x_2), \ldots, (x_n)$. It will be shown that RL_n is a sublattice of the lattice of ideals of $F[x_1, \ldots, x_n]$ and is a regular local Noether lattice. The relationship of $\{RL_n\}$ and arbitrary regular local Noether lattices will be described as follows.

A local Noether lattice L of dimension n is regular if and only if there exists an isomorphism $\varphi: RL_n \to L$ with the property that the images under φ of prime, primary, and principal elements in RL_n are prime, primary, and principal, respectively in L.

A distributive regular local Noether lattice is isomorphic to one of the lattices RL_n.

For $n \geq 2$, it will also be shown that RL_n is not isomorphic to the lattice of all ideals of any Noetherian ring. In fact, an appropriate sublattice of RL_2 provides an example of a Noether lattice for which the usual "converse" to Krull's principal-ideal theorem (a prime of rank one is a minimal prime of some principal ideal) does not hold.

In the process of obtaining these results, the following two theorems which are generalizations of theorems about Noetherian rings will be proved. If L is a regular local Noether lattice, then any join of a subset of a regular system of parameters is a prime. If L is regular, then L/D is regular if and only if D is a join of a subset of a regular system of parameters. (L/D is the set of all elements in L greater than or equal to D and is a Noether lattice [2].

It is possible to obtain fairly concrete information about Noether lattices satisfying the distributive law. It will be shown that a Noether lattice is distributive if and only if $(A \vee B):E = A:E \vee B:E$ for all A, B and all principal elements E in L. A complete characterization of distributive local Noether lattices will be given. Each distributive local Noether lattice may be represented as RL_n/θ for a suitable integer n and a suitable equivalence relation θ. To complete the characterization, a description will be given of those equivalence relations θ such that RL_n/θ is a distributive local Noether lattice.

A Noether-lattice imbedding of L' in L will be defined as an isomorphism of L' into L which preserves prime, primary, and principal elements. The first main result above states that there is a Noether-lattice imbedding of RL_n in each regular local Noether lattice of dimension n. A question which naturally arises is whether every Noether lattice has a Noether lattice imbedding in the lattice of ideals of some ring. This question will be answered by an example of a Noether lattice which cannot be imbedded in the lattice of ideals of any Noetherian ring. This lattice has the trivial multiplication (AB = 0 unless A = I or B = I). The computations used in this example will be generalized to give a proof of the following theorem.

A lattice may be represented as a Noether lattice with the trivial multiplication if and only if it is a finite-dimensional modular lattice with precisely one proper maximal element in which every element but I is a join of atoms.

The final result will be an abstract characterization of RL_3 as follows.

Let L be a regular local Noether lattice with precisely three minimal primes. Then if each minimal prime of L is a member of a regular system of parameters, L is isomorphic to RL_3.

This result lends some credence to the conjecture that every regular local Noether lattice may be imbedded in the lattice of ideals of some Noetherian ring. The depth of the analysis used in proving the result described above indicates that this conjecture is likely to be very difficult either to verify or disprove.

CHAPTER II

PRELIMINARY RESULTS AND EXAMPLES

The following important lemma is an immediate consequence of the intersection theorem.

Lemma 2.1. *If* L *is a local Noether lattice and* $A, B \in L$, *then* $AB = B$ *implies* $A = I$ *or* $B = 0$.

Proof: Let $A \neq I$. Then

$$B = A^k B \leq M^k \quad \text{all } k \quad .$$

Thus $B \leq \bigwedge_k M^k = 0$, so that $B = 0$.

Lemma 2.2. *Let* L *be a local Noether lattice. Then an element of* L *is principal if and only if it is join-irreducible.*

Proof: Clearly, join-irreducible elements are principal; so let E be a principal element in L ($E \neq 0$). Let $E = D_1 \vee \ldots \vee D_n$. Then, by equation (1.1),

$$D_i = (D_i{:}E)E \quad (\text{all } i) \quad .$$

Thus

$$\begin{aligned} E &= (D_1{:}E)E \vee \ldots \vee (D_n{:}E)E \\ &= (D_1{:}E \vee \ldots \vee D_n{:}E)E \quad . \end{aligned}$$

By Lemma 2.1,

$$(D_1:E) \vee \ldots \vee (D_n:E) = I \quad .$$

Since L is local, there must be a j such that $D_j:E = I$; but this implies that $E \leq D_j$, so that $E = D_j$.

Corollary 2.1. Let L be the lattice of ideals of a local Noetherian ring R. Then the principal elements of L are precisely the principal ideals of R.

Applying the Kurosh-Ore theorem to the dual lattice of a local Noether lattice, we obtain the following corollary.

Corollary 2.2. Let L be a local Noether lattice, and let $A \in L$. Then any two minimal representations of A as a join of principal elements have the same number of principal elements. If L is distributive, then each element of L has a unique minimal representation as a join of principal elements.

Of course, the usual replacement properties [1] of the Kurosh-Ore theorem follow also.

Some of the examples and proofs we give use computations with quotient sublattices $L/D = \{A \in L \mid A \geq D\}$. With the multiplication $A \circ B = AB \vee D$, L/D is a Noether lattice; and if F is a principal element of L, then $F \vee D$ is principal in L/D [2].

Lemma 2.3. If L is a local Noether lattice, then the principal elements of L/D are precisely the elements of the form $D \vee E$, where E is principal in L.

Proof: Let E' be a principal element in L/D. There exist $E_1, \ldots, E_k$ in L such that

$$E' = D \vee E_1 \vee \ldots \vee E_k = (D \vee E_1) \vee \ldots \vee (D \vee E_k) .$$

E' is principal in L/D, so that we can apply Lemma 2.2 to L/D to obtain a j such that $E' = D \vee E_j$.

In the proof of the fact that every local ring has a system of parameters, the following lemma is often used. If $P_1, P_2, \ldots, P_n$ are prime ideals of a Noetherian ring R and A is an ideal of R not contained in any P_i, then there exists a principal ideal $(a) \leq A$ such that $(a) \not\leq P_i$ for all i [4, p. 12]. This lemma does not hold for Noether lattices, as the following example shows.

Let RL_2 be the set consisting of (0) and all the ideals of $F[x,y]$ (F a field) of the form

$$(x)^{i(1)}(y)^{j(1)} \vee \ldots \vee (x)^{i(n)}(y)^{j(n)} .$$

It is easily seen that RL_2 is closed under join and multiplication. We shall show that if

$$A = (a_1) \vee \ldots \vee (a_n), \quad \text{and}$$
$$B = (b_1) \vee \ldots \vee (b_n) ,$$

where $a_s = x^{i(s)}y^{j(s)}$ and $b_t = x^{k(t)}y^{h(t)}$, then

$$A \wedge B = \bigvee_{s,t} (\text{l.c.m.}(a_s, b_t)) .$$

Clearly $\text{l.c.m.}(a_s, b_t)$ is in $A \wedge B$; therefore, let $p(x,y)$ be in

$A \wedge B$. Since $p(x,y)$ is in A, there exist polynomials $p_s(x,y)$ such that

$$p(x,y) = p_1(x,y)a_1 + \ldots + p_n(x,y)a_n \quad .$$

Thus each nonzero term of $p(x,y)$ is divisible by a_s for some s. Similarly, each term is divisible by b_t for some t. Therefore $p(x,y)$ is in the join of the principal ideals generated the least common multiples of the a_s and the b_t.

This shows that RL_2 is closed under meet. To show that RL_2 is closed under residuation, observe that since $A:(B \vee C) = A:B \wedge A:C$ and $A:(BC) = (A:B):C$, it is sufficient to show that $A:(x)$ is in RL_2 for all A in RL_2. We assume that $A':(x)$ is in RL_2 if A' is a join of fewer $(x)^i(y)^j$ than A. If $A = (y)^j$, $A:(x) = (y)^j:(x) = (y)^j \in RL_2$, so that we may assume

$$A = (x)^i(y)^j \vee A' \qquad (i \geq 1) \quad .$$

Then using equation (1. 2), we find that

$$A:(x) = (x)^{i-1}(y)^j \vee A':(x) \quad ;$$

the right-hand member is in RL_2, by the induction hypothesis. But now, since principal elements are defined by equations using meet, join, multiplication and residuation, the elements $(x)^i(y)^j$ are principal in RL_2. Thus RL_2 is a Noether lattice.

We note also that RL_2 is distributive, for if $A = (a_1) \vee \ldots \vee (a_n)$, $B = (b_1) \vee \ldots \vee (b_m)$, and $C = (c_1) \vee \ldots \vee (c_k)$ are elements of RL_2, then

$$A \wedge (B \vee C) = \vee \{\text{l. c. m.}(a_i, d_j) \mid d_i = b_i,\ i \le m;\ d_i = c_{i-m}, i > m\}$$
$$= \bigvee_{i,j} (\text{l. c. m.}(a_i, b_j)) \vee \bigvee_{i,j} (\text{l. c. m.}(a_i, c_j))$$
$$= (A \wedge B) \vee (A \wedge C) .$$

It is clear that the only proper prime elements of RL_2 are $(x) \vee (y)$, (x), (y), and (0). However, by Lemma 2. 2, the only principal elements in RL_2 are the elements $(x)^i(y)^j$. Thus every principal element of RL_2 is less than or equal to (x) or (y). Now, with $A = (x) \vee (y)$, $P_1 = (x)$, and $P_2 = (y)$, it is clear that $A \not\leq P_1$ and $A \not\leq P_2$, while every principal element contained in A is contained in either P_1 or P_2. Clearly, though, RL_2 has a system of parameters, and in fact it is regular.

The next example is an example of a local Noether lattice without a system of parameters. Let $L = RL_2/(x)(y)$. By Lemma 2. 3, all principal elements of L are less than or equal to (x) or (y), since $(x)(y)$ is less than both (x) and (y). But since (x) and (y) are primes of rank 0, every principal element of L has rank 0. Thus $(x) \vee (y)$ is a prime of rank 1 containing no principal elements of rank 1, so that the "converse" to the Krull theorem does not hold for L and L has no system of parameters.

These examples show that RL_2 cannot be isomorphic to the lattice of ideals of any ring.

CHAPTER III

BASIC STRUCTURE THEOREMS FOR REGULAR LOCAL NOETHER LATTICES

The concept of a Noether lattice imbedding is used in the main theorem of this section. Let L and L' be Noether lattices. We say that $\varphi: L \to L'$ is a Noether-lattice imbedding of L in L' if φ is an isomorphism of L into L' and the images under φ of prime, primary, and principal elements of L are prime, primary, and principal, respectively, in L'.

Recall that RL_n consists of all joins of products of the ideals (x_1), (x_2), . . . , (x_n) in the lattice of ideals of $F[x_1, \ldots, x_n]$ (F a field). As in the case of RL_2, it is easily verified that RL_n is a regular local Noether lattice that is not isomorphic to the lattice of ideals of any ring. Again, since the meet of two elements in RL_n is the join of the ideals generated by the least common multiples of their generators, we can apply the computation by which we showed that RL_2 is distributive to prove that RL_n is distributive. The main theorem of this chapter states that if L is a local Noether lattice of dimension n, then L is a regular local Noether lattice if and only if there exists a Noether-lattice imbedding of RL_n in L.

In the proof of the main theorem, we shall use two theorems that are generalizations of well-known theorems [4, p. 73], [5, p. 303] about regular local rings. The following lemma and its corollary form the basis of the proof of these two theorems.

Lemma 3.1. Let L be a local Noether lattice, let $A = E_1 \vee \ldots \vee E_r$ be a member of L, and let A have dimension s. Then the dimension of $E_1 \vee \ldots \vee E_{r-1}$ is at most $s + 1$.

Proof: Let $\dim(E_1 \vee \ldots \vee E_{r-1}) = s + i$. Then there exists a chain of primes

$$P_1 > \ldots > P_{s+i} > P_{s+i+1} \geq E_1 \vee \ldots \vee E_{r-1} .$$

Since E_r is a principal element by Lemma 6.4 of [2] there exists a chain of primes

$$P_1 = P_1^* > P_2^* > \ldots > P_{s+i}^* > P_{s+i+1}$$

such that $P_{s+i}^* \geq E_r$. Since $P_{s+i}^* \geq A$ and A has dimension at least $s + i - 1$, we see that $s \geq s + i - 1$, which implies that $i \leq 1$.

Corollary 3.1. Let L be a regular local Noether lattice, and let $E_1, \ldots, E_n$ be a regular system of parameters for L. Then $\dim(E_1 \vee \ldots \vee E_k) = n - k$.

Proof: Apply Lemma 3.1 $n - k$ times to show that $\dim(E_1 \vee \ldots \vee E_k)$ is at most $n - k$. Suppose $\dim(E_1 \vee \ldots \vee E_k)$ is $n - k - i$ $(i \geq 0)$. Then apply Lemma 3.1 k times to show that $\dim(0) \leq n - i$. Thus $n \leq n - i$ which implies $i = 0$.

The usual ring-theoretic proof [4, p. 75] shows that if L is a regular local Noether lattice of dimension one, then every element of

L is of the form E^k, where E is the maximal element of L. Clearly, this implies that 0 is a prime in L; for if $E^k = 0$, then E would be contained in some prime of rank 0, contrary to the relation rank(E) = dim(0) = 1. The next theorem extends this remark, and its proof is a rather natural extension of the simple computation used to prove the remark.

Theorem 3.1. Let L be a regular local Noether lattice. Then any join of a subset of a regular system of parameters is a prime.

Proof: Let $E_1, \ldots, E_n$ be a regular system of parameters for L. The proof uses induction on n - r to show that 0 is a prime in $L/(E_1 \vee \ldots \vee E_r)$. By Lemma 3.1 and the remark above, 0 is a prime in $L/(E_1 \vee \ldots \vee E_{n-1})$.

Assume $E_1 \vee \ldots \vee E_r$ is a prime if $r > i$. Let

$$L' = L/(E_1 \vee \ldots \vee E_i) ,$$

and let X' denote $X \vee E_1 \vee \ldots \vee E_i$ for all X in L. By the induction hypothesis, E_j' is a prime in L' for all $j > i$. Now E_j' must contain a minimal prime of 0', for it is prime. By Corollary 3.1, L' has dimension n - i and E_j' has dimension n - i - 1. By Lemma 6.4 in [2], there exists a chain,

$$(E_1 \vee \ldots \vee E_n)' > P_1^* > \ldots > P_{n-i-1}^* > P_{n-i}'$$

in L' such that $P_{n-i-1}^* \geq E_j'$. Thus $E_j' = P_{n-i-1}^*$, and E_j' is not a minimal prime of 0'.

Now let P' be a minimal prime of $0'$ contained in E_j'. Since $P' = (P':E_j')E_j'$ and P' is prime, $P':E_j' \leq P'$. Therefore, $P' = P':E_j'$, and so $P' = P'E_j'$, which implies that $P' = 0'$, by Lemma 2.1. Therefore, $0'$ is a prime in L', and hence $E_1 \vee \ldots \vee E_i$ is a prime for all i.

Theorem 3.2. Let L be a regular local Noether lattice, and let D be an element of L. Then L/D is regular if and only if D is a join of a subset of a regular system of parameters.

Proof: Corollary 3.1 implies that if L is a regular local Noether lattice and $E_1, \ldots, E_n$ is a regular system of parameters for L, then $L/(E_1 \vee \ldots \vee E_k)$ is regular.

Assume that L/D is regular, and let M be the maximal element of L. Then, by Lemma 2.3, $M = D \vee F_1 \vee \ldots \vee F_k$, where $D \vee F_1, \ldots, D \vee F_k$ is a regular system of parameters in L/D. Let $D = H_1 \vee \ldots \vee H_r$. Since $D \vee F_1, \ldots, D \vee F_k$ is a regular system of parameters, we may assume that by renumbering the H_i and dropping superfluous ones that

$$M = H_1 \vee \ldots \vee H_s \vee F_1 \vee \ldots \vee F_k$$

is a minimal representation of M as a join of principal elements. By Corollary 2.2, $s + k = \dim(L)$; therefore $H_1, \ldots, H_s$ is a subset of a regular system of parameters. Thus $H_1 \vee \ldots \vee H_s = D'$ is a prime. But rank $D' = s$ and $\dim(D) = k = \dim(L) - s$, so that rank $(D) \leq s$, which implies that $D = D'$.

In proving the main theorem, we shall use the fact that in a

Noether lattice A:B = A if and only if no associated prime of A contains B (this can be proved as for rings; see [4, p. 23]). We shall also need the following lemma (the symbol $\cong$ indicates lattice isomorphism).

Lemma 3.2. *Suppose that* L *is a Noether lattice in which* 0 *is a prime, that* A, $E \in L$ (E *principal*), *and that* $A{:}E = A$. *Then*

$$(E \vee A)/(E^i \vee A) \cong I/(E^{i-1} \vee A) \quad .$$

Proof: In the relations

$$\begin{aligned}(E \vee A)/(E^i \vee A) &= [E \vee (E^i \vee A)]/(E^i \vee A) \\ &\cong E/[E \wedge (E^i \vee A)] = E/[E^i \vee (E \wedge A)] \\ &= E/[E^i \vee (A{:}E)E] = E/(E^i \vee EA) \\ &= E/[E^{i-1} \vee A)E] \cong I/(E^{i-1} \vee A) \quad ,\end{aligned}$$

the first isomorphism follows by modularity, the second by Lemma 6.3 of [2].

Theorem 3.3. *Let* L *be a local Noether lattice of dimension* n. *Then* L *is a regular local Noether lattice if and only if there exists a Noether-lattice imbedding of* RL_n *in* L.

Proof: Clearly if RL_n can be imbedded in L, the maximal prime of RL_n maps onto the maximal prime of L. Thus the maximal prime of L is a join of n principal elements, and L is regular.

Now assume that L is a regular local Noether lattice and let $E_1, \ldots, E_n$ be a regular system of parameters for L. Define a map

φ from RL_n into L by defining $\varphi(0) = 0$ and $\varphi[(x_i)] = E_i$, and then extending φ to all of L by preserving product and join. Since by Corollary 2.2 each element of RL_n has a unique minimal representation as a join of principal elements, and since each principal element of RL_n has a unique factorization in terms of the elements (x_i), this method of defining φ does yield a map. Note that RL_k may be considered as a subset of RL_n. We shall use induction to prove that φ is a Noether-lattice imbedding; in particular, we shall show that φ restricted to RL_k is a Noether-lattice imbedding of RL_k in L for $k \leq n$.

The restriction of φ to RL_1 is an isomorphism of RL_1 into L, since the image of φ is a regular local Noether lattice of dimension 1 and is therefore isomorphic to RL_1 (see the remark preceding Theorem 3.1). A simple inductive argument shows that E_1^i is primary; since E_1 is both principal and prime (Theorem 3.1) it follows that this isomorphism is a Noether lattice imbedding.

Now assume that φ restricted to RL_j is a Noether lattice imbedding of RL_j in L, for $j < k \leq n$. Denote the restriction of φ to RL_k by φ'. To show that φ' is an isomorphism, observe first that φ' preserves products and joins. It is evident that φ' preserves meets of principal elements, for

$$E_1^{i(1)} \wedge E_2^{i(2)} \wedge \dots \wedge E_k^{i(k)} = E_1^{i(1)} E_2^{i(2)} \dots E_k^{i(k)} .$$

Thus

$$E_1^{i(1)} E_2^{i(2)} \dots E_k^{i(k)} \wedge E_1^{j(1)} E_2^{j(2)} \dots E_k^{j(k)}$$

$$= E_1^{m(1)} E_2^{m(2)} \dots E_k^{m(k)} ,$$

where $m(t) = \max(i(t), j(t))$.

We shall use computations with residuations to show that φ' preserves arbitrary meets; but first we must prove that φ' preserves residuation in certain special cases. Let

$$A = \bigvee_h E_2^{i(h,2)} E_3^{i(h,3)} \dots E_k^{i(h,k)} \quad .$$

Then, by the induction hypothesis, the element A has a normal decomposition in which all the associated primes are contained in $E_2 \vee \dots \vee E_k$, and since $E_1 \not\leq E_2 \vee \dots \vee E_k$, $A{:}E_1 = A$.

Thus, in view of equation (1. 2),

$$\varphi'[B{:}(x_1)] = \varphi'(B){:}\varphi'[(x_1)] \quad .$$

Since $X{:}(YZ) = (X{:}Y){:}Z$, it follows that

$$\varphi'(B{:}F) = \varphi'(B){:}\varphi'(F)$$

for all $B \in RL_k$ and all principal elements $F \in RL_k$. Also,

$$\varphi'(B \wedge F) = \varphi'[(B{:}F)F] = \varphi'(B{:}F)\varphi'(F)$$
$$= [\varphi'(B){:}\varphi'(F)]\varphi'(F) = \varphi'(B) \wedge \varphi'(F) \quad .$$

We shall now show that φ' preserves all meets. Since RL_k is distributive, $(A \vee B) \wedge F = (A \wedge F) \vee (B \wedge F)$ for all A, B, and principal elements F in RL_k. Then, in L,

$$\varphi'[(A \vee B) \wedge F] = [\varphi'(A) \wedge \varphi'(F)] \vee [\varphi'(B) \wedge \varphi'(F)] \quad .$$

Now let $C = F_1 \vee \dots \vee F_s \in RL_k$. Temporarily, let $\varphi'(X) = X'$. Assume that

(3. 1) $\quad (F_1' \vee \dots \vee F_r') \wedge D' = (F_1' \wedge D') \vee \dots \vee (F_r' \wedge D') \quad ,$

for all $r < s$ and for all D'. Then

$$(F'_1 \vee \dots \vee F'_s) \wedge D' = (F'_1 \vee \dots \vee F'_s) \wedge (F'_s \vee D') \wedge D'$$
$$= \{F'_s \vee [(F'_1 \vee \dots \vee F'_{s-1}) \wedge (F'_s \vee D')]\} \wedge D'$$
$$= \{F'_s \vee [F'_1 \wedge (F'_s \vee D')] \vee \dots \vee [F'_{s-1} \wedge (F'_s \vee D')]\} \wedge D'$$
$$= [F'_s \vee (F'_1 \wedge F'_s) \vee (F'_1 \wedge D') \vee \dots \vee (F'_{s-1} \wedge F'_s) \vee (F'_{s-1} \wedge D')] \wedge D'$$
$$= [F'_s \vee (F'_1 \wedge D') \vee \dots \vee (F'_{s-1} \wedge D')] \wedge D'$$
$$= (F'_s \wedge D') \vee (F'_1 \wedge D') \vee \dots \vee (F'_{s-1} \wedge D') \quad .$$

This shows that Equation (3. 1) holds for all r. Now let $D = H_1 \vee \dots \vee H_t$. Then

$$\varphi'(C) \wedge \varphi'(D) = (F'_1 \vee \dots \vee F'_s) \wedge D'$$
$$= (F'_1 \wedge D') \vee (F'_2 \wedge D') \vee \dots \vee (F'_s \wedge D')$$
$$= \bigvee_{i,j} (F'_i \wedge H'_j) = \bigvee_{i,j} \varphi'(F_i) \wedge \varphi'(H_j)$$
$$= \bigvee_{i,j} \varphi'(F_i \wedge H_j) = \varphi'[\bigvee_{i,j} (F_i \wedge H_j)]$$
$$= \varphi'(C \wedge D) \quad .$$

Now, since $X:(Y \vee Z) = (X:Y) \wedge (X:Z)$ and $\varphi'(A:F) = \varphi'(A):\varphi'(F)$, φ' preserves residuation and is therefore a homomorphism.

But now $\varphi'(RL_k)$ is distributive, and, therefore by Corollary 2. 2, two elements are equal if and only if they are joins of exactly the same principal elements. But this implies that the mapping φ' is one-to-one, since it is clearly one-to-one on principal elements. Thus φ' is an isomorphism.

By Lemma 2. 2, the only principal elements of RL_k are the elements $(x_1)^{i(1)}(x_2)^{i(2)} \dots (x_k)^{i(k)}$; therefore φ' maps principal elements to principal elements. By Theorem 3. 1, φ' maps primes to primes. To show that φ' is a Noether-lattice imbedding, we must show that it preserves primary elements. Since every meet-irreducible element is primary, since the intersection of primaries with the same associated prime is primary, and since every element is an intersection of meet-irreducible elements, it is sufficient to show that φ' preserves meet-irreducible elements.

Since RL_k is distributive, it is easy to see that the only meet-irreducible elements in RL_k are the elements of the form $(x_1)^{i(1)} \vee \ldots \vee (x_h)^{i(h)}$. Now, in L, the elements $E_1 \vee \ldots \vee E_s$ are meet-irreducible, since they are prime by Theorem 3. 1. We shall use induction to show that the elements $E_1^{i(1)} \vee \ldots \vee E_s^{i(s)}$ are irreducible in L. Suppose that $E_1^{i(1)-1} \vee E_2^{i(2)} \vee \ldots \vee E_s^{i(s)}$ is irreducible in L. Then, by Lemma 3. 2, $B = E_1^{i(1)} \vee \ldots \vee E_s^{i(s)}$ is irreducible in B_1/B, where $B_1 = E_1 \vee E_2^{i(2)} \vee \ldots \vee E_s^{i(s)}$. Now assume that $B = C_1 \wedge \ldots \wedge C_r$. Since $B = B \wedge B_1$,

$$B = (B_1 \wedge C_1) \wedge \ldots \wedge (B_1 \wedge C_r) \quad .$$

Since B is irreducible in B_1/B, there exists a j such that $B = B_1 \wedge C_j$, and therefore

$$\begin{aligned} B{:}E_1 &= (B_1 \wedge C_j){:}E_1 = B_1{:}E_1 \wedge C_j{:}E_1 \\ &= C_j{:}E_1 \text{ since } E_1 \leq B_1 \quad . \end{aligned}$$

Thus

$$C_j \leq C_j{:}E_1 = B{:}E_1 = E_1^{i(1)-1} \vee E_2^{i(2)} \vee \ldots \vee E_s^{i(s)} \quad .$$

But this implies that C_j is in B_1/B, hence $C_j = B$. Therefore B is meet-irreducible and φ' preserves meet-irreducible elements. This implies that φ' is a Noether-lattice imbedding of RL_k in L. But now, by induction, φ is a Noether-lattice imbedding of RL_n in L.

CHAPTER IV

DISTRIBUTIVE NOETHER LATTICES: BASIC STRUCTURE THEOREMS AND EXAMPLES

The first result of this chapter is a characterization of distributive Noether lattices in terms of join and residuation.

Theorem 4. 1. *Let* L *be a Noether lattice. Then* L *is distributive if and only if*

(4. 1) $$(A \vee B):E = A:E \vee B:E$$

for all A, B *and all principal elements* E *in* L.

Remark: Note that Equation (4. 1) is equivalent to

(4. 2) $$(A \vee B) \wedge E = (A \wedge E) \vee (B \wedge E)$$

for all A, B and all principal elements E in L. To obtain (4. 2) from (4. 1), multiply both sides of (4. 1) by E and use equation (1. 1). To obtain (4. 1) from (4. 2), residuate both sides of (4. 2) by E; then use the facts that in any multiplicative lattice L, $(X \wedge Y):Y = X:Y$ and $0:Y \leq X:Y$ for all X and Y in L.

Proof of Theorem 4. 1: Suppose L is distributive. Then by the distributive law, (4. 2) holds for all A, B and all principal elements E in L. By the remark above, (4. 1) holds.

Now suppose that (4. 1) holds for all A, B and all principal elements E in L. Since (4.2) also holds, it follows by induction that

$$(A_1 \vee \ldots \vee A_k) \wedge E = (A_1 \wedge E) \vee \ldots \vee (A_k \wedge E)$$

for all $A_1, \ldots, A_k$ in L. Let A, B, and C be in L, and let $C = E_1 \vee \ldots \vee E_n$ be a minimal representation of C as a join of principal elements. Let $C_1 = E_2 \vee \ldots \vee E_n$, and suppose inductively that for all A, B in L,

$$(4.3) \qquad (A \vee B) \wedge C_1 = (A \wedge C_1) \vee (B \wedge C_1) \quad .$$

It is clear that $(A \vee B) \wedge C \geq (A \wedge C) \vee (B \wedge C)$; thus it is necessary to show that $(A \vee B) \wedge C \leq (A \wedge C) \vee (B \wedge C)$. In the relations

$$\begin{aligned}(A \vee B) \wedge C &= (A \vee B) \wedge (A \vee B \vee E_1) \wedge (E_1 \vee C_1)\\ &= (A \vee B) \wedge \{E_1 \vee [(A \vee B \vee E_1) \wedge C_1]\}\\ &= (A \vee B) \wedge [E_1 \vee (A \wedge C_1) \vee (B \wedge C_1) \vee (E_1 \wedge C_1)]\\ &= (A \wedge C_1) \vee (B \wedge C_1) \vee [E_1 \wedge (A \vee B)]\\ &= (A \wedge C_1) \vee (A \wedge E_1) \vee (B \wedge C_1) \vee (B \wedge E_1)\\ &\leq (A \wedge C) \vee (B \wedge C) \quad ,\end{aligned}$$

the second and fourth lines are by modularity, the third line is by (4.3) and induction, and the fifth line is by (4.2). Therefore L is distributive.

By replacing the phrase "principal element" by the phrase "principal ideal" throughout the proof of Theorem 4.1, we obtain the following corollary which gives a sharper result for the case where L is the lattice of ideals of a ring.

Corollary 4.1. *If L is the lattice of ideals of a Noetherian ring R, then L is distributive if and only if equation (4.1) holds for all ideals A, B and all principal ideals E of R.*

For local Noether lattices, the distributive law is a very strong restriction on the lattice structure.

Lemma 4.1. *Let L be a distributive local Noether lattice, and let $M = E_1 \vee \dots \vee E_n$ be a minimal representation of the maximal element as a join of principal elements. Then each nonzero proper principal element of L is a product of powers of the elements E_i.*

Proof: Let $E \neq I$ be a nonzero principal element of L. Then

$$E = E \wedge M = E \wedge (E_1 \vee \dots \vee E_n) = (E \wedge E_1) \vee \dots \vee (E \wedge E_n) \quad ,$$

which implies that $E = E \wedge E_k$ for some k by Lemma 2.2. Thus $E \leq E_k$. Let i(k) be the largest integer such that $E \leq E_k^{i(k)}$. Then

$$E = (E:E_k^{i(k)})\, E_k^{i(k)}$$

by Equation 1.1. Lemma 2.2 implies that $E = FE_k^{i(k)}$ for some principal element $F < E:E_k^{i(k)}$. $F \not\leq E_k$ since $E \not\leq E_k^m$ for any m greater than i(k). Thus $F = I$ or $F \leq E_j$ for some $j \neq k$. If $F = I$, there is nothing more to do. If $F \neq I$, apply the same process to F as we applied to E to obtain

$$E = HE_j^{i(j)}\, E_k^{i(k)}$$

where i(j) is the largest integer such that $F \leq E_j^{i(j)}$. Iterate this process to obtain

$$E = K E_1^{i(1)} E_2^{i(2)} \dots E_n^{i(n)}$$

with $K \not\leq E_m$ for $m = 1, 2, \dots, n$. Then $K = I$ and the lemma is proved.

Theorem 4.2. A distributive regular local Noether lattice is isomorphic to one of the lattices RL_n.

Proof: Let L be a distributive regular local Noether lattice of dimension n. By Theorem 3.3 there is a Noether-lattice imbedding of RL_n in L. Let $E_i = \varphi[(x_i)]$. Since by the definition of a Noether-lattice imbedding $\{E_1, \dots, E_n\}$ is a regular system of parameters, Lemma 4.1 implies that each element of L is a join of products of powers of the elements E_i. Thus φ is an onto map, so it is an isomorphism of RL_n onto L.

From Lemma 4.1 we see that it is possible to define a map from RL_n onto a distributive local Noether lattice whose maximal element has a minimal representation as a join of n principal elements. The next theorem shows that this fact allows us to characterize distributive local Noether lattices in a very concrete manner.

Theorem 4.3. A lattice L is a distributive local Noether lattice if and only if there exist an integer n, an equivalence relation θ on RL_n and an equivalence relation σ on the set of principal elements of RL_n such that L is isomorphic to RL_n/θ and the following conditions are satisfied:

1) $H_1 \vee \ldots \vee H_r \, \theta \, K_1 \vee \ldots \vee K_m$ if and only if for each $i \leq r$ there exists a principal element N in RL_n and $j \leq m$ such that $H_i \, \sigma \, N \leq K_j$, and for each $j \leq m$ there exists a principal element N' in RL_n and $i \leq r$ such that $K_j \, \sigma \, N' \leq H_i$.

2) $H \, \sigma \, N$ implies $HK \, \sigma \, NK$.

3) $HK \not\equiv 0 \pmod{\sigma}$ and $HK \, \sigma \, NK$ implies $H \, \sigma \, N$.

4) $X \, \theta \, I$ implies $X = I$.

Remark: Intuitively this theorem says that each distributive local Noether lattice arises by the identification of certain principal elements and then by the extension of these identifications to the whole lattice by join, multiplication, and cancellation.

Proof of Theorem 4.3. Suppose that L is a distributive local Noether lattice. Let $M = E_1 \vee \ldots \vee E_n$ be a minimal representation of the maximal element of L. By Lemma 4.1, each element of L has a representation as a join of products of powers of the elements E_i. Since two elements of RL_n are equal only if they are joins of the same principal elements by Corollary 2.2, and since, by residuation, two principal elements of RL_n are equal only if they are the same product of the same powers of the elements (x_i), it is possible to define a map φ from RL_n into L by defining $\varphi[(x_i)] = E_i$ and then extending this map to all of RL_n by the rules

$$\varphi(AB) = \varphi(A)\,\varphi(B)$$

$$\varphi(A \vee B) = \varphi(A) \vee \varphi(B) \quad .$$

Define an equivalence relation θ on RL_n by $A \theta B$ if and only if $\varphi(A) = \varphi(B)$. Let $\langle A \rangle$ denote the set of all elements of RL_3 equivalent to A mod θ. Define

(4.4) $\vee \langle A \rangle = \{A' \mid A' \theta A\}$,

(4.5) $\langle A \rangle \leq \langle B \rangle$ if and only if $A \leq \vee \langle B \rangle$,

(4.6) $\langle A \rangle \vee \langle B \rangle = \langle A \vee B \rangle$,

(4.7) $\langle A \rangle \wedge \langle B \rangle = \langle (\vee \langle A \rangle) \wedge (\vee \langle B \rangle) \rangle$, and

(4.8) $\langle A \rangle \langle B \rangle = \langle AB \rangle$.

It is immediate that (4.5) defines a partial ordering on the set L' of equivalence classes modulo θ. Since $\vee \langle X \rangle \theta X$ for all X in RL_n, and since $X \leq Y$ implies $\vee \langle X \rangle \leq \vee \langle Y \rangle$ for all X and Y in RL_n, (4.6) and (4.7) give the meet and join relative to the partial ordering given in (4.5) for each pair of elements in L'. With the multiplication given in (4.8), L' is a complete multiplicative lattice satisfying the ascending chair condition.

Consider the map $\varphi': L' \to L$ defined by $\varphi'(\langle A \rangle) = \varphi(A)$. By the definition of φ' and θ, $\varphi'(\langle A \rangle) = \varphi'(\langle B \rangle)$ implies $\varphi(A) = \varphi(B)$ which means that $\langle A \rangle = \langle B \rangle$. Thus φ' is 1 to 1. If X is in L, then $X = \varphi(Y)$ for some Y in RL_n which implies that $X = \varphi'(\langle Y \rangle)$. Therefore φ' is onto. Since $\langle A \rangle \vee \langle B \rangle = \langle A \vee B \rangle$ and $\langle A \rangle \langle B \rangle = \langle AB \rangle$, φ' preserves the partial order and multiplication of L'. Now assume

$$\varphi'(\langle A \rangle) \leq \varphi'(\langle B \rangle) \quad .$$

Then

$$\varphi'(\langle A\rangle \vee \langle B\rangle) = \varphi'(\langle A\rangle) \vee \varphi'(\langle B\rangle) = \varphi'(\langle B\rangle) \quad ,$$

so that $\langle A\rangle \vee \langle B\rangle = \langle B\rangle$ and $\langle A\rangle \leq \langle B\rangle$. Thus φ' and its inverse are order preserving which implies that φ' is a lattice isomorphism. Since φ' preserves multiplication, L and L' are isomorphic as multiplicative lattices and thus as Noether lattices.

Now let σ be the restriction of θ to the set of principal elements of RL_n. Then Corollary 2.2 implies that condition 1 of the theorem is satisfied. Condition 2 of the theorem follows immediately from the definition of θ. To verify condition 3, suppose $\langle 0\rangle \neq \langle HK\rangle = \langle NK\rangle$. Since L' is a Noether lattice, it follows by Equation (1.2) that

$$\langle H\rangle \vee \langle 0{:}K\rangle = \langle N\rangle \vee \langle 0{:}K\rangle \quad .$$

Because $\langle H\rangle, \langle N\rangle \not\leq \langle 0\rangle{:}\langle K\rangle$, a minimal representation of $\langle H\rangle \vee \langle 0\rangle{:}\langle K\rangle$ has the form

$$\langle H\rangle \vee \langle K_1 \vee \ldots \vee K_t\rangle \quad ,$$

and a minimal representation of $\langle N\rangle \vee \langle 0\rangle{:}\langle K\rangle$ has the form

$$\langle N\rangle \vee \langle K_1' \vee \ldots \vee K_t'\rangle \quad .$$

Then since $\langle H\rangle, \langle N\rangle \not\leq \langle 0\rangle{:}\langle K\rangle$, it follows by Corollary 2.2 that $H \sigma N$. Condition 4 follows from the fact that L' is local.

Now suppose that θ and σ are equivalence relations on RL_n such that conditions (1) - (4) are satisfied. We must verify that RL_n/θ may be regarded as a Neother lattice.

Denote the equivalence class of all elements congruent to A mod θ by $\langle A \rangle$. Define a partial ordering on $L = RL_n/\theta$ by using equations (4. 4) and (4. 5). The fact that (4. 5) does yield a partial ordering follows immediately from the fact that $\vee\langle X \rangle \, \theta \, X$ for all X in RL_3 and the fact that if $X \leq Y$ in RL_3, $\vee \langle X \rangle \leq \vee \langle Y \rangle$. The first of these is clear; to verify the second note that if $X' \, \theta \, X$, then $X' \vee Y \, \theta \, Y$ by condition 1 of the theorem. Then

$$(4.9) \quad \vee \langle X \rangle = \vee \{X' \mid X' \, \theta \, X\} \leq \vee \{X' \vee Y \mid X' \, \theta \, X\} \leq \vee \langle Y \rangle \quad .$$

It follows immediately that (4. 6) and (4. 7) give the join and meet of any two elements of L relative to the partial ordering given in (4. 5). It is clear that L is complete and is a multiplicative lattice with the multiplication given by (4. 8). We shall show that L is a Noether lattice by showing that L is distributive and that every element of L is a join of principal elements.

In order to show that L is distributive it is necessary to use

$$(4.10) \qquad \vee \langle A \vee B \rangle = (\vee \langle A \rangle) \vee (\vee \langle B \rangle) \quad .$$

Clearly $(\vee \langle A \rangle) \vee (\vee \langle B \rangle) \leq \vee \langle A \vee B \rangle$. Thus assume

$$C = K_1 \vee \dots \vee K_r \, \theta \, A \vee B \quad .$$

Let $A = E_1 \vee \dots \vee E_s$ and $B = H_1 \vee \dots \vee H_t$. Then, by condition 1, for each K_i there exist j and N_j such that either

$$K_i \, \theta \, N_i \leq E_j \text{ or } K_i \, \theta \, N_i \leq H_j \quad .$$

Thus by (4. 9) the elements K_i may be divided into two sets such that the join of the first is less than or equal to $\vee \langle A \rangle$, and the join of the second is less than or equal to $\vee \langle B \rangle$. Then $C \leq (\vee \langle A \rangle) \vee (\vee \langle B \rangle)$, and therefore

$$\vee \langle A \vee B \rangle = (\vee \langle A \rangle) \vee (\vee \langle B \rangle) \quad .$$

But this implies that L is distributive, for in the equations

$$\begin{aligned} \langle C \rangle \wedge (\langle A \rangle \vee \langle B \rangle) &= \langle C \rangle \wedge \langle A \vee B \rangle \\ &= (\vee \langle C \rangle) \wedge [(\vee \langle A \rangle) \vee (\vee \langle B \rangle)] \\ &= [(\vee \langle C \rangle) \wedge (\vee \langle A \rangle)] \vee [(\vee \langle C \rangle) \wedge (\vee \langle B \rangle)] \\ &= (\langle C \rangle \wedge \langle A \rangle) \vee (\langle C \rangle \wedge \langle B \rangle) \quad , \end{aligned}$$

the second and fourth lines follow from (4. 7) and the third line follows from the fact that RL_n is distributive.

Using (4. 5), (4. 8), and the definition of residuation we obtain the equations

(4. 11)
$$\begin{aligned} \langle A \rangle : \langle B \rangle &= \vee \{ \langle X \rangle \mid \langle X \rangle \langle B \rangle \leq \langle A \rangle \} \\ &= \vee \{ \langle X \rangle \mid XB \leq \vee \langle A \rangle \} \\ &= \vee \{ \langle X \rangle \mid X \leq (\vee \langle A \rangle) : B \} \\ &= \langle (\vee \langle A \rangle) : B \rangle \quad . \end{aligned}$$

We shall use (4. 11) to show that $\langle E \rangle$ is principal in L for each principal element E in RL_n. To show that $\langle E \rangle$ is meet principal it is necessary to prove that

(4. 12)
$$\langle A \rangle \wedge \langle E \rangle = (\langle A \rangle : \langle E \rangle) \langle E \rangle \quad .$$

Note first that if $H \theta K$ in RL_n, then $\langle H \rangle = \langle K \rangle$ and

$$\langle (\vee \langle A \rangle):H \rangle = \langle A \rangle:\langle H \rangle = \langle A \rangle:\langle K \rangle = \langle (\vee \langle A \rangle):K \rangle .$$

Multiplying the left side of this expression by $\langle H \rangle$ and the right side by $\langle K \rangle$ and applying equation (1.1) in RL_n, we obtain the equation

$$(4.13) \qquad \langle (\vee \langle A \rangle) \wedge H \rangle = \langle (\vee \langle A \rangle) \wedge K \rangle \quad .$$

Note also that since $\vee \langle E \rangle \theta E$, $\vee \langle E \rangle = E \vee K_1 \vee \ldots \vee K_r$ with $K_i \theta H_i \leq E$ by condition 1 of the theorem. Using this fact, equation (4.11), and applying (4.13) and the fact that $\langle A \vee B \rangle = \langle A \rangle \vee \langle B \rangle$ in the third line below, we obtain

$$\begin{aligned}\langle A \rangle \wedge \langle E \rangle &= \langle (\vee \langle A \rangle) \wedge (E \vee K_1 \vee \ldots \vee K_r) \rangle \\ &= \langle [(\vee \langle A \rangle) \wedge E] \vee [(\vee \langle A \rangle) \wedge K_1] \vee \ldots \vee [(\vee \langle A \rangle) \wedge K_r] \rangle \\ &= \langle [(\vee \langle A \rangle) \wedge E] \vee [(\vee \langle A \rangle) \wedge H_1] \vee \ldots \vee [(\vee \langle A \rangle) \wedge H_r] \rangle \\ &= \langle (\vee \langle A \rangle) \wedge (E \vee H_1 \vee \ldots \vee H_r) \rangle \\ &= \langle (\vee \langle A \rangle) \wedge E \rangle = \langle [(\vee \langle A \rangle):E]\, E \rangle \\ &= \langle A:E \rangle \langle E \rangle \quad .\end{aligned}$$

This proves equation (4.12). Now let $A = N_1 \vee \ldots \vee N_K$ in RL_n. In the equations

$$\begin{aligned}(\langle A \rangle \wedge \langle B \rangle : \langle E \rangle) \langle E \rangle &= [(\langle N_1 \rangle \vee \ldots \vee \langle N_K \rangle) \wedge \langle B \rangle : \langle E \rangle] \langle E \rangle \\ &= (\langle N_1 \rangle \wedge \langle B \rangle : \langle E \rangle) \langle E \rangle \vee \ldots \vee (\langle N_K \rangle \wedge \langle B:E \rangle) \langle E \rangle \\ &= \{[(\langle B \rangle : \langle E \rangle) : \langle N_1 \rangle] \langle N_1 \rangle\} \langle E \rangle \vee \ldots \vee \{[(\langle B \rangle : \langle E \rangle) : \langle N_K \rangle] \langle N_K \rangle\} \langle E \rangle\end{aligned}$$

$$= (\langle B\rangle : \langle EN_1\rangle)\langle EN_1\rangle \vee \ldots \vee (\langle B\rangle : \langle EN_K\rangle)\langle EN_K\rangle$$

$$= (\langle B\rangle \wedge \langle EN_1\rangle) \vee \ldots \vee (\langle B\rangle \wedge \langle EN_K\rangle)$$

$$= \langle B\rangle \wedge (\langle EN_1\rangle \vee \ldots \vee \langle EN_K\rangle)$$

$$= \langle B\rangle \wedge \langle E\rangle\langle N_1 \vee \ldots \vee N_K\rangle$$

$$= \langle B\rangle \wedge \langle E\rangle\langle A\rangle \quad ,$$

lines 2 and 6 follow from the distributive law, lines 3 and 5 follow from equation (4. 12), and line 4 follows from the fact that $(X:Y):Z = X:(YZ)$. Thus $\langle E\rangle$ is meet principal.

The first step in proving that $\langle E\rangle$ is join principal is proving

$$(4.14) \qquad (\langle B\rangle\langle E\rangle):\langle E\rangle = \langle B\rangle \vee \langle 0\rangle:\langle E\rangle .$$

If $\langle B\rangle\langle E\rangle = \langle 0\rangle$, then $\langle B\rangle \leq \langle 0\rangle:\langle E\rangle$ and (4. 14) holds. Assume $\langle B\rangle\langle E\rangle \neq 0$. It is always true that

$$(\langle B\rangle\langle E\rangle):\langle E\rangle \geq \langle 0\rangle:\langle E\rangle \vee \langle B\rangle \quad ,$$

so suppose that K is a principal element in RL_n such that $\langle K\rangle\langle E\rangle \leq \langle B\rangle\langle E\rangle$, with $\langle K\rangle\langle E\rangle \neq \langle 0\rangle$. Then $KE \vee BE \;\theta\; BE$ so that by property 1 of the theorem there exists $H' \;\theta\; KE$ such that $H' \leq BE$. But by Equation 1. 1 $H' = H' \wedge E = HE$ with $H = H':E$. Then $KE \;\theta\; HE$ implies $K \theta H \leq B$ by condition 3. Therefore $\langle K\rangle \leq \langle B\rangle$. Thus

$$\vee \{\langle K\rangle \mid \langle K\rangle\langle E\rangle \leq \langle B\rangle\langle E\rangle \text{ and } \langle K\rangle\langle E\rangle \neq \langle 0\rangle\} \leq \langle B\rangle \quad .$$

Therefore Equation (4. 14) holds. It now follows that $\langle E\rangle$ is join principal because in the equations

$$(\langle A\rangle \vee \langle B\rangle\langle E\rangle):\langle E\rangle = [(\langle A\rangle \vee \langle BE\rangle) \wedge \langle E\rangle]:\langle E\rangle$$
$$= [(\langle A\rangle \wedge \langle E\rangle) \vee \langle BE\rangle]:\langle E\rangle$$
$$= [(\langle A\rangle:\langle E\rangle)\langle E\rangle \vee \langle B\rangle\langle E\rangle]:\langle E\rangle$$
$$= [(\langle A\rangle:\langle E\rangle \vee \langle B\rangle)\langle E\rangle]:\langle E\rangle$$
$$= \langle A\rangle:\langle E\rangle \vee \langle B\rangle \vee \langle 0\rangle:\langle E\rangle$$
$$= \langle A\rangle:E \vee \langle B\rangle ,$$

line 1 follows from the fact that $(X \wedge Y):Y = X:Y$, line 2 follows from distributivity, line 3 follows from (4. 12), line 5 follows from (4. 14), and line 6 follows from the fact that $0:X \leq Y:X$ in any Noether lattice. This proves that $\langle E\rangle$ is principal, so that each element of L is a join of principal elements. Thus L is a distributive Noether lattice. L is local, for if $\langle A\rangle \vee \langle B\rangle = \langle I\rangle$, $A \vee B = I$ by condition 4 so that $A = I$ or $B = I$. This proves the theorem.

This theorem may be used to construct interesting examples of Noether lattices. For example, the lattice L obtained by identifying $(x_1)^2$ and $(x_2)^2$ in RL_2 is drawn schematically in Figure 4. 1. The dots indicate that the pattern above them is to be continued.

To verify that L is the lattice shown in Figure 4. 1, let $\langle (x_1)\rangle = E$ and $\langle (x_2)\rangle = H$. Note first that each principal element of L has a factorization of the form E^i or E^iH. Thus every nonprincipal element X of L has the form $X = E^i \vee E^jH$ where i is the smallest integer such that E^i is contained in X and j is the smallest integer such that $E^jH \leq X$. If $j \geq i$, $X = E^i$. If $j < i$,

$$E^i \vee E^jH = E^j(E^{i-j} \vee H) = \begin{cases} E^j(E \vee H) & \text{if } i - j = 1 \\ E^jH & \text{if } i - j > 1 \end{cases} .$$

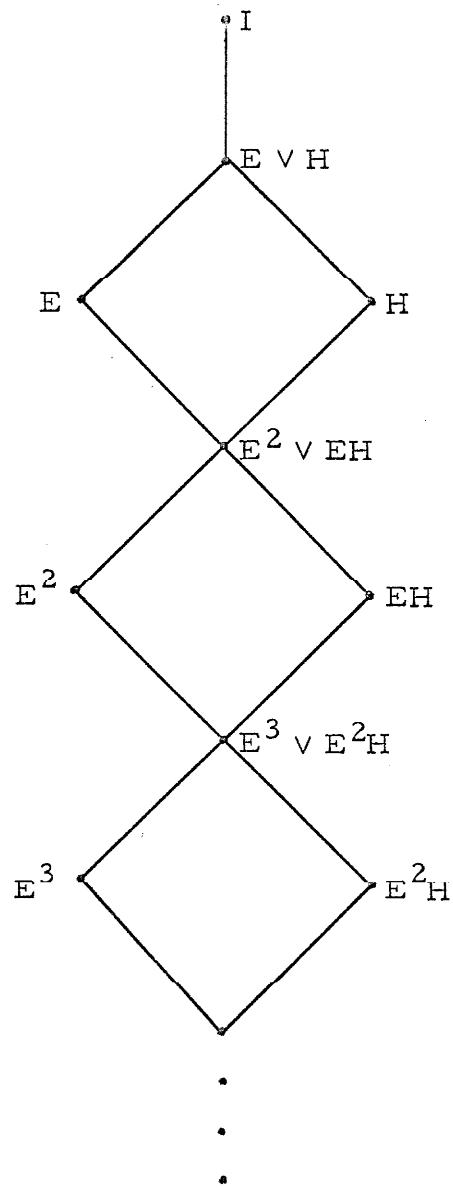

Figure 4. 1

The lattice obtained from RL_2 by identifying $(x_1)^2$ and $(x_2)^2$ as described in Theorem 4. 3.

Therefore each element of L has the form E^i, E^jH, or $E^k(E \vee H)$. Thus Figure 4.1 is the correct diagram for L.

Another interesting example which may be obtained from RL_2 is

$$L' = RL_2 / [(x_1) \vee (x_2)]^2 \quad .$$

This is the lattice obtained from RL_2 by identifying $(x_1)^2$, $(x_1)(x_2)$, and $(x_2)^2$ with 0 as described in Theorem 4.3. Also $L' = L/E^2 \vee EH$ where L is the lattice described above.

L' arises naturally as a quotient sublattice of any distributive local Noether lattice which is not a chain. This is a crucial point in the proof of the following theorem.

<u>Theorem 4.4.</u> <u>If L_R is the lattice of ideals of a local ring</u> R, <u>then L_R is distributive if and only if L_R is a chain.</u>

<u>Proof:</u> If the maximal ideal of R is principal, L_R is a chain. Suppose the maximal ideal of R has a minimal representation of the form $E_1 \vee \ldots \vee E_K$ with each E_i a principal ideal. Let $A = (E_1 \vee E_2)^2 \vee E_3 \vee \ldots \vee E_K$ and let X' denote $X \vee A$. Then it is easily shown that $E_1' \vee E_2'$, E_1', E_2', and 0' are distinct elements of L_R/A. For example, if $E_1' = E_2'$, then

$$\begin{aligned} I &= (E_1 \vee A):E_1 = (E_2 \vee A):E_2 \\ &= (E_1^2 \vee E_2 \vee \ldots \vee E_K):E_1 \\ &= E_1 \vee (E_2 \vee \ldots \vee E_K):E_1 \quad . \end{aligned}$$

Since L_R is local, $(E_2 \vee \ldots \vee E_K):E_1 = I$. This is impossible since it implies that $E_1 \leq E_2 \vee \ldots \vee E_K$.

Now suppose that $X' \neq I$ is an element of L_R/A. Then

$$X' = X' \wedge (E_1' \vee E_2') = (X' \wedge E_1') \vee (X_2' \wedge E_2')$$
$$= (X':E_1')E_1' \vee (X':E_2'):E_2' \quad .$$

But if $X':E_j' \neq I'$, $(X':E_j')E_j' = 0'$. Thus I', $E_1' \vee E_2'$, E_1', E_2', and $0'$ are the only elements of L_R/A. L_R/A is the lattice of ideals of the ring R/A. Since E_1' and E_2' are join-irreducible, they must be principal ideals. Let $E_1' = (x)$ and $E_2' = (y)$. Since $(x) \neq (y)$, $(x+y) \neq 0$. Also $(x+y) \neq (x)$, for if $x + y = rx$, y is in (x) which is impossible. If $(x+y) = (x, y)$, (x, y) is principal and by Corollary 2. 2, $(x, y) = (x)$ or $(x, y) = (y)$. Again this is impossible, so that L_R/A cannot be the lattice of all ideals of any ring. Thus the maximal ideal of R is principal and L_R is a chain.

Theorem 3. 3 asserts that there is a Noether-lattice imbedding of a distributive regular local Noether lattice of dimension n in each regular local Noether lattice of dimension n. This might lead us to hope that there is a Noether-lattice imbedding of a distributive local Noether lattice whose maximal element is a join of n principal elements in each local Noether lattice whose maximal element is a join of n principal elements. Unfortunately this is not the case, as the following example shows. Let F denote the real numbers, and let

$$R = F[x, y]_{(x,y)} / (x, y)(x^2 + y^2) \quad ,$$

following the usual notation for factor rings and rings of quotients [4]. Let L be the lattice of ideals of R. If $A \neq I$ is in L and $A \not\leq (x^2+y^2)$, then $0:A = (x^2+y^2)$, since (x^2+y^2) is a prime. Suppose $E \vee F = (x, y)$. By the exchange properties of the Kurosh-Ore theorem mentioned after Corollary 2.2, $E \not\leq (x^2+y^2)$, $F \not\leq (x^2+y^2)$. If E and F were members of a distributive sublattice L' of L which had a Noether-lattice imbedding in L, then (x^2+y^2) would be in L' also since $(x^2+y^2) = 0:E$. But then since $(x^2+y^2):E = (x^2+y^2)$,

$$\begin{aligned}(x^2+y^2) &= (x^2+y^2) \wedge (E \vee F) = [(x^2+y^2) \wedge E] \vee [(x^2+y^2) \wedge F] \\ &= [(x^2+y^2):E]E \vee [(x^2+y^2):F]F \\ &= (x^2+y^2)E \vee (x^2+y^2)F = 0 \quad .\end{aligned}$$

However $(x^2+y^2) \neq 0$, so that there is no Noether-lattice imbedding in L of a distributive local Noether lattice whose maximal element is a join of two principal elements.

CHAPTER V

NOETHER LATTICES WITH TRIVIAL MULTIPLICATION

On the basis of the results of Chapter III and Chapter IV, one might suspect that every Noether lattice has a Noether-lattice imbedding in the lattice of ideals of some ring. The purpose of this chapter is to show that this is not the case. The example given in this chapter has the trivial multiplication defined by $AB = A$ if $B = I$, $AB = 0$ if $A, B \neq I$. The next theorem classifies all those Noether lattices with this multiplication and will be useful in the construction of the example mentioned.

Theorem 5. 1. A lattice L may be represented as a Noether lattice with the trivial multiplication if and only if L is a finite-dimensional modular lattice in which every element but I is a join of atoms.

Proof: Suppose that L is a Noether lattice with the trivial multiplication. Note first that L is local, for if $I = A \vee B$ and $A, B \neq I$, then $IX = AX \vee BX = 0$ for all $X \neq I$ in L. Thus $A = 0$, $B = 0$ and $I = 0$. Now let $E \neq I$ be principal in L. Then (using equation 1. 1) if $X < E$, $X = (X:E)E = 0$ since $(X:E) \neq I$. Thus the proper nonzero principal elements of L are atoms and every element of L but I is a join of atoms. Let M be the proper maximal element of L. Since $M^2 = 0$, $L = L/M^2$. By Theorem 6. 2 of [2] L/M^2 is finite dimensional. Therefore L is finite dimensional.

Now suppose that L is a finite dimensional modular lattice in which every element but I is a join of atoms. Since I is join-irreducible, the trivial multiplication on L is a commutative, associative, join-distributive multiplication. To show that L is a Noether lattice, we shall show that the atoms of L are principal. In fact every element of L is join-principal. Since the multiplication is trivial, it is clear that

$$(5.1) \qquad X:Y = \begin{cases} M & \text{if} \quad Y \not\leq X \\ I & \text{if} \quad Y \leq X \end{cases}$$

for all X and Y in L, where M is the proper maximal element of L. Now let A and C be elements of L. Then

$$(A \vee IC):C = (A \vee C):C = I = A:C \vee I \quad .$$

If $B \neq I$, then

$$(A \vee BC):C = A:C = A:C \vee B \quad ,$$

since $A:C = I$ or $A:C = M \geq B$. Thus C is join principal. Now let E be an atom of L. If $B \not\geq E$, then $B:E = M$. Therefore

$$(A \wedge B:E)E = (A \wedge M)E = 0 = AE \wedge B \quad ,$$

since $AE \wedge B \leq E \wedge B$ which is zero because E is an atom and $E \not\leq B$. If $B \geq E$,

$$(A \wedge B:E)E = (A \wedge I)E = AE = AE \wedge B \quad ,$$

because $AE \leq E \leq B$. Thus E is meet principal. Since the atoms of

L are principal and every element of L but I is a join of atoms, and since I and 0 are trivially principal, L is a Noether lattice.

We shall now construct a lattice L which cannot be imbedded in the lattice of ideals of any ring. The elements of L are the symbol I and the elements of the lattice of subspaces of a non-Desarguesian projective plane (for example the Moulton plane). The partial ordering in L is given by $I > X$ for all $X \neq I$ and by the partial ordering of the lattice of subspaces of the plane. Let the maximum element of the lattice of subspaces of the plane be M. Then $M/0$ is a finite dimensional complemented modular lattice, so that every element of L but I is a join of atoms and by Theorem 5.1, L may be regarded as a Noether lattice with the trivial multiplication.

Now suppose that φ is a Noether-lattice imbedding of L in L_R, the lattice of ideals of a Noetherian ring R. Since M is prime and each $X \leq M$ is M-primary, the fact that φ is a Noether-lattice imbedding implies that $\varphi(M)$ is prime and $\varphi(X)$ is $\varphi(M)$-primary for all $X \leq M$ in L. Because of this, we may assume without loss of generality that

$$R = [R/\varphi(0)]_{(\varphi(M))} .$$

Thus L_R may be assumed to have the trivial multiplication. But for any ring R with maximal ideal P, P/P^2 is a vector space over the field R/P [4, p. 52]. This means that $\varphi(M)/0$ is a vector space. But the lattice of subspaces of a vector space is Arguesian, so that the lattice of ideals of R has no non-Arguesian sublattices. But L is non-Arguesian, so that L cannot be imbedded in L_R.

CHAPTER VI

AN ABSTRACT CHARACTERIZATION OF RL_3

The example given in the last chapter does not disprove the conjecture that every regular local Noether lattice L has a Noether-lattice imbedding in the lattice of ideals of some ring. The next theorem shows that if L satisfies certain stringent conditions, then the conjecture is valid. In fact, these conditions imply that L is distributive.

Theorem 5.1. Let L be a regular local Noether lattice with precisely 3 minimal (nonzero) primes. If each minimal prime is a regular parameter for L, L is isomorphic to RL_3.

Proof: If L has dimension 3, there is a Noether-lattice imbedding of RL_3 in L, so that $\varphi[(x_1)]$, $\varphi[(x_2)]$, and $\varphi[(x_3)]$ are the minimal primes of L.

Let K be a principal element of L. By the principal element theorem [2], K is contained in one of the three minimal primes in L. K is a multiple of this minimal prime since the minimal prime is principal. Thus if K cannot be factored, K is equal to the minimal prime. If K can be factored, each of its product-irreducible factors may be considered to be principal by Lemma 2.2. Thus K is a product of powers of the minimal primes. This means that the imbedding of RL_3 in L is an onto map so that L is isomorphic to RL_3.

Clearly L cannot have dimension 1, because if it did it would have precisely one minimal nonzero prime. (See the remark before Theorem 3.3).

The remainder of the proof consists of showing that L cannot have dimension 2. Thus suppose L has dimension 2. Denote the minimal primes of RL_3 by E, F, and H, and the minimal primes of L by e, f, and h. As in Theorems 3.3 and 4.3 define a map φ from RL_3 into L by letting $\varphi(E) = e$, $\varphi(F) = f$ and $\varphi(H) = h$, and then extending this map by preserving joins and products. As we remarked above, every principal element of L is a product of powers of e, f, and h, so that φ is an onto map. For the remainder of this proof we will denote the elements of RL_3 by capital letters and the images of these elements under φ by the corresponding small letters with the exception that $\varphi(I) = 1$. Thus e, f, h, k, and n are principal elements in L. The letters p, q, r, s, t, u, and v will be reserved for integers.

Note that again as in Theorem 4.3,

$$a \wedge b = \varphi[(\vee \varphi^{-1}(a)) \wedge (\vee \varphi^{-1}(b))] \quad , \tag{6.1}$$

where $\varphi^{-1}(x)$ is the inverse image of x and $\vee \varphi^{-1}(x)$ is the join of this set.

Let m denote the maximal element of L. By the definition of a regular parameter,

$$e \vee f = m \quad \text{or} \quad e \vee h = m \; ,$$
$$f \vee e = m \quad \text{or} \quad f \vee h = m \; , \quad \text{and}$$
$$h \vee e = m \quad \text{or} \quad h \vee f = m \; .$$

By inspection of the three relations given above it is clear that without loss of generality we may assume that

$$e \vee f = e \vee h = m = e \vee f \vee h \quad . \tag{6.2}$$

Since L/h and L/f are regular of dimension one, they are chains (as described in the remark made before Theorem 3.3) so that there exist integers p and q for which

$$h \vee f = e^p \vee f = e^q \vee h \quad .$$

If $p \leq q$, Equation (1.2) and the fact that h is prime imply that $e^{q-p} \vee h = 1$ which implies that $p = q$. Since the same type of computation may be made if $q \leq p$, $p = q$ and

$$h \vee f = e^q \vee f = e^q \vee h \quad . \tag{6.3}$$

The computations used to prove the theorem will differ for $q = 1$ and $q > 1$, but the following relation holds in both cases.

$$\vee\, \varphi^{-1}(e \vee f \vee h)^r \leq (E \vee F \vee H)^r \quad . \tag{6.4}$$

To prove this, suppose $E^tF^uH^v \leq \vee\, \varphi^{-1}(e \vee f \vee h)^r$. Then $e^tf^uh^v \leq (e \vee f \vee h)^r$, and hence

$$\begin{aligned} f^uh^v \leq (e \vee f)^r : e^t &= (e^r \vee e^{r-1}f \vee \ldots \vee f^r) : e^t \\ &= (e^{r-1} \vee e^{r-2}f \vee \ldots \vee f^{r-1} \vee f^r) : e^{t-1} \\ &= \begin{cases} 1 & \text{if } t \geq r \\ (e \vee f)^{r-t} & \text{if } t < r \end{cases} \end{aligned}$$

Similarly $h^v \leq (e \vee h)^{r-t-u}$ unless $t+u \geq r$. Then unless

$$1 = (e \vee h)^{r-t-u} : h^v = (e \vee h)^{r-t-u-v} \quad ,$$

$t+u+v \geq r$. Thus $t+u+v \geq r$ and $E^t F^u H^v \leq (E \vee F \vee H)^r$, proving (6. 4).

Also,

$$(6.5) \qquad e^r \vee f^r = e^r \vee (e \vee f)^r = e^r \vee (e \vee h)^r = e^r \vee h^r \quad .$$

Now assume $q = 1$. We will show that one of the equations

$$(6.6) \qquad e^5 \vee h^3 = e^5 \vee f^3 \, ,$$

$$(6.7) \qquad e^5 \vee h^2 = e^5 \vee f^2 \, , \qquad \text{or}$$

$$(6.8) \qquad e^5 \vee h = e^5 \vee f$$

must hold. Notice that (6. 6) implies that

$$e^5 \vee h^3 \geq h^3 \vee f^3 \geq (h \vee f)^5 = (e \vee h \vee f)^5 \geq e^4 h \quad .$$

Thus $1 = (e^5 \vee h^3) : e^4 h = e \vee h^2$ by (1. 2) and the fact that e and h are prime. This is impossible, so that (6. 6) cannot hold. Similarly (6. 7) and (6. 8) cannot hold, so showing that one of these must hold will show that $q = 1$ is impossible. Note that one of (6. 6), (6. 7), or (6. 8) would be implied by a relation of the form

$$(6.9) \qquad e^5 \vee h^3 f^2 = e^5 \vee h^u f^v \qquad \text{with} \qquad u + v = 5 \qquad \text{and} \qquad u \neq 3 \, ,$$

which means that it is sufficient to show that (6. 9) holds.

Since every element of RL_3 is a join of principal elements,

$$(6.10) \qquad \vee \varphi^{-1}(e^5 \vee h^3 f^2) = E^5 \vee H^3 F^2 \vee K_1 \vee \dots \vee K_t \quad ,$$

for suitable principal elements $K_1, \ldots, K_t$.

We will show that there exists a p such that $e^5 \vee h^3f^2 = e^5 \vee k_p$ and that k_p must be of the form given in (6. 9).

It is easily checked that

$$(e \vee f)^5 = (e^3 \vee f^3)(e \vee f)^2 = (e^3 \vee f^3)(e \vee h)^2 \quad .$$

Thus expanding this equation and using modularity,

$$\begin{aligned} e^5 \vee h^3f^2 &= (e^5 \vee h^3f^2) \wedge (e \vee f)^5 = (e^5 \vee h^3f^2) \wedge (e^3 \vee f^3)(e \vee h)^2 \\ &= (e^5 \vee h^3f^2) \wedge (e^5 \vee e^2f^3 \vee e^4h \vee ef^3h \vee e^3h^2 \vee f^3h^2) \\ &= e^5 \vee [(e^5 \vee h^3f^2) \wedge (e^2f^3 \vee e^4h \vee ef^3h \vee e^3h^2 \vee f^3h^2)] \\ &= e^5 \vee [(\vee \varphi^{-1}(e^5 \vee h^3f^2)) \wedge A] \quad ; \end{aligned}$$

the last line follows from (6. 1) with $A = \vee \varphi^{-1}(e^2f^3 \vee e^4h \vee ef^3h \vee e^3h^2 \vee f^3h^2)$. Using (6. 10) and the fact that RL_3 is distributive,

$$(6.\,11) \quad e^5 \vee h^3f^2 = e^5 \vee \varphi(H^3F^2 \wedge A) \vee \varphi(K_1 \wedge A) \vee \ldots \vee \varphi(K_n \wedge A) \quad .$$

Define a principal element to be of degree p if it equals $E^rF^sH^t$ or $e^rf^sh^t$ with $r+s+t = p$. We will now show that each principal element contained in $\varphi(H^3F^2 \wedge A)$ has degree 6 or more. If $H^3F^2 \not\leq A$, then this is the case. Thus we wish to show that $\varphi(A):h^3f^2 \neq 1$. In the equations

$$\begin{aligned} \varphi(A):h^3f^2 &= (e^4h \vee e^3h^2 \vee e^2f^3 \vee ef^3h \vee f^3h^2):h^3f^2 \\ &= (e^4h \vee e^3fh \vee e^2f^3 \vee ef^3h \vee f^3h^2):h^3f^2 \\ &= (e^3h \vee e^2f \vee efh \vee fh^2):h^3 \end{aligned}$$

$$= (e^3 \vee ef \vee fh):h^2$$

$$= (eh^2 \vee fh \vee ef):h^2$$

$$= e \vee f \neq 1 \quad ,$$

the second line follows from the relation $e^3h(e \vee h) = e^3h(e \vee f)$, the third and fourth lines follow by repeated application of (1. 2), the fifth line follows from the relation

$$e(e^2 \vee f) = e[(e \vee f)^2 \vee f] = e[(h \vee f)^2 \vee f] = e(h^2 \vee f) \quad ,$$

and the last line follows by repeated application of (1. 2).

Thus if none of the elements K_s have degree five,

$$e^5 \vee h^3f^2 \leq e^5 \vee (e \vee f \vee h)^6 = e^5 \vee (e \vee h)^6 \quad . \tag{6. 12}$$

Using (6. 12) and applying (1. 2) repeatedly, we obtain

$$1 = [e^5 \vee (e \vee h)^6]:h^3f^2 = (e^5 \vee e^4h^2 \vee e^3h^3 \vee e^2h^4 \vee eh^5 \vee h^6):h^3f^2$$

$$= (e^3 \vee e^2h \vee eh^2 \vee h^3):f^2 = (e \vee h)^3:f^2$$

$$= (e \vee f)^3:f^2 = (e^3 \vee e^2f \vee ef^2 \vee f^3):f^2 = e \vee f \quad .$$

This is impossible, so that one of the elements K_s must have degree five by (6. 4), and $\varphi(K_s \wedge A)$ must contain a principal element of degree 5, say n. Let $n_1, \ldots, n_r$ be the principal elements of degree 5 contained in at least one of $\varphi(K_1 \wedge A), \ldots, \varphi(K_t \wedge A)$. Then

$$e^5 \vee h^3f^2 = e^5 \vee n_1 \vee \ldots \vee n_r \vee c \tag{6. 13}$$

where $c \leq (e \vee f \vee h)^6$. Thus since (6. 12) cannot hold, Lemma 2. 2

implies that there is some $n \in \{n_1, \ldots, n_r\}$ such that

$$e^5 \vee h^3f^2 = e^5 \vee n \quad . \tag{6.13}$$

It is clear that $n \leq e$ since $h^3f^2 \not\leq e$. But $n \neq h^3f^2$ because $n \leq \varphi(K_s \wedge A) \leq \varphi(A)$ and $h^3f^2 \not\leq \varphi(A)$. This proves (6.9) and the case $q = 1$ is complete.

Now suppose that $q > 1$. Recall that this means that $e^q \vee f = f \vee h = e^q \vee h$. We shall deal with this case by showing that

$$e^{q^2+1} \vee h^{q-p} = e^{q^2+1} \vee f^{q-p} \qquad \text{with} \qquad q > p \geq 0 \quad . \tag{6.14}$$

For if (6.14) holds, we have

$$\begin{aligned} e^{q^2+1} \vee h^{q-p} &\geq h^{q-p} \vee f^{q-p} \geq (h \vee f)^{2(q-p)-1} \\ &= (e^q \vee h)^{2(q-p)-1} \geq e^{q(q-p)}h^{q-p-1} \quad . \end{aligned}$$

But then, applying (1.2) and the fact that e and h are prime, we have

$$\begin{aligned} 1 &= (e^{q^2+1} \vee h^{q-p}) : e^{q(q-p)}h^{q-p-1} \\ &= e^{1+qp} \vee h \quad , \end{aligned}$$

which is impossible.

An important relation we will use in proving (6.14) is

$$f^q \vee h^q = f^q \vee e^{q^2} \quad . \tag{6.15}$$

By equation (6.5), $e^q \vee f^q = e^q \vee f^q \vee h^q$, so that by modularity

$$h^q \vee f^q = (h^q \vee f^q) \wedge (e^q \vee f^q) = f^q \vee [e^q \wedge (h^q \vee f^q)] \quad . \tag{6.16}$$

Let

$$\vee \varphi^{-1}(h^q \vee f^q) = H^q \vee F^q \vee K_1 \vee \ldots \vee K_t \quad .$$

Since a principal element is contained in e^q if and only if it is a multiple of e^q and since by the uniqueness theorems for primary decompositions [2] each principal element has a unique representation as a product of powers of e, f, and h, $\vee \varphi^{-1}(e^q) = E^q$. Thus (6. 16) becomes

$$h^q \vee f^q = f^q \vee \varphi(E^q H^q) \vee \varphi(K_1 \wedge E^q) \vee \ldots \vee \varphi(K_t \wedge E^q) \quad .$$

Since $h^q \vee f^q \neq f^q \vee e^q h^q$ by residuation by h^q and (1. 2), Lemma 2. 2 implies that there exists an s such that

$$h^q \vee f^q = f^q \vee \varphi(K_s \wedge E^q) \quad .$$

But $K_s \not\leq F$, for if it were, h^q would be contained in f, and $h^q \not\leq f$. Thus $K_s = E^p H^r$. If $r \geq q$ residuation by h^r would give

$$1 = (h^q \vee f^q):h^r = (f^q \vee h^r e^{\max(p,q)}):h^r$$

$$= f^q \vee e^{\max(p,q)} \quad ,$$

and this is impossible. Thus letting $u = \max(p, q)$,

$$h^q \vee f^q = f^q \vee h^r e^u$$

with $r < q$. Residuation of both sides by h^r gives

$$(6.17) \qquad h^{q-r} \vee f^q = f^q \vee e^u \quad .$$

Joining f to both sides of (6. 17) yields

$$h^{q-r} \vee f = f \vee e^{u} \quad ,$$

but since $f \vee h^{q-r} = f \vee (f \vee h)^{q-r} = f \vee (f \vee e^{q})^{q-r}$, $u = q(q-r)$.

Now joining h to both sides of (6.17)

$$\begin{aligned} h \vee f^{q} &= h \vee f^{q} \vee e^{q(q-r)} \\ &= h \vee f^{q} \vee (h \vee e^{q})^{q-r} \\ &= h \vee f^{q} \vee (h \vee f)^{q-r} \\ &= h \vee f^{q-r} \quad . \end{aligned}$$

Residuating both sides by f^{q-r},

$$h \vee f^{r} = 1 \quad ,$$

which implies that $r = 0$. Therefore (6.15) holds.

We shall show that (6.14) holds by demonstrating that

(6.18) $$e^{q^2+1} \vee fh^{q} = e^{q^2+1} \vee f^{u}h^{v}$$

with $u+v = k+1$ and $u \neq 1$. Note that

(6.19) $$\begin{aligned} fh^{q} &\leq (e^{q^2} \vee h^{q})(e \vee f) \\ &= (e^{q^2} \vee f^{q})(e \vee h) = e^{q^2+1} \vee ef^{q} \vee e^{q^2}h \vee f^{q}h \quad , \end{aligned}$$

by substituting (6.15) and (6.2) into the first line.

By the modular law and (6.19),

(6.20) $$\begin{aligned} e^{q^2+1} \vee fh^{q} &= (e^{q^2+1} \vee fh^{q}) \wedge (e^{q^2+1} \vee ef^{q} \vee e^{q^2}h \vee f^{q}h) \\ &= e^{q^2+1} \vee \varphi[(\vee\, \varphi^{-1}(e^{q^2+1} \vee fh^{q})) \wedge A] \end{aligned}$$

where $A = \vee \varphi^{-1}(ef^q \vee e^{q^2}h \vee f^q h)$. Let

$$\vee \varphi^{-1}(e^{q^2+1} \vee fh^q) = E^{q^2+1} \vee FH^q \vee K_1 \vee \ldots \vee K_t \quad .$$

Then substituting into (6. 20),

$$e^{q^2+1} \vee fh^q = e^{q^2+1} \vee \varphi[(E^{q^2+1} \wedge A) \vee (FH^q \wedge H) \vee (K_1 \wedge A) \vee \ldots \vee (K_t \wedge A)]$$

$$= e^{q^2+1} \vee \varphi(FH^q \wedge A) \vee \varphi(K_1 \wedge A) \vee \ldots \vee \varphi(K_t \wedge A) \quad .$$

We shall now show that any principal element contained in $\varphi(FH^q \wedge A)$ has degree at least $q + 2$ by showing that $fh^q \not\leq \varphi(A)$. In the relations

$$\varphi(A):fh^q = (ef^q \vee e^{q^2}h \vee f^q h):fh^q = (ef^{q-1} \vee e^{q^2}h \vee f^{q-1}h):h^q$$

$$= (ef^{q-1} \vee e^{q^2} \vee f^{q-1}):h^{q-1} = (e^{q^2} \vee f^{q-1}):h^{q-1}$$

$$\leq (e^{q^2} \vee f):h^{q-1} = [(e^q \vee f)^q \vee f]:h^{q-1}$$

$$= [(h \vee f)^q \vee f]:h^{q-1} = [h^q \vee f]:h^{q-1}$$

$$= h \vee f \neq 1 \quad ,$$

the first two lines follow by (1. 2) and the third follows from the fact that if, in a multiplicative lattice, $X \leq Y$, then $X:Z \leq Y:Z$. Thus unless there is an s such that $\varphi(K_s \wedge A)$ contains an element of degree $q + 1$ (by (6. 4) it contains no elements of degree q or less),

$$e^{q^2+1} \vee fh^q \leq e^{q^2+1} \vee (e \vee f \vee h)^{q+2} \quad . \tag{6. 21}$$

This however implies that $e \vee fh^q \leq e \vee (f \vee h)^{q+2}$, or by (6. 3)

$$e \vee fh^q \leq e \vee (e^q \vee h)^{q+2} = e \vee h^{q+2}$$

$$= e \vee (e \vee h)^{q+2} = e \vee (e \vee f)(e \vee h)^{q+1} = e \vee fh^{q+1} \quad .$$

Residuating both sides by fh^q we obtain $1 = e \vee h$ which is impossible, so that (6. 21) does not hold.

Because (6. 21) cannot hold,

$$e^{q^2+1} \vee fh^q \neq e^{q^2+1} \vee \varphi(FH^q \wedge A) \quad ,$$

so that by Lemma 2. 2 there exists an s such that

$$e^{q^2+1} \vee fh^q = e^{q^2+1} \vee \varphi(K_s \wedge A) \quad .$$

Since $fh^q \not\leq e$, $\varphi(K_s \wedge A) \not\leq e$. Therefore because (6. 21) cannot hold, there exists $n \not\leq e$ of degree $q + 1$ such that $n \leq \varphi(K_s \wedge A)$ and

$$e^{q^2+1} \vee fh^q = e^{q^2+1} \vee n \quad .$$

But $n \neq fh^q$ since $n \leq \varphi(K_s \wedge A) \leq \varphi(A)$ and $fh^q \not\leq \varphi(A)$. But $n = f^u h^v$ with $u+v = q+1$, so that (6. 18) holds and the case $q > 1$ is impossible. Therefore L cannot have dimension 2 and L is isomorphic to RL_3.

REFERENCES

1. R. P. Dilworth, Note on the Kurosh Ore Theorem, Bull. Amer. Math. Soc., 52 (1946), 659.

2. R. P. Dilworth, Abstract Commutative Ideal Theory, Pacific J. Math., 12 (1962), 481-498.

3. P. J. McCarthy, Note On Abstract Commutative Ideal Theory, Amer. Math. Monthly, 74 (1967), 706-707.

4. D. G. Northcott, Ideal Theory, Cambridge Tracts in Mathematics and Mathematical Physics, 42. Cambridge University Press, Cambridge, 1953.

5. O. Zariski and P. Samuel, Commutative Algebra, Vol. 2. Van Nostrand, Princeton, 1960.